

Generalization of some commutative perturbation results

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Abstract

We study the stability of certain spectra under some algebraic conditions weaker than the commutativity and we generalize many known commutative perturbation results.

1 Introduction

In [6], the authors investigated how to explicitly express the Drazin inverse of the sum $(P + Q)$ of two complex matrices P and Q , under the conditions $PQ \in \text{comm}(P)$ and $QP \in \text{comm}(Q)$, which are weaker than the commutativity of P with Q . A few years later, Huihui Zhu et al. [13] obtained the representations for the pseudo Drazin inverse of the sum and the product of two elements of a complex Banach algebra, under the same conditions. Recently, H. Zou et al. [14] extended the known expressions for the generalized Drazin inverse of the product and the sum of two elements of a complex Banach algebra by considering the same conditions.

In this paper, we study these conditions and other in a ring \mathcal{A} , that are $ab \in \text{comm}(a)$, $ba \in \text{comm}(b)$, $ab \in \text{comm}(b)$ and $ba \in \text{comm}(a)$. After giving some algebraic results, we focus on the Banach algebra of bounded linear operators $L(X)$ acting on the complex Banach algebra X . We generalize some commutative perturbation spectral results, in particular, if N is nilpotent and $N \in \text{comm}_r(T)$ (i.e. $NT \in \text{comm}(T)$ and $TN \in \text{comm}(N)$), then $\sigma_*(T) \setminus \{0\} = \sigma_*(T + N) \setminus \{0\}$, where $\sigma_* \in \{\sigma_p, \sigma_p^0, \sigma_a\}$. If in addition $N \in \text{comm}_w(T)$ (i.e. $N \in \text{comm}_r(T)$ and $N^* \in \text{comm}_r(T^*)$), then we deduce by duality that $\sigma_*(T) = \sigma_*(T + N)$, where $\sigma_* \in \{\sigma_p, \sigma_p^0, \sigma_a, \sigma_s, \sigma\}$. This allows us to show that if K is a power compact operator and $K \in \text{comm}_w(T)$, then $\sigma_*(T) = \sigma_*(T + K)$, where $\sigma_* \in \{\sigma_{uf}, \sigma_{uw}, \sigma_{ub}, \sigma_e, \sigma_w, \sigma_b\}$. Note that here we lose quite a few commutative properties, for example if $S \notin \text{comm}(T)$ and $S \in \text{comm}_w(T)$, then $S \notin \text{comm}_w(T - \lambda I)$ for every $\lambda \neq 0$.

2 Terminology and preliminaries

Let \mathcal{A} be a ring and let $a \in \mathcal{A}$. Denote by $\text{comm}(a)$ the set of all elements that commute with a , by $\text{comm}^2(a) = \text{comm}(\text{comm}(a))$ and by $\text{Nil}(\mathcal{A})$ the nilradical of \mathcal{A} . If in addition \mathcal{A} is a complex Banach algebra with unit e , then we mean by $\sigma(a)$, $\text{acc}\sigma(a)$, $r(a)$ and $\exp(a)$, the spectrum, the accumulation point of $\sigma(a)$, the spectral radius of a and the exponential of a , respectively. We

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say that a is quasi-nilpotent if $r(a) = 0$. In the case of $\mathcal{A} = L(X)$, the algebra of all bounded linear operators acting on an infinite dimensional complex Banach space X , T^* , $\alpha(T)$ and $\beta(T)$ means respectively, the dual of the operator $T \in \mathcal{A}$, the dimension of the kernel $\mathcal{N}(T)$ and the codimension of the range $\mathcal{R}(T)$. Denote further $\mathcal{R}(T^\infty) := \bigcap_{n \geq 0} \mathcal{R}(T^n)$ and $\mathcal{N}(T^\infty) := \bigcup_{n \geq 0} \mathcal{N}(T^n)$. The ascent and the descent of T are defined by $p(T) = \inf\{n \in \mathbb{N} : \mathcal{N}(T^n) = \mathcal{N}(T^{n+1})\}$ (with $\inf \emptyset = \infty$) and $q(T) = \inf\{n \in \mathbb{N} : \mathcal{R}(T^n) = \mathcal{R}(T^{n+1})\}$. A subspace M of X is T -invariant if $T(M) \subset M$ and the restriction of T on M is denoted by T_M . $(M, N) \in \text{Red}(T)$ if M, N are closed T -invariant subspaces and $X = M \oplus N$ ($M \oplus N$ means that $M \cap N = \{0\}$). Let $n \in \mathbb{N}$, denote by $T_{[n]} = T_{\mathcal{R}(T^n)}$ and by $m_T = \inf\{n \in \mathbb{N} : \inf\{\alpha(T_{[n]}), \beta(T_{[n]})\} < \infty\}$ the *essential degree* of T . T is called upper semi-B-Fredholm (resp., lower semi-B-Fredholm) if the *essential ascent* $p_e(T) := \inf\{n \in \mathbb{N} : \alpha(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{p_e(T)+1})$ is closed (resp., the *essential descent* $q_e(T) := \inf\{n \in \mathbb{N} : \beta(T_{[n]}) < \infty\} < \infty$ and $\mathcal{R}(T^{q_e(T)})$ is closed). If T is an upper or a lower (resp., upper and lower) semi-B-Fredholm then T is called *semi-B-Fredholm* (resp., *B-Fredholm*) and its index is defined by $\text{ind}(T) = \alpha(T_{[m_T]}) - \beta(T_{[m_T]})$. T is said to be an upper semi-B-Weyl (resp., a lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) if T is an upper semi-B-Fredholm with $\text{ind}(T) \leq 0$ (resp., T is a lower semi-B-Fredholm with $\text{ind}(T) \geq 0$, T is a B-Fredholm with $\text{ind}(T) = 0$, T is an upper semi-B-Fredholm and $p(T_{[m_T]}) < \infty$, T is a lower semi-B-Fredholm and $q(T_{[m_T]}) < \infty$, $p(T_{[m_T]}) = q(T_{[m_T]}) < \infty$). If T is upper semi-B-Fredholm (resp., lower semi-B-Fredholm, semi-B-Fredholm, B-Fredholm, upper semi-B-Weyl, lower semi-B-Weyl, B-Weyl, left Drazin invertible, right Drazin invertible, Drazin invertible) with essential degree $m_T = 0$, then T is said to be an upper semi-Fredholm (resp., lower semi-Fredholm, semi-Fredholm, Fredholm, upper semi-Weyl, lower semi-Weyl, Weyl, upper semi-Browder, lower semi-Browder, Browder) operator. T is said to be bounded below if T is upper semi-Fredholm with $\alpha(T) = 0$.

$\sigma_p(T)$: point spectrum of T

$\sigma_p^0(T) := \{\lambda \in \sigma_p(T) : \alpha(T - \lambda I) < \infty\}$

$\sigma_a(T)$: approximatif spectrum of T

$\sigma_s(T)$: surjectif spectrum of T

$\sigma_e(T)$: essential spectrum of T

$\sigma_{uf}(T)$: upper semi-Fredholm spectrum of T

$\sigma_{lf}(T)$: lower semi-Fredholm spectrum of T

$\sigma_w(T)$: Weyl spectrum of T

$\sigma_{uw}(T)$: upper semi-Weyl spectrum of T

$\sigma_{lw}(T)$: lower semi-Weyl spectrum of T

$\sigma_b(T)$: Browder spectrum of T

$\sigma_{ub}(T)$: upper semi-Browder spectrum of T

$\sigma_{lb}(T)$: lower semi-Browder spectrum of T

$\sigma_{gd}(T) = \text{acc } \sigma(T)$ is the generalized Drazin spectrum of T

$\sigma_{g_z d}(T) = \text{acc acc } \sigma(T)$ is the g_z -invertible spectrum of T [2]

3 Generalization of Newton formula

We start this section by the next preliminary lemma that will be play a crucial role in the sequel.

Lemma 3.1. Let \mathcal{A} be a ring and $a, b \in \mathcal{A}$. The following statements hold:

- (I) If $ab \in \text{comm}(a)$, then
 - (i) $a^n b \in \text{comm}(a^m)$ for every integers $n, m \geq 1$.
 - (ii) $(ab)^n = a^n b^n$ and $(ba)^n = ba^n b^{n-1} = (ba)(ab)^{n-1} = ba^{n-1} b^{n-1} a$ for every integer $n \geq 2$.
 - (iii) $a(a+b) \in \text{comm}(a)$.
- (II) If $ab \in \text{comm}(b)$, then
 - (i) $ab^n \in \text{comm}(b^m)$ for every integers $n, m \geq 1$.
 - (ii) $(ab)^n = a^n b^n$ and $(ba)^n = a^{n-1} b^n a = (ab)^{n-1} (ba) = ba^{n-1} b^{n-1} a$ for every integer $n \geq 2$.
 - (iii) $(a+b)b \in \text{comm}(b)$.
- (III) If $ab \in \text{comm}(a)$ and $ba \in \text{comm}(b)$, then
 - (i) $a^n b^m \in \text{comm}(a^k)$ for every strictly positive integers n, m and k .
 - (ii) $a^n - b^n + ba^{n-1} - a^{n-1} b = (a^{n-1} + ba^{n-2} + b^2 a^{n-3} + \dots + b^{n-2} a + b^{n-1})(a-b)$ and $a^n - b^n + b^{n-1} a - ab^{n-1} = (a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \dots + ab^{n-2} + b^{n-1})(a-b)$ for every integer $n \geq 2$.
 - (iii) $(a+b)a \in \text{comm}(a+b)$ and $b(a+b) \in \text{comm}(b)$.
- (IV) If $ab \in \text{comm}(b)$ and $ba \in \text{comm}(a)$, then
 - (i) $a^n b^m \in \text{comm}(b^k)$ for every strictly positive integers n, m and k .
 - (ii) $a^n - b^n + ab^{n-1} - b^{n-1} a = (a-b)(a^{n-1} + ba^{n-2} + b^2 a^{n-3} + \dots + b^{n-2} a + b^{n-1})$ and $a^n - b^n + a^{n-1} b - ba^{n-1} = (a-b)(a^{n-1} + a^{n-2} b + a^{n-3} b^2 + \dots + ab^{n-2} + b^{n-1})$ for every integer $n \geq 2$.
 - (iii) $a(a+b) \in \text{comm}(a+b)$ and $(a+b)b \in \text{comm}(b)$.

Proof. (I) (i) Let's use induction with the following statement

$$P_m : a^n b \in \text{comm}(a^m) \text{ for every integer } n \geq 1.$$

Let $n \geq 1$ be an integer such that $a^n b a = a^{n+1} b$. Then $a^{n+1} b a = a(a^n b a) = a(a^{n+1} b) = a^{n+2} b$. So P_1 holds. Assume that P_m holds for some integer $m \geq 1$. Let $n \geq 1$ be an integer, then $(a^n b) a^{m+1} = ((a^n b) a^m) a = (a^m (a^n b)) a = a^m ((a^n b) a) = a^m (a (a^n b)) = a^{m+1} (a^n b)$. So P_{m+1} holds. Consequently, $a^n b \in \text{comm}(a^m)$ for every integers $n, m \geq 1$.

(ii) The equality $(ab)^n = a^n b^n$ is obvious. Let us prove by induction that $(ba)^n = ba^n b^{n-1}$ for every integer $n \geq 2$. For $n = 2$ the equality holds. Assume that $(ba)^n = ba^n b^{n-1}$ for some integer $n \geq 2$, then we get from the first point (i) that $(ba)^{n+1} = (ba)(ba)^n = b(aba^n) b^{n-1} = b(a^{n+1} b) b^{n-1} = ba^{n+1} b^n$. Consequently, the equality holds for every integer $n \geq 2$. On the other hand, since $(ba)^n = b(ab)^{n-1} a$ is always true, then $(ba)^n = ba^n b^{n-1} = (ba)(ab)^{n-1} = b(ab)^{n-1} a = ba^{n-1} b^{n-1} a$ for every integer $n \geq 2$. The point (iii) is trivial.

(II) The proof is identical to that of (I).

(III) (i) Let's use induction with the following statement

$$P_k : a^n b^m \in \text{comm}(a^k) \text{ for every strictly positive integers } n \text{ and } m.$$

P_1 is holds. Indeed, we consider the following statement

$$Q_m : a^n b^m \in \text{comm}(a) \text{ for every strictly positive integer } n.$$

From the point (i) of the statement (I), Q_1 holds. Assume now that Q_m holds for some strictly integer m and let n be a strictly positive integer. Since $bab = b^2a$, we conclude that $a(a^n b^{m+1}) = (a(a^n b^m))b = ((a^n b^m)a)b = a^n(b^m ab) = a^n(b^{m+1}a) = (a^n b^{m+1})a$. So Q_{m+1} holds.

Suppose that P_k holds for some strictly positive integer k and let n and m be two strictly positive integers. The statement P_1 implies that $(a^n b^m)a^{k+1} = ((a^n b^m)a^k)a = (a^k(a^n b^m))a = a^k((a^n b^m)a) = a^k(a(a^n b^m)) = a^{k+1}(a^n b^m)$. Thus P_{k+1} holds and this completes the proof.

The point (ii) is an immediate consequence of (i), and the point (iii) is trivial.

(IV) Goes similarly with (III). □

Throughout this paper we consider on a ring \mathcal{A} the sets defined as follows

$$\begin{aligned} \text{comm}_l(a) &= \{b \in \mathcal{A} : ab \in \text{comm}(a) \text{ and } ba \in \text{comm}(b)\}, \\ \text{comm}_r(a) &= \{b \in \mathcal{A} : ab \in \text{comm}(b) \text{ and } ba \in \text{comm}(a)\}, \\ \text{comm}_w(a) &= \text{comm}_l(a) \cap \text{comm}_r(a). \end{aligned}$$

Example 3.2. Let \mathcal{A} be a ring. Then for every $a, b \in \mathcal{A}$ we have

$$a \in \text{comm}(b) \implies b \in \text{comm}_w(a) \implies b^2 \in \text{comm}(a) \text{ and } a^2 \in \text{comm}(b)$$

However, we show by the following examples that the reverse of these implications are not true in general.

- (i) In the matrix space $\mathcal{M}_2(\mathcal{A})$, where \mathcal{A} is a ring with non-null unit e , the elements $P = \begin{pmatrix} 0 & e \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}$ satisfy $PQP = P^2Q = QP^2 = QPQ = Q^2P \neq PQ^2$, but $PQ \neq QP$. On the other hand, if we consider $S = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}$, then $S \in \text{comm}(P^2)$ and $P \in \text{comm}(S^2)$, but PS does not commute neither with P nor with S , and SP does not commute neither with P nor with S .
- (ii) Hereafter ℓ^2 denotes the Hilbert space $\ell^2(\mathbb{N})$. We consider in the Banach algebra $L(\ell^2)$, the operators T and S defined by $T(x_1, x_2, \dots) = (x_1, x_1, x_3, x_4, \dots)$ and $S(x_1, x_2, \dots) = (x_1, 0, \dots)$. Then $S \in \text{comm}_l(T)$, but ST does not commute with T and TS does not commute with S . On the other hand, $S^* \in \text{comm}_r(T^*)$, but T^*S^* does not commute with T^* and S^*T^* does not commute with S^* . This shows that the conditions assumed in the statements (I) and (II) of Lemma 3.1 are independent. As another example, consider in $\mathcal{M}_2(\mathcal{A})$, $M = \begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}$

and $N = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$. Then $M \in \text{comm}_l(N)$ for every $a, b \in \mathcal{A}$ such that $b \neq 0$ and $a \neq -b$. But NM does not commute with M and MN does not commute with N .

- (iii) Let T and N be the operators defined on ℓ^2 by $T(x_1, x_2, \dots) = (x_2, 0, \dots)$, $N(x_1, x_2, \dots) = (0, x_1, 0, \dots)$. Then $T \in \text{comm}(S^2)$ and $S \in \text{comm}(T^2)$. But $TN \notin \text{comm}(T) \cup \text{comm}(N)$ and $NT \notin \text{comm}(T) \cup \text{comm}(N)$.
- (iv) For the operators N_1 and N_2 defined on ℓ^2 by $N_1(x_1, x_2, \dots) = (0, x_1, x_2, 0, \dots)$, $N_2(x_1, x_2, \dots) = (0, -x_1, 0, \dots)$, we have $N_1 \in \text{comm}_w(N_2)$, but $N_1 \notin \text{comm}(N_2)$.
- (v) In $\mathcal{M}_3(\mathbb{C})$, $P = \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ satisfy $PQP = P^2Q = QP^2 = QPQ = Q^2P = PQ^2 = 0$, so that $P \in \text{comm}_w(Q)$, but $PQ \neq QP$.

A simple check, one can easily obtain the dressed results in the following remark.

Remark 3.3. Let \mathcal{A} be a ring with unit e . For $a, b \in \mathcal{A}$ and $\mu, \lambda \in \mathbb{C}$, the following statements hold:

- (i) If $ab \in \text{comm}(a)$, then $(a - \lambda e)(b - \mu e) \in \text{comm}(a - \lambda e)$ if and only if $a \in \text{comm}(b)$ or $\lambda = 0$.
- (ii) If $ba \in \text{comm}(a)$, then $(b - \mu e)(a - \lambda e) \in \text{comm}(a - \lambda e)$ if and only if $a \in \text{comm}(b)$ or $\lambda = 0$.
- (iii) If $b \in \text{comm}_w(a)$, then $a^n \in \text{comm}(b^m)$ for every integers $n, m \geq 1$ such that $nm \geq 2$.
- (iv) If $b \in \text{comm}_l(a)$ (resp., $b \in \text{comm}_r(a)$, $b \in \text{comm}_w(a)$), then $(a + b) \in \text{comm}_l(b)$ (resp., $(a + b) \in \text{comm}_r(b)$, $(a + b) \in \text{comm}_w(b)$).
- (v) If $aba = a^2b = ba^2$, then $a^n \in \text{comm}(b^m)$ for every $m \geq 1$ and $n \geq 2$.

Let \mathcal{A} be a unital complex Banach algebra and $B \subset \mathcal{A}$. The exponential of $a \in \mathcal{A}$ is defined by $\exp(a) = \sum_{n=0}^{\infty} \frac{a^n}{n!}$. In the next we denote by

$$C_1(B) = \{a \in \mathcal{A} : \forall b \in B, a \in \text{comm}(ab) \cup \text{comm}(ba) \text{ or } b \in \text{comm}(ba)\},$$

$$C_2(B) = \{a \in \mathcal{A} : \forall b \in B, a \in \text{comm}(ab) \cup \text{comm}(ba) \text{ or } b \in \text{comm}(ab)\},$$

$$C_3(B) = \{a \in \mathcal{A} : \forall b \in B, b \in \text{comm}(ab) \cup \text{comm}(ba)\}.$$

Theorem 3.4. *If \mathcal{A} is a unital complex Banach algebra and $\mathcal{A} = C_i(\mathcal{A})$, $i=1, 2$ or 3 , then \mathcal{A} is commutative.*

Proof. Let $a, b \in \mathcal{A}$ and let $\lambda \in \mathbb{C}$. Consider $x_\lambda = \exp(\lambda a)$ and $y_\lambda = \exp(-\lambda a)b$. Assume that $\mathcal{A} = C_1(\mathcal{A})$, then $b \exp(\lambda a) = x_\lambda y_\lambda x_\lambda \in \{x_\lambda^2 y_\lambda, y_\lambda x_\lambda^2\} = \{\exp(\lambda a)b, \exp(-\lambda a)b \exp(2\lambda a)\}$ or $\exp(-\lambda a)b^2 = y_\lambda x_\lambda y_\lambda = y_\lambda^2 x_\lambda = \exp(-\lambda a)b \exp(-\lambda a)b \exp(\lambda a)$. Hence $b^2 = b \exp(-\lambda a)b \exp(\lambda a)$ for all $\lambda \in \mathbb{C}$. Moreover,

$$b \exp(\lambda a)b \exp(-\lambda a) = b \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\sum_{k=0}^n C_n^k a^k b (-a)^{n-k} \right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} b (\delta_a)^n(b),$$

where $C_n^k = \frac{n!}{k!(n-k)!}$ and $\delta_a(b) = ab - ba$. Thus $bab = b^2a$ for every $a, b \in \mathcal{A}$. By the similar arguments, we get $b = \exp(\lambda a)b\exp(-\lambda a) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\sum_{k=0}^n C_n^k a^k b (-a)^{n-k} \right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} (\delta_a)^n(b)$, for all $\lambda \in \mathbb{C}$. Therefore $ab = ba$ and \mathcal{A} is commutative. The case of $\mathcal{A} = C_2(\mathcal{A})$ is analogous. For the case of $\mathcal{A} = C_3(\mathcal{A})$, it suffices to use that same arguments with $x_\lambda = \exp(\lambda a)$ and $y_\lambda = b\exp(-\lambda a)$. \square

Theorem 3.5. *Let \mathcal{A} be a ring. For every $a, b \in \mathcal{A}$ we have*

- (i) *If $b \in \text{comm}_r(a)$, then for every integer $n > 0$, $(a+b)^n = \sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k)$.*
- (ii) *If $b \in \text{comm}_l(a)$, then for every integer $n > 0$, $(a+b)^n = \sum_{k=1}^n C_{n-1}^{k-1} (a^k b^{n-k} + b^k a^{n-k})$.*

Where a^0 designates the unit element of \mathcal{A} .

Proof. (i) For $n = 1$ the statement holds. Assume that the statement holds for some integer $n \geq 1$, then

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n = (a+b) \left(\sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k) \right) \\
&= a \left(\sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k) \right) + b \left(\sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k) \right) \\
&= \sum_{k=1}^n C_{n-1}^{k-1} (a^{n+1-k} b^k + a b^{n-k} a^k) + \sum_{k=1}^n C_{n-1}^{k-1} (b a^{n-k} b^k + b^{n+1-k} a^k) \\
&= \sum_{k=1}^n C_{n-1}^{k-1} (a^{n+1-k} b^k + b^{n+1-k} a^k) + \sum_{k=1}^n C_{n-1}^{k-1} (a b^{n-k} a^k + b a^{n-k} b^k) \\
&= \sum_{k=1}^n C_{n-1}^{k-1} (a^{n+1-k} b^k + b^{n+1-k} a^k) + \sum_{k=1}^n C_{n-1}^{k-1} (b^{n-k} a^{k+1} + a^{n-k} b^{k+1}) \quad (\text{see Lemma 3.1}) \\
&= \sum_{k=1}^n C_{n-1}^{k-1} (a^{n+1-k} b^k + b^{n+1-k} a^k) + \sum_{k=2}^{n+1} C_{n-1}^{k-2} (b^{n+1-k} a^k + a^{n+1-k} b^k) \\
&= \sum_{k=2}^n (C_{n-1}^{k-2} + C_{n-1}^{k-1}) (a^{n+1-k} b^k + b^{n+1-k} a^k) + (a^n b + b^n a) + (a^{n+1} + b^{n+1}) \\
&= \sum_{k=1}^{n+1} C_n^{k-1} (a^{n+1-k} b^k + b^{n+1-k} a^k).
\end{aligned}$$

So the statement holds for $n+1$. Consequently, $(a+b)^n = \sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k)$ for every integer $n > 0$.

(ii) The second statement can be proved similarly. \square

Note that the second point of the previous theorem was firstly proved for complex matrices in

[6, Lemma 2.3]. This result has been extended by Huihui Zhu et al. and Honglin Zou et al. to a Banach algebra, see [13, Lemma 2.3] and [14, Lemma 2.9].

The next corollary gives a generalization to the known Binomial theorem.

Corollary 3.6. *Let \mathcal{A} be a ring. If $a, b \in \mathcal{A}$ such that $b \in \text{comm}_w(a)$, then for every positive integer $n \neq 2$, the following statements hold:*

$$(i) (a+b)^n = \sum_{k=0}^n C_n^k a^k b^{n-k} = \sum_{k=0}^n C_n^k b^k a^{n-k}.$$

$$(ii) a^n - b^n = (a-b) \sum_{k=0}^{n-1} a^k b^{n-k-1} = \left(\sum_{k=0}^{n-1} a^k b^{n-k-1} \right) (a-b) = (a-b) \sum_{k=0}^{n-1} b^{n-k-1} a^k = \left(\sum_{k=0}^{n-1} b^{n-k-1} a^k \right) (a-b).$$

Proof. (i) The cases $n = 0$ and $n = 1$ are trivial. For $n \geq 3$, as $b \in \text{comm}_w(a)$ then Theorem 3.5 and Remark 3.3 imply that

$$\begin{aligned} (a+b)^n &= \sum_{k=1}^n C_{n-1}^{k-1} (a^{n-k} b^k + b^{n-k} a^k) \\ &= \sum_{k=1}^n C_{n-1}^{k-1} a^{n-k} b^k + \sum_{k=1}^n C_{n-1}^{k-1} b^{n-k} a^k \\ &= \sum_{k=1}^n C_{n-1}^{k-1} a^{n-k} b^k + \sum_{k=0}^{n-1} C_{n-1}^{n-k-1} b^k a^{n-k} \\ &= \sum_{k=1}^{n-1} (C_{n-1}^{k-1} + C_{n-1}^{n-k-1}) a^{n-k} b^k + a^n + b^n \\ &= \sum_{k=0}^n C_n^k a^k b^{n-k} = \sum_{k=0}^n C_n^k b^k a^{n-k} \end{aligned}$$

(ii) Follows directly from Lemma 3.1 and Remark 3.3. \square

Let \mathcal{A} be a Banach algebra with unit e . It is well known that $\exp(a+b) = \exp(a)\exp(b)$ for every $a \in \text{comm}(b)$. But this identity can fail for noncommuting a and b . The next corollary shows that if $b \in \text{comm}_w(a)$, then this identity remains true if and only if $a \in \text{comm}(b)$.

Corollary 3.7. *Let \mathcal{A} be a Banach algebra with unit e and let $a, b \in \mathcal{A}$ such that $b \in \text{comm}_w(a)$. Then $\exp(a)\exp(b) - \exp(a+b) = \frac{ab-ba}{2}$. In particular, $\exp(a)\exp(b) - \exp(b)\exp(a) = ab-ba$.*

Proof. The Cauchy product implies that

$$\begin{aligned}
\exp(a)\exp(b) &= \left(\sum_{n=0}^{\infty} \frac{a^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{b^n}{n!} \right) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^n \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!} \right) \\
&= e + (a+b) + \frac{a^2 + 2ab + b^2}{2} + \sum_{n=3}^{\infty} \left(\sum_{k=0}^n \frac{a^k}{k!} \frac{b^{n-k}}{(n-k)!} \right) \\
&= e + (a+b) + \frac{a^2 + 2ab + b^2}{2} + \sum_{n=3}^{\infty} \frac{(a+b)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!} + \frac{ab - ba}{2} \\
&= \exp(a+b) + \frac{ab - ba}{2}. \quad \square
\end{aligned}$$

Recall that an element x of a ring \mathcal{A} is called nilpotent if $x^n = 0$ for some positive integer n . If so then the integer $d(x) = \min\{n \in \mathbb{N} : x^n = 0\}$ is called the degree of x . And the nilradical $\text{Nil}(\mathcal{A})$ is the set consisting of all nilpotent elements of \mathcal{A} , that is, $\text{Nil}(\mathcal{A}) := \{a \in \mathcal{A} \mid a \text{ is nilpotent}\}$.

Lemma 3.8. Let \mathcal{A} be a ring and let $a, b \in \mathcal{A}$. The following assertions hold:

- (i) If $b \in \text{Nil}(\mathcal{A})$ and $ab \in \text{comm}(a) \cup \text{comm}(b)$ or $ba \in \text{comm}(a) \cup \text{comm}(b)$, then ab and ba both belong to $\text{Nil}(\mathcal{A})$. Furthermore, in the first case we have $d(ab) \leq d(b)$ and $d(ba) \leq d(b) + 1$, and in the second case we have $d(ba) \leq d(b)$ and $d(ab) \leq d(b) + 1$.
- (ii) If $a, b \in \text{Nil}(\mathcal{A})$ and $b \in \text{comm}_l(a) \cup \text{comm}_r(a)$, then $a + b \in \text{Nil}(\mathcal{A})$ and

$$\max\{d(a), d(b)\} - \min\{d(a), d(b)\} \leq d(a+b) \leq d(a) + d(b).$$

- (iii) If $b \in \text{comm}_l(a) \cap \text{comm}(a^n)$ for some integer $n > 0$, then $a^m - b^m = (a^{m-1} + ba^{m-2} + b^2a^{m-3} + \dots + b^{m-2}a + b^{m-1})(a - b)$ for all integer $m > n$. Analogously, if $a \in \text{comm}_r(b) \cap \text{comm}(b^n)$ for some integer $n > 0$, then $a^m - b^m = (a - b)(a^{m-1} + ba^{m-2} + b^2a^{m-3} + \dots + b^{m-2}a + b^{m-1})$ for all integer $m > n$.

Proof. (i) Let $n > 0$ be an integer such that $b^n = 0$. If $ab \in \text{comm}(a)$, then Lemma 3.1 implies that $(ab)^n = a^n b^n = 0$ and $(ba)^{n+1} = ba^{n+1} b^n = 0$. And if $ba \in \text{comm}(a)$, we obtain again by Lemma 3.1 that $(ba)^n = b^n a^n = 0$ and $(ab)^{n+1} = b^n a^n b = 0$. The other cases go similarly.

(ii) Let $n, m \geq 1$ two integers such that $a^n = 0$ and $b^m = 0$. If $b \in \text{comm}_l(a)$, from Theorem 3.5 we get $(a+b)^{n+m} = \sum_{k=1}^{n+m} C_{n+m-1}^{k-1} (a^k b^{n+m-k} + b^k a^{n+m-k})$. Thus if $k \geq n$, then $a^k = 0$ and if $k \leq n$, then $n+m-k \geq m$ and so $b^{n+m-k} = 0$. If $k \geq m$, then $b^k = 0$ and if $k \leq m$, then $n+m-k \geq n$, so $a^{n+m-k} = 0$. Hence $(a+b)^{n+m} = 0$ and consequently $d(a+b) \leq d(a) + d(b)$. On the other hand, we have from Remark 3.3 that $(a+b) \in \text{comm}_l(b)$. Hence $\max\{d(a), d(b)\} - \min\{d(a), d(b)\} \leq d(a+b)$. The proof of the case $b \in \text{comm}_r(a)$ goes similarly.

- (iii) Is an immediate consequence of Lemma 3.1. □

Let a be an element of a Banach algebra \mathcal{A} with unit e . The spectral radius $r(a)$ of a can be expressed by the formula $r(a) = \inf\{M > 0 : ((\frac{a}{M})^n)_n \text{ is bounded}\}$.

Proposition 3.9. *Let \mathcal{A} be a complex Banach algebra with unit e and let $a, b \in \mathcal{A}$. The following assertions hold:*

(i) *If $ab \in \text{comm}(a) \cup \text{comm}(b)$, then $r(ab) \leq r(a)r(b)$.*

(ii) *If $a \in \text{comm}_r(b) \cup \text{comm}_l(b)$, then $r(a+b) \leq r(a) + r(b)$.*

Proof. Let $M > r(a)$, $N > r(b)$. (i) From Lemma 3.1 we have $(ab)^n = a^n b^n$. Since the product of two bounded sequences is bounded, then the sequence $\{(\frac{a^p b^q}{M^p N^q})\}_{p,q}$ is bounded. In particular, $\{(\frac{ab}{MN})^n\}_n$ is bounded and hence $r(ab) \leq r(a)r(b)$.

(ii) Assume that $a \in \text{comm}_r(b)$ (the other case goes similarly). From Theorem 3.5, we have

$$\left(\frac{a+b}{M+N}\right)^n = \sum_{k=1}^n C_{n-1}^{k-1} \left(\frac{M^{n-k} N^k}{(M+N)^n} \frac{a^{n-k} b^k}{M^{n-k} N^k} + \frac{N^{n-k} M^k}{(M+N)^n} \frac{b^{n-k} a^k}{N^{n-k} M^k} \right).$$

Hence $\{(\frac{a+b}{M+N})^n\}_n$ is bounded and thus $r(a+b) \leq r(a) + r(b)$. \square

Corollary 3.10. *Let \mathcal{A} be a complex Banach algebra with unit e and let $a, b \in \mathcal{A}$. The following assertions hold:*

(i) *If a or b is quasi-nilpotent and $ab \in \text{comm}(a) \cup \text{comm}(b)$, then ab is quasi-nilpotent.*

(ii) *If a and b are quasi-nilpotent and $a \in \text{comm}_r(b) \cup \text{comm}_l(b)$, then $a+b$ is quasi-nilpotent.*

4 Perturbation results

Throughout this section, we focus on the stability of some spectra of bounded linear operators in the Banach algebra $\mathcal{A} = L(X)$. We start first with some preliminaries results.

Proposition 4.1. *Let $S, T \in L(X)$. The following statements hold:*

(i) *$TS \in \text{comm}(T)$ if and only if $S^*T^* \in \text{comm}(T^*)$.*

(ii) *$TS \in \text{comm}(T)$ if and only if $\mathcal{R}(ST - TS) \subset \mathcal{N}(T)$, and $ST \in \text{comm}(T)$ if and only if $\mathcal{R}(T) \subset \mathcal{N}(ST - TS)$.*

Proof. Obvious. \square

Corollary 4.2. *Let $S, T \in L(X)$. The following statements hold:*

(i) *If T is one-to-one, then $TS \in \text{comm}(T)$ if and only if $S \in \text{comm}(T)$.*

(ii) *If T is onto, then $ST \in \text{comm}(T)$ if and only if $S \in \text{comm}(T)$.*

(iii) *Moreover, if T and S are self-adjoint Hilbert space operators, then $TS \in \text{comm}(T)$ if and only if $ST \in \text{comm}(T)$.*

Example 4.3. Note that if an operator T is not onto and $ST \in \text{comm}(T)$, then we cannot guarantee that S commutes with T even if T is one-to-one. For this, consider the unilateral right shift R and the nilpotent operator N defined on the Hilbert space ℓ^2 by $Rx = (0, x_1, x_2, \dots)$,

$Nx = (0, -x_1, 0, \dots)$, where $x = (x_n)_{n \geq 1} \in \ell^2$. R is one-to-one and not onto, and $NR \in \text{comm}(R)$. But $NR \neq RN$. This entails also from Proposition 4.1 that the condition of the injectivity of T assumed in the first assertion of Corollary 4.2 is crucial.

Recall that the degree of stable iteration of an operator T is defined by $\text{dis}(T) = \inf \Delta(T)$, where

$$\Delta(T) = \{m \in \mathbb{N} : \alpha(T_{[m]}) = \alpha(T_{[r]}), \forall r \in \mathbb{N} \ r \geq m\}.$$

T is said to be semi-regular if $\mathcal{R}(T)$ is closed and $\text{dis}(T) = 0$, and T is said to be essentially semi-regular if $\mathcal{R}(T)$ is closed and there exists a finite-dimensional subspace F such that $\mathcal{N}(T) \subset \mathcal{R}(T^\infty) + F$. For more details about these definitions, one can see [7, 8].

Proposition 4.4. *Let $S, T \in L(X)$ such that $S \in \text{comm}_r(T)$. The following assertions hold:*

- (i) *If $\text{dis}(TS) = 0$, then $\text{dis}(S) = 0$ and $\text{dis}(T) \leq 1$.*
- (ii) *If TS is semi-regular, then S is semi-regular.*
- (iii) *If TS is essentially semi-regular, then S is essentially semi-regular.*

Proof. (i) As $S \in \text{comm}_r(T)$ we then get from Lemma 3.1 that $(TS)^n = T^n S^n = ST^2 S(TS)^{n-2}$ for all integer $n \geq 2$. Moreover, $\text{dis}(TS) = 0$ implies that $\text{dis}((TS)^m) = 0$ for every $m \geq 1$. Hence $\mathcal{N}(S^m) \subset \mathcal{N}((TS)^m) \subset \bigcap_n \mathcal{R}((TS)^{nm}) \subset \mathcal{R}(S)$ for all $m \geq 1$. Hence $\text{dis}(S) = 0$. Let $n \geq 1$. As $\text{dis}(TS) = 0$, then $\mathcal{N}(T^{n+1}) \subset \mathcal{N}(ST^{n+1}) = \mathcal{N}(TST^n) \subset \mathcal{N}(T^n) + \mathcal{R}(TS^{n+1})$. Therefore, $\mathcal{N}(T^{n+1}) \subset \mathcal{N}(T^n) + \mathcal{R}(T)$ and then $\text{dis}(T) \leq 1$. The points (ii) and (iii) are consequences of [7, Lemme 4.15], [8, Corollary 3.4, Theorem 3.5] and the fact that $S(T^2 S) = (TS)^2 = (T^2 S)S$. \square

The next corollary extends [8, Theorem 3.5] and [7, Proposition 3.7, Lemme 4.15].

Corollary 4.5. *If $T, S \in L(X)$ such that $S \in \text{comm}_w(T)$ and TS is semi-regular (resp., essentially semi-regular), then ST, T and S are also semi-regular (resp., essentially semi-regular).*

Proof. As $S \in \text{comm}_w(T)$ then $(TS)^2 = (ST)^2$. Hence TS is semi-regular (resp., essentially semi-regular) if and only if $(TS)^2$ is semi-regular (resp., essentially semi-regular) if and only if $(ST)^2$ is semi-regular (resp., essentially semi-regular) if and only if ST is semi-regular (resp., essentially semi-regular). The rest of the proof follows directly from Proposition 4.4. \square

Proposition 4.6. *Let $T \in L(X)$ and $N \in \text{Nil}(L(X))$. The following holds:*

- (i) *If $N \in \text{comm}_r(T)$, then T is onto if and only if $T + N$ is onto.*
- (ii) *If $N \in \text{comm}_l(T)$, then T is bounded below if and only if $T + N$ is bounded below.*

Proof. Under conditions assumed, Corollary 4.2 implies that $N \in \text{comm}(T)$. And the results are already done. \square

Lemma 4.7. *Let $T, S \in L(X)$ such that $ST \in \text{comm}(T)$ and let $\lambda \neq 0$. Then $M := \mathcal{N}(T - \lambda I)$ is S -invariant and $S_M \in \text{comm}(T_M)$. If in addition $TS \in \text{comm}(S)$, then $B := \mathcal{N}(T + S - \lambda I)$ is S -invariant and $S_B \in \text{comm}(T_B)$.*

Proof. Let $x \in M$, then $(T - \lambda I)S(\lambda x) = (T - \lambda I)ST(x) = ST(T - \lambda I)(x) = 0$. Thus M is S -invariant. On the other hand, as T_M is invertible and $ST \in \text{comm}(T)$, then $S_M \in \text{comm}(T_M)$. If in addition $TS \in \text{comm}(S)$, then $S(T + S) \in \text{comm}(T + S)$ and thus B is S -invariant and T -invariant. Therefore $S_B \in \text{comm}(T_B)$. \square

Theorem 4.8. *Let $T \in L(X)$ and $N \in \text{Nil}(L(X))$ such that $T \in \text{comm}(NT)$. Then*

$$\sigma_p(T) \setminus \{0\} \subset \sigma_p(T + N) \setminus \{0\}.$$

Proof. Let $\lambda \neq 0$ and let $x \in \mathcal{N}(T - \lambda I)$. We show by induction that $((T + N) - \lambda I)^n(x) = N^n(x)$ for any $n \in \mathbb{N}$. Indeed, $((T + N) - \lambda I)(x) = N(x)$. Assume that $((T + N) - \lambda I)^n(x) = N^n(x)$ for some positive integer n . Then $((T + N) - \lambda I)^{n+1}(x) = ((T + N) - \lambda I)(N^n(x)) = TN^n(x) + N^{n+1}(x) - \lambda N^n(x)$. Furthermore, Lemma 4.7 implies that $TN^m(x) = \lambda N^m x$ for all $m \in \mathbb{N}$. Hence $((T + N) - \lambda I)^{n+1}(x) = N^{n+1}(x)$. Let $p \geq 1$ such that $N^p = 0$, then $((T + N) - \lambda I)^p(x) = 0$ and thus $x \in \mathcal{N}(((T + N) - \lambda I)^p)$. This yields $\mathcal{N}(T - \lambda I) \subset \mathcal{N}((T + N) - \lambda I)^p$. Hence $\sigma_p(T) \setminus \{0\} \subset \sigma_p(T + N) \setminus \{0\}$. \square

From the proof of Theorem 4.8, we obtain the next proposition.

Proposition 4.9. *Let $T, N \in L(X)$ such that $T \in \text{comm}(NT)$ and $N^p = 0$ for some strictly positive integer p . Then for every $\lambda \neq 0$, we have $\mathcal{N}(T - \lambda I) \subset \mathcal{N}((T + N) - \lambda I)^p$. If in addition $N \in \text{comm}(TN)$, then $\mathcal{N}((T + N) - \lambda I) \subset \mathcal{N}(T - \lambda I)^p$.*

Note that in the case of $NT \in \text{comm}(T)$ and $N^2 = 0$, the following proposition shows (without the condition $TN \in \text{comm}(N)$) that $\mathcal{N}((T + N) - \lambda I) \subset \mathcal{N}(T - \lambda I)^2$, which implies in turn that the inclusion proved in Theorem 4.8 becomes equality.

Proposition 4.10. *Let $T \in L(X)$ and $N \in \text{Nil}(L(X))$ such that $NT \in \text{comm}(T)$ and $N^2 = 0$. Then $\sigma_p(T) \setminus \{0\} = \sigma_p(T + N) \setminus \{0\}$ and $\sigma_p^0(T) \setminus \{0\} = \sigma_p^0(T + N) \setminus \{0\}$.*

Proof. Let $\lambda \neq 0$ and let us to show that $\mathcal{N}(T + N - \lambda I) \subset \mathcal{N}(T - \lambda I)^2$. Let $x \in \mathcal{N}(T + N - \lambda I)$, then $(T + N)x = \lambda x$. So $\lambda^2(T - \lambda I)^2x = \lambda^2(T - \lambda I)(-N)x = \lambda^2(-TNx + N(\lambda x)) = \lambda^2(-TNx + N(T + N)x) = \lambda^2(N^2x + (NT - TN)x)$. Moreover, we have $\lambda NTx = NT^2x + NTNx = TNTx + NTNx = TN(\lambda I - N)x + NTNx = \lambda TNx - TN^2x + NTNx$ and $\lambda NTNx = NTNTx + NTN^2x = N^2T^2x + NTN^2x$. Hence $\lambda^2(T - \lambda I)^2x = \lambda^2N^2x + N^2T^2x + NTN^2x - \lambda TN^2x = 0$ and then $x \in \mathcal{N}(T - \lambda I)^2$. On the other hand, from proposition 4.9, we have $\mathcal{N}(T + N - \lambda I) \subset \mathcal{N}(T - \lambda I)^2$. Hence $\sigma_p(T) \setminus \{0\} = \sigma_p(T + N) \setminus \{0\}$ and $\sigma_p^0(T) \setminus \{0\} = \sigma_p^0(T + N) \setminus \{0\}$. \square

Corollary 4.11. *Let $T \in L(X)$ and $N \in \text{Nil}(L(X))$. The following assertions hold:*

- (i) *If $T \in \text{comm}(NT) \cap \text{comm}(TN)$, then $\sigma_p(T) \setminus \{0\} \subset \sigma_p(T + N) \setminus \{0\}$ and $\sigma_p(T^*) \setminus \{0\} \subset \sigma_p(T^* + N^*) \setminus \{0\}$.*
- (ii) *If $N \in \text{comm}_r(T)$, then $\sigma_p(T) \setminus \{0\} = \sigma_p(T + N) \setminus \{0\}$ and $\sigma_p^0(T) \setminus \{0\} = \sigma_p^0(T + N) \setminus \{0\}$.*

Proof. (i) is obvious and (ii) is a consequence of Proposition 4.9. \square

The condition assumed in assertions (ii) of the previous corollary cannot guarantee that $\sigma_p(T) = \sigma_p(T + N)$ or $\sigma_p^0(T) = \sigma_p^0(T + N)$, as the following examples shows.

Example 4.12. Let $T, N \in L(\ell^2)$ be the operators defined by $T(x) = (0, \frac{x_1}{2}, \frac{x_2}{3}, \dots)$, $N(x) = (0, \frac{-x_1}{2}, 0, \dots)$ for every $x = (x_n)_{n \geq 1} \in \ell^2$. Clearly, T and $T + N$ are quasi-nilpotent and compact, N is nilpotent. Moreover, $TNT = NT^2 = NT = N^2T = NTN = TN^2 \neq T^2N$ and $\sigma_p(T) = \sigma_p^0(T) = \emptyset \neq \{0\} = \sigma_p^0(T + N) = \sigma_p(T + N)$. If we take the nilpotent operator $Q \in L(\ell^2)$ defined by $Q(x) = (0, \frac{-x_1}{2}, 0, \frac{-x_3}{4}, 0, \frac{-x_5}{6}, \dots)$, then $(T + Q)^2 = 0$, $TQT = QT^2 = QT = Q^2T = QTQ = TQ^2 \neq T^2Q$ and $\sigma_p(T) = \sigma_p^0(T) = \sigma_p^0(T + Q) = \emptyset \neq \{0\} = \sigma_p(T + Q)$.

Remark 4.13. Let $T \in L(X)$. It is well known that $\sigma_p(T) = \sigma_p(T + N)$ for every operator $N \in \text{comm}(T) \cap \text{Nil}(L(X))$. This result cannot be extended for operator $N \in [\text{comm}(T^2) \cap \text{Nil}(L(X))] \cup [\text{Nil}(L(X)) \cap N^{-1}(\text{comm}(T))]$, as the following shows. The nilpotent operators T and N defined in the point (iii) of the Example 3.2 satisfy $\emptyset = \sigma_p(T) \setminus \{0\} = \sigma_p(N) \setminus \{0\} \neq \{-1, 1\} = \sigma_p(T + N) \setminus \{0\}$, although $TN^2 = N^2T = NT^2 = T^2N = 0$. Note also that $\sigma_p(S) = \sigma_a(S) = \sigma_s(S) = \sigma(S)$ for all $S \in \{T, N, T + N\}$.

To give further information about the approximate point spectrum of sums of operators we need to introduce the *Berberian-Quisley extension* [3, 11]. Consider $\ell^\infty(X)$ the Banach space of all bounded sequences $x = (x_n)$ of X by imposing term-by-term linear combination and the supremum norm $\|x\| = \sup \|x_n\|$. Then the quotient space $X_0 = \ell^\infty(X)/c_0(X)$ is a Banach space, where $c_0(X) = \{(x_n) \in X : \lim \|x_n\| = 0\}$. Any operator $T \in L(X)$ generates an operator $T^0 \in L(X_0)$ defined by $T^0(x + c_0(X)) = (Tx_n)_n + c_0(X)$ for every $x = (x_n) \in \ell^\infty(X)$. The mapping $T \rightarrow T^0$ of $L(X)$ into $L(X_0)$ is an isometric isomorphism and $\sigma_a(T) = \sigma_a(T^0) = \sigma_p(T^0)$.

Proposition 4.14. *Let $T \in L(X)$ and let $N \in \text{Nil}(L(X))$. The following assertions hold:*

- (i) *If $T \in \text{comm}(NT)$, then $\sigma_a(T) \setminus \{0\} \subset \sigma_a(T + N) \setminus \{0\}$, and if in addition $N^2 = 0$, then $\sigma_a(T) \setminus \{0\} = \sigma_a(T + N) \setminus \{0\}$. While if $T \in \text{comm}(TN)$, then $\sigma_s(T) \setminus \{0\} \subset \sigma_s(T + N) \setminus \{0\}$, and if in addition $N^2 = 0$, then $\sigma_s(T) \setminus \{0\} = \sigma_s(T + N) \setminus \{0\}$.*
- (ii) *If $T \in \text{comm}(NT) \cap \text{comm}(TN)$, then $\sigma_*(T) \setminus \{0\} \subset \sigma_*(T + N) \setminus \{0\}$, and if in addition $N^2 = 0$, then $\sigma_*(T) \setminus \{0\} = \sigma_*(T + N) \setminus \{0\}$, where $\sigma_* \in \{\sigma_a, \sigma_s, \sigma\}$.*
- (iii) *If $N \in \text{comm}_r(T)$, then $\sigma_a(T) \setminus \{0\} = \sigma_a(T + N) \setminus \{0\}$.*
- (iv) *If $N \in \text{comm}_l(T)$, then $\sigma_s(T) \setminus \{0\} = \sigma_s(T + N) \setminus \{0\}$.*
- (v) *If $N \in \text{comm}_w(T)$, then $\sigma_*(T) = \sigma_*(T + N)$, where $\sigma_* \in \{\sigma_p, \sigma_p^0, \sigma_a, \sigma_s, \sigma\}$.*

Proof. (i) Since $T \in \text{comm}(NT)$ then $T^0N^0T^0 = (TNT)^0 = (NT^2)^0 = N^0(T^0)^2$. So $T^0 \in \text{comm}(N^0T^0)$. Moreover, $\|(N^0)^p\| = \|(N^p)^0\| = 0$ and then N^0 is nilpotent. From Theorem 4.8, $\sigma_a(T) \setminus \{0\} = \sigma_p(T^0) \setminus \{0\} \subset \sigma_p(T^0 + N^0) \setminus \{0\} = \sigma_p((T + N)^0) \setminus \{0\} = \sigma_a(T + N) \setminus \{0\}$. If in addition $N^2 = 0$, then we deduce from Corollary 4.10 that $\sigma_a(T) \setminus \{0\} = \sigma_a(T + N) \setminus \{0\}$. While if $T \in \text{comm}(TN)$, then $T^* \in \text{comm}(N^*T^*)$ and thus $\sigma_s(T) \setminus \{0\} = \sigma_a(T^*) \setminus \{0\} \subset \sigma_a(T^* + N^*) \setminus \{0\} = \sigma_a((T + N)^*) \setminus \{0\} = \sigma_s(T + N) \setminus \{0\}$, and if in addition $N^2 = 0$, then $\sigma_s(T) \setminus \{0\} = \sigma_s(T + N) \setminus \{0\}$. The point (ii) follows directly from the first.

(iii) As $N \in \text{comm}_r(T)$ then $T \in \text{comm}(NT)$ and $(T + N) \in \text{comm}(-N(T + N))$. So the first

point gives the desired result. The proof of (iv) goes similarly with (i), while the proof of (v) is a consequence of (iii), (iv) and Proposition 4.6. \square

Corollary 4.15. *Let \mathcal{A} be an arbitrary unital complex Banach algebra and let $x \in \mathcal{A}$ and $a \in \text{Nil}(\mathcal{A})$. The following statements hold:*

(i) *If $x \in \text{comm}(ax) \cap \text{comm}(xa)$, then $\sigma(x) \setminus \{0\} \subset \sigma(x+a) \setminus \{0\}$. If in addition $a^2 = 0$, then $\sigma(x) \setminus \{0\} = \sigma(x+a) \setminus \{0\}$.*

(ii) *If $x \in \text{comm}_w(a)$, then $\sigma(x) = \sigma(x+a)$.*

Proof. Consider the operators $L_a(y) = ay$ and $L_x(y) = xy$. We have $L_x \in L(\mathcal{A})$, $L_a \in \text{Nil}(L(\mathcal{A}))$ and $\sigma(x) = \sigma(L_x)$. Remark that if $a^2 = 0$, then $L_a^2 = L_{a^2} = 0$. And if $x \in \text{comm}(ax) \cap \text{comm}(xa)$ (resp., $x \in \text{comm}_w(a)$), then $L_x \in \text{comm}(L_a L_x) \cap \text{comm}(L_a L_x)$ (resp., $L_x \in \text{comm}_w(L_a)$). By applying Proposition 4.14, we get the desired results. \square

In this paragraph we present the *Construction of Sadovskii/Buoni, Harte, Wickstead* [4, 9, 12], which will play an important role in the next. The space $m(X)$ consisting of the relatively compact sequences of X is a closed subspace of $\ell^\infty(X)$. Consider $P(T) \in L(\mathcal{P}(X))$ the operator defined by $P(T)(x + m(X)) = (Tx_n)_n + m(X)$, where $x = (x_n) \in \ell^\infty(X)$ and $\mathcal{P}(X) = \ell^\infty(X)/m(X)$. The mapping $T \rightarrow P(T)$ of $L(X)$ into $L(\mathcal{P}(X))$ is a unital homomorphism with kernel $K(X)$ and induces a norm decreasing monomorphism from $L(X)/K(X)$ to $L(\mathcal{P}(X))$. Moreover, $\|P(T)\| \leq \|T\|$, $\sigma_{uf}(T) = \sigma_a(P(T))$, $\sigma_{lf}(T) = \sigma_s(P(T))$ and $\sigma_e(T) = \sigma(P(T))$.

Proposition 4.16. *Let $T, K \in L(X)$ such that K is a power compact operator. The following assertions hold:*

(i) *If $T \in \text{comm}(KT)$, then $\sigma_*(T) \setminus \{0\} \subset \sigma_*(T+K) \setminus \{0\}$, where $\sigma_* \in \{\sigma_{uf}, \sigma_{uw}\}$, and if in addition K^2 is compact, then $\sigma_{**}(T) \setminus \{0\} = \sigma_{**}(T+K) \setminus \{0\}$, where $\sigma_{**} \in \{\sigma_{uf}, \sigma_{uw}, \sigma_{ub}\}$. While if $T \in \text{comm}(TK)$, then $\sigma_+(T) \setminus \{0\} \subset \sigma_+(T+K) \setminus \{0\}$, where $\sigma_+ \in \{\sigma_{lf}, \sigma_{lw}\}$, and if in addition K^2 is compact, then $\sigma_{++}(T) \setminus \{0\} = \sigma_{++}(T+K) \setminus \{0\}$, where $\sigma_{++} \in \{\sigma_{lf}, \sigma_{lw}, \sigma_{lb}\}$.*

(ii) *If $T \in \text{comm}(KT) \cap \text{comm}(TK)$, then $\sigma_*(T) \setminus \{0\} \subset \sigma_*(T+K) \setminus \{0\}$, where $\sigma_* \in \{\sigma_{uf}, \sigma_{uw}, \sigma_{lf}, \sigma_{lw}, \sigma_e, \sigma_w\}$, and if in addition K^2 is compact, then $\sigma_{**}(T) \setminus \{0\} = \sigma_{**}(T+K) \setminus \{0\}$, where $\sigma_{**} \in \{\sigma_{uf}, \sigma_{uw}, \sigma_{ub}, \sigma_{lf}, \sigma_{lw}, \sigma_{lb}, \sigma_e, \sigma_w, \sigma_b\}$.*

(iii) *If $K \in \text{comm}_r(T)$, then $\sigma_*(T) \setminus \{0\} = \sigma_*(T+K) \setminus \{0\}$, where $\sigma_* \in \{\sigma_{uf}, \sigma_{uw}, \sigma_{ub}\}$.*

(iv) *If $K \in \text{comm}_l(T)$, then $\sigma_*(T) \setminus \{0\} = \sigma_*(T+K) \setminus \{0\}$, where $\sigma_* \in \{\sigma_{lf}, \sigma_{lw}, \sigma_{lb}\}$.*

(v) *If $K \in \text{comm}_w(T)$, then $\sigma_*(T) = \sigma_*(T+K)$, where $\sigma_* \in \{\sigma_e, \sigma_w, \sigma_b, \sigma_{gd}, \sigma_{gzd}\}$.*

Proof. (i) $T \in \text{comm}(KT)$ implies that $P(T) \in \text{comm}(P(K)P(T))$. Let $p \geq 1$ such that K^p is compact, we have $P(0) = P(K^p) = P(K)^p$, and so $P(K)$ is nilpotent. From [4, Theorem 2] and Proposition 4.14, $\sigma_{uf}(T) \setminus \{0\} = \sigma_a(P(T)) \setminus \{0\} \subset \sigma_a(P(T) + P(K)) \setminus \{0\} = \sigma_a(P(T+K)) \setminus \{0\} = \sigma_{uf}(T+K) \setminus \{0\}$. Let $\lambda \notin \sigma_{uw}(T+K) \setminus \{0\}$, then $\lambda \notin \sigma_{uf}(T) \setminus \{0\}$. By using the same argument as Oberai in [10, Lemma 2], we get that $\text{ind}(T+K - \lambda I) = \text{ind}(T - \lambda I)$ and thus $\lambda \notin \sigma_{uw}(T) \setminus \{0\}$. Therefore $\sigma_{uw}(T) \setminus \{0\} \subset \sigma_{uw}(T+K) \setminus \{0\}$. Since $\sigma_{ub}(T) = \sigma_{uw}(T) \cup \text{iso } \sigma_a(T)$ then if in addition K^2 is compact, $\sigma_{ub}(T) \setminus \{0\} = \sigma_{ub}(T+K) \setminus \{0\}$. The rest of the proof is clear and is left to the reader. \square

We recall that $T \in L(X)$ is said to have the SVEP at $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ , the function $f \equiv 0$ is the only analytic solution of the equation $(T - \mu I)f(\mu) = 0 \quad \forall \mu \in U_\lambda$.

Lemma 4.17. Let $T \in L(X)$. Then T is quasi-nilpotent if and only if $\sigma_*(T) = \{0\}$, where $\sigma_* \in \{\sigma_a, \sigma_s\}$.

Proof. Since $\partial\sigma(T) \subset \sigma_*$, then the proof follows from [1, Theorem 2.97, Theorem 2.98] and the fact that T and T^* have the SVEP on the boundary $\partial\sigma(T)$. \square

Recall that $T \in L(X)$ is Riesz if $T - \lambda I$ is Browder for all non-zero complex λ , which is equivalent to say that $\pi(T) := T + K(X)$ is quasi-nilpotent in the Calkin algebra $L(X)/K(X)$, where $K(X)$ is the ideal of all compact operators.

Proposition 4.18. Let $T, R \in L(X)$ such that R is Riesz. The following statements hold:

- (i) If $T \in \text{comm}(RT) \cap \text{comm}(TR)$ and T is Fredholm, then $\sigma_e(T) = \sigma_e(T + R)$ and $\sigma_w(T) = \sigma_w(T + R)$.
- (ii) If $T \in \text{comm}(TR)$ and T is upper semi-Fredholm, then $\sigma_{uf}(T) = \sigma_{uf}(T + R)$ and $\sigma_{uw}(T) = \sigma_{uw}(T + R)$.
- (iii) If $T \in \text{comm}(RT)$ and T is lower semi-Fredholm, then $\sigma_{lf}(T) = \sigma_{lf}(T + R)$ and $\sigma_{lw}(T) = \sigma_{lw}(T + R)$.

Proof. (i) Assume that T is Fredholm. By Atkinson theorem we get that $\pi(T)$ is invertible in the Calkin algebra. As $TRT \in \{T^2R, RT^2\}$ then $\pi(T)\pi(R) = \pi(R)\pi(T)$ and thus $TR - RT \in K(X)$. Since R is Riesz then $\pi(R)$ is quasi-nilpotent, and hence $\sigma_e(T) = \sigma(\pi(T)) = \sigma(\pi(T + R)) = \sigma_e(T + R)$. And thus $\sigma_w(T) = \sigma_w(T + R)$.

(ii) If T is upper semi-Fredholm, then from [4] the operator $P(T) \in \ell^\infty(X)/m(X)$ defined above is bounded below. As $TRT = T^2R$, from Corollary 4.2, $P(T)P(R) = P(R)P(T)$. Thus $P(TR - RT) = P(0)$, so that $TR - RT \in K(X)$. On the other hand, since R is Riesz then $\sigma_a(P(R)) = \sigma_{uf}(R) = \{0\}$ and this implies from Lemma 4.17 that $P(R)$ is quasi-nilpotent. Hence $\sigma_{uf}(T) = \sigma_a(P(T)) = \sigma_a(P(T + R)) = \sigma_{uf}(T + R)$. Let $\lambda \notin \sigma_{uw}(T)$, then $\lambda \notin \sigma_{uf}(T) = \sigma_{uf}(T + \mu R)$ for all $\mu \in \mathbb{C}$. From [5, Theorem V.I.8], we deduce that $\alpha(T - \lambda I) = \alpha((T + \mu R) - \lambda I)$ and $\beta(T - \lambda I) = \beta((T + \mu R) - \lambda I)$ for all $\mu \in \mathbb{C}$. Hence $\text{ind}(T - \lambda I) = \text{ind}((T + R) - \lambda I)$ for all $\mu \in \mathbb{C}$. Consequently, $\lambda \notin \sigma_{uw}(T + R)$. The point (iii) goes similarly. \square

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