THE ASYMPTOTIC BEHAVIOUR OF p-CAPACITARY POTENTIALS IN ASYMPTOTICALLY CONICAL MANIFOLDS

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ABSTRACT. We study the asymptotic behaviour of the p-capacitary potential and of the weak Inverse Mean Curvature Flow of a bounded set along the ends of an Asymptotically Conical Riemannian manifolds with asymptotically nonnegative Ricci curvature.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

A natural question in the qualitative study of solutions to partial differential equations regards their behaviour at large distances on complete Riemannian manifolds. For harmonic potentials, a very satisfactory description was achieved in the fairly general framework of complete Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth in [LTW97] and [CM97]. In [LTW97] using the representation formula and in [CM97] employing the monotonicity of the Almgren's frequency function, the authors proved that the harmonic potential u of an open bounded subset Ω with smooth boundary, namely the solution to

$$\begin{cases} \Delta_g u = 0 & \text{on } M \smallsetminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega, \\ u(x) \to 0 & \text{as } d_g(x, \Omega) \to +\infty, \end{cases}$$

is asymptotically equivalent to $d_g(x, \Omega)^{2-n}$, far away from Ω (see also [Din02; AFM20]). In [AFM20] these results were applied to establish the Willmore Inequality in this framework and, consequently, a sharp Isoperimetric Inequality in dimension $n = 3$ with an explicit optimal constant depending only on the Asymptotic Volume Ratio (AVR(g) for short) and the dimension of the manifold. The asymptotic behaviour of harmonic functions played a role also in the proof of the Positive Mass Theorem in [AMO21] in the context of Asymptotically Flat Riemannian manifolds with nonnegative scalar curvature.

In the last few years, it became evident that even stronger geometric conclusions can be drawn from the study of p-harmonic potentials on complete Riemannian manifolds, such as the validity of Minkowski Inequalities [FMP19; AFM22; BFM21] and the Riemannian Penrose Inequality [Ago+22]. Aim of the present work is to provide a detailed analysis of the asymptotic behaviour of these functions in the context of Asymptotically Conical Riemannian manifolds. This study extends some classical results, obtained by Kichenassamy and Véron [KV86] (see also [Col+15]) in the context of the flat Euclidean space.

To state our results, we now introduce some notation and setup. We recall that on a Riemannian manifold (M, g) the p-capacitary potential of a bounded open domain $\Omega \subset M$ is the

solution $u : M \setminus \Omega \to \mathbb{R}$ to

$$
\begin{cases}
\Delta_g^{(p)} u = 0 & \text{on } M \setminus \overline{\Omega}, \\
\quad\; u = 1 & \text{on } \partial\Omega, \\
u(x) \to 0 & \text{as } d_g(x, \Omega) \to +\infty,
\end{cases}
\tag{1.1}
$$

where $\Delta_g^{(p)} u = \operatorname{div}_g(|\mathrm{D}u|^{p-2}\mathrm{D}u)$ is the p-Laplace operator associated with the metric g. Through-out the paper we will systematically work on complete noncompact Riemannian manifolds (M, g) of dimension $n \geq 3$ that are *Asymptotically Conical* with *quadratically asymptotically nonnega-tive Ricci curvature*, that is

$$
\operatorname{Ric}(x) \geq -\frac{(n-1)\kappa^2}{(1 + d_g(x, o))^2}
\tag{1.2}
$$

for some some fixed $o \in M$ and $\kappa \in \mathbb{R}$, and every $x \in M$. In accordance with [CEV17], by the locution *Asymptotically Conical Riemannian manifolds*, we are denoting, manifolds such that, outside of some open bounded subset K, are diffeomorphic to a (truncated) cone $[1, +\infty) \times L$, over a smooth hypersuface L, called the *link* of the cone, and such that the metric is close (in the $\mathscr{C}^{k,\alpha}$-topology for some $k \in \mathbb{N}$ and $\alpha \in [0, 1]$) to the cone metric $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$, where ρ is the radial coordinate on the cone (see Definition 2.13 for details).

Differently from the case $\operatorname{Ric} \geq 0$, where the Cheeger-Gromoll splitting theorem applies, here we possibly have to deal with manifolds that have more than one single end (see [LT92, Definition 0.4 and discussion thereafter] for the notion of ends). Nevertheless, due to the compactness and smoothness of ∂K, the manifold is forced to have a finite number of *ends* $E_1, \ldots, E_N$. We can assume that each end E_i is diffeomorphic to $[1, +\infty) \times L_i$ for every $i = 1, \ldots, N$, being L_i the connected components of L, and consequently each end is Asymptotically Conical. We can define an Asymptotic Volume Ratio on each end as

$$
\operatorname{AVR}(g; E_i) = \lim_{R \to +\infty} \frac{|B(o, R) \cap E_i|}{|\mathbb{B}^n| R^n}.
$$

Indeed, it is not hard to realise that, even if it is not monotone, the ratio $|B(o, R)|/R^n$ has a limit as $R \to +\infty$ in Asymptotically Conical Riemannian manifolds. Moreover, the Asymptotic Volume Ratio $\operatorname{AVR}(g)$ of the manifold splits as

$$
\operatorname{AVR}(g) = \sum_{i=1}^{N} \operatorname{AVR}(g; E_i)
$$

where $0 < \operatorname{AVR}(g) \leq N$ is the Asymptotic Volume Ratio of (M, g).

It is not surprising that the asymptotic behaviour of the solution to (1.1) could be different depending on the end, and in fact it turns out that the behaviour on a given end it is not affected by what happens on the others. More precisely, observing that for large enough T the set $\{u > 1/T\}$ contains K, we can define the *normalised p-capacity of Ω relative to the end E_i* as

$$
\mathrm{C}_p^{(i)}(\Omega) = \left(\frac{p-1}{n-p}\right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int\limits_{\{u=1/t\} \cap E_i} |\mathrm{D}u|^{p-1} \, \mathrm{d}\sigma,
\tag{1.3}
$$

for every $i = 1, \ldots, m$. Observe that, by virtue of the divergence theorem, the right hand side does not depend on $t > T$ and the above formula yields a well posed definition. The *normalised*

p-capacity of Ω (see Definition 2.10) then split as

$$\mathrm{C}_p(\Omega) = \sum_{i=1}^{N} \mathrm{C}_p^{(i)}(\Omega).$$

We are now ready to state our first main result.

Theorem 1.1 (Asymptotic behaviour of the p-capacitary potential). *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying (1.2). Let $E_1, \ldots, E_N$ be the (finitely many) ends of M with respect to the compact K in Definition 2.13. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $u : M \setminus \Omega \to \mathbb{R}$ the solution to the problem (1.1). Then*

$$u(x) = \left(\frac{\mathrm{C}_p^{(i)}(\Omega)}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{p-1}} \rho(x)^{-\frac{n-p}{p-1}} + o\left(\rho(x)^{-\frac{n-p}{p-1}} \right) \tag{1.4}$$

on E_i as $\rho(x) \to +\infty$ for every $i = 1, \ldots, N$.

When the $\mathscr{C}^0$-Asymptotically Conical condition is strengthen to $\mathscr{C}^k$-Asymptotically Conical condition, it is not difficult to deduce corresponding asymptotic behaviours for the derivatives up to order k of u (see Theorem 3.1 below). Our result extends to the nonlinear setting the asymptotic analyses carried out in [AMO22, Theorem 2.2], [AMO21, Lemma 2.2], [HM20, Lemma 4.1] and [MMT20, Lemma A.2], although without refined estimates of the error terms. It would be interesting to deal, building on Theorem 1.1, with such remainders, possibly following the lines of [Chr90].

We emphasise that Theorem 1.1 plays a crucial role in the recent proof of the Riemannian Penrose Inequality through p-harmonic potentials proposed in [Ago+22] and that it will be employed to provide new results in this field under milder asymptotic conditions. To this end, we point out that the requirements above do not involve explicit rates of decay to the reference metric, that are usually assumed when dealing with these topics.

It turns out that our approach, employed for proving Theorem 1.1 and its consequences, appearing in Section 3, happen to fit also the geometric case of the weak Inverse Mean Curvature Flow starting at a bounded $\Omega \subset M$ with smooth boundary. We briefly recall that with this notion, introduced by Huisken and Ilmanen [HI01] as a weak counterpart to the classical evolution by inverse mean curvature [Ger90; Urb90], it is indicated a Lipschitz function $w \in \mathrm{Lip}_{\mathrm{loc}}$ that satisfies

$$\mathrm{div}\left(\frac{\mathrm{D}w}{|\mathrm{D}w|} \right) = |\mathrm{D}w|$$

on $M \setminus \overline{\Omega}$ and such that $\Omega = \{w < 0\}$ in a very geometric nonstandard weak variational sense. By the pioneering work of Moser [Mos07; Mos08] and subsequent extensions to Riemannian manifolds [KN09; MRS21], the solution w can also be interpreted as the locally uniform limit as $p \to 1^+$ of $-(p-1)\log u_p$, where u_p is the p-capacitary potential of Ω.

In analogy with (1.3), we set

$$|\partial\Omega^*|^{(i)} = \frac{|\partial\{w \leq t\} \cap E_i|}{\mathrm{e}^t},$$

for every $i = 1, \ldots, N$ and every $t \geq T$, where T is so chosen that $\{w \geq T\}$ contains K. By means of [HI01, Exponential Growth Lemma 2.6] and the divergence theorem, we have that

the right hand side does not depend on $t > T$ and yields a well posed definition. As for the p-capacity, we have that

$$|\partial\Omega^*| = \sum_{i=1}^{N} |\partial\Omega^*|^{(i)}$$

where Ω^* is the strictly outward minimising hull of Ω [FM20]. We refer to Section 4 for a more detailed discussion. The following theorem provides a description of the asymptotic behaviour for the weak IMCF for large times.

Theorem 1.2 (Asymptotic behaviour of the Inverse Mean Curvature Flow). *Let (M, g) be a complete $\mathscr{C}^1$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying (1.2). Let $E_1, \dots, E_N$ be the (finitely many) ends of M with respect to the compact K in Definition 2.13. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $w : M \smallsetminus \Omega \to \mathbb{R}$ be the weak Inverse Mean Curvature Flow starting at Ω. Then*

$$w(x) = \log\left(\rho(x)^{n-1}\right) - \log\left(\frac{|\partial\Omega^*|^{(i)}}{\mathrm{AVR}(g; E_i)|\mathbb{S}^{n-1}|}\right) + o(1) \tag{1.5}$$

on E_i as $\rho(x) \to +\infty$ for every $i = 1, \dots, N$.

It might be useful to observe that, coherently, multiplying by $-(p-1)$ the logarithm of the right hand side of (1.4) and letting $p \to 1^+$ one exactly recovers the right hand side of (1.5). In fact, as a consequence of [FM20, Theorem 1.2], we have that

$$\lim_{p\to 1^+} \mathrm{C}_p^{(i)}(\Omega) = \frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|} \tag{1.6}$$

holds for every $i = 1, \dots, N$ (see Lemma 4.4 below). Observe that, differently from Theorem 1.1 we required $\mathscr{C}^1$-convergence of the metric. This additional requirement is due to the fact that, up to the authors' knowledge, a Cheng-Yau-type estimate with sharp decay for the gradient of the IMCF is not known. Therefore, in our proof we use the gradient bound [HI01, Weak Existence Theorem 3.1]. In some cases, for example $\mathrm{Ric} \geq 0$, this requirement can be weakened in favour of the $\mathscr{C}^0$-convergence (see the discussion before Proposition 4.2). To our knowledge, Theorem 1.2 with the explicit constant in the expansion (1.5) was known only in the flat case of $\mathbb{R}^n$, and was obtained by completely different means. Indeed, in this setting, the level sets of the weak IMCF become starshaped (and thus smooth) after a sufficiently long time, as a consequence of [HI08, Theorem 2.7]. At this point, (1.5) could be easily deduced by classical results [Ger90; Urb90] for the smooth IMCF. It is worth pointing out that the arguments we employ got an important inspiration also from those in the proof of [HI01, Blowdown Lemma 7.1], that actually helped also in establishing Theorem 1.1. Theorem 1.2 provided a simplified approach to the proof of [HI01, Blowdown Lemma 7.1]. At the same time extend such result to the class of Asymptotically Conical Riemannian manifolds, adding the explicit characterisation of the constant term in the expansion (1.5).

The results above in particular apply to Asymptotically Locally Euclidean spaces (ALE for short) *gravitational instantons*, that are noncompact hyperkhäler Ricci Flat 4-dimensional manifolds playing a role in the study of Euclidean Quantum Gravity Theory, Gauge Theory and String Theory (see [Haw77; EH79; Kro89a; Kro89b; Min09; Min10; Min11]). Moreover, it is not difficult to realise that the completeness assumption can be dropped, and that the above results can be extended to manifolds with boundary. Indeed, the method proposed is completely blind to everything is inside Ω, and, therefore, we can also include Asymptotically Flat Riemannian

manifolds (M, g) with compact boundary ∂M, of fundamental importance in General Relativity.

Summary. The paper is organised as follows. In Section 2, we recall some basic notions about p-harmonic functions and the p-capacitary potential as well as an improvement of Li-Yau-type estimates holding true on Asymptotically Conical Riemannian manifolds satisfying the bound (1.2), with controlled constants as $p \to 1^+$. Section 3 and Section 4 are devoted to the proof of the asymptotic behaviour of the p-capacitary potential and of the (weak) IMCF, that are respectively Theorem 1.1 and Theorem 1.2, together with some other related results. In the last section we prove the sharpness of the Minkowski Inequality in Asymptotically Conical Riemannian manifolds with nonnegative Ricci curvature and its rigidity statement in the general Euclidean volume growth case.

2. Preliminary results on p-capacitary potentials

2.1. p-harmonic functions and regularity.
Before considering the specific case of problem (1.1), we recall the definition of p-harmonic functions, as well as their regularity estimates. Given an open subset U of a complete Riemannian manifold (M, g), we say that $v \in W^{1,p}(U)$ is p-harmonic if

$$\int_U \left\langle |\mathrm{D}v|^{p-2}\mathrm{D}v \,|\, \mathrm{D}\psi \right\rangle \, \mathrm{d}\mu = 0. \tag{2.1}$$

for any test function $\psi \in \mathscr{C}_c^\infty(U)$. With $\langle \cdot \,|\, \cdot \rangle$ we denote as usual the scalar product induced by the underlying Riemannian metric g on the tangent space at each point.

We now precisely recall what is known about the regularity of p-harmonic functions. Suppose by now that $K \Subset U$ is entirely contained in a chart of M. For any $k \in \mathbb{N}$ and $\alpha \in (0, 1)$, we have that

$$g_{ij}(x)\xi^i\xi^j > \lambda\delta_{ij}\xi^i\xi^j \qquad\qquad \|g_{ij}\|_{\mathscr{C}^{k,\alpha}(K)} < \Lambda_{k,\alpha} \tag{2.2}$$

holds for every $x \in K$ and $\xi \in \mathbb{R}^n$ for some positive constants λ and $\Lambda_{k,\alpha}$. Regularity results for p-harmonic functions (see [Tol83; DiB83; Lie88]) yield

$$|\mathrm{D}v(x)| \le \mathrm{C} \qquad\qquad |\mathrm{D}v(x) - \mathrm{D}v(y)| \le \mathrm{C}\left(d_g(x, y)\right)^\beta$$

for every p-harmonic function v with $|v| \le 1$, with C, β depending only on n, p, α, λ and $\Lambda_{0,\alpha}$ and the distance of K from the boundary of U. For a general K, since a compact set can be finitely covered by charts, the result can be extended defining, with abuse of notation, λ and $\Lambda_{k,\alpha}$ as the minimal λ and maximal $\Lambda_{k,\alpha}$ in (2.2) among a family of charts covering K. Hence, the theorem below easily follows by a scaling argument, being that $v/\|v\|_{L^\infty(U)}$ is p-harmonic as well.

Theorem 2.1 (Schauder interior estimates)**.** *Let (M, g) be a complete Riemannian manifold of dimension n, $U \subseteq M$ be an open subset and let $1 < p$. For any $\alpha \in (0, 1)$ and $K \Subset U$, there exists a positive constant $\beta = \beta(n, p, \alpha, \Lambda_{0,\alpha}, \lambda) \in (0, \alpha)$ such that any bounded solution*

$v : U \to \mathbb{R}$ *of the problem* $\Delta_g^{(p)} v = 0$ *on* U *belongs to* $\mathscr{C}^{1,\beta}(K)$. *Moreover, there is a positive constant* $\mathrm{C} = \mathrm{C}(n, p, d_g(K, \partial U), \Lambda_{0,\alpha}, \lambda)$ *such that*

$$|\mathrm{D}v(x)| \leq \mathrm{C} \, \|v\|_{L^\infty(U)} \qquad\qquad |\mathrm{D}v(x) - \mathrm{D}v(y)| \leq \mathrm{C} \, \|v\|_{L^\infty(U)} \left(d_g(x,y) \right)^\beta$$

for any $x, y \in K$.

On the other hand, the classical regularity theory for quasilinear nondegenerate elliptic equations ensures that Sobolev functions satisfying (2.1) are smooth around the points where the gradient does not vanish (see [LSU68, Chapter 4, Section 6]). Moreover, the classical elliptic regularity theory can be applied to get higher order interior estimates (see [GT15, Theorem 6.6]).

Theorem 2.2 (Higher order Schauder estimates). *Let* (M, g) *be a Riemannian manifold of dimension* n, $U \subseteq M$ *be an open smooth subset of* M *and consider* $1 < p < n$. *Then for any* $k \in \mathbb{N}$, $\beta \in (0,1)$ *and* $K \Subset U$, *and every bounded solution* $v : U \to \mathbb{R}$ *of the problem* $\Delta_g^{(p)} v = 0$ *on* U *such that* $|\mathrm{D}v| \geq m > 0$ *on* K, *there exists a constant* $\mathrm{C}_{k,\beta} = \mathrm{C}_{k,\beta}(n, p, d_g(K, \partial U), \Lambda_{k-1,\beta}, \lambda, m)$ *such that*

$$\|v\|_{\mathscr{C}^{k,\beta}(K)} \leq \mathrm{C}_{k,\beta} \, \|v\|_{L^\infty(U)} \, .$$

Given $U \subseteq M$ with Lipschitz boundary, a p-harmonic function $u \in W^{1,p}(U)$ attains some Dirichlet data $g \in L^p(\partial U)$ if u coincides with g on ∂U in the sense of the trace operator. We report in the next remark the issue of the boundary regularity.

Remark 2.3 (Boundary regularity of p-harmonic functions). We point out that, if a p-harmonic function attains some $\mathscr{C}^{1,\alpha}$-Dirichlet data on a $\mathscr{C}^{1,\alpha}$ boundary, then the $\mathscr{C}^{1,\beta}$-estimates of Theorem 2.1 can be extended up to the boundary. This is a major contribution of [Lie88]. Moreover, if its gradient does not vanish at the boundary the function is smooth up to the boundary and Theorem 2.2 extends as well.

We finally retrieve the Comparison Principles [Tol83, Lemma 3.1, Proposition 3.3.2] by Tolksdorf, specialised for our purposes.

Theorem 2.4 (Comparison Principles). *Let* (M, g) *be a complete Riemannian manifold,* $U \subseteq M$ *be an open bounded subset and* $v_1, v_2 : U \to \mathbb{R}$ *be two* p-*harmonic functions.*

- *(Weak) Comparison Principle. If* $v_1, v_2 \in \mathscr{C}^0(\overline{U})$ *and* $v_1 \leq v_2$ *on* ∂U, *then* $v_1 \leq v_2$ *on* U.
- *Strong Comparison Principle. Suppose in addition that* U *is connected,* $v_1 \in \mathscr{C}^1(\overline{U})$, $v_2 \in \mathscr{C}^2(\overline{U})$ *and* $|\nabla v_2| \geq \delta > 0$ *in* U. *If* $v_1 \leq v_2$ *(resp.* $v_1 \geq v_2$*) on* U, *then* $v_1 = v_2$ *or* $v_1 < v_2$ *(resp.* $v_1 > v_2$*) on* U.

To conclude, we want to recall a compactness theorem that holds for p-harmonic functions. It is a natural question whether the limit of a sequence of p-harmonic functions is still p-harmonic. Clearly the weak formulation in (2.1) suggests that $\mathscr{C}^1$-convergence on compact subsets is enough to ensure that also the limit function is p-harmonic. The following theorem relaxes this hypothesis in favour of uniform convergence on compact subsets.

Theorem 2.5 (Compactness Theorem). *Let* $(v_n)_{n \in \mathbb{N}}$ *be a sequence of* p-*harmonic functions on* U *that converges uniformly to* v *on compact subsets of* U *as* $n \to +\infty$. *Then* $v \in W^{1,p}_{\mathrm{loc}}$ *is* p-*harmonic on* U.

Proof. See [HK88, Theorem 3.2]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 2.6. Suppose that $(U_n)_{n\in\mathbb{N}}$ is a sequence of open subsets converging to U open subset as $n \to +\infty$. Let g_n be a metric on U_n for every $n \in \mathbb{N}$ that locally uniformly converges to some metric g on U as $n \to +\infty$. The above theorem still holds if v_n is p-harmonic with respect to the metric g_n. Consequently, v is p-harmonic on U with respect to g.

2.2. **p-nonparabolic manifolds and the p-capacitary potential.** We analyse here the existence and uniqueness of solution u_p to (1.1) on complete Riemannian manifolds. Given a noncompact Riemannian manifold M, we consider the p-capacitary potential of a bounded set with smooth boundary $\Omega \subset M$, that is a function $u \in W^{1,p}_{\text{loc}}(M \smallsetminus \Omega)$ solving (1.1). The regularity results previously discussed ensure that u belongs to $\mathscr{C}^{1,\beta}_{\text{loc}}(M \smallsetminus \Omega)$ and it is smooth near the points where the gradient does not vanish. In particular, by Hopf Maximum Principle [Tol83] the datum on $\partial\Omega$ is attained smoothly.

We now focus on some classical sufficient conditions to ensure the existence of the p-capacitary potential, which turns out to be related to the notion of p-Green's function we are going to recall.

Definition 2.7 (p-Green's function)**.** Let (M, g) be a complete Riemannian manifold. Let $\text{Diag}(M) = \{(x, x) \in M \times M \,|\, x \in M\}$. For $p \geq 1$, we say that $G_p : M \times M \smallsetminus \text{Diag}(M) \to \mathbb{R}$ is a *p-Green's function* for M if it weakly satisfies $\Delta_p G(o, \cdot) = -\delta_o$ for any $o \in M$, where δ_o is the Dirac delta centred at o, that is, if it holds

$$\int\limits_M \left\langle |\mathrm{D}\,G_p(o, \cdot)|^{p-2}\,\mathrm{D}\,G_p(o, \cdot) \,\Big|\, \mathrm{D}\psi \right\rangle \mathrm{d}\mu = \psi(o)$$

for any $\psi \in \mathscr{C}^\infty_c(M)$.

The notion of p-Green's function immediately calls for that of p-nonparabolic Riemannian manifold.

Definition 2.8 (p-nonparabolicity)**.** We say that a complete noncompact Riemannian manifold (M, g) is p-nonparabolic if, for any $o \in M$, there exists a *positive p-Green's function* $G_p : M \smallsetminus \{o\} \to \mathbb{R}$. With the expression *$p$-Green function* we are in fact referring to the positive minimal one.

The notion of p-nonparabolicity is intimately related to existence of a solution to (1.1), in that if the p-Green's function of a p-nonparabolic Riemannian manifold vanishes at infinity, then such solution exists for any open bounded subset $\Omega \subset M$ with smooth boundary. A complete and self contained proof of this fact is provided in the Appendix of [FM20]. We report the statement of such basic though fundamental result.

Theorem 2.9 (Existence of the p-capacitary potential)**.** *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Assume also that the p-Green's function G_p satisfies $G_p(o, x) \to 0$ as $d_g(o, x) \to +\infty$ for some $o \in M$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then, there exists a unique solution u_p to (1.1).*

We want to underline that, the existence of a p-Green's function does not guarantee that it vanishes at infinity. This last property is related to the geometry of all ends. We refer the reader to [Hol90; Hol99] for a detailed discussion on this topic.

It is convenient to recall here the definition of p-capacity of an open bounded subset $\Omega \subset M$ together with a normalised version of it which turns out to be more convenient for our computations.

Definition 2.10 (p-capacity and normalised p-capacity)**.** Let (M, g) be a complete noncompact Riemannian manifold, and let Ω be an open bounded subset of M. The *p-capacity* of Ω is defined

as

$$\mathrm{Cap}_p(\Omega) = \inf \left\{ \int_M |\mathrm{D}v|^p \, \mathrm{d}\mu \; \middle| \; v \in \mathscr{C}_c^\infty(M), \, v \geq 1 \text{ on } \Omega \right\}.$$

On the other hand, the *normalised p-capacity* of Ω is defined as

$$\mathrm{C}_p(\Omega) = \inf \left\{ \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_M |\mathrm{D}v|^p \, \mathrm{d}\mu \; \middle| \; v \in \mathscr{C}_c^\infty(M), \, v \geq 1 \text{ on } \Omega \right\}.$$

A function u solving (1.1) realises the *p*-capacity of the initial set Ω, and actually one can also characterise such quantity with a suitable integral on $\partial\Omega$. We resume these facts in the following statements, whose proof can be found in [BFM21, Section 2.2]

Proposition 2.11. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold. Assume also that the p-Green's function G_p satisfies $G_p(o, x) \to 0$ as $d_g(o, x) \to +\infty$ for some $o \in M$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then the solution u_p to (1.1) realises*

$$\mathrm{C}_p(\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{M \smallsetminus \overline{\Omega}} |\mathrm{D}u_p|^p \, \mathrm{d}\mu$$

Moreover, we have that

$$\mathrm{C}_p(\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\{u_p = 1/t\}} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma. \tag{2.3}$$

holds for almost every $t \in [1, +\infty)$, including any $1/t$ regular value for u_p.

In particular, evaluating (2.3) at $t = 1$, which is a regular value by the Hopf Maximum Principle, we have that

$$\mathrm{C}_p(\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\partial\Omega} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma.$$

Moreover, one can actually relate the capacity of $\Omega_t = \{u > 1/t\} \cup \Omega$ to the capacity of Ω.

Proposition 2.12. *Let (M, g) be a complete noncompact p-nonparabolic Riemannian manifold, for some $p > 1$. Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then the solution u_p to (1.1) realises*

$$\mathrm{C}_p(\Omega_t) = t^{p-1} \mathrm{C}_p(\Omega) \tag{2.4}$$

for every $t \in [1, +\infty)$, where $\Omega_t = \{u > 1/t\} \cup \Omega$. In particular, the map $t \mapsto \mathrm{C}_p(\Omega_t)$ is smooth.

From now on, unless where it is necessary, we fix $1 < p$ and we drop the subscript p when we consider a solution u_p to the problem (1.1).

Given an end $E \subset M$ and a set $\Omega \subset M$ such that $\partial E \subset \Omega$ we define the capacity of Ω restricted to E as

$$\mathrm{Cap}_p(\Omega; E) = \inf \left\{ \int_E |\mathrm{D}v|^p \, \mathrm{d}\mu \; \middle| \; v \in \mathscr{C}_c^\infty(M), \, v \geq 1 \text{ on } \Omega \right\}.$$

The set function $\mathrm{Cap}_p(\,\cdot\,; E)$ has the same properties of the standard *p*-capacity in Definition 2.10.

2.3. **Preliminary properties of Asymptotically Conical manifolds.** We give here the precise definition of Asymptotically Conical Riemannian manifolds according to [CEV17]. For better comprehension, we recall the definition of the Hölder seminorm of a tensor field. A tensor field $T \in \mathcal{T}_s^q(M)$ is α-Hölder continuous at x for some $\alpha \in [0,1]$ if there exists a geodesically convex open neighbourhood U_x centred at x such that

$$\sup_{y \in U_x \smallsetminus \{x\}} \frac{|T(x) - T(y)|_g}{(d_g(x,y))^\alpha} < +\infty$$

is finite, where, to compute the difference between $T(x)$ and $T(y)$, we parallel transport $T(y)$ onto x. The tensor field T is said to be α-Hölder continuous on $U \subset M$ if it is α-Hölder continuous at every $x \in U$. We sometimes omit the subscript g if it is clear the metric we are referring to.

Consider a *cone with link L*, namely $((0, +\infty) \times L, \hat{g})$ where $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$. In this case, let $s > 0$ be such that $B_s(x)$ is geodesically convex in $((0, +\infty) \times L, \hat{g})$ for every $x \in \{1\} \times L$. Then, for every $x \in (0, +\infty) \times L$ the ball of radius $s\rho(x)$ centred at x is still geodesically convex, where $\rho : (0, +\infty) \times L \to (0, +\infty)$ is the projection onto the first coordinate. Given an α-Hölder continuous tensor field T, we define the α-Hölder seminorm of T at x as

$$[T]_{\alpha, \hat{g}}^{(s)}(x) = \sup_{y \in B_{s\rho(x)}(x) \smallsetminus \{x\}} \frac{|T(x) - T(y)|_{\hat{g}}}{(\mathrm{dist}(x,y))^\alpha}.$$

Observe that, if T is bounded (with respect to $|\cdot|_{\hat{g}}$) and $s, t > 0$ satisfy the above assumptions, $[T]_{\alpha, \hat{g}}^{(s)}(x) = [T]_{\alpha, \hat{g}}^{(t)}(x)$ for any $x \in (R, +\infty) \times L$ provided R is large enough. Then, the following definition is well-posed and we can drop the superscript (s).

Definition 2.13 ($\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian manifolds)**.** Let (M, g) be a Riemannian manifold, $k \in \mathbb{N}$ and $\alpha \in [0, 1)$. M is said to be $\mathscr{C}^{k,\alpha}$-*Asymptotically Conical* if there exists an open bounded subset $K \subseteq M$, a closed smooth hypersurface L and a diffeomorphism $\pi : M \smallsetminus K \to [1, +\infty) \times L$ such that

$$\sum_{i=0}^{k} \rho^i \left| \mathrm{D}_{\hat{g}}^{(i)}(\pi_* g - \hat{g}) \right|_{\hat{g}} + \rho^{k+\alpha} \left[\mathrm{D}_{\hat{g}}^{(k)}(\pi_* g - \hat{g}) \right]_{\alpha, \hat{g}} = o(1), \tag{2.5}$$

as $\rho \to +\infty$, where $\rho : [1, +\infty) \times L \to [1, +\infty)$ is the projection map onto the first component and $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ is the cone metric. We use the convention $\mathscr{C}^k = \mathscr{C}^{k,0}$.

Definition 2.13 says that in a $\mathscr{C}^{k,\alpha}$-Asymptotically Conical Manifold the metric g approaches the metric $\hat{g}$ of a *truncated cone with link L* with respect to a scaling invariant $\mathscr{C}^{k,\alpha}$-norm. The diffeomorphism $\pi : M \smallsetminus K \to [1, +\infty) \times L$ identifies the boundary of K with the link L. With abuse of notation, $\pi_* \rho : M \smallsetminus K \to [1, +\infty)$ will be denoted by ρ and $\pi_* \hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ by $\hat{g}$. Moreover, by convention the set $\{\rho < 1\}$ is used to denote K and accordingly $\{\rho \le r\} = M \smallsetminus \{\rho > r\}$ and $\{1 \le \rho \le r\} = M \smallsetminus (\{\rho > r\} \cup K)$. Given any coordinate system $(\vartheta^1, \ldots, \vartheta^{n-1})$ on an open subset U of L, $(\rho, \vartheta^1, \ldots, \vartheta^{n-1})$ are coordinates on $(1, +\infty) \times U \subset M \smallsetminus \overline{K}$. The condition $|g - \hat{g}|_{\hat{g}} = o(1)$ as $\rho \to +\infty$ is equivalent to a condition on the coordinates that can be read as

$$g_{\rho\rho} = 1 + o(1) \qquad\qquad g_{\rho j} = o(\rho) \qquad\qquad g_{ij} = \rho^2 g_{ij}^L + o(\rho^2)$$

for every $i, j = 1, \ldots, n-1$ as $\rho \to +\infty$. By using Cramer's rule to solve the system and Laplace expansion to compute determinants, we obtain

$$g^{\rho\rho} = 1 + o(1) \qquad\qquad g^{\rho j} = o(\rho^{-1}) \qquad\qquad g^{ij} = \rho^{-2} g_L^{ij} + o(\rho^{-2}).$$

The $\mathscr{C}^{0,\alpha}$-Asymptotically Conical condition for $\alpha > 0$ gives, in addition, information on the Hölder seminorm of the components. Indeed, arguing as before we get that

$$[g_{\rho\rho} - 1]_{\alpha,\hat{g}} = o(\rho^{-\alpha}) \qquad [g_{\rho j}]_{\alpha,\hat{g}} = o(\rho^{1-\alpha}) \qquad \left[g_{ij} - \rho^2 g_{ij}^L\right]_{\alpha,\hat{g}} = o(\rho^{2-\alpha})$$

for every $i, j = 1, \ldots, n-1$ as $\rho \to +\infty$. Increasing k in the $\mathscr{C}^{k,\alpha}$-Asymptotically Conical assumption we gain knowledge about the k-th derivative of the components of g. Increasing k in the $\mathscr{C}^{k,\alpha}$-Asymptotically Conical assumption we gain knowledge about the k-th derivative of $(g - \hat{g})$.

Consider for every $s > 0$ the family of diffeomorphism on $(0, +\infty) \times L$ defined as

$$
\begin{aligned}
\omega_s : \quad (0, +\infty) \times L \quad &\longrightarrow \quad (0, +\infty) \times L \\
(\rho, \vartheta^1, \ldots, \vartheta^{n-1}) \quad &\longmapsto \quad (s\rho, \vartheta^1, \ldots, \vartheta^{n-1}),
\end{aligned}
\tag{2.6}
$$

With abuse of language we will also denote by ω_s any restriction of it to some truncated cone. Since ω_s induces a family of diffeomorphisms from $(1/s, +\infty) \times L$ onto $\{\rho \geq 1\} \subset M$ through the composition with π in Definition 2.13, we will also denote by ω_s such map. The condition (2.5) can be also interpreted as the convergence of the family of metrics on the cone $(0, +\infty) \times L$, built for every $s \geq 1$ by pulling the metric g back through the diffeomorphism ω_s and properly rescaling them. This is the content of the following lemma.

Lemma 2.14. *A complete Riemannian manifold (M, g) is $\mathscr{C}^{k,\alpha}$-Asymptotically Conical if and only if the metric $g_{(s)} = s^{-2}\omega_s^* g$ satisfies*

$$\sum_{i=0}^{k} \rho^i \left| D_{\hat{g}}^{(i)}(g_{(s)} - \hat{g}) \right|_{\hat{g}} + \rho^{k+\alpha} \left[D_{\hat{g}}^{(k)}(g_{(s)} - \hat{g}) \right]_{\alpha,\hat{g}} = o(1),$$

as $s \to +\infty$ on $[R, +\infty) \times L$ for every $R > 0$.

Proof. Since $\omega_s^* \, d\rho = r \, d\rho$ and $\omega_s^* \, d\vartheta^i = d\vartheta^i$, is clear that $s^{-2}\omega_s^* \hat{g} = \hat{g}$. Thus the the case of $\mathscr{C}^0$-Asymptotically Conical manifold follows from algebra operations on tensors. The result for $k \in \mathbb{N}$ and $\alpha \in [0, 1]$ follows in the same way from the fact that $D_{\hat{g}} s^{-2} \omega_s^* g = s^{-2} \omega_s^* (D_{\hat{g}} g)$ and $\text{dist}_{\hat{g}}(x, y) = \text{dist}_{\hat{g}}(\omega_s(x), \omega_s(y))/s$ for every $x, y \in (1/s, +\infty) \times L$. $\square$

We want to highlight the relation between the coordinate ρ and the distance induced by g on M.

Lemma 2.15. *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold and $o \in M$. Then*

$$\lim_{d_g(o,x) \to +\infty} \frac{d_g(o, x)}{\rho(x)} = 1. \tag{2.7}$$

Observe that since K is compact there exist a $R > 0$ such that $d_g(o, x) > R$ implies $x \in M \setminus K$, hence (2.7) makes sense. Since π is a diffeomorphism, taking the limit for $d_g(o, x) \to +\infty$ is the same of taking it for $\rho(x) \to +\infty$.

Proof. Since $|D\rho|_g = 1 + o(1)$, for every $\varepsilon > 0$ we can find $R_\varepsilon > 1$ such that $1 - \varepsilon \leq |D\rho|_g \leq 1 + \varepsilon$ on $\{\rho \geq R_\varepsilon\}$. Pick $x \in \{\rho \geq R_\varepsilon\}$ and a curve $\gamma : [R_\varepsilon, \rho(x)] \to M$ which is the solution to the problem

$$\begin{cases} \dot{\gamma}(s) = \frac{D\rho}{|D\rho|^2}(\gamma(s)), \\ \gamma(\rho(x)) = x. \end{cases}$$

Computing the length of γ we get

$$L(\gamma) = \int_{R_\varepsilon}^{\rho(x)} |\dot\gamma(s)|\, \mathrm{d}s = \int_{R_\varepsilon}^{\rho(x)} \frac{1}{|\mathrm{D}\rho|_g}(\gamma(s))\, \mathrm{d}s \leq \frac{\rho(x) - R_\varepsilon}{1 - \varepsilon},$$

which ensures that

$$\limsup_{\rho(x)\to+\infty} \frac{d_g(o,x)}{\rho(x)} \leq \limsup_{\rho(x)\to+\infty} \frac{L(\gamma) + 2\operatorname{diam}(\{\rho \leq R_\varepsilon\})}{\rho(x)} \leq \frac{1}{1-\varepsilon}.$$

Conversely, consider any geodesic $\sigma : [0, L] \to M$, parametrised by arc length, joining $\sigma(0) \in \{\rho = R_\varepsilon\}$ and $\sigma(L) = x$. Then we obtain

$$\rho(x) - R_\varepsilon = \int_0^L \langle \mathrm{D}\rho \mid \dot\sigma(s)\rangle\, \mathrm{d}s \leq \int_0^L |\mathrm{D}\rho|_g(\sigma(s))\, \mathrm{d}s \leq (1 + \varepsilon)L$$

which yields

$$\liminf_{\rho(x)\to+\infty} \frac{d_g(x,o)}{\rho(x)} \geq \liminf_{\rho(x)\to+\infty} \frac{L - 2\operatorname{diam}(\{\rho \leq R_\varepsilon\})}{\rho(x)}$$

$$\geq \liminf_{\rho(x)\to+\infty} \frac{\rho(x) - R_\varepsilon - 2\operatorname{diam}(\{\rho \leq R_\varepsilon\})}{(1+\varepsilon)\rho(x)} = \frac{1}{1+\varepsilon}.$$

By the arbitrariness of $\varepsilon > 0$ we can conclude. $\qquad\square$

In Riemannian manifolds with nonnegative Ricci curvature, in virtue of Bishop-Gromov theorem, one can define an Asymptotic Volume Ratio since

$$\operatorname{AVR}(g) = \lim_{r\to+\infty} \frac{|B(o, R)|}{|\mathbb{B}^n| R^n}$$

exists and does not depend on $o \in M$. Here we relaxed the condition on Ricci curvature so we can not apply Bishop-Gromov theorem, but on the other side we require an asymptotic behaviour for the metric, that allows to define an Asymptotic Volume Ratio as well.

Lemma 2.16. *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Then*

$$\frac{|L|}{|\mathbb{S}^{n-1}|} = \lim_{R\to+\infty} \frac{|\{1 \leq \rho \leq R\}|}{|\mathbb{B}^n| R^n} = \lim_{R\to+\infty} \frac{|\{\rho = R\}|}{|\mathbb{S}^{n-1}| R^{n-1}} \qquad (2.8)$$

where L is the link of the cone (M, g) is asymptotic to.

Proof. One can easily show that $\det(g) = \det(\hat{g})(1 + o(1)) = \rho^{2(n-1)}\det(g_L)(1 + o(1))$. Hence, for every $\varepsilon > 0$ there exists $R_\varepsilon \geq 1$ such that

$$|L|(1 - \varepsilon)\frac{(R^n - R_\varepsilon^n)}{n} \leq |\{R_\varepsilon \leq \rho \leq R\}| \leq |L|(1 + \varepsilon)\frac{(R^n - R_\varepsilon^n)}{n}$$

Dividing each term by $|\mathbb{B}^n| R^n$ one get

$$|L|(1 - \varepsilon)\frac{(R^n - R_\varepsilon^n)}{n|\mathbb{B}^n| R^n} \leq \frac{|\{R_\varepsilon \leq \rho \leq R\}|}{|\mathbb{B}^n| R^n} \leq |L|(1 + \varepsilon)\frac{(R^n - R_\varepsilon^n)}{n|\mathbb{B}^n| R^n}.$$

Since $|\{1 \leq \rho \leq R_\varepsilon\}|/(|\mathbb{B}^n| R^n)$ vanishes as $R \to +\infty$ we obtain

$$(1 - \varepsilon)\frac{|L|}{n|\mathbb{B}^n|} \leq \lim_{R\to+\infty} \frac{|\{1 \leq \rho \leq R\}|}{|\mathbb{B}^n| R^n} \leq (1 + \varepsilon)\frac{|L|}{n|\mathbb{B}^n|}$$

which in turn gives the first identity in (2.8) by arbitrariness of ε. We turn to prove the second identity. Since De L'Hôpital rule gives

$$\lim_{R \to +\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}R}|\{1 \le \rho \le R\}|}{|\mathbb{S}^{n-1}|R^{n-1}} = \lim_{R \to +\infty} \frac{|\{1 \le \rho \le R\}|}{|\mathbb{B}^n|R^n} = \frac{|L|}{|\mathbb{S}^{n-1}|}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}R}|\{1 \le \rho \le R\}| = \frac{\mathrm{d}}{\mathrm{d}R} \int_1^R \int_{\{\rho=s\}} \frac{1}{|\mathrm{D}\rho|} \, \mathrm{d}\sigma \, \mathrm{d}s = |\{\rho = R\}|(1 + o(1)),$$

we conclude the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Coupling this result with Lemma 2.15 one gets that

$$\lim_{R \to +\infty} \frac{|B(o, R)|}{R^n|\mathbb{B}^n|} = \lim_{R \to +\infty} \frac{|\{1 \le \rho \le R\}|}{|\mathbb{B}^n|R^n} = \frac{|L|}{|\mathbb{S}^{n-1}|}$$

for every $o \in M$. Hence the left hand side limit exists and does not depend on the point $o \in M$. We can finely give the following definition.

Definition 2.17. Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. The Asymptotic Volume Ratio of (M, g) is defined as

$$\mathrm{AVR}(g) = \frac{|L|}{|\mathbb{S}^{n-1}|}$$

where L is the link of the cone (M, g) is asymptotic to.

In this case $0 < \mathrm{AVR}(g)$, but in general $\mathrm{AVR}(g)$ could exceed 1 and $\mathrm{AVR}(g) = 1$ does not imply that (M, g) is isometric to the flat Euclidean space.

As already observed in the Introduction, a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold is forced to have a finite number of ends.

Lemma 2.18. *A complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold (M, g) has finitely many ends with connected boundary each of them is diffeomorphic to $[1, +\infty) \times L_i$ where L_i is a connected component of the link of the asymptotic cone.*

Proof. Since L is a compact hypersurface, it has a finite number of connected component. Each end with respect to K is therefore diffeomorphic to a cone on a connected component of L. $\quad\square$

As already mentioned in the Introduction, given $E_1, \ldots, E_N$ the ends of M, each E_i is $\mathscr{C}^0$-Asymptotically Conical and we can define the Asymptotic Volume ratio of E_i as

$$\mathrm{AVR}(g; E_i) = \lim_{R \to +\infty} \frac{|B(o, R) \cap E_i|}{|\mathbb{B}^n|R^n} = \frac{|L_i|}{|\mathbb{S}^{n-1}|}, \tag{2.9}$$

for every $i = 1, \ldots, N$, where as a above L_i denotes a connected component of the link of the asymptotic cone. Moreover, the Asymptotic Volume Ratio of (M, g) splits as

$$\mathrm{AVR}(g) = \sum_{i=1}^N \mathrm{AVR}(g; E_i) \tag{2.10}$$

where $\mathrm{AVR}(g) > 0$ is the Asymptotic Volume Ratio of (M, g).

If $\mathrm{Ric}_L \ge (n - 2)g_L$, then the cone $([1, +\infty) \times L, \hat{g})$ has nonnegative Ricci curvature and in particular $\mathrm{AVR}(g; E_i) \le 1$ for every $i = 1, \ldots, k$. This condition is automatically true if the manifold is $\mathscr{C}^0$-Asymptotically Conical and Ric satisfies (1.2), thanks to the following lemma.

Lemma 2.19. *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Suppose that $\mathrm{Ric}_g \geq -f(\mathrm{dist}(x, o))$ for some nonnegative smooth function $f(t) = o(1)$ as $t \to +\infty$, for some $o \in M$. Then $\mathrm{Ric}_{\hat{g}} \geq 0$, where $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$ is the asymptotical conic metric of g and L is the link of the limit cone. In particular, $\mathrm{AVR}(g) \leq N$ where N is the number of connected component of the link L.*

Proof. By Lemma 2.14 we can assume that f is a function of ρ. Let ω_s be as in (2.6), we denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s, +\infty) \times L$. It is easy to prove that $([1/s, +\infty) \times L, \mathrm{dist}_{g_{(s)}}, x)$ converges in the pointed-Gromov-Hausdorff topology to $([0, +\infty) \times L, \mathrm{dist}_{\hat{g}}, x)$ for some $x \in \{\rho = 2\}$. Since

$$\lim_{s \to +\infty} |B(x, 1)|_{g_{(s)}} = |B(x, 1)|_{\hat{g}}$$

by [DG18, Theorem 1.2] $([1/s, +\infty) \times L, \mathrm{dist}_{g_{(s)}}, \mu_{g_{(s)}}, x)$ converges to $([0, +\infty) \times L, \mathrm{dist}_{\hat{g}}, \mu_{\hat{g}}, x)$ in the pointed-measured-Gromov-Hausdorff topology. By [GMS15, Theorem 7.2] $\mathrm{Ric}_{\hat{g}} \geq -f(s)$ for every s, hence $\mathrm{Ric}_{\hat{g}} \geq 0$. In particular, $|L_i| \leq |\mathbb{S}^{n-1}|$, hence $\mathrm{AVR}(g) \leq N$ by (2.9) and (2.10). $\square$

In [MRS21, Theorem 1.7] the authors guarantee the existence of the (weak) IMCF starting at $\Omega \subseteq M$ with smooth boundary whenever the Ricci curvature satisfies a nondecreasing lower bound and a global L^1-Sobolev Inequality is in force, that is

$$\left(\int_M |\varphi|^{\frac{n}{n-1}} \, \mathrm{d}\mu \right)^{\frac{n-1}{n}} \leq \mathrm{C}_S \int_M |\mathrm{D}\varphi| \, \mathrm{d}\mu \tag{2.11}$$

for every $\varphi \in \mathrm{Lip}_c(M)$ where C_S is some finite constant depending only on the geometry of the manifold.

It is well known that the existence of a finite constant C_S for the L^1-Sobolev Inequality is equivalent to the existence of a positive Isoperimetric constant, which is a $\mathrm{C}_{\mathrm{Iso}} > 0$ such that

$$\mathrm{C}_{\mathrm{Iso}} \leq \frac{|\partial K|^n}{|K|^{n-1}} \tag{2.12}$$

for every compact domain K (see [FF60, Remark 6.6] or [SY94, pp. 89-90]).

Riemannian manifolds with nonnegative Ricci curvature and Euclidean Volume Growth satisfies both (2.11) and (2.12) as observed by [Var85] (see also [Car94] and [Heb99, Theorem 8.4]). It is then plausible that these inequalities hold also when the manifold asymptotically behaves as a cone with nonnegative Ricci curvature.

Proposition 2.20. *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Suppose that $\mathrm{Ric}_g \geq -f(\mathrm{dist}(x, o))$ for some nonnegative smooth function $f(t) = o(1)$ as $t \to +\infty$, for some $o \in M$. Then (M, g) admits a global L^1-Sobolev Inequality (2.11) for some finite constant C_S or equivalently it has a positive isoperimetric constant $\mathrm{C}_{\mathrm{Iso}} > 0$.*

Proof. In virtue of [PST14, Theorem 3.2] it is enough to prove that a L^1-Sobolev Inequality is satisfied outside some compact set. By Lemma 2.18 (M, g) has only a finite number of ends each of them corresponding to one connected component of the link L. Thus, we can assume that (M, g) has only one Asymptotically Conical end E. By Lemma 2.19, E asymptotically behaves as a cone with nonnegative Ricci curvature, that satisfies the L^1-Sobolev Inequality for some constant $\hat{C}$. Suppose by contradiction that for every compact $K \subset E$, the L^1-Sobolev Inequality is not satisfied on $E \smallsetminus K$. Since the metric g converges to the metric $\hat{g} = \mathrm{d}\rho^2 + \rho^2 g_L$, for every

$\varepsilon > 0$ there exists a compact K_ε such that

$$\left| \int_M |\varphi|^{\frac{n}{n-1}} \, \mathrm{d}\mu_g - \int_M |\varphi|^{\frac{n}{n-1}} \, \mathrm{d}\mu_{\hat{g}} \right| \leq \varepsilon \int_M |\varphi|^{\frac{n}{n-1}} \, \mathrm{d}\mu_{\hat{g}}$$

and

$$\left| \int_M |\mathrm{D}\varphi|_g \, \mathrm{d}\mu_g - \int_M |\mathrm{D}\varphi|_{\hat{g}} \, \mathrm{d}\mu_{\hat{g}} \right| \leq \varepsilon \int_M |\mathrm{D}\varphi|_{\hat{g}} \, \mathrm{d}\mu_{\hat{g}}$$

for every $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$. Moreover, for every C there exists a function $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$ such that

$$\left(\int_M |\varphi|_g^{\frac{n}{n-1}} \, \mathrm{d}\mu_g \right)^{\frac{n-1}{n}} > \mathrm{C} \int_M |\mathrm{D}\varphi|_g \, \mathrm{d}\mu_g$$

Then for every $\varepsilon < 1$ and C we have $\varphi \in \mathrm{Lip}_c(E \smallsetminus K_\varepsilon)$ that satisfies

$$\left(\int_M |\varphi|_{\hat{g}}^{\frac{n}{n-1}} \, \mathrm{d}\mu_{\hat{g}} \right)^{\frac{n-1}{n}} \geq \frac{1}{(1+\varepsilon)^{\frac{n-1}{n}}} \left(\int_M |\varphi|_g^{\frac{n}{n-1}} \, \mathrm{d}\mu_g \right)^{\frac{n-1}{n}} > \frac{\mathrm{C}}{(1+\varepsilon)^{\frac{n-1}{n}}} \int_M |\mathrm{D}\varphi|_g \, \mathrm{d}\mu_g$$

$$\geq \mathrm{C} \frac{(1-\varepsilon)}{(1+\varepsilon)^{\frac{n-1}{n}}} \int_M |\mathrm{D}\varphi|_{\hat{g}} \, \mathrm{d}\mu_{\hat{g}}.$$

It is enough to choose $\varepsilon < 1$ and C such that

$$\mathrm{C} \frac{(1-\varepsilon)}{(1+\varepsilon)^{\frac{n-1}{n}}} > \hat{\mathrm{C}}$$

to obtain a contradiction to the L^1-Sobolev Inequality on the asymptotic cone. $\qquad\square$

With some analogy, considering $\Omega \subseteq M$ some open subset with smooth boundary and $u_p : M \smallsetminus \Omega \to \mathbb{R}$ the p-capacitary potential associated to Ω, $1 < p < n$, we recall from the Introduction that

$$\mathrm{C}_p^{(i)}(\Omega) = \left(\frac{p-1}{n-p} \right)^{p-1} \frac{1}{|\mathbb{S}^{n-1}|} \int_{\{u_p = 1/t\} \cap E_i} |\mathrm{D}u_p|^{p-1} \, \mathrm{d}\sigma = \frac{\mathrm{C}_p \left(\{u \leq \frac{1}{t}\} \cap E_i; E_i \right)}{t^{p-1}}. \qquad (2.13)$$

This is well defined since for every $T \in [1, +\infty)$ large enough, $\{u_p > 1/T\}$ contains the compact K in Definition 2.13 and the quantity in (2.13) does not depend on $t \geq T$. Moreover, it is readily checked that, by (2.3), the p-capacity of Ω splits as

$$\mathrm{C}_p(\Omega) = \sum_{i=1}^{N} \mathrm{C}_p^{(i)}(\Omega).$$

The quantity $\mathrm{C}_p^{(i)}(\Omega)$ represents the portion of Ω that contributes to its p-capacity under the influence of the end E_i. In the case Ω is already contains K in Definition 2.17 it is exactly the p-capacity of $\Omega \cap E_i$.

On cones, the p-capacity of the cross section $\{\rho = r\}$ can be easily computed since the function $u = (\rho/r)^{-(n-p)/(p-1)}$ is the p-capacitary potential associated to these sets. In Asymptotically Conical Riemannian manifold one might expect that the p-capacity of $\{\rho = r\}$ approaches the model one for large r. Despite the definition of the p-capacity involves the first order derivatives

of the p-capacitary potential, the convergence is also true even if the metric converges only in the $\mathscr{C}^0$-topology.

Lemma 2.21. *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Let ρ be the radial coordinate. Then*

$$\lim_{r \to +\infty} \frac{\operatorname{Cap}_p(\{\rho \leq r\})}{r^{n-p}|\mathbb{S}^{n-1}|} = \left(\frac{n-p}{p-1}\right)^{p-1} \operatorname{AVR}(g). \tag{2.14}$$

Proof. Since the metric g converges to the metric $\hat{g}$, for every $\varepsilon > 0$ there exists a $R_\varepsilon > 0$ such that for every $r \geq R_\varepsilon$

$$\left| \int_M |\mathrm{D}\varphi|_g^p \, \mathrm{d}\mu_g - \int_M |\mathrm{D}\varphi|_{\hat{g}}^p \, \mathrm{d}\mu_{\hat{g}} \right| \leq \varepsilon \int_M |\mathrm{D}\varphi|_{\hat{g}}^p \, \mathrm{d}\mu_{\hat{g}}$$

holds for every function $\varphi \in \mathscr{C}_c^\infty(\{\rho \geq r\})$ such that $\varphi = 1$ on $\{\rho = r\}$. In particular, we have that

$$(1 - \varepsilon) \int_M |\mathrm{D}\varphi|_{\hat{g}}^p \, \mathrm{d}\mu_{\hat{g}} \leq \int_M |\mathrm{D}\varphi|_g^p \, \mathrm{d}\mu_g \leq (1 + \varepsilon) \int_M |\mathrm{D}\varphi|_{\hat{g}}^p \, \mathrm{d}\mu_{\hat{g}}.$$

The set $\{\rho \geq r\}$ is diffeomorphic to $[r, +\infty) \times L$ where L is the cross section of the cone (M, g) is asymptotic to. Hence, the family of φ considered above are in one-to-one correspondence with the competitors for the p-capacity of $\{\rho \leq r\}$ in the Riemannian cone $[r, +\infty) \times L$. Dividing each side by $|\mathbb{S}^{n-1}|$, recalling the characterisation of $\operatorname{AVR}(g)$ in Definition 2.17 and taking the infimum on each side of the previous chain of inequalities we are left with

$$(1 - \varepsilon)r^{n-p} \operatorname{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1} \leq \frac{\operatorname{Cap}_p(\{\rho \leq r\})}{|\mathbb{S}^{n-1}|} \leq (1 + \varepsilon)r^{n-p} \operatorname{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1}.$$

dividing each term by r^{n-p} and sending to the limit as $r \to +\infty$ we have that

$$(1 - \varepsilon) \operatorname{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1} \leq \lim_{r \to +\infty} \frac{\operatorname{Cap}_p(\{\rho \leq r\})}{r^{n-p}|\mathbb{S}^{n-1}|} \leq (1 + \varepsilon) \operatorname{AVR}(g) \left(\frac{n-p}{p-1}\right)^{p-1}$$

which in turns gives (2.14) by arbitrariness of ε. $\qquad\square$

2.4. Li-Yau-type estimates and gradient bound.

We will provide Li-Yau-type estimates for the p-Green function G_p with a controlled constant as $p \to 1^+$. These estimates will be the starting point for the proof of both Theorems 1.1 and 1.2. We highlight that in [MRS21] the authors provided Li-Yau-type estimates for p-harmonic functions. The upper bound in [MRS21, Theorem 3.6] carried out in a broader setting is actually in term of the distance. Conversely, the lower bound in [MRS21, Corollary 2.8] is in terms of the distance in a model which has the same radial sectional curvature of the lower bound in (1.2). This estimate does not seem sufficient for our aims.

However, since our setting disposes of a precise asymptotic structure we can improve such lower bound by inheriting some techniques coming from [Hol99], and using the natural foliaton of ends induced by the cross-sections of the asymptotic cone. To accomplish this program we first need a global Harnack Inequality holding on each end of (M, g).

Proposition 2.22 (Harnack's Inequality). *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying (1.2) and let $p > 1$. A uniform Harnack Inequality*

holds on every $\{\rho = R\} \cap E$, that is there exists a constant $C_H > 0$ that depends only on the dimension and p but not on R such that

$$\inf_{\{\rho=R\}\cap E} u \geq C_H^{\frac{1}{p-1}} \sup_{\{\rho=R\}\cap E} u \tag{2.15}$$

for every positive p-harmonic function u on E. Moreover, C_H is bounded as $p \to 1^+$

Proof. Proposition 2.20 and [Hol99, Example 2.20] guarantees that the hypotheses of [MRS21, Theorem 3.4] are satisfied on $B(x, 6r)$ for every $x \in M \setminus B(o, 2R)$. Hence we have that

$$\sup_{B(x,R)} u \leq C^{\frac{1}{p-1}} \inf_{B(x,R)} u \tag{2.16}$$

for every positive p-harmonic function u on E, for some constant which is bounded for large R and in p as $p \to 1^+$. Since the $\{\rho = R\} \cap E$ is connected and its diameter increases linearly in R by the asymptotic assumption, it can be covered with N balls of radius R, for some N not depending on R. By chaining (2.16) N times we obtain (2.15). $\square$

As a consequence we can overcome the main issue in [Hol99, Proposition 5.9], that is a control on the bounded components of $E \setminus B(o, r)$. The proof of [Hol99, Proposition 5.9] suggest also an explicit value for the constant.

Proposition 2.23. *Let* (M, g) *be a* $\mathscr{C}^0$*-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying* (1.2) *and* $o \in M$. *For every end* E *of* M *there exists a constant* $C > 0$ *such that*

$$\sup_{\partial E(r)} G(o, x) \geq C^{\frac{1}{p-1}} \int_{2r}^{+\infty} \left(\frac{t}{|B(o,t) \cap E|} \right)^{\frac{1}{p-1}} dt \tag{2.17}$$

holds for every $r > 0$ *where* $E(r)$ *is the unbounded component of* $E \setminus \overline{B(o, r)}$. *Moreover, the constant* $C = a \operatorname{Cap}_p(\overline{K}; E)$ *where* $a > 0$ *is such that* $G_p(x, o) \geq a^{1/(p-1)}$ *on* $M \setminus K$ *and* K *is the compact set in Definition 2.13.*

Proof. We already observed that (M, g) as a finite number of ends with connected boundary in Lemma 2.18. Denote by E one of them. Since E is $\mathscr{C}^0$-Asymptotically Conical, the volume of $B(o, r) \cap E$ grows like r^n, then for some large $R > 0$

$$\int_R^{+\infty} \left(\frac{r}{|B(o,r) \cap E|} \right)^{\frac{1}{p-1}} dr \leq CR^{-\frac{n-p}{p-1}} < +\infty$$

that is E is p-large. E satisfies a weak $(1, p)$-Poincaré Inequality and a volume-doubling property (see [Hol99, Example 2.20]. Moreover, E satisfies the volume comparison condition which means that there exists a constant C_v such that

$$|E \cap B(o, r)| \leq C_v |B(x, r/8)| \tag{2.18}$$

holds for any $r \geq R$ and $x \in \partial B(o, r) \cap E$. Observe that (2.18) holds on the cones. Indeed, since L is compact $|B(x, r/8)| \geq \mu r^n / 8^n$ for some positive $\mu > 0$. If E is merely $\mathscr{C}^0$-Asymptotically Conical, both the volume of $E \cap B(o, r)$ and $B(x, r/8)$ are approaching the corresponding ones on the cone, which proves (2.18).

With the very same arguments as in [Hol99, Proposition 5.9] one can prove (2.17), by restricting all quantities to the given end. A close look to [Hol99, Proposition 5.9] gives also that the constant can be chosen as above, firstly choosing $a > 0$ and following the computations accordingly. $\square$

The above Proposition implies the following double-sided bound on G_p, that will in turn imply an analogous one for u_p. Such double bound will be key for working out the asymptotics.

Theorem 2.24 (Li-Yau-type estimates). *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with* Ric *satisfying* (1.2). *Let $p > 1$, each end of M is p-nonparabolic and, given G_p the p-Green function, then there exist two constants $\mathrm{C}_L, \mathrm{C}_U > 0$*

$$\mathrm{C}_L^{\frac{1}{p-1}} d_g(o, x)^{-\frac{n-p}{p-1}} \leq G_p(o, x) \leq \mathrm{C}_U^{\frac{1}{p-1}} d_g(o, x)^{-\frac{n-p}{p-1}}. \tag{2.19}$$

for every $x \in M \smallsetminus \{o\}$. Moreover, the constant C_U is bounded as $p \to 1^+$ and there exists $R > 0$ not depending on p such that the lower bound in (2.19) *holds with constant $\mathrm{C}_L = \mathrm{C} \, a \, \mathrm{Cap}_p(\overline{K}; E)$ on $M \smallsetminus B(o, R)$, where $a > 0$ is such that $G_p(x, o) \geq a^{1/(p-1)}$ on $M \smallsetminus K$, C does not depend on p and K is the bounded set in Definition* 2.13.

Proof. The upper bound in (2.19) with C_U bounded as $p \to 1^+$ follows from [MRS21, Theorem 3.6]. Indeed, the assumptions are satisfied since a p-Sobolev Inequality holds true as a standard consequence of the L^1-Sobolev inequality in Proposition 2.20. For what it concerns the lower bound, we are in position to apply Proposition 2.23. The main issue is that we do not have control on the bounded components of $E \smallsetminus \overline{B(o, R)}$. Consider the function $R : [1, +\infty) \to \mathbb{R}$ defined as

$$R(t) = \max \{d_g(o, x) \mid x \in \{\rho = t\}\}.$$

Observe that $\{\rho \geq t\} \supset E(2R(t))$. Then applying the Harnack's Inequality (2.15), Comparison Principle and (2.17) one gets that

$$\inf_{\{\rho=t\}} G_p(o, x) \geq \mathrm{C}_H^{\frac{1}{p-1}} \sup_{\{\rho=t\}} G_p(o, x) \geq \mathrm{C}_H^{\frac{1}{p-1}} \sup_{\partial E(2R(t))} G_p(o, x)$$

$$\geq \mathrm{C}_4^{\frac{1}{p-1}} \int_{4R(t)}^{+\infty} \left(\frac{r}{|B(o, r) \cap E|} \right)^{\frac{1}{p-1}} \mathrm{d}r \geq \mathrm{C}_5^{\frac{1}{p-1}} R(t)^{-\frac{n-p}{p-1}}.$$

for every $t \geq T$, for some T large enough not depending on p and $\mathrm{C}_5 = \mathrm{Ca}\mathrm{Cap}_p(\partial K; E)$. By Lemma 2.15 there exists $R_1 \geq R(T)$ such that $\partial B(o, r) \subset \{\rho \leq 2r\}$ and $R(2r) \leq 4r$ hold for every $r \geq R_1$. Then, by the Maximum Principle,

$$\inf_{\partial B(o,r)} G_p(o, x) \geq \inf_{\{\rho=2r\}} G_p(o, x) \geq \mathrm{C}_5^{\frac{1}{p-1}} R(2r)^{-\frac{n-p}{p-1}} \geq \left(\frac{\mathrm{C}_5}{4^{n-p}} \right)^{\frac{1}{p-1}} r^{-\frac{n-p}{p-1}},$$

holds for every $r \geq 1$, since R_1 does not depend on p. The global lower bound follows since it is satisfied near the pole o, but the new constant C_L might go to 0 as $p \to 1^+$ $\qquad\square$

Arguing as done in the proof of [BFM21, Theorem 2.15], it is easy to derive analogous estimates for u_p. We do not take trace of the best constant as in the previous result, since this tool will be only used to derive Theorem 1.1 where $1 < p < n$ is fixed.

Corollary 2.25 (Li-Yau-type estimates for the p-capacitary potential). *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with* Ric *satisfying* (1.2). *Let be $1 < p < n$. There exists a unique solution u_p to* (1.1) *and there exists a positive and finite constant C such that*

$$\mathrm{C}^{-1} d_g(x, o)^{-\frac{n-p}{p-1}} \leq u_p(x) \leq \mathrm{C} d_g(x, o)^{-\frac{n-p}{p-1}} \tag{2.20}$$

for every $x \in M \smallsetminus \Omega$.

Proof. The existence follows from (2.19) and Theorem 2.9. Moreover, In light of (2.19), it suffices to show that there exists a positive finite constant C such that $C^{-1}G_p \leq u_p \leq CG_p$. Choose any $\sup_{\partial\Omega} u_p <$ C. Then, $C^{-1}G_p < u_p$ on $\partial\Omega$. Moreover, since both u_p and G_p vanish at infinity, for any $\delta > 0$ we have $C^{-1}G_p < u_p + \delta$ on $\partial B(o, R)$ for any R big enough. The Comparison Principle applied to the p-harmonic functions $u_p + \delta$ and G_p in $B(o, R) \smallsetminus \overline{\Omega}$ shows that $C^{-1}G_p < u_p + \delta$ in the latter subset. The radius R being arbitrarily big, this implies that, by passing to the limit as $R \to +\infty$, that $C^{-1}G_p < u_p + \delta$ in the whole $M \smallsetminus \Omega$. Letting $\delta \to 0^+$ leaves with $C^{-1}G_p \leq u_p$, and consequently with the lower bound in (2.20). The inequality $u_p \leq CG_p$, yielding the upper bound, is shown the same way. $\qquad\square$

We recall the following Cheng-Yau-type estimate, proved in [WZ10], together with the consequent gradient bound for the p-capacitary potential.

Theorem 2.26 (Cheng-Yau-type estimate). *Let (M, g) a complete Riemannian manifold. Let $v \in W^{1,p}_{\mathrm{loc}} B(o, 2R)$ be a positive p-harmonic function on a geodesic ball $B(o, 2R)$ for some $R > 0$ where $\mathrm{Ric} \geq -(n-1)\kappa^2$. Then there exists a constant $C = C(p, n)$ such that*

$$\sup_{B(o,R)} |\mathrm{D}\log(v)| \leq C\left(\frac{1}{R} + \kappa\right). \tag{2.21}$$

The above local estimate enables us to provide a global gradient bound for u_p, that will be key for its asymptotic analysis.

Proposition 2.27. *Let (M, g) be a complete $\mathscr{C}^0$-Asymptotically Conical manifold with Ric satisfying (1.2). Let $\Omega \subset M$ open bounded with smooth boundary and let u_p be its p-capacitary potential. Then, there exists a constant $C > 0$ such that*

$$|\mathrm{D}\log u_p| \leq \frac{C}{\mathrm{dist}(x, o)} \tag{2.22}$$

holds on the whole $M \smallsetminus \Omega$.

Proof. By the $\mathscr{C}^1$-regularity of u, it clearly suffices to show that (2.22) holds true outside some compact set containing Ω. Let then $o \in \Omega$ and $R > 0$ be such that $\Omega \subset B(o, R)$, and let $x \in M \smallsetminus \overline{B(o, 2R)}$. With this choice, we have $B(x, \mathrm{dist}(o, x) - R) \subset M \smallsetminus \overline{B(o, R)}$. Thus, applying inequality (2.21) to the function u_p in the ball $B(x, \mathrm{dist}(o, x) - R)$ we get

$$\frac{|\mathrm{D}u_p|}{u_p} \leq 2C\left(\frac{1}{\mathrm{dist}(o, x) - R} + \frac{\kappa}{\mathrm{dist}(o, x) + 1}\right) \leq 2C\frac{(2 + \kappa)}{\mathrm{dist}(o, x)}.$$

concluding the proof. $\qquad\square$

The constant in (2.21) and consequently in (2.22) is such that $(p-1)C$ diverges as $p \to 1^+$. Up to the authors' knowledge a Cheng-Yau-type estimate with $(p-1)C$ controlled in p is yet to be discovered. This should lead to various other insights about the weak IMCF and its relations with p-harmonic potentials.

3. ASYMPTOTIC BEHAVIOUR OF THE p-CAPACITARY POTENTIAL

In this section we prove Theorem 1.1. In fact, as anticipated in the Introduction, we prove a more general statement that provides information also about the asymptotic behaviour of the derivatives of u, if the asymptotic structure of the underlying metric is suitably reinforced.

Theorem 3.1 (Asymptotic behaviour of the derivatives of p-capacitary potential)**.** *Let (M, g) be a complete $\mathscr{C}^{k,\alpha}$-Asymptotically Conical Riemannian manifold for some $\alpha > 0$ and $k \in \mathbb{N}$ with* Ric *satisfying* (1.2)*. Let $E_1, \ldots, E_N$ be the (finitely many) ends of M with respect to the compact K in Definition 2.13. Consider $\Omega \subset M$ be an open bounded subset with smooth boundary and $u : M \smallsetminus \Omega \to \mathbb{R}$ a solution of the problem* (1.1)*. Then*

$$\left| \mathrm{D}^\ell u - \left(\frac{\mathrm{C}_p^{(i)}(\Omega)}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{p-1}} \mathrm{D}^\ell \rho^{-\frac{n-p}{p-1}} \right|_{\hat{g}} = o\left(\rho^{-\frac{n-p}{p-1} - \ell} \right) \tag{3.1}$$

on E_i as $\mathrm{dist}(o, x) \to +\infty$ *for every $i = 1, \ldots, N$ and $\ell \leq k+1$.*

For future reference we want to specify the behaviour of partial derivatives coming from (3.1). Given a coordinate system $(\vartheta^1, \ldots, \vartheta^{n-1})$ on L one has that

$$\frac{\partial^{j+|\alpha|} u}{\partial \rho^j \partial \vartheta^\alpha} = \left(\frac{\mathrm{C}_p^{(i)}(\Omega)}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{p-1}} \frac{\partial^{j+|\alpha|}}{\partial \rho^j \partial \vartheta^\alpha} \left(\rho^{-\frac{n-p}{p-1}} \right) + o\left(\rho^{-\frac{n-p}{p-1} - j} \right)$$

as $\rho \to +\infty$, where α is a $(n-1)$-dimensional multi-index such that $j + |\alpha| \leq k$.

Along with the proof, we extend Lemma 2.21, showing that the p-capacity of the p-capacitary potential behaves like the p-capacity of the cross sections approaching infinity.

Proposition 3.2 (Asymptotic behaviour of the p-capacity of level sets)**.** *In the same assumptions and notations of Theorem 1.1, set, for $i = 1, \ldots, N$,*

$$v_i = \left(\frac{\mathrm{C}_p^{(i)}(\Omega)}{\mathrm{AVR}(g; E_i)} \right)^{\frac{1}{n-p}} u^{-\frac{p-1}{n-p}}.$$

Then, we have

$$\lim_{s \to +\infty} \frac{\mathrm{Cap}_p(\{v_i \leq s\} \cap E_i; E_i)}{s^{n-p} |\mathbb{S}^{n-p}|} = \left(\frac{n-p}{p-1} \right)^{p-1} \mathrm{AVR}(g; E_i).$$

Moreover, as a byproduct, we obtain the following uniqueness result on Riemannian cones.

Proposition 3.3. *Let $((0, +\infty) \times L, \hat{g})$ be a Riemannian cone with nonnegative Ricci curvature where L is a closed connected smooth hypersurface. Let u be a nonnegative p-harmonic function on $(0, +\infty) \times L$ satisfying $u(x) \leq \mathrm{C}\rho(x)^{-(n-p)/(p-1)}$ for every $x \in (0, +\infty) \times L$ for some constant $\mathrm{C} \geq 0$. Then, there exists a nonnegative $\gamma \in \mathbb{R}$ such that*

$$u(x) = \gamma \rho(x)^{-\frac{n-p}{p-1}}$$

holds on $(0, +\infty) \times L$.

Proof of Theorems 1.1 and 3.1 and Propositions 3.2 and 3.3. It is enough to prove the theorems in the case M has only one end. The proof of the general case then follows applying the result to each end. We will denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s, +\infty) \times L$, being ω_s the family of diffeomorphisms defined in (2.6). We find convenient to organise the proof in four steps. The first three steps are devoted do prove Theorem 1.1. The second and the third one contain the proof of Proposition 3.3 and Proposition 3.2 respectively. In the last we complete the proof for higher order asymptotic behaviour mainly using Schauder estimates for p-harmonic functions.

Step 1. Suppose that (M, g) is $\mathscr{C}^0$-Asymptotically Conical. Define the family of functions $u_s : [1/s, +\infty) \times L \to \mathbb{R}$ as

$$u_s(x) = s^{\frac{n-p}{p-1}} u \circ \omega_s(x)$$

where ω_s is the map defined in (2.6). The aim of this step is to prove compactness of $(u_s)_{s \geq 1}$ with respect to local uniform convergence on $(0, +\infty) \times L$. In particular, by Theorem 2.5 (see also Remark 2.6) any limit point w for the sequence $(u_s)_{s \geq 1}$ is p-harmonic with respect to the metric $\hat{g}$ on $(0, +\infty) \times L$. Moreover, there exists a positive constant C such that

$$\mathrm{C}^{-1} \rho(x)^{-\frac{n-p}{p-1}} \leq w(x) \leq \mathrm{C} \rho(x)^{-\frac{n-p}{p-1}} \tag{3.2}$$

is satisfied for every $x \in (0, +\infty) \times L$.

By Corollary 2.25 we have that

$$\mathrm{C}_1^{-1} \left(\mathrm{dist}(o, x) \right)^{-\frac{n-p}{p-1}} \leq u(x) \leq \mathrm{C}_1 \left(\mathrm{dist}(o, x) \right)^{-\frac{n-p}{p-1}}.$$

holds on $M \smallsetminus \Omega$. In particular, since by Lemma 2.15 the distance function from o behaves asymptotically as the coordinate ρ, we deduce that there exist $S_2, \mathrm{C}_2 > 0$ such that

$$\mathrm{C}_2^{-1} \rho(x)^{-\frac{n-p}{p-1}} \leq u_s(x) \leq \mathrm{C}_2 \rho(x)^{-\frac{n-p}{p-1}}, \tag{3.3}$$

holds on $[1/s, +\infty) \times L$ for every $s \geq S_2$. In particular, $(u_s)_{s \geq 1}$ is equibounded. By the gradient estimate Proposition 2.27

$$|\mathrm{D}u|(x) \leq \mathrm{C}_3 u(x)^{\frac{n-1}{n-p}} \leq \mathrm{C}_4 \left(\mathrm{dist}(o, x) \right)^{-\frac{n-1}{p-1}}$$

for some positive constants $\mathrm{C}_3, \mathrm{C}_4$. Hence, employing again Lemma 2.15 there exist $S_5, \mathrm{C}_5 > 0$ such that

$$|\mathrm{D}u_s|_{g_{(s)}}(x) \leq \mathrm{C}_5 \rho(x)^{-\frac{n-1}{p-1}} \tag{3.4}$$

holds on $[1/s, +\infty) \times L$ for every $s \geq S_5$. By Lemma 2.14 we have that for some $\varepsilon > 0$ there is $S_6 > 0$ such that

$$|\mathrm{D}u_s|_{\hat{g}} \leq (1 + \varepsilon)|\mathrm{D}u_s|_{g_{(s)}}$$

holds for every $s \geq S_6$. Combining it with (3.4) we obtain that the family $(u_s)_{s \geq 1}$ is equicontinuous. By Arzelà-Ascoli Theorem we conclude that $(u_s)_{s \geq 1}$ is precompact with respect to the local uniform convergence. (3.2) follows from (3.3).

Step 2. Here we prove that any limit point v of the family $(u_s)_{s \geq 1}$ has the form

$$v(x) = \gamma \rho(x)^{-\frac{n-p}{p-1}}, \tag{3.5}$$

for some nonnegative $\gamma \in \mathbb{R}$, proving also Proposition 3.3. Let $v : (0, +\infty) \times L \to \mathbb{R}$ be a nonnegative p-harmonic function satisfying the bound $v(x) \leq \mathrm{C} \rho(x)^{-\frac{n-p}{p-1}}$ on $(0, +\infty) \times L$.

Define the function $\mathfrak{e}_v : (0, +\infty) \to \mathbb{R}$ as

$$\mathfrak{e}_v(t) = \frac{R(t)}{r(t)},$$

where $[r(t), R(t)] \times L$ is the smallest annulus containing $\{v = 1/t\}$ for every $t \in (0, +\infty)$. Observe that, $\mathfrak{e}_v(t) \geq 1$ and $\mathfrak{e}_v(t) = 1$ if and only if $\{v = 1/t\}$ is a cross-section of the cone. By the Comparison Principle Theorem 2.4, using the potentials of $\{\rho = r(t)\}$ and $\{\rho = R(t)\}$ as barriers, we have that

$$r(t) \left(\frac{t}{T} \right)^{\frac{n-p}{p-1}} \leq \rho(x) \leq R(t) \left(\frac{t}{T} \right)^{\frac{n-p}{p-1}} \tag{3.6}$$

holds for every $x \in \{v = 1/T\}$ for every $T \geq t$. Hence, $\mathfrak{e}_v$ is nonincreasing. Moreover, since $(0, +\infty) \times L$ is connected, $\rho(x)^{-\frac{n-p}{p-1}}$ is p-harmonic and $\mathscr{C}^2((0, +\infty) \times L)$ and $\left| D\rho^{-\frac{n-p}{p-1}} \right| \geq \frac{n-p}{p-1} R^{-\frac{n-1}{p-1}}$ holds on $(0, R) \times L$ for every $R > 0$, by the Strong Comparison Principle Theorem 2.4 the inequalities in (3.6) are strict unless $\{v = t\}$ is a cross-section. It is not hard to see that $\mathfrak{e}_v$ is scale invariant

Consider for $s \leq 1$ the family $v_s : [1, +\infty) \times L \to \mathbb{R}$ defined as

$$v_s(x) = s^{\frac{n-p}{p-1}} v \circ \omega_s(x)$$

where ω_s is defined in (2.6). Using the same argument of Step 1 we have that

$$v_s(x) \leq \mathrm{C}\rho(x)^{-\frac{n-p}{p-1}} \qquad \text{and} \qquad |Dv_s|(x) \leq \mathrm{C}\rho(x)^{-\frac{n-1}{p-1}}$$

holds on $(0, +\infty) \times L$ for some constant $\mathrm{C} > 0$. Hence, appealing to the Arzelà-Ascoli Theorem, $(v_s)_{s \leq 1}$ is precompact with respect to the local uniform convergence. Let w be a limit point for $(v_s)_{s \leq 1}$. Since $\mathfrak{e}$ is scale invariant, $\mathfrak{e}_{v_s}(t) = \mathfrak{e}_v(t/s)$. Then, $\mathfrak{e}_w(t)$ is constant equal to some $\mathfrak{e}_w \in [1, +\infty)$ that by monotonicity satisfies $\mathfrak{e}_w = \sup_t \mathfrak{e}_v(t) \in [1, +\infty)$. Suppose by contradiction that $\mathfrak{e}_w > 1$. Then the level $\{w = 1\} \subset [r(1), \mathfrak{e}_w r(1)] \times L$ and $\{w = 1\}$ touches both the cross-sections $\{\rho = r(1)\}$ and $\{\rho = \mathfrak{e}_w r(1)\}$ without being equal to either one. By (3.6) and the Strong Comparison Principle Theorem 2.4

$$r(1)t^{\frac{p-1}{n-p}} < \rho(x) < \mathfrak{e}_w r(1) t^{\frac{p-1}{n-p}}$$

holds for every $x \in \{w = 1/t\}$ for every $t > 1$. We therefore have that $\mathfrak{e}_w(t) < \mathfrak{e}_w$ which is a contradiction. In conclusion $\mathfrak{e}_w$ must be 1 and since $1 \leq \mathfrak{e}_v(t) \leq \mathfrak{e}_w = 1$, v is as in (3.5).

Step 3. By Step 2, any limit point w for the sequence $(u_s)_{s \geq 1}$ given by Step 1 has the form $\gamma \rho^{-\frac{n-p}{p-1}}$, where $\gamma > 0$ by (3.2). We are now going to prove that

$$\gamma = \mathrm{C}_p(\Omega)^{\frac{1}{p-1}} \operatorname{AVR}(g)^{-\frac{1}{p-1}}. \tag{3.7}$$

The characterisation (3.7) ensures that any converging subsequence admits the same limit, proving that the whole family $(u_s)_{s \geq 1}$ locally uniformly converges to $\gamma \rho^{-\frac{n-p}{p-1}}$ as $s \to +\infty$. In particular, for every $\varepsilon > 0$ there exists a $S \geq 1$ such that

$$\sup_{\{\rho = s\}} s^{\frac{n-p}{p-1}} \left| u - \left(\frac{\mathrm{C}_p(\Omega)}{\operatorname{AVR}(g)} \right)^{\frac{1}{p-1}} \rho^{-\frac{n-p}{p-1}} \right| = \sup_{\{\rho = 1\}} \left| u_s - \left(\frac{\mathrm{C}_p(\Omega)}{\operatorname{AVR}(g)} \right)^{\frac{1}{p-1}} \rho^{-\frac{n-p}{p-1}} \right| \leq \varepsilon$$

for every $s \geq S$, proving Theorem 1.1 and Proposition 3.2.

Observe that $\gamma > 0$ by Corollary 2.25. In order to prove (3.7), we find convenient to work with the auxiliary function

$$v = \left(\frac{u}{\gamma} \right)^{-\frac{p-1}{n-p}}.$$

Since w is a limit point for the family $(u_s)_{s \geq 1}$ there is a subsequence $(u_{s_k})_{k \in \mathbb{N}}$, s_k increasing and divergent as $k \to +\infty$, such that $u_{s_k} \to v = \gamma \rho^{-\frac{n-p}{p-1}}$ locally uniformly on $(0, +\infty) \times L$ as $k \to +\infty$. Then for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\left\{ \rho \leq \frac{s_k}{1 + \varepsilon} \right\} \subset \{v \leq s_k\} \subset \left\{ \rho \leq \frac{s_k}{1 - \varepsilon} \right\}$$

By the monotonicity of the p-capacity with respect to the inclusion, we have that

$$\mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\}\right) \leq \mathrm{Cap}_p\left(\{v \leq s_k\}\right) \leq \mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}\right)$$

By (2.4) we can compute the capacity of level sets of v in terms of the capacity of $\partial\Omega$, that is

$$\mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\}\right) \leq \gamma^{-(p-1)} s_k^{n-p} \mathrm{Cap}_p\left(\Omega\right) \leq \mathrm{Cap}_p\left(\left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}\right)$$

Dividing each side by $|\mathbb{S}^{n-1}| s_k^{n-p}$, sending $k \to +\infty$ and using Lemma 2.21 we infer that

$$\left(\frac{n-p}{p-1}\right)^{p-1} \frac{\mathrm{AVR}(g)}{(1+\varepsilon)^{n-p}} \leq \gamma^{-(p-1)} \frac{\mathrm{Cap}_p(\Omega)}{|\mathbb{S}^{n-1}|} \leq \left(\frac{n-p}{p-1}\right)^{p-1} \frac{\mathrm{AVR}(g)}{(1-\varepsilon)^{n-p}}$$

Then (3.7) follows by arbitrariness of $\varepsilon > 0$, keeping in mind the characterisation of $\mathrm{AVR}(g)$ in (2.17) and the relation between the p-capacity and the normalised p-capacity.

Step 4. Suppose now (M, g) is $\mathscr{C}^{0,\alpha}$-Asymptotically Conical for $\alpha > 0$. By Theorem 2.1 $u_s \in \mathscr{C}^{1,\beta}_{\mathrm{loc}}((1/s, +\infty) \times L)$ for some $\beta \in (0, \alpha)$ and for every $K \subset (1/s, +\infty) \times L$ there exists constant $\mathrm{C} > 0$ such that

$$\|u_s\|_{\mathscr{C}^{1,\beta}(K)} \leq \mathrm{C} \|u_s\|_{\mathscr{C}^0((1/s,+\infty)\times L)} . \tag{3.8}$$

Since the metric $g_{(s)}$ locally $\mathscr{C}^{0,\alpha}$-converges to $\hat{g}$ on $(0, +\infty) \times L$ by Lemma 2.14, the constant in (3.8) can be chosen not depending on s. Hence, by Arzelà-Ascoli Theorem, $(u_s)_{s \geq 1}$ $\mathscr{C}^1$-locally converges on $(0, +\infty) \times L$ to the function

$$\left(\frac{\mathrm{C}_p(\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{p-1}} \rho^{-\frac{n-p}{p-1}} \tag{3.9}$$

as $s \to +\infty$. This proves Theorem 3.1 for $k = 0$ and $\ell \leq 1$.

If (M, g) is $\mathscr{C}^{k,\alpha}$-Asymptotically Conical for $k \geq 1$ and $\alpha > 0$, we already proved that (3.1) holds for $\ell \leq 1$. In particular, for every R there exists $S > 0$ such that $|\mathrm{D}u_s| > 0$ holds on every compact $K \subset (R, +\infty) \times L$ for every $s \geq S$. Applying Theorem 2.2, $u_s \in \mathscr{C}^\infty((R, +\infty) \times L)$ for every $s \geq S$. Moreover, for every $K \subset (R, +\infty) \times L)$ there exists a constant $\mathrm{C} > 0$ such that

$$\|u_s\|_{\mathscr{C}^{k+1,\alpha}(K)} \leq \mathrm{C} \|u_s\|_{\mathscr{C}^0((1/s,+\infty)\times L)} . \tag{3.10}$$

Since $g_{(s)}$ locally $\mathscr{C}^{k+1,\alpha}$-converges to $\hat{g}$ on $(0, +\infty) \times L$, the constant in (3.10) can be chosen not depending on s. Since R is arbitrary, $(u_s)_{s \geq 1}$ is precompact with respect to the local $\mathscr{C}^{k+1}$-topology. Hence, $(u_s)_{s \geq 1}$ converges on compact subsets of $(0, +\infty) \times L$ up to its $(k+1)$-th derivative to the function defined in (3.9) as $s \to +\infty$, concluding the proof of Theorem 3.1. $\square$

As a consequence of Theorem 1.1, we extend Lemma 2.16 showing that the volume of level sets of a suitable function of the p-capacitary potential behaves like the volume of geodesic balls approaching infinity. Requiring the assumption of Theorem 3.1 we actually deduce that the same occur for the $(n-1)$-Hausdorff measure of the level sets.

Proposition 3.4 (Asymptotic behaviour of the area of level sets)**.** *In the same assumptions and notations of Theorem 1.1, set, for $i = 1, \dots, N$,*

$$v_i = \left(\frac{\mathrm{C}_p^{(i)}(\Omega)}{\mathrm{AVR}(g; E_i)}\right)^{\frac{1}{n-p}} u^{-\frac{p-1}{n-p}} .$$

Then, we have

$$\mathrm{AVR}(g; E_i) = \lim_{s \to +\infty} \frac{|\{v_i \leq s\} \cap E_i|}{s^n |\mathbb{B}^n|}. \tag{3.11}$$

Moreover, if in addition (M, g) is $\mathscr{C}^{0,\alpha}$-Asymptotically Conical for $\alpha \in (0, 1)$, then

$$\mathrm{AVR}(g; E_i) = \lim_{s \to +\infty} \frac{|\{v_i = s\} \cap E_i|}{s^{n-1} |\mathbb{S}^{n-1}|}. \tag{3.12}$$

Proof of Proposition 3.4. We prove the statement in the case M has only one end, being the general case a direct consequence. We drop the subscript i in the following lines. By Theorem 1.1, for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$(1 - \varepsilon)\rho \leq v \leq (1 + \varepsilon)\rho$$

holds on $\{\rho \geq R_\varepsilon\}$. Thus we have that

$$\left\{\rho \leq \frac{s}{1 + \varepsilon}\right\} \subset \{v \leq s\} \subset \left\{\rho \leq \frac{s}{1 - \varepsilon}\right\}.$$

By the monotonicty of the volume we have that

$$\left|\left\{\rho \leq \frac{s}{1 + \varepsilon}\right\}\right| \leq |\{v \leq s\}| \leq \left|\left\{\rho \leq \frac{s}{1 - \varepsilon}\right\}\right|.$$

dividing each side by $s^n |\mathbb{B}^n|$ and passing to the limit as $s \to +\infty$ we can conclude that

$$\frac{\mathrm{AVR}(g)}{(1 + \varepsilon)^n} \leq \lim_{s \to +\infty} \frac{|\{v \leq s\}|}{s^n |\mathbb{B}^n|} \leq \frac{\mathrm{AVR}(g)}{(1 - \varepsilon)^n}$$

by arbitrariness of $\varepsilon > 0$ we have that the volume of level sets behaves like the one of geodesic balls approaching infinity, proving the first identity in (3.11). A straightforward computation relying on the identity

$$|\mathrm{D}v| = \left(\frac{\mathrm{C}_p(\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{n-p}} \left(\frac{p-1}{n-p}\right) u^{-\frac{n-1}{p-1}} |\mathrm{D}u|$$

permits to write

$$\mathrm{AVR}(g) = \frac{1}{|\mathbb{S}^{n-1}| s^{n-1}} \int\limits_{\{v=s\}} |\mathrm{D}v|^{p-1} \, \mathrm{d}\sigma.$$

If in addition (M, g) is $\mathscr{C}^{0,\alpha}$-Asymptotically Conical for $\alpha \in (0, 1)$, then $|\mathrm{D}v|$ approaches 1 at infinity, hence we have

$$\mathrm{AVR}(g) = \lim_{s \to +\infty} \frac{1}{|\mathbb{S}^{n-1}| s^{n-1}} \int\limits_{\{v=s\}} |\mathrm{D}v|^{p-1} \, \mathrm{d}\sigma = \lim_{s \to +\infty} \frac{|\{v = s\}|}{s^{n-1} |\mathbb{S}^{n-1}|}$$

which concludes the proof of (3.12). □

4. Asymptotic behaviour of the weak IMCF

We give the precise definition of (proper) weak Inverse Mean Curvature Flow (IMCF for short).

Given an open set $U \subseteq M$, we say that $w \in \mathrm{Lip}_{\mathrm{loc}}(U)$ is a weak IMCF if for every $v \in \mathrm{Lip}_{\mathrm{loc}}(U)$ with $\{w \neq v\} \Subset U$ and any compact set $K \subset U$ containing $\{w \neq v\}$, we have

$$J_w^K(w) \leq J_w^K(v)$$

where

$$J_w^K(v) = \int_K |\mathrm{D}v| + v|\mathrm{D}w| \, \mathrm{d}\mu.$$

Moreover, given a bounded open set $\Omega \subset M$ with smooth boundary we say that w is a weak IMCF starting at Ω if for every $v \in \mathrm{Lip}_{\mathrm{loc}}(M)$ with $\{w \neq v\} \Subset M \setminus \Omega$ and any compact set $K \subset M \setminus \Omega$ containing $\{w \neq v\}$ we have

$$J_w^K(w) \leq J_w^K(v),$$

and the set Ω is the 0-sublevel set of w, that is

$$\Omega = \{w < 0\}.$$

We say that w is a proper weak IMCF if the function w is proper, that is, its sublevel sets are precompact. In the following result we gather the existence and the fundamental estimates for w in the Asymptotically Conical setting.

Theorem 4.1. *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold satisfying the Ricci curvature bound* (1.2). *Let $\Omega \subset M$ be an open bounded subset with smooth boundary. Then, there exists a proper weak IMCF w starting at Ω. Moreover, given $o \in \Omega$, the function w satisfies*

$$(n-1)\log \mathrm{dist}(x, o) - \mathrm{C} \leq w \leq (n-1)\log \mathrm{dist}(x, o) + \mathrm{C} \tag{4.1}$$

where $\mathrm{C} = \mathrm{C}(M, n, \Omega)$.

Proof. The existence is guaranteed by [MRS21, Theorem 1.7] whose assumptions are satisfied in virtue of (2.11) and (1.2). The lower bound is the consequence of [MRS21, Theorem 1.7 and 1.3]. Let R be such that

$$-(p-1)\log G_p(x, o) \leq (n-p)\log \mathrm{dist}(x, o) - \log \mathrm{C}_L \tag{4.2}$$

on $M \setminus B(o, R)$ with the constant $\mathrm{C}_L = \mathrm{C}\, a \, \mathrm{Cap}_p(\overline{K}; E)$ described in the statement of Theorem 2.24. By [MRS21], $-(p-1)\log G_p(x, o)$ locally uniformly converges as $p \to 1^+$. Then we can choose a in the constant C_L independent from p so that $G_p(o, x) \geq a^{\frac{1}{p-1}}$ holds on $M \setminus K$ where K is as in Definition 2.13. Moreover, by (1.6), since ∂K is smooth, $\mathrm{Cap}_p(\overline{K}; E)$ is bounded as $p \to 1^+$. Hence the constant C_L in (4.2) does not depend on p. Passing to the limit as $p \to 1^+$, in virtue of the upper bound in [MRS21, Theorem 1.7], we obtain

$$w \leq (n-1)\log \mathrm{dist}(x, o) + \mathrm{C}$$

outside some $B(o, R)$. Since both the left and side and the right hand side are continuous, the bound can be extended to $M \setminus \Omega$. $\qquad\square$

In [MRS21, Remark 4.9] the authors also obtained a gradient bound that reads as

$$|\mathrm{D}w| \leq \frac{\mathrm{C}}{\mathrm{dist}(x, o)^{1/\kappa'}}, \qquad \text{where } \kappa' = \frac{1 + \sqrt{1 + 4\kappa^2}}{2} \geq 1,$$

for some constant $\mathrm{C} > 0$ depending only on Ω, the dimension n and the geometry of the ambient manifold. By [GW79] (see also [MRS21, Remark 4.5]) The exponent κ' can be chosen equal to 1 if the lower bound on the Ricci curvature is of the kind $\mathrm{Ric} \geq -(n-1)f(\mathrm{dist}(x, o))$ for some smooth nonnegative function $f(t)$, such that

$$\int_0^{+\infty} t f(t) \, \mathrm{d}t < +\infty.$$

The Weak Existence Theorem 3.1 [HI01] the function w satisfies

$$|\mathrm{D}w|(x) \leq \sup_{\partial\Omega \cap B(x,r)} \mathrm{H} + \frac{\mathrm{C}}{r} \tag{4.3}$$

for almost every $x \in M \smallsetminus \Omega$ and for every r for which there exists a function $\psi \in \mathscr{C}^2(B(x,r))$ such that $\psi \geq \mathrm{dist}(x, \cdot)^2$, $\psi(x) = 0$, $|\mathrm{D}\psi| \leq 3\,\mathrm{dist}(x, \cdot)$, $\mathrm{D}^2\psi \leq 3g$ and $\mathrm{Ric} \geq -\mathrm{C}/r^2$ in $B(x,r)$. The existence of ψ is guaranteed if a sectional curvature lower bound is ensured. Otherwise, one can require a higher rate of convergence of the metric.

Proposition 4.2. *Let (M,g) be a $\mathscr{C}^1$-Asymptotically Conical Riemannian manifold satisfying the Ricci curvature bound (1.2). Let $\Omega \subset M$ be an open bounded subset with smooth boundary, $o \in M$. There exists a postive constant $\mathrm{C} = \mathrm{C}(n, M, \Omega) > 0$, such that the solution $w \in \mathrm{Lip}(M)$ of the weak IMCF starting at Ω, given by Theorem 4.1 satisfies*

$$|\mathrm{D}w|(x) \leq \frac{\mathrm{C}}{\mathrm{dist}(x,o)} \tag{4.4}$$

for almost every $x \in M \smallsetminus \Omega$.

Proof. By [HI01, Weak Existence Theorem 3.1] the function w satisfies (4.3) for almost every $x \in M \smallsetminus \Omega$. In virtue of the discussion in [HI01, Definition 3.3] (see also the proof of [HI01, Blowdown Lemma 7.1]) there exists a constant $\mathrm{C} > 0$ and $R > 0$ such that $r \geq \mathrm{C}\,\mathrm{dist}(x,o)$ in (4.3) for every $x \in M \smallsetminus B(o, R)$. Then (4.4) follows taking r so that $\partial\Omega \cap B(x,r) = \varnothing$. $\square$

The notion of weak IMCF is intimately tied with that of strictly outward minimising sets [HI01, Minimizing Hulls, p.371]. Leaving the details to [FM20], we recall that a bounded set $E \subset M$ is strictly outward minimising if $P(F) > P(E)$ for any bounded $F \supset E$ with $|F \setminus E| > 0$. The following simple lemma implies the existence of a foliation of strictly outward minimising sets in Asymptotically Conical manifolds.

Lemma 4.3. *Let (M,g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold. Then $\{\rho \leq r\}$ is strictly outward minimising for r large enough.*

Proof. Consider any $\varphi \in \mathscr{C}_c^\infty(\{\rho \geq r\})$, then

$$\int_{\{\rho \geq r\}} \mathrm{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\varphi\,\mathrm{d}\mu = -\int_{\{\rho \geq r\}} \left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\,\middle|\,\mathrm{D}\varphi\right\rangle \mathrm{d}\mu - \int_{\{\rho = r\}} \varphi\,\mathrm{d}\sigma_g.$$

Observe that the right hand side of the previous identity depends only on the coefficient of the metric and not on their derivatives. Since the metric g converges to the metric $\hat{g}$, for every $\varepsilon > 0$ there exists R_ε such that for every $r \geq R_\varepsilon$

$$\left|\int_M \mathrm{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\varphi\,\mathrm{d}\mu_g - \int_M \frac{n-1}{\rho}\varphi\,\mathrm{d}\mu_{\hat{g}}\right| \leq \varepsilon \int_M \frac{n-1}{\rho}\varphi\,\mathrm{d}\mu_{\hat{g}} \tag{4.5}$$

for every $\varphi \in \mathscr{C}_c^\infty(\{\rho \geq r\})$ and

$$\left||E|_g - |E|_{\hat{g}}\right| \leq \varepsilon|E|_{\hat{g}}$$

for every measurable $E \subset \{\rho \geq r\}$. By (4.5) and the density of compactly supported smooth function, for every $E \subset \{\rho \geq r\}$ we have that

$$\int_E \mathrm{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right)\mathrm{d}\mu_g \geq (1-\varepsilon)\frac{n-1}{\sup_E \rho}|E|_{\hat{g}}.$$

Let F be a subset of finite perimeter containing $\{\rho < r\}$, then

$$
\left(\frac{1-\varepsilon}{1+\varepsilon}\right)\left(\frac{n-1}{\sup_F \rho}\right)|F \smallsetminus \{\rho < r\}|_g \le (1-\varepsilon)\frac{n-1}{\sup_F \rho}|F \smallsetminus \{\rho \le r\}|_{\hat{g}} \le \int_M \operatorname{div}\left(\frac{\mathrm{D}\rho}{|\mathrm{D}\rho|}\right) \mathrm{d}\mu_g
$$

$$
\le \int_{\partial^* F} \left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|} \,\middle|\, \nu_{\partial^* F} \right\rangle \mathrm{d}\sigma_g - \int_{\{\rho=r\}} \left\langle \frac{\mathrm{D}\rho}{|\mathrm{D}\rho|} \,\middle|\, \nu_{\{\rho=r\}} \right\rangle \mathrm{d}\sigma_g
$$

$$
\le |\partial^* F|_g - |\{\rho = r\}|_g.
$$

This proves that $|\{\rho = r\}|_g \le |\partial^* F|_g$ and the equality holds true if and only if $|F \smallsetminus \{\rho < r\}|_g = 0$, which gives that $\{\rho \le r\}$ is strictly outward minimising. $\qquad\square$

By [FM20, Theorem 2.16] the existence of an exhaustion of strictly outward minimising sets provides, for any bounded $\Omega \subset M$ with smooth boundary, a suitable bounded *strictly outward minimising hull* Ω^*, that in particular fulfils

$$
|\partial\Omega^*| = \inf\left\{|\partial^* F| \mid F \text{ is bounded and } \Omega \subseteq F\right\}.
$$

Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold defined in Definition 2.13 and denote by $E_1, \ldots, E_N$ the (finitely many) ends. Consider $\Omega \subset M$ open bounded subset with smooth boundary and $w : M \smallsetminus \Omega \to [0, +\infty)$ the weak IMCF w starting at Ω. As we did for the p-capacity we can define the area of the strictly outward minimising hull of Ω with respect to one end E_i. Indeed, there exists a time T such that $\{w \le t\}$ contains the compact K defined in Definition 2.13 for every $t \ge T$. We then define the area of $\partial\Omega^*$ with respect to E_i as

$$
|\partial\Omega^*|^{(i)} = \frac{|\partial\{w \le t\} \cap E_i|}{\mathrm{e}^t}
$$

for some $t \ge T$. Observe that such definition is well posed by [HI01, Exponential Growth Lemma 1.6]. Moreover, it can be checked, substantially using [FM20, Proposition 3.4], that $|\partial\Omega^*|$ splits as

$$
|\partial\Omega^*| = \sum_{i=1}^N |\partial\Omega^*|^{(i)}.
$$

Theorem 1.2 in [FM20] establishes a relation between $\mathrm{C}_p(\Omega)$ and $|\partial\Omega^*|$. The same relation between $\mathrm{C}_p^{(i)}(\Omega)$ and $|\partial\Omega^*|^{(i)}$ holds true.

Lemma 4.4. *Let (M, g) be a $\mathscr{C}^0$-Asymptotically Conical Riemannian manifold with Ricci curvature satisfying (1.2) and N be the number of ends. Let $\Omega \subseteq M$ be an open bounded subset with smooth boundary. Then,*

$$
\lim_{p \to 1^+} \mathrm{C}_p^{(i)}(\Omega) = \frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|}
$$

holds for every $i = 1, \ldots, N$

Proof. Let w be the solution to the weak IMCF starting at Ω and T large enough so that $\{w \le T\}$ contains K in Definition 2.13. Given u_p the solution to (1.1), by [MRS21] we have $-(p-1)\log u_p$ where converges locally uniformly to w as $p \to 1^+$. In particular, for every $t \ge T$ there exists $p_t \in (1, n)$ such that $\{w \le T\} \subseteq \{-(1-p)\log u_p \le t\}$ holds for every $p < p_t$. Arguing as in [FM20, Theorem 1.2], since an Isoperimetric Inequality is in force by Proposition 2.20, we can prove that $|\partial\{w \le t\} \cap E_i| \le \mathrm{C}_{n,p} \operatorname{Cap}_p(\partial\{w \le t\}; E_i)$, for some constant $\mathrm{C}_{n,p}$ such that

$C_{n,p} \to 1$ as $p \to 1^+$. In particular, by the monotonicity of the p-capacity and Proposition 2.12 we have that

$$|\partial\Omega^*|^{(i)} \le C_{n,p}\, e^{-T} \mathrm{Cap}_p\left(\{w \le T\} \cap E_i; E_i\right) \le e^{-T}\mathrm{Cap}_p\left(\left\{u_p \ge C_{n,p}\, e^{-\frac{t}{p-1}}\right\} \cap E_i; E_i\right)$$

$$= C_{n,p}\, e^{t-2T}\mathrm{Cap}_p\left(\left\{u_p \ge e^{-\frac{T}{p-1}}\right\} \cap E_i; E_i\right) \le C_{n,p} e^{t-T}\mathrm{Cap}_p^{(i)}(\Omega)$$

Sending $p \to 1^+$ and then $t \to T^+$ we have that

$$\frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|} \le \lim_{p\to 1^+} C_p^{(i)}(\Omega).$$

If for some $i = 1, \ldots, N$ the inequality is strict, then

$$\frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|} = \sum_{i=1}^{N} \frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|} < \sum_{i=1}^{N} \lim_{p\to 1^+} C_p^{(i)}(\Omega) = \lim_{p\to 1^+} C_p(\Omega) = \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|}$$

which is a contradiction. $\square$

Clearly, we also obtain the analogous of Proposition 3.2.

Proposition 4.5 (Asymptotic behaviour of the area of level sets). *In the same assumptions and notations of Theorem 1.2, set, for $i = 1, \ldots, m$,*

$$v_i = \left(\frac{|\partial\Omega^*|^{(i)}}{|\mathbb{S}^{n-1}|\, \mathrm{AVR}(g; E_i)}\right)^{\frac{1}{n-1}} e^{\frac{w}{n-1}}.$$

Then, we have

$$\lim_{s\to+\infty} \frac{|\{v_i = s\} \cap E_i|}{s^{n-1}|\mathbb{S}^{n-1}|} = \mathrm{AVR}(g).$$

Observe that a similar result for the p-capacitary potential was obtained in (3.12). In that case a first order asymptotic behaviour for the p-capacitary potential was required in the proof. The reason is that the area is linked to the level sets of IMCF in the same way the p-capacity is linked to the level set of p-capacitary potential. A simple $\mathscr{C}^0$-convergence is therefore enough in this case.

As a byproduct we also obtain the counterpart of Proposition 3.3 proving that

$$w(x) = (n-1)\log(\rho(x)) \qquad\qquad \text{with } x \in (0,+\infty) \times L$$

is the unique solution on $(0,+\infty) \times L$ up to a constant.

Proposition 4.6. *Let $((0,+\infty) \times L, \hat{g})$ be a Riemannian cone with nonnegative Ricci curvature where L is a closed connected smooth hypersurface. Let w be a weak IMCF on $(0,+\infty) \times L$ satisfying $w(x) \ge (n-1)\log\rho(x) + C$ for every $x \in (0,+\infty) \times L$ for some constant $C \ge 0$. Then, there exists a $\gamma \in \mathbb{R}$ such that*

$$w(x) = (n-1)\log(\rho(x)) + \gamma \qquad\qquad \text{with } x \in (0,+\infty) \times L,$$

holds on $(0,+\infty) \times L$.

In the case of the flat Euclidean space, this uniqueness result has been obtained in [HI01, Proposition 7.2].

Proof of Theorem 1.2 and Propositions 4.5 and 4.6. The proof follows the same lines of Theorem 1.1. We prove the theorem in the case M has only one end, since the general case follows applying the result to each end. We denote by $g_{(s)}$ the metric $s^{-2}\omega_s^* g$ on $[1/s, +\infty) \times L$, being ω_s the family of diffeomorphism defined in (2.6). We divide the proof in three steps. The second and the third one contains the proof of Propositions 4.5 and 4.6 respectively.

Step 1. Define for every $s \geq 1$ the family of functions $w_s : [1/s, +\infty) \times L \to \mathbb{R}$ as

$$w_s = w \circ \omega_s - (n-1)\log(s)$$

where ω_s is the map defined in (2.6). Employing (4.1) in Theorem 4.1 and Proposition 4.2 as in the proof of Theorem 1.1, it is easy to show that $(w_s)_{s \geq 1}$ is equibounded and equi-Lipschitz. By Arzelà-Ascoli Theorem, $(w_s)_{s \geq 1}$ is precompact with respect to the local uniform convergence on $(0, +\infty) \times L$. Moreover, by [HI01, Compactness Theorem 2.1] every limit point u is a solution to the (weak) IMCF on $(0, +\infty) \times L$ and by (4.1) there exists a positive constant $C > 0$ such that

$$(n-1)\log(\rho(x)) - C \leq u(x) \leq (n-1)\log(\rho(x)) + C$$

is satisfied on $(0, +\infty) \times L$.

Step 2. Here we prove Proposition 4.6, inferring in particular that any limit point v of $(w_s)_{s \geq 1}$ satisfies

$$v(x) = (n-1)\log \rho(x) + \gamma$$

on $(0, +\infty) \times L$ for some $\gamma \in \mathbb{R}$. Let $\mathfrak{e}_v : \mathbb{R} \to \mathbb{R}$ be defined as

$$\mathfrak{e}_v(t) = \frac{R(t)}{r(t)}$$

where, for every $t \in \mathbb{R}$, $[r(t), R(t)] \times L$ is the smallest annulus containing $\{v = t\}$. Arguing as in Step 2 of Theorem 1.1 starting from any weak IMCF v on $(0, +\infty) \times L$ we can produce a function $u : (0, +\infty) \times L \to \mathbb{R}$ such that $\mathfrak{e}_u(t)$ is constant and is equal to $\mathfrak{e}_u = \sup_t \mathfrak{e}_v(t) \in [1, +\infty)$. Suppose by contradiction that $\mathfrak{e}_u > 1$. Then the level $\{u = 0\} \subset [r(0), \mathfrak{e}_u r(0)] \times L$ and touches both the cross-sections $\{\rho = r(0)\}$ and $\{\rho = \mathfrak{e}_u r(0)\}$ without being equal to either one. We aim to compare the weak flow and the two strong flows and prove that the level sets of u detach from the spheres. Perturb $\{\rho \leq r(0)\}$ outward and $\{\rho \leq \mathfrak{e}_u r(0)\}$ inward to obtain D^- and D^+ respectively with the following properties:

- $\{\rho \leq r(0)\} \subset D^- \subset \{u \leq 0\}$ and $\{u \leq 0\} \subset D^+ \subset \{\rho \leq \mathfrak{e}_u r(0)\}$;
- D^- and D^+ are starshaped with smooth strictly mean convex boundary.

Then the smooth IMCF starting at D^+ and D^- exists for all time by [Zho18, Theorem 3.1] and by [HI01, Smooth Flow Lemma 2.3] it coincides with the weak notion of the IMCF. Denote by $(D_t^-)_{t \geq 0}$ and $(D_t^+)_{t \geq 0}$ the sublevel sets of the two weak (and smooth) IMCF starting at D^- and D^+ respectively. By the Strong Comparison Principle for smooth flows we have that

$$\rho(x) > r(0)e^{\frac{t}{n-1}} \text{ for } x \in \partial D_t^- \qquad \text{and} \qquad \rho(x) < \mathfrak{e}_u r(0)e^{\frac{t}{n-1}} \text{ for } x \in \partial D_t^+. \qquad (4.6)$$

On the other hand, by the Weak Comparison theorem [HI01, Theorem 2.2(ii)]

$$D_t^- \subset \{u \leq t\} \subset D_t^+. \qquad (4.7)$$

Coupling (4.6) and (4.7) we have that $\mathfrak{e}_u(t) < \mathfrak{e}_u$ which is the desired contradiction. Then $\mathfrak{e}_u = 1$ that completes, as in Step 2 of Theorem 1.1, the proof of Proposition 4.6.

Step 3. Let $v = (n-1)\log\rho + \gamma$ be a limit point of the family $(w_s)_{s\geq 1}$. We are now going to prove that

$$\gamma = \log\left(\frac{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right). \tag{4.8}$$

The characterisation proves Proposition 4.5 and implies that the limit point is unique, concluding the proof. We work with the auxiliary function

$$u = \mathrm{e}^{\frac{w-\gamma}{n-1}}$$

Since v is a limit point for the family $(w_s)_{s\geq 1}$ there exists a subsequence $(w_{s_k})_{k\in\mathbb{N}}$, s_k increasing and divergent as $k \to +\infty$, such that $w_{s_k} \to v = (n-1)\log\rho + \gamma$ locally uniformly on $(0, +\infty) \times L$ as $k \to +\infty$. Then for any $\varepsilon > 0$ there exists $k_\varepsilon \in \mathbb{N}$ such that

$$\left\{\rho \leq \frac{s_k}{1+\varepsilon}\right\} \subset \{u \leq s_k\} \subset \left\{\rho \leq \frac{s_k}{1-\varepsilon}\right\}$$

holds for every $k \geq k_\varepsilon$. By Lemma 4.3 we can assume k_ε large enough so that both the left most and the right most sets are strictly outward minimising for any $k \geq k_\varepsilon$. Then the perimeter is monotone by inclusion and by [HI01, Exponential Growth Lemma 1.6] we have

$$\left|\left\{\rho = \frac{s_k}{1+\varepsilon}\right\}\right| \leq \mathrm{e}^\gamma s_k^{n-1}|\partial\Omega^*| \leq \left|\left\{\rho = \frac{s_k}{1-\varepsilon}\right\}\right|$$

Dividing both sides by $|\mathbb{S}^{n-1}|s_k^{n-1}$, sending $k \to +\infty$ and using Lemma 2.16 we infer that

$$\frac{\mathrm{AVR}(g)}{(1+\varepsilon)^{n-1}} \leq \mathrm{e}^\gamma \frac{|\partial\Omega^*|}{|\mathbb{S}^{n-1}|} \leq \frac{\mathrm{AVR}(g)}{(1-\varepsilon)^{n-1}}$$

Then (4.8) follows by arbitrariness of $\varepsilon > 0$. $\qquad\square$

Firstly, observe that we do not have the analogous of Theorem 3.1 for the IMCF. The asymptotic behaviour of higher order derivatives of the p-capacitary potential is indeed a consequence of the higher regularity one gets once shown that the gradient does not vanish anymore sufficiently far out. In the case of the weak IMCF, it is not clear to us how to work out this last step.

The result above is to be compared with [HI01, Lemma 7.1]. We obtain here an explicit characterisation of the constants c_λ that is

$$c_\lambda = -(n-1)\log\left(\frac{|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\lambda\right).$$

The constant appearing in (1.5) satisfies

$$\log\left(\frac{\mathrm{AVR}(g)|\mathbb{S}^{n-1}|}{|\partial\Omega^*|}\right) = \lim_{p\to 1^+} -(p-1)\log\left[\left(\frac{\mathrm{Cap}_p(\partial\Omega)}{\mathrm{AVR}(g)}\right)^{\frac{1}{p-1}}\right]$$

thanks to (1.6), which is the constant in (1.4), trasformed accordingly to $w_p = -(p-1)\log u_p$. Hence, even if by Theorem 4.1 $w_p \to w$ only locally uniformly as $p \to 1^+$, the asymptotic behaviour of w is anyway affected by this procedure.

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