A GENERALIZATION OF GEROCH'S CONJECTURE

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Abstract. The Theorem of Bonnet–Myers implies that manifolds with topology $M^{n-1} \times \mathbb{S}^1$ do not admit a metric of positive Ricci curvature, while the resolution of Geroch's conjecture implies that the torus $\mathbb{T}^n$ does not admit a metric of positive scalar curvature. In this work we introduce a new notion of curvature interpolating between Ricci and scalar curvature (so called m-intermediate curvature), and use stable weighted slicings to show that for $n \leq 7$ and $1 \leq m \leq n-1$ the manifolds $N^n = M^{n-m} \times \mathbb{T}^m$ do not admit a metric of positive m-intermediate curvature.

1. Introduction

Closed manifolds with positive Ricci curvature have finite fundamental group due to the Theorem of Bonnet–Myers, in particular manifolds of topological type $N^n = M^{n-1} \times \mathbb{S}^1$ do not admit a metric of positive Ricci curvature. A different proof (at least in dimension $n \leq 7$) can be obtained by minimizing area in a homology class and using the stability inequality with test function $f = 1$.

On the other hand, a conjecture of Geroch asks whether the torus $\mathbb{T}^n$ does admit a metric of positive scalar curvature. This conjecture was resolved by R. Schoen und S.-T. Yau for $3 \leq n \leq 7$ by using minimal hypersurfaces [10], and by M. Gromov and H.-B. Lawson by using spinors for all dimensions [6]. The non-existence result for metrics of positive scalar curvature was extended to closed n-dimensional aspherical manifolds for $n \in \{4, 5\}$ independently by O. Chodosh and C. Li [2] and by M. Gromov [5]. For a more detailed overview on topological obstructions to positive scalar curvature we refer to the recent survey by O. Chodosh and C. Li [3].

The above obstruction for positive Ricci curvature and positive scalar curvature raise the following question: What kind of curvature obstructions can be found for manifolds of topological type $N^n = M^{n-m} \times \mathbb{T}^m$? This is an interesting question even for the case $N^4 = \mathbb{S}^2 \times \mathbb{T}^2$.

To investigate this question we define a family of curvature conditions (for $1 \leq m \leq n-1$) reducing to Ricci curvature for $m = 1$ and to scalar curvature for $m = n-1$ as follows:

Definition 1.1 (Positive m-intermediate curvature).
Suppose (N^n, g) is a Riemannian manifold. For given orthonormal vectors $\{e_1, \dots, e_m\}$ at the point $p \in N$ extend them to an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM. The m-intermediate curvature $\mathcal{C}_m$ of the orthornormal vectors $\{e_1, \dots, e_m\}$ is defined by

$$\mathcal{C}_m(e_1, \dots, e_m) := \sum_{p=1}^{m} \sum_{q=p+1}^{n} \mathrm{Rm}_N(e_p, e_q, e_p, e_q).$$

We say that (N^n, g) has positive m-intermediate curvature at $p \in N$, if we have $\mathcal{C}_m(e_1, \dots, e_m) > 0$ for any choice of orthornormal vectors $\{e_1, \dots, e_m\}$. Moreover, we say that the manifold (N^n, g) has positive m-intermediate curvature, if it has positive m-intermediate curvature for all $p \in M$.

The product manifold $N^n = \mathbb{S}^{n-m} \times \mathbb{T}^m$ (with $1 \leq m \leq n-2$) with the standard metric on both factors has positive $(m+1)$-intermediate curvature, and nonnegative m-intermediate curvature.

We observe that the condition of positive m-intermediate curvature defines a non-empty, open, $O(n)$-invariant convex cone in the space of algebraic curvature tensors for $1 \leq m \leq n-1$. Moreover, under the conditions $2 \leq m \leq n-1$ and $n+2-m \leq k \leq n$ the curvature tensor of $\mathbb{S}^{k-1} \times \mathbb{R}^{n-k+1}$ is contained in this open cone. The general surgery result due to S. Hoelzel [7, Theorem A] then implies that positive m-intermediate curvature is preserved under surgeries of codimension at least $n+2-m$.

Remark 1.2 (Connection to other notions of curvature).
(i) The quantity $\mathcal{C}_m(e_1, \ldots, e_m)$ is a sum of sectional curvatures of planes containing at least one of the vectors $e_1, \ldots, e_m$. In particular, positive m-intermediate curvature is a weaker condition than positive sectional curvature.

(ii) A manifold with positive m-intermediate curvature has positive scalar curvature. Indeed, the sum $\sum_{1 \leq p_1 < \cdots < p_m \leq n} \mathcal{C}_m(e_{p_1}, \ldots, e_{p_m})$ is equal to the scalar curvature, up to a factor.

(iii) There is a connection to the notion of (m,n)-intermediate scalar curvature introduced (as m-curvature) into the literature by M.-L. Labbi [9] *and also studied by M. Burkemper, C. Searle and M. Walsh* [1]. *More precisely, the (m,n)-intermediate scalar curvature defined by*

$$s_{m,n}(e_1, \ldots, e_m) = \sum_{p=m+1}^{n} \sum_{q=m+1}^{n} \mathrm{Rm}(e_p, e_q, e_p, e_q)$$

satisfies the relation

$$s_{m,n}(e_1, \ldots, e_m) + 2\,\mathcal{C}_m(e_1, \ldots, e_m) = \mathrm{R}\,.$$

In particular, $\mathcal{C}_m(e_1, \ldots, e_m)$ depends only on the span of $\{e_1, \ldots, e_m\}$, hence the m-intermediate curvature $\mathcal{C}_m$ can be regarded as a scalar function on the Grassmannian.

(iv) For $m = n-1$, we obtain $s_{n-1,n}(e_1, \ldots, e_{n-1}) = 0$ and $2\,\mathcal{C}_{n-1}(e_1, \ldots, e_{n-1}) = \mathrm{R}$. Hence, the intermediate curvature reduces to the scalar curvature in this case.

The case $m = 2$ (also called bi-Ricci curvature) was studied by Y. Shen and R. Ye in [12, 13]. They proved diameter estimates for stable minimal submanifolds in manifolds of positive bi-Ricci curvature and an estimate on the homology radius.

Our first main theorem concerns obstructions for the existence of metrics of positive m-intermediate curvature. To that end, we consider a notion of stable weighted slicing. Our definition closely resembles the notion of minimal k-slicings by R. Schoen and S.-T. Yau [11].

Definition 1.3 (Stable weighted slicing of order m).
Suppose $1 \leq m \leq n-1$ and let (N^n, g) be an orientable Riemannian manifold of dimension $\dim N = n$. *A stable weighted slicing of order m consists of a collection of orientable and smooth submanifolds Σ_k, $0 \leq k \leq m$, and a collection of positive functions $\rho_k \in C^\infty(\Sigma_k)$ satisfying the following conditions:*

- *$\Sigma_0 = N$ and $\rho_0 = 1$.*
- *For each $1 \leq k \leq m$, Σ_k is an embedded two-sided hypersurface in Σ_{k-1}. Moreover, Σ_k is a stable critical point of the ρ_{k-1}-weighted area*

$$\mathcal{H}^{n-k}_{\rho_{k-1}}(\Sigma) = \int_\Sigma \rho_{k-1}\, d\mu$$

  in the class of hypersurfaces $\Sigma \subset \Sigma_{k-1}$.

- *For each $1 \leq k \leq m$, the function $\frac{\rho_k}{\rho_{k-1}|_{\Sigma_k}} \in C^\infty(\Sigma_k)$ is a first eigenfunction of the stability operator associated with the ρ_{k-1}-weighted area.*

Observe that we use the first eigenfunction of the Jacobi operator of weighted area, while in [11, p. 7] a perturbed version of the weighted stability operator (denoted by Q_k) is used.

It is a classical theorem that manifolds with positive Ricci curvature do not admit stable minimal hypersurfaces. Our first theorem shows that manifolds with m-intermediate curvature do not allow stable weighted slicings of order m.

Theorem 1.4 (m-intermediate curvature and stable weighted slicings)**.**
Assume that $1 \leq m \leq n-1$ and $n(m-2) \leq m^2 - 2$. Suppose (N^n, g) is a closed and orientable Riemannian manifold with positive m-intermediate curvature. Then N does not admit a stable weighted slicing

$$\Sigma_m \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$$

of order m.

The inequality $n(m-2) \leq m^2 - 2$ is automatically satisfied for $m = 1$ (Ricci curvature), $m = 2$ (bi-Ricci curvature), $m = n-2$, and $m = n-1$ (scalar curvature). Moreover, the inequality $n(m-2) \leq m^2 - 2$ holds for all $n \leq 7$ and all $1 \leq m \leq n-1$. Surprisingly, in dimension $n \geq 8$, the inequality $n(m-2) \leq m^2 - 2$ fails for $m = 3$ (tri-Ricci curvature) and $m = 4$ (tetra-Ricci curvature).

The second step, which essentially is given in work of R. Schoen and S.-T. Yau [10], gives a topological condition for the existence of a stable weighted slicing:

Theorem 1.5 (Existence of stable weighted slicings)**.**
Assume $n \leq 7$ and $1 \leq m \leq n-1$. Let N^n be a closed and orientable manifold of dimension n, and suppose that there exists a closed and orientable manifold M^{n-m} and a map $F : N^n \to M^{n-m} \times \mathbb{T}^m$ with non-zero degree. Then for each Riemannian metric g on N^n there exists a stable weighted slicing

$$\Sigma_m \subset \Sigma_{m-1} \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n$$

of order m. In conjunction with Theorem 1.4 we deduce that the manifold N does not admit a metric with positive m-intermediate curvature.

The dimensional restriction allows us to invoke the regularity theory for hypersurfaces minimizing a weighted area. As a consequence of the above theorem we observe the following corollary:

Corollary 1.6 (Nonexistence of metrics of positive m-intermediate curvature)**.**
The product manifolds $N^n = M^{n-m} \times \mathbb{T}^m$ do not admit a metric of positive m-intermediate curvature for $n \leq 7$ and $1 \leq m \leq n-1$.

In particular, the manifold $\mathbb{S}^2 \times \mathbb{T}^2$ does not admit a metric of positive bi-Ricci curvature.

In Section 2 we introduce our notation and recall the first and second variation formula for weighted area. In Section 3, we describe the proof of Theorem 1.4. Afterwards, in Section 4, we give the proof of Theorem 1.5 and establish existence of stable weighted slicings under topological assumptions.

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2. The first and second variation of weighted area

For a Riemannian manifold (N^n, g) we consider its Levi-Civita connection D and its Riemann curvature tensor Rm_N given by the formula

$$\mathrm{Rm}_N(X,Y,Z,W) = -g(D_X D_Y Z - D_Y D_X Z - D_{[X,Y]}Z, W)$$

for vector fields $X, Y, Z, W \in \Gamma(TN)$.
Consider a two-sided embedded submanifold Σ^{n-1}. We denote its induced Levi-Civita connection by D_Σ, its unit normal vector field by $\nu \in \Gamma(N\Sigma)$, its scalar-valued second fundamental form by h_Σ and its mean curvature (the trace of the scalar-valued second fundamental form over Σ) by H_Σ. The gradient of a smooth function on N or Σ is denoted by $D_N f$ or $D_\Sigma f$.

Our arguments employ the first and second variation formula of a suitably weighted area: Consider a Riemannian manifold (N^n, g), a smooth positive function $\rho : N \to \mathbb{R}$, and an embedded two-sided closed manifold $\Sigma \subset N^n$. For a given smooth function $f \in C^\infty(\Sigma)$ we consider a variation $F : (-\epsilon, \epsilon) \times \Sigma \to N$ with $F(0,x) = x$ and $\frac{\partial}{\partial s}F(s,x)\big|_{s=0} = f(x)\,\nu(x)$. In the following, we denote the map $F(s,\cdot)$ by F_s. Moreover, we denote by Σ_s the image of F_s and by ν_s the unit normal vector field to F_s.

By precomposing the maps F_s with suitable tangential diffeomorphisms, we can arrange that the variation is normal in the sense that

$$\frac{\partial}{\partial s}F_s = f_s\,\nu_s,$$

where f_s is a smooth function on Σ_s.

We consider the weighted area defined by

$$\mathcal{H}^{n-1}_\rho(\Sigma) := \int_\Sigma \rho\, d\mu.$$

We recall the classical formulae for the first and second variation of weighted area:

Proposition 2.1 (First variation of weighted area)**.**
The first variation of weighted area is given by

$$\frac{d}{ds}\mathcal{H}^{n-1}_\rho(\Sigma_s)\bigg|_{s=0} = \int_\Sigma \rho f\,(H_\Sigma + \langle D_N \log\rho, \nu\rangle)\, d\mu.$$

Proof. This is a consequence of the first variation formula for area, and the chain rule. □

Corollary 2.2.
Suppose Σ is a critical point of weighted area. Then we have

$$H_\Sigma = -\langle D_N \log\rho, \nu\rangle.$$

For a constant weight we recover the minimal surface equation $H_\Sigma = 0$.

Proposition 2.3 (Second variation formula on critical points)**.**
If Σ is a critical point of the weighted area functional, then the second variation of weighted area is given by

$$\begin{aligned}&\frac{d^2}{ds^2}\mathcal{H}^{n-1}_\rho(\Sigma_s)\bigg|_{s=0}\\ &= \int_\Sigma \rho\left(-f\Delta_\Sigma f - \left(|h_\Sigma|^2 + \mathrm{Ric}_N(\nu,\nu)\right)f^2 + f^2(D^2_N\log\rho)(\nu,\nu) - f\langle D_\Sigma\log\rho, D_\Sigma f\rangle\right)d\mu.\end{aligned}$$

Proof. We use normal variations for our computation, and hence the first derivative is given by

$$\frac{d}{ds}\int_{\Sigma_s}\rho\,d\mu_s=\int_{\Sigma_s}\rho f_s\left(H_{\Sigma_s}+\langle D_N\log\rho,\nu_s\rangle\right)\,d\mu_s.$$

We now differentiate both sides of this equation with respect to s, and evaluate the result at $s=0$. By the variation formulas for hypersurfaces, compare for example with [8], the first order change in the mean curvature is given by

$$\frac{\partial}{\partial s}H_{\Sigma_s}\Big|_{s=0}=-\Delta_\Sigma f-\left(|h_\Sigma|^2+\mathrm{Ric}_N(\nu,\nu)\right)f,$$

whereas the first order change in the normal vector field is given by

$$D_s\nu_s\big|_{s=0}=-D_\Sigma f.$$

This implies

$$\begin{aligned}&\frac{\partial}{\partial s}\left(H_{\Sigma_s}+\langle D_N\log\rho,\nu_s\rangle\right)\Big|_{s=0}\\&=-\Delta_\Sigma f-\left(|h_\Sigma|^2+\mathrm{Ric}_N(\nu,\nu)\right)f+(D_N^2\log\rho)(\nu,\nu)f-\langle D_\Sigma\log\rho,D_\Sigma f\rangle,\end{aligned}$$

hence

$$\begin{aligned}&\frac{d^2}{ds^2}\mathcal{H}^{n-1}_\rho(\Sigma_s)\Big|_{s=0}\\&=\int_\Sigma\rho f\left(-\Delta_\Sigma f-\left(|h_\Sigma|^2+\mathrm{Ric}_N(\nu,\nu)\right)f+(D_N^2\log\rho)(\nu,\nu)f-\langle D_\Sigma\log\rho,D_\Sigma f\rangle\right)\,d\mu.\end{aligned}$$

□

For a constant weight we recover the usual second variation formula for minimal hypersurfaces:

$$\frac{d^2}{ds^2}\mathcal{H}^{n-1}(\Sigma_s)\Big|_{s=0}=\int_\Sigma\left(-f\Delta_\Sigma f-\left(|h_\Sigma|^2+\mathrm{Ric}_N(\nu,\nu)\right)f^2\right)\,d\mu.$$

3. Properties of stable weighted slicings

Let (N^n,g) be a closed and orientable Riemannian manifold of dimension $\dim N=n$. Throughout this section, we assume that we are given a stable weighted slicing of order m. Our goal is to show that the metric g cannot have positive m-intermediate curvature.

By the first variation formula for weighted area, Corollary 2.2, the mean curvature H_{Σ_k} of the slice Σ_k in the manifold Σ_{k-1} satisfies for $1\le k\le m$ the relation

$$H_{\Sigma_k}=-\langle D_{\Sigma_{k-1}}\log\rho_{k-1},\nu_k\rangle.$$

By the second variation formula for weighted area (compare Proposition 2.3) we obtain for $1\le k\le m$ the inequality

$$\begin{aligned}0\le&\int_{\Sigma_k}\rho_{k-1}\left(-\psi\Delta_{\Sigma_k}\psi-\psi\langle D_{\Sigma_k}\log\rho_{k-1},D_{\Sigma_k}\psi\rangle\right)\,d\mu\\&-\int_{\Sigma_k}\rho_{k-1}\Big(|h_{\Sigma_k}|^2+\mathrm{Ric}_{\Sigma_{k-1}}(\nu_k,\nu_k)-(D^2_{\Sigma_{k-1}}\log\rho_{k-1})(\nu_k,\nu_k)\Big)\,\psi^2\,d\mu\end{aligned}$$

for all $\psi \in C^\infty(\Sigma_k)$. By Definition 1.3 we may write $\rho_k = \rho_{k-1}\, v_k$, where $v_k > 0$ is the first eigenfunction of the stability operator for the weighted area functional. The function v_k satisfies

$$\begin{aligned} \lambda_k v_k =& - \Delta_{\Sigma_k} v_k - \langle D_{\Sigma_k} \log \rho_{k-1}, D_{\Sigma_k} v_k \rangle - \big(|h_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k)\big)\, v_k \\ &+ (D^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) v_k, \end{aligned}$$

where $\lambda_k \geq 0$ denotes the first eigenvalue of the stability operator.

By setting $w_k = \log v_k$ we record the following equation:

$$\begin{aligned} \lambda_k =& - \Delta_{\Sigma_k} w_k - \langle D_{\Sigma_k} \log \rho_{k-1}, D_{\Sigma_k} w_k \rangle - \big(|h_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k)\big) \\ &+ (D^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) - |D_{\Sigma_k} w_k|^2. \end{aligned} \tag{1}$$

We next record two lemmata connecting the second derivatives on consecutive slices.

Lemma 3.1 (First slicing identity)**.**
We have for $1 \leq k \leq m$ the identiy

$$\Delta_{\Sigma_k} \log \rho_{k-1} + (D^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) = \Delta_{\Sigma_{k-1}} \log \rho_{k-1} + H^2_{\Sigma_k}.$$

Proof. The above formula follows by applying the formula relating the Laplace operator on a submanifold to the Laplace operator on the ambient space

$$\Delta_{\Sigma_k} f + (D^2_{\Sigma_{k-1}} f)(\nu_k, \nu_k) = \Delta_{\Sigma_{k-1}} f - H_{\Sigma_k} \langle D_{\Sigma_{k-1}} f, \nu_k \rangle.$$

to the function $f = \log \rho_{k-1}$. The gradient term on the right-hand side is rewritten by using the first variation formula for weighted area

$$H_{\Sigma_k} = -\langle D_{\Sigma_{k-1}} \log \rho_{k-1}, \nu_k \rangle.$$

from Corollary 2.2. □

Lemma 3.2 (Second slicing identity)**.**
We have for $1 \leq k \leq m-1$ the identity

$$\begin{aligned} \Delta_{\Sigma_k} \log \rho_k =& \Delta_{\Sigma_k} \log \rho_{k-1} + (D^2_{\Sigma_{k-1}} \log \rho_{k-1})(\nu_k, \nu_k) \\ &- \big(\lambda_k + |h_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k, \nu_k) + \langle D_{\Sigma_k} \log \rho_k, D_{\Sigma_k} w_k \rangle\big)\,. \end{aligned}$$

Proof. This follows from the identity $\log \rho_k = w_k + \log \rho_{k-1}$ together with the equation (1). □

Lemma 3.3 (Stability inequality on the bottom slice)**.**
On the bottom slice Σ_m we have the inequality

$$\int_{\Sigma_m} \rho^{-1}_{m-1} \left(\Delta_{\Sigma_{m-1}} \log \rho_{m-1} + H^2_{\Sigma_m}\right) d\mu \geq \int_{\Sigma_m} \rho^{-1}_{m-1} \left(|h_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m)\right) d\mu.$$

Proof. By the second variation of weighted area (compare Proposition 2.3) the stability inequality on the bottom slice Σ_m gives

$$\begin{aligned} 0 \leq& \int_{\Sigma_m} \rho_{m-1} \left(-\psi \Delta_{\Sigma_m} \psi - \psi \langle D_{\Sigma_m} \log \rho_{m-1}, D_{\Sigma_m} \psi \rangle\right) d\mu \\ &- \int_{\Sigma_m} \rho_{m-1} \Big(|h_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m, \nu_m) - (D^2_{\Sigma_{m-1}} \log \rho_{m-1})(\nu_m, \nu_m)\Big)\, \psi^2\, d\mu \end{aligned}$$

for all $\psi \in C^\infty(\Sigma_m)$. Since the weight ρ_{m-1} is positive, we may use the direction $\psi = \rho_{m-1}^{-1}$ in the stability inequality, and observe

$$-\Delta_{\Sigma_m}\psi = -\Delta_{\Sigma_m}\rho_{m-1}^{-1} = \rho_{m-1}^{-1}\Delta_{\Sigma_m}\log\rho_{m-1} - \rho_{m-1}^{-3}|D_{\Sigma_m}\rho_{m-1}|^2,$$
$$-\langle D_{\Sigma_m}\log\rho_{m-1}, D_{\Sigma_m}\psi\rangle = -\langle D_{\Sigma_m}\log\rho_{m-1}, D_{\Sigma_m}\rho_{m-1}^{-1}\rangle = \rho_{m-1}^{-3}|D_{\Sigma_m}\rho_{m-1}|^2.$$

The gradient terms in the previous formulae cancel, and we obtain by rearrangement

$$\begin{aligned}&\int_{\Sigma_m}\rho_{m-1}^{-1}\Big(\Delta_{\Sigma_m}\log\rho_{m-1} + (D^2_{\Sigma_{m-1}}\log\rho_{m-1})(\nu_m,\nu_m)\Big)\,d\mu\\ \geq &\int_{\Sigma_m}\rho_{m-1}^{-1}\left(|h_{\Sigma_m}|^2 + \mathrm{Ric}_{\Sigma_{m-1}}(\nu_m,\nu_m)\right)\,d\mu.\end{aligned}$$

Finally, we use the first slicing equality from Lemma 3.1 to replace

$$\Delta_{\Sigma_m}\log\rho_{m-1} + (D^2_{\Sigma_{m-1}}\log\rho_{m-1})(\nu_m,\nu_m) = \Delta_{\Sigma_{m-1}}\log\rho_{m-1} + H^2_{\Sigma_m}.$$

□

Lemma 3.4 (Main inequality)**.**
We have the inequality

$$\int_{\Sigma_m}\rho_{m-1}^{-1}\left(\Lambda + \mathcal{R} + \mathcal{E} + \mathcal{G}\right)\,d\mu \leq 0,$$

where the eigenvalue term Λ*, the intrinsic curvature term* $\mathcal{R}$*, the extrinsic curvature term* $\mathcal{E}$*, and the gradient term* $\mathcal{G}$ *are given by*

$$\Lambda = \sum_{k=1}^{m-1}\lambda_k,\ \mathcal{R} = \sum_{k=1}^{m}\mathrm{Ric}_{\Sigma_{k-1}}(\nu_k,\nu_k),\ \mathcal{G} = \sum_{k=1}^{m-1}\langle D_{\Sigma_k}\log\rho_k, D_{\Sigma_k}w_k\rangle,$$
$$\text{and }\ \mathcal{E} = \sum_{k=1}^{m}|h_{\Sigma_k}|^2 - \sum_{k=2}^{m}H^2_{\Sigma_k}.$$

Proof. If we substitute the first slicing equality, Lemma 3.1, into the second slicing equality, Lemma 3.2, we obtain for $1 \leq k \leq m-1$ the identity

$$\Delta_{\Sigma_k}\log\rho_k = \Delta_{\Sigma_{k-1}}\log\rho_{k-1} + H^2_{\Sigma_k} - \left(\lambda_k + |h_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k,\nu_k) + \langle D_{\Sigma_k}\log\rho_k, D_{\Sigma_k}w_k\rangle\right).$$

Summation of the above formula over k from 1 to $m-1$ yields

$$\begin{aligned}\Delta_{\Sigma_{m-1}}\log\rho_{m-1} =&\Delta_{\Sigma_0}\log\rho_0 + \sum_{k=1}^{m-1}H^2_{\Sigma_k}\\ &- \sum_{k=1}^{m-1}\left(\lambda_k + |h_{\Sigma_k}|^2 + \mathrm{Ric}_{\Sigma_{k-1}}(\nu_k,\nu_k) + \langle D_{\Sigma_k}\log\rho_k, D_{\Sigma_k}w_k\rangle\right).\end{aligned}$$

We plug this equation into the stability inequality, Lemma 3.3. Moreover, we observe that the weight ρ_0 is constant, the mean curvature of the top slice H_{Σ_1} vanishes, and that the stability inequality contains the mean curvature term $H^2_{\Sigma_m}$ the extrinsic curvature term $|h_{\Sigma_m}|^2$ and the curvature term $\mathrm{Ric}_{\Sigma_{m-1}}(\nu_m,\nu_m)$. Then the lemma follows by grouping the terms suitably. □

We consider two examples to illustrate the structure of the curvature terms:

Example 3.5 (Positive Ricci curvature and $m = 1$)**.**
In the case $m = 1$ we have the slicing $\Sigma_1 \subset \Sigma_0 = N^n$ and we recover the classic result on the instability of minimal hypersurfaces in positive Ricci curvature $\mathrm{Ric} > 0$. Indeed, we have $\Lambda = \mathcal{G} = 0$, $\mathcal{E} = |h_{\Sigma_1}|^2$, and $\mathcal{R} = \mathrm{Ric}_N(\nu_1, \nu_1)$. Thus $\mathcal{R} + \mathcal{E} > 0$. Combined with the existence theory for stable weighted slicings from Section 4 this implies the non-existence of metrics of positive Ricci curvature on manifolds with topology $N^n = M^{n-1} \times \mathbb{S}^1$ in dimension $\dim N \leq 7$.

Example 3.6 (Positive bi-Ricci curvature and $m = 2$)**.**
In the case $m = 2$ we have the slicing $\Sigma_2 \subset \Sigma_1 \subset \Sigma_0 = N^n$. We moreover observe $\Lambda = \lambda_1 \geq 0$, $\mathcal{G} = |D_{\Sigma_1} w_1|^2 \geq H^2_{\Sigma_2}$, and the curvature terms $\mathcal{E}$ and $\mathcal{R}$ are given by

$$\mathcal{E} = |h_{\Sigma_1}|^2 + |h_{\Sigma_2}|^2 - H^2_{\Sigma_2},$$
$$\text{and } \mathcal{R} = \mathrm{Ric}_N(\nu_1, \nu_1) + \mathrm{Ric}_N(\nu_2, \nu_2) - \mathrm{Rm}_N(\nu_1, \nu_2, \nu_1, \nu_2) - (h^2_{\Sigma_1})(\nu_1, \nu_1).$$

Thus if we assume positive bi-Ricci curvature we have $\Lambda + \mathcal{R} + \mathcal{E} + \mathcal{G} > 0$. This shows a non-existence result for stable weighted slicings of order two. Combined with the existence theory for stable weighted slicings from Section 4 this implies that a manifold with topology $N^n = M^{n-2} \times \mathbb{T}^2$ (with $n \leq 7$) does not admit a metric of positive bi-Ricci curvature.

The eigenvalue term Λ is non-negative, since it is the sum of the non-negative eigenvalues. We will estimate the other terms below.

The first step is to estimate the gradient terms:

Lemma 3.7 (Estimate of gradient terms)**.**
We have the estimate

$$\mathcal{G} \geq \sum_{k=2}^{m} \left(\frac{1}{2} + \frac{1}{2(k-1)} \right) H^2_{\Sigma_k}.$$

Proof. We define for $k \geq 1$ the nonnegative real numbers α_k by

$$\alpha_k = \frac{k-1}{2k}.$$

By direct computation one verifies the identity

$$1 - \alpha_{k-1} = \frac{1}{4\alpha_k}$$

for $k \geq 2$. Using the identity $H_{\Sigma_{k+1}} = -\langle D_{\Sigma_k} \log \rho_k, \nu_{k+1} \rangle$, we obtain

$$\begin{aligned}
&\langle D_{\Sigma_k} \log \rho_k, D_{\Sigma_k} w_k \rangle \\
=&\langle D_{\Sigma_k} \log \rho_k, D_{\Sigma_k} (\log \rho_k - \log \rho_{k-1}) \rangle \\
=&(1 - \alpha_k) |D_{\Sigma_k} \log \rho_k|^2 - \frac{1}{4\alpha_k} |D_{\Sigma_k} \log \rho_{k-1}|^2 \\
&+ \alpha_k \left| D_{\Sigma_k} \log \rho_k - \frac{1}{2\alpha_k} D_{\Sigma_k} \log \rho_{k-1} \right|^2 \\
=&(1 - \alpha_k) H^2_{\Sigma_{k+1}} + (1 - \alpha_k) |D_{\Sigma_{k+1}} \log \rho_k|^2 - (1 - \alpha_{k-1}) |D_{\Sigma_k} \log \rho_{k-1}|^2 \\
&+ \alpha_k \left| D_{\Sigma_k} \log \rho_k - \frac{1}{2\alpha_k} D_{\Sigma_k} \log \rho_{k-1} \right|^2
\end{aligned}$$

for $2 \leq k \leq m-1$. Summation over k from 2 to $m-1$ yields the formula

$$\sum_{k=2}^{m-1} \langle D_{\Sigma_k} \log \rho_k, D_{\Sigma_k} w_k \rangle \geq \sum_{k=2}^{m-1} (1-\alpha_k)\, H^2_{\Sigma_{k+1}} + (1-\alpha_{m-1})\, |D_{\Sigma_m} \log \rho_{m-1}|^2 - |D_{\Sigma_2} \log \rho_1|^2.$$

Moreover, the identity $H_{\Sigma_2} = -\langle D_{\Sigma_1} \log \rho_1, \nu_2 \rangle$ implies

$$\langle D_{\Sigma_1} \log \rho_1, D_{\Sigma_1} w_1 \rangle = |D_{\Sigma_1} \log \rho_1|^2 = H^2_{\Sigma_2} + |D_{\Sigma_2} \log \rho_1|^2.$$

Adding the two inequalities gives

$$\sum_{k=1}^{m-1} \langle D_{\Sigma_k} \log \rho_k, D_{\Sigma_k} w_k \rangle \geq \sum_{k=1}^{m-1} (1-\alpha_k)\, H^2_{\Sigma_{k+1}} + (1-\alpha_{m-1})\, |D_{\Sigma_m} \log \rho_{m-1}|^2.$$

□

In the next step we rewrite the intrinsic curvature terms with the help of the Gauss equations:

Lemma 3.8 (Iterated Gauss equations)**.**
The curvature term $\mathcal{R}$ is given by

$$\mathcal{R} = \mathcal{C}_m(e_1, \dots, e_m) + \sum_{k=1}^{m-1} \sum_{p=k+1}^{m} \sum_{q=p+1}^{n} \left(h_{\Sigma_k}(e_p, e_p) h_{\Sigma_p}(e_q, e_q) - h_{\Sigma_k}(e_p, e_q)^2 \right),$$

where $\mathcal{C}_m$ denotes the m-intermediate curvature of the Riemannian manifold (N^n, g).

Proof. Fix a point $x \in \Sigma_m$ and consider an orthornomal basis $\{e_1, \dots, e_n\}$ of $T_x N$ with $e_j = \nu_j$ for $1 \leq j \leq m$ as above. We observe by the definition of the Ricci curvature on the slice Σ_{k-1}, and by the Gauss equations the formula

$$\begin{aligned} \mathrm{Ric}_{\Sigma_{p-1}}(\nu_p, \nu_p) = \mathrm{Ric}_{\Sigma_{p-1}}(e_p, e_p) &= \sum_{q=p+1}^{n} \mathrm{Rm}_{\Sigma_{p-1}}(e_p, e_q, e_p, e_q) \\ &= \sum_{q=p+1}^{n} \mathrm{Rm}_N(e_p, e_q, e_p, e_q) + \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(h_{\Sigma_k}(e_p, e_p) h_{\Sigma_k}(e_q, e_q) - h_{\Sigma_k}(e_p, e_q)^2 \right). \end{aligned}$$

Summation over p from 1 to m then implies

$$\begin{aligned} \mathcal{R} &= \sum_{p=1}^{m} \mathrm{Ric}_{\Sigma_{p-1}}(\nu_p, \nu_p) \\ &= \sum_{p=1}^{m} \sum_{q=p+1}^{n} \mathrm{Rm}_N(e_p, e_q, e_p, e_q) + \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(h_{\Sigma_k}(e_p, e_p) h_{\Sigma_k}(e_q, e_q) - h_{\Sigma_k}(e_p, e_q)^2 \right) \\ &= \mathcal{C}_m(e_1, \dots, e_m) + \sum_{p=1}^{m} \sum_{q=p+1}^{n} \sum_{k=1}^{p-1} \left(h_{\Sigma_k}(e_p, e_p) h_{\Sigma_k}(e_q, e_q) - h_{\Sigma_k}(e_p, e_q)^2 \right). \end{aligned}$$

If we interchange the order of summation, the assertion follows. □

Remark 3.9 (Observation on full slicing)**.**
In the special case $m = n-1$ the curvature term $\mathcal{R}$ can be rewritten as

$$\begin{aligned}\mathcal{R} &= \mathcal{C}_{n-1}(e_1,\dots,e_{n-1}) + \sum_{k=1}^{n-2}\sum_{p=k+1}^{n-1}\sum_{q=p+1}^{n}\big(h_{\Sigma_k}(e_p,e_p)h_{\Sigma_p}(e_q,e_q) - h_{\Sigma_k}(e_p,e_q)^2\big)\\ &= \frac{1}{2}\,\mathrm{R}_N + \frac{1}{2}\sum_{k=1}^{n-2}\big(H_{\Sigma_k}^2 - |h_{\Sigma_k}|^2\big)\,.\end{aligned}$$

(cf. Remark 1.2 (iv)). Note that the mean curvature of the top slice Σ_1 vanishes, and that $H^2_{\Sigma_{n-1}} = |h_{\Sigma_{n-1}}|^2$ since Σ_{n-1} is one-dimensional. Therefore, for $m = n-1$ we obtain

$$\begin{aligned}\mathcal{R}+\mathcal{E}+\mathcal{G} &= \frac{1}{2}\,\mathrm{R}_N + \frac{1}{2}\sum_{k=1}^{n-1}|h_{\Sigma_k}|^2 - \frac{1}{2}\sum_{k=1}^{n-1}H^2_{\Sigma_k} + \mathcal{G}\\ &\geq \frac{1}{2}\,\mathrm{R}_N + \frac{1}{2}\sum_{k=1}^{n-1}|h_{\Sigma_k}|^2 + \sum_{k=2}^{n-1}\frac{1}{2(k-1)}H^2_{\Sigma_k}.\end{aligned}$$

In the last step we have used the estimate for the gradient terms $\mathcal{G}$ from Lemma 3.7. Hence, we recover a similar result as in the computation of R. Schoen and S.-T. Yau [11].

In the next step we need to analyze the contributions coming from the extrinsic curvature. We fix $m \in \{2,\dots,n-1\}$, and we define for $1 \leq k \leq m$ the extrinsic curvature terms $\mathcal{V}_k$:

$$\begin{aligned}\mathcal{V}_1 =& |h_{\Sigma_1}|^2 + \sum_{p=2}^{m}\sum_{q=p+1}^{n}\big(h_{\Sigma_1}(e_p,e_p)h_{\Sigma_1}(e_q,e_q) - h_{\Sigma_1}(e_p,e_q)^2\big)\,,\\ \mathcal{V}_k =& |h_{\Sigma_k}|^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)}\right)H^2_{\Sigma_k}\\ &+ \sum_{p=k+1}^{m}\sum_{q=p+1}^{n}\big(h_{\Sigma_k}(e_p,e_p)h_{\Sigma_k}(e_q,e_q) - h_{\Sigma_k}(e_p,e_q)^2\big) \text{ for } 2 \leq k \leq m-1,\\ \mathcal{V}_m =& |h_{\Sigma_m}|^2 - \left(\frac{1}{2} - \frac{1}{2(m-1)}\right)H^2_{\Sigma_m}.\end{aligned}$$

By combining Lemma 3.7, with Lemma 3.8, and the above expressions $\mathcal{V}_k$ for the extrinsic curvature terms, we obtain:

Lemma 3.10.
For $2 \leq m \leq n-1$ we have the pointwise estimate

$$\mathcal{R}+\mathcal{E}+\mathcal{G} \geq \mathcal{C}_m(e_1,\dots,e_m) + \sum_{k=1}^{m}\mathcal{V}_k.$$

In the following lemmata we estimate the extrinsic curvature terms $\mathcal{V}_k$. The estimate for $\mathcal{V}_m$ follows from the trace estimate for symmetric two-tensors. The estimate for $\mathcal{V}_1$ uses minimality of the top slice Σ_1. The estimate for $\mathcal{V}_k$ with $2 \leq k \leq m-1$ is the most involved.

Lemma 3.11 (Extrinsic curvature terms on top slice)**.**
For $2 \leq m \leq n-1$ we have the estimate

$$\mathcal{V}_1 \geq \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} \left(\sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p) \right)^2 .$$

Proof. To estimate the term $\mathcal{V}_1$, we begin by discarding the off-diagonal terms of the second fundamental form h_{Σ_1}:

$$\begin{aligned}\mathcal{V}_1 =& |h_{\Sigma_1}|^2 + \sum_{p=2}^{m} \sum_{q=p+1}^{n} \left(h_{\Sigma_1}(e_p, e_p) h_{\Sigma_1}(e_q, e_q) - h_{\Sigma_1}(e_p, e_q)^2 \right) \\ \geq & \sum_{p=2}^{n} h_{\Sigma_1}(e_p, e_p)^2 + \sum_{p=2}^{m} \sum_{q=p+1}^{n} h_{\Sigma_1}(e_p, e_p) h_{\Sigma_1}(e_q, e_q).\end{aligned}$$

The terms on the right hand side can be rewritten as follows:

$$\mathcal{V}_1 \geq \frac{1}{2} \sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p)^2 + \sum_{q=m+1}^{n} h_{\Sigma_1}(e_q, e_q)^2 + \sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p)\, H_{\Sigma_1} - \frac{1}{2} \left(\sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p) \right)^2 .$$

Recall that $H_{\Sigma_1} = 0$. By the Cauchy–Schwarz inequality,

$$\sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p)^2 \geq \frac{1}{m-1} \left(\sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p) \right)^2$$

and

$$\sum_{q=m+1}^{n} h_{\Sigma_1}(e_q, e_q)^2 \geq \frac{1}{n-m} \left(\sum_{q=m+1}^{n} h_{\Sigma_1}(e_q, e_q) \right)^2 = \frac{1}{n-m} \left(\sum_{p=2}^{m} h_{\Sigma_1}(e_p, e_p) \right)^2 ,$$

where in the last step we have used the fact that $H_{\Sigma_1} = 0$. Putting these facts together, the assertion follows. ☐

Lemma 3.12 (Extrinsic curvature terms on intermediate slices)**.**
For $2 \leq m \leq n-1$ and $2 \leq k \leq m-1$ we have the estimate

$$\mathcal{V}_k \geq \frac{m^2 - 2 - n(m-2)}{2(m-1)(n-m)} \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right)^2 .$$

Proof. To estimate the term $\mathcal{V}_k$, we start by discarding the off-diagonal terms:

$$\begin{aligned}\mathcal{V}_k =& |h_{\Sigma_k}|^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)} \right) H_{\Sigma_k}^2 + \sum_{p=k+1}^{m} \sum_{q=p+1}^{n} \left(h_{\Sigma_k}(e_p, e_p) h_{\Sigma_k}(e_q, e_q) - h_{\Sigma_k}(e_p, e_q)^2 \right) \\ \geq & \sum_{p=k+1}^{n} h_{\Sigma_k}(e_p, e_p)^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)} \right) H_{\Sigma_k}^2 + \sum_{p=k+1}^{m} \sum_{q=p+1}^{n} h_{\Sigma_k}(e_p, e_p) h_{\Sigma_k}(e_q, e_q).\end{aligned}$$

The terms on the right hand side can be rewritten as follows:

$$\begin{aligned}\mathcal{V}_k \geq & \frac{1}{2} \sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p)^2 + \sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q)^2 \\ & + \frac{1}{2(k-1)} \left(\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p) \right)^2 - \left(\frac{1}{2} - \frac{1}{2(k-1)} \right) \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right)^2 \\ & + \frac{1}{k-1} \left(\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p) \right) \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right).\end{aligned}$$

The Cauchy–Schwarz inequality gives

$$\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p)^2 \geq \frac{1}{m-k} \left(\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p) \right)^2$$

and

$$\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q)^2 \geq \frac{1}{n-m} \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right)^2.$$

Moreover, Young's inequality implies

$$\begin{aligned}\left(\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p) \right) \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right) \geq & - \frac{m-1}{2(m-k)} \left(\sum_{p=k+1}^{m} h_{\Sigma_k}(e_p, e_p) \right)^2 \\ & - \frac{m-k}{2(m-1)} \left(\sum_{q=m+1}^{n} h_{\Sigma_k}(e_q, e_q) \right)^2.\end{aligned}$$

Putting these facts together, the assertion follows. ☐

Lemma 3.13 (Extrinsic curvature terms on bottom slice)**.**
For $2 \leq m \leq n-1$ we have the estimate

$$\mathcal{V}_m \geq \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} H_{\Sigma_m}^2. \tag{2}$$

Proof. We observe by the the trace estimate for symmetric two-tensors the inequality

$$\begin{aligned}\mathcal{V}_m = |h_{\Sigma_m}|^2 - \left(\frac{1}{2} - \frac{1}{2(m-1)} \right) H_{\Sigma_m}^2 \geq \left(\frac{1}{n-m} - \left(\frac{1}{2} - \frac{1}{2(m-1)} \right) \right) H_{\Sigma_m}^2 \\ = \frac{m^2 - 2 - n(m-2)}{2(n-m)(m-1)} H_{\Sigma_m}^2.\end{aligned}$$

☐

With the above observations we prove our first theorem:

Proof of Theorem 1.4. Assume that $1 \leq m \leq n-1$ and $n(m-2) \leq m^2-2$. Suppose that (N^n, g) is a closed and orientable Riemannian manifold which admits a stable weighted slicing

$$\Sigma_m \subset \Sigma_{m-1} \subset \cdots \subset \Sigma_1 \subset \Sigma_0 = N^n.$$

If $m=1$, the stability inequality implies that (N^n, g) cannot have positive Ricci curvature. Hence, it remains to consider the case when $2 \leq m \leq n-1$ and $n(m-2) \leq m^2-2$. In this case, it follows from Lemma 3.11, Lemma 3.12, and Lemma 3.13 that $\mathcal{V}_k \geq 0$ for all $1 \leq k \leq m$. Using Lemma 3.10, we obtain the pointwise inequality

$$\mathcal{R} + \mathcal{E} + \mathcal{G} \geq \mathcal{C}_m(e_1, \ldots, e_m).$$

If $\mathcal{C}_m(e_1, \ldots, e_m)$ is strictly positive, this contradicts our main inequality, Lemma 3.4. Therefore, the Riemannian manifold (N^n, g) cannot have positive m-intermediate curvature. □

4. Existence of stable weighted slicings

In this section we prove existence of stable weighted slicings of order m. The argument uses the mapping degree and is essentially contained in Theorem 4.5 of [11]. Alternatively, one could also use an argument based on homology, compare with Theorem 4.6 in [11].

Proof of Theorem 1.5. Suppose N^n and M^{n-m} are closed and orientable manifolds, and suppose $F : N^n \to \mathbb{T}^m \times M^{n-m}$ is a map of degree $d \neq 0$. The projection of F onto the factors yields maps $f_0 : N \to M$ and maps $f_1, \ldots, f_m : N \to \mathbb{S}^1$. Let Θ be a top-dimensional form of the manifold M normalized such that $\int_M \Theta = 1$, and let θ be a one-form on the circle $\mathbb{S}^1$ with $\int_{\mathbb{S}^1} \theta = 1$. We define the pull-back forms $\Omega := f_0^* \Theta$ and $\omega_j := f_j^* \theta$. By the normalization condition we deduce that $\int_N \omega_1 \wedge \cdots \wedge \omega_m \wedge \Omega = d$.

We claim that one can construct closed and orientable slices Σ_k and weights ρ_k such that $\int_{\Sigma_k} \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = d$. We prove the claim by induction. The base case $k=0$ holds by the previous observation and by setting $\Sigma_0 := N$ and $\rho_0 := 1$. For the induction step we suppose that we have constructed the slice Σ_{k-1} and the weight ρ_{k-1}, such that $\int_{\Sigma_{k-1}} \omega_k \wedge \cdots \wedge \omega_m \wedge \Omega = d$.

We define a class $\mathcal{A}_k$ by

$$\mathcal{A}_k = \left\{ \Sigma \text{ is an } (n-k) - \text{integer rectifiable current in } \Sigma_k \text{ with } \int_\Sigma \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = d \right\}.$$

The first step is to show that the class $\mathcal{A}_k$ is non-empty. To prove this, let us fix a regular value $p_k \in \mathbb{S}^1$ of the map $f_k|_{\Sigma_{k-1}} : \Sigma_{k-1} \to \mathbb{S}^1$. The existence of a regular value follows from Sard's Theorem.

On the complement $\mathbb{S}^1 \backslash \{p_k\}$ the one-form θ is exact. In other words, there exists a smooth function $\psi_k : \mathbb{S}^1 \backslash \{p_k\} \to \mathbb{R}$, such that $d\psi_k = \theta$. Moreover, due to the normalization condition $\int_{\mathbb{S}^1} \theta = 1$, the function ψ_k jumps by 1 at p_k.

We next consider the pre-image $\tilde{\Sigma}_k = \{x \in \Sigma_{k-1} : f_k(x) = p_k\}$. Since $p_k \in \mathbb{S}^1$ is a regular value of the map $f_k|_{\Sigma_{k-1}} : \Sigma_{k-1} \to \mathbb{S}^1$, it follows that $\tilde{\Sigma}_k$ is a closed and orientable submanifold of Σ_{k-1}. We define a function $\varphi_k : \Sigma_{k-1} \backslash \tilde{\Sigma}_k \to \mathbb{R}$ by setting $\varphi_k := \psi_k \circ f_k$. Since the pull-back commutes with the differential, we deduce $d\varphi_k = f_k^* (d\psi_k) = f_k^* \theta = \omega_k$ on $\Sigma_{k-1} \backslash \tilde{\Sigma}_k$.

The above observation (and the closedness of the forms $\omega_k, \ldots, \omega_m, \Omega$) implies

$$\omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = d(\varphi_k \, \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega). \tag{3}$$

We first consider the case when $\tilde{\Sigma}_k$ is empty. Integrating the identity (3) over Σ_{k-1} gives

$$d = \int_{\Sigma_{k-1}} \omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega = \int_{\Sigma_{k-1}} d(\varphi_k \, \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega) = 0.$$

This is a contradiction.

It remains to consider the case when $\tilde{\Sigma}_k$ is non-empty. In this case $\tilde{\Sigma}_k$ is a smooth, orientable and embedded hypersurface in Σ_{k-1}. We integrate identity (3) over $\Sigma_{k-1} \setminus \tilde{\Sigma}_k$. By Stokes theorem, the integral of the right hand side yields two boundary integrals over $\tilde{\Sigma}_k$. Since the function φ_k jumps by 1 along $\tilde{\Sigma}_k$, we obtain

$$\begin{aligned} d = \int_{\Sigma_{k-1} \setminus \tilde{\Sigma}_k} \omega_k \wedge \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega &= \int_{\Sigma_{k-1} \setminus \tilde{\Sigma}_k} d(\varphi_k \, \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega) \\ &= \pm \int_{\tilde{\Sigma}_k} \omega_{k+1} \wedge \cdots \wedge \omega_m \wedge \Omega, \end{aligned}$$

where the sign depends on the choice of orientation of $\tilde{\Sigma}_k$. Therefore, we can make a choice of orientation so that $\tilde{\Sigma}_k$ belongs to the class $\mathcal{A}_k$. In particular, the class $\mathcal{A}_k$ is non-empty.

We consider the variational problem

$$\sigma_k = \inf \left\{ \mathbb{M}_{\rho_{k-1}, n-k}(\Sigma) : \Sigma \in \mathcal{A}_k \right\},$$

where $\mathbb{M}_{\rho_{k-1}, n-k}$ denotes the ρ_{k-1}-weighted mass functional on $(n-k)$-integer rectifiable currents. By the compactness theory for integer rectifiable currents, compare for example Theorem 7.5.3 in [14], we deduce that there exists an $(n-k)$-integer rectifiable current Σ_{k+1} with mass $\mathbb{M}_{\rho_{k-1}, n-k}(\Sigma_k) = \sigma_k$.

By the regularity theory for integer rectifiable currents, compare for example Theorem 7.5.8 in [14] or the survey [4], and the dimension bound $n \leq 7$ we deduce that Σ_k is a smooth and orientable (and hence two-sided) hypersurface. Moreover, the smooth surface Σ_k is stable with respect to variations of the weighted area, and therefore we can find a positive first eigenfunction v_k of the weighted stability operator. Defining the weight ρ_k by the formula $\rho_k = \rho_{k-1} \cdot v_k$ completes the induction step. □

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