On the lower bound for kissing numbers of ℓ_p-spheres in high dimensions

Chengfei Xie* and Gennian Ge†

July 21, 2022

Abstract

In this paper, we give some new lower bounds for the kissing number of ℓ_p-spheres. These results improve the previous work due to Xu (2007). Our method is based on coding theory.

Keywords: kissing number, Gilbert-Varshamov type bound, ℓ_p-sphere

AMS subject classifications: 52C17, 05B40, 11H71, 05D05

1 Introduction

Let S^{n-1} be the unit sphere in $\mathbb{R}^n$. The (translative) kissing number problem asks the maximum number of nonoverlapping translates $S^{n-1} + \boldsymbol{x}$ that can touch S^{n-1} at its boundary. This is an old and difficult problem in discrete geometry. The exact answer is only known in dimensions $1, 2, 3, 4, 8$, and 24. In dimensions 1 and 2, the problem is trivial; in dimension 3, the problem is known as the Gregory-Newton Problem and was solved by Schütte and van der Waerden [12] (see also [7] for another proof); in dimension 4, the problem was solved by Musin [9] via an extension of Delsarte's method; in dimensions 8 and 24, the problem was solved by Levenšteĭn [8] and Odlyzko and Sloane [10] independently.

*C. Xie is with the School of Mathematical Sciences, Capital Normal University, Beijing 100048, China (email: cfxie@cnu.edu.cn).

†Corresponding author. G. Ge is with the School of Mathematical Sciences, Capital Normal University, Beijing 100048, China (e-mail: gnge@zju.edu.cn). The research of G. Ge is supported by the National Key Research and Development Program of China under Grant Nos. 2020YFA0712100 and 2018YFA0704703, National Natural Science Foundation of China under Grant No. 11971325, and Beijing Scholars Program.

Let $K_2(n)$ be the kissing number of S^{n-1}. The best upper bound for $K_2(n)$ in high dimensions is due to Kabatjanskiĭ and Levenšteĭn [5]: $K_2(n) \le 2^{0.401n(1+o(1))}$. Using a sphere covering argument, Shannon [13] and Wyner [16] obtained a lower bound $K_2(n) \ge c\sqrt{n}(2/\sqrt{3})^n$. Recently, Jenssen et al. [3] improved the lower bound by a linear factor in the dimension. See also Fernández et al. [1] for constant factor improvement.

In this paper, we consider the kissing number of ℓ_p-spheres. For $p \ge 1$, let $S_p^{n-1}(R)$ be the ℓ_p-sphere with radius R and centered at $\mathbf{0}$ in $\mathbb{R}^n$, that is, $S_p^{n-1}(R) := \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_p = R\}$, where the ℓ_p-norm $\|\cdot\|_p$ is defined by $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. We simply write $S_p^{n-1} = S_p^{n-1}(1)$. Let $K_p(n)$ be the kissing number of S_p^{n-1}. Minkowski-Hadwiger theorem [2] implies an upper bound $K_p(n) \le 3^n - 1$. This bound was improved by Sah et al. [11] for $p \ge 2$. Much less is known about the upper bound when p is between 1 and 2.

On the lower bound, Larman and Zong [6] proved that $K_p(n) \ge (9/8)^{n(1+o(1))} = 2^{0.1699n(1+o(1))}$. Xu [17] improved this result for every $p \ge 1$, for instance, $K_3(n) \ge 2^{0.4564n(1+o(1))}$. Our main result is an improvement to the work of Xu. Since our result does not have an explicit formula, we list some numerical results here:

$$K_1(n) \ge 2^{0.1247n(1+o(1))} + 2^{0.1825n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots ;$$

$$K_2(n) \ge 2^{0.2059n(1+o(1))} + 2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots ;$$

$$K_3(n) \ge cn2^{0.4564n(1+o(1))} + 2^{0.1562n(1+o(1))} + 2^{0.0425n(1+o(1))} + \cdots .$$

We give some explanation to our results. In the lower bound for $K_2(n)$, the $2^{0.2059n(1+o(1))}$ term is the same as the lower bound due to Xu, so we improve the lower bound by adding the remainder terms $2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots$. In the lower bound for $K_3(n)$, the $2^{0.4564n(1+o(1))}$ term is the same as the lower bound due to Xu, so we improve the leading term by a factor of n and add some remainder terms.

Our idea comes from coding theory. The translative kissing number $K_p(n)$ is equal to the largest size of an ℓ_p*-spherical code* with minimum distance 1 (see Lemma 2.1). We choose a discrete set X from S_p^{n-1}. Applying ideas from coding theory, we are able to find a large subset of X, in which points have pairwise distance larger than or equal to 1. This gives a lower bound for $K_p(n)$.

2 An improved Gilbert-Varshamov type bound

Let $A_p(n,d)$ be the maximum size of a subset of S_p^{n-1} in which the points have pairwise ℓ_p-distance at least $2d$; that is,

$$A_p(n,d) := \max\{|C| : C \subseteq S_p^{n-1} \text{ and } d_p(\boldsymbol{x}, \boldsymbol{y}) \ge 2d, \forall \boldsymbol{x}, \boldsymbol{y} \in C\},$$

where $d_p(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|_p$ is the ℓ_p-distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. In other words, $A_p(n, d)$ is the largest size of an ℓ_p-spherical code with minimum distance $2d$. The following lemma is an easy observation.

Lemma 2.1. *The translative kissing number $K_p(n)$ of S_p^{n-1} is equal to $A_p(n, 1/2)$.*

Proof. For convenience, let $k_1 = K_p(n)$ and $k_2 = A_p(n, 1/2)$.

Suppose $S_p^{n-1}, S_p^{n-1} + \boldsymbol{x}_1, S_p^{n-1} + \boldsymbol{x}_2, \ldots, S_p^{n-1} + \boldsymbol{x}_{k_1}$ form a kissing configuration. For every i, if $d_p(\mathbf{0}, \boldsymbol{x}_i) > 2$, then $S_p^{n-1} + \boldsymbol{x}_i$ and S_p^{n-1} do not share a common point; if $d_p(\mathbf{0}, \boldsymbol{x}_i) < 2$, then $S_p^{n-1} + \boldsymbol{x}_i$ and S_p^{n-1} are overlapping. Thus, $d_p(\mathbf{0}, \boldsymbol{x}_i) = 2$ and $\frac{1}{2}\boldsymbol{x}_i \in S_p^{n-1}$ for every i. Moreover, $d_p(\boldsymbol{x}_i, \boldsymbol{x}_j) \geq 2$ for $i \neq j$. So $d_p(\frac{1}{2}\boldsymbol{x}_i, \frac{1}{2}\boldsymbol{x}_j) \geq 1$ for $i \neq j$. Therefore, $\{\frac{1}{2}\boldsymbol{x}_1, \frac{1}{2}\boldsymbol{x}_2, \ldots, \frac{1}{2}\boldsymbol{x}_{k_1}\}$ is an ℓ_p-spherical code with minimum distance 1, i.e. $k_2 \geq k_1$.

On the other hand, suppose $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{k_2}\}$ is an ℓ_p-spherical code with minimum distance 1. Then $S_p^{n-1} + 2\boldsymbol{x}_1, S_p^{n-1} + 2\boldsymbol{x}_2, \ldots, S_p^{n-1} + 2\boldsymbol{x}_{k_2}$ are nonoverlapping, and $S_p^{n-1} + 2\boldsymbol{x}_i$ touches S_p^{n-1} at $\boldsymbol{x}_i$ for every i. So $k_1 \geq k_2$. Thus the lemma follows. ☐

For a positive integer $m \leq n$, which will be determined later, we define a family $\mathcal{J}(m, n)$ of subsets of $\mathbb{R}^n$ recursively. Define $m_1 := m$ and

$$J_1(m, n) := \left\{ \boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \{0, \pm 1\}^n : \sum_{i=1}^{n} |u_i|^p = m \right\}.$$

Suppose we have defined m_i and $J_i(m, n)$. Then we define

$$m_{i+1} := \lfloor m_i / 2^p \rfloor \tag{1}$$

and

$$J_{i+1}(m, n) := \left\{ \boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in \{0, \pm(m/m_{i+1})^{1/p}\}^n : \sum_{i=1}^{n} |u_i|^p = m \right\}.$$

This process terminates when $m_r < 2^p$ for some r. So we obtain $\{m_1 > m_2 > \ldots > m_r\}$ and $\mathcal{J}(m, n) = \{J_1(m, n), J_2(m, n), \ldots, J_r(m, n)\}$. And we have the following proposition.

Proposition 2.2. *For $\mathcal{J}(m, n)$ defined above, the following statements hold.*

1. *If $i \neq j$, then $J_i(m, n) \cap J_j(m, n) = \emptyset$.*
2. *For every $1 \leq i \leq r$ and for every $\boldsymbol{u} \in J_i(m, n)$, $\boldsymbol{u}$ has exactly $n - m_i$ zero coordinates.*
3. *For every $1 \leq i \leq r$,*

$$|J_i(m, n)| = \binom{n}{m_i} 2^{m_i}. \tag{2}$$

4. For every $1 \le i \le r$ and for every $\boldsymbol{u} \in J_i(m,n)$, the ℓ_p-norm of $\boldsymbol{u}$ is $m^{1/p}$.

5. If $i \neq j$, then for every $\boldsymbol{u} \in J_i(m,n)$ and $\boldsymbol{v} \in J_j(m,n)$, $d_p(\boldsymbol{u},\boldsymbol{v}) \ge m^{1/p}$.

Proof. The first four statements are trivial.

In order to prove the last statement, let $\boldsymbol{u} = (u_1, u_2, \ldots, u_n) \in J_i(m,n)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_n) \in J_j(m,n)$, where $1 \le i < j \le r$. Without loss of generality, assume that $u_1 = u_2 = \cdots = u_{m_i} = (m/m_i)^{1/p}$ and $u_{m_i+1} = u_{m_i+2} = \cdots = u_n = 0$. In other words, $\boldsymbol{u} = (m/m_i)^{1/p} \cdot 1^{m_i}0^{n-m_i}$. For $1 \le k \le m_i$, we have $v_k \in \{0, \pm(m/m_j)^{1/p}\}$, and

$$\begin{aligned}
|u_k - v_k|^p &\ge \min\left\{|(m/m_i)^{1/p} - 0|^p, |(m/m_i)^{1/p} - (m/m_j)^{1/p}|^p, |(m/m_i)^{1/p} + (m/m_j)^{1/p}|^p\right\} \\
&= \min\left\{|(m/m_i)^{1/p} - 0|^p, |(m/m_i)^{1/p} - (m/m_j)^{1/p}|^p\right\} \\
&= \min\left\{\frac{m}{m_i}, \frac{m}{m_i} \cdot |1 - (m_i/m_j)^{1/p}|^p\right\} \\
&= \frac{m}{m_i} \min\left\{1, |1 - (m_i/m_j)^{1/p}|^p\right\}.
\end{aligned}$$

Since $j > i$, it follows that $m_j \le m_{i+1} = \lfloor \frac{m_i}{2^p} \rfloor \le \frac{m_i}{2^p}$. Thus $m_i/m_j \ge 2^p$, and

$$|u_k - v_k|^p \ge \frac{m}{m_i} \min\left\{1, |1 - (m_i/m_j)^{1/p}|^p\right\} \ge \frac{m}{m_i} \min\left\{1, |1 - (2^p)^{1/p}|^p\right\} = \frac{m}{m_i}.$$

Therefore,

$$d_p(\boldsymbol{u},\boldsymbol{v})^p = \sum_{k=1}^{n} |u_k - v_k|^p \ge \sum_{k=1}^{m_i} |u_k - v_k|^p \ge \sum_{k=1}^{m_i} \frac{m}{m_i} = m.$$

This completes the proof. □

For every i, let $J'_i(m,n)$ be a largest subset of $J_i(m,n)$ with the property that $d_p(\boldsymbol{u},\boldsymbol{v}) \ge m^{1/p}$ for every $\boldsymbol{u},\boldsymbol{v} \in J'_i(m,n)$. Since we have proved that $d_p(\boldsymbol{u},\boldsymbol{v}) \ge m^{1/p}$ if $\boldsymbol{u} \in J'_i(m,n) \subseteq J_i(m,n)$ and $\boldsymbol{v} \in J'_j(m,n) \subseteq J_j(m,n)$ for $i \neq j$, the set

$$\frac{1}{m^{1/p}} \bigcup_{i=1}^{r} J'_i(m,n) := \left\{\boldsymbol{x} \in \mathbb{R}^n : m^{1/p}\boldsymbol{x} \in \bigcup_{i=1}^{r} J'_i(m,n)\right\}$$

is an ℓ_p-spherical code with minimum distance 1. So

$$A_p(n, 1/2) \ge \left|\frac{1}{m^{1/p}} \bigcup_{i=1}^{r} J'_i(m,n)\right| = \left|\bigcup_{i=1}^{r} J'_i(m,n)\right| = \sum_{i=1}^{r} |J'_i(m,n)|. \tag{3}$$

For $1 \le i \le r$ and $\boldsymbol{u} \in J_i(m,n)$, define

$$B_{i,n}(\boldsymbol{u}, m) := \left\{\boldsymbol{v} \in J_i(m,n) : d_p(\boldsymbol{u},\boldsymbol{v}) < m^{1/p}\right\},$$

which is the open ℓ_p-ball centered at $\boldsymbol{u}$ with radius $m^{1/p}$ in the metric space $(J_i(m,n), \|\cdot\|_p)$. Note that the size of $B_{i,n}(\boldsymbol{u},m)$ is independent of $\boldsymbol{u}$. If we write $B_{i,n}(m)$ for the size of $B_{i,n}(\boldsymbol{u},m)$, then

$$B_{i,n}(m) = \sum_{2t+2^p x<m_i} \binom{m_i}{t}\binom{n-m_i}{t}\binom{m_i-t}{x}2^t. \tag{4}$$

Using the above notations, we have the following theorem, which is a Gilbert-Varshamov type bound for $|J_i'(m,n)|$.

Theorem 2.3. *For every $1 \le i \le r$, we have*

$$|J_i'(m,n)| \ge \left\lceil \frac{|J_i(m,n)|}{B_{i,n}(m)} \right\rceil = \left\lceil \frac{\binom{n}{m_i}2^{m_i}}{B_{i,n}(m)} \right\rceil. \tag{5}$$

The following corollary is immediate and it is our main result.

Corollary 2.4.

$$A_p(n,1/2) \ge \max_{1\le m\le n} \sum_{i=1}^{r} \left\lceil \frac{\binom{n}{m_i}2^{m_i}}{B_{i,n}(m)} \right\rceil. \tag{6}$$

Remark 2.5. *In [17, Lemma 2.1], the lower bound for $A_p(n,1/2)$ is given by* $\max_{1\le m\le n} \left\lceil \frac{\binom{n}{m_1}2^{m_1}}{B_{1,n}(m)} \right\rceil$. *So Corollary 2.4 gives an improvement.*

Proof of Theorem 2.3. Let i be given and $J = \left\lceil \frac{|J_i(m,n)|}{B_{i,n}(m)} \right\rceil$. We choose points from $J_i(m,n)$ recursively. At first, we arbitrarily choose $\boldsymbol{u}_1$ in $J_i(m,n)$. Suppose we have chosen $\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_k$ for some $k < J$. The set

$$J_i(m,n) \setminus \left(\bigcup_{j=1}^{k} B_{i,n}(\boldsymbol{u}_j, m) \right)$$

has size at least

$$|J_i(m,n)| - \sum_{j=1}^{k} |B_{i,n}(\boldsymbol{u}_j,m)| = |J_i(m,n)| - kB_{i,n}(m) > 0.$$

So we can choose $\boldsymbol{u}_{k+1}$ from $J_i(m,n) \setminus \left(\bigcup_{j=1}^{k} B_{i,n}(\boldsymbol{u}_j,m)\right)$ and $d_p(\boldsymbol{u}_{k+1}, \boldsymbol{u}_j) \ge m^{1/p}$ for every $1 \le j \le k$. This process continues as long as $k < J$. Therefore, $\{\boldsymbol{u}_1, \boldsymbol{u}_2, \dots, \boldsymbol{u}_J\}$ is a subset of $J_i(m,n)$, in which the points have pairwise distance at least $m^{1/p}$. And hence $|J_i'(m,n)| \ge J$. $\square$

3 Some numerical results for small p

It seems that there does not exist an explicit formula for the lower bound in Corollary 2.4. So we give some numerical results for small p in this section. In [17], Xu gives the lower bound for $\max_{1\le m\le n}\left\lceil \frac{\binom{n}{m_1}2^{m_1}}{B_{1,n}(m)}\right\rceil$. We still need to estimate the rest terms in right hand side of inequality (6).

Define

$$F_p(\sigma)=\frac{\binom{n}{\lfloor\sigma n\rfloor}2^{\lfloor\sigma n\rfloor}}{\sum_{2t+2^px<\lfloor\sigma n\rfloor}\binom{\lfloor\sigma n\rfloor}{t}\binom{n-\lfloor\sigma n\rfloor}{t}\binom{\lfloor\sigma n\rfloor-t}{x}2^t},\sigma\in(0,1).$$

Then by equations (1)-(4) and inequality (6), we have

$$A_p(n,1/2)\ge\max_{0<\sigma<1}\sum_{i=1}^{r}F_p\left(\frac{\sigma}{2^{(i-1)p}}\right).$$

3.1 The value of r

We first estimate the value of r. Suppose $m=\lceil 2^{kp}+2^{(k-1)p}+\cdots+2^p\rceil$ for some k. Then

$$m_1=m=\lceil 2^{kp}+2^{(k-1)p}+\cdots+2^p\rceil\in\left[2^{kp}+2^{(k-1)p}+\cdots+2^p,2^{kp}+2^{(k-1)p}+\cdots+2^p+1\right].$$

We calculate

$$\begin{aligned}m_2=\left\lfloor\frac{m_1}{2^p}\right\rfloor&\in\left[\lfloor 2^{(k-1)p}+2^{(k-2)p}+\cdots+1\rfloor,\lfloor 2^{(k-1)p}+2^{(k-2)p}+\cdots+1+2^{-p}\rfloor\right]\\&\subseteq\left[2^{(k-1)p}+2^{(k-2)p}+\cdots+2^p,2^{(k-1)p}+2^{(k-2)p}+\cdots+1+2^{-p}\right],\end{aligned}$$

and

$$\begin{aligned}m_3=\left\lfloor\frac{m_2}{2^p}\right\rfloor&\in\left[\lfloor 2^{(k-2)p}+2^{(k-3)p}+\cdots+1\rfloor,\lfloor 2^{(k-2)p}+2^{(k-3)p}+\cdots+2^{-p}+2^{-2p}\rfloor\right]\\&\subseteq\left[2^{(k-2)p}+2^{(k-3)p}+\cdots+2^p,2^{(k-2)p}+2^{(k-3)p}+\cdots+2^{-p}+2^{-2p}\right].\end{aligned}$$

So

$$m_k\in\left[2^p,2^p+1+2^{-p}+\cdots+2^{-(k-1)p}\right],$$

and

$$m_{k+1}\in\left[1,1+2^{-p}+2^{-2p}+\cdots+2^{-kp}\right]\subseteq[1,2).$$

Therefore $m_{k+1}=1$ and $r=k+1$ if $m=\lceil 2^{kp}+2^{(k-1)p}+\cdots+2^p\rceil$. Note that $\lceil 2^{kp}+2^{(k-1)p}+\cdots+2^p\rceil\in[2^{kp},2^{(k+1)p})$. On the other hand, if $m\in[2^{kp},\lceil 2^{kp}+2^{(k-1)p}+\cdots+2^p\rceil)$, then m_k may be less than 2^p. So we conclude that $r=\lfloor\log_{2^p}m\rfloor+1$ or $r=\lfloor\log_{2^p}m\rfloor$.

3.2 The behavior of $F_p(\sigma)$

In this subsection, we investigate the asymptotic behavior of $F_p(\sigma)$.

Let $H(\sigma)$ be the entropy function defined as

$$H(\sigma) = \begin{cases} 0, & \text{if } \sigma = 0 \text{ or } \sigma = 1; \\ -\sigma \log_2 \sigma - (1-\sigma)\log_2(1-\sigma), & \text{if } 0 < \sigma < 1. \end{cases}$$

We have the following theorem.

Theorem 3.1 ([17])**.** *We have*

$$\lim_{n\to\infty} \frac{1}{n} \log_2 F_p(\sigma) \geq \min_{0\leq y\leq \min\{\sigma/2,1-\sigma\}} f_p(\sigma, y),$$

where

$$f_p(\sigma, y) = (\sigma - y)\left(1 - H\left(\frac{\sigma - 2y}{2^p(\sigma - y)}\right)\right) + H(\sigma) - \sigma H\left(\frac{y}{\sigma}\right) - (1-\sigma)H\left(\frac{y}{1-\sigma}\right).$$

3.3 Numerical results for some special values of p

Let $g_p(\sigma) = \min_{0\leq y\leq \min\{\sigma/2,1-\sigma\}} f_p(\sigma, y)$. We list some numerical results for special values of p.

For $p = 1$, see left hand side of Figure 1 for the graph of $g_1(\sigma)$. $g_1(\sigma)$ attains its maximum 0.1825 at $\sigma_0 = 0.2605$. So

$$\begin{aligned} A_1(n, 1/2) &\geq \max_{0\leq\sigma\leq 1} \sum_{i=1}^{r} F_1\left(\frac{\sigma}{2^{i-1}}\right) \\ &\geq \sum_{i=1}^{r} F_1\left(\frac{2\sigma_0}{2^{i-1}}\right) \\ &\geq F_1(2\sigma_0) + F_1(\sigma_0) + F_1\left(\frac{\sigma_0}{2}\right) + \cdots \\ &\geq 2^{g_1(2\sigma_0)\cdot n(1+o(1))} + 2^{g_1(\sigma_0)\cdot n(1+o(1))} + 2^{g_1(\sigma_0/2)\cdot n(1+o(1))} + \cdots \\ &= 2^{0.1247n(1+o(1))} + 2^{0.1825n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots. \end{aligned}$$

Although $2^{0.1247n(1+o(1))} + 2^{0.1554n(1+o(1))} + \cdots = o(2^{0.1825n(1+o(1))})$, we still write them explicitly since they improve the previous bound.

Remark 3.2. *In [15], Talata obtained $A_1(n, 1/2) \geq 2^{0.1825n(1+o(1))}$ as well.*

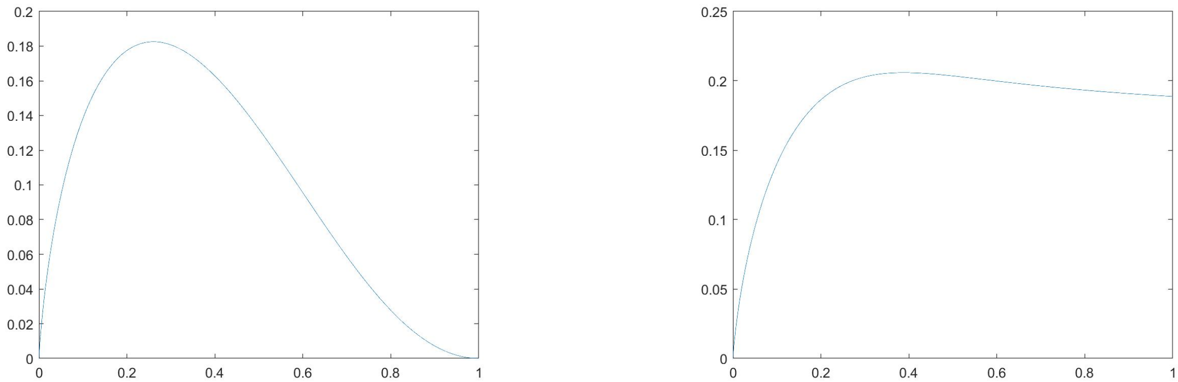

Figure 1: The graphs of $g_1(\sigma)$ and $g_2(\sigma)$

For $p = 2$, see right hand side of Figure 1 for the graph of $g_2(\sigma)$. $g_2(\sigma)$ attains its maximum 0.2059 at $\sigma_0 = 0.3881$. So

$$\begin{aligned}
A_2(n, 1/2) &\geq \max_{0\leq\sigma\leq 1} \sum_{i=1}^{r} F_2\left(\frac{\sigma}{2^{2(i-1)}}\right) \\
&\geq \sum_{i=1}^{r} F_2\left(\frac{\sigma_0}{4^{i-1}}\right) \\
&\geq F_2(\sigma_0) + F_2\left(\frac{\sigma_0}{4}\right) + F_2\left(\frac{\sigma_0}{4^2}\right) + \cdots \\
&\geq 2^{g_2(\sigma_0)\cdot n(1+o(1))} + 2^{g_2(\sigma_0/4)\cdot n(1+o(1))} + 2^{g_2(\sigma_0/16)\cdot n(1+o(1))} + \cdots \\
&= 2^{0.2059n(1+o(1))} + 2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots .
\end{aligned}$$

We also write the $2^{0.1381n(1+o(1))} + 2^{0.0584n(1+o(1))} + \cdots = o(2^{0.2059n(1+o(1))})$ terms explicitly.

For $p = 2.1$, see Figure 2 for the graph of $g_{2.1}(\sigma)$. $g_{2.1}(\sigma)$ attains its maximum 0.2163 at

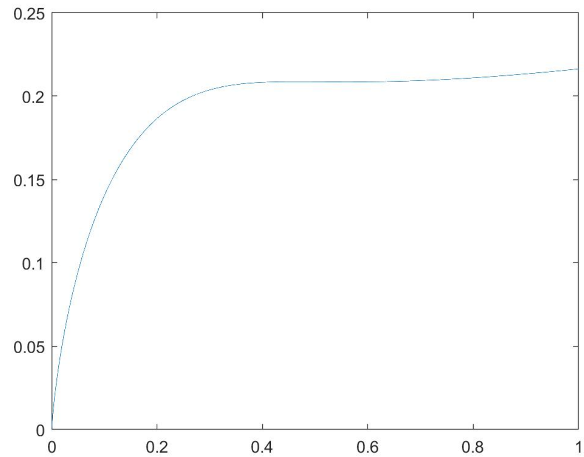

Figure 2: The graph of $g_{2.1}(\sigma)$

$\sigma_0 = 0.9998$. So

$$\begin{aligned}
A_{2.1}(n,1/2) &\geq \max_{0\leq\sigma\leq 1} \sum_{i=1}^{r} F_{2.1}\left(\frac{\sigma}{2^{2.1(i-1)}}\right) \\
&\geq \sum_{i=1}^{r} F_{2.1}\left(\frac{\sigma_0}{4.2871^{i-1}}\right) \\
&\geq F_{2.1}(\sigma_0) + F_{2.1}\left(\frac{\sigma_0}{4.2871}\right) + F_{2.1}\left(\frac{\sigma_0}{4.2871^2}\right) + \cdots \\
&\geq 2^{g_{2.1}(\sigma_0)\cdot n(1+o(1))} + 2^{g_{2.1}(\sigma_0/4.2871)\cdot n(1+o(1))} + 2^{g_{2.1}(\sigma_0/18.3792)\cdot n(1+o(1))} + \cdots \\
&= 2^{0.2163n(1+o(1))} + 2^{0.1944n(1+o(1))} + 2^{0.0995n(1+o(1))} + \cdots .
\end{aligned}$$

We also write the $2^{0.1944n(1+o(1))} + 2^{0.0995n(1+o(1))} + \cdots = o(2^{0.2163n(1+o(1))})$ terms explicitly.

4 Some numerical results for large p

There exists a threshold $p_0 \approx 2.1$ (we do not attempt to calculate the exact value of p_0) such that when $p > p_0$, $F_p(\sigma)$ attains its maximum at $\sigma = 1$. For $\sigma = 1$, i.e. $m = n$, we have another lower bound. Let $m = n$, and recall inequalities (3) and (5). We have

$$\begin{aligned} A_p(n, 1/2) &\geq \sum_{i=1}^{r} |J_i'(n,n)| \\ &= |J_1'(n,n)| + \sum_{i=2}^{r} |J_i'(n,n)| \\ &\geq |J_1'(n,n)| + \sum_{i=2}^{r} \left\lceil \frac{\binom{n}{m_i} 2^{m_i}}{B_{i,n}(n)} \right\rceil \\ &= |J_1'(n,n)| + \sum_{i=2}^{r} F_p\left(\frac{1}{2^{p(i-1)}}\right). \end{aligned}$$

Indeed, we can improve the lower bound for $|J_1'(n,n)|$ slightly.

4.1 An improvement of the lower bound for $|J_1'(n,n)|$

Recall the definition of $J_1(n,n)$ and $J_1'(n,n)$. $J_1(n,n) = \{\pm 1\}^n$ and $J_1'(n,n)$ is a largest subset of $\{\pm 1\}^n$ in which points have pairwise distance larger than or equal to $n^{1/p}$. For $\boldsymbol{u}, \boldsymbol{v} \in \{\pm 1\}^n$, let $d_H(\boldsymbol{u}, \boldsymbol{v}) := |\{i : u_i \neq v_i\}|$ be the Hamming distance between them. The following lemma is an easy observation.

Lemma 4.1. *For every $\boldsymbol{u}, \boldsymbol{v} \in \{\pm 1\}^n$, we have*

$$(d_p(\boldsymbol{u}, \boldsymbol{v}))^p = 2^p \cdot d_H(\boldsymbol{u}, \boldsymbol{v}).$$

By this lemma, it suffices to find a largest subset of $\{\pm 1\}^n$, in which points have pairwise Hamming distance larger than or equal to $\lceil n/2^p \rceil$. Recall the definition of $B_{1,n}(\boldsymbol{u}, n)$ and we have

$$\begin{aligned} B_{1,n}(\boldsymbol{u}, n) &= \left\{ \boldsymbol{v} \in \{\pm 1\}^n : d_p(\boldsymbol{u}, \boldsymbol{v}) < n^{1/p} \right\} \\ &= \{ \boldsymbol{v} \in \{\pm 1\}^n : 2^p \cdot d_H(\boldsymbol{u}, \boldsymbol{v}) < n \} \\ &= \{ \boldsymbol{v} \in \{\pm 1\}^n : d_H(\boldsymbol{u}, \boldsymbol{v}) \leq \lceil n/2^p \rceil - 1 \}. \end{aligned}$$

So $B_{1,n}(n) = |B_{1,n}(\boldsymbol{u}, n)| = \sum_{k=0}^{\lceil n/2^p \rceil - 1} \binom{n}{k}$. We have the following theorem, which gives a better lower bound for $|J_1'(n,n)|$ than that in inequality (5).

Theorem 4.2 ([4])**.** *There exists a positive constant c such that*

$$|J_1'(n,n)| \geq c \frac{2^n}{B_{1,n}(n)} \log_2 B_{1,n}(n).$$

Note that

$$\lim_{n\to\infty} \frac{1}{n} \log_2 B_{1,n}(n) = H\left(\frac{1}{2^p}\right),$$

by Stirling's formula. So

$$|J_1'(n,n)| \geq c\frac{n2^n}{B_{1,n}(n)} = cn2^{n(1-H(2^{-p})+o(1))},$$

for some constant c (maybe depends on p). Although $n = 2^{o(n)}$, we write it explicitly to represent the improvement.

4.2 Numerical results for some special values of p

As before, let $g_p(\sigma) = \min_{0\leq y\leq \min\{\sigma/2, 1-\sigma\}} f_p(\sigma, y)$. We list some numerical results for special values of p.

For $p = 2.2$, see left hand side of Figure 3 for the graph of $g_{2.2}(\sigma)$. We have

$$\begin{aligned} A_{2.2}(n, 1/2) &\geq |J_1'(n,n)| + \sum_{i=2}^{r} F_{2.2}\left(\frac{1}{2^{2.2(i-1)}}\right) \\ &\geq cn2^{n(1-H(2^{-2.2})+o(1))} + F_{2.2}(0.2176) + F_{2.2}(0.0474) + \cdots \\ &\geq cn2^{n(1-H(2^{-2.2})+o(1))} + 2^{g_{2.2}(0.2176)\cdot n(1+o(1))} + 2^{g_{2.2}(0.0474)\cdot n(1+o(1))} + \cdots \\ &= cn2^{0.2442n(1+o(1))} + 2^{0.1913n(1+o(1))} + 2^{0.0915n(1+o(1))} + \cdots . \end{aligned}$$

For $p = 3$, see right hand side of Figure 3 for the graph of $g_3(\sigma)$. We have

$$\begin{aligned} A_3(n, 1/2) &\geq |J_1'(n,n)| + \sum_{i=2}^{r} F_3\left(\frac{1}{2^{3(i-1)}}\right) \\ &\geq cn2^{n(1-H(2^{-3})+o(1))} + F_3(0.1250) + F_3(0.0156) + \cdots \\ &\geq cn2^{n(1-H(2^{-3})+o(1))} + 2^{g_3(0.1250)\cdot n(1+o(1))} + 2^{g_3(0.0156)\cdot n(1+o(1))} + \cdots \\ &= cn2^{0.4564n(1+o(1))} + 2^{0.1562n(1+o(1))} + 2^{0.0425n(1+o(1))} + \cdots . \end{aligned}$$

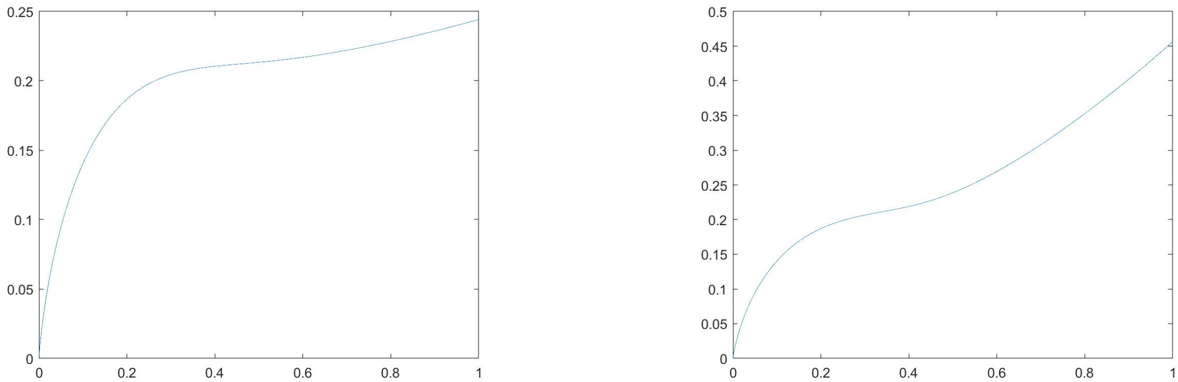

Figure 3: The graphs of $g_{2.2}(\sigma)$ and $g_3(\sigma)$

For $p = 4$, see Figure 4 for the graph of $g_4(\sigma)$. We have

$$\begin{aligned} A_4(n, 1/2) &\geq |J_1'(n,n)| + \sum_{i=2}^{r} F_4\left(\frac{1}{2^{4(i-1)}}\right) \\ &\geq cn2^{n(1-H(2^{-4})+o(1))} + F_4(0.0625) + F_4(0.0039) + \cdots \\ &\geq cn2^{n(1-H(2^{-4})+o(1))} + 2^{g_4(0.0625)\cdot n(1+o(1))} + 2^{g_4(0.0039)\cdot n(1+o(1))} + \cdots \\ &= cn2^{0.6627n(1+o(1))} + 2^{0.1083n(1+o(1))} + 2^{0.0145n(1+o(1))} + \cdots . \end{aligned}$$

5 Further remarks

In [11], Sah et al. obtained an inequality between ℓ_p-spherical codes for different p; that is, $A_p(n, d) \leq A_q(n, d^{p/q})$ for all $1 \leq q \leq p$ and $d \in (0, 1]$. So

$$A_2(n, d) \leq A_p(n, d^{2/p}), \text{ if } 1 \leq p \leq 2, \tag{7}$$

and

$$A_p(n, d) \leq A_2(n, d^{p/2}), \text{ if } p \geq 2. \tag{8}$$

Sah et al. used inequality (8) to obtain an upper bound for $A_p(n, d)$ $(p \geq 2)$.

On the other hand, Swanepoel [14] had used inequality (7) to obtain a lower bound for $A_p(n, 1/2)$ $(1.62107 < p \leq 2)$ before. Because the best lower bound for $A_2(n, d)$ has been improved since then, we update this type of lower bound here. We need the following theorem, which is the best known lower bound for $A_2(n, d)$ $(d \in (0, 1))$.

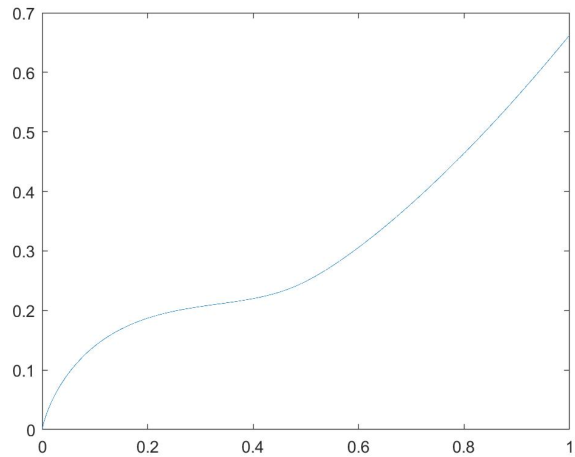

Figure 4: The graph of $g_4(\sigma)$

Theorem 5.1 ([1]). *Let $\theta \in (0, \pi/2)$ be fixed. Then*

$$A_2(n, \sin(\theta/2)) \geq (1 + o(1)) \ln \frac{\sin\theta}{\sqrt{2}\sin(\theta/2)} \cdot n \cdot \frac{\sqrt{2\pi n}\cos\theta}{\sin^{n-1}\theta}.$$

For $1 < p \leq 2$, we have

$$A_p(n, 1/2) \geq A_2(n, (1/2)^{p/2}).$$

Let $\sin(\theta/2) = 2^{-p/2}$. Then $\cos(\theta/2) = \sqrt{1 - 2^{-p}}, \sin\theta = 2^{1-p/2}\sqrt{1 - 2^{-p}}$, and $\cos\theta = 1 - 2^{1-p}$. So

$$\begin{aligned} A_p(n, 1/2) &\geq A_2(n, (1/2)^{p/2}) \\ &= A_2(n, \sin(\theta/2)) \\ &\geq (1 + o(1)) \ln \sqrt{2 - 2^{1-p}} \cdot n \cdot \frac{\sqrt{2\pi n}(1 - 2^{1-p})}{(2^{1-p/2}\sqrt{1 - 2^{-p}})^{n-1}}. \end{aligned} \tag{9}$$

After some numerical calculations, when $p \in (1.9948, 2]$, the lower bound in inequality (9) is better than that in inequality (6).

Acknowledgements

The authors would like to thank Professor Hong Liu for helpful comments on the manuscript.

References

[1] I. G. Fernández, J. Kim, H. Liu, and O. Pikhurko. New lower bounds on kissing numbers and spherical codes in high dimensions. *arXiv e-prints*, page arXiv:2111.01255, Nov. 2021.

[2] H. Hadwiger. Über Treffanzahlen bei translationsgleichen Eikörpern. *Arch. Math.*, 8:212–213, 1957.

[3] M. Jenssen, F. Joos, and W. Perkins. On kissing numbers and spherical codes in high dimensions. *Adv. Math.*, 335:307–321, 2018.

[4] T. Jiang and A. Vardy. Asymptotic improvement of the Gilbert-Varshamov bound on the size of binary codes. *IEEE Trans. Inform. Theory*, 50(8):1655–1664, 2004.

[5] G. A. Kabatjanskiĭ and V. I. Levenšteĭn. Bounds for packings on the sphere and in space. *Problemy Peredači Informacii*, 14(1):3–25, 1978.

[6] D. G. Larman and C. Zong. On the kissing numbers of some special convex bodies. *Discrete Comput. Geom.*, 21(2):233–242, 1999.

[7] J. Leech. The problem of the thirteen spheres. *Math. Gaz.*, 40:22–23, 1956.

[8] V. I. Levenšteĭn. On bounds for packings in n-dimensional euclidean space. *Soviet Math. Dokl.*, 20:417–421, 1979.

[9] O. R. Musin. The kissing number in four dimensions. *Ann. of Math. (2)*, 168(1):1–32, 2008.

[10] A. M. Odlyzko and N. J. A. Sloane. New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. *J. Combin. Theory Ser. A*, 26(2):210–214, 1979.

[11] A. Sah, M. Sawhney, D. Stoner, and Y. Zhao. Exponential improvements for superball packing upper bounds. *Adv. Math.*, 365:107056, 9, 2020.

[12] K. Schütte and B. L. van der Waerden. Das problem der dreizehn kugeln. *Math. Ann.*, 125(1):325–334, 1952.

[13] C. E. Shannon. Probability of error for optimal codes in a Gaussian channel. *Bell System Tech. J.*, 38:611–656, 1959.

[14] K. J. Swanepoel. New lower bounds for the Hadwiger numbers of l_p balls for $p < 2$. *Appl. Math. Lett.*, 12(5):57–60, 1999.

[15] I. Talata. A lower bound for the translative kissing numbers of simplices. *Combinatorica*, 20(2):281–293, 2000.

[16] A. D. Wyner. Capabilities of bounded discrepancy decoding. *Bell System Tech. J.*, 44:1061–1122, 1965.

[17] L. Xu. A note on the kissing numbers of superballs. *Discrete Comput. Geom.*, 37(3):485–491, 2007.