ON BI-LIPSCHITZ ISOMORPHISMS OF SELF-SIMILAR JORDAN ARCS

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Abstract. We find the conditions for bi-Lipschitz equivalence of self-similar Jordan arcs which are the attractors of self-similar zippers.

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A system $\mathcal{S} = \{S_i, i \in J\}$ (where $J = \{1, ..., m\}$) of contracting similarities of $\mathbb{R}^n$, is called a *self-similar zipper* with vertices $\{z_0, \ldots, z_m\}$ and signature $\boldsymbol{\varepsilon} \in \{0,1\}^m$, if for any $i \in J$, $S_i(z_0) = z_{i-1+\varepsilon_i}$ and $S_i(z_m) = z_{i-\varepsilon_i}$ [1]. A compact set $K \subset \mathbb{R}^n$ is called the *attractor of the zipper* $\mathcal{S}$, if $K = S_1(K) \cup .... \cup S_m(K)$.

If $0 = t_0 < t_1 < ... < t_m = 1$ is a set of points of the segment $I = [0,1] \subset \mathbb{R}$, then a self-similar zipper $\mathcal{T} = \{T_1, ..., T_m\}$ with vertices $\{t_0, ..., t_m\}$ and signature $\boldsymbol{\varepsilon} \in \{0,1\}^m$ is called *linear*. The attractor of a linear zipper is the segment I.

For any zipper $\mathcal{S} = \{S_1, ..., S_m\}$ with vertices $\{z_0, \ldots, z_m\}$ and for any linear zipper with the same signature $\boldsymbol{\varepsilon}$ there is a unique continuous map $g : I \to K(\mathcal{S})$, for which $g(t_i) = z_i$ and $S_i \circ g = g \circ T_i$ for every $i \in J$. Moreover, the mapping g is Hölder continuous and $g(I) = K(\mathcal{S})$. Such mappings g are called *structural parametrizations* of the attractor of the zipper $\mathcal{S}$. A zipper $\mathcal{S}$ is called a *Jordan zipper* if and only if one (and therefore any) of the structural parametrizations of its attractor is a homeomorphism of the segment $I = [0,1]$ to $K(\mathcal{S})$[1]. Indeed, as it was shown by Z.-Y. Wen and L.-F. Xi [3], it is possible that the homeomorphism g^{-1}, inverse to the structural parametrization, does not satisfy the Hölder condition for any value of the Hölder exponent.

Note an important property of the subarcs of the attractor γ of a Jordan zipper $\mathcal{S}$. Let $J^* = \{\boldsymbol{j} = j_1...j_n, j_k \in J, n \in \mathbb{N}\}$ be the set of all multi-indices over J. Let $S_{\boldsymbol{j}} = S_{j_1} \circ .... \circ S_{j_n}$, and $\gamma_{\boldsymbol{j}} = S_{\boldsymbol{j}}(\gamma)$. The systems $\{\gamma_{\boldsymbol{j}}, \boldsymbol{j} \in J^n, n \in \mathbb{N}\}$ form a refining sequence of partitions of the arc γ, in which $\gamma_{\boldsymbol{i}} \supseteq \gamma_{\boldsymbol{j}}$ if and only if $\boldsymbol{i} \sqsubseteq \boldsymbol{j}$. If $\boldsymbol{i} \not\sqsubseteq \boldsymbol{j}$ and $\boldsymbol{i} \not\sqsupseteq \boldsymbol{j}$, then $\gamma_{\boldsymbol{i}} \cap \gamma_{\boldsymbol{j}}$ is either empty or is the common endpoint of subarcs $\gamma_{\boldsymbol{i}}$ and $\gamma_{\boldsymbol{j}}$.

We say that a Jordan arc $\gamma \subset \mathbb{R}^n$ is an *arc of bounded turning* [2, p.100], if there exists $M > 0$ such that for any $x, y \in \gamma$ the diameter $|\gamma_{xy}|$ of the subarc $\gamma_{xy} \subset \gamma$

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with endpoints x, y is not greater than $M\|x-y\|$.

Let $\mathcal{S}=\{S_i, i\in J\}$ and $\mathcal{S}'=\{S'_i, i\in J\}$ be self-similar Jordan zippers, and γ and γ' be their attractors. A homeomorphism $f:\gamma\to\gamma'$ *agrees* with the systems $\mathcal{S}$ and $\mathcal{S}'$, if for any $i\in J$, $f\circ S_i = S'_i\circ f$. In this case, we say f defines an isomorphism of the zippers $\mathcal{S}$ and $\mathcal{S}'$. Note that for any $\boldsymbol{j}\in J^*$, $f(\gamma_{\boldsymbol{j}})=\gamma'_{\boldsymbol{j}}$.
The uniqueness and inversibility of the structural parametrization of Jordan zippers imply the following proposition.

PROPOSITION 1. *Let self-similar Jordan zippers $\mathcal{S}$ and $\mathcal{S}'$ with attractors γ and γ', have the same signature. Then there is a unique homeomorphism $f:\gamma\to\gamma'$, which agrees with $\mathcal{S}$ and $\mathcal{S}'$.* ■

We say that the zippers $\mathcal{S}$ and $\mathcal{S}'$ *are bi-Hölder (resp. bi-Lipschitz) isomorphic* if the homeomorphism f is bi-Hölder (resp. bi-Lipschitz).
We prove the following statement:

THEOREM 1. *Let self-similar zippers $\mathcal{S}=\{S_i, i\in J\}$, $\mathcal{S}'=\{S'_i, i\in J\}$ have attractors γ and γ' which are Jordan arcs of bounded turning. Let be p_i (resp. q_i) be the similarity ratios of the mappings S_i(resp. S'_i).Then $\mathcal{S}$ and $\mathcal{S}'$ are bi-Hölder isomorphic if and only if their signatures are equal. Moreover, the Hölder exponents of homeomorphisms f and f^{-1} are greater or equal to $\alpha=\min\{\log p_i/\log q_i, \log q_i/\log p_i, i\in J\}$.*

Proof. Let $p_{min}=\min\{p_1,...,p_m\}$. Let $f:\gamma\to\gamma'$ be a homeomorphism which agrees with $\mathcal{S}$ and $\mathcal{S}'$. Let $x,y\in\gamma$, and $x'=f(x), y'=f(y)$. There is $M>0$ such that the arcs $\gamma_{xy}\subset\gamma$ and $f(\gamma_{xy})=\gamma'_{x'y'}$ obey the inequalities

$$|\gamma_{xy}|\le M\|x-y\| \text{ and } |\gamma'_{x'y'}|\le M\|x'-y'\|. \tag{1}$$

Due to the properties of the families $\{\gamma_{\boldsymbol{j}}, \boldsymbol{j}\in J^*\}$ and $\{\gamma'_{\boldsymbol{j}}, \boldsymbol{j}\in J^*\}$ there are only two possibilities for the arcs $\gamma_{xy}, \gamma'_{xy}$:
1. There are $\boldsymbol{i}=i_1...i_k\in J^*$ and $i_{k+1}\in I$ such that $\gamma_{\boldsymbol{i}i_{k+1}}\subset\gamma_{xy}\subset\gamma_{\boldsymbol{i}}$ and $\gamma'_{\boldsymbol{i}i_{k+1}}\subset\gamma'_{x'y'}\subset\gamma'_{\boldsymbol{i}}$. These inclusions imply the inequalities $p_{\boldsymbol{i}i_{k+1}}|\gamma|\le|\gamma_{xy}|\le p_{\boldsymbol{i}}|\gamma|$ and $q_{\boldsymbol{i}i_{k+1}}|\gamma'|\le|\gamma'_{x'y'}|\le q_{\boldsymbol{j}}|\gamma'|$. From the conditions (1) it follows that $p_{\boldsymbol{i}}\le\dfrac{|\gamma_{xy}|}{|\gamma|p_{min}}\le\dfrac{M\|x-y\|}{|\gamma|p_{\min}}$. Since for any multi-index $\boldsymbol{k}$, $q_{\boldsymbol{k}}\le p_{\boldsymbol{k}}^\alpha$, we get

$$\|x'-y'\|\le|\gamma'_{x'y'}|\le p_{\boldsymbol{i}}^\alpha|\gamma'|\le\frac{M^\alpha|\gamma'|}{p_{\min}^\alpha|\gamma|^\alpha}\|x-y\|^\alpha.$$

2. There are multi-indices $\boldsymbol{i}=i_1...i_k$, $\boldsymbol{j}=j_1...j_l$ and indices i_{k+1}, j_{l+1} such that $\gamma_{\boldsymbol{i}i_{k+1}}\cup\gamma_{\boldsymbol{j}j_{l+1}}\subset\gamma_{xy}\subset\gamma_{\boldsymbol{i}}\cup\gamma_{\boldsymbol{j}}$, and $\gamma_{\boldsymbol{i}i_{k+1}}\cap\gamma_{\boldsymbol{j}j_{l+1}}=\gamma_{\boldsymbol{i}}\cap\gamma_{\boldsymbol{j}}$ is a singleton; similar relations hold for $\gamma'_{x'y'}$.

From the inequalities $\max(p_{\boldsymbol{i}i_{k+1}}, p_{\boldsymbol{j}j_{l+1}})|\gamma|\le|\gamma_{xy}|\le(p_{\boldsymbol{i}}+p_{\boldsymbol{j}})|\gamma|$ and $|\gamma'_{x'y'}|\le(q_{\boldsymbol{i}}+q_{\boldsymbol{j}})|\gamma'|$ we deduce that $\|x'-y'\|\le|\gamma'_{x'y'}|\le 2\max(p_{\boldsymbol{i}}, p_{\boldsymbol{j}})^\alpha|\gamma'|$. As $\max(p_{\boldsymbol{i}}, p_{\boldsymbol{j}})\le\dfrac{M\|x-y\|}{|\gamma|p_{\min}}$, we get

$$\|x'-y'\|\le 2\frac{M^\alpha|\gamma'|}{p_{\min}^\alpha|\gamma|^\alpha}\|x-y\|^\alpha. \tag{2}$$

If an index $i \in J$ is such that $\alpha = \log q_i / \log p_i$, then for pairs of points $x = S_i^k(z_0), y = S_i^k(z_m)$ we have the equality $\|x' - y'\| = M_0 \|x - y\|^\alpha$, where $M_0 = \|z'_m - z'_0\| / (\|z_m - z_0\|)^\alpha$. The same argument is valid for the mapping $f^{-1} : \gamma' \to \gamma$. Thus, the mappings f and f^{-1} satisfy the Hölder condition with exponent α and this value α is minimal. ■

Considering the case when $\alpha = 1$, we obtain the condition for bi-Lipschitz equivalence of self-similar zippers.

THEOREM 2. *Let* $\mathcal{S} = \{S_1, ..., S_m\}$ *and* $\mathcal{S}' = \{S'_1, ..., S'_m\}$ *be self-similar Jordan zippers in* $\mathbb{R}^n$ *with attractors* γ *and* γ'*, signatures* $\boldsymbol{\varepsilon}$ *and* $\boldsymbol{\varepsilon}'$*, and similarity coefficients* $\operatorname{Lip} S_i, \operatorname{Lip} S'_i, i \in J$ *respectively.*
1. If $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'$ *and for any* $i \in J$, $\operatorname{Lip} S_i = \operatorname{Lip} S'_i$*, and* γ, γ' *are the arcs of bounded turning, then* $\mathcal{S}$ *and* $\mathcal{S}'$ *are bi-Lipschitz isomorphic.*
2. If $\mathcal{S}$ *and* $\mathcal{S}'$ *are bi-Lipschitz isomorphic, and* γ *is an arc of bounded turning, then* $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'$*, and for any* $i \in J$, $\operatorname{Lip} S_i = \operatorname{Lip} S'_i$*, and* γ' *is of bounded turning.*

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