

# ON COMEAGER SETS OF METRICS WHOSE RANGES ARE DISCONNECTED

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ABSTRACT. For a metrizable space  $X$ , we denote by  $\text{Met}(X)$  the space of all metric that generate the same topology of  $X$ . The space  $\text{Met}(X)$  is equipped with the supremum distance. In this paper, for every strongly zero-dimensional metrizable space  $X$ , we prove that the set of all metrics whose ranges are closed totally disconnected subsets of the line is a dense  $G_\delta$  subspace in  $\text{Met}(X)$ . As its application, we show that some sets of universal metrics are meager in spaces of metrics.

## 1. INTRODUCTION

A metric  $d$  on  $X$  is said to be *ultrametric* if it satisfies  $d(x, y) \leq d(x, z) \vee d(z, y)$  for all  $x, y, z \in X$ , where  $\vee$  is the maximum operator on  $\mathbb{R}$ . A topological space is said to be *metrizable* (resp. *ultrametrizable*) if there exists a metric (resp. ultrametric) that generates the same topology of the space. Let  $X$  be a metrizable space. Let  $S$  be a subset of  $[0, \infty)$  with  $0 \in S$ . We denote by  $\text{Met}(X; S)$  (resp.  $\text{UMet}(X; S)$ ) the set of all metrics (resp. ultrametrics) that generate the same topology of  $X$  taking values in  $S$ . We often write  $\text{Met}(X) = \text{Met}(X; [0, \infty))$ . We define a map  $\mathcal{D}_X: \text{Met}(X)^2 \rightarrow [0, \infty]$  by  $\mathcal{D}_X(d, e) = \sup |d(x, y) - e(x, y)|$ . Then  $\mathcal{D}_X$  is a metric on  $\text{Met}(X)$  taking values in  $[0, \infty]$ . As in the case of ordinary metric spaces, using open balls, we can introduce the topology on  $(\text{Met}(X), \mathcal{D}_X)$ . We can also define an ultrametric  $\mathcal{UD}_X^S$  on  $\text{UMet}(X; S)$ . We omit its definition since we do not use it in this paper.

A topological space  $X$  is said to be *strongly 0-dimensional* if for every pair  $A, B$  of disjoint closed subsets of  $X$ , there exists a clopen set  $V$  such that  $A \subset V$  and  $V \cap B = \emptyset$ . Such a space is sometimes said to be *ultranormal*. Note that a topological space  $X$  is ultrametrizable if and only if it is metrizable and strongly 0-dimensional (see [7, Theorem II]).

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In [17], [19] and [18], the author investigated geometric properties in the space  $(\text{Met}(X), \mathcal{D}_X)$  of metrics and the space  $(\text{UMet}(X; S), \mathcal{UD}_X^S)$  of ultrametrics.

There are some researches concerning the range of a metric and the topology of an underlying set. Dovgoshey–Shcherbak [10] proved that an ultrametrizable space  $X$  is separable if and only if for every  $d \in \text{UMet}(X; \mathbb{R}_{\geq 0})$ , the set  $\{d(x, y) \mid x, y \in X\}$  is countable.

Broughan [4] proved that the following statements are equivalent to each other:

- (1) The space  $X$  is ultrametrizable;
- (2)  $\text{Met}(X; H) \neq \emptyset$ , where  $H = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_{\geq 1}\}$ ;
- (3) There exists a decreasing sequence  $\{a_i\}_{i \in \mathbb{Z}_{\geq 1}}$  converging to 0, such that  $\text{Met}(X; B) \neq \emptyset$ , where  $B = \{0\} \cup \{a_i \mid i \in \mathbb{Z}_{\geq 1}\}$ .

A metric  $d$  on  $X$  is said to be *gap-like* if for every  $p \in X$ , the set  $\{d(p, x) \mid x \in X\}$  is not dense in any neighborhood of 0 in  $[0, \infty)$ . Broughan [5, Theorem 7] proved that the Euclidean metric on  $\mathbb{R} \setminus \mathbb{Q}$  is a uniform limit of gap-like metrics on  $\mathbb{R} \setminus \mathbb{Q}$ . We improve Broughan's result on approximation of a metric by gap-like metrics.

A topological space is said to be *totally disconnected* if every its connected component is a singleton. We denote by  $\mathcal{Z}$  the set of all closed totally disconnected subsets of  $[0, \infty)$  containing 0. We define  $\text{DC}(X) = \bigcup_{S \in \mathcal{Z}} \text{Met}(X; S)$ . Namely,  $\text{DC}(X)$  is the set of all metrics whose ranges are closed and totally disconnected. The following is our first result:

**Theorem 1.1.** *Let  $X$  be a strongly 0-dimensional metrizable space. Then the set  $\text{DC}(X)$  is dense  $G_\delta$  in the space  $(\text{Met}(X), \mathcal{D}_X)$ .*

A subset of topological space is said to be *nowhere dense* if its closure has no interior points. A subset of a topological space is *meager* if it is a countable union of nowhere dense subsets of this space. A subset of a topological space is said to be *comeager* if its complement is meager. Note that if a subset of a topological space  $X$  contains a dense  $G_\delta$  set, then it is comeager. If  $X$  is a Baire space, then the converse is true.

Let  $X$  be a metrizable space. We denote by  $\text{GL}(X)$  the set of all gap-like metrics in  $\text{Met}(X)$ . As a consequence of Theorem 1.1, we have:

**Theorem 1.2.** *Let  $X$  be a strongly 0-dimensional metrizable space. Then the set  $\text{GL}(X)$  is comeager in the space  $(\text{Met}(X), \mathcal{D}_X)$ .*

Theorem 1.1 and 1.2 will be proven in Section 2. Our main results (Theorems 1.1 and 1.2) can be considered as generalizations of Broughan's theorem [5, Theorem 7].

*Remark 1.1.* The author does not know whether the set  $\text{DC}(X)$  has the anti-transmissible property defined in [17] or not.

In Section 3, as applications of Theorem 1.1, we show that some sets of universal metrics are meager in space of metrics.

## 2. PROOFS OF THEOREMS

For a metric space  $(X, d)$  and a subset  $A$ , we denote by  $\text{diam}_d(A)$  the diameter of  $A$  with respect to  $d$ . We begin with an amalgamation of metrics.

**Proposition 2.1.** *Let  $I$  be a set. Let  $(X, d)$  be a metric space. Let  $\{B_i\}_{i \in I}$  be a covering of  $X$  consisting of mutually disjoint clopen subsets of  $X$ . Let  $P = \{p_i\}_{i \in I}$  be points with  $p_i \in B_i$ . Let  $\{e_i\}_{i \in I}$  be a set of metrics such that  $e_i \in \text{Met}(B_i)$ . Let  $h$  be a metric on  $P$  generating the discrete topology on  $P$ . We define a function  $D : X^2 \rightarrow [0, \infty)$  by*

$$D(x, y) = \begin{cases} e_i(x, y) & \text{if } x, y \in B_i; \\ e_i(x, p_i) + h(p_i, p_j) + e_j(p_j, y) & \text{if } x \in B_i \text{ and } y \in B_j. \end{cases}$$

*Then  $D \in \text{Met}(X)$  and  $D|_{B_i^2} = e_i$  for all  $i \in I$ . Moreover, if for every  $i \in I$  we have  $\text{diam}_d(B_i) \leq \epsilon$  and  $\text{diam}_{e_i}(B_i) \leq \epsilon$ , then  $\mathcal{D}_X(D, d) \leq 4\epsilon + \mathcal{D}_P(d|_{P^2}, h)$ .*

*Proof.* The statement that  $D$  is a metric can be proven in a similar way to [19, Proposition 3.1]. We now prove that  $D$  generates the same topology of  $X$ . Take  $x \in X$ . Then there exists a unique element  $i \in I$  with  $x \in B_i$ . Since  $h$  generates the discrete topology on  $P$ , there exists  $\delta_i \in (0, \infty)$  such that  $\delta_i \leq h(p_i, p_j)$  for all  $j \in I$  with  $i \neq j$ .

Take  $r \in (0, \infty)$  with  $r < \delta_i$ . By the definition of  $D$ , the open ball centered at  $x$  with radius  $r$  with respect to  $D$  is contained in  $B_i$ . Since  $X$  is homeomorphic to the direct sum of  $\{B_i\}_{i \in I}$ , we conclude that  $D \in \text{Met}(X)$ .

To prove the latter part, we take  $x, y \in X$ . If  $x, y \in B_i$  for some  $i \in I$ , then, by  $\text{diam}_{e_i}(B_i) \leq \epsilon$  and  $\text{diam}_d(B_i) \leq \epsilon$ , we have

$$|D(x, y) - d(x, y)| = |e_i(x, y) - d(x, y)| \leq 2\epsilon < 4\epsilon + \mathcal{D}_P(d|_{P^2}, h).$$

If  $x \in B_i$  and  $y \in B_j$  for some distinct  $i, j \in I$ , then we have

$$\begin{aligned} |D(x, y) - d(x, y)| &\leq e(x, p_i) + e(y, p_j) + |h(p_i, p_j) - d(x, y)| \\ &\leq 2\epsilon + |h(p_i, p_j) - d(x, y)|. \end{aligned}$$

We also have

$$\begin{aligned} |h(p_i, p_j) - d(x, y)| &\leq |d(x, y) - d(p_i, p_j)| + |h(p_i, p_j) - d(p_i, p_j)| \\ &\leq d(x, p_i) + d(x, p_j) + \mathcal{D}_P(d|_{P^2}, h) \leq 2\epsilon + \mathcal{D}_P(d|_{P^2}, h). \end{aligned}$$

This implies that  $|D(x, y) - d(x, y)| \leq 4\epsilon + \mathcal{D}_P(d|_{P^2}, h)$ . Therefore we conclude that  $\mathcal{D}_X(D, d) \leq 4\epsilon + \mathcal{D}_P(d|_{P^2}, h)$ .  $\square$

The following is deduced from [11, Proposition 1.2 and Corollary 1.4].

**Lemma 2.2.** *Let  $X$  be a strongly 0-dimensional metrizable space. Then every open cover  $\{U_i\}_{i \in I}$  of  $X$  has a refinement  $\{E_j\}_{j \in J}$  such that each*

$E_j$  is open in  $X$ , and  $\bigcup_{j \in J} E_j = X$ , and  $E_j \cap E_{j'} = \emptyset$  if  $j \neq j'$ . In this case, each  $E_j$  is clopen.

For each  $x \in \mathbb{R}$ , we denote by  $\lceil x \rceil$  the minimum integer of all  $k \in \mathbb{Z}$  with  $x \leq k$ . The following two lemmas are related to *metric-preserving functions* (see [8]).

**Lemma 2.3.** *For all  $x, y \in \mathbb{R}$ , we have  $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$ .*

*Proof.* By  $x \leq \lceil x \rceil$  and  $y \leq \lceil y \rceil$ , we have  $x + y \leq \lceil x \rceil + \lceil y \rceil$ . By  $\lceil x \rceil + \lceil y \rceil \in \mathbb{Z}$  and by the definition of  $\lceil x + y \rceil$ , we conclude that  $\lceil x + y \rceil \leq \lceil x \rceil + \lceil y \rceil$ .  $\square$

The following is a reformulation of [9, Theorem 4.1] or [8, Theorem 1 and Proposition 1].

**Theorem 2.4.** *Let  $f: [0, \infty) \rightarrow [0, \infty)$  be an increasing function such that  $f^{-1}(0) = 0$ . Then the following statements are equivalent to each other:*

- (1) *For every set  $X$  and for every metric  $d$  on  $X$ , the function  $f \circ d: X \times X \rightarrow [0, \infty)$  is a metric on  $X$ .*
- (2) *The function  $f$  is subadditive, i.e., the inequality  $f(x + y) \leq f(x) + f(y)$  holds for all  $x, y \in [0, \infty)$ .*

*Remark 2.1.* [9, Theorem 4.1] is proven using only the property that  $\mathbb{R}$  is a linearly ordered Abelian group. Thus, Theorem 2.4 is still true for metrics taking values in linearly ordered Abelian groups. For more discussion of generalized metrics, we refer the readers to [20].

For  $\eta \in (0, \infty)$ , we define  $\eta \cdot \mathbb{Z} = \{\eta \cdot n \mid n \in \mathbb{Z}\}$  and  $\eta \cdot \mathbb{Z}_{\geq 0} = (\eta \cdot \mathbb{Z}) \cap \mathbb{R}_{\geq 0}$ .

**Proposition 2.5.** *Let  $X$  be a discrete topological space,  $\eta \in (0, \infty)$  and  $d \in \text{Met}(X)$ . Then, there exists a metric  $e \in \text{Met}(X)$  such that  $\mathcal{D}_X(d, e) \leq \eta$  and  $e(x, y) \in \eta \cdot \mathbb{Z}_{\geq 0}$  and  $\eta \leq e(x, y)$  for all distinct  $x, y \in X$ .*

*Proof.* We define  $e: X \times X \rightarrow \mathbb{R}$  by  $e(x, y) = \eta \cdot \lceil \eta^{-1} \cdot d(x, y) \rceil$  if  $x \neq y$ ; otherwise  $e(x, x) = 0$ . According to Theorem 2.4, the map  $e$  is a metric on  $X$ . Since  $d$  generates the discrete topology on  $X$ , by the definition of  $\lceil x \rceil$ , we conclude that  $\eta \leq e(x, y)$  for all distinct  $x, y \in X$ . Thus,  $e \in \text{Met}(X)$  is a required metric.  $\square$

Let  $A$  and  $B$  be subsets of  $\mathbb{R}$ . We define the sum of  $A$  and  $B$  by  $A + B = \{a + b \mid a \in A, b \in B\}$ .

**Definition 2.1.** Let  $\eta \in (0, \infty)$  and  $u \in (0, 1)$ . Let us define the sets  $\mathbb{O}(\eta, u)$  and  $\mathbb{E}(\eta, u)$  as

$$\mathbb{O}(\eta, u) = \{0\} \cup \{\eta \cdot u^n \mid n \in \mathbb{Z}_{\geq 0}\}$$

and

$$\mathbb{E}(\eta, u) = \eta \cdot \mathbb{Z}_{\geq 0} + \mathbb{O}(\eta, u) + \mathbb{O}(\eta, u).$$

Note that the set  $\mathbb{E}(\eta, u)$  is the closure of the set  $\{\eta \cdot (l + u^n + u^m) \mid l, n, m \in \mathbb{Z}_{\geq 0}\}$ .

**Lemma 2.6.** *Let  $\eta \in \mathbb{R}_{\geq 0}$  and  $u \in (0, 1)$ . Then we have  $\mathbb{E}(\eta, u) \in \mathcal{Z}$ .*

*Proof.* Since  $\mathbb{E}(\eta, u)$  is countable, it is totally disconnected. Since  $\mathbb{E}(\eta, u)$  is the sum of the compact set  $\mathbb{O}(\eta, u) + \mathbb{O}(\eta, u)$  and the closed set  $\eta \cdot \mathbb{Z}_{\geq 0}$ , it is closed (see [3, Corollary 1, § 4.1, p.251] or [1, Theorem 1.4.30]). Thus  $\mathbb{E}(\eta, u) \in \mathcal{Z}$ .  $\square$

**Theorem 2.7.** *Let  $X$  be a strongly 0-dimensional metrizable space. Let  $d \in \text{Met}(X)$ . Let  $\epsilon \in (0, \infty)$ . Then there exist  $r \in (0, 1)$  and a metric  $D$  on  $X$  such that:*

- (1)  $D \in \text{Met}(X; \mathbb{E}(\epsilon/5, r))$ ;
- (2)  $\mathcal{D}_X(d, D) \leq \epsilon$ .

*In particular, the set  $\text{DC}(X)$  is dense in  $\text{Met}(X)$ .*

*Proof.* We put  $\eta = \epsilon/5$  and  $r = \min\{1/2, \epsilon/10\}$ . Applying Lemma 2.2 to a covering of  $X$  consisting of open balls with radius  $r$ , we can find a clopen covering  $\{E_i\}_{i \in I}$  of  $X$ . For each  $i \in I$ , we take a point  $p_i \in E_i$ . In this case, we have  $\text{diam}_d(E_i) \leq 2r \leq \eta$ . Put  $Y = \{p_i\}_{i \in I}$ . By Proposition 2.5, there exists  $h \in \text{Met}(Y; \eta \cdot \mathbb{Z}_{\geq 0})$  such that  $h$  generates the discrete topology on  $Y$  and  $\mathcal{D}_Y(d|_{Y^2}, h) \leq \eta$ . Put  $R = \{0\} \cup \{\eta \cdot r^n \mid n \in \mathbb{Z}_{\geq 0}\} (= \mathbb{O}(\eta, r))$ . According to [18, Proposition 2.14], we can take a metric  $e_i \in \text{UMet}(E_i; R)$ . Then we have  $\text{diam}_{e_i}(E_i) \leq \eta$ . We define

$$D(x, y) = \begin{cases} e_i(x, y) & \text{if } x, y \in E_i; \\ e_i(x, p_i) + h(p_i, p_j) + e_j(p_j, y) & \text{if } x \in E_i \text{ and } y \in E_j. \end{cases}$$

Applying Proposition 2.1 to  $\{p_i\}_{i \in I}$ ,  $\{E_i\}_{i \in I}$ ,  $\{e_i\}_{i \in I}$ ,  $h$ , and  $\eta$ , we have  $D \in \text{Met}(X)$  and we obtain  $\mathcal{D}_X(D, d) \leq 4\eta + \mathcal{D}_Y(d|_{Y^2}, h) \leq 5\eta = \epsilon$ . By the definition of  $D$ , we have  $D(x, y) \in \mathbb{E}(\eta, r)$  for all  $x, y \in X$ . This complete the proof of Theorem 2.7.  $\square$

**Definition 2.2.** Let  $q \in \mathbb{Z}_{\geq 0}$ . We say that a closed subset  $A$  of  $[0, \infty)$  is a  $q$ -*nebula* if there exists a family  $\{I_i\}_{i=0}^k$  of intervals satisfying the the following conditions:

- (1) we have  $0 \in I_0$  and  $A = \bigcup_{i=1}^k I_i$ ;
- (2) each  $I_i$  is a closed interval in  $[0, \infty)$ ;
- (3) for all  $i \in \{0, \dots, k-1\}$ , we have  $\text{diam}(I_i) < 2^{-q}$ , where ‘‘diam’’ stands for the diameter with respect to the Euclidean metric;
- (4) the set  $I_k$  is unbounded and  $I_k \subset (q, \infty)$ ;
- (5) if  $i \neq j$ , then we have  $I_i \cap I_j = \emptyset$ .

**Lemma 2.8.** *Let  $\{A_q \mid q \in \mathbb{Z}_{\geq 0}\}$  be a family of subsets of  $[0, \infty)$  such that  $A_q$  is a  $q$ -nebula for every  $q \in \mathbb{Z}_{\geq 0}$ . Then the set  $\bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$  belongs to  $\mathcal{Z}$ .*

*Proof.* Put  $S = \bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$ . Then  $S$  is closed. Since every  $q \in \mathbb{Z}_{\geq 0}$  satisfies  $0 \in A_q$ , we have  $0 \in S$ . For the sake of contradiction, we assume that  $S$  has a connected component containing at least two points. Then there is  $a, b \in S$  such that  $[a, b] \subset S$ . Take  $q \in \mathbb{Z}_{\geq 0}$  so that  $b < q$  and  $2^{-q} \leq |b - a|$ . Then  $[a, b] \subset A_q$ . This is a contradiction to the definition of the  $q$ -nebula. Thus,  $S$  is totally disconnected. Hence  $S \in \mathcal{Z}$ .  $\square$

**Lemma 2.9.** *Let  $S$  be a subset of  $\mathbb{R}$ . Let  $a, b \in \mathbb{R}$ . If  $S$  is totally disconnected and  $a < b$ , then  $[a, b] \setminus S \neq \emptyset$ .*

*Proof.* If  $[a, b] \subset S$ , then  $S$  has a connected component containing at least two points.  $\square$

**Lemma 2.10.** *Let  $S \in \mathcal{Z}$ . Then for every  $q \in \mathbb{Z}_{\geq 0}$ , there exists a  $q$ -nebula  $A$  such that  $S \subset A$  and each compact connected component of  $A$  intersects  $S$ .*

*Proof.* Let  $M = (q + 1)2^{q+1}$ . For each  $m \in \{0, \dots, M\}$ , we put  $C_m = m \cdot 2^{-(q+1)}$ . Put  $\eta = 2^{-(q+3)}$ . Put  $A_m = C_m - \eta$  and  $B_m = C_m + \eta$ . Put  $t_0 = 0$ . According to Lemma 2.9, for each  $m \in \{1, \dots, M\}$ , we can take  $t_m \in [A_m, B_m] \setminus S$ . Then, by the definitions of  $A_m, B_m$ , and  $\eta$ , we obtain  $|t_m - t_{m+1}| < 2^{-q}$  for all  $m$  and  $q < t_M$ . Take  $k \in \mathbb{Z}_{\geq 0}$  and a map  $\phi: \{0, \dots, k\} \rightarrow \{0, \dots, M\}$  such that  $\phi(0) = 0$  and  $\phi(k) = M$  and  $[t_{\phi(i)}, t_{\phi(i+1)}] \cap S \neq \emptyset$  for all  $i \in \{0, \dots, k-1\}$  and  $S \subset [t_{\phi(k)}, \infty) \cup \bigcup_{i=0}^{k-1} [t_{\phi(i)}, t_{\phi(i+1)}]$ . Since  $S \cap [0, t_{\phi(k)}]$  is compact, for each  $i \in \{0, \dots, k-1\}$ , we can define  $a_i$  and  $b_i$  by  $a_i = \min S \cap [t_{\phi(i)}, t_{\phi(i+1)}]$  and  $b_i = \max S \cap [t_{\phi(i)}, t_{\phi(i+1)}]$ . We define  $a_k = \min S \cap [t_{\phi(k)}, \infty)$  if that set is non-empty; otherwise  $a_k = t_{\phi(k)}$ . Then we have  $[a_0, b_0] \subset [0, t_1]$  and  $[a_i, b_i] \subset (t_{\phi(i)}, t_{\phi(i+1)})$  for all  $i \in \{1, \dots, k-1\}$ , and  $[a_k, \infty) \subset (q, \infty)$ . For each  $i \in \{0, \dots, k-1\}$ , we put  $I_i = [a_i, b_i]$  and  $I_k = [a_k, \infty)$ . Put  $A = \bigcup_{i=0}^k I_i$ . Then  $A$  is a  $q$ -nebula with  $S \subset A$ . For each  $i \in \{0, \dots, k-1\}$ , we have  $I_i \cap S \neq \emptyset$ . Thus the set  $A$  is a desired one.  $\square$

**Proposition 2.11.** *Let  $S \in \mathcal{Z}$ . Then there exists a family  $\{A_q\}_{q \in \mathbb{Z}_{\geq 0}}$  such that each  $A_q$  is a  $q$ -nebula and  $S = \bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$ .*

*Proof.* According to Lemma 2.10, for each  $q \in \mathbb{Z}_{\geq 0}$ , we can take a  $q$ -nebula  $A_q$  such that  $S \subset A_q$  and each compact connected component intersects  $S$ . Then we obtain  $S \subset \bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$ . We shall show the converse inclusion. Let  $x$  belong to  $\bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$  and let  $N \in \mathbb{Z}_{\geq 0}$  satisfy the inequality  $N > x$ . Then, for every  $i > N$ , the point  $x$  belongs to a compact connected component of  $A_i$ , say  $C_i$ . Note that  $\text{diam } C_i < 2^{-i}$ .

By the assumption on compact connected components of  $\{A_q\}_{q \in \mathbb{Z}_{\geq 0}}$ , we have  $S \cap C_i \neq \emptyset$  for all  $i > N$ . We can take  $y_i \in S \cap C_i$ . By  $\text{diam } C_i < 2^{-i}$ , we have  $\lim_{i \rightarrow \infty} y_i = x$ . Since  $S$  is closed, we obtain  $x \in S$ . Therefore we conclude that  $S = \bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$ .  $\square$

We omit the proof of the following lemma since it is elementary.

**Lemma 2.12.** *Let  $a, b \in \mathbb{R}$  and  $\epsilon \in (0, \infty)$ . If  $x \in [a, b]$  and  $|x - y| \leq \epsilon$ , then  $y \in [a - \epsilon, b + \epsilon]$ .*

**Definition 2.3.** Let  $X$  be a metrizable space. We denote by  $\text{Neb}_q(X)$  the set of all metrics  $d \in \text{Met}(X)$  such that there exists a  $q$ -nebula  $A$  with  $d \in \text{Met}(X; A)$ .

**Proposition 2.13.** *Let  $X$  be a metrizable space. Then for every  $q \in \mathbb{Z}_{\geq 0}$ , the set  $\text{Neb}_q(X)$  is open in  $\text{Met}(X)$ .*

*Proof.* Take  $d \in \text{Neb}_q(X)$ . Then there exists a  $q$ -nebula  $A$  with  $d \in \text{Met}(X; A)$ . Let  $\{I_i\}_{i=0}^k$  be a family of closed interval appearing in the definition of nebulae such that  $A = \bigcup_{i=0}^k I_i$ . For  $i \in \{1, \dots, k-1\}$ , we put  $I_i = [a_i, b_i]$ , and  $I_0 = [0, b_0]$  and  $I_k = [a_k, \infty)$ . We can take  $c \in (0, \infty)$  such that if  $i \neq j$ , then for all  $x \in I_i$  and  $y \in I_j$ , we have  $c < d(x, y)$ . We take a sufficient small  $\epsilon \in (0, \infty)$  so that  $|b_i - a_i| + 2\epsilon < 2^{-q}$  for all  $i \in \{0, \dots, k-1\}$ , and  $\epsilon < a_k - q$  and  $\epsilon < c/4$ . We define  $J_0 = [0, b_0 + \epsilon]$ . For each  $i \in \{1, \dots, k-1\}$ , we define  $J_i = [a_i - \epsilon, b_i + \epsilon]$ . We define  $J_k = [a_k - \epsilon, \infty)$ . Then the set  $B = \bigcup_{i=0}^k J_i$  is a  $q$ -nebula. Take  $e \in \text{Met}(X)$  with  $\mathcal{D}_X(d, e) < \epsilon$ . Then, by the definition of  $\epsilon$  and by Lemma 2.12, we have  $e \in \text{Met}(X; B)$ , and hence  $e \in \text{Neb}_q(X)$ . This means that  $\text{Neb}_q(X)$  is open in  $\text{Met}(X)$ .  $\square$

*Proof of Theorem 1.1.* Let  $X$  be a strongly 0-dimensional metrizable space. By Lemma 2.6 and Theorem 2.7, the set  $\text{DC}(X)$  is dense in  $\text{Met}(X)$ . Put  $L = \bigcap_{q \in \mathbb{Z}_{\geq 0}} \text{Neb}_q(X)$ . We shall prove that  $\text{DC}(X) = L$ . Take  $d \in \text{DC}(X)$ . Then there exists  $S \in \mathcal{Z}$  with  $d \in \text{Met}(X; S)$ . By Proposition 2.11, there exists a sequence  $\{A_q\}_{q \in \mathbb{Z}_{\geq 0}}$  such that each  $A_q$  is a  $q$ -nebula and  $\bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q = S$ . For each  $q \in \mathbb{Z}_{\geq 0}$ , we have  $d \in \text{Neb}_q(X)$ , we obtain  $d \in L$ . Thus  $\text{DC}(X) \subset L$ . To prove the converse inclusion, we take  $d \in L$ . Then there exists a sequence  $\{A_q\}$  such that each  $A_q$  is a  $q$ -nebula and  $d \in \text{Met}(X; A_q)$ . Put  $T = \bigcap_{q \in \mathbb{Z}_{\geq 0}} A_q$ . By Lemma 2.8, the set  $T \in \mathcal{Z}$ . We also have  $d \in \text{Met}(X; T)$ , and hence  $d \in \text{DC}(X)$ . Therefore we conclude that  $\text{DC}(X) = L$ . Since Proposition 2.13 states that each  $\text{Neb}_q(X)$  is open in  $\text{Met}(X)$ , the set  $\text{DC}(X)$  is  $G_\delta$  in  $\text{Met}(X)$ . This finishes the proof of Theorem 1.1.  $\square$

*Proof of Theorem 1.2.* Since each  $S \in \mathcal{Z}$  is not dense in any neighborhood of 0, we have  $\text{DC}(X) \subset \text{GL}(X)$ . This finishes the proof of Theorem 1.2.  $\square$

In [17, Lemma 5.1], it is proven that for a second countable locally compact metrizable space  $X$ , the space  $(\text{Met}(X), \mathcal{D}_X)$  is a Baire space. Using this fact, we obtain the following corollary.

**Corollary 2.14.** *Let  $X$  be a strongly 0-dimensional  $\sigma$ -compact locally compact metrizable space. Then the sets  $\text{DC}(X)$  and  $\text{GL}(X)$  have the second category in  $(\text{Met}(X), \mathcal{D}_X)$  in the sense of Baire.*

### 3. APPLICATIONS

In this section, as applications of Theorem 1.1, we will show that some sets of universal metrics are meager.

**3.1. Constructions of universal metrics.** Let  $\mathcal{M}$  be a class of metric spaces. A metric space  $(X, d)$  is *universal for  $\mathcal{M}$* , or  *$\mathcal{M}$ -universal* if for every  $(Y, e) \in \mathcal{M}$ , there exists an isometric embedding  $I: (Y, e) \rightarrow (X, d)$ . In this case we will say that the metric  $d: X \times X \rightarrow [0, \infty)$  also is  $\mathcal{M}$ -universal.

A metric space is called a *finite metric space* if its cardinality is finite. Let  $\mathcal{F}$  be the class of all finite metric spaces.

We shall construct some universal metrics. The following theorem is proven by Holsztynski [15] (see also [23]).

**Theorem 3.1.** *There exists an  $\mathcal{F}$ -universal metric in  $\text{Met}(\mathbb{R})$ .*

In this paper, we prove a generalization of Theorem 3.1.

Denote by  $\Gamma$  be the Cantor set. We say that a topological space is a *Cantor space* if it is homeomorphic to  $\Gamma$ . We say that a topological space is a *punctured Cantor space* if it is homeomorphic to  $\Gamma \setminus \{p\}$  for some  $p \in \Gamma$ . Note that a topological space is a punctured Cantor space if and only if it is homeomorphic to the countable disjoint union of Cantor spaces.

We prove that a punctured Cantor space has an  $\mathcal{F}$ -universal metric.

Let  $r \in (0, \infty)$ . A subset  $S$  of a metric space  $(X, d)$  is said to be  *$r$ -separated* if  $r \leq d(x, y)$  for all distinct  $x, y \in S$ .

The following lemmas (Lemmas 3.2 and 3.3) are proven in [23]. For the sake of self-containedness, we provide proofs.

**Lemma 3.2.** *Let  $r \in (0, \infty)$ . Let  $(X, d)$  and  $(Y, e)$  be metric spaces. Let  $f: X \rightarrow Y$  be a surjective continuous map. We define a function  $\rho: X \times X \rightarrow [0, \infty)$  by*

$$\rho(x, y) = \min\{d(x, y), r\} \vee e(f(x), f(y)),$$

*where the symbol  $\vee$  stands for the maximal operator on  $\mathbb{R}$ . Then  $\rho \in \text{Met}(X)$  and the metric space  $(X, \rho)$  is universal for all  $r$ -separated finite subspace of  $Y$ .*

*Proof.* Since  $\min\{d(x, y), r\} \in \text{Met}(X)$  and  $f$  is continuous, we have  $\rho \in \text{Met}(X)$ . We next prove the universality of  $\rho$ . Take an arbitrary



$r$ -separated subset  $E = \{p_i \mid i \in \{0, \dots, k\}\}$  of  $Y$ . For each  $i$ , we take  $t_i \in X$  with  $f(t_i) = p_i$ . We define  $I: (E, e|_{E^2}) \rightarrow (X, \rho)$  by  $I(p_i) = t_i$ . Since  $E$  is  $r$ -separated, we have  $r \leq e(f(t_i), f(t_j))$  for distinct  $i, j$ . Then we have  $\rho(I(p_i), I(p_j)) = \rho(t_i, t_j) = e(f(t_i), f(t_j)) = e(p_i, p_j)$  for all  $i, j$ . Thus  $I$  is an isometric embedding. Therefore we conclude that  $\rho$  is universal for all  $r$ -separated finite subspaces of  $Y$ .  $\square$

Let  $n \in \mathbb{Z}_{\geq 1}$ . We denote by  $\mathcal{C}_n$  the class of all  $(n^{-1})$ -separated finite metric spaces  $(X, d)$  satisfying that  $\text{diam}_d(X) \leq n$ , and  $\text{Card}(X) \leq n$ , where ‘‘Card’’ stands for the cardinality.

**Lemma 3.3.** *Let  $n \in \mathbb{Z}_{\geq 1}$ . Then the space  $[0, n]^n$  equipped with the  $\ell^\infty$ -Euclidean metric is  $\mathcal{C}_n$ -universal.*

*Proof.* Let  $(X, d)$  be a metric space in  $\mathcal{C}_n$ . We represent  $X = \{p_i \mid i \in \{1, \dots, k\}\}$ , where  $k \leq n$ . For each  $i \in \{1, \dots, k\}$ , we define  $F_i: \{1, \dots, n\} \rightarrow [0, n]$  by  $F_i(m) = d(p_i, p_m)$  if  $i \leq k$ ; otherwise,  $F_i = 0$ . We define  $\phi: X \rightarrow [0, n]^n$  by  $\phi(p_i) = (F_1(i), \dots, F_n(i))$ . As is the case of the Fréchet embedding, using the triangle inequality, we conclude that  $\phi$  is an isometric embedding.  $\square$

According to Lemmas 3.2 and 3.3, we obtain:

**Corollary 3.4.** *Let  $X$  be a metrizable space. Let  $n \in \mathbb{Z}_{\geq 1}$ . If there exists a continuous surjective map  $f: X \rightarrow [0, n]^n$ , then there exists a  $\mathcal{C}_n$ -universal metric  $e \in \text{Met}(X)$ .*

**Proposition 3.5.** *Let  $\Lambda$  be a punctured Cantor space. There exists an  $\mathcal{F}$ -universal metric in  $\text{Met}(\Lambda)$ .*

*Proof.* Take a sequence  $\{K_i\}_{i \in \mathbb{Z}_{\geq 0}}$  of subsets of  $\Lambda$  satisfying that  $\Lambda = \coprod_{i \in \mathbb{Z}_{\geq 0}} K_i$  and each  $K_i$  is a Cantor space. Since every compact metrizable space is a continuous image of a Cantor space (see, for example, [27, Theorem 30.7]), we can take a continuous surjective map  $f_i: K_i \rightarrow [0, n]^n$ . Then, by Corollary 3.4, we can take a  $\mathcal{C}_n$ -universal metric  $e_i \in \text{Met}(K_i)$ . For each  $i \in \mathbb{Z}_{\geq 0}$ , we take  $p_i \in K_i$ . We define a metric  $h$  on  $\{p_i\}_{i \in \mathbb{Z}_{\geq 0}}$  by  $h(p_i, p_j) = 1$  if  $i \neq j$ . Applying Proposition 2.1 to  $\{p_i\}_{i \in \mathbb{Z}_{\geq 0}}$ ,  $\{K_i\}_{i \in \mathbb{Z}_{\geq 0}}$ ,  $\{e_i\}_{i \in \mathbb{Z}_{\geq 0}}$ , and  $h$ , we can take  $D \in \text{Met}(\Lambda)$  such that  $D|_{K_i^2} = e_i$ . Since every metric space in  $\mathcal{F}$  belongs to  $\mathcal{C}_n$  for some  $n \in \mathbb{Z}_{\geq 1}$ , we conclude that the metric  $D$  is  $\mathcal{F}$ -universal.  $\square$

The next is Hausdorff’s metric extension theorem [13] (see also [25]).

**Theorem 3.6.** *For a metrizable space  $X$ , and for a closed subset  $A$  of  $X$ , and for every  $d \in \text{Met}(A)$ , there exists  $D \in \text{Met}(X)$  such that  $D|_{A^2} = d$ .*

Using Proposition 3.5 and Theorem 3.6 we obtain the following generalization of Theorem 3.1.

**Corollary 3.7.** *If a metrizable space  $X$  contains a punctured Cantor space as a closed subset, then there exists an  $\mathcal{F}$ -universal metric in  $\text{Met}(X)$ .*

*Remark 3.1.* If a metrizable topological space  $X$  is Polish and there is an unbounded metric  $d \in \text{Met}(X)$  such that the complement of every bounded subspace of  $(X, d)$  has uncountable cardinality, then  $X$  contains a punctured Cantor space as closed subset. This follows from the fact that every uncountable Polish space contains a Cantor space (see [21, Corollary 6.5]).

Let  $S$  be a subset of  $[0, \infty)$  with  $0 \in S$ . We denote by  $\mathcal{T}(S)$  the class of all two-point metric spaces whose metrics take values in  $S$ .

Some examples and properties of  $\mathcal{T}(S)$ -universal metric spaces can be found in [2] and [14].

**Proposition 3.8.** *Let  $S$  be a countable subset of  $[0, \infty)$  with  $0 \in S$ . Let  $X$  be a countable discrete space. Then there exists a  $\mathcal{T}(S)$ -universal metric in  $\text{Met}(X)$ .*

*Proof.* Put  $S = \{0\} \cup \{s_i\}_{i \in \mathbb{Z}_{\geq 0}}$ . Take subsets  $A, B$  of  $X$  such that  $A \cap B = \emptyset$  and  $X = A \sqcup B$  and  $A$  and  $B$  are countable. Put  $A = \{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$  and  $B = \{b_i\}_{i \in \mathbb{Z}_{\geq 0}}$ . For each  $i \in \mathbb{Z}_{\geq 0}$ , we define a metric  $e_i$  on  $\{a_i, b_i\}$  by  $e_i(a_i, b_i) = s_i$ . We define a metric  $h$  on  $A$  such that  $h(x, y) = 1$  if  $x \neq y$ . Applying Proposition 2.1 to  $\{a_i\}_{i \in \mathbb{Z}_{\geq 0}}$ ,  $\{\{a_i, b_i\}\}_{i \in \mathbb{Z}_{\geq 0}}$ ,  $\{e_i\}_{i \in \mathbb{Z}_{\geq 0}}$ , and  $h$ , we obtain a metric  $D \in \text{Met}(X)$  such that  $D(a_i, b_i) = e_i(a_i, b_i) = s_i$ . Then  $D$  is a desired one.  $\square$

Proposition 3.8 and Theorem 3.6 implies:

**Corollary 3.9.** *Let  $S$  be a countable subset of  $[0, \infty)$  with  $0 \in S$ . If a topological space  $X$  contains a countable discrete space as a closed subset, then there exists a  $\mathcal{T}(S)$ -universal metric in  $\text{Met}(X)$ .*

**3.2. The meagerness of sets of universal metrics.** Let  $X$  be a metrizable space. Let  $\text{T}(X; S)$  be the set of all  $\mathcal{T}(S)$ -universal metrics in  $\text{Met}(X)$ . Let  $\text{CT}(X; S)$  be the closure of  $\text{T}(X; S)$  in  $\text{Met}(X)$ .

**Lemma 3.10.** *Let  $S$  be a dense subset of  $[0, \infty)$  with  $0 \in S$ . Let  $X$  be a metrizable space. Then for every  $d \in \text{CT}(X; S)$ , the set  $\{d(x, y) \mid x, y \in X\}$  is dense in  $[0, \infty)$ .*

*Proof.* By the definition of  $\text{CT}(X; S)$ , for  $d \in \text{CT}(X; S)$ , we can take a sequence  $\{e_n\}_{n \in \mathbb{Z}_{\geq 0}}$  in  $\text{T}(X; S)$  satisfying that  $\mathcal{D}_X(d, e_n) \leq 2^{-n}$  for all  $n \in \mathbb{Z}_{\geq 0}$ . Take arbitrary  $q \in [0, \infty)$  and  $\epsilon \in (0, \infty)$ . Take  $r \in S$  with  $|q - r| \leq \epsilon/2$  and take  $k \in \mathbb{Z}_{\geq 0}$  with  $2^{-k} \leq \epsilon/2$ . Since  $e_k$  is  $\mathcal{T}(S)$ -universal, we can take  $x, y \in X$  such that  $e_k(x, y) = r$ . Then we have  $|d(x, y) - r| \leq 2^{-k}$ , and hence  $|d(x, y) - q| \leq \epsilon$ . Thus the set  $\{d(x, y) \mid x, y \in X\}$  is dense in  $[0, \infty)$ .  $\square$

The following is an application of Theorem 1.1.

**Theorem 3.11.** *Let  $S$  be a dense subset of  $[0, \infty)$  with  $0 \in S$ . Let  $X$  be a strongly 0-dimensional metrizable space. Then the set  $\text{CT}(X; S)$  is meager in  $\text{Met}(X)$ .*

*Proof.* Since every  $S \in \mathcal{Z}$  is not dense in  $[0, \infty)$ , Lemma 3.10 implies that  $\text{CT}(X; S) \cap \text{DC}(X) = \emptyset$ . Since  $\text{DC}(X)$  is comeager (see Theorem 1.1), we conclude that  $\text{CT}(X; S)$  is meager.  $\square$

Let  $\mathcal{M}$  be a class of finite metric spaces. We say that a metric space  $(X, d)$  is  $\mathcal{M}$ -*injective* if for every metric space  $(A, m) \in \mathcal{M}$  and for every  $B \subset A$ , every isometric embedding  $\phi: (B, m|_{B^2}) \rightarrow (X, d)$  can be extended to an isometric embedding  $\Phi: (A, m) \rightarrow (X, d)$ .

Let  $R$  be a subset of  $[0, \infty)$ . Let  $\mathcal{N}(R)$  be the class of all finite ultrametric spaces whose metrics take values in  $R$ . Let  $\mathcal{Q}$  be the class of all finite metric spaces whose metrics take values in  $\mathbb{Q}$ .

For each subset  $R$  of  $[0, \infty)$  with  $0 \in R$ , there exists a complete  $\mathcal{N}(R)$ -injective ultrametric space. If  $R$  is countable, then it is unique up to isometry, and it is called the  *$R$ -Urysohn universal ultrametric space* (see [12] and [26]). A countable  $\mathcal{Q}$ -injective metric space uniquely exists up to isometry, and it is called the *rational Urysohn universal metric space* (see, for example, [22]).

*Remark 3.2.* The completion of the rational Urysohn universal metric spaces is a complete separable  $\mathcal{F}$ -injective metric space, and it is called the *Urysohn universal metric space*. For more discussions on this space, we refer the readers to, for example, [22], [16] and [24].

In this paper, we use only the fact that  $\mathcal{N}(R)$ -injective ultrametric spaces and the rational Urysohn universal space are  $\mathcal{T}(R)$ -universal and  $\mathcal{T}(\mathbb{Q}_{\geq 0})$ -universal, respectively. By Theorem 3.11, we obtain the following two corollaries.

**Corollary 3.12.** *Let  $X$  be a strongly 0-dimensional metrizable space. Let  $R$  be a dense subset of  $[0, \infty)$  with  $0 \in R$ . Then the following subsets of  $\text{Met}(X)$  are meager in  $\text{Met}(X)$ :*

- (1) *The set of all  $\mathcal{F}$ -universal metrics in  $\text{Met}(X)$ .*
- (2) *The set of all metrics  $d \in \text{Met}(X)$  such that  $(X, d)$  is an  $\mathcal{N}(R)$ -injective ultrametric space.*

*Remark 3.3.* It can happen that the two sets appearing in Corollary 3.12 are empty.

Note that by the Sierpiński's characterization of the rational numbers (see, for example, [6]), the rational Urysohn universal space is homeomorphic to the space  $\mathbb{Q}$  of rational numbers.

**Corollary 3.13.** *The set of all metric  $d$  such that  $(\mathbb{Q}, d)$  is rational Urysohn universal space is meager in  $\text{Met}(\mathbb{Q})$ .*

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