

EXTENDING PROPER METRICS

YOSHITO ISHIKI

ABSTRACT. We first prove a version of Tietze-Urysohn’s theorem for proper functions taking values in non-negative real numbers defined on σ -compact locally compact Hausdorff spaces. As an application, we prove an extension theorem of proper metrics, which states that if X is a σ -compact locally compact Hausdorff space, A is a closed subset of X , and d is a proper metric on A that generates the same topology of A , then there exists a proper metric D on X such that D generates the same topology of X and $D|_A = d$. Moreover, if A is a proper retract, we can choose D so that (A, d) is quasi-isometric to (X, D) . We also show analogues of the theorems explained above for ultrametrizable spaces.

1. INTRODUCTION

Tietze–Urysohn’s theorem states that every continuous function on a closed subset of a normal space can be extended to the whole space as a continuous function. This theorem has played an important role in topology and analysis (for example, the existence of a partition of unity). There are many generalizations of Tietze–Urysohn’s theorem (see for instance [10], [23], [16] and [12]).

For a metrizable space X , we denote by $\text{Met}(X)$ the set of all metrics on X generating the same topology of X . Hausdorff’s extension theorem states that for every metrizable space X , for every closed subset A of X , and for every $d \in \text{Met}(A)$, there exists a metric $D \in \text{Met}(X)$ such that $D|_A = d$. This theorem can be considered as an analogue of Tietze–Urysohn’s theorem for metric spaces, and some variants have been investigated by some authors (see [2], [9], [8], and [15]).

Tietze–Urysohn’s theorem and Hausdorff’s extension theorem are not only analogous, but also logically connected with each other. In fact, according to [1], Hausdorff’s extension theorem can be proven using Dugundji’s theorem (see [10]), which is an improvement of Tietze–Urysohn’s theorem. For more discussion on connections between extensions of maps and metrics, we refer the readers to [14].

Date: December 27, 2022.

2020 Mathematics Subject Classification. Primary 54C10, Secondary 54E35, 54E45.

Key words and phrases. Tietze-Urysohn’s theorem, Proper maps, Proper metrics, Proper retracts, Extension of metrics, Ultrametrics.

In [15], the author proved an ultrametric analogue of Hausdorff's extension theorem using the method described above, namely, using the property that every continuous function on a closed subsets of an ultrametrizable space can be extended to the whole space. Remark that since every non-empty closed set in an ultrametric space is a retract of the whole space (see [3, Theorem 2.9]), all extension problems of continuous maps defined on a closed subsets of an ultrametric space are solved affirmatively.

Let X and Y be topological spaces. A map $f: X \rightarrow Y$ is said to be *proper* if for every compact subset K of Y , the inverse image $f^{-1}(K)$ is compact. A metric d on X is said to be *proper* if all bounded closed subsets of (X, d) are compact. In this case, for a fixed point $p \in X$, the function defined by $x \mapsto d(p, x)$ is a proper map. These two concepts are the main subjects of this paper.

In the present paper, we prove a new variant of Hausdorff's extension theorem for proper metrics. The key idea is, as mentioned above, that an extension theorem of continuous maps implies an extension theorem of metrics. We first show Tietze–Urysohn's theorem for proper functions (Theorem 2.3), and then, as an application, we prove Hausdorff's extension theorem for proper metrics (Theorem 3.9).

To show Tietze–Urysohn's theorem for proper functions, we use the so-called controlling Tietze–Urysohn's theorem (see [12] and [23]), which includes not only an extension of a given function, but also an extension of the zero set of the function.

Similarly, using the fact that every non-empty closed subset of an ultrametric space is a retract of the whole space, we prove an extension theorem of proper ultrametrics (Theorem 3.15).

In this paper, we also prove an extension theorem of proper metrics focusing on large scale structures of metric spaces (Theorem 4.8) using Michael's continuous selection theorems, which are also generalizations of Tietze–Urysohn's theorem. More precisely, we prove that for every σ -compact locally compact space X , and for every closed subset A of X , if A is a proper retract of X , then for every proper metric $d \in \text{Met}(X)$, there exists a proper metric $D \in \text{Met}(X)$ such that (A, D) is quasi-isometric to (X, D) . We also prove an ultrametric version of this extension theorem (Theorem 4.9).

2. EXTENSION OF PROPER FUNCTIONS

A main purpose of this section is to prove Tietze–Urysohn's theorem for proper functions (Theorem 2.3).

A topological space is said to be σ -compact if it is a countable union of compact subspaces. A topological space is said to be *locally compact* if every point in the space has a compact neighborhood.

Remark 2.1. As a consequence of Urysohn's metrization theorem (see [19, Theorem 34.1]), all σ -compact locally compact Hausdorff spaces are metrizable. Indeed, they are second countable and regular.

The following theorem is deduced from Yamazaki's theorem [23, Corollary 2.1] or Frantz's theorem [12, Theorem 1].

Theorem 2.1. *Let X be a normal space, A a closed subset of X , and Z a closed G_δ subset of X . Assume that $f: A \rightarrow [0, 1]$ is a continuous function such that $Z \cap A = f^{-1}(0)$. Then there exists a continuous function $F: X \rightarrow [0, 1]$ satisfying that $F|_A = f$ and $F^{-1}(0) = Z$.*

For a σ -compact locally compact Hausdorff space X , we put $\tilde{\alpha}X = X \sqcup \{\infty\}$. We define a topology on $\tilde{\alpha}X$ by declaring neighborhood systems of $\tilde{\alpha}X$ as follows: If $p \in X$, then the neighborhood system of p in $\tilde{\alpha}X$ is the family of all subsets V of X such that $V \setminus \{\infty\}$ is a neighborhood of p in X , and the neighborhood system of ∞ in $\tilde{\alpha}X$ is the set of all subsets V of $\tilde{\alpha}X$ satisfying that $\tilde{\alpha}X \setminus V$ is a relatively compact subset of X . In what follows, we always consider that $\tilde{\alpha}X$ is equipped with this topology. If X is non-compact, then $\tilde{\alpha}X$ coincides with the one-point compactification of X . If X is compact, then $\tilde{\alpha}X$ is nothing but the topological direct sum of X and the point ∞ . Remark that if $X = \emptyset$, then $\tilde{\alpha}X = \{\infty\}$.

Let X and Y be σ -compact locally compact Hausdorff spaces. For a map $f: X \rightarrow Y$, we define an induced map $\tilde{\alpha}f: \tilde{\alpha}X \rightarrow \tilde{\alpha}Y$ by $\tilde{\alpha}f|_X = f$ and $\tilde{\alpha}f(\infty) = \infty$.

Proposition 2.2. *Let X and Y be σ -compact locally compact Hausdorff spaces. Then the following statements hold:*

- (1) *For every proper map $f: X \rightarrow Y$, the map $\tilde{\alpha}f: \tilde{\alpha}X \rightarrow \tilde{\alpha}Y$ is continuous.*
- (2) *If a continuous map $F: \tilde{\alpha}X \rightarrow \tilde{\alpha}Y$ satisfies $F^{-1}(\infty) = \{\infty\}$, then the restriction $F|_X: X \rightarrow Y$ is proper.*

Proof. We first prove (1). Let A be a closed subset of $\tilde{\alpha}Y$. Then A is compact and it is contained in Y , or $A = B \cup \{\infty\}$ for some closed subset B of Y . In any case, the inverse image $(\tilde{\alpha}f)^{-1}(A)$ is closed. Thus $\tilde{\alpha}f$ is continuous.

To prove (2), we take an arbitrary compact subset K of Y . Since $\infty \notin K$, we have $F^{-1}(\infty) \cap F^{-1}(K) = \emptyset$. By $F^{-1}(\infty) = \{\infty\}$, we obtain $\infty \notin F^{-1}(K)$. This means that $F^{-1}(K)$ is compact in X . Thus $F|_X$ is proper. \square

Theorem 2.3. *Let X be a σ -compact locally compact Hausdorff space, and A a closed subset of X . If $f: A \rightarrow [0, \infty)$ is a continuous proper function, then there exists a continuous proper function $F: X \rightarrow [0, \infty)$ such that $F|_A = f$.*

Proof. Notice that $\tilde{\alpha}[0, \infty)$ is homeomorphic to $[0, \infty]$. According to (1) in Proposition 2.2, the map $\tilde{\alpha}f: \tilde{\alpha}A \rightarrow [0, \infty]$ is continuous. The space $\tilde{\alpha}A$ can be considered as a closed subset of $\tilde{\alpha}X$. Since X is σ -compact, the singleton $\{\infty\}$ is a closed G_δ set in $\tilde{\alpha}X$. The space $[0, \infty]$ is homeomorphic to $[0, 1]$. Since $\tilde{\alpha}X$ is compact and Hausdorff, it is normal. Thus, due to Theorem 2.1, there exists a continuous map $h: \tilde{\alpha}X \rightarrow [0, \infty]$ such that $h|_{\tilde{\alpha}A} = \tilde{\alpha}f$ and $h^{-1}(\infty) = \{\infty\}$. By (2) in Proposition 2.2, the function $F = h|_X: X \rightarrow [0, \infty)$ is proper and satisfies $F|_A = f$. This finishes the proof of Theorem 2.3. \square

Remark 2.2. In Theorem 2.3, it is important that the target space is $[0, \infty)$. In general, a proper function $f: A \rightarrow \mathbb{R}$ can not be extended to the ambient space as a proper function. For example, if we define a map $f: \mathbb{Z} \rightarrow \mathbb{R}$ by $f(n) = (-1)^n \cdot n$, then f is proper. However, for any continuous extension $F: \mathbb{R} \rightarrow \mathbb{R}$ of f , the set $F^{-1}(0)$ is non-compact by the intermediate value theorem.

The following proposition is well-known. However, for the sake of self-containedness, we provide a proof.

Proposition 2.4. *A Hausdorff space is σ -compact and locally compact if and only if there exists a continuous proper function $f: X \rightarrow [0, \infty)$.*

Proof. We first assume that X is σ -compact and locally compact. Applying Theorem 2.3 to $A = \emptyset$ and the empty map from \emptyset into $[0, \infty)$, we obtain a proper function from X into $[0, \infty)$.

Next assume that there exists a continuous proper function $f: X \rightarrow [0, \infty)$. By $X = \bigcup_{i=0}^{\infty} f^{-1}([0, i])$, the space X is σ -compact. Since $X = \bigcup_{i=0}^{\infty} f^{-1}([0, i])$ and each $f^{-1}([0, i])$ is open and relatively compact, the space X is locally compact. \square

3. EXTENSION OF PROPER METRICS

In this section, we shall prove two extension theorems of proper metrics and ultrametrics (Theorems 3.9 and 3.15).

A metric d on X is said to be *ultrametric* if it satisfies $d(x, y) \leq d(x, z) \vee d(z, y)$ for all $x, y, z \in X$, where \vee is the maximum operator on \mathbb{R} . A topological space is said to be *metrizable* (resp. *ultrametrizable*) if there exists a metric (resp. ultrametric) that generates the same topology of the space. Let X be a metrizable space, and S a subset of $[0, \infty)$ with $0 \in S$. We denote by $\text{Met}(X; S)$ (resp. $\text{UMet}(X; S)$) the set of all metrics (resp. ultrametrics) that generate the same topology of X taking values in S . We often write $\text{Met}(X) = \text{Met}(X; [0, \infty))$.

A topological space X is said to be *ultranormal* if for every pair A and B of disjoint closed subsets of X , there exists a clopen set V such that $A \subset V$ and $V \cap B = \emptyset$. Note that a topological space X is ultrametrizable if and only if it is metrizable and ultranormal (see [5, Theorem II]). For a topological space X , a pair of subsets A and B of X

is said to be *completely separated* if there exists a continuous function $f: X \rightarrow [0, 1]$ such that $f^{-1}(0) = A$ and $f^{-1}(1) = B$. A topological space X is *strongly 0-dimensional* if X is completely regular and any two completely separated subsets of X are separated by a clopen subset of X . Remark that the class of ultranormal spaces coincides with the class of strongly 0-dimensional normal spaces. In particular, a metrizable space is ultranormal if and only if it is strongly 0-dimensional.

The next is Hausdorff's extension theorem [13] (see also [2] and [21]).

Theorem 3.1. *Let X be a metrizable space, and A a closed subset A of X . If $d \in \text{Met}(A)$, then there exists $D \in \text{Met}(X)$ such that $D|_{A^2} = d$.*

A subset S of $[0, \infty)$ is said to be *characteristic* if $0 \in S$ and if for all $r \in (0, \infty)$, there exists $s \in S \setminus \{0\}$ with $s \leq r$.

We next explain the author's extension theorem of ultrametrics [15, Theorem 1.2], which is an analogue of Hausdorff's extension theorem for ultrametrics:

Theorem 3.2. *Let S be a characteristic subset of $[0, \infty)$. Let X be an ultrametrizable space, and A a closed subset of X . If $d \in \text{UMet}(A; S)$, then there exists $D \in \text{UMet}(X; S)$ such that $D|_{A^2} = d$.*

The following proposition can be considered as a 0-dimensional analogue of Proposition 2.4.

Proposition 3.3. *Let S be an unbounded subset of $[0, \infty)$, and X an ultranormal σ -compact locally compact Hausdorff space. Then there exists a continuous proper function $f: X \rightarrow S$.*

Proof. Let $\{U_i\}_{i \in I}$ be an open covering of X consisting of relatively compact subsets. Since X is paracompact and ultranormal, using [11, Corollary 1.4], we obtain an open covering $\{V_j\}_{j \in J}$ of X refining $\{U_i\}_{i \in I}$ such that $V_j \cap V_{j'} = \emptyset$ if $j \neq j'$. In this case, each V_j is clopen and compact. Since X is σ -compact, the set J is at most countable. We may assume that $J \subset \mathbb{Z}_{\geq 0}$. Take a strictly increasing sequence $\{a_j\}_{j \in \mathbb{Z}_{\geq 0}}$ taking values in S such that $\lim_{j \rightarrow \infty} a_j = \infty$. We define a map $f: X \rightarrow S$ by $f(x) = a_j$ if $x \in V_j$. From the fact that $\{V_j\}_{j \in J}$ is a mutually disjoint clopen covering of X , it follows that the map f is continuous. Since each V_j is compact, we conclude that f is proper. \square

Recall that the symbol \vee stands for the maximum operator on \mathbb{R} . Namely, $x \vee y = \max\{x, y\}$.

Definition 3.1. Let S be a subset of $[0, \infty)$ with $0 \in S$. We define an ultrametric M_S by

$$M_S(x, y) = \begin{cases} 0 & \text{if } x = y; \\ x \vee y & \text{if } x \neq y. \end{cases}$$

Remark 3.1. The construction of the metric M_S was given by Delhommé–Laflamme–Pouzet–Sauer [6, Proposition 2], which also can be found in [15] and [7].

Let (X, d) be a metric space, $x \in X$, and $\epsilon \in (0, \infty)$. We denote by $U(x, \epsilon; d)$ (resp. $B(x, \epsilon; d)$) the open (resp. closed) ball centered at x with radius ϵ .

A subset S of $[0, \infty)$ is said to be *sporadic* if there exists a sequence $\{s_n\}_{n \in \mathbb{Z}}$ such that $S = \{0\} \cup \{s_n \mid n \in \mathbb{Z}\}$, $\lim_{n \rightarrow -\infty} s_n = 0$, $\lim_{n \rightarrow \infty} s_n = \infty$ and $s_i < s_{i+1}$ for all $i \in \mathbb{Z}$. Note that every sporadic subset of $[0, \infty)$ is unbounded and characteristic in $[0, \infty)$, and every unbounded characteristic subset of $[0, \infty)$ contains a sporadic subset.

Lemma 3.4. *Let S be a sporadic subset of $[0, \infty)$. Then the Euclidean topology on S coincides with that induced from M_S .*

Proof. For all $x \in S \setminus \{0\}$, we have $U(x, x; M_S) = \{x\}$ and $U(0, x; M_S) = S \cap [0, x)$. This proves the lemma. \square

Definition 3.2. Let X be a topological space, and $f: X \rightarrow \mathbb{R}$ a continuous map. We define a pseudo-metric $E[f]$ on X by $E[f](x, y) = |f(x) - f(y)|$. Let S be a subset of $[0, \infty)$. Let $f: X \rightarrow S$ be a continuous map. We also define a pseudo-metric $M_S[f]$ on X by $M_S[f](x, y) = M_S(f(x), f(y))$.

Definition 3.3. Let X be a set and $d, e: X^2 \rightarrow \mathbb{R}$ be arbitrary maps. We define $d \vee e: X^2 \rightarrow \mathbb{R}$ by $(d \vee e)(x, y) = d(x, y) \vee e(x, y)$. Notice that if d is a metric on X and e is a pseudo-metric on X , then $d \vee e$ is a metric on X .

Note that a metric d on X is proper if and only if all closed balls of (X, d) are compact.

Lemma 3.5. *Let X be a metrizable space. Let $f: X \rightarrow [0, \infty)$ be a continuous proper function, and $d \in \text{Met}(X)$. Then the map $d \vee E[f]$ is a proper metric in $\text{Met}(X)$.*

Proof. Since f is continuous, the map $E[f]: X^2 \rightarrow [0, \infty)$ is also continuous. Then the assumption $d \in \text{Met}(X)$ yields $d \vee E[f] \in \text{Met}(X)$. For all $r \in (0, \infty)$ and $p \in X$, we notice that $B(p, r; d \vee E[f]) \subset f^{-1}([f(p) - r, f(p) + r])$. Since f is proper, the set $B(p, r; d \vee E[f])$ is compact. Thus, we conclude that $d \vee E[f]$ is a proper metric. \square

Lemma 3.6. *Let S be an unbounded characteristic subset of $[0, \infty)$, and T a sporadic subset of $[0, \infty)$ with $T \subset S$. Let X be an ultrametrizable space. Assume that $f: X \rightarrow T$ is a continuous proper function, and $d \in \text{UMet}(X; S)$. Then the map $d \vee M_T[f]$ is a proper metric in $\text{UMet}(X; S)$.*

Proof. Lemma 3.4 implies that $M_T[f]: X^2 \rightarrow T$ is continuous. Thus, by $d \in \text{UMet}(X; S)$, and by $T \subset S$, we have $d \vee M_T[f] \in \text{UMet}(X; S)$.

For all $r \in (0, \infty)$ and $p \in X$, we obtain $B(p, r; d \vee M_T[f]) \subset f^{-1}([0, r] \cup \{f(p)\})$. Since f is proper, the set $B(p, r; d \vee M_T[f])$ is compact. Therefore $d \vee M_T[f]$ is a proper metric. This completes the proof. \square

Lemma 3.5 gives a new proof of the following well-known corollary:

Corollary 3.7. *Let X be a σ -compact locally compact Hausdorff space. Then there exists a proper metric in $\text{Met}(X)$. In particular, the space X is completely metrizable.*

Proof. Take $d \in \text{Met}(X)$ and take a proper continuous function $f: X \rightarrow [0, \infty)$ (see Proposition 2.4). By Lemma 3.5, the map $d \vee E[f]$ is a proper metric in $\text{Met}(X)$. This proves the first part of the corollary. The latter part follows from the fact that every proper metric is complete. \square

Corollary 3.8. *Let S be an unbounded characteristic subset of $[0, \infty)$, and X an ultranormal σ -compact locally compact Hausdorff space. Then there exists a proper metric in $\text{UMet}(X; S)$. In particular, the space X is completely ultrametrizable.*

Proof. Since S is unbounded and characteristic, there exists a sporadic set of T such that $T \subset S$. According to Proposition 3.3, there exists a continuous proper function $f: X \rightarrow T$. Using [15, Proposition 2.14] or applying Theorem 3.2 to $A = \emptyset$, we can take $d \in \text{UMet}(X; S)$. Then, Lemma 3.6 implies that $d \vee M_T[f]$ is a proper metric in $\text{UMet}(X; S)$. \square

Using Theorem 2.3, we obtain an extension theorem of proper metrics.

Theorem 3.9. *Let X be a σ -compact locally compact Hausdorff space, and A a non-empty closed subset of X . If $d \in \text{Met}(A)$ is a proper metric, then there exists a proper metric $D \in \text{Met}(X)$ with $D|_{A^2} = d$.*

Proof. Fix $p \in A$ and define a map $f: A \rightarrow [0, \infty)$ by $f(x) = d(x, p)$. Then f is a continuous proper function. According to Theorem 2.3, there exists a continuous proper function $F: X \rightarrow [0, \infty)$ with $F|_A = f$. Due to Hausdorff's extension theorem (Theorem 3.1), we can take a metric $e \in \text{Met}(X)$ such that $e|_{A^2} = d$. We define a map $D: X^2 \rightarrow [0, \infty)$ by

$$D(x, y) = e(x, y) \vee E[F](x, y)$$

Lemma 3.5 implies that the map D is a proper metric in $\text{Met}(X)$. We shall prove that $D|_{A^2} = d$. If $x, y \in A$, we have $e(x, y) = d(x, y)$ and $E[F](x, y) = |F(x) - F(y)| = |d(x, p) - d(y, p)|$. The triangle inequality yields $|d(x, p) - d(y, p)| \leq d(x, y)$. Thus, we obtain $E[F](x, y) \leq d(x, y)$ for all $x, y \in A$. Therefore, by the definition of D , we have $D|_{A^2} = d$. This completes the proof. \square

Let X be a topological space. A subset A of X is said to be a *retract* if there exists a continuous map $r: X \rightarrow A$ such that $r(a) = a$ for all

$a \in A$. In this case, the continuous map r is said to be a *retraction*. A subset A is said to be a *proper retract* if there exists a retraction $r: X \rightarrow A$, which is a proper map. For more discussion of proper retracts, we refer the readers to [18].

The next lemma follows from the strong triangle inequality.

Lemma 3.10. *Let X be a set, and d be an ultrametric on X . Then for all $x, y, z \in X$, the inequality $d(x, z) < d(y, z)$ implies $d(y, z) = d(x, y)$.*

Let (X, d) and (Y, e) be metric spaces, and $f: X \rightarrow Y$ be a map. We say that f is *metrically proper* if the inverse image $f^{-1}(A)$ is bounded in (X, d) for every bounded subset A of Y .

The proof of the following theorem is presented in [3, Theorem 2.9].

Theorem 3.11. *Let (X, d) be an ultrametric space, and A be a closed subset of X . Let $\tau \in (1, \infty)$. Then there exists a τ^2 -Lipschitz retraction $r: X \rightarrow A$. Moreover, if A is unbounded, the retraction r associated with A can be chosen to be metrically proper.*

By proving the existence of a proper ultrametric on an ultranormal σ -compact locally compact Hausdorff space (Corollary 3.8), we show that a non-compact closed subset of an ultranormal σ -compact locally compact Hausdorff space is not only just a retract, but also a proper retract.

Theorem 3.12. *Let X be an ultranormal σ -compact locally compact Hausdorff space, and A a non-empty non-compact closed subset of X . Then A is a proper retract of X .*

Proof. Using Corollary 3.8, we can take a proper ultrametric $d \in \text{UMet}(X; [0, \infty))$. Since A is non-compact and d is proper, it is unbounded in (X, d) . The latter part of Theorem 3.11 implies that there exists a metrically proper retraction $r: X \rightarrow A$ with respect to d . To prove that r is proper, we take an arbitrary compact subset K of A . Since K is bounded, and since r is metrically proper, the inverse image $r^{-1}(K)$ is bounded and closed. Since d is a proper metric, the set $r^{-1}(K)$ is compact, and hence r is proper. This finishes the proof of Theorem 3.12. \square

Before proving the following corollary, notice that the composition of two proper maps is proper.

Corollary 3.13. *Let X be an ultranormal σ -compact locally compact Hausdorff space, and A a non-empty closed subset of X . If Y is a non-compact metrizable space, then every continuous proper map $f: A \rightarrow Y$ can be extended into a continuous proper map $F: X \rightarrow Y$.*

Proof. We divide the proof into two cases.

Case 1. (A is non-compact): Theorem 3.12 guarantees the existence of a proper retraction $r: X \rightarrow A$. Put $F = f \circ r$. Then $F: X \rightarrow Y$ is a desired extension.

Case 2. (A is compact): In this case, let Z be the countable discrete space. Fix $\omega \in A$. Put $Z = \{a_i \mid i \in \mathbb{Z}_{\geq 0}\}$, where $a_*: \mathbb{Z}_{\geq 0} \rightarrow Z$ is injective. Note that $X \times Z$ is an ultranormal non-compact σ -compact locally compact Hausdorff space. Put $C = A \times \{a_0\} \cup \{\omega\} \times Z$. Then C is a non-compact closed subset of $X \times Z$. Since Y is non-compact, we can take a countable closed discrete subset $\{b_i \mid i \in \mathbb{Z}_{\geq 1}\}$ of Y . We define a map $g: C \rightarrow Y$ by $g((x, a_0)) = f(x)$ for all $x \in A$ and $g((\omega, a_i)) = b_i$ for all $i \in \mathbb{Z}_{\geq 1}$. Then g is continuous and proper. Thus, using Case 1, we can take a continuous proper map $G: X \times Z \rightarrow Y$ such that $G|_C = g$. We define a map $F: X \rightarrow Y$ by $F(x) = G(x, a_0)$. Then F is a continuous proper map and satisfies $F|_A = f$. \square

Proposition 3.14. *Let S be an unbounded characteristic subset of $[0, \infty)$, and T a sporadic subset of $[0, \infty)$ with $T \subset S$. Let X be an ultranormal σ -compact locally compact Hausdorff space. If $d \in \text{UMet}(X; S)$, then there exists a metric $w \in \text{UMet}(X; T)$ such that $w(x, y) \leq d(x, y)$ for all $x, y \in X$. Moreover, if d is proper, so is w .*

Proof. Take a real sequence $\{a_n\}_{n \in \mathbb{Z}}$ such that $T = \{0\} \cup \{a_n \mid n \in \mathbb{Z}\}$, $\lim_{n \rightarrow \infty} a_n = \infty$, $\lim_{n \rightarrow -\infty} a_n = 0$, and $a_i < a_{i+1}$ for all $i \in \mathbb{Z}$. We define a map $\psi: [0, \infty) \rightarrow [0, \infty)$ by

$$\psi(x) = \begin{cases} 0 & \text{if } x = 0; \\ a_i & \text{if } a_i \leq x < a_{i+1}. \end{cases}$$

Put $w = \psi \circ d$. According to [15, Lemma 2.2], we observe that $w \in \text{UMet}(X; T)$. By the definition of ψ , we have $w(x, y) \leq d(x, y)$ for all $x, y \in X$. This completes the first part of the proposition. To prove the latter part, assume that d is proper and take $p \in X$ and $r \in (0, \infty)$. Put $\psi(r) = a_i$. Then we have $B(p, r; w) = B(p, a_i; w) \subset B(p, a_{i+1}; d)$. Since d is proper, the set $B(p, r; w)$ is compact. Thus w is proper. \square

Theorem 3.12 provides an ultrametric version of Theorem 3.9.

Theorem 3.15. *Let S be an unbounded characteristic subset of $[0, \infty)$. Let X be an ultranormal σ -compact locally compact Hausdorff space, and A a non-empty closed subset of X . If $d \in \text{UMet}(A; S)$ is proper, then there exists a proper metric $D \in \text{UMet}(X; S)$ such that $D|_{A^2} = d$.*

Proof. The proof is similar to that of Theorem 3.9. Fix $p \in A$. Take a sporadic subset T of $[0, \infty)$ with $T \subset S$. Using Proposition 3.14, we can take $w \in \text{UMet}(X; T)$ with $w(x, y) \leq d(x, y)$ for all $x, y \in X$. We define a map $f: A \rightarrow T$ by $f(x) = w(p, x)$. Then f is a continuous proper function. According to Corollary 3.13, we can take a continuous proper function $F: X \rightarrow T$ such that $F|_A = f$. By Theorem 3.2, there exists a metric $e \in \text{UMet}(X; S)$ such that $e|_{A^2} = d$. We define a map $D: X^2 \rightarrow S$ by

$$D(x, y) = e(x, y) \vee M_T[F](x, y).$$

Lemma 3.6 implies that the map D is a proper metric in $\text{UMet}(X; S)$. We shall prove $D|_{A^2} = d$. Take $x, y \in A$. We may assume that $w(p, x) \leq w(p, y)$. If $w(p, x) < w(p, y)$, Lemma 3.10 yields $w(x, y) = w(p, y)$. Thus $M_T[F](x, y) = w(x, y) \leq d(x, y)$. If $w(p, x) = w(p, y)$, then, by the definition of M_T , we have $M_T[F](x, y) = 0 \leq d(x, y)$. Thus, due to $e|_{A^2} = d$ and the definition of D , we obtain $D|_{A^2} = d$. This finishes the proof of Theorem 3.15. \square

4. PROPER METRICS AT LARGE SCALES

For a topological space Y , we denote by $P(Y)$ the set of all non-empty subsets of Y . For topological spaces X and Y , we say that a map $\phi: X \rightarrow P(Y)$ is *lower semi-continuous* if for every open subset O of Y , the set $\{x \in X \mid \phi(x) \cap O \neq \emptyset\}$ is open in X . For a map $\phi: X \rightarrow P(Y)$, a map $f: X \rightarrow Y$ is said to be a *selection of ϕ* if it is continuous and satisfies $f(x) \in \phi(x)$ for all $x \in X$.

The following proposition from E. Michael [16, Proposition 1.4] states that the existence of a selection of a set-valued map is equivalent to the extension of a selection defined on a closed subset of the domain.

Proposition 4.1. *Let X and Y be topological space. If \mathcal{S} is a subset of $P(Y)$ containing all one-point subsets of Y , then the following statements are equivalent to each other:*

- (1) *For all lower semi-continuous map $\phi: X \rightarrow \mathcal{S}$, there exists a selection of ϕ .*
- (2) *If A is a closed subset of X , and $\phi: X \rightarrow \mathcal{S}$ is a lower semi-continuous map, and if $f: X \rightarrow Y$ is a selection of the restricted map $\phi|_A: A \rightarrow \mathcal{S}$, then there exists a map $F: X \rightarrow Y$, which is a selection of $\phi: X \rightarrow \mathcal{S}$ such that $F|_A = f$.*

Let V be a Banach space. We denote by $\mathcal{CC}(V)$ the set of all non-empty closed convex subsets of V . The next theorem is known as Michael's selection theorem on paracompact spaces (see [16, Theorem 3.2'']):

Theorem 4.2. *Let X be a paracompact space, and V a Banach space. If $\phi: X \rightarrow \mathcal{CC}(V)$ is a lower semi-continuous map, then there exists a selection of ϕ .*

For a topological space Z , we denote by $\mathcal{C}(Z)$ the set of all non-empty closed subsets of Z . Recall that every ultranormal paracompact space is 0-dimensional (i.e., it has covering dimension 0). The following theorem is known as the 0-dimensional Michael selection theorem (see [17, Theorem 2]):

Theorem 4.3. *Let X be a 0-dimensional paracompact space, Z a completely metrizable space. If $\phi: X \rightarrow \mathcal{C}(Z)$ is a lower semi-continuous map, then there exists a selection of ϕ .*

For every Banach space V (resp. completely metrizable space Z), the set $\mathcal{CC}(V)$ (resp. $\mathcal{C}(Z)$) contains all one-point sets of V (resp. Z). Thus, we can apply Proposition 4.1 to Theorems 4.2 and 4.3, and then we obtain the next two theorems on extending selections:

Theorem 4.4. *Let X be a paracompact space, and A a closed subset of X . Let V be a Banach space, and $\phi: X \rightarrow \mathcal{CC}(V)$ a lower semi-continuous map. If $f: A \rightarrow V$ is a selection of $\phi|_A: A \rightarrow \mathcal{CC}(V)$, then there exists a selection $F: X \rightarrow V$ of ϕ such that $F|_A = f$.*

Theorem 4.5. *Let X be an ultranormal paracompact space, and A a closed subset of X . Let Z be a completely metrizable space, and $\phi: X \rightarrow \mathcal{C}(Z)$ a lower semi-continuous map. If $f: A \rightarrow Z$ is a selection of $\phi|_A: A \rightarrow \mathcal{C}(Z)$, then there exists a selection $F: X \rightarrow Z$ of ϕ such that $F|_A = f$.*

Propositions 4.6 can be deduced from [20, Theorem 0.48] or [22, Lemma 1.4.6]. The proof of Proposition 4.7 is presented in [15, Corollary 2.24]. The definition of ultra-normed modules can be found in [15].

Proposition 4.6. *Let X be a topological space, and let $(V, \|\cdot\|)$ be a Banach space. Let $H: X \rightarrow V$ be a continuous map and $r \in (0, \infty)$. Then the map $\phi: X \rightarrow \mathcal{CC}(V)$ defined by $\phi(x) = B(H(x), r; \|\cdot\|)$ is lower semi-continuous.*

Proposition 4.7. *Let X be a topological space, R be a commutative ring, and let (V, h) be an ultra-normed R -module. Let $H: X \rightarrow V$ be a continuous map and $r \in (0, \infty)$. Then a map $\phi: X \rightarrow \mathcal{C}(V)$ defined by $\phi(x) = B(H(x), r; h)$ is lower semi-continuous.*

Let (Z, h) be a metric space and $\eta \in (0, \infty)$. A subset E of Z is said to be η -dense in (Z, h) if for all $x \in Z$, there exists $y \in E$ such that $h(x, y) \leq \eta$.

Theorem 4.8. *Let $\eta \in [0, \infty)$. Let X be a σ -compact locally compact Hausdorff space, and A a proper retract of X . If $d \in \text{Met}(X)$ is a proper metric, then there exists a proper metric $D \in \text{Met}(X)$ such that $D|_{A^2} = d$ and A is η -dense in (X, D) .*

Proof. We first take a Banach space $(V, \|\cdot\|)$ and an isometric embedding $l: (A, d|_{A^2}) \rightarrow (V, \|\cdot\|)$. For example, we can choose $(V, \|\cdot\|)$ as the space of all real-valued bounded continuous functions on A , and $l: A \rightarrow V$ as the Kuratowski embedding defined by $l(x)(y) = d(x, y) - d(\xi, y)$ for a fixed point $\xi \in A$.

We take a proper retraction $r: X \rightarrow A$ and define $\phi: X \rightarrow \mathcal{CC}(V)$ by $\phi(x) = B(l(r(x)), \eta; \|\cdot\|)$. Applying Proposition 4.6 to $H = l \circ r$, we can assert that the map ϕ is lower semi-continuous. For all $a \in A$, the equality $r(a) = a$ implies that $l(a) \in \phi(a)$, namely, the map $l: A \rightarrow V$

is a selection of $\phi|_A: A \rightarrow \mathcal{CC}(V)$. Then Theorem 4.4 guarantees the existence of a selection $L: X \rightarrow V$ of ϕ such that $L|_A = l$.

Due to Hausdorff's extension theorem (Theorem 3.1), we can take $e \in \text{Met}(X)$ with $e|_{A^2} = d$. We define a map $u: X^2 \rightarrow [0, \infty)$ by $u(x, y) = \min\{e(x, y), \eta\}$. Then $u \in \text{Met}(X)$. We also define a map $v: X^2 \rightarrow [0, \infty)$ by $v(x, y) = \|L(x) - L(y)\| \vee u(x, y)$. Since $u \in \text{Met}(X)$ and L is continuous, we have $v \in \text{Met}(X)$. Using $L|_A = l$, we obtain $\|L(a) - L(b)\| = d(a, b)$ for all $a, b \in A$. Then, from $u(x, y) \leq e(x, y)$ for all $x, y \in X$, and $e|_{A^2} = d$, it follows that $v|_{A^2} = d$.

Next we fix $p \in A$ (note that $A \neq \emptyset$). We define a continuous proper function $f: A \rightarrow [0, \infty)$ by $f(x) = d(p, x)$, and define $F = f \circ r$. Then $F: X \rightarrow [0, \infty)$ is a continuous proper function with $F|_A = f$. We also define a metric D on X by $D(x, y) = v(x, y) \vee E[F](x, y)$.

Lemma 3.5 implies that D is a proper metric in $\text{Met}(X)$. In a similar way to the proof of Theorem 3.9, we obtain $D|_{A^2} = d$.

We now show that A is η -dense in (X, D) . Take an arbitrary point $x \in X$. The relations $L(r(x)) = l(r(x))$ and $L(x) \in \phi(x)$ yield

$$(4.1) \quad \|L(x) - L(r(x))\| \leq \eta.$$

From (4.1), the inequality $u(x, r(x)) \leq \eta$, and the definition of v , it follows that

$$(4.2) \quad v(x, r(x)) \leq \eta.$$

Since r is a retraction, we have $r(r(x)) = r(x)$. Thus $E[F](x, r(x)) = |F(x) - F(r(x))| = |f(r(x)) - f(r(x))| = 0$, and hence

$$(4.3) \quad E[F](x, r(x)) = 0.$$

Therefore, by (4.2), (4.3), and the definition of D , we conclude that

$$(4.4) \quad D(x, r(x)) \leq \eta.$$

Since $r(x) \in A$, and $x \in X$ is arbitrary, the inequality (4.4) proves that A is η -dense in (X, D) . This completes the proof of Theorem 4.8. \square

The proof of Theorem 4.9 is analogous with Theorems 4.8.

Theorem 4.9. *Let $\eta \in (0, \infty)$, and S an unbounded characteristic subset of $[0, \infty)$. Let X be an ultranormal σ -compact locally compact Hausdorff space, and A a non-empty non-compact closed subset of X . If $d \in \text{UMet}(A; S)$ is proper, then there exists a proper metric $D \in \text{UMet}(X; S)$ such that $D|_{A^2} = d$ and A is η -dense in (X, D) .*

Proof. We put $R = \mathbb{Z}/2\mathbb{Z}$. However, as long as R is an integral domain, the choice of R does not affect the proof of the theorem. We first verify that there exists an isometric embedding $(A, d|_{A^2})$ into a complete ultra-normed R -module. Let (Y, m) be the completion of $(A, d|_{A^2})$. Since the set $d(A^2)$ is invariant under the completion (see (12) in [4, Theorem 1.6]), we have $m(Y^2) = d(A^2)$, and hence $m \in \text{UMet}(Y; S)$. According to [15, Theorem 1.1], we can take a complete ultra-normed

R -module (V, h) with $h \in \text{UMet}(V; S)$ and an isometric embedding $J: (Y, m) \rightarrow (V, h)$. We put $l = J|_A: A \rightarrow V$.

Theorem 3.12 enables us to take a proper retraction $r: X \rightarrow A$. Since S is characteristic, we can also take $\theta \in S \setminus \{0\}$ with $\theta \leq \eta$. We define a map $\phi: X \rightarrow \mathcal{C}(V)$ by $\phi(x) = B(l(r(x)), \theta; h)$. Applying Proposition 4.7 to $H = l \circ r$, we notice that the map ϕ is lower semi-continuous. For all $a \in A$, the equality $r(a) = a$ implies that $l(a) \in \phi(a)$, namely, the map $l: A \rightarrow V$ is a selection of $\phi|_A: A \rightarrow \mathcal{C}(V)$. Using Theorem 4.5, there exists a selection $L: X \rightarrow V$ of ϕ such that $L|_A = l$.

Due to Theorem 3.2, we can take $e \in \text{UMet}(X; S)$ such that $e|_{A^2} = d$. Put $u(x, y) = \min\{e(x, y), \theta\}$. Then $u \in \text{UMet}(X; S)$. We define a map $v: X^2 \rightarrow [0, \infty)$ by $v(x, y) = h(L(x), L(y)) \vee u(x, y)$. From $u \in \text{UMet}(X; S)$ and the continuity of L , it follows that $v \in \text{UMet}(X; S)$. Using $L|_A = l$, we have $h(L(a), L(b)) = d(a, b)$ for all $a, b \in A$. Then, by $u(x, y) \leq e(x, y)$ for all $x, y \in X$, and by $e|_{A^2} = d$, we obtain $v|_{A^2} = d$.

Next we fix $p \in A$ and take a sporadic subset T of $[0, \infty)$ with $T \subset S$. Due to Proposition 3.14, there exists $w \in \text{UMet}(A; T)$ with $w(a, b) \leq d(a, b)$ for all $a, b \in A$. We define a continuous proper function $f: A \rightarrow T$ by $f(x) = w(p, x)$, and define a map $F = f \circ r$. Then $F: X \rightarrow T$ is a continuous proper function with $F|_A = f$. We also define a metric D on X by $D(x, y) = v(x, y) \vee M_T[F](x, y)$.

Lemma 3.6 implies that D is a proper ultrametric in $\text{UMet}(X; S)$. In a similar way to the proof of Theorem 3.15, we obtain $D|_{A^2} = d$.

We now show that A is η -dense in (X, D) . Take an arbitrary point $x \in X$. The relations $L(r(x)) = l(r(x))$ and $L(x) \in \phi(x)$ yield

$$(4.5) \quad h(L(x), L(r(x))) \leq \theta.$$

From (4.5), the inequalities $u(x, r(x)) \leq \theta$ and $\theta \leq \eta$, and the definition of v , it follows that

$$(4.6) \quad v(x, r(x)) \leq \eta.$$

Since r is a retraction, we have $r(r(x)) = r(x)$. Then $M_T[F](x, r(x)) = M_T(F(x), F(r(x))) = M_T(f(r(x)), f(r(x))) = 0$, and hence

$$(4.7) \quad M_T[F](x, r(x)) = 0.$$

Therefore, by (4.6), (4.7), and the definition of D , we conclude that

$$(4.8) \quad D(x, r(x)) \leq \eta.$$

Since $r(x) \in A$ and $x \in X$ is arbitrary, the inequality (4.8) proves that A is η -dense in (X, D) . This completes the proof of Theorem 4.9. \square

Acknowledgements. The author would like to thank the referee for helpful comments and suggestions.

REFERENCES

1. R. Arens, *Extension of functions on fully normal spaces*, Pacific J. Math. **2** (1952), 11–22.
2. R. H. Bing, *Extending a metric*, Duke. Math. J. **14** (1947), no. 3, 511–519.
3. N. Brodskiy, J. Dydak, J. Higes, and A. Mitra, *Dimension zero at all scales*, Topology Appl. **154** (2007), no. 14, 2729–2740.
4. A. B. Comicheo and K. Shamseddine, *Summary on non-Archimedean valued fields*, Advances in Ultrametric Analysis (A. Escassut, C. Perez-Garcia, and K. Shamseddibe, eds.), Contemp. Math., vol. 704, Amer. Math. Soc., Providence, RI, 2018, pp. 1–36.
5. J. de Groot, *Non-Archimedean metrics in topology*, Proc. Amer. Math. Soc. **7** (1956), 948–953.
6. C. Delhommé, C. Laflamme, M. Pouzet, and N. Sauer, *Indivisible ultrametric spaces*, Topology Appl. **155** (2008), no. 14, 1462–1478.
7. D. Dordovskiy, O. Dovgoshey, and E. Petrov, *Diameter and diametrical pairs of points in ultrametric spaces*, *p*-Adic Numbers Ultrametric Anal. Appl. **3** (2011), no. 4, 253–262.
8. O. Dovgoshey, O. Martio, and M. Vuorinen, *Metrization of weighted graphs*, Ann. Comb. **17** (2013), no. 3, 455–476.
9. O. Dovgoshey and E. A. Petrov, *A subdominant pseudoultrametric on graphs*, Mat. Sb. **204** (2013), no. 8, 51–72.
10. J. Dugundji, *An extension of Tietze’s theorem*, Pacific J. Math. **1** (1951), 353–367.
11. R. L. Ellis, *Extending continuous functions on zero-dimensional spaces*, Math. Ann. **186** (1970), no. 2, 114–122.
12. M. Frantz, *Controlling Tietze-Urysohn extensions*, Pacific J. Math. **169** (1995), no. 1, 53–73.
13. F. Hausdorff, *Erweiterung einer Homöomorphie*, Fund. Math. **16** (1930), 353–360.
14. M. Hušek, *Extension of mappings and pseudometrics*, Extracta Math. **25** (2010), no. 3, 277–308.
15. Y. Ishiki, *An embedding, an extension, and an interpolation of ultrametrics*, *p*-Adic Numbers Ultrametric Anal. Appl. **13** (2021), no. 2, 117–147.
16. E. Michael, *Continuous selections. I*, Ann. of Math. (2) **63** (1956), 361–382.
17. ———, *Selected selection theorems*, Amer. Math. Monthly **63** (1956), no. 4, 233–238.
18. ———, *Closed retracts and perfect retracts*, Topology Appl. **121** (2002), no. 3, 451–468.
19. J. R. Munkres, *Topology*, 2nd ed., Pearson modern classic, Pearson, New York, 2018, Originally published in 2000 [2018 reissue].
20. D. Repovš and P. V. Semenov, *Continuous Selections of Multivalued Mappings*, Mathematics and its Applications, vol. 455, Kluwer Academic Publishers, Dordrecht, 1998.
21. H. Toruńczyk, *A short proof of Hausdorff’s theorem on extending metrics*, Fund. Math. **77** (1972), no. 2, 191–193.
22. J. van Mill, *Infinite-Dimensional Topology: Prerequisites and Introduction*, North-Holland Mathematical Library, vol. 43, North-Holland Publishing Co., Amsterdam, 1988.
23. K. Yamazaki, *Controlling extensions of functions and C-embedding*, Topology Proc. **26** (2001/02), no. 1, 323–341.

(Yoshito Ishiki)

PHOTONICS CONTROL TECHNOLOGY TEAM
RIKEN CENTER FOR ADVANCED PHOTONICS
2-1 HIRASAWA, WAKO, SAITAMA 351-0198, JAPAN
Email address: yoshito.ishiki@riken.jp