A characterization of the ellipsoid by planar grazes

I. Gonzalez-García[1], J. Jerónimo-Castro[2],
D. J. Verdusco-Hernández[3], E. Morales-Amaya[4]
[1,2]Facultad de Ingeniería
Universidad Autónoma de Querétaro, México
[3,4]Facultad de Matemáticas-Acapulco,
Universidad Autónoma de Guerrero, México

Dedicated to David George Larman

Abstract

In this paper we proved the following: *Let $K, L \subset \mathbb{R}^3$ be two O-symmetric convex bodies with $L \subset \text{int}K$ strictly convex. Suppose that from every x in* $\text{bd}K$ *the graze $\Sigma(L, x)$ is a planar curve and K is almost free with respect to L. Then L is an ellipsoid.*

1 Introduction

A very important problem in Geometric Tomography is to establish properties of a given convex body, i.e., a compact and convex set with non-empty interior, if we know some properties over its sections or its projections. An interesting result was given by C. A. Rogers [12]:

Let $K \subset \mathbb{R}^n$ be a convex body, $n \geq 3$, and p be a point in $\mathbb{R}^n$. If all the 2-dimensional sections of K through p have a centre of symmetry, then K also has a centre of symmetry.

In the same article, Rogers also conjectured that K must be an ellipsoid if we also have the condition that p is not the centre of symmetry of K. This conjecture was first proved by P. W. Aitchison, C. M. Petty, and C. A. Rogers [1] and is known as the False Centre Theorem. Later, L. Montejano and E. Morales-Amaya gave a simpler proof of the False Centre Theorem in [10]. In the following theorem, due to S. Olovjanishnikov [11], the considered sections of K are not concurrent:

A convex body $K \subset \mathbb{R}^n$, $n \geq 3$, is an ellipsoid provided all hyperplanar sections of K that divide the volume of K in a given ratio $\lambda \neq 1$ have a centre of symmetry.

In [2], J. A. Barker, and D. G. Larman conjectured the following, which is a variant of Olovjanishnikov's theorem: *Suppose $K, L \subset R^n$ are convex bodies, with $n \geq 3$ and $L \subset \text{int}K$. Suppose that whenever H is a hyperplane supporting L the section $H \cap K$ of K is centrally symmetric. Then K is an ellipsoid.*

Another variant of Olovjanishnikov's theorem would be the following conjecture by G. Bianchi, and P. M. Gruber [3]: *Let K be a convex body in $\mathbb{R}^n$, $n \geq 3$, and let δ be a continuous real function on $\mathbb{S}^{n-1}$ such that for each vector $u \in \mathbb{S}^{n-1}$ the hyperplane $\{x : \langle x, u\rangle = \delta(u)\}$ intersects the interior of K. If any such intersection is centrally symmetric and (with, possibly, a few exceptions) does not contain a possibly existing centre of K, then K is an ellipsoid.*

There is another kind of characterizations of the ellipsoid considering properties of the intersections of cones or cylinders with the body. Given a point $x \in \mathbb{R}^n \setminus K$ we denote the cone generated by K with apex x by $\mathrm{C}(K, x)$, i.e., $\mathrm{C}(K, x) := \{x + \lambda(y - x) : y \in K, \lambda \geq 0\}$. The boundary of $\mathrm{C}(K, x)$ is denoted by $\mathrm{S}(K, x)$, in other words, $\mathrm{S}(K, x)$ is the support cone of K from the point x. We denote the *graze* of K from x by $\Sigma(K, x)$, i.e., $\Sigma(K, x) := \mathrm{S}(K, x) \cap \partial K$. Using this notion of grazes, A. Marchaud proved the following in [9]:

Let $K \subset \mathbb{R}^3$ be a convex body and H be a plane which is either disjoint from K or meets K at a single point. Then K is an ellipsoid if for every point $x \in H \setminus K$, the graze $\Sigma(K, x)$ contains a planar convex curve γ such that $\operatorname{conv}\gamma \cap \operatorname{int} K \neq \emptyset$.

However, if we change the plane H for a surface Γ which encloses K it is not known whether K is an ellipsoid or not. Marchaud's theorem is also true if the apexes of the cones are points at infinity. In this case the grazes are called *shadow boundaries* and are obtained by intersections of ∂K with circumscribed cylinders. The first proof that a convex body is an ellipsoid if and only if every shadow boundary lies in a hyperplane is due to H. Blaschke [4]. We suspect the following is true.

Conjecture 1. *Let $L \subset \operatorname{int} K \subset \mathbb{R}^n$ be convex bodies such that for every point $x \in \partial K$ it holds that $\Sigma(L, x)$ lies in a hyperplane. Then L is an ellipsoid.*

With the additional condition that the grazes are ellipses, it was proved in [5] that L is an ellipsoid. In this work we give another progress in order to prove this conjecture, and was very unexpected that under the hypotheses of Theorem 1 the grazes of the body L result to be centrally symmetric (Lemma 1). Before we give the statement of the main result in this work, we give some more definitions and notation. Let L and K be two O-symmetric convex bodies in $\mathbb{R}^3$, with $L \subset \text{int}K$. We say that the points $x, y \in \text{bd}K$ are *free with respect to* L if the line through x and y, $\ell(x, y)$, does not meet L. Suppose that for every point $x \in \text{bd}K$, the graze $\Sigma(L, x)$ is a planar curve and denote the plane where it is contained by Δ_x. The body K is said to be *almost free with respect to* L if for each $z \in \text{bd}K$ and $w \in \Pi_z \cap \text{bd}K$, where Π_z is a plane through O parallel to Δ_z, the points z and w are free with respect to L (see Fig. 1).

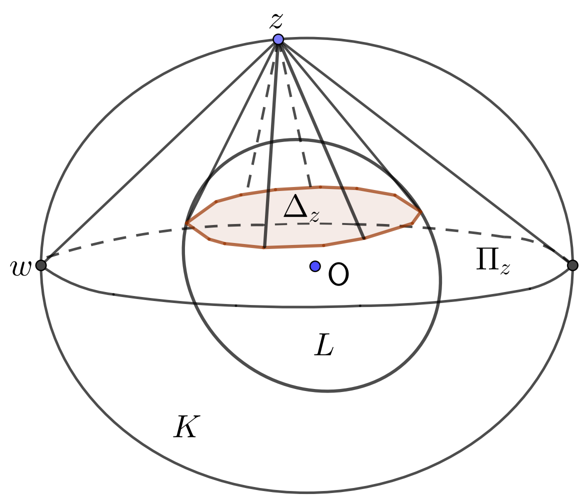

Figure 1: K is almost free with respect to L

The main result of this article is the following.

Theorem 1. *Let $K, L \subset \mathbb{R}^3$ be two O-symmetric convex bodies with $L \subset \text{int}K$ strictly convex. Suppose that from every x in $\text{bd}K$ the graze $\Sigma(L, x)$ is a planar curve and K is almost free with respect to L. Then L is an ellipsoid.*

It is worth to notice that Theorem 1 is not only evidence for the veracity of Conjecture 1. Once we have the conclusion of Lemma 1 we arrive to a particular cases of, for one hand, the Bianchi and Gruber's conjecture and, on the other hand, the Baker and Larman's conjecture mentioned before, and Theorem 1 gives a positive answers to this cases.

2 Main result

We first prove two lemmas.

Lemma 1. *For each $x \in \text{bd}K$ the graze $\Sigma(L, x)$ is centrally symmetric with centre at the point $O_x := \ell(x, -x) \cap \Delta_x$.*

Proof. Let $\Omega_x := S(L, x) \cap S(L, -x)$. By Lemma 2.2 in [8] we have that Ω_x is a simple and closed curve, moreover, since L is centred at O we have that Ω_x is centrally symmetric with centre at O. Let $a \in \Sigma(L, x)$ be any point and let z be the point where the line $\ell(x, a)$ intersects Ω_x (see Fig. 2). Consider the point $b \in \Sigma(L, x)$ such that $[a, b]$ is an affine diameter of $\Sigma(L, x)$. Suppose that $O_x \notin [a, b]$ and let a' be the point where $[-z, x]$ intersects $\Sigma(L, x)$. Let H_a be a support plane of L through the points x and z, and let $\ell_a := \Delta_x \cap H_a$, and let ℓ_z be the support lines of $\Sigma(L, x)$ and Ω_x, through a and z, respectively, with ℓ_z parallel to ℓ_a. Let ℓ_b be the support line of $\Sigma(L, x)$ through b and parallel to ℓ_a (this line exist since $[a, b]$ is an affine diameter of $\Sigma(L, x)$), and let H_b be the support plane of L through ℓ_b and x.

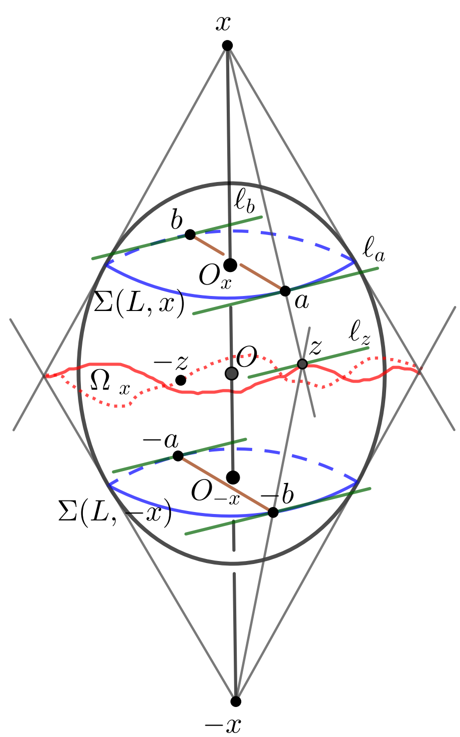

Figure 2: Ω_x is a planar and closed curve

Now, since Ω_x is an O-symmetric set, the line $-\ell_z$ is a support line of Ω_x through $-z$. The plane $H_{a'}$ through $-\ell_z$ and x contains the line $\ell(x,-z)$, hence, the line $\ell_{a'} := \Delta_x \cap H_{a'}$ is a support line of $\Sigma(L,x)$, parallel to ℓ_b. This can happen only if $a' = b$, otherwise we obtain that one of $H_{a'}$ or H_b is not a support plane of L. We have proved that the affine diameter $[a,b]$ passes through O_x, and the same happens for any other affine diameter of $\Sigma(L,x)$, by a theorem of P. C. Hammer [6] we have that $\Sigma(L,x)$ has centre of symmetry at O_x. □

Lemma 2. *For every $x \in \mathrm{bd}K$ we have that Ω_x is a planar curve parallel to $\Sigma(L,x)$.*

Proof. We use the notation of Lemma 1 and Fig. 2. Since the points x, b, $-z$, z, a, are coplanar and O_x and O are the midpoints of the segments $[b,a]$ and $[-z,z]$, respectively, by elementary Geometry we have that $[-z,z]$ must be parallel to $[b,a]$. It follows that Ω_x is parallel to $\Sigma(L,x)$, indeed, they are homothetic with centre of homothety at x. □

Remark 1. *If K is a Euclidean ball then we can prove at this point that L is also a Euclidean ball. It is not difficult to prove that for every $x \in \mathrm{bd}K$, by Lemma 2, the intersection $S(L,x) \cap S(L,-x)$ is a planar curve contained in $x^{\perp}$. It was proved in [5] (Theorem 2) that under this condition the body L is a Euclidean ball.*

Lemma 3. *For every $u \in \mathbb{S}^2$ there exists $v(u) \in \mathbb{S}^2$ such that for every $x \in u^{\perp} \cap \mathrm{bd}K$, the plane Δ_x is parallel to $v(u)$.*

Proof. Consider a point $x \in u^{\perp} \cap \mathrm{bd}K$, and let $y, -y \in \mathrm{bd}K$ be such that the planes Δ_y, and Δ_{-y} are parallel to $u^{\perp}$. We claim that Δ_x is parallel to $\ell(y,-y)$. Suppose to the contrary that Δ_x is not parallel to $\ell(y,-y)$. By virtue of Lemma 1 we have that

$$\Sigma(L,-y) = \Sigma(L,y) - 2 \cdot O_y. \tag{1}$$

Let $\mu := \mathrm{conv}(\Sigma(L,x) \cap \Sigma(L,y))$ be the chord with extreme points in the intersection $\Sigma(L,x) \cap \Sigma(L,y)$. Notice that the chord μ is well defined since K is almost free with respect to L. By (1), the chord $\mu - 2 \cdot O_y$ belongs to Δ_{-y} but, since Δ_x is not parallel to $\ell(y,-y)$, this chord is not contained in Δ_x. On the other hand, by virtue that Δ_y and Δ_{-y} are at the same distance from $u^{\perp}$ and given that $\Sigma(L,x)$ has centre at $O_x \in u^{\perp}$, the image ℓ of μ, under the central symmetry with respect to O_x restricted to the plane Δ_x, is in Δ_{-y} (see Fig. 3). Applying the same argument for $-x$, it follows that $\Sigma(L,-y)$ has four parallel chords of the same length which contradicts the strict convexity of L. Thus Δ_x is parallel to $\ell(y,-y)$. Finally we define $v(u)$ as the unit vector parallel to $\ell(y,-y)$. □

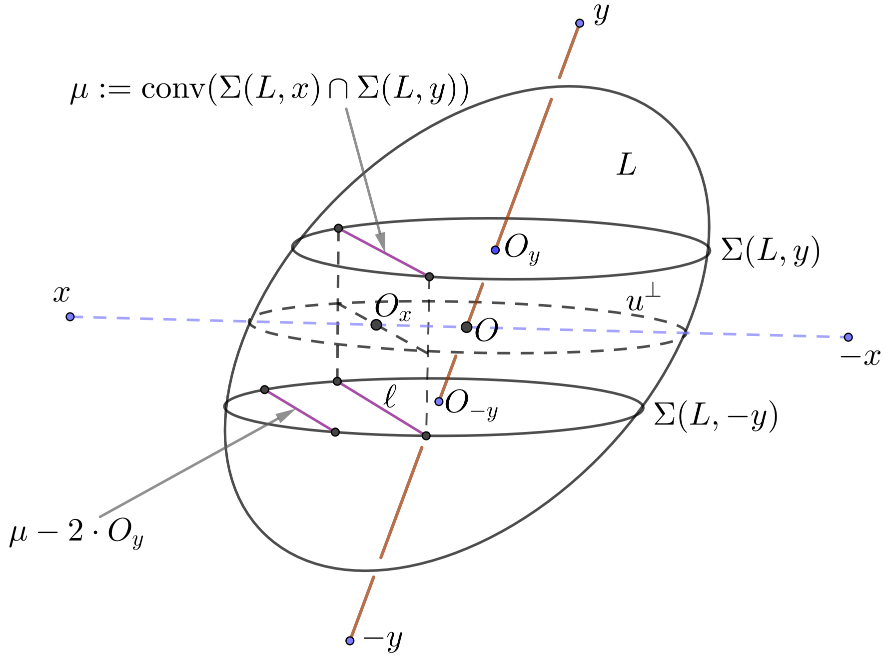

Figure 3: Δ_x is parallel to $\ell(y,-y)$

Proof of Theorem 1. Let x be a point in bd$K$ and we define the unit vector $z = \frac{x}{|x|}$. Let $u \in z^\perp \cap \mathbb{S}^2$. We are going to prove that $u^\perp \cap \Delta_x$ is a line of affine symmetry of $\Sigma(L,x)$. In order to show this, we are going to prove that through the extreme points of the chords of $\Sigma(L,x)$ parallel to $v(u)$, $v(u)$ given by Lemma 3, there passes support lines of $\Sigma(L,x)$ that intersect each other in a point in $u^\perp \cap \Delta_x$. By this property and since one of the chords pass through the centre of $\Sigma(L,x)$, we have by Lemma 3 in [7] that $u^\perp \cap \Delta_x$ is a line of affine symmetry for $\Sigma(L,x)$.

Let $y \in \mathrm{bd}(u^\perp \cap K)$ be a point such that line $\ell(x,y)$ is contained in $u^\perp \backslash C(L,x)$ (see Fig. 4). We denote by Γ_1, Γ_2 the supporting planes of L containing the line $\ell(x,y)$, by a,b the common points between bd$L$ and Γ_1, Γ_2, respectively. Since Γ_1, Γ_2 are supporting planes of $C(L,x)$ it follows $a,b \in \Sigma(L,x)$. Analogously we conclude that $a,b \in \Sigma(L,y)$. Hence $\ell(a,b) = \Delta_x \cap \Delta_y$. By Lemma 3, Δ_x and Δ_y are parallel to $v(u)$. Thus the line $\Delta_x \cap \Delta_y$ is parallel to $v(u)$. We denote by L_1, L_2 the lines $\Gamma_1 \cap \Delta_x, \Gamma_2 \cap \Delta_x$, respectively, and by c the point $\ell(x,y) \cap \Delta_x$. It is clear that $c \in u^\perp \cap \Delta_x$ and that L_1, L_2 are supporting lines of $\Sigma(L,x)$ passing through c and a and c and b, respectively. Varying $y \in \mathrm{bd}(u^\perp \cap K)$, such that line $\ell(x,y)$ is contained in $u^\perp \backslash C(L,x)$ it follows that $\Sigma(L,x)$ satisfies the required property and, consequently, $u^\perp \cap \Delta_x$ is a line of affine symmetry of $\Sigma(L,x)$. Finally varying $u \in (z^\perp \cap \mathbb{S}^2)$ we conclude that $\Sigma(L,x)$ is an ellipse. Finally, we conclude, using Theorem 5 in [5] that L is an ellipsoid. □

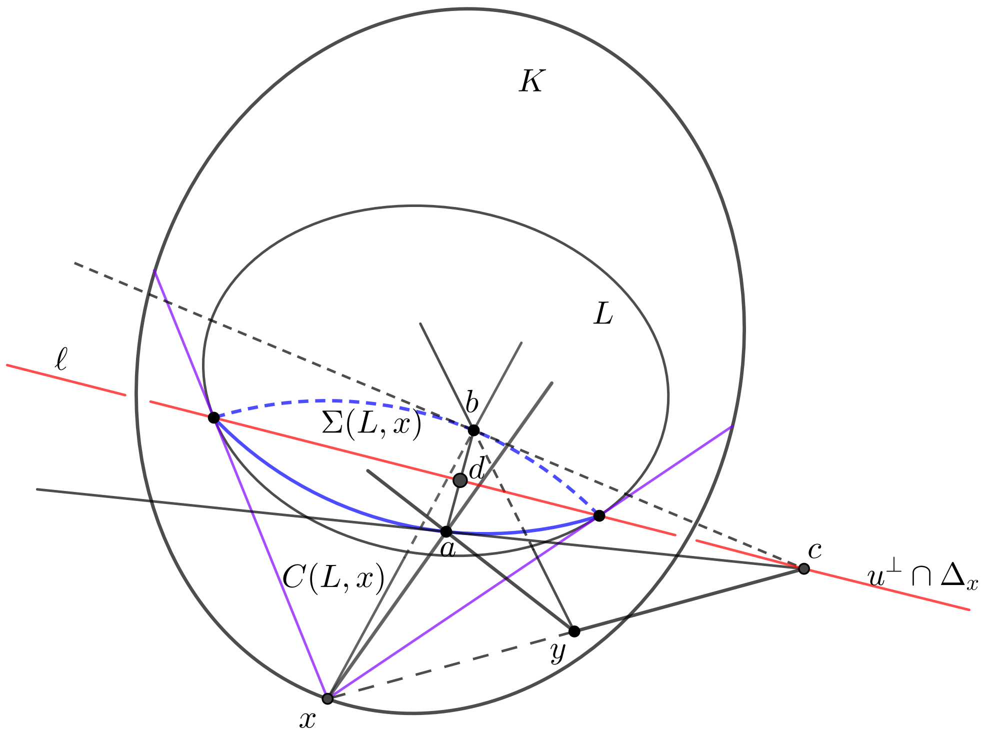

Figure 4: $u^{\perp} \cap \Delta_x$ is a line of affine symmetry for $\Sigma(L, x)$

References

[1] P. W. Aitchison, C. M. Petty, and C. A. Rogers, *A convex body with a false centre is an ellipsoid.* Mathematika **18** (1971), 50 – 59.

[2] J. A. Barker, and D. G. Larman, *Determination of convex bodies by certain sets of sectional volumes.* Discrete Math. **241** (2001), 79 – 96.

[3] G. Bianchi, and P. M. Gruber, *Characterizations of ellipsoids.* Arch. Math. (Basel) **49** (1987), no. 4, 344 – 350.

[4] W. Blaschke, *Kreis und Kugel.* Göschen, Leipzig, (1916).

[5] I. González-García, J. Jerónimo-Castro, E. Morales-Amaya, and D. J. Verdusco-Hernández, *Sections and projections of nested convex bodies.* Aequat. Math. **96** (2022), 885 – 900.

[6] P. C. Hammer, *Diameters of convex bodies.* Proc. Amer. Math. Soc. **5** (1954), 304 − 306.

[7] J. Jerónimo-Castro, *A characterization of the ellipse related to illumination bodies.* Elem. Math. **70** (2015), 95 − 102.

[8] J. Jerónimo-Castro, and T. B. McAllister, *Two characterizations of ellipsoidal cones.* J. Convex Anal. **20** (2013), 1181 − 1187.

[9] A. Marchaud, *Un théoreme sur les corps convexes.* Ann. Scient. École Norm. Supér. **76** (1959), 283 − 304.

[10] L. Montejano, and E. Morales-Amaya, *Variations of classic characterizations of ellipsoids and a short proof of the false centre theorem.* Mathematika **54** (2007), no. 1-2, 35 − 40.

[11] S. P. Olovjanishnikov, *On a characterization of the ellipsoid.* Ucen. Zap. Leningrad. State Univ. Ser. Mat. **83** (1941), 114 − 128.

[12] C. A. Rogers, *Sections and projections of convex bodies.* Port. Math. **24** (1965), 99 − 103.