SMALL BALL PROBABILITIES FOR THE FRACTIONAL STOCHASTIC HEAT EQUATION DRIVEN BY A COLORED NOISE

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ABSTRACT. We consider the fractional stochastic heat equation on the d-dimensional torus $\mathbb{T}^d := [-1,1]^d$, $d \geq 1$, with periodic boundary conditions:

$$\partial_t u(t,x) = -(-\Delta)^{\alpha/2} u(t,x) + \sigma(t,x,u)\dot{F}(t,x) \quad x \in \mathbb{T}^d, t \in \mathbb{R}^+,$$

where $\alpha \in (1,2]$ and $\dot{F}(t,x)$ is a white in time and colored in space noise. We assume that σ is Lipschitz in u and uniformly bounded. We provide small ball probabilities for the solution u when $u(0,x) \equiv 0$.

1. Introduction

In this paper we consider small ball probabilities for solutions to the fractional stochastic heat equation of the type:

$$\partial_t u(t,\mathbf{x}) = -(-\Delta)^{\alpha/2} u(t,\mathbf{x}) + \sigma(t,\mathbf{x},u)\dot{F}(t,\mathbf{x}) \quad \mathbf{x} \in \mathbb{T}^d, t \in \mathbb{R}^+, \tag{1.1}$$

with given initial profile $u(0,\cdot) = u_0 : \mathbb{T}^d \to \mathbb{R}$ where $\mathbb{T}^d := [-1,1]^d$ is a d-dimensional torus. The operator $-(-\Delta)^{\alpha/2}$, where $1 < \alpha \leq 2$, is the fractional power Laplacian on $\mathbb{T}^d$. The centered Gaussian noise $\dot{F}$ is white in time and colored in space, i.e.,

$$\mathbb{E}\left(\dot{F}(t,\mathbf{x}), \dot{F}(s,\mathbf{y})\right) = \delta_0(t-s)\Lambda(\mathbf{x}-\mathbf{y}),$$

where δ_0 is the Dirac delta generalized function and $\Lambda : \mathbb{T}^d \to \mathbb{R}_+$ is a nonnegative generalized function whose Fourier series is given by

$$\Lambda(\mathbf{x}) = \sum_{\mathbf{n}\in\mathbb{Z}^d} \lambda(\mathbf{n}) \exp(\pi i \mathbf{n}\cdot\mathbf{x}) \tag{1.2}$$

where $\mathbf{n}\cdot\mathbf{x}$ represents the dot product of two d-dimensional vectors. We will need the following two assumptions on the function $\sigma : \mathbb{R}_+ \times \mathbb{T}^d \times \mathbb{R} \to \mathbb{R}$.

Hypothesis 1.1. *There exists a constant $\mathcal{D} > 0$ such that for all $t \geq 0$, $\mathbf{x} \in \mathbb{T}^d$, $u, v \in \mathbb{R}$,*

$$|\sigma(t,\mathbf{x},u) - \sigma(t,\mathbf{x},v)| \leq \mathcal{D}|u-v|. \tag{1.3}$$

Hypothesis 1.2. *There exist constants $\mathcal{C}_1$, $\mathcal{C}_2 > 0$ such that for all $t \geq 0$, $\mathbf{x} \in \mathbb{T}^d$, $u \in \mathbb{R}$,*

$$\mathcal{C}_1 \leq \sigma(t,\mathbf{x},u) \leq \mathcal{C}_2. \tag{1.4}$$

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In fact, (1.1) is not well-posed since the solution u is not differentiable and $\dot{F}$ exists as a generalized function. However, under the assumptions (1.3) and (1.4), we define the mild solution $u(t,\mathbf{x})$ to (1.1) in the sense of Walsh [Wal86] satisfying

$$u(t,\mathbf{x}) = \int_{\mathbb{T}^d} \bar{p}(t,\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y} + \int_{[0,t]\times\mathbb{T}^d} \bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y},u(s,\mathbf{y}))F(dsd\mathbf{y}), \tag{1.5}$$

where $\bar{p} : \mathbb{R}_+ \times \mathbb{T}^d \to \mathbb{R}_+$ is the fundamental solution of the fractional heat equation on $\mathbb{T}^d$

$$\begin{aligned} \partial_t \bar{p}(t,\mathbf{x}) &= -(-\Delta)^{\alpha/2}\bar{p}(t,\mathbf{x}) \\ \bar{p}(0,\mathbf{x}) &= \delta_0(\mathbf{x}). \end{aligned} \tag{1.6}$$

Following [Dal99], it is well known (see also [DKM+09]) that if $\lambda(\mathbf{n})$, the Fourier coefficients of $\Lambda(\mathbf{x})$, satisfy

$$\sum_{\mathbf{n}\in\mathbb{Z}^d} \frac{\lambda(\mathbf{n})}{1+|\mathbf{n}|^\alpha} < \infty, \tag{1.7}$$

where $|\cdot|$ is the Euclidean norm, then there exists a unique random field solution $u(t,\mathbf{x})$ to equation (1.5). Examples of spatial correlation satisfying (1.7) are:

1. The Riesz kernel $\Lambda(\mathbf{x}) = |\mathbf{x}|^{-\beta}$, $0<\beta<d$. In this case, there exist positive constants c_1, c_2 such that for all $\mathbf{n}\in\mathbb{Z}^d$,
$$c_1|\mathbf{n}|^{-(d-\beta)} \le \lambda(\mathbf{n}) \le c_2|\mathbf{n}|^{-(d-\beta)}, \tag{1.8}$$
and it is easy to check that condition (1.7) holds whenever $\beta<\alpha$.
2. The space-time white noise case $\Lambda(\mathbf{x}) = \delta_0(\mathbf{x})$. In this case, $\lambda(\mathbf{n})$ is a constant and (1.7) is only satisfied when $\alpha > d$, that is, $d=1$ and $1<\alpha\le 2$.

Small ball probability problems have a long history, and one can see [LS01] for more surveys. In short, we are interested in the probability that a stochastic process X_t starting at 0 stays in a small ball for a long time period, i.e.,

$$P\left(\sup_{0\le t\le T}|X_t| < \varepsilon\right)$$

where $\varepsilon>0$ is small. A recent paper [AJM21] has studied this problem when X_t is the solution of the stochastic heat equation with $d=1$, $\alpha=2$ and $\Lambda=\delta_0$. The objective of this paper is to generalize their results with the Riesz kernel.

2. Main Result

Theorem 2.1. *Under the assumptions* (1.3) *and* (1.4)*, if* $u(t,\mathbf{x})$ *is the solution to* (1.1) *with* $u_0(\mathbf{x})\equiv 0$*, then there are positive constants* $\boldsymbol{C}_0, \boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \mathcal{D}_0$ *depending only on* $\mathcal{C}_1,\mathcal{C}_2$*,* α*,* β *and* d*, such that for all* $\mathcal{D}<\mathcal{D}_0, \varepsilon_0>\varepsilon>0$*,* $T>1$*, we have*

(a) *when* $d=1$ *and* $\alpha\ge 2\beta$*,*

$$\boldsymbol{C}_0 \exp\left(-\frac{\boldsymbol{C}_1 T}{\varepsilon^{\frac{2(2\alpha-\beta)}{\beta}}}\right) < P\left(\sup_{\substack{0\le t\le T\\ \mathbf{x}\in\mathbb{T}^d}}|u(t,\mathbf{x})| \le \varepsilon\right) < \boldsymbol{C}_2 \exp\left(-\frac{\boldsymbol{C}_3 T}{\varepsilon^{\frac{2(\alpha+\beta)}{\alpha-\beta}}}\right),$$

(b) *or in other cases,*

$$0 \leq P\left(\sup_{\substack{0\leq t\leq T\\ \mathbf{x}\in\mathbb{T}^d}} |u(t,\mathbf{x})| \leq \varepsilon\right) < \mathbf{C_2}\exp\left(-\frac{\mathbf{C_3}T}{\varepsilon^{\frac{2\alpha}{\alpha-\beta}\left(\left(1+\frac{\beta}{\alpha d}\right)\wedge\left(\frac{2\alpha-\beta}{\alpha}\right)\right)}}\right).$$

Here we make a couple of remarks. These could be of independent interests.

Remark 2.1. (a) *The lack of lower bound for small ball probability in part (b) is due to an exponential growth number of grids in space.*

(b) *When $d \geq 2$ and $\Lambda(\mathbf{x}) = \delta_0(\mathbf{x})$, the solution exists as a distribution. Is there a way to estimate the small probability for some norm of this solution?*

(c) *The small ball probability estimation has a close relation with the Chung's type Law of the Iterated Logarithm (see* [LS01] *for more details). Can we follow the idea from* [LX21] *to get a similar result for non-Gaussian random fields/strings?*

Here is the organization of this paper. In Section 3 we state the key proposition and how this proposition relates to the main result. In Section 4 we give some useful estimations. In Section 5 we prove the key proposition.

Throughout the entire paper, C and C' denote positive constants whose values may vary from line to line. The dependence of constants on parameters will be denoted by mentioning the parameters in parenthesis.

3. Key Proposition

We decompose $[-1,1]$ into intervals of length ε^2 on each dimension and divide $[0,T]$ into intervals of length $c_0\varepsilon^4$ where c_0 satisfies

$$0 < c_0 < \min\left\{1, \left(\frac{C_6}{36\mathcal{C}_2^2 \ln C_5}\right)^{\frac{\alpha}{\alpha-\beta}}\right\} \tag{3.1}$$

where C_5, C_6 are constants specified in Lemma 4.5. Moreover, for $\forall\varepsilon > 0$ and $\mathcal{C}$ is specified in Lemma 5.2, we require

$$0 < c_0 < \mathcal{C}\varepsilon^{\frac{2\alpha d-4\beta}{\beta}}. \tag{3.2}$$

Remark 3.1. *Unlike the white noise case in* [AJM21]*, c_0 needs to be selected depending on ε in this paper. Indeed, c_0 does not appear in the bounds for small ball probability.*

Define $t_i = ic_0\varepsilon^4, x_j = j\varepsilon^2$ and

$$n_1 := \min\{n \in \mathbb{Z} : n\varepsilon^2 > 1\},$$

where $i \in \mathbb{N}$ and $j \in \mathbb{Z}$. Consider a sequence of sets $R_{i,j} \subset \mathbb{R}\times\mathbb{R}^d$ as

$$R_{i,j} = \left\{(t_i, x_{j_1}, x_{j_2}, ...x_{j_d})\,|\, -n_1+1 \leq j_k \leq j, k = 1,2,...,d\right\}. \tag{3.3}$$

By symmetry, $(x_{j_1}, x_{j_2}, ...x_{j_d})$ lies in $[-1,1]^d$ when

$$-n_1 + 1 \leq j_k \leq n_1 - 1 \text{ for } k = 1,2,...,d \tag{3.4}$$

For $n \geq 0$, we define a sequence of events that we can use for the upper bound in Theorem 2.1,

$$F_n = \left\{ |u(t, \mathbf{x})| \leq t_1^{\frac{\alpha-\beta}{2\alpha}} \text{ for all } (t, \mathbf{x}) \in R_{n,n_1-1} \right\}. \tag{3.5}$$

In addition, let $E_{-1} = \Omega$ and for $n \geq 0$, we define a sequence of events that we can use for the lower bound in Theorem 2.1,

(3.6)

$$E_n = \left\{ |u(t_{n+1}, \mathbf{x})| \leq \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \text{ and } |u(t, \mathbf{x})| \leq \varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \text{ for all } t \in [t_n, t_{n+1}], \mathbf{x} \in [-1, 1]^d \right\}.$$

The following proposition along with the Markov property will lead to Theorem 2.1.

Proposition 3.1. *Consider the solution to* (1.1) *with* $u_0(\mathbf{x}) \equiv 0$. *Then, there exist* $\varepsilon_1 > 0$ *and* $\mathbf{C_4}, \mathbf{C_5}, \mathbf{C_6}, \mathbf{C_7}, \mathcal{D}_0 > 0$ *depending only on* $\mathcal{C}_1$, $\mathcal{C}_2$, α, β, *and* d *such that for any* $0 < \varepsilon < \varepsilon_1$ *and* $\mathcal{D} < \mathcal{D}_0$,

(a)

$$P\left(F_n \middle| \bigcap_{k=0}^{n-1} F_k \right) \leq \mathbf{C_4} \exp\left(-\frac{\mathbf{C_5}}{\varepsilon^2 + \mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}} \right),$$

(b) *and when* $d = 1$ *and* $\alpha \geq 2\beta$,

$$P\left(E_n \middle| \bigcap_{k=-1}^{n-1} E_k \right) \geq \mathbf{C_6} \exp\left(-\frac{\mathbf{C_7}}{\varepsilon^{\frac{4(\alpha-\beta)^2}{\alpha\beta}}} \right).$$

Next we show how Theorem 2.1 follows from Proposition 3.1.

Proof of Theorem 2.1: The event F_n deals with $u(t, \mathbf{x})$ at the time t_n, so putting these events together indicates

$$F := \bigcap_{n=0}^{\left\lfloor \frac{T}{t_1} \right\rfloor} F_n \supset \left\{ |u(t, \mathbf{x})| \leq t_1^{\frac{\alpha-\beta}{2\alpha}}, t \in [0, T], \mathbf{x} \in [-1, 1]^d \right\},$$

and

$$P(F) = P\left(\bigcap_{n=0}^{\left\lfloor \frac{T}{t_1} \right\rfloor} F_n \right) = P(F_0) \prod_{n=1}^{\left\lfloor \frac{T}{t_1} \right\rfloor} P\left(F_n \middle| \bigcap_{k=0}^{n-1} F_k \right).$$

With $u_0(\mathbf{x}) \equiv 0$, Proposition 3.1 immediately yields

$$\begin{aligned} P(F) &\leq \left[\mathbf{C_4} \exp\left(-\frac{\mathbf{C_5}}{\varepsilon^2 + \mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}} \right) \right]^{\left\lfloor \frac{T}{t_1} \right\rfloor} \leq \mathbf{C_4'} \exp\left(-\frac{\mathbf{C_5'} T}{\varepsilon^2 t_1 + \mathcal{D}^2 t_1^{\frac{2\alpha-\beta}{\alpha}}} \right) \\ &\leq \mathbf{C_2} \exp\left(-\frac{\mathbf{C_3} T}{t_1^{\left(1+\frac{\beta}{\alpha d}\right) \wedge \left(\frac{2\alpha-\beta}{\alpha}\right)}} \right). \end{aligned}$$

The last inequality follows from the inequality of c_0 in (3.2) and $\mathcal{D} < \mathcal{D}_0$. Therefore we have

$$P\left(\left\{|u(t,\mathbf{x})| \le t_1^{\frac{\alpha-\beta}{2\alpha}}, t \in [0,T], \mathbf{x} \in [-1,1]^d\right\}\right) < \mathbf{C_2} \exp\left(-\frac{\mathbf{C_3}T}{t_1^{\left(1+\frac{\beta}{\alpha d}\right)\wedge\left(\frac{2\alpha-\beta}{\alpha}\right)}}\right),$$

then replacing $t_1^{\frac{\alpha-\beta}{2\alpha}}$ with ε and adjusting ε_1 to ε_0 give the upper bound in Theorem 2.1.

For the lower bound, the event E_n deals with $u(t,\mathbf{x})$ in the time interval $[t_n, t_{n+1}]$, so putting these events together indicates

$$E := \bigcap_{n=-1}^{\left\lfloor \frac{T}{t_1}\right\rfloor - 1} E_n \subset \left\{|u(t,\mathbf{x})| \le \varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, t \in [0,T], \mathbf{x} \in [-1,1]^d\right\},$$

and

$$P(E) = P\left(\bigcap_{n=-1}^{\left\lfloor \frac{T}{t_1}\right\rfloor - 1} E_n\right) = P(E_{-1}) \prod_{n=0}^{\left\lfloor \frac{T}{t_1}\right\rfloor - 1} P\left(E_n \middle| \bigcap_{k=-1}^{n-1} E_k\right).$$

With $u_0(\mathbf{x}) \equiv 0$, Proposition 3.1 immediately yields

$$P(E) \ge \left[\mathbf{C_6} \exp\left(-\frac{\mathbf{C_7}}{\varepsilon^{\frac{4(\alpha-\beta)^2}{\alpha\beta}}}\right)\right]^{\frac{T}{t_1}} \ge \mathbf{C_0} \exp\left(-\frac{\mathbf{C_1}T}{t_1 \varepsilon^{\frac{4(\alpha-\beta)^2}{\alpha\beta}}}\right).$$

Therefore, from the inequality of c_0 in (5.27), we have

$$P\left(\left\{|u(t,\mathbf{x})| \le \varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, t \in [0,T], \mathbf{x} \in [-1,1]^d\right\}\right) > \mathbf{C_0} \exp\left(-\frac{\mathbf{C_1}T}{\varepsilon^{\frac{4(\alpha-\beta)(2\alpha-\beta)}{\alpha\beta}}}\right),$$

then replacing $\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$ with ε and adjusting ε_1 to ε_0 give the lower bound in Theorem 2.1.

4. Preliminary

In this section, we provide some preliminary results that are used to prove the key proposition 3.1.

4.1. **Heat Kernel Estimates.** For $\mathbf{x} \in \mathbb{R}^d$, $p(t,\mathbf{x})$ is the smooth function determined by its Fourier transform in $\mathbf{x}$

$$\hat{p}(t,\nu) := \int_{\mathbb{R}^d} p(t,\mathbf{x}) \exp(2\pi i \nu \cdot \mathbf{x}) d\mathbf{x} = \exp(-t(2\pi|\nu|)^\alpha), \quad \nu \in \mathbb{R}^d.$$

For $\mathbf{x} \in \mathbb{T}^d$, from the standard Fourier decomposition we have

$$\bar{p}(t,\mathbf{x}) = 2^{-d} \sum_{\mathbf{n}\in\mathbb{Z}^d} \exp\left(-\pi^\alpha |\mathbf{n}|^\alpha t\right) \exp(\pi i \mathbf{n} \cdot \mathbf{x}). \tag{4.1}$$

The following lemma gives an estimation on heat kernel $\bar{p}(t,\mathbf{x})$, which is similar to Lemma 2.1 and Lemma 2.2 in [Li17].

Lemma 4.1. *For all $t \geq s > 0$ and $\mathbf{x}, \mathbf{y} \in \mathbb{T}^d$, there exist constants $C, C' > 0$ depending only on α, d such that*

$$\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{y}-\mathbf{x}) - \bar{p}(t, \mathbf{y})| d\mathbf{y} \leq C \left(\frac{|\mathbf{x}|}{t^{1/\alpha}} \wedge 1 \right), \tag{4.2}$$

$$\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{x}) - \bar{p}(s, \mathbf{x})| d\mathbf{x} \leq C' \left(\log\left(\frac{t}{s}\right) \wedge 1 \right). \tag{4.3}$$

Proof. We begin with inequality (4.2),

$$\begin{aligned}
\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{y}-\mathbf{x}) - \bar{p}(t, \mathbf{y})| d\mathbf{y} &= \int_{\mathbb{T}^d} \left| \sum_{\mathbf{n} \in \mathbb{Z}^d} [p(t, \mathbf{y}-\mathbf{x}+2\mathbf{n}) - p(t, \mathbf{y}+2\mathbf{n})] \right| d\mathbf{y} \\
&\leq \int_{\mathbb{T}^d} \sum_{\mathbf{n} \in \mathbb{Z}^d} |p(t, \mathbf{y}-\mathbf{x}+2\mathbf{n}) - p(t, \mathbf{y}+2\mathbf{n})| \, d\mathbf{y} \\
&= \int_{\mathbb{R}^d} |p(t, \mathbf{y}-\mathbf{x}) - p(t, \mathbf{y})| d\mathbf{y} \\
&\leq \int_{\mathbb{R}^d} |\mathbf{x}| \cdot \sup_{c_0 \in [0,1]} |\nabla_{\mathbf{z}} p(t, \mathbf{y} - c_0 \mathbf{x})| \, d\mathbf{y}.
\end{aligned} \tag{4.4}$$

By Lemma 5 in [BJ07] and (2.3) of [JS16], we have

$$|\nabla_{\mathbf{z}} p(t, \mathbf{z})| \leq C(d, \alpha) |\mathbf{z}| \left(\frac{t}{|\mathbf{z}|^{d+2+\alpha}} \wedge t^{-(d+2)/\alpha} \right) \leq C(d, \alpha) \frac{t|\mathbf{z}|}{(t^{1/\alpha} + |\mathbf{z}|)^{d+2+\alpha}}. \tag{4.5}$$

We put (4.5) into (4.4) to get

$$\begin{aligned}
\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{y}-\mathbf{x}) - \bar{p}(t, \mathbf{y})| d\mathbf{y} &\leq C(d, \alpha) |\mathbf{x}| \int_{\mathbb{R}^d} \frac{t|\mathbf{y}|}{(t^{1/\alpha} + |\mathbf{y}|)^{d+2+\alpha}} d\mathbf{y} \\
&\leq C(d, \alpha) |\mathbf{x}| \int_0^\infty \frac{tx}{(t^{1/\alpha} + x)^{d+2+\alpha}} x^{d-1} dx \\
&= \frac{C(d, \alpha)|\mathbf{x}|}{t^{1/\alpha}} \int_0^\infty \frac{w^d}{(1+w)^{d+2+\alpha}} dw \\
&\leq \frac{C(d, \alpha)|\mathbf{x}|}{t^{1/\alpha}}.
\end{aligned}$$

Clearly, $\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{y}-\mathbf{x}) - \bar{p}(t, \mathbf{y})| d\mathbf{y} \leq 2$, so that (4.2) follows. For inequality (4.3), we have

$$\begin{aligned}
\int_{\mathbb{T}^d} |\bar{p}(t, \mathbf{x}) - \bar{p}(s, \mathbf{x})| d\mathbf{x} &= \int_{\mathbb{T}^d} \left| \sum_{\mathbf{n} \in \mathbb{Z}^d} [p(t, \mathbf{x}+2\mathbf{n}) - p(s, \mathbf{x}+2\mathbf{n})] \right| d\mathbf{x} \\
&\leq \int_{\mathbb{T}^d} \sum_{\mathbf{n} \in \mathbb{Z}^d} |p(t, \mathbf{x}+2\mathbf{n}) - p(s, \mathbf{x}+2\mathbf{n})| \, d\mathbf{x} \\
&= \int_{\mathbb{R}^d} |p(t, \mathbf{x}) - p(s, \mathbf{x})| d\mathbf{x} \\
&\leq \int_{\mathbb{R}^d} \int_s^t |\partial_r p(r, \mathbf{x})| dr d\mathbf{x}.
\end{aligned} \tag{4.6}$$

Proposition 2.1 in [VdPQR17] shows

$$|(-\Delta)^{\alpha/2}p(r,\mathbf{x})| \le \frac{C(d,\alpha)}{(r^{2/\alpha}+|\mathbf{x}|^2)^{\frac{d+\alpha}{2}}}. \tag{4.7}$$

Applying (1.6), (4.7) and Fubini's theorem to (4.6) yields that

$$\begin{aligned}\int_{\mathbb{T}^d}|\bar{p}(t,\mathbf{x})-\bar{p}(s,\mathbf{x})|d\mathbf{x} &\le C(d,\alpha)\int_{\mathbb{R}^d}\int_s^t \frac{1}{(r^{2/\alpha}+|\mathbf{x}|^2)^{\frac{d+\alpha}{2}}}drd\mathbf{x}\\ &= C(d,\alpha)\int_s^t\int_0^\infty \frac{x^{d-1}}{(r^{2/\alpha}+x^2)^{\frac{d+\alpha}{2}}}dxdr\\ &= C(d,\alpha)\int_s^t \frac{dr}{r}\int_0^\infty \frac{w^{d-1}}{(1+w^2)^{\frac{d+\alpha}{2}}}dw\\ &\le C(d,\alpha)\left(\log(t)-\log(s)\right).\end{aligned}$$

Similarly, $\int_{\mathbb{T}^d}|\bar{p}(t,\mathbf{x})-\bar{p}(s,\mathbf{x})|d\mathbf{x} \le 2$, so that (4.3) follows. □

4.2. **Noise Term Estimates.** We denote the second integral of (1.5), i.e. noise term, by

$$N(t,\mathbf{x}) := \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y},u(s,\mathbf{y}))F(dsd\mathbf{y}), \tag{4.8}$$

We will now estimate the regularity of $N(t,\mathbf{x})$ in the following two lemmas.

Lemma 4.2. *There exists a constant $C>0$ depending only on α,β,d and $\mathcal{C}_2$ in (1.4) such that for any $\xi \in \left(0,\frac{1}{\alpha}\wedge\frac{\alpha-\beta}{\alpha}\right)$, $t\in[0,1]$ and $\mathbf{x},\mathbf{y}\in\mathbb{T}^d$, we have*

$$\mathbb{E}\left[(N(t,\mathbf{x})-N(t,\mathbf{y}))^2\right] \le C\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}.$$

Proof. To simplify our computation, fix $t,s,\mathbf{x},\mathbf{y}$ and we denote

$$K(\mathbf{z}) := \bar{p}(t-s,\mathbf{x}-\mathbf{z})-\bar{p}(t-s,\mathbf{y}-\mathbf{z}).$$

Using Fubini's theorem, (1.4) and the triangle inequality, we have

$$\begin{aligned}&\mathbb{E}\left[(N(t,\mathbf{x})-N(t,\mathbf{y}))^2\right]\\ &=\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}K(\mathbf{z})K(\mathbf{w})\Lambda(\mathbf{w}-\mathbf{z})\mathbb{E}[\sigma(s,\mathbf{z},u(s,\mathbf{z}))\sigma(s,\mathbf{w},u(s,\mathbf{w}))]d\mathbf{w}d\mathbf{z}ds\\ &\le \sup_{r,\mathbf{u}}\mathbb{E}\left[\sigma(r,\mathbf{u},u(r,\mathbf{u}))^2\right]\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|K(\mathbf{z})||K(\mathbf{w})|\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}d\mathbf{z}ds\\ &\le \mathcal{C}_2^2\int_0^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|K(\mathbf{z})|[\bar{p}(t-s,\mathbf{x}-\mathbf{w})+\bar{p}(t-s,\mathbf{y}-\mathbf{w})]\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}d\mathbf{z}ds.\end{aligned} \tag{4.9}$$

Then we use the standard Fourier decomposition (4.1) to estimate the spatial convolution,

$$\begin{aligned}
\int_{\mathbb{T}^d} \bar{p}(t-s,\mathbf{x}-\mathbf{w})\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w} &= C(d)\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\exp(-\pi^\alpha|\mathbf{n}|^\alpha(t-s))\exp(\pi i\mathbf{n}\cdot(\mathbf{x}-\mathbf{z})) \\
&\le C(d)\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\exp(-\pi^\alpha|\mathbf{n}|^\alpha(t-s)) \\
&\le C(d)\sum_{\mathbf{n}\in\mathbb{Z}^d}|\mathbf{n}|^{-d+\beta}\exp(-\pi^\alpha|\mathbf{n}|^\alpha(t-s)) \\
&\le C(d)\int_0^\infty x^{-d+\beta}\exp(-\pi^\alpha x^\alpha(t-s))x^{d-1}dx \\
&= C(d)(t-s)^{-\beta/\alpha}\int_0^\infty x^{\frac{\beta-\alpha}{\alpha}}\exp(-x)dx \\
&= C(\alpha,\beta,d)(t-s)^{-\beta/\alpha}.
\end{aligned} \tag{4.10}$$

We can get a similar result for $\int_{\mathbb{T}^d}\bar{p}(t-s,\mathbf{y}-\mathbf{w})\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}$. Applying (4.10), Lemma 4.1 to (4.9) and since $1\wedge x < x^{\alpha\xi}$ for all $x>0, \xi\in(0,1/\alpha)$, we get

$$\begin{aligned}
\mathbb{E}\left[(N(t,\mathbf{x})-N(t,\mathbf{y}))^2\right] &\le C(\alpha,\beta,d)\mathcal{C}_2^2\int_0^t\int_{\mathbb{T}^d}|K(\mathbf{z})|(t-s)^{-\beta/\alpha}d\mathbf{z}ds \\
&\le C(\alpha,\beta,d)\mathcal{C}_2^2\int_0^t(t-s)^{-\beta/\alpha}\left(\frac{|\mathbf{x}-\mathbf{y}|}{(t-s)^{1/\alpha}}\wedge 1\right)ds \\
&\le C(\alpha,\beta,d)\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}\int_0^t(t-s)^{-\xi-\beta/\alpha}ds \\
&\le C(\alpha,\beta,d)\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}.
\end{aligned}$$

Note that the integral $\int_0^t(t-s)^{-\xi-\beta/\alpha}ds$ converges provided $\xi\in\left(0,\frac{1}{\alpha}\wedge\frac{\alpha-\beta}{\alpha}\right)$. □

Lemma 4.3. *There exists a constant $C>0$ depending only on α,β,d and $\mathcal{C}_2$ in (1.4) such that for any $\zeta\in\left(0,\frac{\alpha-\beta}{\alpha}\right)$, $1\ge t\ge s>0$ and $\mathbf{x},\mathbf{y}\in\mathbb{T}^d$, we have*

$$\mathbb{E}\left[(N(t,\mathbf{x})-N(s,\mathbf{x}))^2\right]\le C\mathcal{C}_2^2|t-s|^\zeta.$$

Proof. Using Fubini's theorem, (1.4) and the triangle inequality, we have

$$\begin{aligned}
&\mathbb{E}[(N(t,\mathbf{x})-N(s,\mathbf{x}))^2]\\
&=\mathbb{E}\left[\left(\int_0^s\int_{\mathbb{T}^d}\left[\bar{p}(t-r,\mathbf{x}-\mathbf{z})-\bar{p}(s-r,\mathbf{x}-\mathbf{z})\right]\sigma(r,\mathbf{z},u(r,\mathbf{z}))F(d\mathbf{z}dr)\right.\right.\\
&\qquad\left.\left.+\int_s^t\int_{\mathbb{T}^d}\bar{p}(t-r,\mathbf{x}-\mathbf{z})\sigma(r,\mathbf{z},u(r,\mathbf{z}))F(d\mathbf{z}dr)\right)^2\right]\\
&\le\sup_{r,\mathbf{u}}\mathbb{E}\left[\sigma(r,\mathbf{u},u(r,\mathbf{u}))^2\right]\left(\int_0^s\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}|\bar{p}(t-r,\mathbf{x}-\mathbf{z})-\bar{p}(s-r,\mathbf{x}-\mathbf{z})|\cdot\right.\\
&\qquad[\bar{p}(t-r,\mathbf{x}-\mathbf{w})+\bar{p}(s-r,\mathbf{x}-\mathbf{w})]\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}d\mathbf{z}dr\\
&\qquad\left.+\int_s^t\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\bar{p}(t-r,\mathbf{x}-\mathbf{z})\bar{p}(t-r,\mathbf{x}-\mathbf{w})\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}d\mathbf{z}dr\right)\\
&=:\mathcal{C}_2^2\left(I_1+I_2\right).
\end{aligned}\tag{4.11}$$

Applying (4.10) and Lemma 4.1 to I_1 and since $1\wedge\log(1+x)<x^\zeta$ for all $x>0,\zeta\in(0,1)$, we get

$$\begin{aligned}
I_1&\le C(\alpha,\beta,d)\int_0^s\left(\log\left(\frac{t-r}{s-r}\right)\wedge 1\right)\cdot[(t-r)^{-\beta/\alpha}+(s-r)^{-\beta/\alpha}]dr\\
&\le C(\alpha,\beta,d)\int_0^s\left(\log\left(\frac{t-s+x}{x}\right)\wedge 1\right)x^{-\beta/\alpha}dx\\
&\le C(\alpha,\beta,d)(t-s)^\zeta\int_0^s x^{-\beta/\alpha-\zeta}dx\\
&\le C(\alpha,\beta,d)(t-s)^\zeta.
\end{aligned}\tag{4.12}$$

Note that the integral $\int_0^s x^{-\beta/\alpha-\zeta}ds$ converges provided $\zeta\in\left(0,\frac{\alpha-\beta}{\alpha}\right)$. In order to estimate I_2, we use the standard Fourier decomposition (4.1) to bound the spatial convolution,

$$\begin{aligned}
\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\bar{p}(t-r,\mathbf{x}-\mathbf{z})\bar{p}(t-r,\mathbf{x}-\mathbf{w})\Lambda(\mathbf{w}-\mathbf{z})d\mathbf{w}&=C(d)\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\exp(-2\pi^\alpha|\mathbf{n}|^\alpha(t-r))\\
&\le C(d)\sum_{\mathbf{n}\in\mathbb{Z}^d}|\mathbf{n}|^{-d+\beta}\exp(-2\pi^\alpha|\mathbf{n}|^\alpha(t-r))\\
&\le C(d)\int_0^\infty x^{-d+\beta}\exp(-2\pi^\alpha x^\alpha(t-r))x^{d-1}dx\\
&=C(d)(t-r)^{-\beta/\alpha}\int_0^\infty x^{\frac{\beta-\alpha}{\alpha}}\exp(-x)dx\\
&=C(\alpha,\beta,d)(t-r)^{-\beta/\alpha}.
\end{aligned}$$

Then for I_2 in (4.11), we have

$$I_2\le C(\alpha,\beta,d)\int_s^t(t-r)^{-\beta/\alpha}dr=C(\alpha,\beta,d)(t-s)^{\frac{\alpha-\beta}{\alpha}}.\tag{4.13}$$

By (4.11), (4.12) and (4.13), we conclude

$$\mathbb{E}\left[(N(t,\mathbf{x})-N(s,\mathbf{x}))^2\right]\le C\mathcal{C}_2^2(t-s)^\zeta.$$

□

Lemma 4.4. *There exist constants $C_1, C_2, C_3, C_4 > 0$ depending only on α, β, d and $\mathcal{C}_2$ in (1.4) such that for all $0 \le s < t \le 1$, $\mathbf{x}, \mathbf{y} \in \mathbb{T}^d$, $\xi \in \left(0, \frac{1}{\alpha} \wedge \frac{\alpha-\beta}{\alpha}\right), \zeta \in \left(0, \frac{\alpha-\beta}{\alpha}\right)$, and $\kappa > 0$,*

$$P(|N(t,\mathbf{x}) - N(t,\mathbf{y})| > \kappa) \le C_1 \exp\left(-\frac{C_2\kappa^2}{\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}}\right), \tag{4.14}$$

$$P(|N(t,\mathbf{x}) - N(s,\mathbf{x})| > \kappa) \le C_3 \exp\left(-\frac{C_4\kappa^2}{\mathcal{C}_2^2|t-s|^{\zeta}}\right). \tag{4.15}$$

Proof. For a fixed t, define

$$N_t(s,\mathbf{x}) := \int_{[0,s]\times\mathbb{T}^d} \bar{p}(t-r, \mathbf{x}-\mathbf{y})\sigma(r,\mathbf{y},u(r,\mathbf{y}))F(drd\mathbf{y}).$$

Note that $N_t(t,\mathbf{x}) = N(t,\mathbf{x})$ and $N_t(s,\mathbf{x})$ is a continuous $\mathcal{F}_s^F$ adapted martingale in $s \le t$ since the integrand does not depend on s. For fixed $t, \mathbf{x}$ and $\mathbf{y}$, let

$$M_s := N_t(s,\mathbf{x}) - N_t(s,\mathbf{y}) = \int_{[0,s]\times\mathbb{T}^d} [\bar{p}(t-r,\mathbf{x}-\mathbf{z}) - \bar{p}(t-r,\mathbf{y}-\mathbf{z})]\sigma(r,\mathbf{z},u(r,\mathbf{z}))F(drd\mathbf{z}),$$

and it is easy to check that $M_t = N(t,\mathbf{x}) - N(t,\mathbf{y})$. As M_s is a continuous $\mathcal{F}_s^F$ adapted martingale with $M_0 = 0$, it is a time changed Brownian motion. In particular, we have

$$M_t = B_{\langle M\rangle_t},$$

and Lemma 4.2 gives a uniform bound on the time change as

$$\langle M\rangle_t \le C\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}.$$

Therefore, by the reflection principle for the Brownian motion $B_{\langle M\rangle_t}$,

$$\begin{aligned} P(N(t,\mathbf{x}) - N(t,\mathbf{y}) > \kappa) &= P(M_t > \kappa) = P(B_{\langle M\rangle_t} > \kappa) \\ &\le P\left(\sup_{s \le C\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}} B_s > \kappa\right) = 2P\left(B_{C\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}} > \kappa\right) \\ &\le C_1 \exp\left(-\frac{C_2\kappa^2}{\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}}\right). \end{aligned}$$

Switching $\mathbf{x}$ and $\mathbf{y}$ gives

$$\begin{aligned} P(-N(t,\mathbf{x}) + N(t,\mathbf{y}) > \kappa) = P(M_t < -\kappa) &\le 2P\left(B_{C\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}} < -\kappa\right) \\ &\le C_1 \exp\left(-\frac{C_2\kappa^2}{\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}}\right). \end{aligned}$$

Consequently, for $\forall \xi \in \left(0, \frac{1}{\alpha} \wedge \frac{\alpha-\beta}{\alpha}\right)$,

$$P(|N(t,\mathbf{x}) - N(t,\mathbf{y})| > \kappa) \le C_1 \exp\left(-\frac{C_2\kappa^2}{\mathcal{C}_2^2|\mathbf{x}-\mathbf{y}|^{\alpha\xi}}\right),$$

which completes the proof of (4.14). For a fixed $\mathbf{x}$, we define

$$U_{q_1} = \int_{[0,q_1]\times\mathbb{T}^d} [\bar{p}(t-r,\mathbf{x}-\mathbf{y}) - \bar{p}(s-r,\mathbf{x}-\mathbf{y})]\sigma(r,\mathbf{y},u(r,\mathbf{y}))F(drd\mathbf{y})$$

where $0 \leq q_1 \leq s$. Note U_{q_1} is a continuous $\mathcal{F}^F_{q_1}$ adapted martingale with $U_0 = 0$. Also define

$$V_{q_2} = \int_{[s,s+q_2]\times\mathbb{T}^d} \bar{p}(t-r, \mathbf{x}-\mathbf{y})\sigma(r,\mathbf{y},u(r,\mathbf{y}))F(drd\mathbf{y})$$

where $0 \leq q_2 \leq t-s$. Note V_{q_2} is a continuous $\mathcal{F}^F_{q_2}$ adapted martingale with $V_0 = 0$. Thus, both U_{q_1} and V_{q_2} are time changed Brownian motions, i.e.,

$$U_t = B_{\langle U\rangle_t} \text{ and } V_{t-s} = B'_{\langle V\rangle_{t-s}}$$

where B, B' are two different Brownian motions. Note that $N(t,\mathbf{x}) - N(s,\mathbf{x}) = U_t + V_{t-s}$, then

$$P(N(t,\mathbf{x}) - N(s,\mathbf{x}) > \kappa) \leq P(U_t > \kappa/2) + P(V_{t-s} > \kappa/2).$$

Lemma 4.3 provides a uniform bound on the time changes as

$$\langle U\rangle_t \leq C\mathcal{C}_2^2(t-s)^\zeta \text{ and } \langle V\rangle_{t-s} \leq C\mathcal{C}_2^2(t-s)^\zeta.$$

By the reflection principle for the Brownian motions $B_{\langle U\rangle_t}$ and $B'_{\langle V\rangle_{t-s}}$,

$$\begin{aligned}
P(N(t,\mathbf{x}) - N(s,\mathbf{x}) > \kappa) &\leq P\left(B_{\langle U\rangle_t} > \kappa/2\right) + P\left(B'_{\langle V\rangle_{t-s}} > \kappa/2\right) \\
&\leq 2P\left(\sup_{r\leq C\mathcal{C}_2^2|t-s|^\zeta} B_r > \frac{\kappa}{2}\right) = 4P\left(B_{C\mathcal{C}_2^2|t-s|^\zeta} > \frac{\kappa}{2}\right) \\
&\leq C_3\exp\left(-\frac{C_4\kappa^2}{\mathcal{C}_2^2|t-s|^\zeta}\right).
\end{aligned}$$

In addition,

$$\begin{aligned}
P(-N(t,\mathbf{x}) + N(s,\mathbf{x}) > \kappa) &\leq P(U_t < -\kappa/2) + P(V_{t-s} < -\kappa/2) \\
&\leq 4P\left(B_{C\mathcal{C}_2^2|t-s|^\zeta} < -\frac{\kappa}{2}\right) \\
&\leq C_3\exp\left(-\frac{C_4\kappa^2}{\mathcal{C}_2^2|t-s|^\zeta}\right).
\end{aligned}$$

Consequently, for $\forall\zeta \in \left(0, \frac{\alpha-\beta}{\alpha}\right)$,

$$P(|N(t,\mathbf{x}) - N(s,\mathbf{x})| > \kappa) \leq C_3\exp\left(-\frac{C_4\kappa^2}{\mathcal{C}_2^2|t-s|^\zeta}\right),$$

which completes the proof of (4.15). □

Definition 4.1. *Given a grid*

$$\mathbb{G}_n = \left\{\left(\frac{j}{2^{2n}}, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) : 0 \leq j \leq 2^{2n}, 0 \leq k_1, ..., k_d \leq 2^n, j, k_1, ..., k_d \in \mathbb{Z}\right\},$$

we write

$$\left(t_j^{(n)}, x_{k_1}^{(n)}, ..., x_{k_d}^{(n)}\right) = \left(\frac{j}{2^{2n}}, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right).$$

Two points $\left(t_j^{(n)}, x_{k_1}^{(n)}, ..., x_{k_d}^{(n)}\right), \left(t_{j'}^{(n)}, x_{k'_1}^{(n)}, ..., x_{k'_d}^{(n)}\right)$ *are called* ***nearest neighbors*** *if either*

1. $j = j', |k_i - k'_i| = 1$ *for only one* i *and* $k_l = k'_l$ *for the other indices* l, *or*

2. $|j - j'| = 1$ *and* $k_i = k'_i$ $\forall i$.

The following lemma generalizes the Lemma 3.4 in [AJM21], which plays a key role in estimating the small ball probability.

Lemma 4.5. *There exist constants $C_5, C_6 > 0$ depending on α, β, d and $\mathcal{C}_2$ in (1.4) such that for all $\gamma, \kappa, \varepsilon > 0$ and $\gamma\varepsilon^4 \leq 1$, we have*

$$P\left(\sup_{\substack{0\leq t\leq \gamma\varepsilon^4\\ \mathbf{x}\in[0,\varepsilon^2]^d}} |N(t,\mathbf{x})| > \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right) \leq \frac{C_5}{1\wedge\sqrt{\gamma^d}}\exp\left(-\frac{C_6\kappa^2}{\mathcal{C}_2^2\gamma^{\frac{\alpha-\beta}{\alpha}}}\right).$$

Proof. Fix $\gamma \geq 1$, and consider the grid

$$\mathbb{G}_n = \left\{\left(\frac{j}{2^{2n}}, \frac{k_1}{2^n}, ..., \frac{k_d}{2^n}\right) : 0 \leq j \leq \gamma\varepsilon^4 2^{2n}, 0 \leq k_1, ..., k_d \leq \varepsilon^2 2^n, j, k_1, ..., k_d \in \mathbb{Z}\right\}.$$

Let

$$n_0 = \left\lceil \log_2\left(\gamma^{-1/2}\varepsilon^{-2}\right)\right\rceil, \tag{4.16}$$

and for $n < n_0$, $\mathbb{G}_n$ contains only the origin. For $n \geq n_0$, the grid $\mathbb{G}_n$ has at most $(\gamma\varepsilon^4 2^{2n}+1)\cdot(\varepsilon^2 2^n+1)^d \leq 2^{d+1+(2+d)n}\varepsilon^{2d+4}\gamma \leq 2^{2d+3}2^{(2+d)(n-n_0)}$ many points. We will choose two parameters $0 < \delta_1(\alpha,\beta) < \delta_0(\alpha,\beta) < \frac{\alpha-\beta}{\alpha}$ satisfying the following constraint

$$2\zeta \wedge \alpha\xi = \frac{2(\alpha-\beta)}{\alpha} + 2\delta_1 - 2\delta_0, \tag{4.17}$$

where $\xi \in \left(0, \frac{1}{\alpha}\wedge\frac{\alpha-\beta}{\alpha}\right), \zeta \in \left(0, \frac{\alpha-\beta}{\alpha}\right)$. Fix the constant

$$\mathcal{M} = \frac{1-2^{-\delta_1}}{(3+d)2^{(\delta_0-\delta_1)n_0}},$$

and consider the event

$$A(n,\kappa) = \left\{|N(p)-N(q)| \leq \kappa\mathcal{M}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}2^{-\delta_1 n}2^{\delta_0 n_0} \text{ for all } p,q\in\mathbb{G}_n \text{ nearest neighbors}\right\}.$$

If $p, q \in \mathbb{G}_n$ are the case **1** nearest neighbors in the Definition 4.1, (4.14) implies

$$P\left(|N(p)-N(q)| > \kappa\mathcal{M}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}2^{-\delta_1 n}2^{\delta_0 n_0}\right) \leq C_1\exp\left(-\frac{C_2\kappa^2\mathcal{M}^2\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{2^{-n\alpha\xi}\mathcal{C}_2^2}2^{-2\delta_1 n}2^{2\delta_0 n_0}\right).$$

If $p, q \in \mathbb{G}_n$ are the case **2** nearest neighbors in the Definition 4.1, (4.15) implies

$$P\left(|N(p)-N(q)| > \kappa\mathcal{M}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}2^{-\delta_1 n}2^{\delta_0 n_0}\right) \leq C_3\exp\left(-\frac{C_4\kappa^2\mathcal{M}^2\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{2^{-2n\zeta}\mathcal{C}_2^2}2^{-2\delta_1 n}2^{2\delta_0 n_0}\right).$$

Therefore, a union bound gives

$$\begin{aligned}
P(A^c(n,\kappa)) &\leq \sum_{\substack{p,q\in\mathbb{G}_n \\ \text{nearest neighbors}}} P\left(|N(p)-N(q)| > \kappa\mathcal{M}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}2^{-\delta_1 n}2^{\delta_0 n_0}\right) \\
&\leq C2^{(2+d)(n-n_0)}\exp\left(-\frac{C'\kappa^2\mathcal{M}^2\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{\mathcal{C}_2^2}2^{n(2\zeta\wedge\alpha\xi)}2^{-2\delta_1 n}2^{2\delta_0 n_0}\right) \\
&= C2^{(2+d)(n-n_0)}\exp\left(-\frac{C'\kappa^2\mathcal{M}^2}{\mathcal{C}_2^2}\left(\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}2^{\frac{2n_0(\alpha-\beta)}{\alpha}}\right)2^{n(2\zeta\wedge\alpha\xi)}2^{-2\delta_1 n}2^{2\delta_0 n_0}\right) \\
&\leq C2^{(2+d)(n-n_0)}\exp\left(-\frac{C'\kappa^2\mathcal{M}^2}{\mathcal{C}_2^2\gamma^{\frac{(\alpha-\beta)}{\alpha}}}2^{(2\zeta\wedge\alpha\xi-2\delta_1)(n-n_0)}\right),
\end{aligned}$$

where C, C' are positive constants depending only on α, β, T, d. The last inequality follows from that $\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}2^{\frac{2n_0(\alpha-\beta)}{\alpha}} \geq \gamma^{-\frac{(\alpha-\beta)}{\alpha}}$ by the definition of n_0 in (4.16), and our choice of δ_0, δ_1 in (4.17). Let $A(\kappa) = \bigcap_{n\geq n_0} A(n,\kappa)$ and we can bound $P(A^c(\kappa))$ by summing all $P(A^c(n,\kappa))$,

$$\begin{aligned}
P\left(A^c(\kappa)\right) \leq \sum_{n\geq n_0} P\left(A^c(n,\kappa)\right) &\leq \sum_{n\geq n_0} C2^{(2+d)(n-n_0)}\exp\left(-\frac{C'\kappa^2\mathcal{M}^2}{\mathcal{C}_2^2\gamma^{\frac{(\alpha-\beta)}{\alpha}}}2^{(2\zeta\wedge\alpha\xi-2\delta_1)(n-n_0)}\right) \\
&\leq C_5\exp\left(-\frac{C_6\kappa^2\mathcal{M}^2}{\mathcal{C}_2^2\gamma^{\frac{(\alpha-\beta)}{\alpha}}}\right).
\end{aligned}$$

Now we consider a point (t,x), which is in a grid $\mathbb{G}_n$ for some $n \geq n_0$. From arguments similar to page 128 of [DKM+09], we can find a sequence of points from the origin to $(t,\mathbf{x})$ as $(0,\mathbf{0}) = p_0, p_1, ..., p_k = (t,\mathbf{x})$ such that each pair is the nearest neighbor in some grid $\mathbb{G}_m, n_0 \leq m \leq n$, and at most $(3+d)$ such pairs are nearest neighbors in any given grid. On the event $A(\kappa)$, we have

$$|N(t,\mathbf{x})| \leq \sum_{j=0}^{k-1}|N(p_j)-N(p_{j+1})| \leq (3+d)\sum_{n\geq n_0}\kappa\mathcal{M}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}2^{-\delta_1 n}2^{\delta_0 n_0} \leq \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}.$$

Points in $\mathbb{G}_n$ are dense in $[0,\gamma\varepsilon^4]\times[0,\varepsilon^2]$, and we may extend $N(t,\mathbf{x})$ to a continuous version. Therefore, for $\gamma \geq 1$,

$$P\left(\sup_{\substack{0\leq t\leq\gamma\varepsilon^4 \\ \mathbf{x}\in[0,\varepsilon^2]^d}}|N(t,\mathbf{x})| > \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right) \leq C_5\exp\left(-\frac{C_6\kappa^2}{\mathcal{C}_2^2\gamma^{\frac{\alpha-\beta}{\alpha}}}\right).$$

For $0 < \gamma < 1$, a union bound and stationarity in $\mathbf{x}$ imply that

$$\sqrt{\gamma^d}P\left(\sup_{\substack{0\leq t\leq\gamma\varepsilon^4 \\ \mathbf{x}\in[0,\varepsilon^2]^d}}|N(t,\mathbf{x})| > \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right) \leq P\left(\sup_{\substack{0\leq t\leq\gamma\varepsilon^4 \\ \mathbf{x}\in[0,\sqrt{\gamma}\varepsilon^2]^d}}|N(t,\mathbf{x})| > \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right)$$

$$= P\left(\sup_{\substack{0\le t\le(\sqrt{\gamma}\varepsilon^2)^2\\ \mathbf{x}\in[0,\sqrt{\gamma}\varepsilon^2]^d}} |N(t,\mathbf{x})| > \frac{\kappa}{\gamma^{\frac{\alpha-\beta}{2\alpha}}}\left(\gamma^{1/4}\varepsilon\right)^{\frac{2(\alpha-\beta)}{\alpha}}\right) \le C_5 \exp\left(-\frac{C_6\kappa^2}{\mathcal{C}_2^2\gamma^{\frac{\alpha-\beta}{\alpha}}}\right).$$

As a result,

$$P\left(\sup_{\substack{0\le t\le\gamma\varepsilon^4\\ \mathbf{x}\in[0,\varepsilon^2]^d}} |N(t,\mathbf{x})| > \kappa\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right) \le \frac{C_5}{1\wedge\sqrt{\gamma^d}} \exp\left(-\frac{C_6\kappa^2}{\mathcal{C}_2^2\gamma^{\frac{\alpha-\beta}{\alpha}}}\right).$$

□

Remark 4.1. *If we suppose σ in* (4.8) *satisfies $|\sigma(s,\mathbf{y},u(s,\mathbf{y}))| \le C\left(\gamma\varepsilon^4\right)^{\frac{\alpha-\beta}{2\alpha}}$, then the probability in Lemma 4.5 is bounded above by*

$$\frac{C_5}{1\wedge\sqrt{\gamma^d}} \exp\left(-\frac{C_6\kappa^2}{C^2\left(\gamma\varepsilon^2\right)^{\frac{2(\alpha-\beta)}{\alpha}}}\right),$$

which can be proved similarly to the above lemma.

5. Proof of Proposition 3.1

The following lemma gives a lower bound for variance of the noise term $N(t_1,\mathbf{x})$ and an upper bound on the decay of covariance between two random variables $N(t_1,\mathbf{x})$, $N(t_1,\mathbf{y})$ as $|\mathbf{x}-\mathbf{y}|$ increases.

Lemma 5.1. *Consider noise terms $N(t_1,\mathbf{x})$, $N(t_1,\mathbf{y})$ with a deterministic $\sigma(t,\mathbf{x},u) = \sigma(t,\mathbf{x})$, then there exist constants $C_7, C_8 > 0$ depending only on $\mathcal{C}_1$, $\mathcal{C}_2$, d, α, and β such that*

$$C_7 t_1^{\frac{\alpha-\beta}{\alpha}} \le \mathrm{Var}[N(t_1,\mathbf{x})],$$

$$\mathrm{Cov}[N(t_1,\mathbf{x}), N(t_1,\mathbf{y})] \le C_8 t_1 \left|\mathbf{x}-\mathbf{y}\right|^{-\beta}.$$

Proof. We use the Fubini's theorem, (1.8) the expression (1.2) and (4.1) to show that

$$\begin{aligned}
\mathrm{Var}[N(t_1,\mathbf{x})] &= \int_0^{t_1}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \bar{p}(t_1-s,\mathbf{x}-\mathbf{y})\bar{p}(t_1-s,\mathbf{x}-\mathbf{z})\sigma(s,\mathbf{y})\sigma(s,\mathbf{z})\Lambda(\mathbf{y}-\mathbf{z})d\mathbf{y}d\mathbf{z}ds \\
&\ge C(d)\mathcal{C}_1^2\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\int_0^{t_1} e^{-2\pi^\alpha|\mathbf{n}|^\alpha(t_1-s)}ds\right) \\
&= C(d)\mathcal{C}_1^2\left(\lambda(\mathbf{0})t_1 + \sum_{\mathbf{n}\in\mathbb{Z}^d,\mathbf{n}\neq\mathbf{0}}\lambda(\mathbf{n})\frac{1-e^{-2\pi^\alpha|\mathbf{n}|^\alpha t_1}}{2\pi^\alpha|\mathbf{n}|^\alpha}\right) \\
&\ge C(d,\mathcal{C}_1)\int_1^\infty \frac{1-e^{-2\pi^\alpha x^\alpha t_1}}{2\pi^\alpha x^{d+\alpha-\beta}}x^{d-1}dx.
\end{aligned}$$

The last inequality follows from that $\int_0^{t_1} e^{-2\pi^\alpha|\mathbf{n}|^\alpha(t_1-s)}ds$ decreases as $|\mathbf{n}|$ increases. Changing variable to $w = 2\pi^\alpha x^\alpha t_1$ yields

$$\int_1^\infty \frac{1-e^{-2\pi^\alpha x^\alpha t_1}}{2\pi^\alpha x^{\alpha-\beta+1}}dx = C(\alpha,\beta,d)t_1^{\frac{\alpha-\beta}{\alpha}}\int_{2\pi^\alpha t_1}^\infty \frac{1-e^{-w}}{w^{2-\beta/\alpha}}dw \geq C(\alpha,\beta,d)t_1^{\frac{\alpha-\beta}{\alpha}}\int_{2\pi^\alpha}^\infty \frac{1-e^{-w}}{w^{2-\beta/\alpha}}dw.$$

The last integral converges with $0<\beta<\alpha\wedge d$, which completes the proof of the first part. In addition, we use the definition of $\Lambda(x)$ in (1.2) and the fact $1-e^{-x}\leq x$ to derive the upper bound of covariance between $N(t_1,\mathbf{x})$ and $N(t_1,\mathbf{y})$ when $\mathbf{x}\neq\mathbf{y}$,

$$\begin{aligned}
&\mathrm{Cov}[N(t_1,\mathbf{x}),N(t_1,\mathbf{y})] = \mathbb{E}[N(t_1,\mathbf{x})N(t_1,\mathbf{y})]\\
&\leq C(d)\mathcal{C}_2^2\left(\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\exp(\pi i\mathbf{n}\cdot(\mathbf{x}-\mathbf{y}))\int_0^{t_1}e^{-2\pi^\alpha|\mathbf{n}|^\alpha(t_1-s)}ds\right)\\
&\leq C_8 t_1\sum_{\mathbf{n}\in\mathbb{Z}^d}\lambda(\mathbf{n})\exp(\pi i\mathbf{n}\cdot(\mathbf{x}-\mathbf{y})) = C_8 t_1\,|\mathbf{x}-\mathbf{y}|^{-\beta}.
\end{aligned}$$

□

Proof of Proposition 3.1(a) The Markov property of $u(t,\cdot)$ (see page 247 in [DPZ14]) implies

$$P\left(F_j|\sigma\{u(t_i,\cdot)\}_{0\leq i<j}\right) = P\left(F_j|u(t_{j-1},\cdot)\right).$$

If we can prove that $P\left(F_j|u(t_{j-1},\cdot)\right)$ has a uniform bound $\mathbf{C_4}\exp\left(-\frac{\mathbf{C_5}}{\varepsilon^2+\mathcal{D}^2t_1^{\frac{\alpha-\beta}{\alpha}}}\right)$, then it is still a bound for the conditional probability $P\left(F_j|\bigcap_{k=0}^{j-1}F_k\right)$, which is conditioned on a realization of $u(t_k,\cdot), 0\leq k<j$. Thus, it is enough to show that

$$P\left(F_1\right)\leq \mathbf{C_4}\exp\left(-\frac{\mathbf{C_5}}{\varepsilon^2+\mathcal{D}^2t_1^{\frac{\alpha-\beta}{\alpha}}}\right),$$

where $\mathbf{C_4}$, $\mathbf{C_5}$ do not depend on u_0. Consider the truncated function

$$f_\varepsilon(\mathbf{x}) = \begin{cases}\mathbf{x} & |\mathbf{x}|\leq t_1^{\frac{\alpha-\beta}{2\alpha}}\\ \frac{\mathbf{x}}{|\mathbf{x}|}\cdot t_1^{\frac{\alpha-\beta}{2\alpha}} & |\mathbf{x}|> t_1^{\frac{\alpha-\beta}{2\alpha}}\end{cases},$$

and, particularly, we have $|f_\varepsilon(\mathbf{x})|\leq t_1^{\frac{\alpha-\beta}{2\alpha}}$. Consider the following two equations

$$\partial_t v(t,\mathbf{x}) = -(-\Delta)^{\alpha/2}v(t,\mathbf{x})+\sigma(t,\mathbf{x},f_\varepsilon(v(t,\mathbf{x})))\dot{F}(t,\mathbf{x}),$$

and

$$\partial_t v_g(t,\mathbf{x}) = -(-\Delta)^{\alpha/2}v_g(t,\mathbf{x})+\sigma(t,\mathbf{x},f_\varepsilon(u_0(\mathbf{x})))\dot{F}(t,\mathbf{x})$$

with the same initial $u_0(\mathbf{x})$. We can decompose $v(t,\mathbf{x})$ by

$$v(t,\mathbf{x}) = v_g(t,\mathbf{x})+D(t,\mathbf{x})$$

with

$$D(t,\mathbf{x}) = \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})[\sigma(s,\mathbf{y},f_\varepsilon(v(s,\mathbf{y})))-\sigma(s,\mathbf{y},f_\varepsilon(u_0(\mathbf{y})))]F(dsd\mathbf{y}).$$

The Lipschitz property on the third variable of $\sigma(t,\mathbf{x},u)$ in (1.3) gives

$$\begin{aligned}|\sigma(s,\mathbf{y},f_\varepsilon(v(s,\mathbf{y})))-\sigma(s,\mathbf{y},f_\varepsilon(u_0(\mathbf{y})))| &\le \mathcal{D}|f_\varepsilon(v(s,\mathbf{y}))-f_\varepsilon(u_0(\mathbf{y}))| \\ &\le 2\mathcal{D}t_1^{\frac{\alpha-\beta}{2\alpha}}.\end{aligned} \tag{5.1}$$

Recall that $R_{i,j}$ in (3.3) and define a new sequence of events,

$$H_j=\left\{|v(t,\mathbf{x})|\le t_1^{\frac{\alpha-\beta}{2\alpha}},\forall(t,\mathbf{x})\in R_{1,j}\setminus R_{1,j-1}\right\}.$$

Clearly, the property of $f_\varepsilon(\mathbf{x})$ and (3.5) imply

$$F_1=\bigcap_{j=-n_1+1}^{n_1-1}H_j.$$

Also, define another two sequences of events

$$A_j=\left\{|v_g(t,\mathbf{x})|\le 2t_1^{\frac{\alpha-\beta}{2\alpha}},\forall(t,\mathbf{x})\in R_{1,j}\setminus R_{1,j-1}\right\},$$

$$B_j=\left\{|D(t,\mathbf{x})|> t_1^{\frac{\alpha-\beta}{2\alpha}},\exists(t,\mathbf{x})\in R_{1,j}\setminus R_{1,j-1}\right\}.$$

It is straightforward to check that

$$H_j^c\supset A_j^c\cap B_j^c,$$

which implies

$$\begin{aligned}P(F_1)=P\left(\bigcap_{j=-n_1+1}^{n_1-1}H_j\right)&\le P\left(\bigcap_{j=-n_1+1}^{n_1-1}[A_j\cup B_j]\right)\\ &\le P\left(\left(\bigcap_{j=-n_1+1}^{n_1-1}A_j\right)\bigcup\left(\bigcup_{j=-n_1+1}^{n_1-1}B_j\right)\right)\\ &\le P\left(\bigcap_{j=-n_1+1}^{n_1-1}A_j\right)+P\left(\bigcup_{j=-n_1+1}^{n_1-1}B_j\right)\\ &\le P\left(\bigcap_{j=-n_1+1}^{n_1-1}A_j\right)+\sum_{j=-n_1+1}^{n_1-1}P(B_j).\end{aligned} \tag{5.2}$$

The second inequality can be showed by using induction. Moreover, for $j=-n_1+1$,

$$B_j\subseteq\left\{\sup_{\substack{0\le s\le c_0\varepsilon^4\\ \mathbf{y}\in[(-n_1+1)\varepsilon^2,(-n_1+2)\varepsilon^2]^d}}|D(s,\mathbf{y})|>t_1^{\frac{\alpha-\beta}{2\alpha}}\right\}, \tag{5.3}$$

and for $j > -n_1 + 1$,

$$B_j \subseteq \left\{ \sup_{(t,\mathbf{x})\in R_{1,j}\setminus R_{1,j-1}} |D(t,\mathbf{x})| > t_1^{\frac{\alpha-\beta}{2\alpha}} \right\} \tag{5.4}$$
$$\subseteq \bigcup_{(t,\mathbf{x})\in R_{1,j-1}\setminus R_{1,j-2}} \left\{ \sup_{\substack{0\le s\le c_0\varepsilon^4 \\ \mathbf{y}\in\mathbf{x}+[0,\varepsilon^2]^d}} |D(s,\mathbf{y})| > t_1^{\frac{\alpha-\beta}{2\alpha}} \right\}.$$

From (5.1) and Remark 4.1, we get

$$P\left(\sup_{\substack{0\le s\le c_0\varepsilon^4 \\ \mathbf{y}\in\mathbf{x}+[0,\varepsilon^2]^d}} |D(s,\mathbf{y})| > t_1^{\frac{\alpha-\beta}{2\alpha}} \right) \le \frac{C_5}{1\wedge\sqrt{c_0}^d}\exp\left(-\frac{C_6}{4\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}} \right), \tag{5.5}$$

where the proof does not rely on $\mathbf{x}$ since $u_0(\mathbf{x}) \equiv 0$. Therefore, (5.3) implies

$$P(B_{-n_1+1}) \le P\left(\sup_{\substack{0\le s\le c_0\varepsilon^4 \\ \mathbf{y}\in[0,\varepsilon^2]^d}} |D(s,\mathbf{y})| > t_1^{\frac{\alpha-\beta}{2\alpha}} \right), \tag{5.6}$$

and (5.4) implies, for $j > -n_1 + 1$,

$$P(B_j) \le \left[(j+n_1-1)^d - (j+n_1-2)^d\right] P\left(\sup_{\substack{0\le s\le c_0\varepsilon^4 \\ \mathbf{y}\in[0,\varepsilon^2]^d}} |D(s,\mathbf{y})| > t_1^{\frac{\alpha-\beta}{2\alpha}} \right). \tag{5.7}$$

Hence, using (5.5), (5.6) and (5.7), we conclude that

$$\sum_{j=-n_1+1}^{n_1-1} P(B_j) \le \left((2n_1-2)^d+1\right) \le \frac{C(d)}{\varepsilon^{2d}} \cdot \frac{C_5}{1\wedge\sqrt{c_0}^d}\exp\left(-\frac{C_6}{4\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}} \right). \tag{5.8}$$

To compute the upper bound for $P\left(\bigcap_{j=-n_1+1}^{n_1-1} A_j\right)$, we define a sequence of events involving v_g,

$$I_j = \left\{ |v_g(t,\mathbf{x})| \le 2t_1^{\frac{\alpha-\beta}{2\alpha}}, \forall (t,\mathbf{x})\in R_{1,j} \right\} \quad \text{and} \quad I_{-n_1} = \Omega.$$

Then we can write $P\left(\bigcap_{j=-n_1+1}^{n_1-1} A_j\right)$ in terms of conditional probability as

$$P\left(\bigcap_{j=-n_1+1}^{n_1-1} A_j \right) = P(I_{n_1-1}) = P(I_{-n_1}) \prod_{j=-n_1+1}^{n_1-1} \frac{P(I_j)}{P(I_{j-1})} = \prod_{j=-n_1+1}^{n_1-1} P(I_j|I_{j-1}). \tag{5.9}$$

Let $\mathcal{G}_j$ be the $\sigma-$algebra generated by

$$N_\varepsilon(t,\mathbf{x}) = \int_0^t \int_{\mathbb{T}^d} p(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y},f_\varepsilon(u_0(\mathbf{y})))F(d\mathbf{y}ds),\ (t,\mathbf{x})\in R_{1,j}.$$

If we can show that there is a uniform bound for $P\left(I_j|\mathcal{G}_{j-1}\right)$, then it is still a bound for the conditional probability $P\left(I_j|I_{j-1}\right)$. Notice that $\sigma(s,\mathbf{y},f_\varepsilon(u_0(\mathbf{y})))$ is deterministic and uniformly bounded, then by Lemma 5.1, we have

$$\mathrm{Var}[N_\varepsilon(t_1,\mathbf{x})] \geq C_7 t_1^{\frac{\alpha-\beta}{\alpha}}, \tag{5.10}$$

and for $(t,\mathbf{x}) \in R_{1,j} \setminus R_{1,j-1}$, one can decompose

$$v_g(t,\mathbf{x}) = \int_{\mathbb{T}^d} p(t,\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y} + X + Y, \tag{5.11}$$

where $X = \mathbb{E}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}]$ is a Gaussian random variable, which can be written as

$$X = \sum_{(t,\mathbf{x})\in R_{1,j-1}} \eta^{(j)}(t,\mathbf{x})N_\varepsilon(t,\mathbf{x}), \tag{5.12}$$

for some coefficients $\left(\eta^{(j)}(t,\mathbf{x})\right)_{(t,\mathbf{x})\in R_{1,j-1}}$. Then the conditional variance equals

$$\begin{aligned}\mathrm{Var}(Y|\mathcal{G}_{j-1}) &= \mathbb{E}[(N_\varepsilon(t,\mathbf{x}) - X|\mathcal{G}_{j-1})^2] - (\mathbb{E}[N_\varepsilon(t,\mathbf{x}) - X|\mathcal{G}_{j-1}])^2 \\ &= \mathbb{E}[(N_\varepsilon(t,\mathbf{x}) - \mathbb{E}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}]|\mathcal{G}_{j-1})^2] = \mathrm{Var}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}].\end{aligned}$$

Since $Y = N_\varepsilon(t,\mathbf{x}) - X$ is independent of $\mathcal{G}_{j-1}$, we write $\mathrm{Var}(Y)$ as

$$\mathrm{Var}(Y) = \mathrm{Var}(Y|\mathcal{G}_{j-1}) = \mathrm{Var}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}].$$

In fact, for a Gaussian random variable $Z \sim N(\mu,\sigma^2)$ and any $a > 0$, the probability $P(|Z| \leq a)$ is maximized when $\mu = 0$, thus

$$\begin{aligned}P\left(I_j|\mathcal{G}_{j-1}\right) &\leq P\left(|v_g(t,\mathbf{x})| \leq 2t_1^{\frac{\alpha-\beta}{2\alpha}}, (t,\mathbf{x}) \in R_{1,j} \setminus R_{1,j-1} \middle| \mathcal{G}_{j-1}\right) \\ &\leq P\left(|Z'| \leq \frac{2t_1^{\frac{\alpha-\beta}{2\alpha}}}{\sqrt{\mathrm{Var}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}]}}\right)\end{aligned}$$

where $Z' \sim N(0,1)$. Let's use the notation SD to denote the standard deviation of a random variable. By the Minkowski inequality,

$$\mathrm{SD}(X) \leq \sum_{(t,\mathbf{x})\in R_{1,j-1}} \left|\eta^{(j)}(t,\mathbf{x})\right| \cdot \mathrm{SD}[N_\varepsilon(t,\mathbf{x})],$$

and

$$\mathrm{SD}[N_\varepsilon(t,\mathbf{x})] \leq \mathrm{SD}(X) + \mathrm{SD}(Y).$$

If we can control coefficients by restricting

$$\sum_{(t,\mathbf{x})\in R_{1,j-1}} \left|\eta^{(j)}(t,\mathbf{x})\right| \leq \frac{1}{2},$$

then the standard deviation of X is less than one half the standard deviation of $N_\varepsilon(t,\mathbf{x})$,

$$\mathrm{SD}[N_\varepsilon(t,\mathbf{x})] \leq \mathrm{SD}(X) + \mathrm{SD}(Y) \leq \frac{1}{2}\mathrm{SD}[N_\varepsilon(t,\mathbf{x})] + \mathrm{SD}(Y).$$

From (5.10), $\mathrm{Var}(Y)$ is bounded below by $C_7 t_1^{\frac{\alpha-\beta}{\alpha}}$, so that we can derive the uniform upper bound of $P(I_j|\mathcal{G}_{j-1})$,

$$\begin{aligned} P(I_j|\mathcal{G}_{j-1}) &\le P\left(|Z'| \le \frac{2t_1^{\frac{\alpha-\beta}{2\alpha}}}{\sqrt{\mathrm{Var}[N_\varepsilon(t,\mathbf{x})|\mathcal{G}_{j-1}]}}\right) \\ &\le P\left(|Z'| \le \frac{2t_1^{\frac{\alpha-\beta}{2\alpha}}}{\sqrt{C_7 t_1^{\frac{\alpha-\beta}{\alpha}}}}\right) \\ &= P(|Z'| \le C') < 1, \end{aligned}$$

where C' depends only on $\mathcal{C}_1$, d, α, and β. A bound (3.4) on j and (5.9) together yield

$$P\left(\bigcap_{j=-n_1+1}^{n_1-1} A_j\right) \le C^{2\varepsilon^{-2}} = C\exp\left(-\frac{C'}{\varepsilon^2}\right), \tag{5.13}$$

where C, C' depends only on $\mathcal{C}_1$, d, α, and β. The following lemma shows how to select c_0 to make $\sum_{(t,\mathbf{x})\in R_{1,j-1}} |\eta^{(j)}(t,\mathbf{x})| \le \frac{1}{2}$, which completes the proof.

Lemma 5.2. *For a given $\varepsilon > 0$, we may choose $c_0 > 0$ in (3.2) such that*

$$\sum_{(t,\mathbf{x})\in R_{1,j-1}} |\eta^{(j)}(t,\mathbf{x})| \le \frac{1}{2}.$$

Proof. Let X and Y be random variables defined in (5.11) and (5.12). Since Y and $\mathcal{G}_{j-1}$ are independent, for $\forall (t,\mathbf{x}) \in R_{1,j-1}$,

$$\mathrm{Cov}[Y, N_\varepsilon(t,\mathbf{x})] = 0$$

and for $(t,\mathbf{y}) \in R_{1,j} \setminus R_{1,j-1}$, we have

$$\mathrm{Cov}[N_\varepsilon(t,\mathbf{x}), N_\varepsilon(t,\mathbf{y})] = \mathrm{Cov}[N_\varepsilon(t,\mathbf{x}), X] = \sum_{(t,\mathbf{x}')\in R_{1,j-1}} \eta^{(j)}(t,\mathbf{x}')\mathrm{Cov}[N_\varepsilon(t,\mathbf{x}), N_\varepsilon(t,\mathbf{x}')]. \tag{5.14}$$

We write the equation (5.14) in a matrix form as

$$\mathbf{X} = \mathbf{\Sigma}\eta, \tag{5.15}$$

where the vector $\eta = \left(\eta^{(j)}(t,\mathbf{x})\right)^T_{(t,\mathbf{x})\in R_{1,j-1}}$, the vector $\mathbf{X} = \{\mathrm{Cov}[N_\varepsilon(t,\mathbf{x}), N_\varepsilon(t,\mathbf{y})]\}^T_{(t,\mathbf{x})\in R_{1,j-1}}$, and $\mathbf{\Sigma}$ is the covariance matrix of $\left(N_\varepsilon(t,\mathbf{x})\right)_{(t,\mathbf{x})\in R_{1,j-1}}$. Let $||\cdot||_{1,1}$ be the matrix norm induced by the $||\cdot||_{l_1}$ norm, that is for a matrix $\mathbf{A}$,

$$||\mathbf{A}||_{1,1} := \sup_{\mathbf{x}\neq\mathbf{0}} \frac{||\mathbf{A}\mathbf{x}||_{l_1}}{||\mathbf{x}||_{l_1}}.$$

It can be shown that $||\mathbf{A}||_{1,1} = \max_j \sum_{i=1}^n |a_{ij}|$ (see page 259 of [RB00]). Therefore, we have

$$||\eta||_{l_1} = ||\mathbf{\Sigma}^{-1}\mathbf{X}||_{l_1} \le ||\mathbf{\Sigma}^{-1}||_{1,1}||\mathbf{X}||_{l_1}.$$

We rewrite $\mathbf{\Sigma} = \mathbf{DTD}$, where $\mathbf{D}$ is a diagonal matrix with diagonal entries $\sqrt{\text{Var}[N_\varepsilon(t,\mathbf{x})]}$, and $\mathbf{T}$ is the correlation matrix with entries

$$e_{\mathbf{xx'}} = \frac{\text{Cov}[N_\varepsilon(t,\mathbf{x}), N_\varepsilon(t,\mathbf{x'})]}{\sqrt{\text{Var}[N_\varepsilon(t,\mathbf{x})]} \cdot \sqrt{\text{Var}[N_\varepsilon(t,\mathbf{x'})]}}.$$

Thanks to Lemma 5.1, for $\mathbf{x} \neq \mathbf{x'}$, $|e_{\mathbf{xx'}}|$ can be bounded above by

$$|e_{\mathbf{xx'}}| \leq \frac{C_8 t_1 \, |\mathbf{x} - \mathbf{x'}|^{-\beta}}{C_7 t_1^{1-\beta/\alpha}}.$$

Define $\mathbf{A} = \mathbf{I} - \mathbf{T}$. Because $\mathbf{A}$ has zero diagonal entries, we can bound $||\mathbf{A}||_{1,1}$ by

$$\begin{aligned} ||\mathbf{A}||_{1,1} &= \max_{\mathbf{x}} \sum_{\mathbf{x}\neq\mathbf{x'}} |t_{\mathbf{xx'}}| \leq \sum_{(t,\mathbf{x})\in R_{1,n_1-1}} |e_{\mathbf{0x}}| = \frac{C_8 t_1^{\beta/\alpha}}{C_7} \sum_{(t,\mathbf{x})\in R_{1,n_1-1}} |\mathbf{x}|^{-\beta} \\ &\leq \frac{C(d) C_8 t_1^{\beta/\alpha}}{C_7 \varepsilon^{2\beta}} \int_0^{\sqrt{d}\varepsilon^{-2}} r^{d-\beta-1} dr = \frac{C(d,\beta) C_8}{C_7} \cdot \frac{(c_0 \varepsilon^4)^{\beta/\alpha}}{\varepsilon^{2d}}. \end{aligned}$$

For any $\varepsilon > 0$, we denote $\mathcal{C} = \left(\frac{C_7}{3C(d,\beta)C_8}\right)^{\alpha/\beta}$ and choose $c_0 < \mathcal{C}\varepsilon^{\frac{2\alpha d - 4\beta}{\beta}}$ in (3.2), which makes $||\mathbf{A}||_{1,1} < \frac{1}{3}$. Therefore, summing the geometric series gives that

$$||\mathbf{T}^{-1}||_{1,1} = ||(\mathbf{I} - \mathbf{A})^{-1}||_{1,1} \leq \frac{1}{1 - ||\mathbf{A}||_{1,1}} \leq \frac{3}{2},$$

and $||\mathbf{\Sigma}^{-1}||_{1,1} \leq ||\mathbf{D}^{-1}||_{1,1} \cdot ||\mathbf{T}^{-1}||_{1,1} \cdot ||\mathbf{D}^{-1}||_{1,1} \leq \frac{3}{2} C_7^{-1} t_1^{-\frac{\alpha-\beta}{\alpha}}$. Substituting the bounds into (5.15) and choosing c_0 as in (3.2), we obtain

$$||\eta||_{l_1} \leq \frac{3}{2} C_7^{-1} t_1^{-\frac{\alpha-\beta}{\alpha}} ||\mathbf{X}||_{l_1} < \frac{3}{2} \cdot \frac{1}{3} = \frac{1}{2}.$$

□

Combining (5.2), (5.8) and (5.13) yields

$$\begin{aligned} P(F_1) &\leq \frac{C(d) C_5}{(1 \wedge \sqrt{c_0}^d)\varepsilon^{2d}} \exp\left(-\frac{C_6}{4\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}}\right) + C \exp\left(-\frac{C'}{\varepsilon^2}\right) \\ &\leq C_5' \exp\left(-\frac{d}{2} \ln t_1 - \frac{C_6}{4\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}}\right) + C \exp\left(-\frac{C'}{\varepsilon^2}\right) \end{aligned}$$

We choose a $\mathcal{D}_0$ depending only on α, β and d such that for any $\mathcal{D} < \mathcal{D}_0$,

$$\begin{aligned} P(F_1) &\leq C_5' \exp\left(-\frac{C_6'}{\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}}\right) + C \exp\left(-\frac{C'}{\varepsilon^2}\right) \\ &\leq \mathbf{C_4} \exp\left(-\frac{\mathbf{C_5}}{\varepsilon^2 + \mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}}\right). \end{aligned}$$

which completes the proof of Proposition 3.1 (a).

Proof of Proposition 3.1 (b) We first state the Gaussian correlation inequality, which is crucial to proof Proposition 3.1 (b).

Lemma 5.3. *For any convex symmetric sets K, L in $\mathbb{R}^d$ and any centered Gaussian measure μ on $\mathbb{R}^d$, we have*

$$\mu(K \cap L) \geq \mu(K)\mu(L).$$

Proof. See in paper [Roy14], [LM17]. □

By the Markov property of $u(t,\cdot)$, the behavior of $u(t,\cdot)$ in the interval $[t_n, t_{n+1}]$ depends only on $u(t_n,\cdot)$ and $\dot{F}(t,\mathbf{x})$ on $[t_n,t]\times[-1,1]^d$. Therefore, it is enough to show that

$$P(E_0) \geq \mathbf{C_6} \exp\left(-\frac{\mathbf{C_7}}{\varepsilon^{\frac{2d(\alpha d-\beta)}{\beta}}}\right),$$

where $\mathbf{C_6}$, $\mathbf{C_7}$ do not depend on u_0 and $|u_0(x)| \leq \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$. Now we are ready to compute the lower bound for the small ball probability with a smooth and deterministic $\sigma(s,y,u) = \sigma(s,y)$, which is a Gaussian case. For $n \geq 0$, define a sequence of events

$$D_n = \left\{|u(t_{n+1},\mathbf{x})| \leq \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, \text{and } |u(t,\mathbf{x})| \leq \frac{2}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}},\ \forall t\in[t_n,t_{n+1}], \mathbf{x}\in[-1,1]^d\right\}. \tag{5.16}$$

Denote

$$\bar{p}_t(u_0)(\mathbf{x}) = \bar{p}(t,\cdot) * u_0(\mathbf{x}) = \int_{\mathbb{T}^d} \bar{p}(t,\mathbf{x}-\mathbf{y})u_0(\mathbf{y})d\mathbf{y},$$

and we have

$$\bar{p}_t(u_0)(\mathbf{x}) \leq \sup_{\mathbf{x}} |u_0(\mathbf{x})| \leq \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}. \tag{5.17}$$

We consider the measure Q given by

$$\frac{dQ}{dP} = \exp\left(Z_{t_1} - \frac{1}{2}\langle Z\rangle_{t_1}\right)$$

where

$$Z_{t_1} = -\int_{[0,t_1]\times\mathbb{T}^d} f(s,\mathbf{y})F(dsd\mathbf{y}).$$

If Z_{t_1} satisfies Novikov's condition in [All98], then

$$\widetilde{F}(t,\mathbf{x}) := F(t,\mathbf{x}) - \langle F(\cdot,\mathbf{x}), Z\rangle_t$$

is a centered spatially homogeneous Wiener process under the measure Q (see [All98] for more details). Therefore, for $\mathbf{x}\in[-1,1]^d$, Fubini's Theorem with the covariance structure of $\dot{F}(t,\mathbf{x})$ gives

$$\dot{\widetilde{F}}(t,\mathbf{x}) = \dot{F}(t,\mathbf{x}) + \int_{\mathbb{T}^d} f(t,\mathbf{y})\Lambda(\mathbf{x}-\mathbf{y})d\mathbf{y},$$

which is a colored noise under measure Q. Since $\frac{\bar{p}_t(u_0)(\mathbf{x})}{t_1\sigma(t,\mathbf{x})}$ is smooth and bounded function, and $\Lambda(\mathbf{x})$ is the Riesz kernel on $\mathbb{T}^d$, [RS16] shows that there is a continuous formula for the fractional Laplacian of $\frac{\bar{p}_t(u_0)(\mathbf{x})}{t_1\sigma(t,\mathbf{x})}$ on $\mathbb{T}^d$, so that one may assume that there is a function $f(t,\mathbf{y})$ such that,

$$\int_{\mathbb{T}^d} f(t,\mathbf{y})\Lambda(\mathbf{x}-\mathbf{y})d\mathbf{y} = \frac{\bar{p}_t(u_0)(\mathbf{x})}{t_1\sigma(t,\mathbf{x})}.$$

Moreover, $\Lambda(\mathbf{x}) \geq d^{-\beta/2}$ and (5.17) imply

(5.18)
$$\begin{aligned}
&\mathbb{E}\left[(Z_{t_1})^2\right] = \mathbb{E}\left[\int_{[0,t_1]\times\mathbb{T}^d}\int_{[0,t_1]\times\mathbb{T}^d} f(s,\mathbf{y})f(t,\mathbf{z})F(dsd\mathbf{y})F(dtd\mathbf{z})\right] \\
&= \int_0^{t_1}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} f(s,\mathbf{y})f(s,\mathbf{z})\Lambda(\mathbf{y}-\mathbf{z})d\mathbf{y}d\mathbf{z}ds = \int_0^{t_1}\int_{\mathbb{T}^d} f(s,\mathbf{y})\frac{\bar{p}_s(u_0)(\mathbf{y})}{t_1\sigma(s,\mathbf{y})}d\mathbf{y}ds \\
&\leq \int_0^{t_1}\int_{\mathbb{T}^d} f(s,\mathbf{y})\left(\frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right)\cdot\frac{1}{t_1\mathcal{C}_1}d\mathbf{y}ds \leq \frac{C(d,\beta)\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}}{\mathcal{C}_1 t_1}\int_0^{t_1}\int_{\mathbb{T}^d} f(s,\mathbf{y})\Lambda(\mathbf{1}-\mathbf{y})d\mathbf{y}ds \\
&= \frac{C(d,\beta)\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}}{\mathcal{C}_1 t_1}\int_0^{t_1}\frac{\bar{p}_s(u_0)(\mathbf{1})}{t_1\sigma(s,\mathbf{1})}ds \leq C(d,\beta,\mathcal{C}_1)\frac{\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1} < \infty,
\end{aligned}$$

which satisfies Novikov's condition with a deterministic f. Thus, we can rewrite equation (1.1) with deterministic σ as

$$\begin{aligned}
u(t,\mathbf{x}) &= \bar{p}_t(u_0)(\mathbf{x}) + \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\left[\widetilde{F}(dsd\mathbf{y}) - \frac{\bar{p}_s(u_0)(\mathbf{y})}{t_1\sigma(s,\mathbf{y})}dsd\mathbf{y}\right] \\
&= \bar{p}_t(u_0)(\mathbf{x}) - \frac{t\bar{p}_t(u_0)(\mathbf{x})}{t_1} + \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\widetilde{F}(dsd\mathbf{y}) \\
&= \left(1-\frac{t}{t_1}\right)\bar{p}_t(u_0)(\mathbf{x}) + \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\widetilde{F}(dsd\mathbf{y}).
\end{aligned}$$

The first term is 0 at t_1, and $|u_0(\mathbf{x})| \leq \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$, we have

$$\left|\left(1-\frac{t}{t_1}\right)\bar{p}_t(u_0)(\mathbf{x})\right| \leq \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, \mathbf{x}\in[-1,1]^d, t<t_1. \tag{5.19}$$

Define

$$\widetilde{N}(t,\mathbf{x}) = \int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})\sigma(s,\mathbf{y})\widetilde{F}(d\mathbf{y}ds),$$

and suppose $c_0 < 1$, then applying Lemma 4.5 to $\widetilde{F}$ gives

$$Q\left(\sup_{\substack{0\leq t\leq c_0\varepsilon^4 \\ \mathbf{x}\in[0,c_0\varepsilon^2]^d}}|\widetilde{N}(t,\mathbf{x})| > \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right) \leq C_5\exp\left(-\frac{C_6}{36\mathcal{C}_2^2 c_0^{\frac{\alpha-\beta}{\alpha}}}\right),$$

where $\gamma = c_0^{-1} > 1$ and $\kappa = \left(6c_0^{\frac{\alpha-\beta}{\alpha}}\right)^{-1}$. To make sure that the right hand side is strictly less than 1, we require

$$c_0 < \min\left\{1, \left(\frac{C_6}{36\mathcal{C}_2^2\ln C_5}\right)^{\frac{\alpha}{\alpha-\beta}}\right\},$$

which is mentioned in (3.1). By the Gaussian correlation inequality in Lemma 5.3, we obtain

$$Q\left(\sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}}|\widetilde{N}(t,\mathbf{x})|\le\frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right)\ge Q\left(\sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[0,c_0\varepsilon^2]^d}}|\widetilde{N}(t,\mathbf{x})|\le\frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right)^{\left(\frac{2}{c_0\varepsilon^2}\right)^d}$$
$$\ge\left[1-C_5\exp\left(-\frac{C_6}{36\mathcal{C}_2^2c_0^{\frac{\alpha-\beta}{\alpha}}}\right)\right]^{\left(\frac{2}{c_0\varepsilon^2}\right)^d}.$$

From (5.16) and (5.19), we get

$$Q(D_0)\ge Q\left(\sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}}|\widetilde{N}(t,\mathbf{x})|\le\frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}\right),$$

and if we replace $f(s,y)$ with $2f(s,y)$ for Z_{t_1},

$$1=\mathbb{E}\left[\frac{dQ}{dP}\right]=\mathbb{E}\left[\exp\left(Z_{t_1}-\frac{1}{2}\langle Z\rangle_{t_1}\right)\right]=\mathbb{E}[\exp\left(2Z_{t_1}-2\langle Z\rangle_{t_1}\right)]. \tag{5.20}$$

Because $f(s,y)$ is deterministic, we may estimate the Radon-Nikodym derivative,

$$\mathbb{E}\left[\left(\frac{dQ}{dP}\right)^2\right]=\mathbb{E}[\exp\left(2Z_{t_1}-\langle Z\rangle_{t_1}\right)]=\mathbb{E}[\exp\left(2Z_{t_1}-2\langle Z\rangle_{t_1}\right)\cdot\exp(\langle Z\rangle_{t_1})]$$
$$\le\exp\left(C(d,\beta,\mathcal{C}_1)\frac{\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1}\right).$$

The last inequality follows from (5.18) and (5.20). The Cauchy-Schwarz inequality implies

$$Q(D_0)\le\sqrt{\mathbb{E}\left[\left(\frac{dQ}{dP}\right)^2\right]}\cdot\sqrt{P(D_0)},$$

and as a consequence, we get

$$P(D_0)\ge\exp\left(-\frac{C\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1}\right)\exp\left(\frac{C'}{c_0^d\varepsilon^{2d}}\ln\left[1-C_5\exp\left(-\frac{C_6}{36\mathcal{C}_2^2c_0^{\frac{\alpha-\beta}{\alpha}}}\right)\right]\right), \tag{5.21}$$

where C,C' depend only on $d,\beta,\mathcal{C}_1$. For the lower bound with a non-deterministic $\sigma(t,\mathbf{x},u)$, we write

$$u(t,\mathbf{x})=u_g(t,\mathbf{x})+D(t,\mathbf{x})$$

where $u_g(t,\mathbf{x})$ satisfies the equation

$$\partial_t u_g(t,\mathbf{x})=-(-\Delta)^{\alpha/2}u_g(t,\mathbf{x})+\sigma(t,\mathbf{x},u_0(\mathbf{x}))\dot{F}(t,\mathbf{x})$$

and

$$D(t,\mathbf{x})=\int_{[0,t]\times\mathbb{T}^d}\bar{p}(t-s,\mathbf{x}-\mathbf{y})[\sigma(s,\mathbf{y},u(s,\mathbf{y}))-\sigma(s,\mathbf{y},u_0(\mathbf{y}))]F(ds d\mathbf{y})$$

with an initial profile u_0. Since u_g is Gaussian, for an event defined as

$$\widetilde{D}_0 = \left\{ |u_g(t_1, \mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, \text{and } |u_g(t, \mathbf{x})| \le \frac{2}{3}\varepsilon^{\frac{2(\alpha-\beta)}{4}} \ \forall t \in [0, t_1], \mathbf{x} \in [-1,1]^d \right\},$$

we can apply (5.21) to it and get

$$P(\widetilde{D}_0) \ge \exp\left(-\frac{C\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1} \right) \exp\left(\frac{C'}{c_0^d \varepsilon^{2d}} \ln\left[1 - C_5 \exp\left(-\frac{C_6}{36\mathcal{C}_2^2 c_0^{\frac{\alpha-\beta}{\alpha}}} \right) \right] \right). \tag{5.22}$$

Define the stopping time

$$\tau = \inf\left\{ t : |u(t, \mathbf{x}) - u_0(\mathbf{x})| > 2\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \text{ for some } \mathbf{x} \in [-1,1]^d \right\},$$

and clearly we have $\tau > t_1$ on the event E_0 in (3.6) since $|u_0(x)| \le \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$, and $|u(t,x)| \le \varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$ for $\forall t \in [0, t_1]$ on the event E_0. We make another definition

$$\widetilde{D}(t, x) = \int_{[0,t]\times\mathbb{T}^d} \bar{p}(t-s, \mathbf{x}-\mathbf{y})[\sigma(s, \mathbf{y}, u(s \wedge \tau, \mathbf{y})) - \sigma(s, \mathbf{y}, u_0(\mathbf{y}))] F(dsd\mathbf{y}),$$

and $D(t, \mathbf{x}) = \widetilde{D}(t, \mathbf{x})$ for $t \le t_1$ on the event $\{\tau > t_1\}$. Moreover, from (3.6), we have

$$\begin{aligned} P(E_0) &\ge P\left(\widetilde{D}_0 \bigcap \left\{ \sup_{\substack{0\le t\le t_1 \\ \mathbf{x}\in[-1,1]^d}} |D(t,\mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right\} \right) \\ &= P\left(\left(\widetilde{D}_0 \bigcap \left\{ \sup_{\substack{0\le t\le t_1 \\ \mathbf{x}\in[-1,1]^d}} |D(t,\mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right\} \bigcap \{\tau > t_1\} \right) \right. \\ &\qquad \left. \bigcup \left(\widetilde{D}_0 \bigcap \left\{ \sup_{\substack{0\le t\le t_1 \\ \mathbf{x}\in[-1,1]^d}} |D(t,\mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right\} \bigcap \{\tau \le t_1\} \right) \right). \end{aligned} \tag{5.23}$$

On the event $\{\tau > t_1\}$, we have

$$\sup_{\substack{0\le t\le t_1 \\ \mathbf{x}\in[-1,1]^d}} |D(t,\mathbf{x})| = \sup_{\substack{0\le t\le t_1 \\ \mathbf{x}\in[-1,1]^d}} |\widetilde{D}(t,\mathbf{x})|,$$

and on the event $\widetilde{D}_0 \cap \{\tau \le t_1\}$, we have, for some $\mathbf{x}$,

$$|u_g(\tau, \mathbf{x})| \le \frac{2}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, \ |u(\tau, \mathbf{x}) - u_0(\mathbf{x})| > 2\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \text{ and } |u_0(\mathbf{x})| \le \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}.$$

The above leads to

$$\begin{aligned} \sup_{\mathbf{x}} |D(\tau, \mathbf{x})| &= \sup_{\mathbf{x}} |u(\tau, \mathbf{x}) - u_g(\tau, \mathbf{x})| \ge \sup_{\mathbf{x}} (|u(\tau, \mathbf{x})| - |u_g(\tau, \mathbf{x})|) \\ &\ge \sup_{\mathbf{x}} |u(\tau, \mathbf{x})| - \frac{2}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \ge 2\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} - \frac{1}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} - \frac{2}{3}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \\ &\ge \varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} > \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}, \end{aligned}$$

which implies

$$\widetilde{D}_0 \cap \left\{ \sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}} |D(t,\mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right\} \cap \{\tau \le t_1\} = \phi.$$

Combining the above with (5.23) yields

$$\begin{aligned} P(E_0) &\ge P\left(\widetilde{D}_0 \bigcap \left\{ \sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}} |\widetilde{D}(t,\mathbf{x})| \le \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right\} \bigcap \{\tau > t_1\} \right) \\ &\ge P(\widetilde{D}_0) - P\left(\sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}} |\widetilde{D}(t,\mathbf{x})| > \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right), \end{aligned} \tag{5.24}$$

and $|u(t,\mathbf{x}) - u_0(\mathbf{x})| \le 2\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}}$ for all $t \in [0,t_1]$ and $\mathbf{x} \in [-1,1]^d$. We apply the Lipschitz property on the third variable of $\sigma(t,\mathbf{x},u)$ in (1.3) to Remark 4.1 and use a union bound from (5.8) to get

$$P\left(\sup_{\substack{0\le t\le t_1\\ \mathbf{x}\in[-1,1]^d}} |\widetilde{D}(t,\mathbf{x})| > \frac{1}{6}\varepsilon^{\frac{2(\alpha-\beta)}{\alpha}} \right) \le \frac{C_5}{(c_0\varepsilon^4)^{d/2}} \exp\left(-\frac{C_6}{144\mathcal{D}^2 (c_0\varepsilon^4)^{\frac{\alpha-\beta}{\alpha}}} \right). \tag{5.25}$$

Consequently, from (5.22), (5.24) and (5.25), we conclude that,

$$\begin{aligned} P(E_0) &\ge \exp\left(-\frac{C\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1} \right) \exp\left(\frac{C'}{c_0^d \varepsilon^{2d}} \ln\left[1 - C_5 \exp\left(-\frac{C_6}{36\mathcal{C}_2^2 c_0^{\frac{\alpha-\beta}{\alpha}}} \right) \right] \right) \\ &\qquad - \frac{C_5}{(c_0\varepsilon^4)^{d/2}} \exp\left(-\frac{C_6}{144\mathcal{D}^2 (c_0\varepsilon^4)^{\frac{\alpha-\beta}{\alpha}}} \right) \\ &= \exp\left(-\frac{C\varepsilon^{\frac{4(\alpha-\beta)}{\alpha}}}{t_1} + \frac{C'}{c_0^d\varepsilon^{2d}} \ln\left[1 - C_5 \exp\left(-\frac{C_6}{36\mathcal{C}_2^2 c_0^{\frac{\alpha-\beta}{\alpha}}} \right) \right] \right) \\ &\qquad - C_5 \exp\left(-\frac{d}{2}\ln t_1 - \frac{C_6}{144\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}} \right). \end{aligned} \tag{5.26}$$

When $d = 1$ and $\alpha \ge 2\beta$, we may choose c_0 in (3.2) satisfying

$$\mathcal{C}'\varepsilon^{\frac{4\alpha-8\beta}{\beta}} < c_0 < \mathcal{C}\varepsilon^{\frac{2\alpha-4\beta}{\beta}}, \tag{5.27}$$

where $0 < \mathcal{C}' < \mathcal{C}$ and ε is small enough. Then choose a $\mathcal{D}_0$ depending only on α, β and d such that for any $\mathcal{D} < \mathcal{D}_0$, we have

$$\begin{aligned} P(E_0) &\geq \exp\left(-C\varepsilon^{\frac{-4(\alpha-\beta)^2}{\alpha\beta}}\right) - C_5' \exp\left(-\frac{C_6'}{\mathcal{D}^2 t_1^{\frac{\alpha-\beta}{\alpha}}}\right) \\ &\geq C\exp\left(-C'\varepsilon^{\frac{-4(\alpha-\beta)^2}{\alpha\beta}}\right) - C_5' \exp\left(-\frac{C_6'}{\mathcal{D}^2 \varepsilon^{\frac{4(\alpha-\beta)^2}{\alpha\beta}}}\right) \\ &\geq \mathbf{C_6} \exp\left(-\frac{\mathbf{C_7}}{\varepsilon^{\frac{4(\alpha-\beta)^2}{\alpha\beta}}}\right). \end{aligned}$$

When $d = 1$ and $\alpha < 2\beta$, we may choose c_0 in (3.1). However, the second term could exceed the first term in (5.26) for small enough ε and we may not achieve a lower bound for small probability for any $0 < \varepsilon < \varepsilon_0$. Similarly, for $d \geq 2$, the first term decays exponentially from (5.22) and the second term grows exponentially from (5.25), hence we cannot achieve a lower bound for small probability for any $0 < \varepsilon < \varepsilon_0$, which completes the proof.

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