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Commutation relations of $\mathfrak{g}_2$ and the incidence geometry of the Fano plane

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Abstract

We continue our study and classification of structures on the Fano plane $\mathcal{F}$ and its dual $\mathcal{F}^*$ involved in the construction of octonions and the Lie algebra $\mathfrak{g}_2(\mathbb{F})$ over a field $\mathbb{F}$. These are a "composition factor" $\epsilon : \mathcal{F} \times \mathcal{F} \to \{-1, 1\}$,hhhh inducing an octonion multiplication, and a function $\delta^* : \mathrm{Aut}(\mathcal{F}) \times \mathcal{F}^* \to \{-1, 1\}$ such that $g \in \mathrm{Aut}(\mathcal{F})$ can be lifted to an automorphism of the octonions *iff* $\delta^*(g, \cdot)$ is the Radon transform of a function on $\mathcal{F}$. We lift the action of $\mathrm{Aut}(\mathcal{F})$ on $\mathcal{F}$ to the action of a non-trivial eight-fold covering $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ on a two-fold covering $\hat{\mathcal{F}}_\epsilon$ of $\mathcal{F}$ contained in the octonions. This extends tautologically to an action on the octonions by automorphism. Finally, we associate to incident point-line pairs a generating set of $\mathfrak{g}_2(\mathbb{F})$ and express brackets in terms of the incidence geometry of $\mathcal{F}$ and ϵ.

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We continue our study and classification of structures on the Fano plane $\mathcal{F}$ and its dual $\mathcal{F}^*$ involved in the construction of octonions and the Lie algebra $\mathfrak{g}_2(\mathbb{F})$ over a field $\mathbb{F}$. These are a "composition factor" : $\mathcal{F} \times \mathcal{F} \to \{-1, 1\}$, inducing an octonion multiplication, and a function $\delta^* : Aut(\mathcal{F}) \times \mathcal{F}^* \to \{-1, 1\}$ such that $g \in Aut(\mathcal{F})$ can be lifted to an automorphism of the octonions iff $\delta^*(g, \cdot)$ is the Radon transform of a function on $\mathcal{F}$. We lift the action of $Aut(\mathcal{F})$ on $\mathcal{F}$ to the action of a non-trivial eight-fold covering $Aut(\mathcal{F})$ on a twofold covering $\hat{\mathcal{F}}$ of $\mathcal{F}$ contained in the octonions. This extends tautologically to an action on the octonions by automorphism. Finally, we associate to incident point-line pairs a generating set of $\mathfrak{g}_2(\mathbb{F})$ and express brackets in terms of the incidence geometry of $\mathcal{F}$ and .

In 1900 F. Engel [1] realised the exceptional complex Lie group $G_2(\mathbb{C})$ as the linear isotropy group of a generic three-form in seven dimensions, and in 1907 his student W. Reichel [2] realised the real compact exceptional Lie group $G_2^c(\mathbb{R})$ as the linear isotropy group of a real three-form in seven dimensions. É. Cartan [3] showed that the automorphism group of the octonions is the exceptional Lie group $G_2^c(\mathbb{R})$ and, to the best of the authors' knowledge, H. Freudenthal [4] was the first to point out a close relationship between the Fano plane $\mathcal{F}$ and the octonions. For a more detailed historical perspective see [5]. In this article we give a systematic study of additional structures on $\mathcal{F}$, over and above the projective structure, which are relevant to the construction of the octonions over any field $\mathbb{F}$ of characteristic not two (see also [6]). In particular we give explicit generators of $\mathfrak{g}_2(\mathbb{F})$, the Lie algebra of $G_2^c(\mathbb{F})$, together with their brackets in terms of "augmented" incidence relations of $\mathcal{F}$. We also give simple formulæ for the $\mathfrak{g}_2(\mathbb{F})-$invariant three and four forms on the space of imaginary octonions in terms of these relations.

Our starting point is $\mathrm{Aut}(\mathcal{F})$, the group of automorphisms of $\mathcal{F}$. This is a simple group of order 168 isomorphic to $GL(3,\mathbb{Z}_2)$ or $PSL(2,\mathbb{Z}_7)$ [7]. Elements of order seven, which we consider as "orientations" of $\mathcal{F}$, fall into two conjugacy classes which we consider as defining two "pre-orientations" of $\mathcal{F}$. Two pre-orientations are distinguished by the description of lines in $\mathcal{F}$ in terms of any compatible orientation.

To an orientation of $\mathcal{F}$ one can associate a composition factor on $\mathcal{F}$. A composition factor is the structure allowing a generalisation of Freudenthal's construction of a composition algebra product on an eight-dimensional vector space $\mathbb{O}_{\mathcal{F}}$ canonically associated to $\mathcal{F}$. Our first result completes the classification of composition factors given in [6]. We show that for the action of $\mathrm{Aut}(\mathcal{F})$ on composition factors there are two orbits, each containing eight elements. Given a composition factor ϵ on $\mathcal{F}$ it is then natural to ask whether the action of $\mathrm{Aut}(\mathcal{F})$ on $\mathcal{F}$ extends to an action by automorphisms on $\mathbb{O}_{\mathcal{F}}$ equipped with the product corresponding to ϵ. This is not the case but it turns out that this action lifts to an action of an eight-fold non-split extension of $\mathrm{Aut}(\mathcal{F})$ [8]. In order to understand this phenomenon we first associate to $g \in\mathrm{Aut}(\mathcal{F})$ a function $\delta^*(g,\cdot)$ on $\mathcal{F}^*$, the space of lines in $\mathcal{F}$. We then show that solving the lifting problem for g is equivalent to finding a function on $\mathcal{F}$ whose Radon transform is $\delta^*(g,\cdot)$ and in this way solve it.

In the last part of the paper we associate to each incident pair $(P,D) \in \mathcal{F}\times\mathcal{F}^*$ an element $X_{P,D}$ of $\mathfrak{g}_2(\mathbb{F})$. This gives twenty-one elements which span the fourteen-dimensional vector space $\mathfrak{g}_2(\mathbb{F})$. We express the brackets of the $X_{P,D}$ in terms of the incidence relations of $\mathcal{F}$ and the composition factor ϵ. Finally, this allows us to identify various geometric subalgebras of $\mathfrak{g}_2(\mathbb{F})$ associated to points and lines of $\mathcal{F}$. In particular, to each point we associate a Cartan subalgebra and an $\mathfrak{su}(3)$(or $\mathfrak{sl}(3)$) subalgebra containing it. Dually, we associate to each line an $\mathfrak{so}(3)$ and an $\mathfrak{so}(3)\times\mathfrak{so}(3)$ subalgebra containing it as an ideal.

We now give a more detailed description of the contents of this paper. In Section 1 we recall the definition and basic properties of the Fano plane $\mathcal{F}$ and its dual $\mathcal{F}^*$. We introduce the notion of an orientation of $\mathcal{F}$ and show that such an orientation induces an orientation of each line and of each point. In Section 2, having recalled relevant definitions and results from [6], we first show that the action of $\mathrm{Aut}(\mathcal{F})$ on composition factors has two orbits. We then consider the Radon transform from functions on $\mathcal{F}$ to functions on $\mathcal{F}^*$, and identify its kernel and image. The exponential version of the Radon transform turns out to be very useful in the context of this article. Fixing a composition factor ϵ the next question we address is whether the obvious linear action of $g \in \mathrm{Aut}(\mathcal{F})$ on the associated composition algebra $(\mathbb{O}_{\mathcal{F}}, \epsilon)$ is by automorphism. A necessary and sufficient condition for this is that for any line D of $\mathcal{F}$, g induces an isomorphism of the subalgebras associated to D and $g \cdot D$. The function $\delta^* : \mathrm{Aut}(\mathcal{F}) \times \mathcal{F}^* \to \{-1, 1\}$ which detects this property is introduced and for fixed g classified (Theorem 2.31): there are eight possibilities, each of which is realised by twenty-one elements of $\mathrm{Aut}(\mathcal{F})$. In particular the linear action of $g \in \mathrm{Aut}(\mathcal{F})$ on $(\mathbb{O}_{\mathcal{F}}, \epsilon)$ is by automorphism *iff* $\delta^*(g, P) = 1, \forall P \in \mathcal{F}$. To conclude Section 2 we introduce a "double covering" $\hat{\mathcal{F}}_\epsilon$ of the pair $(\mathcal{F}, \epsilon)$ (see also p. 205 in [9]). The group of automorphisms of $(\mathbb{O}_{\mathcal{F}}, \epsilon)$ is canonically isomorphic to the group of automorphisms of $\hat{\mathcal{F}}_\epsilon$ by restriction, and we show that $\pi : \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon) \to \mathrm{Aut}(\mathcal{F})$ is a non-split extension by $\mathbb{Z}_2^3$ (see also [8, 9]). The main point here is to show that finding a lift of $g \in \mathrm{Aut}(\mathcal{F})$ that acts by automorphism on $\hat{\mathcal{F}}_\epsilon$ is equivalent to finding a function on $\mathcal{F}$ whose Radon transform is $\delta^*(g, \cdot)$.

In Section 3 we fix a composition factor and our starting point is the observation that, notwithstanding non-associativity, left multiplication by purely imaginary octonions gives a representation of the Clifford algebra of $\mathrm{Im}(\mathbb{O}_{\mathcal{F}})$ acting on $\mathbb{O}_{\mathcal{F}}$ (see for example [10]). This means one can realise the Lie algebra $\mathfrak{g}_2$ as the annihilator in $\mathfrak{so}(\mathrm{Im}(\mathbb{O}_{\mathcal{F}}))$ of $1 \in \mathbb{O}_{\mathcal{F}}$ [11]. To each incident pair $(P, D) \in \mathcal{F} \times \mathcal{F}^*$ we associate an element $X_{P,D}$ of $\mathfrak{g}_2(\mathbb{F})$ realised in this way. In order to describe the action of $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ on the $X_{P,D}$ we introduce a function $\delta : \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon) \times \mathcal{F} \to \{-1, 1\}$. As $\hat{g}$ varies in $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ this gives rise to sixty-four functions on $\mathcal{F}$, each of which is realised by twenty-one elements of $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$. It is a surprising fact that the exponential of the Radon transform of $\delta(\hat{g}, \cdot)$ is exactly $\delta^*(\pi(g), \cdot)$ for $\hat{g} \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$. To get explicit formulæ for brackets of the $X_{P,D}$ we find simple normal forms for the orbits of the action of $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ on pairs of incident pairs $\big((P, D), (P', D')\big) \in \big(\mathcal{F} \times \mathcal{F}^*\big)^2$. These formulæ are Fano geometric in the sense that they only involve the incidence geometry of $\mathcal{F}$ and the composition factor (Theorem 3.18). The point here is that it easy to guess these brackets up to a sign but rather more subtle to determine the signs precisely. Furthermore, using the formulæ we associate to each point of $\mathcal{F}$ a Cartan subalgebra and an $\mathfrak{su}(3)$ (or $\mathfrak{sl}(3)$) subalgebra containing it. Dually, to each line in $\mathcal{F}$ we associate an $\mathfrak{so}(3)$ subalgebra and an $\mathfrak{so}(3) \times \mathfrak{so}(3)$ subalgebra containing it as an ideal. In this way any rank two subalgebra of $\mathfrak{g}_2(\mathbb{F})$ can be realised (up to conjugation) as a subalgebra associated either to a line or to a point.

Throughout this paper: $\mathbb{F}$ denotes a field of characteristic not two, $\mathbb{Z}_n$ denotes the ring $\mathbb{Z}/n\mathbb{Z}$ and S_2 denotes the group $\{-1, 1\}$. The elements of $\mathbb{Z}_2$ will be denoted $0, 1$ *(not* $[0], [1]$*) when there is no ambiguity.*

1 Properties of the Fano plane

In this section we recall without proof the basic properties of the Fano plane we need in the rest of the paper.

1.1 The Fano plane and its dual

A projective plane is a set of points together with a collection of subsets called lines such that two distinct lines intersect in a unique point and two distinct points are contained in a unique line. Given a projective plane $\mathcal{P}$ the dual plane $\mathcal{P}^*$ is defined as the set whose points are the lines of $\mathcal{P}$, and whose lines are the subsets of $\mathcal{P}^*$ consisting of concurrent lines. It is easily checked that $\mathcal{P}^*$ is also a projective plane.

If V is a three-dimensional vector space over a field $\mathbb{F}$, $P(V)$, the set of lines passing through the origin in V together with the collection of subsets consisting of lines contained in a fixed plane passing through the origin of V, defines a projective plane.

The simplest example of a projective plane consists of seven points and seven lines arranged as below, each line contains exactly three points and each point is contained in exactly three lines (see Figure 1).

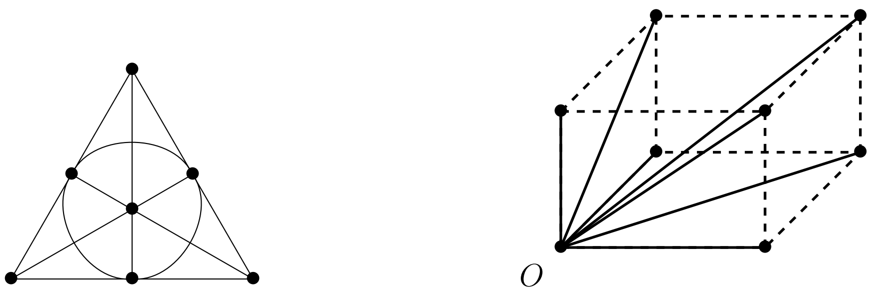


Figure 1: The Fano plane $\mathcal{F}$ and the Fano cube $V_{\mathcal{F}}$.

Any projective plane containing seven points and seven lines can be obtained by projectivising a canonically associated vector space:

Definition 1.1 *Let $\mathcal{F}$ be a projective plane consisting of seven points and seven lines. Let $V_{\mathcal{F}} = \mathcal{F} \cup \{0\}$ be the set obtained by formally adding a point 0 to $\mathcal{F}$. Define the symmetric map $+ : V_{\mathcal{F}} \times V_{\mathcal{F}} \to V_{\mathcal{F}}$ by*

$$P + Q = \begin{cases} R & \text{if} \quad P \neq Q \text{ and } P, Q, R \text{ are aligned} \, ; \\ 0 & \text{if} \quad P = Q \, ; \\ P & \text{if} \quad Q = 0 \, . \end{cases}$$

With respect to the obvious scalar multiplication by $\mathbb{Z}_2$ this defines the structure of a three-dimensional $\mathbb{Z}_2-$vector space on $V_{\mathcal{F}}$ with zero element 0. The natural bijection $P(V_{\mathcal{F}}) \cong \mathcal{F}$ is an isomorphism of projective planes and if P, Q, R are three non-zero points in $V_{\mathcal{F}}$, then P, Q, R are coplanar iff $P + Q + R = 0$.

The set of lines of $\mathcal{F}$ will be denoted $\mathcal{F}^*$. This is a projective plane whose 'lines' are triples of concurrent lines so in particular three lines D_1, D_2, D_3 are concurrent iff $D_1 + D_2 + D_3 = 0$.

Any two three-dimensional vector spaces over $\mathbb{Z}_2$ are isomorphic so any two projective planes consisting of seven points and seven lines are isomorphic in the sense that there is a bijection between them which sends lines to lines. From now on we will refer to any projective plane consisting of seven lines and seven points as a Fano plane and any three-dimensional vector space over $\mathbb{Z}_2$ as a Fano cube.

1.2 Automorphisms of the Fano plane

Definition 1.2 *Let $\mathcal{F}$ be a Fano plane. We set*

$$\mathrm{Aut}(\mathcal{F}) = \left\{ f : \mathcal{F} \to \mathcal{F} \;\; \text{s.t. } f \text{ is a bijection and } f \text{ sends lines to lines} \right\} .$$

This group is also called the group of collineations of $\mathcal{F}$.

Any bijection f: $\mathcal{F} \to \mathcal{F}$ extends to a unique bijection $\hat{f} : V_{\mathcal{F}} \setminus \{0\} \to V_{\mathcal{F}} \setminus \{0\}$, and clearly this defines a natural group isomorphism

$$\mathrm{Aut}(\mathcal{F}) \cong GL\big(V_{\mathcal{F}}\big) .$$

It is well-known [7] that this general linear group is simple of order 168, and generated by two elements a and b satisfying the relations

$$a^2 = b^3 = (ab)^3 = (aba^{-1}b^{-1})^4 = 1 . \tag{1.1}$$

We now summarise the main properties of $\mathrm{Aut}(\mathcal{F})$.

Proposition 1.3

1. *Let $f \in \mathrm{Aut}(\mathcal{F})$ such that $f \neq 1$. Then f is of order $2, 3, 4$ or 7.*

2. *Let $f \in \mathrm{Aut}(\mathcal{F})$ be of order two. Then there exist a line $L \in \mathcal{F}^*$ and a point $P \in \mathcal{F}$ such that $f(R) = R, \forall R \in L$ and $R + f(R) + P = 0$. $\forall R \notin L$, Conversely, given $L \in \mathcal{F}^*$ and $P \in L$ there exists a unique $f \in \mathrm{Aut}(\mathcal{F})$ of order two such that $f(R) = R, \forall R \in L$ and $R + f(R) + P = 0, \forall R \notin L$. Two elements of order two are conjugate.*

3. *Let $f \in \mathrm{Aut}(\mathcal{F})$ be of order three. Then there exists a unique triangle stable by f. Conversely every triangle is obtained in this way from exactly two order three elements of $\mathrm{Aut}(\mathcal{F})$. Two elements of order three are conjugate.*

4. *Let $f \in \mathrm{Aut}(\mathcal{F})$ be of order four. Then there exists $P \in \mathcal{F}$ and $L \in \mathcal{F}^*$ containing P such that $f(P) = P$, $f(L) = L$ and f is of order two on L, and for all $R \notin L$, $R + f(R) + P \neq 0$. Conversely, given $P \in \mathcal{F}$ and $L \in \mathcal{F}^*$ containing P, there exists exactly two elements of order four such that $f(P) = P$ and $f(L) = L$. Two elements of order four are conjugate.*

5. *Let $f \in GL\big(V_{\mathcal{F}}\big)$ be of order seven. Then the minimal polynomial of f is either $x^3 + x^2 + 1$ or $x^3 + x + 1$. Two elements of order seven are conjugate iff they have the same minimal polnomial.*

Corollary 1.4 *In $GL\big(V_{\mathcal{F}}\big)$ there are respectively $21, 56$, 42 and 48 elements respectively of order respectively two, three, four and seven.*

Recall that if $n \in \mathbb{N}$ the Legendre symbol $\left(\frac{n}{7}\right)$ can be defined by: $\left(\frac{n}{7}\right) = n^3 \mod 7$.

Corollary 1.5 *Let $f \in GL\big(V_{\mathcal{F}}\big)$ be of order seven and let n, m be positive integers. Then f^n is conjugate to f^m iff $\left(\frac{n}{7}\right) = \left(\frac{m}{7}\right)$.*

Example 1.6 *Recall that according to* (1.1) *Aut*($\mathcal{F}$) *can be generated by* a, b *satisfying* $a^2 = b^3 = (ab)^7 = (aba^{-1}b^{-1})^4 = 1$*. Denoting the points of* $\mathcal{F}$ *temporarily by the numbers* $1, \cdots, 7$*, an explicit example (see Figure 2) is given by*

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 4 & 6 & 5 & 3 \end{pmatrix} \ , \quad b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 7 & 4 & 6 & 5 & 3 & 1 \end{pmatrix} \ .$$

It the follows that

$$ab = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix} = \tau \ , \quad aba^{-1}b^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 6 & 1 & 7 & 5 & 3 \end{pmatrix} \ .$$

1.3 Orientation, orientation type and lines

Definition 1.7

1. *Let* $\mathcal{F}$ *be a Fano plane. An orientation of* $\mathcal{F}$ *is an element* $\tau \in$ *Aut*($\mathcal{F}$) *of order seven.*

2. *Let* $(\mathcal{F}, \tau)$ *and* $(\mathcal{F}', \tau')$ *be oriented Fano planes. We say* $(\mathcal{F}, \tau)$ *and* $(\mathcal{F}', \tau')$ *are isomorphic iff there exists an isomorphism of Fano planes* $f : \mathcal{F} \to \mathcal{F}'$ *s.t.* $f \circ \tau \circ f^{-1} = \tau'$*.*

It is well-known that the lines of a Fano plane can be described in one of two ways with respect to an orientation.

Proposition 1.8 *Let* $(\mathcal{F}, \tau)$ *be an oriented Fano plane. Then we have one of the following:*

(i) for all P *in* $\mathcal{F}$*, the triple* $D_P = \left\{P, \tau(P), \tau^3(P)\right\}$ *is a line and each line can be written uniquely in this way;*

(ii) for all P *in* $\mathcal{F}$*, the triple* $D_P = \left\{P, \tau^2(P), \tau^3(P)\right\}$ *is a line and each line can be written uniquely in this way.*

Definition 1.9 *An oriented Fano plan* $(\mathcal{F}, \tau)$ *is of type* $(0, 1, 3)$ *if the lines are as in (i) above and of type* $(0, 2, 3)$ *if the lines are as in (ii) above.*

Since there are only two conjugacy classes of elements of order seven in Aut($\mathcal{F}$) it follows that:

Corollary 1.10 *Two oriented Fano planes are isomorphic iff they are of the same type.*

If $(\mathcal{F}, \tau)$ is an oriented Fano plane, the induced map $\tau^* : \mathcal{F}^* \to \mathcal{F}^*$ is an orientation of the dual Fano plane $\mathcal{F}^*$.

Proposition 1.11 *Let* $(\mathcal{F}, \tau)$ *be an oriented Fano plane and let* $P \in \mathcal{F}$*. Then the three lines containing* P *are* $D_P, D_{\tau^{-1}(P)}$ *and* $D_{\tau^{-3}(P)}$ *in case (i) above, and* $D_P, D_{\tau^{-2}(P)}$ *and* $D_{\tau^{-3}(P)}$ *in case (ii) above. In particular,* $(\mathcal{F}, \tau)$ *and* $(\mathcal{F}^*, \tau^{*-1})$ *are isomorphic whereas* $(\mathcal{F}, \tau)$ *and* $(\mathcal{F}^*, \tau^*)$ *are not isomorphic.*

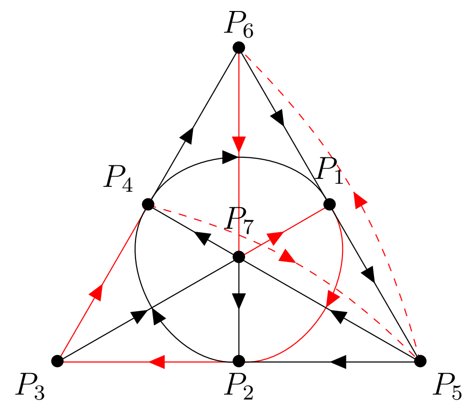


Figure 2: Orientations of lines in the Fano plane induced by τ (red arrows).

An orientation τ of a Fano plane $\mathcal{F}$ induces an orientation (*i.e.*, an order three bijection) on every line D in $\mathcal{F}$ as follows (see Figure 2). If τ is of type $(0,1,3)$ (resp. $(0,2,3)$) there is a unique $P \in \mathcal{F}$ such that $D = \Big\{P, \tau(P), \tau^3(P)\Big\}$ (resp. $D = \Big\{P, \tau^2(P), \tau^3(P)\Big\}$). The orientation induced by τ on D is then the cyclic permutation

$$(P, \tau(P), \tau^3(P)) \qquad \text{(resp. } (P, \tau^2(P), \tau^3(P))) \ . \tag{1.2}$$

For example if $P \in \mathcal{F}$ and τ is of type $(0,1,3)$, the orientation $(\tau^*)^{-1}$ of $\mathcal{F}^*$ induces the orientation on the line $L_P = \{D_P, D_{\tau^{-1}(P)}, D_{\tau^{-3}(P)}\}$ in $\mathcal{F}^*$ given by the cyclic permutation of lines in $\mathcal{F}$

$$(D_P, D_{\tau^{-1}(P)}, D_{\tau^{-3}(P)}) \ . \tag{1.3}$$

Proposition 1.12 *Let $(\mathcal{F}, \tau)$ and $(\mathcal{F}', \tau')$ be oriented Fano planes, and let $f : \mathcal{F} \to \mathcal{F}'$ be an isomorphism of Fano planes. If $(\mathcal{F}, \tau)$ and $(\mathcal{F}', \tau')$ are isomorphic (resp. not isomorphic) then for all $D \in \mathcal{F}^*$, the restriction of f to D is orientation preserving (resp. reversing) with respect to the induced orientations.*

2 Action of Aut($\mathcal{F}$) on composition factors and the augmented Fano plane

In this section we first recall the definition and classification of composition factors which are the structures on $\mathcal{F}$ needed to define an eight-dimensional composition algebra [6]. We then show that the action of Aut($\mathcal{F}$) on composition factors has two orbits, each containing eight elements. These actions of Aut($\mathcal{F}$) $\cong GL(V_{\mathcal{F}})$ are thus clearly not isomorphic to the standard action of $GL(V_{\mathcal{F}})$ on $V_{\mathcal{F}}$. We introduce a Radon transform taking functions on $\mathcal{F}$ to functions on $\mathcal{F}^*$. We show that its image (essentially) consists of functions measuring the extent to which elements of Aut($\mathcal{F}$) preserves line orientations induced by composition factors. Finally, we investigate the problem of lifting automorphisms of $\mathcal{F}$ to automorphisms of the associated composition algebra [8].

2.1 Norms and multiplication factors

Definition 2.1 *1. A norm on $\mathcal{F}$ is a function $N : \mathcal{F} \to \{-1, 1\}$ such that*

$$N(P+Q) = N(P)N(Q)\ , \quad P \neq Q\ .$$

2. Let $\mathcal{F}_0^2 = \{(P,Q) \in \mathcal{F}^2 \quad s.t \quad P \neq Q\}$. A multiplication factor is a map $\epsilon : \mathcal{F}_0^2 \to \{-1, 1\}$ such that $\epsilon_{PQ} + \epsilon_{QP} = 0$. For $P \in \mathcal{F}$ the future (resp. past) $\overrightarrow{P_\epsilon}$ (resp. $\overleftarrow{P_\epsilon}$) of P is defined by: $\overrightarrow{P_\epsilon} = \{Q \in \mathcal{F}$ s.t. $\epsilon_{PQ} = 1\}$ (resp. $\overleftarrow{P_\epsilon} = \{Q \in \mathcal{F}$ s.t. $\epsilon_{PQ} = -1\}$).

3. Let $\mathbb{O}_\mathcal{F}$ be the set of $\mathbb{F}-$valued functions on the Fano cube $V_\mathcal{F}$.

Remark 2.2 *It is easy to see that if N is a norm then the set $\Big\{P \in \mathcal{F}$ s.t. $N(P) = 1\Big\}$ is either $\mathcal{F}$ or a line.*

For $P \in V_\mathcal{F}$ define $e_P \in \mathbb{O}_\mathcal{F}$ by

$$e_P(Q) = \begin{cases} 1 & \text{if } Q = P \\ 0 & \text{if } Q \neq P\ . \end{cases}$$

Then $\{e_P \text{ s.t. } P \in V_\mathcal{F}\}$ is an $\mathbb{F}-$basis of $\mathbb{O}_\mathcal{F}$ and

$$\mathbb{O}_\mathcal{F} = \mathbb{F}e_0 \oplus \mathrm{Vect}\Big\langle e_P \text{ s.t. } P \in \mathcal{F}\Big\rangle\ .$$

A norm and a multiplication factor on $\mathcal{F}$ allow us to endow $\mathbb{O}_\mathcal{F}$ with a norm and a multiplication:

Definition 2.3 *Let $\mathcal{F}$ be a Fano plane equipped with a norm N and a multiplication factor ϵ.*

1. The multiplication $\cdot_\epsilon : \mathbb{O}_\mathcal{F} \times \mathbb{O}_\mathcal{F} \to \mathbb{O}_\mathcal{F}$ is the unique bilinear map such that

   (a) For all $P \neq Q \in \mathcal{F}$: $\quad e_P \cdot_\epsilon e_Q = \epsilon_{PQ} e_{P+Q}$;

   (b) For all $P \in \mathcal{F}$: $\quad e_P \cdot_\epsilon e_P = -N(P)e_0$;

   (c) For all $P \in V_\mathcal{F}$: $\quad e_0 \cdot_\epsilon e_P = e_P \cdot_\epsilon e_0 = e_P \quad$ (and we henceforth denote e_0 by 1).

2. The norm $N_{\mathbb{O}_\mathcal{F}} : \mathbb{O}_\mathcal{F} \to \mathbb{F}$ is the quadratic form: $N_{\mathbb{O}_\mathcal{F}}(\lambda^0 e_0 + \sum_{P\in\mathcal{F}} \lambda^P e_P) = (\lambda^0)^2 + \sum_{P\in\mathcal{F}} (\lambda^P)^2 N(P)$.

*We denote by **1** the quadratic form on $\mathbb{O}_\mathcal{F}$ associated to the trivial norm on $\mathcal{F}$. By definition the triple $(\mathbb{O}_\mathcal{F}, N_{\mathbb{O}_\mathcal{F}}, \epsilon)$ is a composition algebra iff $N_{\mathbb{O}_\mathcal{F}}(Z \cdot_\epsilon W) = N_{\mathbb{O}_\mathcal{F}}(Z)N_{\mathbb{O}_\mathcal{F}}(W), \forall Z, W \in \mathbb{O}_\mathcal{F}$ and in this case we say that ϵ is a composition factor.*

The following proposition gives a necessary and sufficient condition for $(\mathbb{O}_\mathcal{F}, N_{\mathbb{O}_\mathcal{F}}, \epsilon)$ to be a composition algebra (see [6]).

Proposition 2.4 *With the notation above $(\mathbb{O}_\mathcal{F}, N_{\mathbb{O}_\mathcal{F}}, \epsilon)$ is a composition algebra iff:*

(i)	$N(P+R)\epsilon_{PQ}\epsilon_{QR} = 1$	*for any line $\{P, Q, R\}$,*
(ii)	$N(P+Q)\ \epsilon_{PQ}\epsilon_{QR}\epsilon_{RS}\epsilon_{SP}\ N(P+S) = -1$	*for any quadrilateral $\{P, Q, R, S\}$.*

Corollary 2.5 *If $(\mathbb{O}_{\mathcal{F}}, \mathbf{1}, \epsilon)$ is a composition algebra each line $\{P, Q, R = P+Q\}$ of $\mathcal{F}$ is canonically oriented by: $\epsilon_{PQ} = \epsilon_{QR} = \epsilon_{RP} = 1$.*

Example 2.6 *An oriented Fano plane $(\mathcal{F}, \tau)$ has a canonical composition factor ϵ^τ for the norm* $\mathbf{1}$ *defined using the Legendre symbol as follows. Fix $P_0 \in \mathcal{F}$ and for $R, S \in \mathcal{F}$ set*

$$\epsilon^\tau_{RS} \quad = \quad \left(\frac{j-i}{7}\right) \quad = \quad (j-i)^3 \pmod 7, \tag{2.4}$$

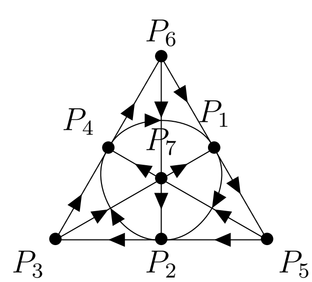
where $R = \tau^i(P_0)$ and $S = \tau^j(P_0)$. This is well-defined, independent of the choice of P_0 and $\epsilon^\tau = \epsilon^{\tau^k}$ (resp. $\epsilon^\tau = -\epsilon^{\tau^k}$) if $k = 1, 2, 4$ (resp. $k = 3, 5, 6$). Geometrically (see Figure 2) this is equivalent to:

$$\epsilon^\tau_{P_iP_j} \quad = \quad \begin{cases} \;\;1 & \text{if there is an arrow from } P_i \text{ to } P_j \\ -1 & \text{if there is an arrow from } P_j \text{ to } P_i \end{cases} .$$

One sees that every line is $\epsilon^\tau-$orientable and that there are seven $\epsilon^\tau-$orientable triangles $\{P_i, P_{i+2}, P_{i+3}\}$ $(i = 1, \cdots, 7)$. Notice that the complement of any line contains exactly one orientable triangle. In fact these geometric properties are true for any composition factor for the norm $\mathbf{1}$ *as shown in [6]. Notice also that the orientation induced on each line by τ (see Section 1.3) is the same as the orientation on each line by ϵ^τ (see Corollary 2.5).*

The multiplication table corresponding to ϵ^τ is then given in Table 1.

Table 1: Octonion multiplication and the oriented Fano plane.



↷	1	e_{P_1}	e_{P_2}	e_{P_3}	e_{P_4}	e_{P_5}	e_{P_6}	e_{P_7}
1	1	e_{P_1}	e_{P_2}	e_{P_3}	e_{P_4}	e_{P_5}	e_{P_6}	e_{P_7}
e_{P_1}	e_{P_1}	-1	e_{P_4}	e_{P_7}	$-e_{P_2}$	e_{P_6}	$-e_{P_5}$	$-e_{P_3}$
e_{P_2}	e_{P_2}	$-e_{P_4}$	-1	e_{P_5}	e_{P_1}	$-e_{P_3}$	e_{P_7}	$-e_{P_6}$
e_{P_3}	e_{P_3}	$-e_{P_7}$	$-e_{P_5}$	-1	e_{P_6}	e_{P_2}	$-e_{P_4}$	e_{P_1}
e_{P_4}	e_{P_4}	e_{P_2}	$-e_{P_1}$	$-e_{P_6}$	-1	e_{P_7}	e_{P_3}	$-e_{P_5}$
e_{P_5}	e_{P_5}	$-e_{P_6}$	e_{P_3}	$-e_{P_2}$	$-e_{P_7}$	-1	e_{P_1}	e_{P_4}
e_{P_6}	e_{P_6}	e_{P_5}	$-e_{P_7}$	e_{P_4}	$-e_{P_3}$	$-e_{P_1}$	-1	e_{P_2}
e_{P_7}	e_{P_7}	e_{P_3}	e_{P_6}	$-e_{P_1}$	e_{P_5}	$-e_{P_4}$	$-e_{P_2}$	-1

Later on we will need the following proposition whose proof is immediate.

Proposition 2.7

1. *For any $P \in \mathcal{F}$, e_P generates a two-dimensional composition subalgebra of $\mathbb{O}_{\mathcal{F}}$.*

2. *If $(P, Q, R) \in \mathcal{F}^3$ are distinct aligned points then $Vect\langle 1, e_P, e_Q, e_R\rangle$ is an associative subalgebra isomorphic to a quaternion subalgebra (which we also denote $\mathbb{H}_D$ where $D = \{P, Q, R\}$.)*

3. *If $(P,Q,R) \in \mathcal{F}^3$ are three non-aligned points then e_P, e_Q, e_R generate $\mathbb{O}_{\mathcal{F}}$.*

In [6] it was shown that if $(\mathbb{O}_{\mathcal{F}}, \mathbf{1}, \epsilon)$ is a composition algebra then either $\overleftarrow{P}$ is a line for all $P \in \mathcal{F}$ or $\overrightarrow{P}$ is a line for all $P \in \mathcal{F}$. Denoting $\mathbb{O}_{\mathcal{F}_1}^+$ (resp. $\mathbb{O}_{\mathcal{F}_1}^-$) the set of all composition algebras $(\mathbb{O}_{\mathcal{F}}, \mathbf{1}, \epsilon)$ such that $\overrightarrow{P}$ (resp. $\overleftarrow{P}$) is a line for all $P \in \mathcal{F}$, it was shown that $\mathbb{O}_{\mathcal{F}_1}^+$ (resp. $\mathbb{O}_{\mathcal{F}_1}^-$) is an affine space for $\mathcal{S}_0^2(V_{\mathcal{F}}^*)$, the space of bilinear forms B on $V_{\mathcal{F}}$ satisfying $B(P,P) = 0, \forall P \in \mathcal{F}$.

In fact, by the following lemma $\mathbb{O}_{\mathcal{F}_1}^+$ and $\mathbb{O}_{\mathcal{F}_1}^-$ are affine $V_{\mathcal{F}}-$spaces.

Lemma 2.8 *The vector spaces $\mathcal{S}_0^2(V_{\mathcal{F}}^*)$ and $V_{\mathcal{F}}$ are $GL(V_{\mathcal{F}})$ equivariantly isomorphic.*

Proof. Let $\wedge : V_{\mathcal{F}} \times V_{\mathcal{F}} \to (V_{\mathcal{F}})^*$ be the unique bilinear form such that

$$P \wedge Q = \begin{cases} 0 & \text{if } \; P = Q \; , \\ \text{the unique non-zero linear form} & \\ \text{which vanishes at } P \text{ and } Q & \text{if } \; P \neq Q \; . \end{cases}$$

One checks that the map $V_{\mathcal{F}} \to \mathcal{S}_0^2(V_{\mathcal{F}}^*)$ given by

$$P \mapsto \Big[(Q,R) \mapsto \big(Q \wedge R\big)(P)\Big]$$

is a $GL(V_{\mathcal{F}})$ equivariant isomorphism. QED

2.2 The action of Aut($\mathcal{F}$) on composition factors for the norm 1

In the rest of the paper we will assume that the norm $N = \mathbf{1}$.

Definition 2.9 (see [6]) *An oriented map is a map $\alpha : \mathcal{F} \to V_{\mathcal{F}}^*$ such that*

(a) For all $P \in \mathcal{F}, \alpha_P(P) = 1$.

(b) If $P \neq Q \in \mathcal{F}$ then: $\quad \alpha_P(Q) + \alpha_Q(P) = 1$.

We denote $\mathcal{F}_0$ the set of all oriented maps.

It was shown in [6] that 'exponentiation' defines a bijection from $\mathcal{F}_0$ to $\mathbb{O}_{\mathcal{F}_1}^+$ (resp. $\mathbb{O}_{\mathcal{F}_1}^-$). Recall the exponential map $e : \mathbb{Z}_2 \to S_2$ and its inverse $\ell : S_2 \to \mathbb{Z}_2$ are defined by

$$e^0 = 1 \quad , \quad e^1 = -1 \;\; \text{and} \;\; \ell(1) = 0 \quad , \quad \ell(-1) = 1 \; .$$

Definition 2.10 *Let $\mathcal{F}$ be a Fano plane, let α be an oriented map, and let ϵ be a composition factor and $g \in$Aut($\mathcal{F}$).*

1. *The oriented map $g \cdot \alpha$ is defined by*

$$(g \cdot \alpha)_P(Q) = \alpha_{g^{-1} \cdot P}(g^{-1} \cdot Q) \;\; \forall P, Q \in \mathcal{F} \; .$$

2. *The composition factor $g \cdot \epsilon$ is defined by*

$$(g \cdot \epsilon)_{PQ} = \epsilon_{g^{-1}\cdot P g^{-1}\cdot Q} \ , \quad \forall P \neq Q \in \mathcal{F} \quad .$$

This gives left actions of $Aut(\mathcal{F})$ on $\mathcal{F}_0$, $\mathbb{O}^{\pm}_{\mathcal{F}}$ that commute with the exponentiation of Theorem 4.9 [6].

Theorem 2.11 *The actions of $Aut(\mathcal{F})$ on $\mathcal{F}_0$, $\mathbb{O}_{\mathcal{F}1}^{+}$ and $\mathbb{O}_{\mathcal{F}1}^{-}$ are transitive.*

Proof. To prove the theorem it is is enough to prove that the action of $\mathrm{Aut}(\mathcal{F})$ on $\mathcal{F}_0$ is transitive. We begin by by proving a series of lemmas.

Lemma 2.12 *Let $P_0 \in \mathcal{F}$ and let $D_{P_0} = \{L \in \mathcal{F}^*, \ \ s.t. \ \ P_0 \notin L\}$. Then the map $E_{P_0} : \mathcal{F}_0 \to D_{P_0}$ defined by*

$$E_{P_0}(\alpha) = \alpha_{P_0}$$

is two-to-one surjective.

Proof. Let $\alpha, \beta \in \mathcal{F}_0$ be such that $E_{P_0}(\alpha) = E_{P_0}(\beta)$. By definition this means that

$$\alpha_{P_0}(P) = \beta_{P_0}(P) \ \ \forall P \in \mathcal{F} \ . \tag{2.5}$$

Since $\mathcal{F}_0$ is an affine $V_{\mathcal{F}}-$space (see Lemma 2.8) there exists a unique $Q \in V_{\mathcal{F}}$ such that $\alpha = \beta + P_0$, *i.e*,

$$\alpha_{P_0}(P) = \beta_{P_0}(P) + P_0 \wedge P(Q) \ \ \forall P \in \mathcal{F} \ . \tag{2.6}$$

By Eqs.[2.5-2.6] we have $P_0 \wedge P(Q) = 0$ for all $P \in \mathcal{F}$, and by Lemma 2.8 this implies that either $Q = 0$ or $Q = P_0$. However $\mathrm{Card}(\mathcal{F}_0) = 8$ and $\mathrm{Card}(D_{P_0}) = 4$ so this proves the lemma. QED

Lemma 2.13 *Let $P_0 \in \mathcal{F}$ and let $\alpha, \beta \in \mathcal{F}_0$ be such that $\alpha(P_0) = \beta(P_0)$. Then there exists $g \in \mathrm{Aut}(\mathcal{F})$ such that $\beta = g \cdot \alpha$.*

Proof. Since $\alpha(P_0) = \beta(P_0)$ by Lemma 2.12 either $\beta = \alpha$ or $\beta = \alpha + P_0$. In the first case $\beta = \mathrm{Id} \cdot \alpha$ so we can suppose that $\beta = \alpha + P_0$. Let $L = \{P_0, Q, R\}$ be any line through P_0 such $\alpha_{P_0}(Q) = 0$. Define $g \in \mathrm{Aut}(\mathcal{F})$ by

$$g \cdot P = \begin{cases} P & \text{if } P \in L \\ P + Q & \text{if } P \notin L \ . \end{cases}$$

Then one checks that $g \in \mathrm{Aut}(\mathcal{F})$ and that $\beta = \alpha + P_0 = g \cdot \alpha$. QED

We now prove the theorem. Let $\alpha, \beta \in \mathcal{F}_0$ and let $P_0 \in \mathcal{F}$. If $\alpha(P_0) = \beta(P_0)$ there exists $g \in \mathrm{Aut}(\mathcal{F})$ such $\beta = g \cdot \alpha$ by Lemma 2.13. If $\alpha(P_0) \neq \beta(P_0)$, since $\alpha(P_0)$ and $\beta(P_0)$ are two lines not containing P_0 there exists $g \in \mathrm{Aut}(\mathcal{F})$ such that $g \cdot P_0 = P_0$ and g maps the line $\alpha(P_0)$ to the line $\beta(P_0)$. Then β and $g \cdot \alpha$ are two elements of $\mathcal{F}_0$ having the same value at P_0. Again by Lemma 2.13 there exists $h \in \mathrm{Aut}(\mathcal{F})$ such that $\beta = h \cdot \alpha$ and this completes the proof of the theorem. QED

Remark 2.14 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane and fed The spaces $V_{\mathcal{F}}$ and $\mathcal{F}_0$ are affine $V_{\mathcal{F}}-$spaces and the action of* $\mathrm{Aut}(\mathcal{F})$ *on both is affine. However the action on the former is not transitive whereas the action on the latter is transitive. This implies that there are two non-conjugate embeddings of* $\mathrm{Aut}(\mathcal{F}) \cong GL(V_{\mathcal{F}})$ *in the affine group of $V_{\mathcal{F}}$. This phenomenon does not occur over $\mathbb{R}$ or $\mathbb{C}$, only over certain finite fields in certain dimensions.*

Remark 2.15 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane and let ϵ^τ be the canonical composition factor (see Example 2.6). By Theorem 2.11 the isotropy group $\mathcal{I}_{\epsilon^\tau} = \{g \in \mathrm{Aut}(\mathcal{F}) \text{ s.t. } \ g \cdot \epsilon^\tau = \epsilon^\tau\}$ has* $168/8 = 21$ *elements of which six are of of seven, fourteen are of order three and the identity. The subgroup $\mathbb{Z}_7(\epsilon^\tau)$ generated by τ is the cyclic group of order seven and one can show that $\mathcal{I}_{\epsilon^\tau}$ is the normaliser of $\mathbb{Z}_7(\epsilon^\tau)$ in Aut($\mathcal{F}$). There is an exact sequence*

$$1 \to \mathbb{Z}_7(\epsilon^\tau) \to \mathcal{I}_{\epsilon^\tau} \to \mathbb{Z}_3 \to 1 \ ,$$

and elements of order three fall into two classes: (1) $z \in \mathcal{I}_{\epsilon^\tau}$ such that $z \cdot g \cdot z^{-1} = g^2, \forall g \in \mathbb{Z}_7(\epsilon^\tau)$ and (2) $z \in \mathcal{I}_{\epsilon^\tau}$ such that $z \cdot g \cdot z^{-1} = g^4, \forall g \in \mathbb{Z}_7(\epsilon^\tau)$. For example one can take

$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 5 & 2 & 6 & 3 & 7 \end{pmatrix} \ , \quad z^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 4 & 6 & 1 & 3 & 5 & 7 \end{pmatrix} \ , \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \end{pmatrix}$$

which satisfy

$$z \cdot \tau \cdot z^{-1} = \tau^4 \ .$$

2.3 Radon transform

In this subsection we introduce the Radon transform associated to the incidence diagram of points and lines in the Fano plane.

Definition 2.16 *Let $\mathcal{F}$ be a Fano plane. The incidence space $\mathcal{I}$ is defined by*

$$\mathcal{I} = \Big\{(P, D) \in \mathcal{F} \times \mathcal{F}^* \quad s.t \quad P \in D\Big\} \ .$$

We denote $\mathcal{S}(\mathcal{F})$ (resp. $\mathcal{S}(\mathcal{F}^*)$) the set of functions from $\mathcal{F}$ to $\mathbb{Z}_2$ (resp. $\mathcal{F}^*$ to $\mathbb{Z}_2$) and set $\mathcal{S}_0(\mathcal{F}) = \big\{f \in \mathcal{S}(\mathcal{F}) \quad s.t. \quad \sum_{P \in \mathcal{F}} \ f(P) = 0\big\}$.

We have the incidence diagram

$$\begin{array}{ccc} & \mathcal{I} & \\ \swarrow & & \searrow \\ \mathcal{F} & & \mathcal{F}^* \end{array}$$

where the two maps are given by projections onto the first and second components respectively.

Definition 2.17 *If $f \in \mathcal{S}(\mathcal{F})$ we define its Radon transform $f^\star \in \mathcal{S}(\mathcal{F}^*)$ by*

$$f^\star(D) = \sum_{P \in D} f(P) \ .$$

We set $T(\mathcal{F}) = \big\{f \in \mathcal{S}(\mathcal{F}) \quad s.t. \quad f^\star \equiv 0\big\}$.

Remark 2.18 *The Radon transform is a linear but not multiplicative map.*

The next proposition characterises the image and the kernel of the Radon transform.

Proposition 2.19

1. *As a $\mathbb{Z}_2-$vector space the kernel $T(\mathcal{F})$ of the Radon transform is of dimension three.*
2. *For each $D \in \mathcal{F}^*$ define $T_D : \mathcal{F} \to \mathbb{Z}_2$ by*
$$T_D(P) = \begin{cases} 0 & \text{if} \quad P \in D \\ 1 & \text{if} \quad P \notin D \end{cases} .$$
Then $T(\mathcal{F}) = \Big\{ T_D \quad s.t \quad D \in \mathcal{F}^* \Big\} \cup \Big\{ 0 \Big\}$.
3. *Let $f \in \mathcal{S}(\mathcal{F}^\star)$. Then there exists $g \in \mathcal{S}(\mathcal{F})$ s.t. $g^\bigstar = f$ iff for any two triples (D_1, D_2, D_3) and (D'_1, D'_2, D'_3) of distinct concurrent lines, we have $\sum_{i=1}^{3} f(D_i) = \sum_{i=1}^{3} f(D'_i)$.*
4. *The image I of the Radon transform $\star : \mathcal{S}(\mathcal{F}) \to \mathcal{S}(\mathcal{F}^*)$ is given by*
$$I = \Big\{ T_P, T_P + 1 \quad s.t. \quad P \in \mathcal{F} \Big\} \cup \Big\{ 0, 1 \Big\}.$$
Here T_P is the function defined above corresponding to the line P of the Fano plane $\mathcal{F}^*$.
5. *Let $f \in \mathcal{S}(\mathcal{F})$. There exists $P \in \mathcal{F}$ such that $f^\bigstar = T_P$ or $f^\bigstar \equiv 0$ iff $f \in \mathcal{S}_0(\mathcal{F})$. In particular*
$$\mathcal{S}_0(\mathcal{F})^\bigstar = \Big\{ f \in \mathcal{S}(\mathcal{F}^*) \quad s.t. \quad \sum_{i=1}^{3} f(D_i) = 0 \ \text{ for distinct concurrent lines } D_1, D_2, D_3 \Big\} .$$

Proof. 1. Let f be in $T(\mathcal{F})$. Since its ´´average´´ on any line is zero it is easy to see that f is completely determined by its values at three non-collinear points (draw a picture).

2. Straightforward.

3. By the rank theorem the image of the Radon transform is of dimension 4 and cardinal 16. Let f be in $\mathcal{S}(\mathcal{F})$. If (D_1, D_2, D_3) are three distinct concurrent lines, we have

$$\sum_{i=1}^{3} f^\bigstar(D_i) = 3f(D_1 \cap D_2 \cap D_3) + \sum_{P \in \mathcal{F} \setminus (D_1 \cap D_2 \cap D_3)} f(P) = \sum_{P \in \mathcal{F}} f(P) \tag{2.7}$$

and therefore this sum is independent of the triple (D_1, D_2, D_3). However this sum is either equal to zero or to one and there are only $8 + 8$ such functions in $\mathcal{S}(\mathcal{F}^*)$ by the argument of 1.

4. Straightforward.

5. This follows from (2.7). QED

From now on we will use a multiplicative version of the Radon transform which we now explain. By abuse of language we refer to both the additive and multiplicative versions as the Radon transform. In the statement of the following proposition we use the mutually inverse group isomorphisms e and ℓ given in (2.5).

Corollary 2.20 *Define* $\mathcal{R}$ *and* $\mathcal{R}^\bigstar$ *by:*

- $\mathcal{R} = e(\mathcal{S}_0(\mathcal{F})) = \left\{h : \mathcal{F} \to S_2 \text{ s.t. } \prod_{P\in\mathcal{F}} h(P) = 1\right\}$;
- $\mathcal{R}^\bigstar = \left\{h : \mathcal{F}^* \to S_2, \text{ s.t. } \prod_{i=1}^{3} h(D_i) = 1 \text{ for distinct concurrent lines } D_1, D_2, D_3\right\}$.

Then $e \circ {}^\bigstar \circ \ell : \mathcal{R} \to \mathcal{R}^\bigstar$ *is a group homomorphism and defines an exact sequence*

$$1 \to e(T(\mathcal{F})) \to \mathcal{R} \to \mathcal{R}^\bigstar \to 1 \ .$$

In particular $\mathcal{R}$ *is of cardinal 64 and* $\mathcal{R}^\bigstar$ *of cardinal 8.*

Proof. After applying e, this follows from Proposition 2.19 (1) and (5). QED

Remark 2.21 *The groups* $e(T(\mathcal{F}))$ *and* $\mathcal{R}^\bigstar$ *are isomorphic to* $(\mathbb{Z}_2)^3$.

The eight functions of $\mathcal{R}^\bigstar$ can be represented by the diagrams in Figure 3.

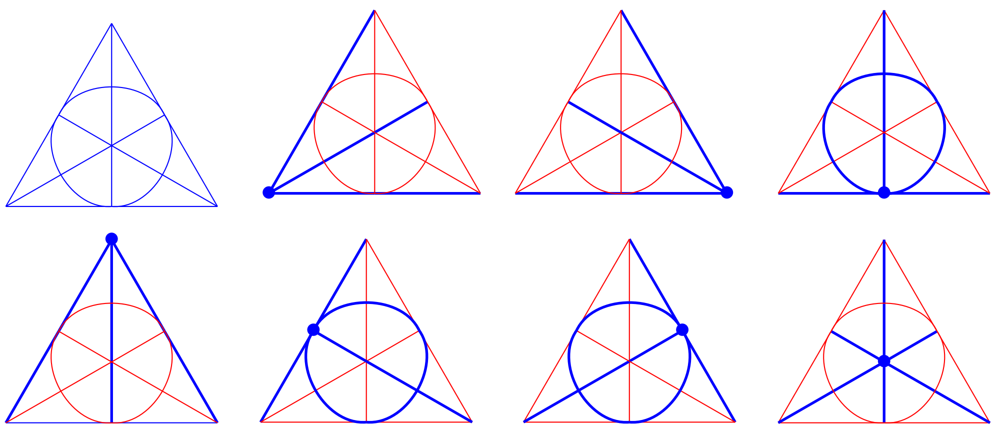


Figure 3: The eight functions of $\mathcal{R}^\bigstar$ where, for $e(T_P), P \in \mathcal{F}$, the point P is emphasised (blue lines: $e(T_P) = 1$, red lines: $e(T_P) = -1$).

Remark 2.22 *From Figure 3 we see that there is a natural one-to-one correspondence between* $\mathcal{R}^\bigstar$ *and* $V_\mathcal{F}$, *and with respect to the natural* $\mathbb{Z}_2-$*vector space structures this is a vector space isomorphism. We have already seem that* $\mathcal{F}_0$ *(see Definition 2.9) is affine space for* $V_\mathcal{F}$ *and so is also an affine space for* $\mathcal{R}^\bigstar$. *This action is as follows: for* $\alpha \in \mathcal{F}_0, f \in \mathcal{R}^\bigstar$ *and* $P \neq Q$, *change* $\alpha_P(Q)$ *only if* f *takes the value* -1 *on the line through* P *and* Q.

Example 2.23 *If* $P \in \mathcal{F}$ *denotes the point at top of the triangle, the fifth diagram in Figure 3 corresponds to the function* $e(T_P)$. *The eight functions in* $\mathcal{R}$ *whose Radon transform is* $e(T_P)$ *can be represented by the diagrams in Figure 4. The point* P *can be recovered from the diagrams representing these functions as follows (see Figure 4):*

1. *In the second diagram* P *is the unique point where the function is equal to* 1 *(blue).*
2. *In the next four diagrams* P *is the sum of the three blue points.*
3. *In the last three diagrams* P *is the sum of the two red points.*

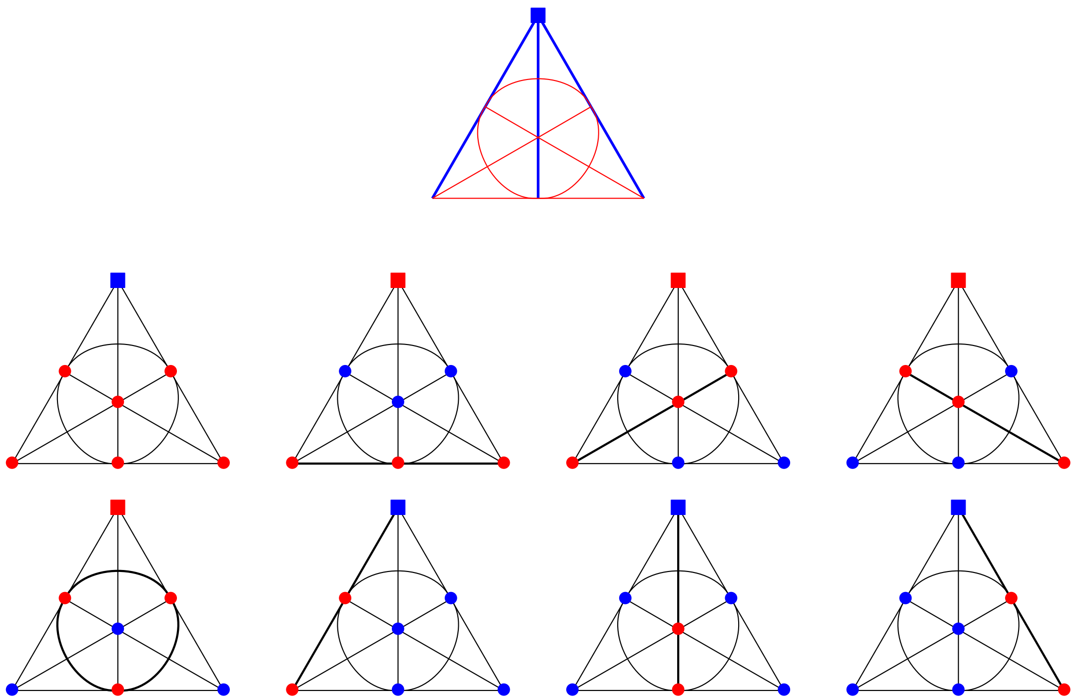


Figure 4: A function $f \in \mathcal{R}^\star$ and its eight inverse images under the Radon transform. (Blue = 1, red = −1.)

2.4 The line orientation invariant δ^* of a composition factor

Let $\mathcal{F}$ be a Fano plane and ϵ a composition factor. In this section we introduce an invariant which measures whether or not an element of $\mathrm{Aut}(\mathcal{F})$ preserves the line orientations induced by ϵ (see (1.2)).

Let g be an element of $\mathrm{Aut}(\mathcal{F})$ and let D be a line in $\mathcal{F}$. If P, Q are two distinct points of D we temporarily set:

$$\delta^*(g, P, Q) = \epsilon_{PQ}\epsilon_{g\cdot Pg\cdot Q} \ ,$$

This number is either 1 or −1, and we now show that it is independent of the choice of $P, Q \in D$.

Lemma 2.24 *With the notation above if P', Q' are two distinct points of D, then $\delta^*(g, P, Q) = \delta^*(g, P', Q')$,*

Proof. Let $D = \{P, Q, R\}$. Without loss of generality we can suppose that $P' = Q$ and $Q' = R$. From Proposition 2.4 we have

$$\epsilon_{PQ}\epsilon_{QR} = \epsilon_{g\cdot Pg\cdot Q}\epsilon_{g\cdot Qg\cdot R} = 1 \ .$$

Since ϵ only takes the values ±1 we have

$$\epsilon_{PQ}\epsilon_{g\cdot Pg\cdot Q} = \epsilon_{QR}\epsilon_{g\cdot Qg\cdot R} \ ,$$

which proves the lemma QED

This lemma shows that the following definition makes sense.

Definition 2.25 *Let $\mathcal{F}$ be a Fano plane and ϵ a composition factor on $\mathcal{F}$. Define $\delta^* : \mathrm{Aut}(\mathcal{F}) \times \mathcal{F}^* \to S_2$ by*

$$\delta^*(g, D) = \epsilon_{PQ}\epsilon_{g\cdot Pg\cdot Q} \ ,$$

for any distinct points (P, Q) of D.

Remark 2.26 *It is immediate from the definition above that:*

$$\delta^*(g_2 g_1, D) = \delta^*(g_2, g_1 \cdot D)\delta^*(g_1, D) \ .$$

The algebraic interpretation of δ^* is as follows: if $g \in$Aut$(\mathcal{F})$ and $D \in \mathcal{F}^*$ then $\delta^*(g, D) = 1$ *iff* g defines an inclusion of composition algebras $\mathbb{H}_D \hookrightarrow \mathbb{O}_{\mathcal{F}}$. The geometrical interpretation is: $\delta^*(g, D) = 1$ *iff* g induces an orientation preserving map from D to $g \cdot D$ (recall ϵ induces an orientation of D and $g \cdot D$ see Eq.[1.3]).

The function $(g, D) \mapsto \delta^*(g, D)$ is not arbitrary. In fact we will show that there are only eight possibilities as a consequence of the following two propositions.

Proposition 2.27 *Let $\mathcal{F}$ be a Fano plane, ϵ a composition factor on $\mathcal{F}$ and $g \in Aut(\mathcal{F})$. Then* $\det g := \prod_{D\in\mathcal{F}^*} \delta^*(g, D) = 1$.

Proof. Let $\mathcal{F}^* = \{D_1, \cdots, D_7\}$. Let $g, h \in \mathrm{Aut}(\mathcal{F})$ and for each line D_i choose distinct $P_i, Q_i \in D_i$. Then

$$\begin{aligned}\det g &= \prod_{i=1}^{7} \frac{\epsilon_{P_iQ_i}}{\epsilon_{g\cdot P_i g\cdot Q_i}},\\ \det h &= \prod_{i=1}^{7} \frac{\epsilon_{P_iQ_i}}{\epsilon_{h\cdot P_i h\cdot Q_i}}.\end{aligned}$$

Choosing the point $(g \cdot P_i, g \cdot Q_i)$ on the line $g \cdot D_i$, the expression for $\det h$ can be written

$$\det h = \prod_{i=1}^{7} \frac{\epsilon_{g\cdot P_i g\cdot Q_i}}{\epsilon_{hg\cdot P_i hg\cdot Q_i}}.$$

Thus

$$\det g \det h = \det(hg) \ ,$$

and det defines a group homomorphism from Aut$(\mathcal{F})$ to S_2. Since Aut$(\mathcal{F})$ is a finite simple group [7], this implies that $\det g = 1$ for any $g \in$Aut$(\mathcal{F})$. QED

Proposition 2.28 *Let $\mathcal{F}$ be a Fano plane, ϵ be a composition factor on $\mathcal{F}$, $g \in Aut(\mathcal{F})$ and $\{P, Q, R, S\}$ be a quadrilateral in $\mathcal{F}$. Then*

$$\delta^*\big(g, P \wedge Q\big)\delta^*\big(g, Q \wedge R\big) = \delta^*\big(g, P \wedge S\big)\delta^*\big(g, S \wedge R\big) \ .$$

Proof. By the quadrilateral rule (Proposition 2.4):

$$\epsilon_{PQ}\epsilon_{QR}\epsilon_{RS}\epsilon_{SP} = \epsilon_{g\cdot Pg\cdot Q}\epsilon_{g\cdot Qg\cdot R}\epsilon_{g\cdot Rg\cdot S}\epsilon_{g\cdot Sg\cdot P} = -1$$

which, using the definition of δ^*, implies

$$\delta^*\big(g, P \wedge Q\big)\delta^*\big(g, Q \wedge R\big) = \delta^*\big(g, P \wedge S\big)\delta^*\big(g, S \wedge R\big) \ .$$

QED

This proposition can be expressed in an equivalent more geometric form. For this let us remark the following property of lines in the Fano plane: if L is any line in the Fano plane there is a unique partition $\{L_1, L_2\}, \{L_3, L_4\}, \{L_5, L_6\}$ of the six remaining lines $L_1, \cdots, L_6$ such that $L = \{L_1 \cap L_2, L_3 \cap L_4, L_5 \cap L_6\}$.

Proposition 2.29 *Let $g \in Aut(\mathcal{F})$. Let L be a line of the Fano plane and let $L_1, \cdots, L_6$ be as above. Then,*

1. $\delta^*(g, L_1)\delta^*(g, L_2) = \delta^*(g, L_3)\delta^*(g, L_4) = \delta^*(g, L_5)\delta^*(g, L_6)$.
2. *Let P and P' be two points of the Fano plane and let D_1, D_2, D_3 (resp. D'_1, D'_2, D'_3) be the three lines passing through P (resp. P'). Then*

$$\delta^*(g, D_1)\delta^*(g, D_2)\delta^*(g, D_3) = \delta^*(g, D'_1)\delta^*(g, D'_2)\delta^*(g, D'_3) \ .$$

Proof. 1: When the line L is removed from the Fano plane $\mathcal{F}$, the six remaining lines are exactly the six sides of the quadrilateral $\mathcal{F} \setminus L$, and the partition above (L_1, L_2), (L_3, L_4), (L_5, L_6) is obtained by grouping opposite sides of the quadrilateral. We can certainly label the vertices of the quadrilateral $\mathcal{F} \setminus L$ in such a way that:

$$L_1 = P \wedge Q \ , \quad L_2 = R \wedge S \ , \quad L_3 = Q \wedge R \ , \quad L_4 = S \wedge P \ , \quad L_5 = R \wedge P \ , \quad L_6 = Q \wedge S$$

By the proposition above we have

$$\delta^*(g, P \wedge Q)\delta^*(g, Q \wedge R) = \delta^*(g, P \wedge S)\delta^*(g, S \wedge R) \ ,$$

which implies

$$\delta^*(g, L_1)\delta^*(g, L_3) = \delta^*(g, L_4)\delta^*(g, L_2) \ .$$

Multiplying this equation by $\delta^*(g, L_3)\delta^*(g, L_2)$ gives

$$\delta^*(g, L_1)\delta^*(g, L_2) = \delta^*(g, L_4)\delta^*(g, L_3) \ .$$

This proves one of the desired equalities, the others follow in a similar fashion.

2: It is always possible to find two points R and S such that $\{P, P', R, S\}$ is a quadrilateral, *i.e.*, the complement of a line. The three lines passing through P are $P \wedge P', P \wedge R, P \wedge S$ and the lines passing through P' are $P \wedge P', P' \wedge R, P' \wedge S$. To prove 2 we have to show that

$$\delta^*(g, P \wedge P')\delta^*(g, P \wedge R)\delta^*(g, P \wedge S) = \delta^*(g, P \wedge P')\delta^*(g, P' \wedge S)\delta^*(g, P' \wedge R) \ ,$$

which is equivalent to

$$\delta^*(g, P \wedge R)\delta^*(g, P \wedge S) = \delta^*(g, P' \wedge S)\delta^*(g, P' \wedge R) \ ,$$

and this follows from 1. QED

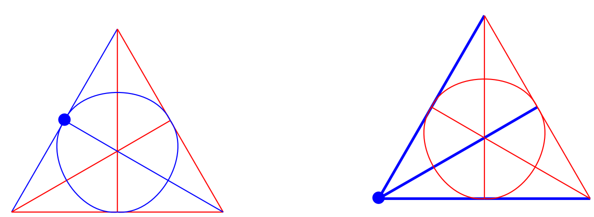
Corollary 2.30 *Let $P \in \mathcal{F}$, let D_1, D_2, D_3 be the three lines passing through P and let $g \in$Aut$(\mathcal{F})$. Then $\delta^*(g, D_1)\delta^*(g, D_2)\delta^*(g, D_3) = 1$.*

Proof. Suppose for contradiction that $\delta^*(g, D_1)\delta^*(g, D_2)\delta^*(g, D_3) = -1$. It follows from Proposition 2.29 2 that the corresponding product $\eta_{P'}$ for any point P' in $\mathcal{F}$ is also equal to -1 and hence that $\prod_{P' \in \mathcal{F}} \eta_{P'} = -1$. Since this product is also equal to $\det(g)^3$ this contradicts Proposition 2.27 and the corollary is proved. QED

Theorem 2.31 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane, let ϵ^τ be the canonical composition factor and let $g \in$ Aut$(\mathcal{F})$. Then $\delta^*(g, \quad) : \mathcal{F}^* \to S_2$ is given by one of the eight diagrams in Figure 3.*

Proof. By Corollary 2.30 the function $\delta^*(g, \dot{\ }) : \mathcal{F}^* \to S_2$ is in $\mathcal{R}^\bigstar$. By Proposition 2.20 there are exactly eight possibilities, which are the functions $e(T_P)$ for $P \in \mathcal{F}$, and the constant function equal to one. QED

Example 2.32 *For the generators a, b in (1.1) the functions $\delta^*(a, \cdot), \delta^*(b, \cdot)$ for the canonical composition factor ϵ^τ are given in Figure 5.*



$\delta^*(a, D)$ $\qquad$ $\delta^*(b, D)$

Figure 5: Values of δ^* for the generators a, b (blue lines: $\delta^* = 1$, red lines: $\delta^* = -1$).

Recall that the group Aut$(\mathcal{F})$ has 168 elements and that for fixed g in Aut$(\mathcal{F})$ there are 8 possibilities for the function $\delta^*(g, \cdot) : \mathcal{F}^* \to S_2$ given by the diagrams in Figure 3. We now show that for each diagram there are exactly 21 elements of Aut$(\mathcal{F})$ corresponding to the same diagram.

Proposition 2.33 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane and let ϵ^τ be the canonical composition factor. Let $\mathcal{I}_{\epsilon^\tau} = \{g \in Aut(\mathcal{F}) : g \cdot \epsilon^\tau = \epsilon^\tau\}$ (see Remark 2.15).*

1. *If $g \in \mathcal{I}_{\epsilon^\tau}$ then $\delta^*(g, D) = 1, \forall D \in \mathcal{F}^*$.*
2. *If $g \in \mathcal{I}_{\epsilon^\tau}$ and $h \in Aut(\mathcal{F})$ then $\delta^*(gh, D) = \delta^*(h, D)$, and $\delta^*(hg, D) = \delta^*(h, g \cdot D), \forall D \in \mathcal{F}^*$.*

Proof. 1: This follows since

$$\begin{cases} g \cdot \epsilon^\tau = \epsilon^\tau \ , \\ \delta^*(g, P \wedge Q) = \epsilon^\tau_{PQ} \epsilon^\tau_{g \cdot P g \cdot Q} \ . \end{cases}$$

2: This follows from the easily checked "multiplier" formula

$$\delta^*(gh, D) = \delta^*(g, h \cdot D)\delta^*(h, D) \ , \tag{2.8}$$

and 1 above. QED

Corollary 2.34 *For each of the 8 diagrams in Figure 3 there are exactly 21 elements $g \in Aut(\mathcal{F})$ whose function $\delta^*(g, \cdot)$ corresponds to that diagram.*

Proof. By Proposition 2.33 (2) the map δ^* factors to a map from the left-coset space $\mathcal{I}_{\epsilon^\tau} \backslash \mathrm{Aut}(\mathcal{F})$ to the set of diagrams. Note that the coset space has 8 elements since $|\mathcal{I}_{\epsilon^\tau}| = 21$ (see Remark 2.15). To show that this map is a bijection it is sufficient to show that it is surjective.

By Proposition 2.33, we have $\delta^*(\tau, D) = 1, \forall D \in \mathcal{F}^*$ and hence $\delta^*(\tau, \cdot)$ corresponds to the first diagram in Figure 3. The other seven diagrams are characterised by the fact that their is a unique point P such that $\delta^*(g, D) = 1$ iff $P \in D$. From this point of view the function $\delta^*(a, \cdot)$ corresponds to the point P_4 (See Figures 5). However, by 2.33 (2), the function $\delta^*(a\tau^k, \cdot)$ corresponds to the diagram with distinguished point $\tau^{-k}(P_4)$, and, since τ is of order seven, we obtain all seven diagrams in this way. QED

Remark 2.35 *By the multiplier formula above and Eq.[1.1] the function $\delta^*(\cdot, \cdot)$ is completely determined by $\delta^*(a, \cdot)$ and $\delta^*(b, \cdot)$.*

We can now prove the converse of Proposition 1.12:

Proposition 2.36 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane and let ϵ^τ be the canonical composition factor. If τ' is an automorphism of order seven which induces the same orientation as τ on every line $D \in \mathcal{F}^*$, then $\tau' = \tau, \tau^2$ or τ^4.*

Proof. If τ' induces the same orientation as τ on every line then $\delta^*(\tau', D) = \delta^*(\tau, D) = 1$ for all $D \in \mathcal{F}^*$. Since there are 21 elements of $\mathrm{Aut}(\mathcal{F})$ with this δ^* (see Corollary 2.34) and since $|\mathcal{I}_{\epsilon^\tau}| = 21$, it follows from Proposition 2.33 (1) that $\tau' \in \mathcal{I}_{\epsilon^\tau}$. The only elements of order seven in $\mathcal{I}_{\epsilon^\tau}$ are the six powers of τ (see Remark 2.15) and only $\tau' = \tau, \tau^2$ or τ^4 induce the same orientation on every line (Proposition 1.12). QED

2.5 Automorphisms of the augmented Fano plane

In this section we investigate to what extent automorphisms of the Fano plane can be lifted to automorphisms of the octonions.

Proposition 2.37 *Let $\mathcal{F}$ be a Fano plane and let ϵ be a composition factor. Let $g \in \mathrm{Aut}(\mathcal{F})$ and denote also by $g \in \mathrm{End}(\mathbb{O}_{\mathcal{F}})$ the unique element such that*

$$g \cdot e_P = e_{g \cdot P} \quad \forall P \in \mathcal{F} \ .$$

Let $D \in \mathcal{F}^*$ and let $\mathbb{H}_D$ denote the subalgebra of $\mathbb{O}_{\mathcal{F}}$ generated by the points of D (see Proposition 2.7 (2)). Then the following are equivalent:

1. *$g|_{\mathbb{H}_D}$ is an algebra homomorphism;*
2. *$\delta^*(g, D) = 1$.*

Proof. The proposition is a consequence of the following equivalences:

$$\begin{array}{ll} & g|_{\mathbb{H}_D} \ \text{ is an algebra homomorphism} \\ \Leftrightarrow & g(e_P \cdot e_Q) = g(e_P) \cdot g(e_Q) \ , \forall P, Q \in D \\ \Leftrightarrow & \epsilon_{PQ} e_{g\cdot(P+Q)} = \epsilon_{g\cdot P g \cdot Q} e_{g\cdot P + g\cdot Q} \qquad (cf.\ \text{Definition } 2.3) \\ \Leftrightarrow & \epsilon_{PQ} = \epsilon_{g\cdot P g\cdot Q} \\ \Leftrightarrow & \delta^*(g, D) = 1 \qquad (cf.\ \text{Definition } 2.25) \end{array} \tag{2.9}$$

QED

This means that the canonical lift of $g \in$Aut$(\mathcal{F})$ acts by automorphism on $\mathbb{O}_{\mathcal{F}}$ *iff* $\delta^*(g, D) = 1, \forall D \in \mathcal{F}^*$. By Corollary 2.34 there certainly exists $g \in$Aut$(\mathcal{F})$ and $D \in \mathcal{F}^*$ such that $\delta^*(g, D) \neq 1$. However, we now exhibit a finite group which acts on the octonions by automorphism and which is an eightfold non-trivial covering of Aut$(\mathcal{F})$. To this end we introduce the augmented Fano plane.

Definition 2.38 *Let $\mathcal{F} = \{P_1, \cdots, P_7\}$ be a Fano plane, ϵ be a composition factor and $\cdot$ be the associated composition product on $\mathbb{O}_{\mathcal{F}}$.*

1. *The augmented Fano plane $\hat{\mathcal{F}}_\epsilon$ is the subset of Im$(\mathbb{O}_{\mathcal{F}})$ defined by*

$$\hat{\mathcal{F}}_\epsilon = \Big\{ \pm e_{P_1}, \cdots, \pm e_{P_7} \Big\} \ .$$

2. *The group Aut$(\hat{\mathcal{F}}_\epsilon)$ of automorphisms of the augmented Fano plane is defined by*

$$Aut(\hat{\mathcal{F}}_\epsilon) = \Big\{ \hat{g} : \hat{\mathcal{F}}_\epsilon \to \hat{\mathcal{F}}_\epsilon \ s.t. \quad (i) \quad \hat{g}(-e_P) = -\hat{g}(e_P) \ \ \forall P \in \mathcal{F} \ ; \quad (ii) \quad \hat{g}(e_P \cdot e_Q) = \hat{g}(e_P) \cdot \hat{g}(e_Q) \ \ \forall P \neq Q \in \mathcal{F} \Big\} \ .$$

Remark 2.39 *Any $\hat{g} : \hat{\mathcal{F}}_\epsilon \to \hat{\mathcal{F}}_\epsilon$ satisfying (i) above extends first to an invertible linear map $\tilde{g} : \mathrm{Im}(\mathbb{O}_{\mathcal{F}}) \to \mathrm{Im}(\mathbb{O}_{\mathcal{F}})$, and then to an invertible linear map $g_{\mathbb{O}_{\mathcal{F}}} : \mathbb{O}_{\mathcal{F}} \to \mathbb{O}_{\mathcal{F}}$ by setting $g_{\mathbb{O}_{\mathcal{F}}}(1) = 1$. It is clear that $g \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ iff $g_{\mathbb{O}_{\mathcal{F}}}$ is an automorphism of $\mathbb{O}_{\mathcal{F}}$.*

There is a natural homomorphism π from $\mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ to $\mathrm{Aut}(\mathfrak{F})$ such that

$$\begin{array}{ccc} \hat{\mathfrak{F}}_\epsilon & \xrightarrow{\hat{g}} & \hat{\mathfrak{F}}_\epsilon \\ \big\downarrow /\mathbb{Z}_2 & & \big\downarrow /\mathbb{Z}_2 \\ \mathfrak{F} & \xrightarrow{\pi(\hat{g})} & \mathfrak{F} \end{array} \tag{2.10}$$

is commutative and we have the following proposition:

Proposition 2.40 *Let $g \in \mathrm{Aut}(\mathfrak{F})$ and let $\tilde{\delta} : \mathfrak{F} \to S_2$. We define $\hat{g} : \hat{\mathfrak{F}}_\epsilon \to \hat{\mathfrak{F}}_\epsilon$ by*

$$\hat{g} \cdot e_P = \tilde{\delta}(P) e_{g \cdot P} \ .$$

Then $\hat{g} \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ iff for any line $D = \{P, Q, R\}$ we have

$$\delta^*(g, D) = \tilde{\delta}(P)\tilde{\delta}(Q)\tilde{\delta}(R) \ , \tag{2.11}$$

and in this case $\pi(\hat{g}) = g$.

Proof. Let $P, Q \in \mathfrak{F}$ be two distinct points and let $R = P + Q$. Without loss of generality we can suppose that $\epsilon_{PQ} = 1$ and since $e_P \cdot e_Q = e_R$ we have

$$e_{g \cdot P} \cdot e_{g \cdot Q} = \epsilon_{rg \cdot P \, g \cdot Q} \; e_{g \cdot R} = \delta^*(g, D) e_{g \cdot R}$$

where D is the line $\{P, Q, R\}$. It follows that

$$\big(\hat{g} \cdot e_P\big) \cdot \big(\hat{g} \cdot e_Q\big) = \tilde{\delta}(P)\tilde{\delta}(Q) \; e_{g \cdot P} \cdot e_{g \cdot Q} = \tilde{\delta}(P)\tilde{\delta}(Q)\delta^*(g, D) e_{g \cdot R} \ .$$

On the other hand,

$$\hat{g} \cdot \big(e_P \cdot e_Q\big) = \hat{g} \cdot e_R = \tilde{\delta}(R) e_{g \cdot R} \ ,$$

which proves the result. QED

Corollary 2.41 *Let $\mathfrak{F}$ be a Fano plane and ϵ be a composition factor.*

1. *For each $g \in \mathrm{Aut}(\mathfrak{F})$ there exist exactly eight elements $\hat{g} \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ such that $\pi(\hat{g}) = g$.*
2. *For $D \in \mathfrak{F}^*$ we define $t_D : \hat{\mathfrak{F}}_\epsilon \to \hat{\mathfrak{F}}_\epsilon$ by (see also [8])*

$$t_D(\pm e_P) = \pm e \circ T_D(P) \; e_P = \begin{cases} \pm e_P & \text{if } P \in D \\ \mp e_P & \text{if } P \notin D \ . \end{cases}$$

(Here $e : \mathbb{Z}_2 \to S_2$ in $e \circ T_D$ denotes the exponential map.) Then $t_D \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ and $\mathrm{Ker}(\pi) = \big\{\mathrm{Id}\big\} \cup \big\{t_D \;\; s.t. \;\; D \in \mathfrak{F}^*\big\}$.

Proof. (1) Given $g\in$Aut$(\mathcal{F})$, finding $\hat{g}\in$Aut$(\hat{\mathcal{F}}_\epsilon)$ such $\pi(\hat{g})=g$ is equivalent to finding a function $\tilde{\delta}:\mathcal{F}\to S_2$ such that Eq.[2.11] is satisfied. But in the language of Corollary 2.20 this means precisely that the multiplicative Radon transform of $\tilde{\delta}$ is $\delta^*(g,\cdot)$. However, $\delta^*(g,\cdot)\in\mathcal{R}^\bigstar$ (see Corollary 2.30) and again by Corollary 2.20 any element of $\mathcal{R}^\bigstar$ is the Radon transform of eight distinct elements of $\mathcal{R}$ (see Eq.[2.7]).

(2) We have $\delta^*(\mathrm{Id},D)=1$ for all $D\in\mathcal{F}^*$ and we know that by Corollary 2.20 $E\circ{}^\bigstar\circ L(f)=1$ *iff* $f\in e(T(\mathcal{F}))$. This proves the result. QED

Remark 2.42 *Recall from Example 1.6 that the elements*

$$a=\begin{pmatrix}1&2&3&4&5&6&7\\1&2&7&4&6&5&3\end{pmatrix}\ ,\quad b=\begin{pmatrix}1&2&3&4&5&6&7\\2&7&4&6&5&3&1\end{pmatrix}$$

generate Aut($\mathcal{F}$). One checks (with the obvious abuse of notation) that

$$\hat{a}=\begin{pmatrix}1&2&3&4&5&6&7\\1&2&7&4&-6&5&-3\end{pmatrix}\ ,\quad \hat{b}=\begin{pmatrix}1&2&3&4&5&6&7\\-2&7&4&6&5&3&-1\end{pmatrix}\tag{2.12}$$

are elements of Aut($\hat{\mathcal{F}}_\epsilon$) satisfying $\pi(\hat{a})=a$ and $\pi(\hat{b})=b$.

Remark 2.43 *A natural question is whether or not the exact sequence*

$$1\to Ker(\pi)\to \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)\to\mathrm{Aut}(\mathcal{F})\to 1$$

is split. In fact it is known that it is not split [12] and the easiest way to see this as follows. Consider the order four element $c\in\mathrm{Aut}(\mathcal{F})$ given by

$$c=\begin{pmatrix}1&2&3&4&5&6&7\\4&2&6&1&7&5&3\end{pmatrix}\ .$$

One checks that

$$\hat{c}=\begin{pmatrix}1&2&3&4&5&6&7\\-4&2&6&1&7&5&-3\end{pmatrix}\ ,$$

is an order 8 element of $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ satisfying $\pi(\hat{c})=c$. The other elements of $\pi^{-1}(c)$ are $t_D\circ\hat{c}$ where D is any line of $\mathcal{F}$ (see Proposition 2.41) and, as one also checks, they too are of order 8. This shows that the exact sequence above is not split.

Note that if $g\in\mathrm{Aut}(\mathcal{F})$ is of order 2 (resp. 3) then $\pi^{-1}(g)$ consists of four elements of order 2 and four elements of order 4 (resp. four elements of order 3 and four elements of order 6). If $g\in\mathrm{Aut}(\mathcal{F})$ is of order 7 then $\pi^{-1}(g)$ consists of elements of order 7.

3 The Lie algebra $\mathfrak{g}_2(\mathbb{F})$

Throughout this section $\mathfrak{so}(n,\mathbb{F})$ denotes the Lie algebra of $n\times n$ skew-symmetric matrices over $\mathbb{F}$. Recall that $\mathfrak{so}(4,\mathbb{F})$ is isomorphic to $\mathfrak{so}(3,\mathbb{F})\times\mathfrak{so}(3,\mathbb{F})$. If $\mathbb{F}$ admits a non-trivial quadratic *extension* $\hat{\mathbb{F}}$ *then we denote by* $\mathfrak{su}(n, \mathbb{F})$ *the* $\mathbb{F}-$*Lie algebra of traceless, anti-hermitian* $n \times n$ *matrices over* $\hat{\mathbb{F}}$.

In this section we fix an oriented Fano plane $(\mathcal{F}, \tau)$ *of type* $(0, 1, 3)$ and the associated canonical composition factor will be denoted ϵ. It is well-known that when $\mathbb{F} = \mathbb{R}$ the group of automorphisms of $\mathbb{O}_{\mathcal{F}}$ is a simple, compact exceptional Lie group usually denoted G_2 (see for instance [11]). R. Wilson in [8] gave an "elementary" construction of $\mathfrak{g}_2$ (the Lie algebra of G_2) together with an action by automorphism of a finite group of order 1344. This is exactly the $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ of Section 2.5. In this section we shall express the commutation relations of $\mathfrak{g}_2$ in terms of the incidence relations of the oriented Fano plane. As a consequence we will show that to each point P of the Fano plane one can associate a Cartan subalgebra $\mathfrak{h}_P$ of $\mathfrak{g}_2(\mathbb{F})$. This association has the following remarkable properties which reflect the geometry of the Fano plane $\mathcal{F}$:

(a) there is a decomposition: $\mathfrak{g}_2(\mathbb{F}) = \bigoplus\limits_{P \in \mathcal{F}} \mathfrak{h}_P$ and if $P \neq Q$ then $[\mathfrak{h}_P, \mathfrak{h}_Q] = \mathfrak{h}_{P+Q}$;

(b) for each line D in $\mathcal{F}$ we have $\mathfrak{s}_D = \bigoplus\limits_{P \in D} \mathfrak{h}_P$ is a Lie subalgebra of $\mathfrak{g}_2(\mathbb{F})$ isomorphic to $\mathfrak{so}(3, \mathbb{F}) \times \mathfrak{so}(3, \mathbb{F})$ and such that $\mathfrak{h}_P$ is a Cartan subalgebra for $P \in D$;

(c) for each point P in $\mathcal{F}$ we have

$$\mathfrak{s}_P = \big\{g \in \mathfrak{g}_2(\mathbb{F}) \ \text{ s.t. } \ g \cdot P = 0\big\} \ ,$$

is a Lie subalgebra isomorphic to $\mathfrak{su}(3, \mathbb{F})$ or $\mathfrak{sl}(3, \mathbb{F})$ containing $\mathfrak{h}_P$ as a Cartan subalgebra;

(d) if $P_1, P_2, P_3 \in \mathcal{F}$ are not aligned then $\mathfrak{h}_{P_1} \oplus \mathfrak{h}_{P_2} \oplus \mathfrak{h}_{P_3}$ generates $\mathfrak{g}_2(\mathbb{F})$;

(e) to the three lines containing a point P, one can associate three elements of of $\mathfrak{h}_P$ whose sum is zero and which generate the root diagram of $\mathfrak{g}_2(\mathbb{F})$.

If $\mathfrak{g}$ is a simple Lie algebra, a decomposition $\mathfrak{g} = \oplus_i \mathfrak{h}_i$ as an orthogonal sum of Cartan subalgebras satisfying $[\mathfrak{h}_i, \mathfrak{h}_j] \subseteq \mathfrak{h}_{k(ij)}$ is called a multiplicative orthogonal decomposition [8, 9].

To prove these results our main tool is the observation that $\mathbb{O}_{\mathcal{F}}$ is a space of spinors of $\mathrm{Im}(\mathbb{O}_{\mathcal{F}})$ with respect to octonion multiplication. This enables us to realise $\mathfrak{g}_2$ inside a Clifford algebra, and use Clifford algebra techniques to perform calculations.

3.1 Octonions, spinors of $\mathfrak{so}(7, \mathbb{F})$ and the Lie algebra $\mathfrak{g}_2(\mathbb{F})$

Let $\mathbb{O}_{\mathcal{F}}$ be the octonions defined by Table 1, let $\mathrm{Im}(\mathbb{O}_{\mathcal{F}})$ be the purely imaginary octonions and let B be the symmetric bilinear form associated to the norm **1**. In this subsection we realise the spinor representation of $\mathfrak{so}(\mathrm{Im}(\mathbb{O}_{\mathcal{F}}), -B)$ in terms of octonion multiplication (see *e.g.* [11] p. 121).

Define $\rho : \mathrm{Im}(\mathbb{O}_{\mathcal{F}}) \to \mathrm{End}(\mathbb{O}_{\mathcal{F}})$ by

$$\rho(x)(o) = x \cdot o \ , \quad \forall x \in \mathrm{Im}(\mathbb{O}_{\mathcal{F}}) \ , \quad \forall o \in \mathbb{O}_{\mathcal{F}} \ .$$

Octonion multiplication is not associative but it is alternate (*i.e.*, the associator is antisymmetric) and this implies

$$(\rho(x))^2 = -B(x,x)\mathrm{Id}$$

and hence ρ extends to a surjective associative algebra homomorphism $\hat{\rho}$ with kernel $\mathcal{J}$ of dimension 2^6:

$$1 \to \mathcal{J} \to C(\mathrm{Im}(\mathbb{O}_{\mathcal{F}}), -B) \xrightarrow{\hat{\rho}} \mathrm{End}(\mathbb{O}_{\mathcal{F}}) \to 1 \ .$$

With respect to the well known decomposition

$$C(\mathrm{Im}(\mathbb{O}_{\mathcal{F}}), -B) = \bigoplus_{k=0}^{7} C^k \ .$$

we have $\mathcal{J} \cap C^k = \{0\}$. Furthermore, C^1 is canonically identified with $\mathrm{Im}(\mathbb{O}_{\mathcal{F}})$ and C^2, spanned by elements of the form $xy - yx\, (x, y \in C^1)$, is closed for the commutator and isomorphic to $\mathfrak{so}(7, \mathbb{F})$ as a Lie algebra. The bracket of C^2 with C^1 defines an action of the Lie algebra C^2 on C^1 which is isomorphic to the vector representation of $\mathfrak{so}(7, \mathbb{F})$.

Definition 3.1 *Set*

$$\mathfrak{g}_2(\mathbb{F}) = \left\{ c \in C^2 \ \ \text{s.t.} \ \ \hat{\rho}(c)(1) = 0 \right\} \ .$$

This is a Lie subalgebra of $\mathfrak{so}(7, \mathbb{F})$.

Remark 3.2 *The Clifford algebra* $C(Im(\mathbb{O}_{\mathcal{F}}), -B)$ *has a two-dimensional centre Vect*$(1, \epsilon)$ *where* $\epsilon^2 = 1$. *This implies that* $1/2(1+\epsilon)\big(\mathfrak{g}_2(\mathbb{F})\big)$ *is a Lie subalgebra of* $C(Im(\mathbb{O}_{\mathcal{F}}), -B)$ *isomorphic to* $\mathfrak{g}_2(\mathbb{F})$. *This embedding is used in [13].*

Remark 3.3 *It can be shown (see e.g. [11] p. 122) that this Lie algebra is isomorphic to the Lie algebra of the group of automorphisms of* $\mathbb{O}_{\mathcal{F}}$.

Recall that $\mathcal{F} = \{P_i \ , i \in \mathbb{Z}_7\}$ where for all i, j in $\mathbb{Z}_7$ we have $P_{i+j} = \tau^j(P_i)$ (see Figure 2). The e_{P_i} form an orthonormal basis of $\mathrm{Im}(\mathbb{O}_{\mathcal{F}})$ and recall that the action of C^2 on C^1 in this basis is given by

$$\begin{aligned} \left[e_{P_iP_j}, e_{P_k}\right] &= \delta_{ik} e_{P_j} - \delta_{jk} e_{P_i} \ , \\ \left[e_{P_iP_j}, e_{P_kP_\ell}\right] &= \delta_{ik} e_{P_jP_\ell} - \delta_{jk} e_{P_iP_\ell} + \delta_{i\ell} e_{P_kP_j} - \delta_{j\ell} e_{P_kP_i} \ . \end{aligned} \tag{3.13}$$

where $e_{P_iP_j} = \frac{1}{2}\, e_{P_i} e_{P_j}, i \neq j$. To simplify notation in what follows we set

$$e_{P_i,P_j} = \hat{\rho}(e_{P_iP_j}) = \frac{1}{4}\hat{\rho}[e_{P_i}, e_{P_j}] = \frac{1}{4}\big(\hat{\rho}(e_{P_i})\hat{\rho}(e_{P_j}) - \hat{\rho}(e_{P_j})\hat{\rho}(e_{P_i})\big), \ \ 1 \leq i \neq j \leq 7 \ .$$

To each point P_i of the Fano plane we now associate a Cartan subalgebra $\mathfrak{h}_{P_i}$ of $\mathfrak{g}_2(\mathbb{F})$. The idea is that each of the three lines containing P_i enables us to write e_{P_i} as a product. For instance we have (see Table 1)

$$e_{P_1} = e_{P_5} \cdot e_{P_6} = e_{P_3} \cdot e_{P_7} = e_{P_2} \cdot e_{P_4} \ .$$

Hence

$$\big(e_{P_3,P_7} - e_{P_5,P_6}\big)(1) = \big(e_{P_5,P_6} - e_{P_2,P_4}\big)(1) = \big(e_{P_2,P_4} - e_{P_3,P_7}\big)(1) = 0 \ ,$$

and $e_{P_3,P_7} - e_{P_5,P_6}, e_{P_5,P_6} - e_{P_2,P_4}, e_{P_2,P_4} - e_{P_3,P_7}$ are in $\mathfrak{g}_2(\mathbb{F})$ by Definition 3.1.

The calculations above motivates the following definition in which we define a function $X : \mathcal{I} \to \mathfrak{g}_2(\mathbb{F})$ where $\mathcal{I} = \{(P, D) \in \mathcal{F} \times \mathcal{F}^* \ \text{ s.t. } \ P \in D\}$ is the incidence space:

Definition/Proposition 3.4 *Given a point P_i in $\mathcal{F}$ recall that the three lines containing P_i (see Proposition 1.11) are:*

$$D_i = \{P_i, P_{i+1}, P_{i+3}\} \ , \quad D_{i-1} = \{P_{i-1}, P_i, P_{i+2}\} \ , \ \ D_{i-3} = \{P_{i-3}, P_{i-2}, P_i\} \ .$$

Define $X_{P_i,D_i}, X_{P_i,D_{i-1}}, X_{P_i,D_{i-3}} \in \hat{\rho}(C^2)$ by

$$\begin{array}{rcl} X_{P_i,D_i} & = & e_{P_{i+2},P_{i-1}} - e_{P_{i-3},P_{i-2}} \ , \\ X_{P_i,D_{i-1}} & = & e_{P_{i-3},P_{i-2}} - e_{P_{i+1},P_{i+3}} \ , \\ X_{P_i,D_{i-3}} & = & e_{P_{i+1},P_{i+3}} - e_{P_{i+2},P_{i-1}} \ . \end{array} \tag{3.14}$$

Then

1. *The action of $X_{P,D}$ on e_Q is given by*

$$\big[X_{P,D}, e_Q\big] = \left\{ \begin{array}{ll} 0 & \text{if } Q \in D \\ \epsilon_{P,Q}\ \epsilon^*_{(PQ),D}\ e_{P+Q} & \text{if } Q \notin D \ , \end{array} \right.$$

   where ϵ^* is the canonical composition factor of $(\mathcal{F}^*, \tau^{*\,-1})$ (see Proposition 1.11).

2. *$X_{P_i,D_i} + X_{P_i,D_{i-i}} + X_{P_i,D_{i-3}} = 0$;*

3. *$X_{P_i,D_i}, X_{P_i,D_{i-1}}$ and $X_{P_i,D_{i-3}}$ are elements of $\mathfrak{g}_2(\mathbb{F})$;*

3′. *$\mathfrak{g}_2(\mathbb{F}) = Vect\Big\langle X_{P_i,D_i}, X_{P_i,D_{i-1}}, X_{P_i,D_{i-3}} : i \in \mathbb{Z}_7\Big\rangle$;*

4. *The subspace $\mathfrak{h}_{P_i} = \mathrm{Vect}\Big\langle X_{P_i,D_i}, X_{P_i,D_{i-1}}, X_{P_i,D_{p-i}}\Big\rangle$ is a Cartan subalgebra of $\mathfrak{g}_2(\mathbb{F})$.*

Proof. Straightforward. QED

Remark 3.5 *Using only the incidence geometry of $\mathcal{F}$ and the composition factor ϵ, on the $\mathbb{F}-$vector space $Im(\mathbb{O}_{\mathcal{F}})$ we can define the three-form*

$$\omega = \sum_{D \in \mathcal{F}^*} \epsilon_{PQ}\ \epsilon_{QR}\ \epsilon_{RP}\ e^P \wedge e^Q \wedge e^R$$

where $\{e^1, \cdots, e^7\}$ is the basis dual to $\{e_1 \cdots, e_7\}$ and $D = \{P, Q, R\}$. Note that each term in the above sum is independent of the order of the points chosen in the corresponding line. This is *the unique three-form invariant under G_2, normalised such that $B(\Omega,\Omega)=7$ and oriented such that*

$$i_v\omega\wedge i_w\omega\wedge\omega=-6B(v,w)\,e^1\wedge\cdots\wedge e^7\ .$$

These are the same conventions as [14] but opposite to [15]. Dually, again using only the incidence geometry of $\mathcal{F}$ and the composition factor ϵ, we can define the four-form

$$\Omega=\sum_{K\in\mathcal{Q}}\epsilon_{PQ}\,\epsilon_{RS}\,e^P\wedge e^Q\wedge e^R\wedge e^S$$

where $\mathcal{Q}$ is the set of (seven) quadrilaterals of $\mathcal{F}$ and $K=\{P,Q,R,S\}$. Again each term in the above sum is independent of the order of the points chosen in the corresponding quadrilateral. This is the unique G_2-invariant four-form satisfying

$$\Omega\wedge\omega=-7e^1\wedge\cdots\wedge e^7\ .$$

These are the same conventions as [15] but opposite to [14].

Remark 3.6 *Not only the $X_{P,D}$ but also their action on $Im(\mathbb{O}_\mathcal{F})$ have now been given in terms of the incidence geometry of $\mathcal{F}$ and the composition factors ϵ and ϵ^*.*

The three generators of the Cartan subalgebra associated to a point of $\mathcal{F}$ (see Definition/Proposition 3.4) give rise to a root system of type G_2 in the following sense

Proposition 3.7 *Let $X_{P_i,D_i},X_{P_i,D_{i-1}},X_{P_i,D_{i-3}}$ and $\mathfrak{h}_{P_i}$ be as in 3.4 and set:*

$$\begin{array}{lllll} Y_{P_i,D_i} &=& X_{P_i,D_{i-1}}-X_{P_i,D_{i-3}} &=& e_{P_{i-3},P_{i-2}}+e_{P_{i+2},P_{i-1}}-2e_{P_{i+1},P_{i+3}}\\ Y_{P_i,D_{i-1}} &=& X_{P_i,D_{i-3}}-X_{P_i,D_i} &=& e_{P_{i-3},P_{i-2}}+e_{P_{i+1},P_{i+3}}-2e_{P_{i+2},P_{i-1}}\\ Y_{P_i,D_{i-3}} &=& X_{P_i,D_i}-X_{P_i,D_{i-1}} &=& e_{P_{i+1},P_{i+3}}+e_{P_{i+2},P_{i-1}}-2e_{P_{i-3},P_{i-2}}\ . \end{array} \tag{3.15}$$

Then

$$\mathcal{W}=\left\{\pm X_{P_i,D_i},\pm X_{P_i,D_{i-1}},\pm X_{P_i,D_{i-3}},\pm Y_{P_i,D_i},\pm Y_{P_i,D_{i-1}},\pm Y_{P_i,D_{i-3}}\right\}\subset\mathfrak{h}_{P_i}$$

is a root system of type G_2. For instance if $\alpha=X_{P_i,D_i}$ and $\beta=Y_{P_i,D_{i-1}}$ then

$$\mathcal{W}=\left\{\pm\alpha,\pm\beta,\pm(\alpha+\beta),\pm(\beta+2\alpha),\pm(\beta+3\alpha),\pm(2\beta+3\alpha)\right\}\ .$$

Proof. This follows from the fact that $\{e_{ij},\ \ 1\le i<j\le 7\}$ is an orthonormal basis of C^2. QED

3.2 Action of Aut$(\hat{\mathcal{F}}_\epsilon)$ on the $X_{P,D}$

In this section we analyse the action of the group $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ (see Definition 2.38) on the elements $X_{P,D}$ of $\mathfrak{g}_2(\mathbb{F})$ introduced above. Recall first that the $X_{P.D}$ are elements of $\mathrm{End}(\mathbb{O}_\mathcal{F})$ (*c.f.* Section 3.1) and the elements of the group $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ act naturally on this vector space (*c.f.* Remark 2.39). This means that it makes sense to conjugate elements of $\hat\rho(C^2)$ by $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$.

We start by making the following observation:

Lemma 3.8 *Let $P \in \mathcal{F}, D \in \mathcal{F}^*$ and let $\hat{g} \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ be such $\pi(\hat{g}) = g \in \mathrm{Aut}(\mathcal{F})$. Then* <u>*up to a sign:*</u>

$$\hat{g}\, X_{P,D}\, \hat{g}^{-1} = X_{g\cdot P, g\cdot D} \ .$$

Proof. Let Q be in $\mathcal{F}$. Then up to a sign, the LHS acting on e_Q is given by (see Eq.2.10)

$$\hat{g}\, X_{P,D}\, \hat{g}^{-1}(e_Q) = \begin{cases} 0 & \text{if } \ g^{-1}\cdot Q \in D \\ \hat{g}\cdot e_{g^{-1}\cdot Q+P} = e_{Q+g\cdot P} & \text{if } \ g^{-1}\cdot Q \notin D \ . \end{cases}$$

On the other hand up to a sign, the RHS acting on e_Q is given by

$$X_{g\cdot P, g\cdot D}(e_Q) = \begin{cases} 0 & \text{if } \ Q \in g\cdot D \\ e_{Q+g\cdot P} & \text{if } \ Q \notin g\cdot D \ . \end{cases}$$

Comparing these two expressions proves the lemma. QED

This lemma means the following definition is legitimate:

Definition 3.9 *Let $(P,D) \in \mathcal{I}$ be an incident pair and let $\hat{g} \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ be such $\pi(\hat{g}) = g \in \mathrm{Aut}(\mathcal{F})$. Define $\delta : \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon) \times \mathcal{I} \to \mathbb{Z}_2$ by*

$$\hat{g}\, X_{P,D}\, \hat{g}^{-1} = \delta(\hat{g}, P, D) X_{g\cdot P, g\cdot D} \ .$$

We now show that in fact δ does not depend on the line D.

Proposition 3.10 *Let $P \in \mathcal{F}$ let $D, D' \in \mathcal{F}^*$ be such that $P \in D \cap D'$. Let $\hat{g} \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ be such $\pi(\hat{g}) = g \in \mathrm{Aut}(\mathcal{F})$. Then $\delta(\hat{g}, P, D) = \delta(\hat{g}, P, D')$.*

Proof. Suppose $D \neq D'$. By proposition 3.4 if $D'' = D + D'$ we have

$$\begin{aligned} X_{P,D} + X_{P,D'} + X_{P,D''} &= 0 \\ X_{g\cdot P, g\cdot D} + X_{g\cdot P, g\cdot D'} + X_{g\cdot P, g\cdot D''} &= 0 \ . \end{aligned}$$

Conjugating the first equation above by $\hat{g}$ we obtain

$$\delta(\hat{g}, P, D) X_{g\cdot P, g\cdot D} + \delta(\hat{g}, P, D') X_{g\cdot P, g\cdot D'} + \delta(\hat{g}, P, D'') X_{g\cdot P, g\cdot D''} = 0$$

and comparing this with the second equation above it follows that $\delta(\hat{g}, P, D) = \delta(\hat{g}, P, D') = \delta(\hat{g}, P, D'')$. QED

From now on we write $\delta(\hat{g}, P)$ for $\delta(\hat{g}, P, D)$.

Example 3.11 *For $D \in \mathcal{F}^*$ and $t_D \in Aut(\hat{\mathcal{F}}_\epsilon)$ (see Corollary 2.41) one checks easily that*

$$\delta(t_D, P) = \begin{cases} 1 & \text{if } \ P \in D \\ -1 & \text{if } \ P \notin D \end{cases} = e^{T_D(P)}$$

where T_D is defined in Proposition 2.19. We denote by T the set of all T_D together with the constant function equal to one. Note that T is a group for pointwise multiplication.

We now establish a connection between the line orientation δ^* (see Section 2.4) and the function δ.

Proposition 3.12 *Let $(P, D) \in \mathcal{F} \times \mathcal{F}^*$ and $\hat{g} \in \mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ be such that $\pi(\hat{g}) = g$. Then*

$$\delta^*(g, D) = \prod_{P \in D} \delta(\hat{g}, P) \ . \tag{3.16}$$

Hence $\delta^*(g, \cdot)$ is the (multiplicative) Radon transform of $\delta(\hat{g}, \cdot)$.

Proof. The proposition is a consequence of the following two lemmas and the fact that $\mathrm{Aut}(\hat{\mathcal{F}}_\epsilon)$ is generated by $\hat{a}, \hat{b}$ given in (2.12) and the t_{D_i} introduced in Proposition 2.41.

Lemma 3.13 *If the formula (3.16) is true for $\hat{g}$ and $\hat{h}$, then it is true for $\hat{g}\hat{h}$.*

Proof. By the multiplier property of δ^* (see Remark 2.26) we have

$$\delta^*(gh, D) = \delta^*(g, h \cdot D)\delta^*(h, D)$$

which is equal to

$$\prod_{P \in D} \delta(\hat{g}, h \cdot P) \prod_{P \in D} \delta(\hat{h}, P)$$

since formula (3.16) is true for $\hat{g}$ and $\hat{h}$. By the analogous multiplier property of δ this is in turn equal to

$$\prod_{P \in D} \delta(\hat{g}\hat{h}, P) \ .$$

QED

Lemma 3.14 *The formula (3.16) is true for*

$$\hat{a} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 4 & -6 & 5 & -3 \end{pmatrix} \ , \quad \hat{b} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ -2 & 7 & 4 & 6 & 5 & 3 & -1 \end{pmatrix}$$

and all translations t_{D_i}.

Proof. To calculate $\delta(\hat{a}, \cdot)$ and $\delta(\hat{b}, \cdot)$ one has to use Definition 3.9 and unfortunately this is tedious if straightforward. The computation of $\delta^*(a, \cdot)$ and $\delta^*(b, \cdot)$ is much easier but we omit the details of both calculations. The results are given in Figure 6 from which the lemma follows. QED

Formula (3.16) in the general case follows from the lemmas above. QED

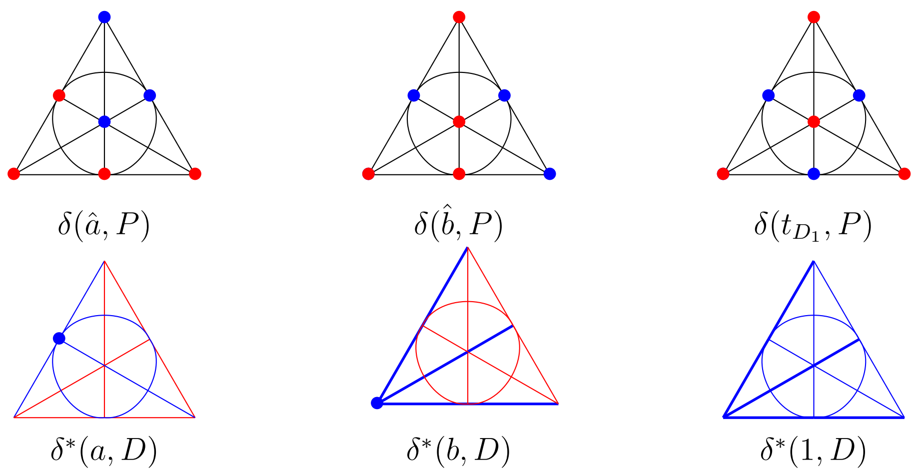


Figure 6: Values of δ (resp. δ^*) for $\hat{a}, \hat{b}, t_D$ (resp. $a, b, 1$); blue points: $\delta = 1$, red points: $\delta = -1$ (resp. blue lines: $\delta^* = 1$, red lines: $\delta^* = -1$).

Corollary 3.15 *For all $\hat{g}$ in Aut($\hat{\mathfrak{F}}_\epsilon$) we have $\delta(g, \cdot) \in \mathfrak{R}$, i.e.,*

$$\prod_{P \in \mathfrak{F}} \delta(\hat{g}, P) = 1 \ .$$

Proof. This follows from the proposition above and equation (2.7). QED

Remark 3.16 *Given $\hat{g} \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ such that $\pi(\hat{g}) = g$ we have seen that the Radon transform of $\delta(\hat{g}, \cdot)$ is $\delta^*(g, \cdot)$. According to Theorem 2.40, this means that that if we define $\hat{g}^{'} : \hat{\mathfrak{F}}_\epsilon \to \hat{\mathfrak{F}}_\epsilon$ by*

$$\hat{g}^{'} \cdot e_P = \delta(\hat{g}, P) e_{g \cdot P} \ ,$$

then $\hat{g}^{'} \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ and $\pi(\hat{g}^{'}) = g$. Thus by Corollary 2.41 either $\hat{g}' = \hat{g}$ or there exists $D \in \mathfrak{F}^*$ such that

$$\hat{g}^{'} = t_D \circ \hat{g} \ .$$

Both cases occur. For example if $\hat{g} = t_D$ then by Example 3.11, we have $\hat{g}^{'} = t_D$. On the other hand if

$$\hat{a} \;\; = \;\; \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 7 & 4 & -6 & 5 & -3 \end{pmatrix}$$

then $\delta(\hat{a}, \cdot)$ is given by the first diagram in Figure 6 and therefore

$$\hat{a}^{'} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & -2 & -7 & -4 & -6 & 5 & 3 \end{pmatrix} = t_{D_5} \circ \hat{a} \ .$$

3.3 Commutation relations for $\mathfrak{g}_2$

The incidence properties of the oriented Fano plane define multiplication of the octonions (see Table 1). In an analogous fashion they also define the Lie bracket of $\mathfrak{g}_2(\mathbb{F})$. In this section we will express all brackets of the $X_{P,D}$'s in $\mathfrak{g}_2(\mathbb{F})$ in terms of the incidence relations of the oriented Fano plane. To begin we need the following lemma:

Lemma 3.17 *Let $\mathcal{I}$ be the incidence space*

$$\mathcal{I} = \Big\{(P,D) \in \mathcal{F} \times \mathcal{F}^* \quad s.t. \quad P \in D\Big\} .$$

Then the orbits of $Aut(\mathcal{F})$ acting on $\mathcal{I} \times \mathcal{I}$ are given by

$$\mathcal{I} \times \mathcal{I} = \mathcal{D} \cup \mathcal{O}_1 \cup \mathcal{O}_2 \cup \mathcal{O}'_3 \cup \mathcal{O}_3 \cup \mathcal{O}_4 \ ,$$

where

$$\begin{aligned}
\mathcal{D} &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P = P', D = D'\Big\} \\
\mathcal{O}_1 &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P = P', D \neq D'\Big\} \\
\mathcal{O}_2 &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P \neq P', D = D'\Big\} \\
\mathcal{O}_3 &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P \neq P', D \neq D', P \in D'\Big\} \\
\mathcal{O}'_3 &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P \neq P', D \neq D', P' \in D\Big\} \\
\mathcal{O}_4 &= \Big\{\Big((P,D),(P',D')\Big) \in \mathcal{I} \times \mathcal{I} : \ P \neq P', D \neq D', P' \notin D, P' \notin D\Big\} .
\end{aligned}$$

These sets contain respectively $21, 42, 42, 84, 84$ *and* 168 *elements.*

Proof.

• $\mathcal{D}$: we shall show that the stabiliser of a point in $\mathcal{D}$ is of order 8. Let $S \subset$Aut$(\mathcal{F})$ be the stabiliser of $\Big(P,D\Big) \in \mathcal{F}$. There is an sequence of group homomorphisms

$$1 \to \mathrm{Ker}(\Phi) \to S \to S_2 \to 1$$

where S_2 is identified with the subgroup of permutations of D which fix P and $\Phi(s) \in S_2$ is the induced action of $s \in S$ on D. By Proposition 1.3 (2), Ker(Φ) is of order four and by Proposition 1.3 (4), Φ is surjective (the complement of Ker(Φ) in S consists of two elements of order two and two elements of order four). Hence S is of order eight and Aut$(\mathcal{F})/S$ has $21 = 168/8$ elements.

A straightforward count shows that $\mathcal{D}$ has 21 elements and it follows that $\mathcal{D}$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$.

• $\mathcal{O}_2$: we shall show that the stabiliser S of a point in $\mathcal{O}_2$ of order four. Let $\Big((P,D),(P',D)\Big) \in \mathcal{O}_2$ and suppose $g \in$Aut($\mathcal{F}$) fixes $\Big((P,D),(P',D)\Big)$, *i.e.*,

$$g \cdot P = P \ , \quad g \cdot P' = P' \ , \quad g \cdot D = D \quad .$$

Since g fixes P and P' its fixes P'' the third point of the line D, and hence by Proposition 1.3 since it fixes three points, g is either of order one or two. In fact by Proposition 1.3 (2), either g is the identity or g can be taken to permute two pairs of points of the complement of D. There are only three elements of this type and together with the identity it is clear that they form a subgroup of order four. Hence $\mathrm{Aut}(\mathcal{F})/S$ has $42 = 168/4$ elements. A straightforward count shows that $\mathcal{O}_2$ has 42 elements and it follows that $\mathcal{O}_2$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$. Similarly $\mathcal{O}_1$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$ containing 42 elements.

• $\mathcal{O}_3$: we shall show that the stabiliser S of a point in $\mathcal{O}_3$ is $\mathbb{S}_2$. Let $\Big((P,D),(P',D')\Big) \in \mathcal{O}_3$ and suppose $g \in$Aut($\mathcal{F}$) fixes $\Big((P,D),(P',D')\Big)$, *i.e.*,

$$g \cdot P = P \ , \quad g \cdot D = D \ , \quad g \cdot P' = P' \ , \quad g \cdot D' = D' \ .$$

Since g fixes P and P' its fixes P'' the third point of the line $D' = P \wedge P'$ and hence by Proposition 1.3, since it fixes three points, g is either of order one or two. Then, either g fixes each point of D in which case (*cf* Proposition 1.3) g is the identity, or g permutes the two points of $D \setminus \{P\}$ and also the two points $\mathcal{F} \setminus \big(D \cup D'\big)$. Hence $\mathrm{Aut}(\mathcal{F})/S$ has $84 = 168/2$ elements. A straightforward count shows that $\mathcal{O}_3$ has 84 elements and it follows that $\mathcal{O}_3$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$. Similarly $\mathcal{O}'_3$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$ containing 84 elements.

• $\mathcal{O}_4$: we shall show that the stabiliser S of an element of $\mathcal{O}_4$ is trivial. First let $\Big((P,D),(P',D')\Big) \in \mathcal{O}_4$ and suppose $g \in$Aut($\mathcal{F}$) fixes $\Big((P,D),(P',D')\Big)$, *i.e.*,

$$g \cdot P = P \ , \quad g \cdot D = D \ , \quad g \cdot P' = P' \ , \quad g \cdot D' = D' \ .$$

Since g fixes P and P' its fixes P'' the third point of the line $P \wedge P'$. Similarly, since g fixes D and D', it fixes the point $D \cap D'$ which is distinct from P and P' ($P \notin D'$ and $P' \notin D$). The point $D \cap D'$ is also distinct from P'' since $D \neq D'$. We have now shown that g fixes four distinct points of $\mathcal{F}$ and it follows from Proposition 1.3 that g is the identity. Hence $\mathrm{Aut}(\mathcal{F})/S$ has 168 elements. A straightforward count shows that $\mathcal{O}_4$ has 168 elements and it follows that $\mathcal{O}_4$ is a single orbit of $\mathrm{Aut}(\mathcal{F})$. QED

Theorem 3.18 *Let $\mathcal{F}$ be an Fano plane and let ϵ be a composition factor. Let P, P' be disctinct point in $\mathcal{F}$ and let D, D' be distinct lines in the Fano plane.*

1. *If $\Big((P,D),(P,D')\Big) \in \mathcal{O}_1$ then*

$$[X_{P,D}, X_{P,D'}] = 0 \ .$$

2. *If* $\Big((P,D),(P',D)\Big) \in \mathcal{O}_2$ *then*

$$[X_{P,D}, X_{P',D}] = 2\epsilon_{PP'} X_{P+P',P\wedge P'} \ .$$

3. *If* $\Big((P,D),(P',D')\Big) \in \mathcal{O}_3 \cup \mathcal{O}'_3$ *(i.e., $P \in D', P' \notin D$ or $P \notin D', P' \in D$) then*

$$[X_{P,D}, X_{P',D'}] = -\epsilon_{PP'} X_{P+P',P\wedge P'} \ .$$

4. *If* $\Big((P,D),(P',D')\Big) \in \mathcal{O}_4$ *(i.e., $P \notin D', P' \notin D$) then*

$$[X_{P,D}, X_{P',D'}] = -\epsilon_{PP'} X_{P+P',D+D'} \ ,$$

where $D + D'$ is the third line containing the point $D \wedge D'$.

Proof. Parts (1) of the theorem have already been proved see Definition/Proposition 3.4 (3)) and part (2) is a straightforward calculation using (3.13). To prove the rest of the theorem we first give the transformation law of brackets of two $X_{P,D}$ under the action of $\mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$. We then explicitly calculate the bracket corresponding to particular elements of each of the orbits $\mathcal{O}_i$ and then, using the transformation law we deduce the formulæ for brackets in the general case.

Lemma 3.19 *Let $(P,D),(P',D') \in \mathcal{I} \times \mathcal{I}$ be such that $P \neq P'$. Suppose that*

$$[X_{P,D}, X_{P',D'}] = \alpha X_{P+P',D''} \tag{3.17}$$

where $\alpha \in \mathbb{F}$, and $D'' \in \mathcal{F}^*$. Then for all $g \in \mathrm{Aut}(\mathcal{F})$

$$[X_{g\cdot P,g\cdot D}, X_{g\cdot P',g\cdot D'}] \ = \ \alpha\delta^*(g, P\wedge P') X_{g\cdot(P+P'),g\cdot D''} \ ,$$

where $P \wedge P'$ is the line passing through P and P'.

Proof. Let $\hat{g} \in \mathrm{Aut}(\hat{\mathfrak{F}}_\epsilon)$ be such that $\pi(\hat{g}) = g$ (see Proposition 2.41 (2)). Recall that (*cf* Section 3.2) we have

$$\hat{g} X_{Q,L} \hat{g}^{-1} = \delta(\hat{g}, Q) X_{g\cdot Q, g\cdot L} \quad \forall\, (Q,L) \in \mathcal{I} \ .$$

Hence conjugating (3.17) by $\hat{g}$ we get

$$\delta(\hat{g},P)\delta(\hat{g},P')[X_{g\cdot P,g\cdot D}, X_{g\cdot P',g\cdot D'}] \ = \ \alpha\delta(\hat{g}, P+P') X_{g\cdot(P+P'),g\cdot D''} \ ,$$

which can be rewritten

$$[X_{g\cdot P,g\cdot D}, X_{g\cdot P',g\cdot D'}] \ = \ \alpha\delta(\hat{g},P)\delta(\hat{g},P')\delta(\hat{g}, P+P') X_{g\cdot(P+P'),g\cdot D''} \ .$$

However using Proposition 3.12 this reduces to

$$[X_{g\cdot P,g\cdot D}, X_{g\cdot P',g\cdot D'}] = \alpha\delta^*(g,(P,P')) X_{g\cdot(P+P'),g\cdot D''} \ .$$

QED

To choose a particular element of $\mathcal{O}_3$ we use the notation of Figure 2. Consider $\Big((P = P_1, D = D_1), (P' = P_3, D' = D_7)\Big) \in \mathcal{O}_3$ in which case $(P + P'' = P_7, D'' = D_7)$. Using (3.13) and (3.14) we obtain by explicit calculation

$$[X_{P_1,D_1}, X_{P_3,D_7}] = -X_{P_7,D_7} \ ,$$

and by the lemma above this proves (3) since any element of $\mathcal{O}_3$ is conjugate to $\Big((P_1, D_1), (P_3, D_7)\Big)$. By symmetry this also proves (4).

To prove (5) consider $\Big((P = P_4, D = D_1), (P' = P_5, D' = D_2)\Big) \in \mathcal{O}_4$ in which case $(P + P' = P_7, D + D' = D_7)$. Using (3.13) and (3.14) we obtain by explicit calculation

$$[X_{P_4,D_1}, X_{P_5,D_2}] = -X_{P_7,D_6} \ ,$$

and by the lemma above this proves (5) since any element of $\mathcal{O}_4$ is conjugate to $\Big((P_4, D_1), (P_5, D_2)\Big)$. QED

Corollary 3.20 *Let $(\mathcal{F}, \tau)$ be an oriented Fano plane and let ϵ be the canonical composition factor.*

1. *There is a decomposition as a direct sum of Cartan subalgebras:*

$$\mathfrak{g}_2(\mathbb{F}) = \bigoplus_{P \in \mathcal{F}} \mathfrak{h}_P \ .$$

2. *If P, Q, R are non-aligned then $\mathfrak{h}_P \oplus \mathfrak{h}_Q \oplus \mathfrak{h}_R$ generates $\mathfrak{g}_2(\mathbb{F})$.*

3. *If P, Q, R are three distinct points on the line D such that $\epsilon_{PQ} = \epsilon_{QR} = \epsilon_{RP} = 1$ then:*

$$\begin{aligned} \big[X_{P,D}, X_{Q,D}\big] &= 2X_{R,D} \ , \\ \big[Y_{P,D}, Y_{Q,D}\big] &= -2Y_{R,D} \ , \\ \big[X_{S_1,D}, Y_{S_2,D}\big] &= 0 \ , \quad \forall S_1, S_2 \in D \ . \end{aligned}$$

   In particular:

   - $[\mathfrak{h}_P, \mathfrak{h}_Q] = \mathfrak{h}_R$.
   - $\mathfrak{g}_D = \mathfrak{h}_P \oplus \mathfrak{h}_Q \oplus \mathfrak{h}_R$ *is a Lie subalgebra of* $\mathfrak{g}_2(\mathbb{F})$ *isomorphic to* $\mathfrak{so}(4, \mathbb{F})$.
   - *If we define*

$$\begin{aligned} \mathcal{I}_{X,D} &= \mathrm{Vect}\Big\langle X_{P,D}, X_{Q,D}, X_{R,D} \Big\rangle \ , \\ \mathcal{I}_{Y,D} &= \mathrm{Vect}\Big\langle Y_{P,D}, Y_{Q,D}, Y_{R,D} \Big\rangle \ , \end{aligned}$$

     then $\mathcal{I}_{X,D}$ and $\mathcal{I}_{Y,D}$ are the two ideals of $\mathfrak{g}_D$ each isomorphic to $\mathfrak{so}(3, \mathbb{F})$.

- *The decomposition*

$$Im(\mathbb{O}_{\mathcal{F}}) = Vect\Big\langle e_P,\ P\in D\Big\rangle \oplus Vect\Big\langle e_P,\ P\notin D\Big\rangle\ ,$$

  is stable under the action of $\mathfrak{so}(4,\mathbb{F})$ *and* $Vect\Big\langle e_P,\ P\notin D\Big\rangle$ *is isomorphic to the defining representation. The representation* $Vect\Big\langle e_P,\ P\in D\Big\rangle$ *is trivial for the action of* $\mathfrak{I}_{X,D}$ *and isomorphic to the adjoint representation of* $\mathfrak{I}_{Y,D}$*.*

Proof. All parts of the Corollary are direct consequences of the Theorem above. QED

Remark 3.21 *As a consequence of this proposition one can associate to each line of the oriented Fano plane an* $\mathfrak{so}(4,\mathbb{F})$ *Lie subalgebra of* $\mathfrak{g}_2(\mathbb{F})$ *just as by Proposition 2.7 one can associate to each line of the oriented Fano plane a quaternion subalgebra of* $\mathbb{O}_{\mathcal{F}}$*.*

We now show that one can associate a rank two simple Lie algebra to each point of the oriented Fano plane.

Corollary 3.22 *For any point* $P\in\mathcal{F}$*, the Lie algebra*

$$\mathfrak{s}_P = \text{Vect}\Big\{X_{Q,D}\in\mathfrak{g}_2(\mathbb{F})\ \text{s.t.}\ X_{Q,D}\cdot e_P = 0\Big\}$$

is isomorphic to $\mathfrak{sl}(3,\mathbb{F})$ *if* $\sqrt{-1}\in\mathbb{F}$ *and to* $\mathfrak{su}(3,\mathbb{F})$ *if* $\sqrt{-1}\notin\mathbb{F}$ *(see the beginning of Section 2 for the definition of* $\mathfrak{su}(3,\mathbb{F})$*).*

Proof. By Definition/Proposition 3.4 (1) if $P\in D$ then $[X_{Q,D},e_P]=0$. Using the notation of Figure 2, without loss of generality we can suppose $P=P_1$. Thus $[X_{Q,D},e_{P_1}]=0$ if

$$X_{Q,D} = \left\{\begin{array}{lll} X_{P_1,D_1}\ , & X_{P_2,D_1}\ , & X_{P_4,D_1}\ ,\\ X_{P_7,D_7}\ , & X_{P_1,D_7}\ , & X_{P_3,D_7}\ ,\\ X_{P_5,D_5}\ , & X_{P_6,D_5}\ , & X_{P_1,D_5}\ ,\end{array}\right.$$

with $X_{P_1,D_1}+X_{P_1,D_7}+X_{P_1,D_5}=0$ (see Definition/Proposition 3.4 (3)). Thus we have eight linearly independent generators which clearly span a Lie subalgebra of $\mathfrak{g}_2(\mathbb{F})$. Again from Definition/Proposition 3.4 (4), X_{P_1,D_1},X_{P_1,D_7} span the Cartan subalgebra $\mathfrak{h}_{P_1}$. We now introduce (in $\mathfrak{g}_2(\mathbb{F})$ if $\sqrt{-1}\in\mathbb{F}$ or in $\mathfrak{g}_2\Big(\mathbb{F}[\sqrt{-1}]\Big)$ if $\sqrt{-1}\notin\mathbb{F}$) the elements:

$$\begin{array}{c} h_{P_1,D_1} = -\sqrt{-1}X_{P_1,D_1}\ ,\\ h_{P_1,D_7} = -\sqrt{-1}X_{P_1,D_7}\ ,\end{array}$$

which generate a Cartan subalgebra, together with the elements

$$\begin{array}{rcl} e^{\pm}_{P_1,D_1} &=& X_{P_2,D_1}\mp\sqrt{-1}X_{P_4,D_1}\ ,\\ e^{\pm}_{P_1,D_7} &=& X_{P_3,D_7}\mp\sqrt{-1}X_{P_7,D_7}\ .\end{array}$$

Then from Theorem 3.18 we have

$$\begin{aligned} \left[h_1, e^{\pm}_{P_1,D_1}\right] &= \pm 2 e^{\pm}_{P_1,D_1} \ , & \left[h_1, e^{\pm}_{P_1,D_7}\right] &= \mp e^{\pm}_{P_1,D_7} \ , \\ \left[h_2, e^{\pm}_{P_1,D_1}\right] &= \mp e^{\pm}_{P_1,D_1} \ , & \left[h_2, e^{\pm}_{P_1,D_7}\right] &= \pm 2 e^{\pm}_{P_1,D_7} \ , \end{aligned}$$

and

$$\left[e^{+}_{P_1,D}, e^{-}_{P_1,D'}\right] = \begin{cases} -4h_{P_1,D} & \text{if} \quad D = D' \ , \\ 0 & \text{if} \quad D \neq D' \ . \end{cases}$$

Thus the Cartan matrix is given by

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \ .$$

The last two generators of $\mathfrak{s}_P$ are given by

$$[e^{\pm}_{P_1,D_1}, e^{\pm}_{P_1,D_7}] = -2e^{\pm}_{P_1,D_5} = -2(X_{P_5,D_5} \pm \sqrt{-1} X_{P_7,D_5}) \ ,$$

and satisfy (again by Theorem 3.18)

$$\left[h_1, e^{\pm}_{P_1,D_5}\right] = \pm e^{\pm}_{1,D_5} \ , \qquad \left[h_2, e^{\pm}_{P_1,D_5}\right] = \pm e^{\pm}_{P_1,D_5} \ ,$$

and

$$\left[e^{+}_{P_1,D_5}, e^{-}_{P_1,D_5}\right] = -4(h_{P_1,D_1} + h_{P_1,D_7}) \ .$$

This ends the proof. QED

Remark 3.23 *The Lie algebra $\mathfrak{s}_P$ has a natural representation on the vector space:*

$$V = \mathrm{Vect}\{e_Q \text{ s.t. } Q \in \mathcal{F} \ , Q \neq P\} \ .$$

This six-dimensional vector space is stable under $\mathfrak{s}_P$ and carries a natural $\mathfrak{s}_P-$invariant "almost-complex structure" $J \in \mathrm{End}(V)$ defined by

$$J(e_Q) = \epsilon_{QP} \ e_{P+Q} \ .$$

It also carries the $\mathfrak{s}_P-$invariant non-degenerate quadratic form

$$g(e_Q, e_R) = \delta_{QR}$$

for which, one checks, J is an isometry. The Lie algebra $\mathfrak{s}_P$ is then characterised as the Lie subalgebra of $\mathrm{End}(V)$ which preserves g and J. This representation is irreducible iff $\sqrt{-1} \notin \mathbb{F}$.

Proposition 2.7 associates composition subalgebras of the octonions to certain configurations of points in the Fano plane. Analogously, one can associate Lie subalgebras of $\mathfrak{g}_2(\mathbb{F})$ to elements of $\mathcal{J} \times \mathcal{J}$ in the following way:

$\mathcal{O}_1$: For each element $\Big((P,D),(P,D')\Big)$ of $\mathcal{O}_1$ the Lie subalgebra of $\mathfrak{g}_2(\mathbb{F})$ generated by $X_{P,D}, X_{P,D'}$ is a Cartan subalgebra denoted $\mathfrak{h}_P$ in Proposition 3.4.

$\mathcal{O}_2$: For each element $\Big((P,D),(P',D)\Big)$ of $\mathcal{O}_2$ the Lie subalgebra of $\mathfrak{g}_2(\mathbb{F})$ generated by $X_{P,D}, X_{P',D}$ is isomorphic to $\mathfrak{so}(3,\mathbb{F})$ by Proposition (3.20) (2).

$\mathcal{O}_3$: For each element $\Big((P,D),(P',D')\Big)$ of $\mathcal{O}_3$ the Lie subalgebra of $\mathfrak{g}_2(\mathbb{F})$, generated by $X_{P,D}, X_{P',D'}$ is isomorphic to $\mathfrak{so}(3,\mathbb{F}) \times \mathfrak{so}(2,\mathbb{F}))$. For instance, in the case of X_{P_1,D_1} and X_{P_7,D_7} we have:

$$\begin{aligned}
\big[X_{P_1,D_1}, X_{P_7,D_7}\big] &= X_{P_3,D_7} \\
\big[X_{P_1,D_1}, X_{P_3,D_7}\big] &= -X_{P_7,D_7} \\
\big[X_{P_1,D_1}, X_{P_1,D_7}\big] &= 0\ .
\end{aligned}$$

It follows that $X_{P_1,D_1} + \frac{1}{2}X_{P_1,D_7} = -\frac{1}{2}Y_{P_1,D_7}$ is in the centre of this Lie algebra which for brevity we shall denote $\mathfrak{s}$. The $X_{P_7,D_7}, X_{P_3,D_7}, X_{P_1,D_7}$ generate an ideal of $\mathfrak{s}$ isomorphic to $\mathfrak{so}(3,\mathbb{F})$. The Lie subalgebra $\mathfrak{s}$ can also be obtained by adding an element of one of the ideals of the $\mathfrak{so}(4,\mathbb{F})$ associated to D_7 to the other ideal (see Proposition 3.20 (2)). Under the action of $\mathfrak{s}$ we have the decompositions:

$$\begin{aligned}
\mathrm{Im}(\mathbb{O}_{\mathcal{F}}) &= \mathrm{Vect}\Big\langle e_P,\ P \in D_7\Big\rangle \oplus \mathrm{Vect}\Big\langle e_P,\ P \notin D_7\Big\rangle \\
&= \Big\langle e_{P_7}, e_{P_1}, e_{P_3}\Big\rangle \oplus \Big\langle e_{P_2}, e_{P_4}, e_{P_5}, e_{P_6}\Big\rangle \\
&= \begin{cases} \Big(\mathbf{1_0} \oplus \mathbf{2}\Big) \oplus \mathbf{4} & \text{if } -1 \text{ is not a square in } \mathbb{F} \\ \Big(\mathbf{1_0} \oplus \mathbf{1} \oplus \mathbf{1'}\Big) \oplus \Big(\mathbf{2} \oplus \mathbf{2'}\Big) & \text{if } -1 \text{ is a square in } \mathbb{F} \end{cases}
\end{aligned}$$

where $\mathbf{1_0}$ is the trivial one-dimensional representation, $\mathbf{1}, \mathbf{1'}$ are irreducible one-dimensional representations, $\mathbf{2}, \mathbf{2'}$ are irreducible two-dimensional representations and $\mathbf{4}$ is an irreducible four-dimensional representation.

$\mathcal{O}'_3$: Same as $\mathcal{O}_3$.

$\mathcal{O}_4$: Finally for each element $\Big((P,D),(P',D')\Big)$ of $\mathcal{O}_4$ the Lie algebra generated by $X_{P,D}, X_{P',D'}$ is $\mathfrak{g}_2(\mathbb{F})$.

References

[1] F. Engel, "Ein neues, dem linearen Complexe analoges Gebilde." Leipz. Ber. 52, 63-76, 220-239 (1900), 1900.

[2] W. Reichel, "Über trilineare alternierende Formen in sechs und sieben Veränderlichen und die durch sie definierten geometrischen Gebilde." Greifswald. 59 S. (1907), 1907.

[3] E. Cartan, "Les groupes réels simples, finis et continus," *Ann. Sci. Éc. Norm. Supér. (3)* **31** (1914) 263–356.

[4] H. Freudenthal, *Oktaven, Ausnahmegruppen und Oktavengeometrie*. Mathematisch Instituut der Rijksuniversiteit te Utrecht, Utrecht, 1951.

[5] I. Agricola, "Old and new on the exceptional group G_2," *Notices Amer. Math. Soc.* **55** (2008) no. 8, 922–929.

[6] M. Rausch de Traubenberg and M. Slupinski, "Incidence geometry of the Fano plane and Freudenthal's ansatz for the construction of (split) octonions," *Innovations in Incidence Geometry (submited)* (2022) .

[7] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, *Atlas of finite groups*. Oxford University Press, Eynsham, 1985. Maximal subgroups and ordinary characters for simple groups, With computational assistance from J. G. Thackray.

[8] R. A. Wilson, "On the compact real form of the Lie algebra $\mathfrak{g}_2$.," *Math. Proc. Camb. Philos. Soc.* **148** (2010) no. 1, 87–91.

[9] A. I. Kostrikin and P. H. Tiep, *Orthogonal decompositions and integral lattices*, vol. 15. Berlin: De Gruyter, 1994.

[10] R. A. Wilson, T. Dray, and C. A. Manogue, "An octonionic construction of E_8 and the Lie algebra magic square," arXiv:2204.04996 [math.GR].

[11] R. Harvey and H. B. Lawson, "Calibrated geometries," *Acta Math.* **148** (1982) 47–157. https://doi.org/10.1007/BF02392726.

[12] G. Sandlöbes, "Perfect groups of order less than 10^4," *Commun. Algebra* **9** (1981) 477–490.

[13] F. Toppan, "The octonionically-induced $\mathcal{N} = 7$ exceptional $G(3)$ superconformal quantum mechanics," arXiv:1912.05596 [hep-th].

[14] S. Karigiannis, *Introduction to G_2- Geometry*, pp. 3–50. Springer US, New York, NY, 2020. https://doi.org/10.1007/978-1-0716-0577-6_1.

[15] R. L. Bryant, "Some remarks on G_2-structures," arXiv:math/0305124 [math.DG].