

Topics in Polar Actions

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Abstract

These are the notes for a series of lectures at the Institute of Geometry and Topology of the University of Stuttgart, Germany, in July 13-15, 2022. We wish to thank Uwe Semmelmann and Andreas Kollross for the invitation to give these lectures. We assume basic knowledge of isometric actions on Riemannian manifolds, including the normal slice theorem and the principal orbit type theorem. Lecture 1 introduces polar actions and culminates with Heintze, Liu and Olmos's argument to characterize them in terms of integrability of the distribution of normal spaces to the principal orbits. The other two lectures are devoted to two of Lytchak and Thorbergsson's results. In Lecture 2 we briefly review Riemannian orbifolds from the metric point of view, and explain their characterization of orbifold points in the orbit space of a proper and isometric action in terms of polarity of the slice representation above. In Lecture 3 we present their proof of the fact that variationally complete actions in the sense of Bott and Samelson on non-negatively curved manifolds are hyperpolar. The appendix contains explanations of some results used in the lectures, namely: a criterion for the polarity of isometric actions on symmetric spaces, a discussion of Cartan's and Hermann's criteria for the existence of totally geodesic submanifolds, and a more or less self-contained derivation of Wilking's transversal Jacobi equation.

Introduction

J. Dadok [Dad85] considered orthogonal representations of compact Lie groups with the property that there is a subspace meeting all orbits, and always orthogonally. He pointed out that they resemble generalized polar coordinates and introduced the name *polar representations* for them. In the same paper he proved a number of basic properties, and especially classified polar representations up to orbit-equivalence¹ using highest weight theory. As an aftermath,

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¹Two isometric actions are said to be *orbit-equivalent* if they have the same orbits, up to an isometric identification between the target spaces.

he realized that every polar representation of a compact connected Lie group is orbit-equivalent to the isotropy representation of a (Riemannian) symmetric space.

Polar actions had already been considered before, in one way or another. One form of É. Cartan’s maximal torus theorem states that a maximal torus with a bi-invariant metric meets every adjoint orbit in the group, orthogonally, and similarly, the Lie algebra of the maximal torus meets every adjoint orbit in the Lie algebra, orthogonally. A generalization of this result applies to the (resp. linear) isotropy action of a symmetric space and the (resp. Cartan subspaces) maximal flats. L. Conlon [Con71], building on work of Bott and Samelson and Hermann, for an action of compact connected Lie group K on a complete Riemannian manifold M , considered his so-called “ K -transversal domain”, namely, a flat closed connected totally geodesic embedded submanifold meeting every K -orbit, and orthogonal at every point of intersection. J. Szenthe, coming from a background on transformation groups, and initially unaware of Conlon’s work, studied the generalized Weyl group for isometric actions of compact Lie groups admitting “orthogonally transversal manifolds”, and proved that such submanifolds are automatically totally geodesic. Palais and Terng [PT87], initially unaware of Dadok’s and Conlon’s works, defined a *section* of an isometric action of a compact Lie group on a connected complete Riemannian manifold M to be a connected closed regularly embedded smooth submanifold Σ of M that meets all orbits orthogonally. They also note that the compactness assumption on G can be replaced by the hypothesis that the action is proper, without substantial changes in the results. They especially took the differential geometric viewpoint and emphasized the relation to Riemannian geometry of submanifolds (more specifically, isoparametric submanifolds, another area with important contributions by Cartan). In the same paper, they mention applications to invariant theory and calculus of variations. These are further exposed in their book [PT88]. Much more recently, later developments in the area of polar actions are collected in [BCO16], which has also an extensive list of bibliographic references.

Singular Riemannian foliations form a class of foliations that generalize the foliations by orbits of an isometric group action. Much of the theory of polar actions has been generalized to “singular Riemannian sections with sections”, or “polar foliations” as they are now called, but the lack of group action causes some difficulties. We will not discuss them here and instead refer to [AB15, Rad17, Tho22] for discussions and references.

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1 Lecture 1: Polar actions

1.1 Sections

A proper isometric action of a Lie group G on a complete Riemannian manifold M is called *polar* if there exists a complete connected immersed submanifold Σ of M which intersects all the orbits and such that Σ is perpendicular to every orbit it meets². Such a submanifold is called a *section*.

A number of basic properties of polar actions is listed in Proposition 1.1.2 below. First, we prove a related result about general proper isometric actions.

1.1.1 Lemma *Let (G, M) be a proper isometric action. Then, for every $p \in M$, the subset $\exp_p(\nu_p(Gp))$ meets all the orbits of G .*

Proof. Fix an arbitrary orbit N of G and a point $q \in N$. Since the action is proper, N is a properly embedded, thus closed submanifold of M . By completeness of M , there exists a minimizing geodesic $\gamma : [0, 1] \rightarrow M$ joining q to Gp , so $\gamma(0) = q$ and $\gamma(1) \in Gp$. Due the first variation of length formula, γ is perpendicular to Gp at $\gamma(1)$. Write $\gamma(1) = gp$ for some $g \in G$. Then $\tilde{\gamma} = g^{-1} \circ \gamma$

²It is possible to consider a more general situation in which Σ is replaced by an isometric immersion $\iota : \Sigma \rightarrow M$, non-necessarily injective. In [BCO16] such actions are called *locally polar*, and the term “polar” is reserved to those actions with an embedded section. It is clear from the discussion below that ι can fail to be injective at p only if $\iota(p)$ is a singular point of the action. The brothers Alekseevsky [AA93] proved that an isometrically immersed section of dimension 1 must be injective, but the case of higher dimension remains open. For simplicity of exposition, herein we restrict to injectively immersed sections and just make some comments in the general case.

is a geodesic joining $g^{-1}q \in N$ to p , and it is perpendicular to Gp at p . Therefore $g^{-1}q = \exp_p v$ where $v = -\tilde{\gamma}'(1) \in \nu_p(Gp)$. This proves that $\exp_p(\nu_p(Gp))$ meets N at $g^{-1}q$. \square

1.1.2 Proposition *Let (G, M) be a polar action. Then:*

- a. *If Σ is a section of (G, M) and $g \in G$, then $g\Sigma$ is a section of (G, M) . In other words, any G -translate of a section is a section.*
- b. *There exists a section of (G, M) through every point of M .*
- c. *The dimension of a section of (G, M) equals the co-homogeneity of the action.*
- d. *Any section of (G, M) contains an open and dense subset consisting of regular points of the action.*
- e. *A section of (G, M) is totally geodesic in M .*
- f. *There exists a unique section of (G, M) through a regular point $p \in M$, and it is given by $\exp_p(\nu_p(Gp))$.*
- g. *If Σ_1 and Σ_2 are two sections of (G, M) , then there exists $g \in G$ such that $g\Sigma_1 = \Sigma_2$. In other words, any two sections differ by an element of the group.*

Proof. (a) If Σ meets a given orbit N at a point p , then $g\Sigma$ meets N at the point gp . This shows that $g\Sigma$ meets all the orbits. Moreover, if $g\Sigma$ meets N at a point q , then it is perpendicular there, because Σ meets N at $g^{-1}q$ and this is perpendicular and G acts by isometries. It follows that $g\Sigma$ satisfies the two defining conditions of a section.

(b) Let Σ be a section of (G, M) . Given $p \in M$, the orbit Gp meets Σ in a point gp for some $g \in G$ by the definition of a section. Then $g^{-1}\Sigma$ is a section by (a) and $p \in g^{-1}\Sigma$.

(c) Let Σ be a section. Then

$$T_p\Sigma \subset \nu_p(Gp)$$

for every $p \in \Sigma$ by definition of a section. Denote by $\Sigma_{reg} = \Sigma \cap M_{reg}$ the open set of regular points of M that lie in Σ . Since $\dim \nu_p(Gp)$ equals the co-homogeneity of (G, M) for $p \in \Sigma_{reg}$, the above inclusion implies that $\dim \Sigma$ is not larger than this co-homogeneity. Recall the submersion $\pi : M_{reg} \rightarrow M_{reg}/G$. Since Σ intersects all the orbits, the restriction $\pi|_{\Sigma_{reg}} : \Sigma_{reg} \rightarrow M_{reg}/G$ is surjective. It follows that $\dim \Sigma_{reg} \geq \dim M_{reg}/G$. Since $\dim M_{reg}/G$ is equal to the co-homogeneity of (G, M) , we conclude that $\dim \Sigma$ is also equal to this co-homogeneity.

(d) It is clear that the set of regular points in Σ is open. Suppose, on the contrary, that there exists a non-empty open subset V of Σ that does not contain regular points of (G, M) . Let $p \in V$ be a point whose isotropy subgroup G_p

has the minimal dimension and the smallest number of connected components among the points in V . By the normal slice theorem, $(G_p) = (G_q)$ for $q \in V$. It follows that $GV \approx Gp \times V$ is a submanifold of M . If S is the normal slice at p , then $T_p S = \nu_p(Gp)$ and $T_p(GV) = T_p(Gp) \oplus T_p \Sigma$. It follows that GV is transversal to S at p . By shrinking V , we can assume $S \cap GV$ is a submanifold W , where $\dim W = \dim \Sigma$. G_p cannot fix all the points of S because p is not regular, but it fixes all the points of W , so the co-homogeneity of (G_p, S) is at least $\dim W + 1 = \dim \Sigma + 1$. The cohomogeneity of a slice is also the co-homogeneity of (G, M) , which contradicts part (c).

(e) Let Σ be a section. By part (d), Σ_{reg} is dense in Σ . Thus, by continuity, it suffices to prove that the second fundamental form of Σ in M vanishes along Σ_{reg} . Let $p \in \Sigma_{reg}$ and consider a normal vector $u \in \nu_p \Sigma$. Since p is a regular point, $\nu_p \Sigma = T_p(Gp)$, so we can find an element X in the Lie algebra of G such that $X_p^* = u$. Owing to the polarity of the action, X^* is perpendicular to Σ everywhere along Σ . Therefore the Weingarten operator A of Σ can be written as $\langle A_u v, v \rangle = -\langle \nabla_v X^*, v \rangle = 0$ for all $v \in T_p M$, where we have used that X^* is a Killing field. Hence A vanishes along Σ_{reg} , as we wanted.

(f) Let $p \in M$ be a regular point and let Σ be a section through p . We have seen that $T_p \Sigma = \nu_p(Gp)$ and Σ is totally geodesic, so $\Sigma \supset \exp_p(T_p \Sigma) = \exp_p(\nu_p(Gp))$. For the converse inclusion, let $q \in \Sigma$. By part (e) and completeness of Σ , there exists a minimizing geodesic of M , $\gamma : [0, 1] \rightarrow \Sigma$, with $\gamma(0) = p$ and $\gamma(1) = q$. Then $q = \exp_p(\gamma'(0))$ where $\gamma'(0) \in T_p \Sigma$, proving that $\Sigma \subset \exp_p(T_p \Sigma) = \exp_p(\nu_p(Gp))$.

(g) Let $p \in \Sigma_1$ be a regular point. There exists $g \in G$ such that $gp \in \Sigma_2$. Now $g\Sigma_1$ and Σ_2 are two sections through the regular point gp , so they must coincide by part (f). \square

The next result shows that the property of being polar is inherited by the slice representations.

1.1.3 Proposition *Let (G, M) be a polar action, and let $p \in M$. Then the slice representation at p is also polar. In fact, if Σ is a section of (G, M) through p , then $T_p \Sigma$ is a section of $(G_p, \nu_p(Gp))$.*

Proof. Set $K = G_p$ and $V = \nu_p(Gp)$ for convenience. The co-homogeneity of the slice representation is the same as that of the slice action of K on the normal slice S at p . This shows that $T_p \Sigma$ has the right dimension of a section of (K, V) .

We claim that $T_p \Sigma$ contains regular points of (K, V) . In fact, let ξ be a principal orbit of G , and choose a connected component β of $\xi \cap \Sigma$. Let γ be a minimizing geodesic in Σ from $\gamma(0) = p$ to $\gamma(1) \in \beta$. Then $\gamma'(0) \in T_p \Sigma$ is a regular point of (K, V) . Next, if we can prove that $T_p \Sigma$ is perpendicular to Kv for every $v \in T_p \Sigma$, this will finish the proof, for it will follow that, for a (K, V) -regular point $w \in T_p \Sigma$, $T_p \Sigma$ coincides with the normal space of Kw in V , and hence $T_p \Sigma$ meets all the K -orbits in V owing to Lemma 1.1.1.

So let $v \in T_p \Sigma$. The Lie algebra \mathfrak{k} consists of the elements X of \mathfrak{g} such that $X_p^* = 0$. We also have that \mathfrak{k} induces Killing fields on V via the action

of (K, V) ; denote them with $()^{**}$. The tangent space $T_v K v$ is spanned by the vectors $X_v^{**} \in V$, where $X \in \mathfrak{k}$. Let γ be an integral curve of v through p . In view of

$$\begin{aligned}
X_v^{**} &= \left. \frac{\nabla}{dt} \right|_{t=0} d(\exp(tX))_p(v) \\
&= \left. \frac{\nabla}{dt} \right|_{t=0} \left. \frac{d}{ds} \right|_{s=0} (\exp(tX))(\gamma(s)) \\
&= \left. \frac{\nabla}{ds} \right|_{s=0} \left. \frac{d}{dt} \right|_{t=0} (\exp(tX))(\gamma(s)) \\
&= \left. \frac{\nabla}{ds} \right|_{s=0} X_{\gamma(s)} \\
&= \nabla_v X,
\end{aligned}$$

for $u \in T_p \Sigma$ we have

$$\langle X_v^{**}, u \rangle = \langle (\nabla_v X)_p, u \rangle = -\langle A_{X_p^*} v, u \rangle = 0.$$

This shows that Kv is perpendicular to $T_p \Sigma$ and completes the proof. \square

1.1.4 Corollary *Let (G, M) be a polar action, and let $p \in M$. Then the isotropy subgroup G_p acts transitively on the set of sections of (G, M) through p .*

Proof. Let Σ_1 and Σ_2 be two sections containing the point p . According to Proposition 1.1.3, the slice representation $(G_p, \nu_p(Gp))$ is polar and $T_p \Sigma_1, T_p \Sigma_2$ are two of its sections. By Proposition 1.1.2(g), there exists $g \in G_p$ such that $dg_p(T_p \Sigma_1) = T_p \Sigma_2$. Since Σ_1 and Σ_2 are totally geodesic, this implies that $g\Sigma_1 = \Sigma_2$, as wished. \square

1.2 The generalized Weyl group

Let (G, M) be a proper isometric action. By the normal slice theorem, locally, the study of the orbit space near an orbit Gp is reduced to the study of the orbit space of the action of G_p on the normal slice S_p . Next, we explain how this reduction can be done globally in the case in which (G, M) is a polar action.

Let (G, M) be a polar action, and let Σ be a section. Denote by $N(\Sigma)$ the normalizer of Σ in G , namely, the subgroup of G consisting of the elements that restrict to isometries of Σ . Then the action of G on M restricts to an action of $N(\Sigma)$ on Σ . In the following, it will be interesting to consider the effectivized action of $N(\Sigma)$ on Σ ; we say an action is *effective* if the only group element that acts as the identity map is the identity element in the group. For that purpose, denote by $Z(\Sigma)$ the centralizer of Σ in G , namely, the subgroup of G consisting of the elements that restrict to the identity on Σ . Take any regular point $p \in \Sigma$. It is obvious that $Z(\Sigma) \subset G_p$, and the reverse inclusion is a consequence of the fact that the slice representation at a regular point is trivial. In particular, $Z(\Sigma) = G_p$ is a closed subgroup of G . Note also that $N(\Sigma)$ is a subgroup of the normalizer of G_p in G , $N(\Sigma) \subset N_G(G_p)$.

The *generalized Weyl group* of the polar action (G, M) with respect to the section Σ is defined to be the quotient group

$$W(\Sigma) = N(\Sigma)/Z(\Sigma).$$

In the following proposition we collect a number of properties related to the generalized Weyl group.³

1.2.1 Proposition *Let (G, M) be a polar action admitting a section Σ .*

- a. *The generalized Weyl group $W(\Sigma)$ is a discrete closed subgroup of $N(G_{pr})/G_{pr}$, for some principal isotropy group G_{pr} of (G, M) . In particular, $W(\Sigma)$ acts properly on Σ .*
- b. *The inclusion $\iota : \Sigma \rightarrow M$ induces a map $\bar{\iota} : \Sigma/W(\Sigma) \rightarrow M/G$, which takes a $N(\Sigma)$ -orbit of a point in Σ to the G -orbit of that point. This map is a homeomorphism between the quotient topological spaces.*
- c. *For every $p \in \Sigma$, $Gp \cap \Sigma = W(\Sigma)p$.*
- d. *If Σ_1 and Σ_2 are sections, then there exists an isomorphism between the generalized Weyl groups $W(\Sigma_1)$ and $W(\Sigma_2)$ which is uniquely defined up to an inner automorphism of $W(\Sigma_1)$.*

Proof. (a) Let $p \in \Sigma$ be a regular point and write $G_p = G_{pr}$. Let S be the normal slice at p . Then S is an open neighborhood of p in Σ . The continuity of the action implies that $gp \in S$ for an element $g \in N(\Sigma)$ sufficiently close to the identity of $N(\Sigma)$. Since Gp is a principal orbit, S meets every orbit near p at a unique point, so $gp = p$, namely, $g \in G_p = Z(\Sigma)$. This argument thus shows that $Z(\Sigma)$ contains an open neighborhood of the identity in $N(\Sigma)$, and this is equivalent to saying that $Z(\Sigma)$ is an open subgroup of $N(\Sigma)$. Hence the quotient $N(\Sigma)/Z(\Sigma)$ is a discrete Lie group. Now every discrete subgroup of a Hausdorff topological group is closed. The properness of the $W(\Sigma)$ -action on Σ is immediate from this and the properness of the G -action on M .

(b) Since Σ meets all the orbits of G in M , this map is surjective. We claim that the map $\bar{\iota}$ is also injective. In order to prove this claim, suppose that $p, q \in \Sigma$ lie in the same G -orbit; we need to prove that they lie in the same $N(\Sigma)$ -orbit, too. We can write $q = gp$ for some $g \in G$. Then q lies in $g\Sigma$, so Σ and $g\Sigma$ are two sections through the point q . By Corollary 1.1.4, there exists $h \in G_q$ such that $hg\Sigma = \Sigma$. It follows that $q = hgq = hgp$ where $hg \in N(\Sigma)$, and this proves the claim.

We already know that $\bar{\iota}$ is a continuous and bijective map, so now we need only to prove that it is an open map. For that purpose, let U be an open set of $\Sigma/W(\Sigma)$ and denote by $\pi_\Sigma : \Sigma \rightarrow \Sigma/W(\Sigma)$ and $\pi_M : M \rightarrow M/G$ the

³In the case of an isometrically immersed section $\iota : \Sigma \rightarrow M$, $W(\iota)$ is defined as the normalizer of the image of ι , quotiented by its centralizer. One then shows that the $W(\iota)$ -action on $\iota(\Sigma)$ lifts to an action on Σ : this is clear on the G -regular points of Σ , and follows on the other points by continuity.

projections. By the definition of quotient topology, we know that $\pi_\Sigma^{-1}(U)$ is open and we want to show that this implies that $\pi_M^{-1}\bar{\iota}(U)$ is open. Since G acts by homeomorphisms on M , this is a consequence of the following relation that we prove in the sequel:

$$\pi_M^{-1} \circ \bar{\iota}(U) = G\pi_\Sigma^{-1}(U).$$

In fact, we have that a point $p \in M$ belongs to the left hand side if and only if $\pi_M(p) \in \bar{\iota}(U)$, and this means that $\pi_M(p) = \bar{\iota}(\pi_\Sigma(q)) = \pi_M(\iota(q))$ for some $q \in \pi_\Sigma^{-1}(U)$. But the latter is equivalent to having p lying in the same G -orbit as a point $q \in \pi_\Sigma^{-1}(U)$, which is exactly the meaning that $p \in G\pi_\Sigma^{-1}(U)$.

(c) One inclusion is obvious, and the other one is the injectivity of the map $\bar{\iota}$ proved in part (b).

(d) By Proposition 1.1.2(g), there exists an element $g \in G$ such that $g\Sigma_1 = \Sigma_2$. It is easy to see that $gN(\Sigma_1)g^{-1} = N(\Sigma_2)$ and $gZ(\Sigma_1)g^{-1} = Z(\Sigma_2)$. So the conjugation by g induces an isomorphism $W(\Sigma_1) \rightarrow W(\Sigma_2)$. If $g' \in G$ is another element satisfying $g'\Sigma_1 = \Sigma_2$, then $g^{-1}g' \in N(\Sigma_1)$, so this element defines an inner automorphism of $W(\Sigma_1)$ and the conjugations by g and g' induce isomorphisms $W(\Sigma_1) \rightarrow W(\Sigma_2)$ that differ by that inner automorphism. \square

1.3 The orbit space

Let (G, M) be a proper isometric action, where M is assumed connected. The set of manifold points of $M^* = M/G$ is open and dense, but M^* is, in general, not a manifold. For this reason, it is interesting to consider a natural metric space structure on M^* .

Let $x, y \in M^*$. Define the distance $d(x, y)$ to be the distance between the G -orbits $\pi^{-1}(x)$ and $\pi^{-1}(y)$ in M . Since the G -action is proper, its orbits are properly embedded submanifolds of M , and therefore $d(x, y) > 0$ for $x \neq y$. It is now clear that d defines a structure of metric space on M^* .

1.3.1 Remark Note that $d(x, y)$ is equal to the length of a geodesic of M joining a point in $\pi^{-1}(x)$ to a point in $\pi^{-1}(y)$, which is horizontal in the sense that it is orthogonal to every G -orbit that it meets; the initial point of the geodesic in one of the two orbits can be any chosen point, by G -invariance, but this determines the endpoint in the other orbit.

Consider the metric space structures thus obtained on M/G and Σ/W , where $W = W(\Sigma)$. It is clear that the quotient spaces $\Sigma/N = \Sigma/W$, where $N = N(\Sigma)$, and that the inclusion⁴ $\iota : \Sigma \rightarrow M$ induces a continuous map $\bar{\iota} : \Sigma/W \rightarrow M/G$ that maps a N -orbit of a point in Σ to the G -orbit of that point. In this respect, we have the following proposition.

1.3.2 Proposition *The map $\bar{\iota} : \Sigma/W(\Sigma) \rightarrow M/G$ is an isometry of metric spaces.*

⁴Or isometric immersion.

Proof. The map $\bar{\iota}$ is non-expanding (or 1-Lipschitz), namely,

$$d(\bar{\iota}(x), \bar{\iota}(y)) \leq d(x, y)$$

for all $x, y \in \Sigma/W$, since every geodesic in Σ is a geodesic in M .

We already know from Proposition 1.2.1 that $\bar{\iota}$ is injective. In view of the continuity of $\bar{\iota}$ and the denseness of G -regular points in Σ , to finish the proof we need only show that $\bar{\iota}$ is an isometry on Σ_{reg} . In fact let $x = Np$, $y = Nq$ where $p, q \in \Sigma_{reg}$. The minimizing geodesic γ in M from p to Gq is entirely contained in Σ . Let $r \in \Sigma \cap Gq$ be the endpoint of γ . Clearly γ minimizes the distance from Np to Nr . Since $Gr = Gq$, by the injectivity of $\bar{\iota}$ we have $Nr = Nq$. Hence $d(\bar{\iota}(x), \bar{\iota}(y)) = \text{Length}(\gamma) = d(x, y)$ as desired. \square

Recall that a *Riemannian orbifold* is a length space locally isometric to the quotient of a Riemannian manifold by a finite group of isometries (cf. Lecture 2). For a section Σ of a polar action (G, M) , the generalized Weyl group $W(\Sigma)$ is a discrete group acting properly on Σ (Proposition 1.2.1(a)); thus its isotropy subgroups are finite. It follows from Proposition 1.3.2 that the orbit space of a polar action is a Riemannian orbifold.

Due to Proposition 1.3.2, the action of $(W(\Sigma), \Sigma)$ can also be seen as a “reduction” of the action (G, M) to a discrete group action, namely, one can recover the same orbit space from a much simpler action of a discrete group action. It easily follows from O’Neill’s equation (3.2.4) and Theorem 1.5.2 that a proper and isometric action admits a reduction to a discrete group if and only if it is a polar action.

Consider for instance the case of an orthogonal representation (G, V) . In invariant theory, if (H, W) is a reduction of (G, V) , that is, W/H is isometric to V/G , it is a natural question to ask if the invariant rings of these representations are isomorphic. In some special cases this question has an affirmative answer, namely, polar representations (by Chevalley’s theorem) and the reduction to the principal isotropy group (by Luna-Richardson’s theorem [LR79]). In [AR15] it is shown that the answer is also positive in case the isometry preserves the codimension of the orbits, and in [AL11] it is remarked to hold for infinitesimally polar actions (cf. section 2); see also [Men21] for the special case of isometries of V/G . In full generality, the question remains open.

The reduction principle for orthogonal representations was apparently first stated in [Str94]. In [GL14] a systematic study of reductions of orthogonal representations was initiated, going much beyond polar representations.

1.4 Examples and classification

We first discuss the linear case. The standard examples of polar representations are the isotropy representations of symmetric spaces without an Euclidean factor, sometimes called *s-representations*. Let $M = G/K$ be a such a symmetric space, and consider the associated decomposition on the Lie algebra level $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ into the ± 1 -eigenspaces of the involution. Then the isotropy representation of M is equivalent to the adjoint representation of K on \mathfrak{p} . Recall that

a symmetric space of compact type and its noncompact dual have equivalent isotropy representations. By passing to a covering, we may assume that K is connected, and M is irreducible. Then the metric is proportional to the Killing form B on \mathfrak{p} . Let \mathfrak{a} be a maximal Abelian subalgebra of \mathfrak{p} (\mathfrak{a} exponentiates to a maximal flat of G/K through the basepoint). Let $X \in \mathfrak{k}$, $Y, Z \in \mathfrak{a}$ be arbitrary. Then $[X, Y]$ is an arbitrary tangent vector to the K -orbit through Y at the point Y , and

$$B([X, Y], Z) = B(X, [Y, Z]) = 0,$$

where we have used the ad-invariance of the Killing form, and the fact that \mathfrak{a} is Abelian. This shows that \mathfrak{a} is orthogonal to $K(Y)$ at Y . If Y is a K -regular point, then $\dim K(Y)$ equals the codimension of \mathfrak{a} , so in this case \mathfrak{a} coincides with the normal space $\nu_Y(K(Y))$. It follows from Lemma 1.1.1 that (K, \mathfrak{p}) is a polar representation. Here the generalized Weyl group coincides with the (little) Weyl group of the symmetric space.

The classification theorem of Dadok [Dad85] implies that every polar representation of a compact connected Lie group is orbit-equivalent to such an s-representation, for some symmetric space (see also [Kol03] for an alternative approach, and [EH99b] for a geometric proof in case of cohomogeneity bigger than two). The list of representations orbit-equivalent to an s-representation is not difficult to compile. Besides the cohomogeneity one case which was previously known, in the irreducible case there is a short list [EH99a], and in the general case there is a description [Ber01].

Note that the orthogonal conjugacy of real symmetric matrices to diagonal matrices proved in basic Linear Algebra courses is the polarity of the s-representation associated to $SL(n, \mathbb{R})/SO(n)$.

Moving to polar actions on more general Riemannian manifolds, for a proper and isometric action, a geodesic orthogonal to an orbit remains orthogonal to every orbit it meets, due to Clairaut's lemma. It follows that cohomogeneity one actions form a class of polar actions with a flat section.⁵ It is interesting to remark that flat sections of polar actions on symmetric spaces of compact type equipped with metrics coming from the Cartan-Killing form were shown to be compact tori (hence properly embedded) in [HPTT95].⁶

A polar action with flat sections is sometimes called *hyperpolar*. In [Her60], Hermann constructed examples of "variationally complete" actions (in the sense of Bott and Samelson, cf. Lecture 3), which were later found to be hyperpolar, as follows. Let (G, H) and (G, K) be two symmetric pairs where, say, G is compact. Then G/H and G/K are compact symmetric spaces. The left-action

⁵Either due to the brothers Alekseevsky result, or to the broader definition of a section, cf. footnote 1.

⁶A cohomogeneity one action on a torus can of course have as section a dense irrational torus. Even in the case of a cohomogeneity one action on a compact symmetric space without flat factor, taking a metric non-proportional to the Cartan-Killing form in the reducible case, one can have a non-embedded section: one such example is the diagonal action of $SO(3)$ on the product of spheres $S^2(1) \times S^2(R)$ where R^2 is irrational. (This follows because $\gamma(t) = ((\cos t, \sin t, 0), (R \sin \frac{t}{R^2}, R \cos \frac{t}{R^2}, 0))$ is a geodesic normal to the orbits.)

of H on G/K , now called a *Hermann action*, is hyperpolar⁷ (cf. section 4.1).

Kollross classified hyperpolar actions (in particular, cohomogeneity one actions) on compact irreducible symmetric spaces in his thesis, which is published as [Kol02]. Later in [Kol17] he showed that one can remove the irreducibility assumption if the cohomogeneity of the action is bigger than one.⁸ His result is that an indecomposable⁹ hyperpolar action of cohomogeneity at least two on a compact symmetric space is orbit-equivalent to a Hermann action.

There are easy examples of polar actions with nonflat sections on compact symmetric spaces of rank one. A simple example is the action of the maximal torus T^n of the isometry group $SU(n+1)$ of complex projective space $\mathbb{C}P^n$, which is polar with sections isometric to a totally geodesic $\mathbb{R}P^n$. According to the classification result of Podestà and Thorbergsson [PT02], the polar actions on classical symmetric spaces of rank one are induced from certain polar actions on spheres using the Hopf fibration. In the case of the Cayley projective plane, there is no Hopf fibration, and their analysis found four polar actions with cohomogeneity one, and four polar actions with cohomogeneity two; a further polar action of cohomogeneity two which was overlooked in [PT02] was later found to be polar in [GK16] (it is given by a maximal subgroup of the isometry group of $\mathbb{O}P^2$ whose Lie algebra is not regular).

No examples of polar actions with nonflat sections on irreducible compact symmetric spaces of rank bigger than one were known, so eventually the question of their existence became a folklore problem. Many special cases were examined, most notably in the case of a Hermitian symmetric space in [Bil06]. Finally, Lytchak and Kollross proved that they do not exist in [KL13].

One direction in which to extend the above results is to pass from compact symmetric spaces to compact non-negatively curved manifolds. Using the theory of Tits buildings, Fang, Grove and Thorbergsson proved in [FGT17] that a polar action of a compact Lie group on a simply connected compact positively curved manifold of cohomogeneity at least two is equivariantly diffeomorphic to a polar action on a compact rank-one symmetric space.

Another direction to go is to consider polar actions on symmetric spaces of non-compact type. Here much fewer results are known. On real hyperbolic space $\mathbb{R}H^n$ the classification is the work of Wu [Wu92]. On complex hyperbolic space $\mathbb{C}H^n$ it has been achieved by Díaz-Ramos, Domínguez-Vázquez and Kollross [DDK17] (previously, the case of $\mathbb{C}H^2$ was dealt with in [BD13]). Next, there is a general description and a partial classification of cohomogeneity one actions on irreducible symmetric spaces of non-compact type, due to Berndt and Tamaru [BT13]. They fall into two cases, namely, either there is a unique singular orbit and the other orbits are distance tubes around the singular or-

⁷The left- and right-action of $H \times K$ on G is also hyperpolar. These actions generalize the isotropy action $(K, G/K)$, and the left- and right-action $(K \times K, G)$, which had been previously considered by Bott and Samelson.

⁸The classification of cohomogeneity one actions on reducible symmetric spaces is still open.

⁹Here an isometric action on a Riemannian manifold is called *decomposable* if the manifold can be written as a Riemannian product, and the action is orbit equivalent to the product of isometric actions on the factors, and *indecomposable* otherwise.

bit, or there are no singular orbits and the orbits define a regular Riemannian foliation. More generally, hyperpolar actions with no singular orbits on symmetric spaces of non-compact type are classified in [BDT10]. Regarding polar actions on higher rank symmetric spaces of non-compact type, there is a classification of those polar actions with a fixed point [DK11]. In another work, Kollross [Kol11] used Cartan duality between symmetric spaces of compact type and non-compact type, which maps an action of reductive algebraic group to a dual action, and showed that the dual action shares many properties in common with the original action, for instance (hyper)polarity; in this way, he obtained a number of new results on polar and hyperpolar actions on noncompact symmetric spaces. See the survey [DDS21] for more details on polar actions on symmetric spaces of non-compact type.

All examples of polar actions discussed above turn out to be on symmetric spaces. In [GZ12] an algorithm to recover a polar action, up to equivariant diffeomorphism, from the specification of the orbit space and the isotropy groups along the strata, which must satisfy certain compatibility conditions, is given, which allows to construct polar actions on a variety of non-symmetric manifolds. This algorithm was used in [Goz15] to classify polar actions on compact simply-connected manifolds up to dimension 5, up to equivariant diffeomorphism (see also [Muc11] for examples of polar actions on non-symmetric manifolds).

1.5 Polarity and the integrability of the distribution of normal spaces to the principal orbits

For a polar action (G, M) , the sections are clearly integral manifolds of the tangent distribution \mathcal{H} on the regular set M_{reg} , defined by $\mathcal{H}_p = \nu_p(Gp)$. Note that this is the horizontal distribution of the Riemannian submersion $M_{reg} \rightarrow M_{reg}/G$. As early as 1987, Palais and Terng conjectured that if \mathcal{H} is integrable then its integral manifolds can be extended to sections, see [PT87, Remark 3.3], where they call \mathcal{H} the *principal horizontal distribution*. As of now, the conjecture has been verified and there are different proofs in the literature, using different techniques, and with different degrees of generality. We shall now sketch the approach of Heintze, Liu and Olmos [HLO06, Appendix A].

For ease of presentation, we start with the linear case.

1.5.1 Proposition *Let (G, V) be an orthogonal representation of a compact Lie group G on an Euclidean vector space V . If the principal horizontal distribution \mathcal{H} is integrable, then (G, V) is polar.*

Proof. Let L be a leaf of \mathcal{H} . It follows from the argument in Proposition 1.1.2(e) that L is totally geodesic. Therefore it is a non-empty open subset of an affine subspace Σ of V . We claim that Σ is a section of (G, V) . In fact, $\Sigma = T_p L = \nu_p(Gp)$ for $Gp \in L$, so Σ meets all G -orbits by Lemma 1.1.1. To see that it meets always orthogonally, let $X \in \mathfrak{g}$, and let γ be any horizontal geodesic with $\gamma(0) = p \in L$. Then the image of γ is contained in Σ and

$J := X^* \circ \gamma$ is a Jacobi field along γ . Since Σ is totally geodesic, also the horizontal component J^h of J is a Jacobi field. Now J^h vanishes on a neighborhood of $t = 0$, so J^h vanishes identically. This shows that $X_{\gamma(t)}^*$ is orthogonal to Σ for all $t \in \gamma^{-1}(V_{reg})$. Since γ is arbitrary, Σ is orthogonal to all principal orbits it meets, and hence to all orbits it meets, by denseness. \square

In the case of an arbitrary complete Riemannian manifold M , we can start the proof with the same argument, but the main issue is the completeness of the leaf L of \mathcal{H} . For that, we shall use Hermann's extension of Cartan's criterion for the existence of a totally geodesic submanifolds with given tangent space at one point (cf. section 4.2).

1.5.2 Theorem *Let (G, M) be a proper isometric action, where M is connected and complete. Assume that the principal horizontal distribution \mathcal{H} is integrable. Then (G, M) is polar in the broader sense (cf. footnote 1).*

Proof. Fix a G -regular point $p \in M$. Let L be maximal leaf of \mathcal{H} through p , and put $S = T_p L$. Consider a S -admissible once-broken geodesic $\gamma : [0, \ell] \rightarrow M$ emanating from p (cf. section 4.2 for this terminology) that does not meet the set of singular points, that is, $\gamma(t)$ lies either in a principal or in an exceptional G -orbit, for all $t \in [0, \ell]$. Then \mathcal{H} is defined along γ , and it is an auto-parallel distribution; in particular, $\mathcal{H}_{\gamma(t)}$ is invariant under $R_{\gamma(t)}$ for all t . It follows that (4.2.5) is satisfied along γ . Next, we consider a S -admissible once-broken geodesic emanating from p which meets the singular set at finitely many points. It follows from Lemma 1.5.3 below that (4.2.5) is satisfied along γ . Finally, we observe that the classes above are dense in the set of S -admissible once-broken geodesic emanating from p . We deduce that (4.2.5) is satisfied in general, by continuity. Now Theorem 4.2.4 yields a complete totally geodesic isometric immersion $\Sigma \rightarrow M$ such that $T_p \Sigma = S$, which is clearly a section. \square

For $v \in TM$, write $\gamma_v(t) = \exp(tv)$.

1.5.3 Lemma *Assume \mathcal{H} is integrable, and let $q \in M$ be a singular point. Then the slice representation at q is polar and, for any section Σ_0 of it, and any G_q -regular $v \in \Sigma_0$, Σ_0 is the parallel transport of $\mathcal{H}_{\gamma_v(t)}$ along γ_v from $\gamma_v(t)$ to q , for small $t > 0$.*

Proof. We first prove that the slice representation at q is polar. Let γ_v be any geodesic such that $v \in \nu_q(Gq)$ is a regular point for the slice representation. Then $\gamma_v(t)$ is G -regular for small $t > 0$. Take a sequence $t_n \searrow 0$; by compactness, we may pass to a subsequence and assume that $\mathcal{H}_{\gamma_v(t_n)}$ converges to a subspace $\Sigma_0 \in \nu_q(Gq)$. Note that $v \in \Sigma_0$. Now $\langle \nabla_u X^*|_{\gamma_v(t_n)}, w \rangle = -\langle A_{X_{\gamma_v(t_n)}^*} u, w \rangle = 0$, for all $X \in \mathfrak{g}$, $u, w \in \mathcal{H}_{\gamma_v(t_n)}$. By continuity, $\langle X_u^{**}, w \rangle = \langle \nabla_u X^*|_q, w \rangle = 0$ for all $X \in \mathfrak{g}_q$, $u, w \in \Sigma_0$. It follows that Σ_0 is a section for the slice representation.

Next, recall that $\exp_q : \nu_q^{\leq \epsilon}(Gq) \rightarrow S_q$ is a G_q -equivariant diffeomorphism, where S_q is the normal slice and $\epsilon > 0$ is small. It follows that

$$(1.5.4) \quad d(\exp_q)_{tv}(T_v G_q(v)) = T_{\gamma_v(t)}(G_q \gamma_v(t))$$

for $t > 0$ small. Put $\Sigma = \exp_q(\Sigma_0)$. Since the metric $(\exp_q^* g)_{tv} \rightarrow g_q$ uniformly on compact subsets as $t \rightarrow 0$, taking orthogonal complements in (1.5.4) we obtain that the distance in $\Lambda^c(T_{\gamma_v(t)}S_q)$ satisfies $d(T_{\gamma_v(t)}\Sigma, \mathcal{H}_{\gamma_v(t)}) \rightarrow 0$ as $t \rightarrow 0$, where c is the cohomogeneity of (G, M) . Together with $T_{\gamma_v(t)}\Sigma \rightarrow \Sigma_0$, this implies that $\mathcal{H}_{\gamma_v(t)} \rightarrow \Sigma_0$, and hence Σ_0 is the parallel transport of $\mathcal{H}_{\gamma_v(t)}$, by continuity. \square

2 Lecture 2: Orbifold points

Historically, orbifolds first arose as manifolds with singular points long before they were formally defined. Definitions of orbifolds were given by Satake in 1950s in the context of automorphic forms (“V-manifolds”), by Thurston in the 1970’s in the context of 3-manifolds (when together with students he coined the name orbifold; the notions of orbifold coverings and orbifold fundamental groups are also due to him), and by Haefliger in the 1980’s in the context of $\text{CAT}(\kappa)$ -spaces (“orbihedrons”). Formally speaking, today there are two accepted ways to define orbifolds: by means of orbifold atlases, and this can be done in the topological, differentiable or Riemannian category; or by means of Lie groupoids, which yields a slightly more general definition, albeit less geometrical. A more direct definition, in the restricted Riemannian setting, has been proposed by Lytchak (see [LT10]). Herein we follow this approach and sketch some of the main ideas. A fuller account can be found in [Lan20].

2.1 Riemannian orbifolds

An intrinsic metric space X is called a *Riemannian orbifold* if every point $x \in X$ admits a neighborhood U isometric to a quotient M/Γ , where M is a Riemannian manifold and Γ is a finite group of isometries. In this definition, M is endowed with the induced intrinsic metric, and M/Γ with the corresponding quotient metric, which measures the distance between orbits in M . Recall that a *intrinsic metric space* is a special kind of metric space in which the distance between any pair of points can be realized as the infimum of the lengths of all rectifiable paths connecting these points.

A Riemannian orbifold is called *good* if it is globally isometric to the quotient space of a Riemannian manifold by a discrete group of isometries, and *bad* otherwise.

Lemma 2.1.2 below shows that a Riemannian orbifold X is locally represented as a quotient in a unique way. We first recall a special case proved in [Swa02]. Let V be an Euclidean space and denote by $S(V)$ its unit sphere. If Γ, Γ' are two finite subgroups of $O(V)$ which are conjugate, then the orbit spaces $V/\Gamma, V/\Gamma'$ are isometric. In fact, if $\Gamma' = f\Gamma f^{-1}$ for some $f \in O(V)$, then

there is an induced isometry

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ \pi \downarrow & & \downarrow \pi' \\ V/\Gamma & \xrightarrow{\bar{f}} & V/\Gamma' \end{array}$$

given by $\bar{f}(\Gamma v) = \Gamma' f(v)$. Conversely:

2.1.1 Lemma *If $V/\Gamma, V/\Gamma'$ are isometric then Γ, Γ' are conjugate in $O(V)$.*

Proof. We proceed by induction on $n = \dim V$. In the initial case of $n = 1$, $V \cong \mathbb{R}$ and the only possibilities for Γ are $\{1\}$ and $\{\pm 1\}$, which yield \mathbb{R} and $[0, +\infty)$, resp., non-isometric orbit spaces, so we are done.

Assume now $n \geq 2$. It is enough to work with $X = S(V)/\Gamma, X' = S(V)/\Gamma'$. Suppose $F : X \rightarrow X'$ is an isometry. Let $x \in X$ be such that $\Gamma \cdot x$ and $\Gamma' \cdot F(x)$ are principal orbits. Choose points $p \in \pi^{-1}(x), p' \in \pi'^{-1}(F(x))$ and open neighborhoods $U_p, U_{p'}, U_x, U_{x'}$ such that $\pi|_{U_p} : U_p \rightarrow U_x, \pi'|_{U_{p'}} : U_{p'} \rightarrow U_{x'}$ are isometries and $F(U_x) = U_{x'}$. Then $(\pi'|_{U_{p'}})^{-1}F\pi : U_p \rightarrow U_{p'}$ is an isometry, where we view $\pi : S(V) \rightarrow X, \pi' : S(V) \rightarrow X'$; since $S(V)$ is a space of constant curvature, we can (uniquely) extend it to a global isometry $f : S(V) \rightarrow S(V)$. Let $\bar{f} : X'' \rightarrow X'$ be the isometry induced on the level of quotients, where $X'' := f^{-1}\Gamma'f$ and $X'' = S(V)/\Gamma''$. Then $\pi'' = \bar{f}^{-1}\pi'f : S(V) \rightarrow X''$. Therefore, identifying $X \cong X''$ using the isometry $\bar{f}^{-1}F$, we get $\pi''|_{U_p} = \pi|_{U_p}$. We will show that $\Gamma'' = \Gamma$ as subgroups of $O(V)$.

It suffices to prove that:

- (a) π is completely determined by its restriction to an open neighborhood U_p of a Γ -regular point p .
- (b) Γ is completely determined by π .

Since π is a local isometry on the principal stratum, it is determined along any unit speed geodesic γ in $S(V)$ emanating from p , until γ reaches a non-regular point, say $q = \gamma(t_0)$ for some $t_0 > 0$. Now $\dot{\gamma}(t_0)$ belongs to the unit sphere S_q of $T_q S(V)$. The space of directions $\Sigma_y X$ for $y = \pi(q) \in X$ is isometric to S_q/Γ_q . Since $\dim T_q S(V) = n - 1$, the action of Γ_q on S_q is known by the induction hypothesis. It follows that the exit direction of $\pi \circ \gamma$ from y is known and thus π is determined along γ beyond t_0 ; this proves (a). Finally, the elements of Γ are in bijective correspondence with the points in $\pi^{-1}(x)$ via the map $\gamma \mapsto \gamma(p)$. For each $\gamma \in \Gamma$, we have a commutative diagram:

$$\begin{array}{ccc} U_p & & \\ \pi \downarrow & \searrow \gamma & \\ U_x & \xleftarrow{\pi} & \gamma(U_p) = U_{\gamma(p)} \end{array}$$

Since γ is an isometry of $S(V)$, using (a) this completely determines it as an element of $O(V)$. Hence (b) is proved. \square

2.1.2 Lemma *Every isometry $F : M/\Gamma \rightarrow M'/\Gamma'$ is locally induced by a locally defined isometry $f : M \rightarrow M'$. Namely, for every $x \in M/\Gamma$, there exist connected open neighborhoods U, U' of $x, x' = F(x)$ of the form $V/\Gamma_p, V'/\Gamma'_{p'}$, where V, V' are normal neighborhoods of $p \in \pi^{-1}(x), p' \in \pi'^{-1}(x')$, resp., and $U' = F(U)$ ($\pi : V \rightarrow V/\Gamma_p, \pi' : V' \rightarrow V'/\Gamma'_{p'}$ denote the canonical projections). Moreover $F \circ \pi = \pi' \circ f$ for some isometry $f : V \rightarrow V'$ with $f(p) = p'$ and $\Gamma'_{p'} = f\Gamma_p f^{-1}$.*

Proof. Let $G = \Gamma_p, G' = \Gamma'_{p'}$. By restriction to slices we have an isometry $F : V/G \rightarrow V'/G'$ with $F(x) = x'$. Here V, V' can be taken to be metric balls of the same, small radius, around p, p' , resp. Consider the actions of G, G' on $T_p V, T_{p'} V'$, resp. Then there is an isometry

$$T_p V/G \cong T_x(V/G) \rightarrow T_{x'}(V'/G') \cong T_{p'} V'/G',$$

which we denote by dF_x . By Lemma 2.1.1, there is an isometry $\varphi : T_p V \rightarrow T_{p'} V'$ such that

$$\begin{array}{ccc} T_p V & \xrightarrow{\varphi} & T_{p'} V' \\ d\pi_p \downarrow & & \downarrow d\pi'_{p'} \\ T_p V/G & \xrightarrow{dF_x} & T_{p'} V'/G' \end{array}$$

is commutative. Since the Riemannian exponential maps $\exp_p : T_p V \rightarrow V, \exp_{p'} : T_{p'} V' \rightarrow V'$ are G, G' -equivariant diffeomorphisms, resp., we can define an equivariant diffeomorphism $f : V \rightarrow V'$ by

$$\begin{array}{ccc} T_p V & \xrightarrow{\varphi} & T_{p'} V' \\ \exp_p \downarrow & & \downarrow \exp_{p'} \\ V & \xrightarrow{f} & V' \end{array}$$

Finally, there is an induced map

$$\begin{array}{ccc} V & \xrightarrow{f} & V' \\ \pi \downarrow & & \downarrow \pi' \\ V/G & \xrightarrow{\bar{f}} & V'/G' \end{array}$$

We claim that $\bar{f} = F$. In fact, for a geodesic $\gamma(t) = \exp_p t\dot{\gamma}(0)$ in V :

$$(2.1.3) \quad \begin{aligned} \bar{f}\pi\gamma(t) &= \bar{f}\pi \exp_p t\dot{\gamma}(0) \\ &= \pi' \exp_{p'} t\varphi(\dot{\gamma}(0)) \end{aligned}$$

by commutativity of the last two diagrams. Since x is a fixed point of G , $\pi \circ \gamma$ is a metric geodesic of V/G . Thus it is mapped under F to a metric geodesic emanating from x' , namely, $\pi' \circ \gamma'$, where $\dot{\gamma}'(0) = \varphi(\dot{\gamma}(0))$:

$$(2.1.4) \quad F\pi\gamma(t) = \pi'\gamma'(t) = \pi' \exp_{p'} t\varphi(\dot{\gamma}(0)).$$

Comparison of (2.1.3) and (2.1.4) proves the claim.

It follows $f : V \rightarrow V'$ is a local isometry on the regular set and thus, by continuity, an isometry everywhere. Now the groups G' , fGf^{-1} acting on V' are orbit-equivalent. If not coincident, they generate a strictly larger group with the same (finite) orbits and thus non-trivial principal isotropy groups, a contradiction (since the slice representation at regular points must be trivial). Hence $G' = fGf^{-1}$. \square

Let X be a Riemannian orbifold and let $x \in X$. Locally represent X around x as a quotient M/Γ and write $x = \Gamma p$ for some $p \in M$. Since the isotropy group Γ_p acts by isometries on M , it can be viewed as a subgroup of the orthogonal group $O(T_p M)$. Moreover, it follows from Lemma 2.1.2 that the congruence class of Γ_p is independent of the local representation of X as a quotient. After identification $T_p M \cong \mathbb{R}^n$, we get a congruence class of subgroups of $O(n)$, called the *local group* of X at x and denoted by $\text{Iso}_x(X)$. A point $x \in X$ is called a *manifold point* of X if $\text{Iso}_x(X) = \{1\}$.

2.1.5 Example Let a Lie group G act by isometries on a Riemannian manifold. Then the orbit space has a canonical structure of Riemannian orbifold in the following two cases:

- (a) G is discrete and the action is proper (such orbifolds are called *good* or *developable*; non-good orbifolds are also called *bad*);
- (b) G is compact and connected and all orbits have the same dimension.

An *orbi-covering* is a continuous map $\pi : Y \rightarrow X$ between Riemannian orbifolds where every $x \in X$ admits a neighborhood $U \cong \tilde{U}/G$ such that every component V of $\pi^{-1}(U)$ must be of the form \tilde{V}/H , with $H \subset G$, and $\pi|_V : V \rightarrow U$ lifts to a H -equivariant homeomorphism $\tilde{V} \rightarrow \tilde{U}$. Here G is the local group $\text{Iso}_x(X)$ of x and H is the local group $\text{Iso}_y(Y)$ at a point $y \in \pi^{-1}(x)$. If we identify $\tilde{V} \cong \tilde{U}$ via this map, the local representation of the covering map is $\tilde{U}/H \rightarrow \tilde{U}/G$ for H a subgroup of G .

It is a fact that every connected orbifold X admits a *universal orbi-covering* \tilde{X} , unique up to equivalence, with the property that it orbi-covers any other orbi-covering space of X . The *orbifold fundamental group* of X is the group $\pi_1^{\text{orb}}(X)$ of deck transformations of the universal orbi-covering; it acts simply transitively on the fibers of this orbi-covering. In general, one can write a presentation of the orbifold fundamental group in terms of its usual fundamental group and its strata of codimension 1 and 2 (cf. [Dav10]). The orbifold fundamental group is a refinement of the usual fundamental group in the sense that an orbifold can be simply-connected in the topological sense without being simply-connected in the orbifold sense.

2.1.6 Example Let the cyclic group \mathbb{Z}_m act by rotations around a fixed axis on the sphere S^2 . The orbit space X is a Riemannian orbifold and topologically a 2-sphere but $\pi_1^{orb}(X) \cong \mathbb{Z}_m$.

2.1.7 Example There is a Riemannian orbifold structure $X_{m,n}$ on S^2 with exactly two non-manifold (conical) points whose local groups are respectively \mathbb{Z}_m and \mathbb{Z}_n . Then $\pi_1^{orb}(X_{m,n}) = \mathbb{Z}_d$ where d is the greatest common divisor of m and n . The orbifold $X_{m,n}$ is good if and only if $m = n$. In particular, $X_{m,1}$ for $m > 1$ is called a *teardrop*, and the bad 2-orbifold depicted in Fig. 1 is the quotient of $X_{2,1}$ by a reflection.

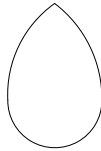


FIG. 1: A bad 2-orbifold

2.2 Orbifold points in orbit spaces

Consider a proper and isometric action of a Lie group G on a Riemannian manifold M , and let $X = M/G$ be its orbit space. The set X_{reg} of regular points of X is exactly the set of points that have neighborhoods isometric to Riemannian manifolds, whereas the slightly larger set X_{orb} of orbifold points of X consists of the set of points that have neighborhoods isometric to quotients of Riemannian manifolds by finite groups of isometries. The main goal of this lecture is to prove the following result [LT10]:

2.2.1 Theorem (Lythak and Thorbergsson) *Let (G, M) be a proper isometric action of a Lie group G on a complete Riemannian manifold M , and consider its orbit space $X = M/G$. A point $x = Gp \in X$ is an orbifold point if and only if the slice representation at $p \in M$ is polar.*

It follows from this theorem that X_{orb} contains all strata of codimension at most two, since representations of cohomogeneity at most two are always polar [HL71].

2.2.2 Lemma *Let (M, g) be a Riemannian manifold, fix a point $p \in M$, and consider the locally defined family of homotheties $\{h_\lambda\}_{0 \leq \lambda \leq 1}$, given by*

$$h_\lambda(\exp_p v) = \exp_p(\lambda v)$$

for $v \in T_p M$. Then the Riemannian metrics $g_\lambda := \frac{1}{\lambda^2} h_\lambda^ g$ converge smoothly on compact neighborhoods of p to $g_0 := (\exp_p^{-1})^* g_p$ as $\lambda \rightarrow 0$.*

Proof. The proof reduces to a standard calculation in normal coordinates. Write $h_\lambda = \exp_p \circ \tilde{h}_\lambda \circ \exp_p^{-1}$, where $\tilde{h}_\lambda(v) = \lambda v$. It is equivalent to show that the metrics $\frac{1}{\lambda^2} \tilde{h}_\lambda^* \exp_p^* g = (\exp_p^* g) \circ \tilde{h}_\lambda$ smoothly converge to the flat metric g_p on compact neighborhoods of 0 in $T_p M$.

For $v \in T_p M$ and e_1, \dots, e_n an orthonormal basis of $(T_p M, g_p)$, we have $(\exp_p^* g)_{\lambda v}(e_i, e_j) = g_{ij}(\lambda v) =: G_{ij}(\lambda, v)$, where g_{ij} are the coefficients of the metric in the induced normal coordinates. But on a compact neighborhood K of 0 and all i, j , the smooth function G_{ij} and all its partial derivatives in v converge uniformly to their value at $\lambda = 0$ as $\lambda \rightarrow 0$, that is, $g_{ij} \circ \tilde{h}_\lambda$ converges in the C^∞ -topology on K to the constant function δ_{ij} . \square

Proof of Theorem 2.2.1. Let S be a normal slice at p , and note that the homotheties h_λ restrict to S and induce isometries $h_\lambda : (S, g_\lambda) \rightarrow (h_\lambda(S), \frac{1}{\lambda^2} g)$, where we have used the notation of Lemma 2.2.2. The G -action preserves the metrics g_λ . For a regular point $q \in S$, denote by $\bar{\kappa}_{g_\lambda}(q)$ the supremum of all sectional curvatures of the metric induced by g_λ on the local quotient GS/G at Gq . Owing to Lemma 2.2.2, $\bar{\kappa}_{g_\lambda}(q) \rightarrow \bar{\kappa}_{g_0}(q)$ as $\lambda \rightarrow 0$. On the other hand,

$$\begin{aligned} \bar{\kappa}_{g_\lambda}(q) &= \bar{\kappa}_{\frac{1}{\lambda^2} g}(h_\lambda(q)) \\ &= \lambda^2 \bar{\kappa}_g(h_\lambda(q)) \\ &= \frac{1}{\|v\|^2} d(h_\lambda(q), p)^2 \bar{\kappa}_g(h_\lambda(q)), \end{aligned}$$

where $q = \exp_p v$ for $v \in \nu_p(Gp)$, so $d(h_\lambda(q), p) = \lambda d(q, p) = \lambda \|v\|$.

If x is an orbifold point, the sectional curvatures of M_{reg}/G near x are locally bounded, so

$$\bar{\kappa}_{g_0}(q) = \frac{1}{\|v\|^2} \lim_{\lambda \rightarrow 0} d(h_\lambda(q), p)^2 \bar{\kappa}_g(h_\lambda(q)) = 0.$$

It follows that the orbit space of the slice representation at p is flat at regular points. By O'Neill's formula (3.2.4), the principal horizontal distribution of the slice representation is integrable, and hence the slice representation is polar.

Conversely, assume that the slice representation at p is polar, and let Σ be a section with associated Weyl group W . Let $N = \exp_p(\Sigma) \cap S$, where S is a normal slice at p . Then W acts on N , and we shall define a W -invariant Riemannian metric \tilde{g} on N such that N/W is isometric to a neighborhood of x in X .

For $q \in S$, put $\mathcal{V}_q = (T_q N)^{\perp_{g_0}}$ and $\mathcal{H}_q = (\mathcal{V}_q)^{\perp_g}$. These are smooth distributions, with $\mathcal{V}_q \supset T_q(Gq)$ (since Σ is g_p -orthogonal to $T_v(G_p v)$ for $v \in \Sigma$), $\mathcal{H}_q \subset T_q(Gq)^{\perp_g}$, and the latter inclusion is an equality if q is a regular point. Let P_q denote the orthogonal projection of $T_q M$ onto \mathcal{H}_q , and define

$$\tilde{g}_q(u, v) = g(P_q(u), P_q(v))$$

for $u, v \in T_q N$. Since $P_q : (T_q N, \tilde{g}_q) \rightarrow (\mathcal{H}_q, g)$ is an isometry, the projection $\pi : (N, \tilde{g}) \rightarrow M/G$ preserves the lengths of all curves contained in the regular set of (G, M) . Since π is W -invariant, the action of W on N preserves the length

of curves in $N \cap M_{reg}$; by continuity, W acts by isometries on N . It follows that $N/W \rightarrow M/G$ is an isometric embedding onto a neighborhood of x , where N/W is a Riemannian orbifold. \square

2.3 Applications

A Riemannian orbifold comes along with a canonical stratification given by the connected components of the set of points with the same local group. Each stratum is a connected Riemannian manifold, which is locally convex with respect to the ambient metric. The closure of any stratum is a union of strata. Any Riemannian orbifold B can be written as a quotient of a Riemannian manifold (the orthonormal frame bundle of B) by an almost free isometric action of a compact Lie group. The canonical stratification of B is then exactly the stratification by orbit type.

Let $X = M/G$ be the orbit space of a proper and isometric action (G, M) . The boundary ∂X of X (in the sense of Alexandrov geometry) is the closure of the union of strata of codimension 1. Since X_{orb} contains all strata of X of codimension at most 2, X_{orb} has non-empty boundary if and only if X has non-empty boundary. A Riemannian orbifold with a non-empty boundary can be doubled. It follows that a Riemannian orbifold B has $\partial B \neq \emptyset$ if and only if $\pi_1^{orb}(B)$ contains a reflection.

2.3.1 Example Let X be the quotient of S^2 by the reflection across the equator. Then $X_{orb} = X$ and $\pi_1^{orb}(X) \cong \mathbb{Z}_2$.

2.3.2 Remark Let (G, M) be a proper and isometric action, where G is connected. Denote by B_0 the subset of points in $X = M/G$ representing non-singular G -orbits (that is, principal and exceptional G -orbits), so that $B_0 = M_0/G$, where M_0 is the union of principal and exceptional G -orbits in M . Then B_0 has the structure of a Riemannian orbifold. We lift (G, M) to a proper and isometric action of the simply-connected covering Lie group \tilde{G} of G on the simply-connected Riemannian covering \tilde{M} of M [Bre72, Thm. I.9.1]. Then $B_0 = \tilde{M}_0/\tilde{G}$. Since all \tilde{G} -orbits have the same dimension in \tilde{M}_0 , there is an epimorphism $\pi_1(\tilde{M}_0) \rightarrow \pi_1^{orb}(B_0)$ [Sal88]. Since the union of singular orbits in \tilde{M}_0 has codimension at most 2 [Bre72, Thm. IV.3.8], $\pi_1(\tilde{M}_0) = \pi_1(\tilde{M}) = \{1\}$, so also $\pi_1^{orb}(B_0) = \{1\}$.

A proof of the following result, probably folklore, can be found in [Lyt10].

2.3.3 Proposition *Let G be a connected compact Lie group of isometries of a simply-connected complete Riemannian manifold, consider the orbit space $X = M/G$, and the subset X_{orb} of orbifold points of X . Then $X_0 := X_{orb} \setminus \partial X_{orb}$ is exactly the set of non-singular (that is, principal and exceptional) G -orbits. Furthermore $\pi_1^{orb}(X_0) = \{1\}$.*

Proof. Due to Theorem 2.2.1, $B = X_{orb}$ consists precisely of the projections of points in M where the slice representation is polar. Let us show that all points $x \in B \setminus \partial B$ represent non-singular G -orbits in M . In fact, let $x \in B$ represent a singular orbit. Choose $p \in M$ projecting to x . Since the slice representation at p is polar, for the normal slice S at p we have that S/G_p is isometric to a Weyl chamber. The projection $S \rightarrow X$ induces an open isometric embedding $S/G_p \rightarrow B$. Since the Weyl chamber has non-empty boundary, so does its image in B . Hence any neighborhood of x in B contains boundary points. Since the boundary is closed, we deduce that $x \in \partial B$.

Under the assumption that M is simply-connected, we now show that all points in ∂B represent singular orbits. Indeed due to Remark 2.3.2, the subset of non-singular orbits $B_0 \subset B$ has $\pi_1^{orb}(B_0) = \{1\}$, so B_0 cannot contain strata of codimension 1. But B_0 is open, so, if it has a point in ∂B , then it has a point lying in a stratum of codimension 1, and hence the whole stratum is contained in B_0 , a contradiction. We have shown that $X_0 = B_0$, as desired. \square

We mention two further instances of major applications of Theorem 2.2.1. A polar action is called *infinitesimally polar* if all of its slice representations are polar. Due to Proposition 1.1.3, every polar representation is infinitesimally polar. Theorem 2.2.1 was the main tool in [GL16] to classify the infinitesimally polar actions on Euclidean spheres. In particular:

2.3.4 Theorem (Gorodski-Lytchak) *An isometric quotient X of the unit sphere by a compact Lie group is isometric to a Riemannian orbifold if and only if the universal orbi-covering \tilde{X} of X is a weighted complex or quaternionic projective space, or \tilde{X} has constant curvature 1 or 4.*

Further, in [GK16] this classification of quotients isometric to Riemannian orbifolds was extended to compact rank one symmetric spaces.

3 Lecture 3: Variationally complete actions

In the 1950s Bott and Samelson introduced the concept of variationally completeness as a means of studying the topology of symmetric spaces and their loop spaces [BS58]. Roughly speaking, a proper and isometric action on a complete Riemannian manifold is variationally complete if it produces enough Jacobi fields along geodesics to determine the multiplicities of the focal points to the orbits. In modern terminology, they proved that the orbits of a variationally complete action are taut submanifolds of the ambient space, in the sense that the energy functional on the space of curves¹⁰ joining a generic point to an arbitrary, fixed orbit is a perfect Morse function. This establishes strong relations between the topology of the ambient manifold and the topology of the orbits.

¹⁰Of H^1 -Sobolev class.

3.1 Bott and Samelson's and related results

Let a Lie group G act properly and isometrically on a complete Riemannian manifold M . A geodesic γ of M is called *horizontal* if it is orthogonal to one orbit (and hence to every orbit it meets). Fix a G -orbit N and a horizontal geodesic γ meeting N at time $t = 0$. A Jacobi field along γ is called an *N -Jacobi field* if it is the variational field of a variation of γ through horizontal geodesics starting at N . Finally, the action (G, M) is called *variationally complete* if for every G -orbit N and every horizontal geodesic γ starting at N , every N -Jacobi field that vanishes for some $t > 0$ is the restriction of a G -Killing vector field along γ .¹¹

The motivation of Bott and Samelson to consider variationally complete actions of G on M was to construct an explicit basis of cycles in the \mathbb{Z}_2 -homology of the path space $\Omega(M; x, N)$, where N is an arbitrary G -orbit, $x \in M$, and the paths start at x and end at a point in N . In modern terminology, we can state their result as follows:

3.1.1 Theorem (Bott-Samelson) *The orbits of a variationally complete action are taut submanifolds (with respect to \mathbb{Z}_2 -coefficients).*

Here a submanifold N of M is called *taut* if, for every nonfocal point x , the energy functional $E : \Omega(M; x, N) \rightarrow \mathbb{R}$, $E(\gamma) = \frac{1}{2} \int \|\gamma'\|^2 ds$, is a perfect Morse function, that is, every critical point (geodesic) of E corresponds to a basis element of $H_*(\Omega(M; x, N))$. Indeed, Bott and Samelson provide an algorithm to construct an explicit cycle for each critical point. In the same paper, for a symmetric space G/K , they prove that the isotropy action of K on G/K , the $K \times K$ -action on G by left and right-multiplication, and the linear isotropy action of K on $T_{x_0}(G/K) \cong \mathfrak{p}$ are variationally complete. Soon thereafter, Hermann [Her60] found a more general family of variationally complete actions on symmetric spaces. Namely, if K and H are both symmetric subgroups of the compact Lie group G , then the action of H on G/K is variationally complete.

L. Conlon was a student of Conlon. In [Con71] he proved the following theorem:

3.1.2 Theorem (Conlon) *A hyperpolar action of a compact Lie group G on a complete Riemannian manifold M is variationally complete.*

Proof. Let $N = Gp$ be a fixed orbit and let q be a focal point of N (that is, a critical value of the normal exponential map) along a geodesic $\gamma : [0, \ell] \rightarrow M$ with $\gamma(0) = p$ and $\gamma(\ell) = q$. Then there exists a Jacobi field J along γ satisfying $J(0) \in T_p N$, $J'(0) + A_{\gamma'(0)} J(0) \in \nu_p N$ and $J(\ell) = 0$; denote by V the space of Jacobi fields satisfying the first two of these conditions, and note that $\dim V = \dim M$.

Fix $s_0 \in (0, \ell)$ such that $p_0 = \gamma(s_0)$ is a regular point for the action of G and p_0 is not a focal point of N . There exists a unique section Σ passing through

¹¹It is equivalent to require that every N -Jacobi field that is tangent to another orbit is the restriction of a G -Killing vector field.

p_0 . Of course, Σ is flat and contains the image of γ . Since p_0 is not a focal point of N , the map $J \in V \mapsto J(s_0) \in T_{p_0}M$ is a linear isomorphism.

Decompose $J = J^v + J^h$ where J^h is the orthogonal projection of J on Σ . Due to the total-geodesicness of Σ , both J^v and J^h are Jacobi fields along γ . Since J^h vanishes at $s = 0$ and $s = \ell$ and Σ is flat, we have $J^h \equiv 0$. Since p_0 is a regular point, $J^v(s_0) \in T_{p_0}(Gp_0)$. Let $X \in \mathfrak{g}$ be such that $X_{p_0}^* = J^v(s_0)$. Owing to $X^* \circ \gamma \in V$, we have $X^* \circ \gamma = J^v = J$, finishing the proof. \square

3.2 The converse results

It was proved in [GT00], by means of classification, that a variationally complete representation is orbit-equivalent to the isotropy representation of a symmetric space, and hence is polar. In [DO01], a direct proof of this result was provided. Since the idea of the proof is very simple and geometric, we present it in the sequel.

3.2.1 Theorem (Di Scala-Olmos) *A variationally complete representation of a compact Lie group G on an Euclidean space V is polar.*

Proof. Let $p \in V$ be a regular point so that $N = Gp$ is a principal orbit. Owing to Lemma 1.1.1, $\Sigma := \nu_p N$ meets all orbits.

Choose $\xi \in \nu_p N$ such that the Weingarten operator A_ξ has all eigenvalues nonzero. This is possible, since $A_p = -\text{id}$, and indeed the set of such vectors is open and dense in $\nu_p N$. Consider the geodesic $\gamma(s) = p + s\xi$, normal to N , and fix $s_1 > 0$ such that $N_1 = Gq$, $q = \gamma(s_1)$, is also a principal orbit. Due to the assumption of variational completeness, q is not a focal point of N along γ . We will show that $T_p N = T_q N_1$ as subspaces of V .

Each eigenvector $u \in T_p N$ of A_ν , with corresponding eigenvalue $\lambda \neq 0$, gives rise to a Jacobi field $J(s) = (1 - \lambda s)u$ along the geodesic $\gamma(s) = p + s\xi$, associated to the variation $\gamma_t(s) = c(t) + s\hat{\xi}(t)$, where c is a smooth curve in N with $c(0) = p$ and $c'(0) = u$, and $\hat{\xi}$ is the parallel extension of ξ to a normal vector field along c . Since $J(0) = u \in T_p N$ and $J(\frac{1}{\lambda}) = 0$, the assumption of variational completeness yields a Killing vector field X induced by G such that $X \circ \gamma = J$. In particular, $J(s) \in T_{\gamma(s)}(G\gamma(s))$ for all s . Since q is not a focal point of N along γ , $s_1 \neq \frac{1}{\lambda}$ and therefore $u \in T_q N_1$. As the eigenvectors of A_ξ span $T_p N$, this shows $T_q N_1 = T_p N$.

We have seen that Σ is orthogonal to all principal orbits passing through an open and dense subset of itself. By a continuity argument, Σ is orthogonal to every orbit it meets. This finishes the proof. \square

An isometric action of a compact Lie group on a compact symmetric space can be lifted to a proper and Fredholm action of a Hilbert-Lie group on a Hilbert space via the so-called ‘‘holonomy map’’, see [TT95]. This idea was combined with the basic idea of the proof of Theorem 3.2.1 to prove the following result in [GT02]:

3.2.2 Theorem (Gorodski-Thorbergsson) *A variationally complete action of a compact Lie group on a compact symmetric space is hyperpolar.*

The following result was proved in [LT07] and generalizes Theorems 3.2.1 and 3.2.2. Its proof is the main goal of this lecture.

3.2.3 Theorem (Lytchak-Thorbergsson) *A variationally complete action on a complete Riemannian manifold with nonnegative sectional curvature is hyperpolar.*

A special case of this theorem is when the group is discrete. It says that a complete Riemannian manifold without conjugate points and with nonnegative sectional curvature is flat. We shall explain the proof of Theorem 3.2.3 in the sequel. The main tool is Wilking’s transversal Jacobi equation [Wil07], which can be viewed as an extension of the methods of Morse theory from the case of Riemannian submersions to the case of singular Riemannian foliations and, in particular, isometric actions.

Let $\pi : M \rightarrow B$ be a Riemannian submersion with horizontal and vertical distributions \mathcal{H} and \mathcal{V} , respectively. Then one of O’Neill’s equation says that the sectional curvature of a horizontal 2-plane $\sigma \subset \mathcal{H}$ and its projection $\sigma^* = d\pi(\sigma) \subset TB$ are related by

$$(3.2.4) \quad K(\sigma^*) = K(\sigma) + 3\|A_X Y\|^2,$$

where $A : \mathcal{H} \times \mathcal{H} \rightarrow \mathcal{V}$ is one of O’Neill’s tensors associated to π , namely,

$$A_X Y = (\nabla_{X^h} Y^h)^v + (\nabla_{X^h} Y^v)^h,$$

for all $X, Y \in TM$. The following properties of A are easily established:

- a. $A_X \mathcal{H} \subset \mathcal{V}$ and $A_X \mathcal{V} \subset \mathcal{H}$ for all $X \in TM$.
- b. A_X is skew-symmetric on $T_p M$ for all $p \in M$.
- c. $A_X Y = \frac{1}{2}[X, Y]^v$ for all $X, Y \in \mathcal{H}$.

In particular \mathcal{H} is integrable if and only if $A_X \mathcal{H} = 0$ for all $X \in \mathcal{H}$.

Let now X, Y be an orthonormal basis of the horizontal 2-plane σ . Then the right hand-side of O’Neill’s equation (3.2.4) reads

$$\langle R(Y, X)X, Y \rangle + 3\langle A_X Y, A_X Y \rangle = \langle (R(Y, X)X)^h - 3A_X^2 Y, Y \rangle.$$

Every projectable Jacobi field J along a horizontal geodesic γ projects to a Jacobi field along $\bar{\gamma} = \pi \circ \gamma$, and this projection induces an isomorphism between space of projectable Jacobi fields modulo the vertical Jacobi fields along γ , and the space of Jacobi fields along $\bar{\gamma}$. It follows that for every projectable Jacobi field J along γ , the horizontal component J^h satisfies the “transversal Jacobi equation”

$$(J^h)'' + (R(J^h, \gamma')\gamma')^h - 3A_{\gamma'}^2 J^h = 0.$$

Let now (G, M) be a proper and isometric action. On the regular part there is a Riemannian submersion $\pi : M_{reg} \rightarrow M_{reg}/G$, to which the above considerations apply. Moreover, Wilking explained how to overcome the difficulties that arise when the horizontal geodesic γ passes through singular points of the action; here we assume γ is a regular complete geodesic in the sense that it passes through regular points; it follows that the singular points along γ form a discrete set of parameters.

The first step is to extend the principal horizontal distribution \mathcal{H} along the regular part of the horizontal geodesic γ to a smooth distribution defined everywhere along γ , namely $\mathcal{H}_t \subset T_{\gamma(t)}M$ for all $t \in \mathbb{R}$, where $\mathcal{H}_t = (T_{\gamma(t)}(G\gamma(t)))^\perp$ if $\gamma(t)$ is a regular point of the G -action. Let Λ be the space of N_t -Jacobi fields along γ , where $N_t = G\gamma(t)$, and let Υ be the subspace of vertical Jacobi fields. We put $\mathcal{H}_t := (\mathcal{V}_t)^\perp$, where we define

$$\mathcal{V}_t := \{J(t)|J \in \Upsilon\} \oplus \{J'(t)|J \in \Upsilon, J(t) = 0\}.$$

3.2.5 Proposition \mathcal{V}_t is a smooth vector bundle of rank $\dim \Upsilon$ along γ . Moreover, $\mathcal{V}_t = T_{\gamma(t)}N_t$ if $\gamma(t)$ is regular.

Proof. It is easy to see that $\{J(t)|J \in \Upsilon\} = T_{\gamma(t)}N_t$ for all t . Indeed denote the Lie algebra of G by \mathfrak{g} . Then $T_{\gamma(t)}N_t = \text{span}\{X_{\gamma(t)}^*|X \in \mathfrak{g}\}$ and, for each $X \in \mathfrak{g}$, the induced Killing field X^* on M restricts to a Jacobi field J along γ which is tangent to the G -orbits everywhere, that is, $J \in \Upsilon$.

Suppose now $\gamma(t_0)$ is regular for some t_0 . Then $\{J'(t_0)|J \in \Upsilon, J(t_0) = 0\}$ is trivial. Indeed if there was $J \in \Upsilon$ with $J(t_0) = 0$ and $J'(t_0) \neq 0$, then $\gamma(t_0)$ would be a focal point of all N_t , but this is impossible as N_{t_0} is a principal orbit.

Finally, if Y is a smooth vector field along γ that has an isolated zero at t_1 , then the vector field \tilde{Y} , given by

$$\tilde{Y}(t) := \begin{cases} \frac{1}{t-t_1}Y(t), & \text{if } t \neq t_1, \\ Y'(t_1), & \text{if } t = t_1, \end{cases}$$

is smooth, and the span of $Y(t)$, $\tilde{Y}(t)$ is one-dimensional on a neighborhood of t_1 . This proves all statements. \square

The second step is to extend the definition of $A_{\gamma'(t)}$ to all t , namely, a skew-symmetric operator A_t on $T_{\gamma(t)}M$ such that A_t coincides with the O'Neill tensor $A_{\gamma'(t)}$ if $\gamma(t)$ is regular. Let Y be a smooth vector field along γ . Set

$$A_t Y(t) := ((Y^h)'(t))^v + ((Y^v)'(t))^h.$$

The tensor A_t clearly satisfies the requirements.

The last step is to write the differential equation along γ that vector fields of the form $Y = J^h$ for some $J \in \Lambda$ must satisfy. This equation was derived in [Wil07], and reads

$$(3.2.6) \quad \frac{(\nabla^h)^2}{dt^2}Y + (R(Y, \gamma')\gamma')^h - 3A_t^2 Y = 0.$$

where $\frac{\nabla^h}{dt}$ is the connection induced on horizontal vector fields along γ (see subsection 4.3).

The idea of the proof of Theorem 3.2.3 is roughly as follows. Owing to Theorem 1.5.2, it suffices to show that \mathcal{H} is integrable over M_{reg} . This, in turn, follows if A_t vanishes identically. On the other hand, if A_t is not identically zero, due to nonnegative curvature of M , we obtain that M/G “has positive curvature” somewhere and thus “conjugate points”. But variationally completeness of (G, M) more or less means “absence of conjugate points” in M/G , leading to a contradiction.

For each t the operator

$$\mathcal{R}(t) : v \mapsto (R(v, \gamma'(t))\gamma'(t))^h - 3A_t^2 v$$

is self-adjoint and positive semidefinite on \mathcal{H}_t . Therefore

$$(3.2.7) \quad Y''(t) + \mathcal{R}(t)Y(t) = 0$$

is a differential equation of Morse-Sturm type (generalization of the Jacobi equation), where the “prime” refers to $\frac{\nabla^h}{dt}$, to which is associated an index form

$$I_{a,b}(X, Y) = \int_a^b \langle X', Y' \rangle - \langle \mathcal{R}(t)X, Y \rangle dt,$$

for each $a < b$, where X and Y are piecewise smooth sections of the horizontal distribution along $\gamma|_{[a,b]}$ vanishing at a and b . Suppose, to the contrary, that there is t_0 such that $A_{t_0} \neq 0$. Then there is a unit vector $v_0 \in \mathcal{H}_{t_0}$ such that $\langle \mathcal{R}(t_0)v_0, v_0 \rangle > 0$. Let Z_0 be the ∇^h -parallel vector field along γ such that $Z_0(t_0) = v_0$. Then

$$(3.2.8) \quad C := \int_{t_0-1}^{t_0+1} \langle \mathcal{R}(t)Z_0(t), Z_0(t) \rangle dt > 0.$$

Let $\varphi : \mathbb{R} \rightarrow [0, 1]$ be a (smooth) bump function with support contained in $[t_0 - N, t_0 + N]$ and satisfying $\varphi|_{[t_0-1, t_0+1]} \equiv 1$, for some $N > 1$. In fact, we can take N as large as needed to further have

$$\int_{t_0-N}^{t_0+N} \varphi'(t)^2 dt < C.$$

Set $Z(t) = \varphi(t)Z_0(t)$. For $a = t_0 - N$ and $b = t_0 + N$, our choices yield

$$\begin{aligned} I_{a,b}(Z, Z) &= \int_{t_0-N}^{t_0+N} \|Z'(t)\|^2 - \langle \mathcal{R}(t)Z(t), Z(t) \rangle dt \\ &= \int_{t_0-N}^{t_0+N} \varphi'(t)^2 dt - \int_{t_0-N}^{t_0+N} \varphi(t)^2 \langle \mathcal{R}(t)Z_0(t), Z_0(t) \rangle dt \\ &< C - \int_{t_0-1}^{t_0+1} \langle \mathcal{R}(t)Z_0(t), Z_0(t) \rangle dt \\ &= 0. \end{aligned}$$

This shows that $I_{a,b}$ has negative index. By the Sturm oscillation theorem, which is a generalization of the Morse index theorem, there is a nonzero solution Y of (3.2.7) such that $Y(a) = Y(c) = 0$ for some $c \in (a, b)$ (a “conjugate point”). By continuity of the index, if necessary, we may perturb a slightly so that $\gamma(a)$ and $\gamma(c)$ become regular points of the G -action and remain conjugate points.

Finally, we show that Y has the form $Y = J^h$ for some $J \in \Lambda$. There is an N_a -Jacobi field J along γ with initial conditions $J(a) = 0$ and $J'(a) = \frac{\nabla^h Y}{dt}(a)$. Set $\hat{Y} = J^h$. Then Y and \hat{Y} are both solutions of the same differential equation (3.2.6), with the same initial conditions at a . It follows that they coincide. This yields a Jacobi field J along γ , tangent to orbits at $t = a$ and $t = c$, which is not tangent to orbits everywhere, leading to a contradiction with the assumption that (G, M) is a variationally complete action.

We have shown that $A_t \equiv 0$ for all t , which says that \mathcal{H} is integrable over M_{reg} , and this implies that its integral manifolds can be extended to sections of (G, M) , due to Theorem 1.5.2. It only remains to check that sections are flat. Let Σ be a section of (G, M) . Since Σ is totally geodesic, it is nonnegatively curved. Suppose, to the contrary, that Σ has a tangent 2-plane σ with positive curvature at a point $p \in \Sigma$, which we can assume is a regular point of the G -action, by denseness. Let γ be a horizontal geodesic with $\gamma(0) = p$ and $\gamma'(0) \in \sigma$. Then we can find $v_0 \in \mathcal{H}_0$ such that $\langle \mathcal{R}(0)v_0, v_0 \rangle > 0$, and proceed as in (3.2.8) to reach a contradiction. Hence Σ is flat and this completes the proof of Theorem 3.2.3.

4 Appendix

4.1 An algebraic criterion for polar actions on symmetric spaces

The following criterion was proved in [Gor04].

4.1.1 Proposition (Gorodski) *Let $M = G/K$ be a symmetric space without Euclidean factor endowed with a Riemannian metric induced from some $\text{Ad}(G)$ -invariant inner product on the Lie algebra \mathfrak{g} of G . Consider a closed, connected subgroup $H \subset G$. By replacing H by a conjugate, if necessary, we may assume that $\bar{1} = 1K \in G/K$ is a regular point. Write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the Cartan decomposition, denote by \mathfrak{h} the Lie algebra of H , and define $\mathfrak{m} = \mathfrak{p} \cap \mathfrak{h}^\perp$. Then the action of H on M is polar if and only if the following two conditions hold:*

- (i) \mathfrak{m} is a Lie triple system (LTS), that is $[\mathfrak{m}, [\mathfrak{m}, \mathfrak{m}]] \subset \mathfrak{m}$; and
- (ii) $[\mathfrak{m}, \mathfrak{m}] \perp \mathfrak{h}$.

Proof. Let $\pi : G \rightarrow G/K$ be the canonical projection, and for $a \in G$ write $\bar{a} = \pi(a) = aK \in G/K$. We have that $H \times K$ acts on G by left and right translations, H acts on G/K by left translations, and π is an equivariant

Riemannian submersion. Therefore the tangent space to the orbit $H \cdot \bar{a}$ at the point \bar{a} is

$$T_{\bar{a}}(H \cdot \bar{a}) = d\pi_a(\mathfrak{h} \cdot a + a \cdot \mathfrak{k}),$$

so that

$$a^{-1} \cdot T_{\bar{a}}(H \cdot \bar{a}) = \pi_*(\text{Ad}_{a^{-1}}\mathfrak{h})$$

where $\pi_* : \mathfrak{g} \rightarrow \mathfrak{p}$ is the projection. Taking orthogonal complements we get that

$$a^{-1} \cdot \nu_{\bar{a}}(H \cdot \bar{a}) = \mathfrak{p} \cap \text{Ad}_{a^{-1}}\mathfrak{h}^\perp$$

where $\nu_{\bar{a}}(H \cdot \bar{a})$ denotes the normal space to $H \cdot \bar{a}$ at \bar{a} . In particular we have

$$\nu_{\bar{1}}(H \cdot \bar{1}) = \mathfrak{m}.$$

Since $\bar{1}$ is a regular point, the action of H on M is polar if and only if $\Sigma = \text{Exp}_{\bar{1}}[\mathfrak{m}]$ is a section, where Exp denotes the Riemannian exponential map of M .

If Σ is a section, then it is totally geodesic in M . Now the fact that $\mathfrak{m} = T_{\bar{1}}\Sigma$ is a LTS is a standard fact from the theory of symmetric spaces, see [Hel78], p. 224. This proves condition (i) and shows that $\mathfrak{s} = [\mathfrak{m}, \mathfrak{m}] + \mathfrak{m}$ is a subalgebra of \mathfrak{g} . Let S denote the corresponding connected subgroup of G . Now $\pi(S) = \Sigma$ and the elements of S induce isometries of Σ . Let $a \in S$. Since Σ intersects the orbits of H orthogonally, we have that $T_{\bar{a}}\Sigma \subset \nu_{\bar{a}}(H \cdot \bar{a})$ and therefore

$$\mathfrak{m} = T_{\bar{1}}\Sigma = a^{-1} \cdot T_{\bar{a}}\Sigma \subset a^{-1} \cdot \nu_{\bar{a}}(H \cdot \bar{a}) = \mathfrak{p} \cap \text{Ad}_{a^{-1}}\mathfrak{h}^\perp.$$

This proves that $\text{Ad}_a\mathfrak{m} \subset \mathfrak{h}^\perp$, so by taking $X, Y \in \mathfrak{m}$ arbitrary and $a = \exp tX$ we get that $\text{Ad}_{\exp tX}Y \in \mathfrak{h}^\perp$ and hence $[X, Y] = \frac{d}{dt}\big|_{t=0} \text{Ad}_{\exp tX}Y \in \mathfrak{h}^\perp$. This gives condition (ii) and proves half the proposition.

Conversely, if conditions (i) and (ii) hold, and \mathfrak{s}, S are as above, then $\mathfrak{s} \subset \mathfrak{h}^\perp$ so $\mathfrak{s} = \text{Ad}_{a^{-1}}\mathfrak{s} \subset \text{Ad}_{a^{-1}}\mathfrak{h}^\perp$ for $a \in S$ and then $\mathfrak{m} = \mathfrak{s} \cap \mathfrak{p} \subset \text{Ad}_{a^{-1}}\mathfrak{h}^\perp \cap \mathfrak{p}$. This implies that $T_{\bar{a}}\Sigma \subset \nu_{\bar{a}}(H \cdot \bar{a})$ for $\bar{a} \in \Sigma$ and hence Σ intersects the orbits of H orthogonally. The fact that Σ intersects *all* the orbits of H follows from Lemma 1.1.1, and this finishes the proof of the proposition. \square

4.1.2 Corollary *The action of H on M is hyperpolar if and only if \mathfrak{m} is an Abelian subalgebra of \mathfrak{p} .*

Proof. If \mathfrak{m} is an Abelian subalgebra of \mathfrak{p} then the criterion of the proposition is satisfied, and Σ is flat by the curvature formula for symmetric spaces. Conversely, if the action of H on M is hyperpolar, then Σ is flat so $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] = 0$. But

$$\langle [\mathfrak{m}, \mathfrak{m}], [\mathfrak{m}, \mathfrak{m}] \rangle = \langle [[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}], \mathfrak{m} \rangle = 0.$$

Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$ and the Killing form of \mathfrak{g} is negative definite on \mathfrak{k} , we deduce that \mathfrak{m} is Abelian, as wished. \square

4.1.3 Corollary (Hermann actions) *If H is a symmetric subgroup of G , then its action on M is hyperpolar.*

Proof. By replacing H by a conjugate, we may assume that $\bar{1}$ is a regular point. Write $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ for the involutive decomposition relative to the symmetric pair (G, H) . Then \mathfrak{m} as in the proposition is given by $\mathfrak{p} \cap \mathfrak{q}$, and we only need to see that it is Abelian.

The orbit through $\bar{1}$ is $HK/K = H/K \cap H$, and $\nu_{\bar{1}}(HK/K) = \mathfrak{m}$. Since $\bar{1}$ is a regular point, the slice representation $(K \cap H, \mathfrak{m})$ is trivial. In particular $[\mathfrak{k} \cap \mathfrak{h}, \mathfrak{m}] = 0$. Since $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k} \cap \mathfrak{h}$, we see that $[[\mathfrak{m}, \mathfrak{m}], \mathfrak{m}] = 0$, and hence $[\mathfrak{m}, \mathfrak{m}] = 0$ as in Corollary 4.1.2. \square

4.2 Cartan's and Hermann's theorems

Let M be a Riemannian manifold. In his book on Riemannian geometry, Cartan states a criterion for the local existence of a totally geodesic submanifold in M with a given tangent space at a given point.

4.2.1 Theorem (Cartan) *Let M be a Riemannian manifold, fix a point $p \in M$ and a subspace $S \subset T_p M$. Assume that there is a normal ball*

$$V = \exp_p(B(0_p, \epsilon))$$

such that for every unit speed radial geodesic $\gamma : [0, \ell] \rightarrow V$ emanating from p ($\ell < \epsilon$),

$$(4.2.2) \quad R(P_\gamma(u), P_\gamma(v))P_\gamma(w) \in P_\gamma(S),$$

for every $u, v, w \in S$, where P_γ denotes the parallel transport along γ , from 0 to ℓ . Then there exists a totally geodesic submanifold N of M such that $T_p N = S$.

Proof. Let $N = \exp_p(S \cap B(0_p, \epsilon))$. We will explain why N is totally geodesic. It suffices to see that parallel transport in M , along piecewise smooth curves in N , preserves the tangent spaces of N .

In the case of a radial geodesic $\gamma(t) = \exp_p(tv)$ with $v \in S$ and $\|v\| < \epsilon$, $t \in [0, 1]$, this follows from the Jacobi equation. In fact consider $q = \gamma(t_0)$ for some $t_0 \in (0, 1)$. Recall that the Jacobi field J along γ with $J(0) = 0$ and $J'(0) = u \in S$ is given by $J(t) = d(\exp_p)_{t_0 v}(t_0 u)$. Let $E_1 = \gamma'$, E_2, \dots, E_n be a parallel orthonormal frame along γ , where $E_1(0) \dots, E_k(0)$ are tangent to S and $E_{k+1}(0) \dots, E_n(0)$ are normal to S . Write $J = \sum_i a_i E_i$. Then $-a_i'' + \sum_j \langle R(E_1, E_j)E_1, E_i \rangle a_j = 0$ for all i , where $a_i(0) = 0$ for all i , and $a_i'(0) = 0$ for $i > k$. Owing to (4.2.2), $\langle R(E_1, E_j)E_1, E_i \rangle \equiv 0$ for $i > k$ and $j \leq k$, so we deduce that $a_i \equiv 0$ vanishes identically for $i > k$. Therefore J is everywhere tangent to the parallel translates of S along γ . Since $T_q N = d(\exp_p)_{t_0 v}(S)$, this proves that the tangent spaces of N along γ are parallel along γ .

In the case of an arbitrary piecewise smooth curve $\eta : [0, 1] \rightarrow V$, we join each point $\eta(s)$ to p by a radial geodesic, so as to obtain a parametrized surface $f(s, t)$, $(s, t) \in [0, 1] \times [0, 1]$, where $\gamma_s = f(s, \cdot)$ is a radial geodesic for each s ,

and $f(s, 0) = p$, $f(s, 1) = \eta(s)$ for all s . Consider the vector fields along f given by

$$\frac{\bar{\partial}}{\partial s} = f_* \frac{\partial}{\partial s} \quad \text{and} \quad \frac{\bar{\partial}}{\partial t} = f_* \frac{\partial}{\partial t}.$$

Then $\frac{\bar{\partial}}{\partial s}$ is a Jacobi field along each γ_s , and we already know from the argument in the previous paragraph that it is everywhere tangent to N and sits in the parallel translate of S along γ_s , whose value at $f(s, t)$ we denote by $S_{s,t}$; note that $T_{f(s,t)}N = S_{s,t}$. Further, it is clear that $\frac{\bar{\partial}}{\partial t}$ is everywhere tangent to N . Let $z \in T_q N = S_{0,1}$ be arbitrary, where $q = \eta(0) = f(0, 1)$. We parallel translate z along γ_0 from q to p to obtain $z_0 \in S$, and then we parallel translate z_0 along each γ_s from p to $\eta(s)$ to obtain a vector field Z along f . By the previous paragraph we know that $Z(s, t) \in S_{s,t}$. The main calculation now is

$$(4.2.3) \quad R\left(\frac{\bar{\partial}}{\partial t}, \frac{\bar{\partial}}{\partial s}\right)Z = \frac{\nabla}{dt} \frac{\nabla}{ds} Z,$$

since Z is parallel along each γ_s . Let $w \in T_p M$ be normal to S and extend it to a vector field W along f and parallel along each γ_s . Then $W(s, t) \perp S_{s,t}$ for all (s, t) . We take the inner product of (4.2.3) throughout with W . Since the left hand-side of (4.2.3) lies in $S_{s,t}$ by (4.2.2), we obtain

$$0 = \left\langle \frac{\nabla}{dt} \frac{\nabla}{ds} Z, W \right\rangle = \frac{d}{dt} \left\langle \frac{\nabla}{ds} Z, W \right\rangle.$$

Now $\langle \frac{\nabla}{ds} Z, W \rangle$ is constant in t . Since it vanishes at $t = 0$, it must also vanish at $t = 1$. This proves that $\{S_{s,1}\}_{s \in [0,1]}$ is a parallel family of subspaces along η , that is the tangent spaces to N are parallel along η . \square

We next introduce some terminology. Let M be a Riemannian manifold, fix a point $p \in M$ and a subspace $S \subset T_p M$. A once-broken geodesic $\gamma : [0, \ell] \rightarrow M$, emanating from p and broken at $t_0 \in (0, \ell)$, is called *S-admissible* if $\gamma'(0) \in S$, $\gamma'(t_0^+)$ lies in the parallel transport of S along γ from p to $\gamma(t_0)$, and $\gamma|_{[t_0, \ell]}$ sits in a convex neighborhood of $\gamma(t_0)$.

4.2.4 Theorem (Hermann) *Let M be a Riemannian manifold, fix a point $p \in M$ and a subspace $S \subset T_p M$. Assume that for every S-admissible once-broken geodesic $\gamma : [0, \ell] \rightarrow M$ emanating from p ,*

$$(4.2.5) \quad R(P_\gamma(u), P_\gamma(v))P_\gamma(w) \in P_\gamma(S),$$

for every $u, v, w \in S$, where P_γ denotes the parallel transport along γ , from 0 to ℓ . Then there exists a complete totally geodesic isometric immersion of Riemannian manifold N into M with $p \in N$ and $T_p N = S$.

Proof. By Cartan's local existence theorem 4.2.1, there exists a totally geodesic immersion of a Riemannian manifold N into a normal neighborhood of p such that $p \in N$ and $T_p N = S$. We take N maximal and assume, by

contradiction, that N is not complete. Then there is a geodesic $\gamma : [0, 1) \rightarrow N$ such that $\lim_{t \rightarrow 1^-} \gamma(t)$ does not exist in N . By completeness of M , γ can be continued past $t = 1$ to a complete geodesic $\tilde{\gamma} : [0, +\infty) \rightarrow M$. Let \tilde{S} be the parallel translate of S along $\tilde{\gamma}$ to $q = \tilde{\gamma}(1)$. Due to 4.2.5 and using again Theorem 4.2.1, there is totally geodesic submanifold \tilde{N} with $q \in \tilde{N}$ and $T_{\tilde{q}}\tilde{N} = \tilde{S}$. Of course the parallel translate of S along $\tilde{\gamma}$ from p to $\gamma(t_1)$ for $t_1 \in (0, 1)$ coincides with the parallel transport of \tilde{S} along $\tilde{\gamma}$ from q to $\gamma(t_1)$. This shows that the tangent spaces of N and \tilde{N} coincide at $\gamma(t)$ for $t = 1 - \delta$ with $\delta > 0$ sufficiently small, and hence implies that N and \tilde{N} coincide on a neighborhood of $\gamma(0, 1)$. Since $q \in \tilde{N}$, this contradicts the maximality of N . \square

4.3 Wilking's transversal Jacobi equation

Let (G, M) be a proper and isometric action, and let $\pi : M \rightarrow M/G$ be the projection. In this appendix we reproduce the derivation of Wilking's transversal Jacobi equation [Wil07] in this special case:

$$(4.3.1) \quad \left(\frac{\nabla^h}{dt} \right)^2 Y + (R(Y, \gamma')\gamma')^h - 3A_t^2 Y = 0,$$

along a horizontal geodesic γ for vector fields of the form $Y = J^h$, where J belongs to the space Λ of N_t -Jacobi fields along γ , $N_t = G\gamma(t)$.

We denote by \mathcal{G} the space of all Jacobi fields along γ . Consider the skew-symmetric bilinear form $\omega(J_1, J_2) = \langle J_1', J_2 \rangle - \langle J_1, J_2' \rangle$, for $J_1, J_2 \in \mathcal{G}$. Then ω is constant and defines a symplectic form on \mathcal{G} such that Λ is a Lagrangian subspace and Υ is an isotropic subspace. Indeed an N_t -Jacobi field J satisfies that $J(t)$ is vertical and $J'(t) + S_\xi J(t)$ is horizontal for this t , where $\xi = \gamma'(t)$ and S_ξ denotes the shape operator in the direction of a normal vector ξ to the orbit N_t , so

$$\begin{aligned} \omega(J_1, J_2) &= \omega(J_1(t), J_2(t)) \\ &= \langle J_1'(t), J_2(t)^v \rangle - \langle J_1(t)^v, J_2'(t) \rangle \\ &= -\langle S_\xi J_1(t), J_2(t) \rangle + \langle J_1(t), S_\xi J_2(t) \rangle \\ &= 0 \end{aligned}$$

for $J_1, J_2 \in \Lambda$; in addition, $\dim \Lambda = \frac{1}{2} \dim \mathcal{G}$, and $\Upsilon \subset \Lambda$.

By continuity it suffices to check equation (4.3.1) at a point t_0 such that N_{t_0} is a principal orbit. Since we can add a vertical Jacobi field to J without changing Y , we may assume that $J(t_0) \in \mathcal{H}_{t_0}$. Let E_1, \dots, E_n be a ∇^h -parallel orthonormal frame field of \mathcal{H} along γ with $E_1(t_0) = J(t_0)$. We first claim that

$$(4.3.2) \quad (J'(t_0))^v = -A_{t_0} J(t_0).$$

Indeed, for any vertical Jacobi field V along γ ,

$$\begin{aligned}\langle J'(t_0), V(t_0) \rangle &= \langle J(t_0), V'(t_0) \rangle \\ &= \langle J(t_0), (V'(t_0))^h \rangle \\ &= \langle J(t_0), A_{t_0} V(t_0) \rangle \\ &= -\langle A_{t_0} J(t_0), V(t_0) \rangle,\end{aligned}$$

where we have used that Λ is Lagrangian and A_{t_0} is skew-symmetric.

The second claim is that

$$(4.3.3) \quad E'_i(t) = AE_i(t)$$

for all t . In fact, for any vertical Jacobi field V along γ ,

$$\begin{aligned}\langle E'_i(t), V(t) \rangle &= -\langle E_i(t), V'(t) \rangle \\ &= -\langle E_i(t), (V'(t))^h \rangle \\ &= -\langle E_i(t), A_t V(t) \rangle \\ &= \langle A_t E_i(t), V(t) \rangle,\end{aligned}$$

proving the claim.

Now we can finish the proof of (4.3.1) as follows. First note that

$$\begin{aligned}\langle E_1, E''_k \rangle_{t_0} &= -\langle E'_1, E'_k \rangle_{t_0} + \left. \frac{d}{dt} \right|_{t=t_0} \langle E_1, E'_k \rangle \\ &= -\langle E'_1, E'_k \rangle_{t_0},\end{aligned}$$

since E_1 is horizontal and E'_k is vertical. Now

$$\begin{aligned}\left\langle \left(\frac{\nabla^h}{dt} \right)^2 J^h, E_k \right\rangle &= \left. \frac{d^2}{dt^2} \right|_{t=t_0} \langle J^h, E_k \rangle \\ &= \left. \frac{d^2}{dt^2} \right|_{t=t_0} \langle J, E_k \rangle \\ &= \langle J'', E_k \rangle_{t_0} + 2\langle J', E'_k \rangle_{t_0} + \langle J, E''_k \rangle_{t_0} \\ &= -\langle R(J, \gamma')\gamma', E_k \rangle_{t_0} + 2\langle J', E'_k \rangle_{t_0} + \langle E_1, E''_k \rangle_{t_0} \\ &= -\langle R(J^h, \gamma')\gamma', E_k \rangle_{t_0} + 2\langle J', E'_k \rangle_{t_0} - \langle E'_1, E'_k \rangle_{t_0} \\ &= -\langle R(J^h, \gamma')\gamma', E_k \rangle_{t_0} - 2\langle A_{t_0} E_1, A_{t_0} E_k \rangle - \langle A_{t_0} E_1, A_{t_0} E_k \rangle \\ &= -\langle R(J^h, \gamma')\gamma', E_k \rangle_{t_0} + 3\langle A_{t_0}^2 J^h, E_k \rangle,\end{aligned}$$

as wished.

5 References

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