Central Extensions of Restricted Affine Nilpotent Lie Algebras $n_+(A_1^{(1)})(p)$

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We dedicate this paper to Karl Hofmann on the occasion of his 90th birthday.

Abstract

Consider the maximal nilpotent subalgebra $n_+(A_1^{(1)})$ of the simplest affine algebra $A_1^{(1)}$ which is one of the $\mathbb{N}$-graded Lie algebras with minimal number of generators. We show truncated versions of this algebra in positive characteristic admit the structure of a family of restricted Lie algebras. We compute the ordinary and restricted 1- and 2-cohomology spaces with trivial coefficients by giving bases. With these we explicitly describe the restricted 1-dimensional central extensions.

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1 Introduction

$\mathbb{N}$-graded Lie algebras over a field $\mathbb{F}$ are a special important class of infinite dimensional Lie algebras. By $\mathbb{N}$-graded we mean that the Lie algebra $\mathfrak{g}$ is the direct sum of subspaces $\mathfrak{g}_i, i \in \mathbb{N}$, such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. Such algebras obviously must have at least two generators. A special class of these algebras are Lie algebras of maximal class, where $\dim \mathfrak{g}_1 = 2$, for $i \geq 2$, $\dim \mathfrak{g}_i = 1$ and $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$ for $i \geq 1$. These are also called filiform Lie algebras, and it is known (see [8]) that exactly three of them are $\mathbb{N}$-graded. If we denote the basis elements by $e_i, i \in \mathbb{N}$, then these three are the maximal nilpotent subalgebra L_1 of the Witt algebra with nonzero brackets $[e_i, e_j] = (j-i)e_{i+j}$, the algebra $\mathfrak{m}_0$ with nonzero brackets $[e_1, e_i] = e_{i+1}$ for $i \geq 2$ and the Lie algebra $\mathfrak{m}_2$ with nonzero brackets $[e_1, e_i] = e_{i+1}$ for all $i \geq 2$ and $[e_2, e_j] = e_{j+2}$ for all $j \geq 3$.

The filiform algebras $L_1, \mathfrak{m}_0$ and $\mathfrak{m}_2$ are well-studied, and some of their important invariants are computed, such as the 1- and 2-cohomology spaces with trivial and adjoint coefficients. Finite dimensional versions of these algebras are also studied in characteristic $p > 0$. Namely, in [3, 4, 5] we considered restricted algebras of these types, and computed the ordinary and restricted 1- and 2-cohomology spaces with trivial coefficients.

There are other types of $\mathbb{N}$-graded Lie algebras with the minimal number of generators e_1 and e_2. One of these is the maximal nilpotent part $n_+(A_1^{(1)})$ of the simplest affine Lie algebra $A_1^{(1)}$ (see [14]) with Cartan matrix

$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

which has a basis $\{e_i \mid i \in \mathbb{N}\}$ with brackets

$$[e_i, e_j] = \begin{cases} e_{i+j} & \text{if } j - i \equiv 1 \pmod 3; \\ 0 & \text{if } j - i \equiv 0 \pmod 3; \\ -e_{i+j} & \text{if } j - i \equiv -1 \pmod 3. \end{cases} \tag{1}$$

The other $\mathbb{N}$-graded nilpotent affine Lie algebra is the maximal nilpotent subalgebra of $n_+(BA_2)$ (see [14]), which is defined by the Cartan matrix

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Here there is a basis $\{e_i \mid i \in \mathbb{N}\}$ with brackets

$$[e_i, e_j] = b_{ij} e_{i+j}$$

where the coefficients b_{ij} depend on the residue obtained when dividing i and j by 8 according to the rule

$$b_{ij} + b_{i'j'} = 0 \text{ if } i + i' \text{ and } j + j' \text{ are divisible by } 8.$$

The Table below gives the numbers b_{ij} (the others are determined from $b_{ij} = -b_{ji}$ and $b_{ij} + b_{8-i,8-j} = 0$).

		i (mod 8)						
		1	2	3	4	5	6	7
j (mod 8)	0	1	-2	-1	0	1	2	-1
	1	0	1	-1	3	-2	0	1
	2	-1	0	0	0	1	-1	0
	3	1	0	0	3	1	1	-2

In addition to these five $\mathbb{N}$-graded algebras with two generators there is a family of Lie algebras with countably many parameters $\lambda_{4k} \in \mathbb{FP}^1$, see [8].

The cohomology with trivial coefficients of the affine subalgebras $n_+(A_1^{(1)})$ and $n_+(BA_1)$ are computed, but nothing is known about restricted cohomology, even though it gives an important invariant of those algebras.

As in the case of the three filiform algebras with two generators, it is interesting to consider whether there is a restricted Lie algebra structure on the truncated algebras $n_+(A_1^{(1)})(p)$ where $p > 0$ is a prime, and to describe their central extensions, as we did for the three filiform cases [3, 4, 5].

In this paper we concentrate on the algebras $n_+(A_1^{(1)})(p)$. The goal of the paper is to classify the restricted one-dimensional central extensions of these restricted algebras. To do this we compute the ordinary and restricted 2-cohomology spaces with trivial coefficients. For completeness, we also compute the the ordinary and restricted 1-cohomology spaces.

The structure of the paper is as follows. In Section 1 we define finite dimensional truncated versions of the algebra $n_+(A_1^{(1)})$ in positive characteristic $p > 0$ and show that they admit the structure of (a family of) restricted Lie algebras. Explicit formulas for the Lie brackets and $[p]$-operators are given. In Section 2 we compute the ordinary 1- and 2-cohomology spaces with trivial coefficients giving descriptions of the bases. Section 3 contains the computation of the restricted 1- and 2-cohomology spaces with trivial coefficients, again with descriptions of the bases. Finally, in Section 4 we explicitly describe the restricted one-dimensional central extensions corresponding to the non-trivial restricted 2-cocycles.

2 Restricted Lie Algebra Structures

Let p be a prime integer and $\mathbb{F}$ a field of characteristic p. For $i, j \in \mathbb{N}$, define

$$a_{i,j} = \begin{cases} -1 & \text{if } j-i \equiv -1 \pmod{3}; \\ 0 & \text{if } j-i \equiv 0 \pmod{3}; \\ 1 & \text{if } j-i \equiv 1 \pmod{3}. \end{cases} \tag{2}$$

Throughout the paper, we will let $[m]_3$ denote the congruence class of an integer m modulo 3 so $a_{i,j} = [j-i]_3$. Let $\mathfrak{g} = n_+(A_1^{(1)}) = \bigoplus_{i=1}^{\infty} \mathbb{F}e_i$ with bracket $[e_i, e_j] = a_{i,j}e_{i+j}$, $\imath = (e_{p+1})$ the ideal generated by e_{p+1} and

$$\mathfrak{g}(p) = n_+(A_1^{(1)})(p) = n_+(A_1^{(1)})/\imath.$$

Then $\mathfrak{g}(p)$ is a finite dimensional, $\mathbb{N}$-graded Lie algebra with basis $\{e_1, \ldots, e_p\}$ and k-th graded component $\mathfrak{g}_k(p) = \mathbb{F}e_k$, $1 \le k \le p$ and $\mathfrak{g}_k(p) = 0$, $k > p$. If $\alpha_i, \beta_i \in \mathbb{F}$, $g = \sum_{i=1}^{p} \alpha_i e_i$ and $h = \sum_{i=1}^{p} \beta_i e_i$, then

$$[g, h] = \sum_{i=1}^{(p-1)/2} \sum_{j=i+1}^{p-i} a_{i,j}(\alpha_i\beta_j - \beta_i\alpha_j)e_{i+j}. \tag{3}$$

For $g_1, \ldots, g_n \in \mathfrak{g}(p)$, denote the n-fold bracket by

$$[[\cdots[[g_1, g_2], g_3]\cdots], g_n] := [g_1, g_2, \ldots, g_n].$$

Note that $[g_1, g_2, \ldots, g_p] = 0$ for all $g_1, \ldots, g_p \in \mathfrak{g}(p)$ so for all $g, h \in \mathfrak{g}(p)$, we have

$$(\operatorname{ad} g)^p(h) = [h, \underbrace{g, \ldots, g}_{p}] = 0.$$

If, for each $1 \le k \le p$, we choose an element $e_k^{[p]}$ in the center of $\mathfrak{g}(p)$, we have

$$\operatorname{ad} e_k^{[p]} = 0 = (\operatorname{ad} e_k)^p,$$

so the operator $-^{[p]} : \mathfrak{g}(p) \to \mathfrak{g}(p)$ gives $\mathfrak{g}(p)$ the structure of a restricted Lie algebra [11, 15]. Since p-fold brackets are 0 in $\mathfrak{g}(p)$, we have $(g+h)^{[p]} = g^{[p]} + h^{[p]}$ for all $g, h \in \mathfrak{g}(p)$. The center $Z(\mathfrak{g}(p))$ is given by

$$Z(\mathfrak{g}(p)) = \begin{cases} \mathbb{F}e_p & \text{if } p \not\equiv 2 \pmod{3}; \\ \mathbb{F}e_{p-1} \oplus \mathbb{F}e_p & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

If $p \not\equiv 2 \pmod 3$, then choosing $e_k^{[p]} \in Z(\mathfrak{g}(p))$ is equivalent to choosing $\lambda_k \in \mathbb{F}$ and setting $e_k^{[p]} = \lambda_k e_p$ hence $\lambda \in \mathbb{F}^p$ determines a restricted Lie algebra structure on $\mathfrak{g}(p)$ which we denote by $\mathfrak{g}^\lambda(p)$. Conversely, if $-^{[p]} : \mathfrak{g}(p) \to \mathfrak{g}(p)$ is any restricted Lie algebra structure on $\mathfrak{g}(p)$, then $\operatorname{ad} e_k^{[p]} = (\operatorname{ad} e_k)^p = 0$ for all $1 \le k \le p$ so that $e_k^{[p]} = \lambda_k e_p$ for some $\lambda_k \in \mathbb{F}$. For any such restricted Lie algebra structure, if $g = \sum_{i=1}^p \alpha_i e_i$, then

$$g^{[p]} = \left(\sum_{i=1}^p \alpha_i^p \lambda_i \right) e_p. \tag{4}$$

Likewise, if $p \equiv 2 \pmod 3$, then the restricted Lie algebra structures on $\mathfrak{g}(p)$ correspond to pairs of elements $\mu, \lambda \in \mathbb{F}^p$ where $e_k^{[p]} = \mu_k e_{p-1} + \lambda_k e_p$. Denote this restricted Lie algebra by $\mathfrak{g}^{\mu,\lambda}(p)$. If $g = \sum_{i=1}^p \alpha_i e_i$, then

$$g^{[p]} = \left(\sum_{i=1}^p \alpha_i^p \mu_i \right) e_{p-1} + \left(\sum_{i=1}^p \alpha_i^p \lambda_i \right) e_p. \tag{5}$$

If $p = 2$, then $n_+(A_1^{(1)})(2) = \mathfrak{m}_0(2)$, and if $p = 3$, $n_+(A_1^{(1)})(3) = \mathfrak{m}_0(3) = \mathfrak{m}_2(3) = L_1(3)$ and hence the possible restricted structures are the same. These filiform algebras were already studied in [3, 4] so everywhere below we assume $p \ge 5$.

3 Ordinary 1- and 2-Cohomology

Our primary interest is in classifying (restricted) one-dimensional central extensions, so we describe only the (graded) cochain spaces $C^q = C^q(\mathfrak{g}(p)) = C^q(\mathfrak{g}(p), \mathbb{F})$ for $q = 0, 1, 2, 3$ and differentials $d^q : C^q(\mathfrak{g}(p)) \to C^{q+1}(\mathfrak{g}(p))$ for $q = 0, 1, 2$ (for more details on this cochain complex we refer the reader to [1, 9]). Set $C^0 = \mathbb{F}$ and $C^q = (\wedge^q \mathfrak{g}(p))^*$ for $q = 1, 2, 3$. We will use the following bases, ordered lexicographically, throughout the paper.

$$\begin{aligned} C^0 &: \{1\} \\ C^1 &: \{e^k \mid 1 \le k \le p\} \\ C^2 &: \{e^{i,j} \mid 1 \le i < j \le p\} \\ C^3 &: \{e^{u,v,w} \mid 1 \le u < v < w \le p\} \end{aligned}$$

Here e^k, $e^{i,j}$ and $e^{u,v,w}$ denote the dual vectors of the basis vectors $e_k \in \mathfrak{g}(p)$, $e_{i,j} = e_i \wedge e_j \in \wedge^2\mathfrak{g}(p)$ and $e_{u,v,w} = e_u \wedge e_v \wedge e_w \in \wedge^3\mathfrak{g}(p)$, respectively. The differentials $d^q : C^q \to C^{q+1}$ are defined for $\psi \in C^1$, $\varphi \in C^2$ and $g, h, f \in \mathfrak{g}$ by

$$
\begin{aligned}
&d^0 : C^0 \to C^1,\ d^0 = 0\\
&d^1 : C^1 \to C^2,\ d^1(\psi)(g \wedge h) = \psi([g,h])\\
&d^2 : C^2 \to C^3,\ d^2(\varphi)(g \wedge h \wedge f) = \varphi([g,h] \wedge f) - \varphi([g,f] \wedge h) + \varphi([h,f] \wedge g).
\end{aligned}
$$

The cochain spaces $C^q(\mathfrak{g}(p))$ are graded, and

$$
\begin{aligned}
&C^1_k(\mathfrak{g}(p)) = \operatorname{span}\{e^k \mid 1 \le k \le p\}\\
&C^2_k(\mathfrak{g}(p)) = \operatorname{span}\{e^{i,j} \mid 1 \le i < j \le p, i+j = k, 3 \le k \le 2p-1\}\\
&C^3_k(\mathfrak{g}(p)) = \operatorname{span}\{e^{u,v,w} \mid 1 \le u < v < w \le p, u+v+w = k, 6 \le k \le 3p-3\}
\end{aligned}
$$

We have $\dim(C^1_k) = 1$ for $1 \le k \le p$, $\dim(C^2_k) = s(k)$ for $3 \le k \le p+1$ and $\dim(C^2_k) = \dim(C^2_{2p+2-k})$ for $p+2 \le k \le 2p-1$ where the map $s : \mathbb{N} \to \mathbb{N}$ is defined by

$$
s(k) = \begin{cases} \frac{k}{2} - 1 & \text{if } k \text{ is even;}\\ \frac{k-1}{2} & \text{if } k \text{ is odd.} \end{cases}
$$

The differentials $d^q_k : C^q_k \to C^{q+1}_k$ are graded maps so $H^q = \bigoplus_k H^q_k$ for $q = 1, 2$. We note that $d^1_1 = d^1_2 = 0$ and $d^2_3 = d^2_4 = d^2_5 = 0$. If $3 \le k \le p$, direct computation shows that

$$
d^1_k(e^k) = \sum_{i=1}^{s(k)} a_{i,k-i} e^{i,k-i}. \tag{6}
$$

The expression (6) together with $d^1_1 = d^1_2 = 0$ immediately gives the following

Theorem 3.1. *A basis for* $\ker d^1$ *is* $\{e^1, e^2\}$*, hence* $\dim H^1(\mathfrak{g}(p)) = 2$*, and the classes of* $\{e^1, e^2\}$ *form a basis.*

The situation for H^2_k is more complicated. If $6 \le k \le 2p-1$, $e^{i,j} \in C^2_k(\mathfrak{g}(p))$ (so $i + j = k$), then the formula for the differential d^2 and the restricted bracket (3) gives

$$d_k^2(e^{i,j}) = \sum_{n=1}^{s(i)} a_{n,i-n} e^{n,i-n,k-i} - \sum_{n=k-2i+1}^{s(k-i)} a_{n,k-i-n} e^{n,k-i-n,i} \\ + \sum_{n=1}^{i-1} a_{n,k-i-n} e^{n,i,k-i-n} - \sum_{n=i+1}^{p-2} a_{n,k-i-n} e^{i,n,k-i-n}. \tag{7}$$

We will use the following notation to describe index ranges in our computations:

$$M(p,k) = \begin{cases} 1 & \text{if } 3 \le k \le p+1; \\ k-p & \text{if } p+2 \le k \le 2p-1, \end{cases}$$

$$G(p,k) = \begin{cases} 1 & \text{if } 6 \le k \le 2p; \\ 1+k-2p & \text{if } 2p+1 \le k \le 3p-3, \end{cases}$$

$$F(p,k) = \begin{cases} \frac{k}{3} - 1 & \text{if } k \equiv 0 \pmod 3; \\ \frac{k-1}{3} - 1 & \text{if } k \equiv 1 \pmod 3; \\ \frac{k-2}{3} - 1 & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

If $e^{u,v,k-v-u} \in C_k^3$, then $G(p,k) \le u \le F(p,k)$ and $u+1 \le v \le s(k-u)$. If

$$d_k^2 \left(\sum_{\substack{1 \le i < j \le p \\ i+j=k}} \alpha_{i,j} e^{ij} \right) = \sum_{\substack{1 \le i < j \le p \\ i+j=k}} \alpha_{i,j} \sum_{\substack{1 \le u < v < w \le p \\ u+v+w=k}} A_{uvw}^{ij} e^{u,v,w} = 0,$$

then the term $e^{u,v,k-v-u}$ gives the "*basic equation*"

$$\boxed{\underbrace{-\alpha_{u,k-u} a_{v,k-v-u}}_{k-u \le p} + \underbrace{\alpha_{v,k-v} a_{u,k-v-u}}_{k-v \le p} + \underbrace{\underbrace{\alpha_{u+v,k-v-u} a_{u,v}}_{u+v<k-v-u \le p} - \underbrace{\alpha_{k-v-u,u+v} a_{u,v}}_{k-v-u<u+v \le p}}_{\text{never both non-zero}} = 0.} \tag{8}$$

The kernel of d_k^2 is the space of solutions to the system (8) where $G(p,k) \le u \le F(p,k)$ and $u+1 \le v \le s(k-u)$. For $3 \le k \le 2p-1$, define the cocycles

$$\varphi_k = \sum_{i=M(p,k)}^{s(k)} a_{i,k-i} e^{i,k-i}. \tag{9}$$

The next Theorem is the main result of this section.

Theorem 3.2. *A basis for* $\ker d^2$ *is*

$$\{e^{1,4}, e^{2,5}, \varphi_3, \varphi_4, \ldots, \varphi_{p+1}\}$$

so $\dim\ker d^2 = p+1$. *Moreover,* $\varphi_k = d^1(e^k)$ *for* $3 \leq k \leq p$, *hence* $\dim H^2(\mathfrak{g}(p)) = 3$ *and the classes of* $\{e^{1,4}, e^{2,5}, \varphi_{p+1}\}$ *form a basis.*

We have already remarked that the cochain spaces and differentials are graded, so Theorem 3.2 follows from

Theorem 3.3. *If* $k \geq p+2$, *then* $\ker d_k^2 = 0$. *If* $k < p+2$, *then* $\ker d_k^2 = \{\varphi_k\}$ *unless* $k=5$ *or* $k=7$. *For all* $p \geq 5$, $\ker d_5^2 = \{e^{1,4}, \varphi_5\}$, *and* $\ker d_7^2 = \{e^{2,5}\}$ *when* $p=5$ *and* $\ker d_7^2 = \{e^{2,5}, \varphi_7\}$ *when* $p>5$. *Moreover* $\varphi_k = d^1(e^k)$ *for* $3 \leq k \leq p$, *hence* $H_k^2(\mathfrak{g}(p)) = 0$ *unless* $k = 5, 7, p+1$. *Bases for* $H_5^2(\mathfrak{g}(p))$, $H_7^2(\mathfrak{g}(p))$ *and* $H_{p+1}^2(\mathfrak{g}(p))$ *consist of the classes of* $\{e^{1,4}\}$, $\{e^{2,5}\}$ *and* $\{\varphi_{p+1}\}$, *respectively.*

Proof. We have already noted that $d_3^2 = d_4^2 = d_5^2 = 0$, and hence $\{e^{1,2}\}$, $\{e^{1,3}\}$ and $\{e^{1,4}, e^{1,5}\}$ form bases for $\ker d_k^2$ for $k = 3,4,5$, respectively. A direct computation using formula (7) for d^2 gives a basis for $\ker d_6^2$ which is $\{\varphi_6\}$, a basis for $\ker d_7^2$ is $\{e^{2,5}\}$ when $p=5$, and a basis for $\ker d_7^2$ is $\{e^{2,5}, \varphi_7\}$ when $p>5$.

For the remainder of the proof we assume $p \geq 5$ and $k \geq 8$. The proof is combinatorial and consists of three parts:

(i) $k > p+2$;

(ii) $k = p+2$;

(iii) $k < p+2$.

In each part, we will treat the cases $k = 6t+r$ where $0 < t$ and $0 \leq r < 6$ separately, and only use the *basic equations* (8) with $u=1$ or $u=2$. We have included examples for $p=23$ in the Appendix to illustrate the computations.

(i) $k > p+2 \Longrightarrow \ker d_k^2 = 0$.

If $u=1$ and $v=i-1$, then (8) reduces to

$$\begin{aligned} &\alpha_{k-p,p}[r-p+1]_3 = 0 && (i = k-p) \\ &\alpha_{i-1,k-i+1}[r-i-1]_3 + \alpha_{i,k-i}[i+1]_3 = 0 && (k-p+1 \leq i \leq s(k)), \end{aligned} \tag{10}$$

and if $u = 2$ and $v = i-2$, then (8) reduces to

$$\begin{aligned} &\alpha_{k-p,p}[r-p-1]_3 = 0 && (i = k-p) \\ &\alpha_{k-p+1,p-1}[r-p]_3 = 0 && (i = k-p+1) \\ &\alpha_{i-2,k-i+2}[r-i+1]_3 + \alpha_{i,k-i}[i-1]_3 = 0 && (k-p+2 \le i \le s(k)). \end{aligned} \tag{11}$$

Suppose $[r]_3 = 1$. If $[i]_3 = 0$, then (10) shows $\alpha_{i,k-i} = 0$. If $[i]_3 = 1$, then $[i-1]_3 = 0$ so that (10) again shows $\alpha_{i,k-i} = 0$. If $[i]_3 = -1$, then (11) shows $\alpha_{i,k-i} = 0$.

Suppose that $[r]_3 = -1$. If $[i]_3 = 1$, then (10) shows $\alpha_{i,k-i} = 0$. If $[i]_3 = 0$, then (10) shows $\alpha_{i-1,k-i+1} + \alpha_{i,k-i} = 0$ and (11) shows $\alpha_{i,k-i} = 0$, hence $\alpha_{i-1,k-i+1} = 0$ as well. (If $r = 5$, then $[s(k)]_3 = -1 \neq 0$, but $[s(k)-2]_3 = 0$ so again (11) implies $\alpha_{s(k),k-s(k)} = 0$.)

Finally suppose $[r]_3 = 0$. If $[i]_3 = 0$, then (10) applied to i and $i+1$ gives

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{12}$$

Moreover, using (11) and (12), we see that if $k-p+2 \le i$ and $\alpha_{i-2,k-i+2} = 0$, then $\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1} = 0$. If $[p]_3 = -1$, then (10) shows $\alpha_{k-p,p} = 0$ and (11) shows $\alpha_{k-p+1,p-1} = 0$, and hence $\alpha_{i,k-i} = 0$ for all i by the previous remark. If $[p]_3 = 1$, then (11) shows $\alpha_{k-p,p} = 0$ so again, $\alpha_{i,k-i} = 0$ for all i. This completes the proof of part (i).

(ii) $k = p+2 \Longrightarrow \ker d_k^2 = 0$

Since $p \ge 5$ is a prime, $p+2 = k = 6t+r$ implies $r = 1$ or $r = 3$. Moreover, $r = 1$ if and only if $[p]_3 = -1$ and $r = 3$ if and only if $[p]_3 = 1$. Suppose that $r = 1$. If $u = 1$ and $v = i-1$, then the *basic equation* (8) still reduces to (10). If $u = 2$ and $v = i-2$, then (8) reduces to

$$\begin{gathered} -\alpha_{2,p}[i]_3 + \alpha_{i-2,k-i+2}[r-i+1]_3 + \alpha_{i,k-i}[i-1]_3 = 0 \\ (k-p+3 \le i \le s(k)) \end{gathered} \tag{13}$$

In particular, taking $i = 5$ and $i = 7$, we have, respectively,

$$\alpha_{2,p} + \alpha_{5,p-3} = 0$$

and

$$-\alpha_{2,p} + \alpha_{5,p-3} = 0.$$

It follows that $\alpha_{2,p} = 0$ and hence (13) reduces to (11), and the argument for $[r]_3 = 1$ in part (i) shows that $\alpha_{i,k-i} = 0$ for all i.

Suppose $r = 3$. If $u = 1$ and $v = i - 1$, then the *basic equation* (8) again reduces to (10), and applying (10) to i and $i + 1$ for $[i]_3 = 0$, we have

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{14}$$

Let $A_m = \alpha_{3m,k-3m}$ for $1 \le m \le t$. Now, we use (13) for all $[i]_3 = 1$ to get a system of equations

$$\begin{aligned} -A_1 - A_1 + A_2 &= 0 \\ -A_1 - A_m + A_{m+1} &= 0 \quad (2 \le m \le t-1) \\ -A_1 - A_t - A_t &= 0. \end{aligned} \tag{15}$$

These equations imply $A_t = -tA_1$ and $(2t - 1)A_1 = 0$ so that $A_1 = A_t = 0$. From this it follows that $A_m = 0$ for all m and hence (14) gives $\alpha_{i,k-i} = 0$ for all i. This completes the proof of part (ii).

(iii) $k < p + 2 \Longrightarrow \ker d_k^2 = \operatorname{span}_{\mathbb{F}}\{\varphi_k\}$

Recall $\dim C_k^2(\mathfrak{g}(p)) = s(k)$ for $k < p+2$. A direct calculation shows that if $k < p + 2$, then the element φ_k defined in (9) is a solution of the *basic equation* (8) so that $\varphi_k \in \ker(d_k^2)$. We complete the proof by showing that the set

$$\{d_k^2(e^{i,k-i}) \mid 1 \le i \le s(k) - 1\} \tag{16}$$

is linearly independent.

If $u = 1$ and $v = i$ satisfies $2 \le i \le s(k) - 1$, then (8) reduces to

$$-\alpha_{1,k-1}[r + i - 1]_3 + \alpha_{i,k-i}[r - i + 1]_3 + \alpha_{i+1,k-i-1}[i - 1]_3 = 0. \tag{17}$$

If $u = 2$ and $v = i$ satisfies $3 \le i \le s(k) - 2$, then (8) reduces to

$$-\alpha_{2,k-2}[r + i + 1]_3 + \alpha_{i,k-i}[r - i - 1]_3 + \alpha_{i+2,k-i-2}[i - 2]_3 = 0. \tag{18}$$

We begin by showing that if $[r]_3 = 1$ or 2, then $\alpha_{1,k-1} = \alpha_{2,k-2} = 0$ and hence (17) and (18) reduce to (10) and (11), respectively. The proof in (i) then shows $\alpha_{i,k-i} = 0$ for $2 \le i \le s(k)-1$ and the set (16) is linearly independent.

The following Table lists selected values of v for each value of r and the resulting simplified equations from (8). Since we are considering only linear combinations of the elements of the set (16), these equations show $\alpha_{1,k-1} = \alpha_{2,k-2} = 0$ by inspection.

$u = 1$		
$r = 1$	$v = s(k) - 1$	$\alpha_{1,k-1} + \alpha_{s(k),k-s(k)} = 0$
$r = 2, 4$	$v = s(k)$	$-\alpha_{1,k-1} + \alpha_{s(k),k-s(k)}[r/2 - 1]_3 = 0$
$r = 5$	$v = s(k) - 2$	$-\alpha_{1,k-1} - \alpha_{s(k)-1,k-s(k)+1} = 0$
	$v = s(k) - 1$	$\alpha_{1,k-1} - \alpha_{s(k)-1,k-s(k)+1} = 0$
$u = 2$		
$r = 1$	$v = s(k) - 3$	$\alpha_{2,k-2} + \alpha_{s(k)-1,k-s(k)+1} = 0$
	$v = s(k) - 1$	$-\alpha_{2,k-2} + \alpha_{s(k)-1,k-s(k)+1} = 0$
$r = 2$	$v = s(k) - 2$	$-\alpha_{2,k-2} + \alpha_{s(k),k-s(k)} = 0$
$r = 4$	$v = s(k) - 1$	$-\alpha_{2,k-2} = 0$
$r = 5$	$v = s(k) - 1$	$-\alpha_{2,k-2} + \alpha_{s(k),k-s(k)} = 0$

If $r = 0$, then selecting $u = 1$ and $v = s(k)$ shows $\alpha_{1,k-1} = 0$ as above. Using this, (17) reduces to (10), and applying (10) to i and $i+1$ for $[i]_3 = 0$, we again have

$$\alpha_{i-1,k-i+1} = \alpha_{i,k-i} = \alpha_{i+1,k-i-1}. \tag{19}$$

Let $A_m = \alpha_{3m,k-3m}$ for $1 \le m \le t-1$. Letting $u = 2$ and $v = i$ ($3 \le i \le s(k) - 2$) in the *basic equation* (8), we have

$$-\alpha_{2,k-2}[i+1]_3 - \alpha_{i,k-i}[i+1]_3 + \alpha_{i+2,k-i-2}[i+1]_3 = 0. \tag{20}$$

Using (20) for all $[i]_3 = 0$, we get a system of equations

$$\begin{aligned} -A_1 - A_m + A_{m+1} &= 0 \quad (1 \le m \le t-2) \\ -A_1 - A_{t-1} &= 0. \end{aligned}$$

These equations imply $tA_1 = 0$ and hence $A_m = 0$ for all m. This together with (19) implies $\alpha_{i,k-i} = 0$ for all $1 \le i \le s(k) - 1$.

Finally, we assume $r = 3$. Letting $u = 1$ and $u = 2$, respectively, with $v = i = s(k) - 1$ in (8), we have

$$\begin{aligned} \alpha_{1,k-1} + \alpha_{s(k)-1,k-s(k)+1} - \alpha_{s(k),k-s(k)} &= 0 \\ -\alpha_{2,k-2} - \alpha_{s(k)-1,k-s(k)+1} - \alpha_{s(k),k-s(k)} &= 0 \end{aligned}$$

We are considering linear combinations of elements of (16) hence these equations show that

$$\alpha_{1,k-1} = -\alpha_{s(k)-1,k-s(k)+1} = \alpha_{2,k-2}. \tag{21}$$

Now, for $u = 1$ and $u = 2$ respectively, we have

$$\begin{aligned} \alpha_{i+1,k-i-1}[i-1]_3 &= \alpha_{1,k-1}[i-1]_3 + \alpha_{i,k-i}[i-1]_3 \ (2 \le i \le s(k)-1) \\ \alpha_{i+2,k-i-}[i+1]_3 &= \alpha_{2,k-2}[i+1]_3 + \alpha_{i,k-i}[i+1]_3 \ (3 \le i \le s(k)-2) \end{aligned} \tag{22}$$

Together, (21), (22) and an induction shows that

$$\alpha_{i,k-i} = c(i)\alpha_{1,k-1}, 2 \le i \le s(k) - 1.$$

Moreover, $c(i) \le i$ for all i, hence, in particular,

$$c(s(k)-1) \le s(k) - 1 = 3t < 6t + 1 \le p - 1.$$

This implies that

$$\alpha_{s(k)-1,k-s(k)+1} = -\alpha_{1,k-1} \text{ and } \alpha_{s(k)-1,k-s(k)+1} = c(s(k)-1)\alpha_{1,k-1}$$

with $c(s(k)-1) \ne -1$ in $\mathbb{F}$. This is a contradiction unless $\alpha_{1,k-1} = 0$, so (16) is linear independent and the proof is complete. □

Remark 1. *It is interesting to note that* $\dim H^2(n_+(A_1^{(1)})(p)) = 3$ *is independent of the prime p, just as for the filiform algebras* $\mathfrak{m}_2(p)$ *[3]. On the other hand,* $\dim H^2(\mathfrak{m}_0(p))$ *depends on p [4]. It would be interesting to see whether or not the dimension of the 2-cohomology for the other affine algebras* $n_+(BA_2)(p)$ *depends on p. Of course, in all cases, the dimension of the ordinary 1-cohomology spaces is 2 as all of these algebras are 2-generated.*

4 Restricted Cohomology

For general information on restricted cohomology of restricted Lie algebras, we refer the reader to [7, 12, 13]. Where no confusion can arise, for notational simplicity we denote by C_*^q both the spaces $C_*^q(\mathfrak{g}^\lambda(p)) = C_*^q(\mathfrak{g}^\lambda(p);\mathbb{F})$ and $C_*^q(\mathfrak{g}^{\mu,\lambda}(p)) = C_*^q(\mathfrak{g}^{\mu,\lambda}(p);\mathbb{F})$. We describe only the restricted cochain spaces C_*^q for $q = 0, 1, 2, 3$ and the restricted differentials $d_*^q : C_*^q \to C_*^{q+1}$ for $q = 0, 1, 2$, and refer the reader to [6] for a more detailed description of this partial cochain complex. The computations of the restricted cohomology spaces $H^q_*(\mathfrak{g}^\lambda(p);\mathbb{F})$ and $H^q_*(\mathfrak{g}^{\mu,\lambda}(p);\mathbb{F})$ for $q=1,2$ are carried out in separate subsections.

Given $\varphi \in C^2$, a map $\omega : \mathfrak{g}(p) \to \mathbb{F}$ is **φ-compatible** (in [6], the authors use the term ω *has the $*$-property with respect to* φ) if for all $g, h \in \mathfrak{g}(p)$ and all $\alpha \in \mathbb{F}$, $\omega(\alpha g) = \alpha^p \omega(g)$ and

$$\omega(g+h) = \omega(g) + \omega(h) + \sum_{\substack{g_i = g \text{ or } h \\ g_1 = g, g_2 = h}} \frac{1}{\#(g)} \varphi([g_1, g_2, g_3, \dots, g_{p-1}] \wedge g_p). \quad (23)$$

Note that $\omega : \mathfrak{g}(p) \to \mathbb{F}$ is 0-compatible if and only if ω is p-semilinear.

If $\zeta \in C^3$, then a map $\eta : \mathfrak{g}(p) \times \mathfrak{g}(p) \to \mathbb{F}$ is **ζ-compatible** (in [6], η *has the $**$-property with respect to* ζ) if for all $\alpha \in \mathbb{F}$ and all $g, h, h_1, h_2 \in \mathfrak{g}(p)$ we have $\eta(\cdot, h)$ linear in the first coordinate, $\eta(g, \alpha h) = \alpha^p \eta(g, h)$ and

$$\begin{aligned} \eta(g, h_1 + h_2) = \eta(g, h_1) + \eta(g, h_2) - \\ \sum_{\substack{l_1, \dots, l_p = 1 \text{or} 2 \\ l_1 = 1, l_2 = 2}} \frac{1}{\#\{l_i = 1\}} \zeta(g \wedge [h_{l_1}, \cdots, h_{l_{p-1}}] \wedge h_{l_p}). \end{aligned}$$

Define the restricted cochain spaces as $C^0_* = C^0$, $C^1_* = C^1$,

$$C^2_* = \{(\varphi, \omega) \mid \varphi \in C^2, \omega : \mathfrak{g} \to \mathbb{F} \text{ is } \varphi\text{-compatible}\}$$

$$C^3_* = \{(\zeta, \eta) \mid \zeta \in C^3, \eta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F} \text{ is } \zeta\text{-compatible}\}.$$

If $\varphi \in C^2$, we can assign values $\omega(e_k)$ arbitrarily to elements e_k of a basis $\{e_1, \dots, e_p\}$ for $\mathfrak{g}(p)$, set $\omega(\alpha e_k) = \alpha^p \omega(e_k)$ for all $\alpha \in \mathbb{F}$ and use (23) to determine a unique φ-compatible map $\omega : \mathfrak{g}(p) \to \mathbb{F}$ [4]. In particular, we can define $\tilde{\varphi}(e_k) = 0$ for all k and use (23) to determine a unique φ-compatible map $\widetilde{\varphi} : \mathfrak{g}(p) \to \mathbb{F}$. Note that, in general, $\widetilde{\varphi} \neq 0$ but $\widetilde{\varphi}(0) = 0$.

Lemma 4.1. *If $\varphi_1, \varphi_2 \in C^2$ and $\alpha \in \mathbb{F}$, then $\widetilde{(\alpha\varphi_1 + \varphi_2)} = \alpha\widetilde{\varphi}_1 + \widetilde{\varphi}_2$.*

Proof. An easy computation shows $(\alpha\widetilde{\varphi}_1 + \widetilde{\varphi}_2)(e_k) = 0$ for all k and $\alpha\widetilde{\varphi}_1 + \widetilde{\varphi}_2$ satisfies the compatibility condition (23) with $\varphi = \alpha\varphi_1 + \varphi_2$. The result now follows from uniqueness. ∎

For $1 \le k \le p$, define $\overline{e}^k : \mathfrak{g}(p) \to \mathbb{F}$ by

$$\overline{e}^k\left(\sum_{n=1}^{p} \alpha_n e_n\right) = \alpha_k^p.$$

It is shown in [6] that

$$\dim C_*^2 = \binom{p+2-1}{2} = \binom{p+1}{2} = \binom{p}{2} + p,$$

and hence it follows that

$$\{(e^{i,j}, \widetilde{e^{i,j}}) \mid 1 \le i < j \le p\} \cup \{(0, \overline{e}^k) \mid 1 \le k \le p\}$$

is a basis for C_*^2. We use this basis in all computations that follow.

Remark 2. *The $(p-1)$-fold bracket in (23) always gives a multiple of e_p. For $p \ge 5$, if $k < p+1$, φ_k is identically zero on $e_p \wedge \mathfrak{g}^\lambda(p)$ (or $e_p \wedge \mathfrak{g}^{\mu,\lambda}(p)$), and hence $\widetilde{\varphi_k} = 0$ because $\widetilde{\varphi_k}(e_i) = 0$ for all i. Likewise $\widetilde{e^{1,4}} = 0$ and $\widetilde{e^{2,5}} = 0$, unless $p = 5$. The restriction of φ_{p+1} to $e_p \wedge \mathfrak{g}^\lambda(p)$ (or $e_p \wedge \mathfrak{g}^{\mu,\lambda}(p)$) is equal to $e^{1,p}$, hence $\widetilde{\varphi_{p+1}} = \widetilde{e^{1,p}}$. We can then compute, using (23):*

$$\widetilde{e^{2,5}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) = \frac{1}{2}\alpha_1^3 \alpha_2^2 \quad (p \neq 5),$$

$$\widetilde{\varphi_{p+1}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) = \widetilde{e^{1,p}}\left(\sum_{i=1}^{p} \alpha_i e_i\right) = \alpha_1^{p-1}\alpha_2.$$

For $\psi \in C_*^1$, define the map $\operatorname{ind}^1(\psi) : \mathfrak{g}(p) \to \mathbb{F}$ by

$$\operatorname{ind}^1(\psi)(g) = \psi(g^{[p]}).$$

The map $\operatorname{ind}^1(\psi)$ is $d^1(\psi)$-compatible for all $\psi \in C_*^1$, and the differential $d_*^1 : C_*^1 \to C_*^2$ is defined by $d_*^1(\psi) = (d^1(\psi), \operatorname{ind}^1(\psi))$. For $(\varphi, \omega) \in C_*^2$, define the map $\operatorname{ind}^2(\varphi, \omega) : \mathfrak{g}(p) \times \mathfrak{g}(p) \to \mathbb{F}$ by the formula

$$\operatorname{ind}^2(\varphi, \omega)(g, h) = \varphi(g \wedge h^{[p]}).$$

The map $\operatorname{ind}^2(\varphi, \omega)$ is $d^2(\varphi)$-compatible for all $\varphi \in C^2$, and the differential $d_*^2 : C_*^2 \to C_*^3$ is defined by $d_*^2(\varphi, \omega) = (d^2(\varphi), \operatorname{ind}^2(\varphi, \omega))$ (see [6] for details). We note that (with trivial coefficients) if ω_1 and ω_2 are both φ-compatible, then $\operatorname{ind}^2(\varphi, \omega_1) = \operatorname{ind}^2(\varphi, \omega_2)$.

Lemma 4.2. *If* $(\varphi, \omega) \in C^2_*$ *and* $\varphi = d^1(\psi)$ *with* $\psi \in C^1$*, then* $(\varphi, \mathrm{ind}^1(\psi)) \in C^2_*$ *and* $\mathrm{ind}^2(\varphi, \omega) = \mathrm{ind}^2(\varphi, \mathrm{ind}^1(\psi))$.

Proof. We know that $\mathrm{ind}^1(\psi)$ is $d^1(\psi)$-compatible for all $\psi \in C^1 = C^1_*$ [6]. If $\varphi = d^1(\psi)$, then $(\varphi, \mathrm{ind}^1(\psi)) = (d^1(\psi), \mathrm{ind}^1(\psi)) \in C^2_*$, and ind^2 depends only on φ by the last sentence in the previous paragraph. □

We will use the following notation everywhere in the next two subsections: let $g = \sum \alpha_i e_i$, $h = \sum \beta_i e_i$, $\psi = \sum \gamma_i e^i$ and $\varphi = \sum \sigma_{i,j} e^{i,j}$.

4.1 Restricted Cohomology for $p \not\equiv 2 \pmod 3$

Assume $p \geq 5$ is a prime and $p \not\equiv 2 \pmod 3$. Using the $[p]$-operator (4) and the definitions of the maps ind^1 and ind^2, we have

$$\mathrm{ind}^1(\psi)(g) = \gamma_p \left(\sum_{i=1}^{p} \alpha_i^p \lambda_i \right) \tag{24}$$

and

$$\mathrm{ind}^2(\varphi, \omega)(g, h) = \left(\sum_{i=1}^{p} \beta_i^p \lambda_i \right) \left(\sum_{i=1}^{p-1} \sigma_{i,p} \alpha_i \right). \tag{25}$$

Theorem 4.3. $H^1_*(\mathfrak{g}^\lambda(p)) = H^1(\mathfrak{g}(p))$*, and the classes of the cocycles* $\{e^1, e^2\}$ *form a basis.*

Proof. Recall that H^1_* is the subspace of H^1 consisting of those cocycles $\psi \in C^1$ for which $\mathrm{ind}^1(\psi) = 0$ (see [10] or [6]). By Theorem 3.1, $\psi \in C^1$ is a cocycle if and only if $\gamma_k = 0$ for $3 \leq k \leq p$. Since $p \geq 5$, then (24) shows that $\mathrm{ind}^1(\psi) = 0$ for all cocycles ψ. □

Theorem 4.4. *The set*

$$\{(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}}), (\varphi_3, \widetilde{\varphi_3}), \ldots, (\varphi_{p+1}, \widetilde{\varphi_{p+1}}), (0, \overline{e}^1), \ldots, (0, \overline{e}^p)\} \tag{26}$$

forms a basis for the kernel of d^2_**. Moreover,* $\dim H^2_*(\mathfrak{g}^\lambda(p)) = p + 3$ *and the classes of*

$$\{(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}}), (\varphi_{p+1}, \widetilde{\varphi_{p+1}}), (0, \overline{e}^1), \ldots, (0, \overline{e}^p)\}$$

form a basis.

Proof. If $\lambda = 0$, then (25) shows that $\mathrm{ind}^2(\varphi, \omega) = 0$ for all $(\varphi, \omega) \in C^2_*$. Therefore, if $\varphi \in C^2$ is an ordinary cocycle, $d^2_*(\varphi, \omega) = (0, 0)$ for any φ-compatible map ω. Moreover, if $d^2_*(\varphi, \omega) = (0, 0)$, then φ must be an ordinary cocycle. This, together with our result on the ordinary cohomology (Theorem 3.2) shows that (26) is a basis for the kernel of d^2_*. If $3 \le k \le p$, then $\varphi_k = d^1(e^k)$, and Lemma 4.2 shows that we can replace $\widetilde{\varphi_k}$ with $\mathrm{ind}^1(e^k)$, so $d^1_*(e^k) = (\varphi_k, \mathrm{ind}^1(e^k))$. This shows that $\dim H^2_*(\mathfrak{g}^\lambda(p)) = p + 3$ and the classes of

$$\{(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}}), (\varphi_{p+1}, \widetilde{\varphi_{p+1}}), (0, \overline{e}^1), \ldots, (0, \overline{e}^p)\}$$

form a basis.

If $\lambda \neq 0$, then $\lambda_n \neq 0$ for some $1 \le n \le p$. If $(\varphi, \omega) \in C^2_*$, then (25) shows

$$\mathrm{ind}^2(\varphi, \omega)(e_m, e_n) = \lambda_n \sigma_{m,p}$$

for all $1 \le m \le p - 1$. It follows that $d^2_*(\varphi, \omega) = (0, 0)$ if and only if $d^2(\varphi) = 0$ and $\sigma_{1,p} = \sigma_{2,p} = \cdots = \sigma_{p-1,p} = 0$. By Theorem 3.2, if $d^2(\varphi) = 0$, then $i + j \le p + 1$ for all i, j so $\sigma_{2,p} = \cdots = \sigma_{p-1,p} = 0$. Moreover, the coefficient of $e^{1,p}$ in φ_{p+1} is $\sigma_{1,p} = a_{1,p} = p - 1 \pmod 3 = 0 \pmod 3$. This shows that $d^2_*(\varphi, \omega) = (0, 0)$ if and only if $d^2(\varphi) = 0$ and hence (26) is a basis for $\ker d^2_*$ in this case as well. The proof now proceeds as in the case $\lambda = 0$. □

4.2 Restricted Cohomology for $p \equiv 2 \pmod 3$

Using formula (5) for the p-operator and the definitions of the maps ind^1 and ind^2, we have

$$\mathrm{ind}^1(\psi)(g) = \gamma_{p-1}\left(\sum_{i=1}^{p} \alpha_i^p \mu_i\right) + \gamma_p\left(\sum_{i=1}^{p} \alpha_i^p \lambda_i\right) \tag{27}$$

and

$$\begin{aligned}\mathrm{ind}^2(\varphi, \omega)(g, h) = &\left(\sum_{i=1}^{p} \beta_i^p \mu_i\right)\left(\sum_{i=1}^{p-2} \sigma_{i,p}\alpha_i - \sigma_{p-1,p}\alpha_p\right) \\ &+ \left(\sum_{i=1}^{p} \beta_i^p \lambda_i\right)\left(\sum_{i=1}^{p-1} \sigma_{i,p}\alpha_i\right).\end{aligned} \tag{28}$$

The congruence class of p modulo 3 has no affect on the 1-cohomology, hence the proof of Theorem 4.3 also shows

Theorem 4.5. *$H^1_*(\mathfrak{g}^{\mu,\lambda}(p)) = H^1(\mathfrak{g}(p))$, and the classes of the cocycles $\{e^1, e^2\}$ form a basis.*

Theorem 4.6. *If $\mu + \lambda = 0$, then $H^2_*(\mathfrak{g}^{\mu,\lambda}(p)) = H^2_*(\mathfrak{g}^{\lambda}(p))$.*

Proof. If $(\varphi, \omega) \in C^2_*$, then (28) shows for $1 \leq n \leq p$

$$\operatorname{ind}(\varphi,\omega)(e_m, e_n) = \begin{cases} (\mu_n + \lambda_n)\sigma_{m,p} & \text{if } 1 \leq m \leq p-2 \\ (-\mu_n + \lambda_n)\sigma_{m,p} & \text{if } m = p-1 \\ 0 & m = p \end{cases} \tag{29}$$

If $\mu + \lambda = 0$, it follows that $d^2_*(\varphi, \omega) = (0,0)$ if and only if $d^2(\varphi) = 0$ and $\sigma_{p-1,p} = 0$. But $d^2(\varphi) = 0$ implies $\sigma_{p-1,p} = 0$ by Theorem 3.2 so $d^2_*(\varphi,\omega) = (0,0)$ if and only if $d^2(\varphi) = 0$ and (26) is a basis for $\ker d^2_*$. The argument now procedes precisely as in the proof of Theorem 4.4. □

Theorem 4.7. *If $\mu + \lambda \neq 0$, then $\dim H^2_*(\mathfrak{g}^{\mu,\lambda}(p)) = p + 2$ and the classes of*

$$\{(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}}), (0, \overline{e}^1), \ldots, (0, \overline{e}^p)\}$$

form a basis.

Proof. If $\mu + \lambda \neq 0$, then looking at (29), we see $d^2_*(\varphi,\omega) = (0,0)$ implies $d^2(\varphi) = 0$ and $\sigma_{1,p} = \cdots = \sigma_{p-2,p} = 0$. Theorem 3.2 shows that if $d^2(\varphi) = 0$, then $\sigma_{2,p} = \cdots = \sigma_{p-1,p} = 0$. Moreover for φ_{p+1}, $\sigma_{1,p} = a_{1,p} = p - 1 \not\equiv 0 \pmod 3$, and hence $d^2_*(\varphi_{p+1}, \widetilde{\varphi_{p+1}}) \neq (0,0)$. For any other ordinary cocycle φ, we have $\sigma_{1,p} = 0$. Therefore the kernel of d^2_* is

$$\{(e^{1,4}, \widetilde{e^{1,4}}), (e^{2,5}, \widetilde{e^{2,5}}), (\varphi_3, \widetilde{\varphi_3}), \ldots, (\varphi_p, \widetilde{\varphi_p}), (0, \overline{e}^1), \ldots, (0, \overline{e}^p)\},$$

and we again procede as in the proof of Theorem 4.4. □

5 One-Dimensional Central Extensions

One-dimensional central extensions $E = \mathfrak{g} \oplus \mathbb{F}c$ of an ordinary Lie algebra $\mathfrak{g}$ are parameterized by the cohomology space $H^2(\mathfrak{g})$ [9, Ch. 1, Sec. 4.6], and restricted one-dimensional central extensions of a restricted Lie algebra $\mathfrak{g}$ with $c^{[p]} = 0$ are parameterized by the restricted cohomology space $H^2_*(\mathfrak{g})$

[10, Theorem 3.3]. If $(\varphi, \omega) \in C^2_*(\mathfrak{g})$ is a restricted 2-cocycle, then the corresponding restricted one-dimensional central extension $E = \mathfrak{g} \oplus \mathbb{F}c$ has Lie bracket and $[p]$-operation defined by

$$\begin{aligned} [g,h] &= [g,h]_{\mathfrak{g}} + \varphi(g \wedge h)c; \\ [g,c] &= 0; \\ g^{[p]} &= p^{[p]_{\mathfrak{g}}} + \omega(g)c; \\ c^{[p]} &= 0, \end{aligned} \tag{30}$$

where $[\cdot,\cdot]_{\mathfrak{g}}$ and $\cdot^{[p]_{\mathfrak{g}}}$ denote the Lie bracket and $[p]$-operation in $\mathfrak{g}$, respectively. We can use (30) to explicitly describe the restricted one-dimensional central extensions corresponding to the restricted cocycles in Theorems 4.4, 4.6 and 4.7. For the rest of this section, let $g = \sum \alpha_i e_i$ and $h = \sum \beta_i e_i$ denote two arbitrary elements of $\mathfrak{g}^\lambda(p)$ or $\mathfrak{g}^{\mu,\lambda}(p)$.

Just as in [2, 3, 4], if $E_k = g^\lambda(p) \oplus \mathbb{F}c$ denotes the one-dimensional restricted central extension of $\mathfrak{g}^\lambda(p)$ (or $\mathfrak{g}^{\mu,\lambda}(p)$) determined by the cohomology class of the restricted cocycle $(0, \overline{e}^k)$, then (30) gives the bracket and $[p]$-operation in E_k:

$$\begin{aligned} [g,h] &= [g,h]_{\mathfrak{g}^\lambda(p)}; \\ [g,c] &= 0; \\ g^{[p]} &= g^{[p]_{\mathfrak{g}^\lambda(p)}} + \alpha_k^p c; \\ c^{[p]} &= 0. \end{aligned}$$

For the restricted cocycles $(e^{1,4}, \widetilde{e^{1,4}})$, $(e^{2,5}, \widetilde{e^{2,5}})$ and $(\varphi_{p+1}, \widetilde{\varphi_{p+1}})$, we summarize the corresponding restricted one-dimensional central extensions E in the following Table (see Remark 2). Everywhere in the Table, we omit the brackets $[g,c] = 0$ and $[p]$-operation $c^{[p]} = 0$ for brevity.

Table 1: Restricted one-dimensional central extensions with $p \equiv 2 \pmod 3$ and $\mu + \lambda = 0$	
$p > 5$	
$(e^{1,4}, 0)$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + (\alpha_1\beta_4 - \alpha_4\beta_1)c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}}$
$(e^{2,5}, 0)$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + (\alpha_2\beta_5 - \alpha_5\beta_2)c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}}$
$(\varphi_{p+1}, \widetilde{\varphi_{p+1}})$	$[g,h] = [g,h]_{\mathfrak{g}^{\mu,\lambda}(p)} + \left(\sum_{i=1}^{s(p+1)} a_{i,p+1-i}(\alpha_i\beta_{p+1-i} - \alpha_{p+1-i}\beta_i)\right) c$ $g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}} + \alpha_1^{p-1}\alpha_2 c$

If $p = 5$, the extensions $(e^{1,4}, 0)$ and $(\varphi_{p+1}, \widetilde{\varphi_{p+1}})$ are as in Table 1, but the map $\widetilde{e^{2,5}} \neq 0$ and the $[p]$-operator for the extension $(e^{2,5}, \widetilde{e^{2,5}})$ is given by

$$g^{[p]} = g^{[p]_{\mathfrak{g}^{\mu,\lambda}(p)}} + \frac{1}{2}\alpha_1^3\alpha_2^2.$$

The bracket for $(e^{2,5}, \widetilde{e^{2,5}})$ is unchanged.

If $\mu + \lambda \neq 0$, then Theorem 4.7 states $(\varphi_{p+1}, \widetilde{\varphi_{p+1}})$ is not a cocycle. The other two restricted one-dimensional extensions in Table 1 are unchanged.

If $p \not\equiv 2 \pmod 3$, then $p > 5$ and Theorem 4.6 implies $H_*^2(\mathfrak{g}^{\mu,\lambda}(p)) = H_*^2(\mathfrak{g}^{\lambda}(p))$ so that the restricted one-dimensional central extensions are the same as those in Table 1 with $p > 5$ and $\mathfrak{g}^{\mu,\lambda}(p)$ replaced with $\mathfrak{g}^{\lambda}(p)$.

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A Appendix to Proof of Theorem 3.3

The examples below illustrate the computations in the proof of Theorem 3.3 for the prime $p = 23$ and values of $k = 6t + r$ for $r = 0, 1, 2, 3, 4, 5$. The examples are listed in the order they appear in parts (i), (ii) and (iii) of the proof. In part (ii), the example for $r = 3$ uses $p = 19$ because necessarily $p \equiv 1 \pmod 3$ in this case.

In each matrix, only the rows for $u = 1$ and $u = 2$ are shown. The circles indicate entries for which there is exactly one non-zero entry in the row which shows the corresponding coefficient $\alpha_{i,k-i} = 0$. The shading indicates terms from the simplified equations from (12).

(i) $([r]_3 = 1)$ $k = 28 = 6 \cdot 4 + 4$ **with** $r = 4$**.**

	$e^{5,23}$	$e^{6,22}$	$e^{7,21}$	$e^{8,20}$	$e^{9,19}$	$e^{10,18}$	$e^{11,17}$	$e^{12,16}$	$e^{13,15}$
$e^{1,4,23}$	0								
$e^{1,5,22}$	0	$\textcircled{1}$							
$e^{1,6,21}$		-1	-1						
$e^{1,7,20}$			1	0					
$e^{1,8,19}$				0	$\textcircled{1}$				
$e^{1,9,18}$					-1	-1			
$e^{1,10,17}$						1	0		
$e^{1,11,16}$							0	$\textcircled{1}$	
$e^{1,12,15}$								-1	-1
$e^{1,13,14}$									1
$e^{2,3,23}$	$\textcircled{1}$								
$e^{2,4,22}$		-1							
$e^{2,5,21}$	1		0						
$e^{2,6,20}$		0		$\textcircled{1}$					
$e^{2,7,19}$			-1		-1				
$e^{2,8,18}$				1		0			
$e^{2,9,17}$					0		$\textcircled{1}$		
$e^{2,10,16}$						-1		-1	
$e^{2,11,15}$							1		0
$e^{2,12,14}$								0	
$\vdots$									

(i) $([r]_3 = -1)$ $k = 29 = 6 \cdot 4 + 5$ **with** $r = 5$**.**

	$e^{6,23}$	$e^{7,22}$	$e^{8,21}$	$e^{9,20}$	$e^{10,19}$	$e^{11,18}$	$e^{12,17}$	$e^{13,16}$	$e^{14,15}$
$e^{1,5,23}$	1								
$e^{1,6,22}$	0	-1							
$e^{1,7,21}$		-1	0						
$e^{1,8,20}$			1	1					
$e^{1,9,19}$				0	-1				
$e^{1,10,18}$					-1	0			
$e^{1,11,17}$						1	1		
$e^{1,12,16}$							0	-1	
$e^{1,13,15}$								-1	0
$e^{2,4,23}$	-1								
$e^{2,5,22}$		0							
$e^{2,6,21}$	1		1						
$e^{2,7,20}$		0		-1					
$e^{2,8,19}$			-1		0				
$e^{2,9,18}$				1		1			
$e^{2,10,17}$					0		-1		
$e^{2,11,16}$						-1		0	
$e^{2,12,15}$							1		1
$e^{2,13,14}$								0	1
$\vdots$									

(i) ($[r]_3 = 0$ and $[p]_3 = -1$) $k = 30 = 6 \cdot 5 + 0$ with $r = 0$.

	$e^{7,23}$	$e^{8,22}$	$e^{9,21}$	$e^{10,20}$	$e^{11,19}$	$e^{12,18}$	$e^{13,17}$	$e^{14,16}$
$e^{1,6,23}$	-1							
$e^{1,7,22}$	0	0						
$e^{1,8,21}$		-1	1					
$e^{1,9,20}$			1	-1				
$e^{1,10,19}$				0	0			
$e^{1,11,18}$					-1	1		
$e^{1,12,17}$						1	-1	
$e^{1,13,16}$							0	0
$e^{1,14,15}$								-1
$e^{2,5,23}$	0							
$e^{2,6,22}$		1						
$e^{2,7,21}$	1		-1					
$e^{2,8,20}$		0		0				
$e^{2,9,19}$			-1		1			
$e^{2,10,18}$				1		-1		
$e^{2,11,17}$					0		0	
$e^{2,12,16}$						-1		1
$e^{2,13,15}$							1	
$\vdots$								

(ii) $(r = 1$ **so** $[p]_3 = -1)$ $k = 25 = 6 \cdot 4 + 1$**.**

	$e^{2,23}$	$e^{3,22}$	$e^{4,21}$	$e^{5,20}$	$e^{6,19}$	$e^{7,18}$	$e^{8,17}$	$e^{9,16}$	$e^{10,15}$	$e^{11,14}$	$e^{12,13}$
$e^{1,2,22}$	0	1									
$e^{1,3,21}$		−1	−1								
$e^{1,4,20}$			1	0							
$e^{1,5,19}$				0	1						
$e^{1,6,18}$					−1	−1					
$e^{1,7,17}$						1	0				
$e^{1,8,16}$							0	1			
$e^{1,9,15}$								−1	−1		
$e^{1,10,14}$									1	0	
$e^{1,11,13}$										0	1
$e^{2,3,20}$	1	0		1							
$e^{2,4,19}$	0		−1		−1						
$e^{2,5,18}$	−1			1		0					
$e^{2,6,17}$	1				0		1				
$e^{2,7,16}$	0					0		−1			
$e^{2,8,15}$	−1						1		0		
$e^{2,9,14}$	1							0		1	
$e^{2,10,13}$	0								−1		−1
$e^{2,11,12}$	−1									1	0
⋮											

(ii) $(r = 3$ **so** $[p]_3 = 1,\ p = 19),\ k = 21 = 6 \cdot 3 + 3$**.**

	$e^{2,19}$	$e^{3,18}$	$e^{4,17}$	$e^{5,16}$	$e^{6,15}$	$e^{7,14}$	$e^{8,13}$	$e^{9,12}$	$e^{10,11}$
$e^{1,2,18}$	−1	1							
$e^{1,3,17}$		1	−1						
$e^{1,4,16}$			0	0					
$e^{1,5,15}$				−1	1				
$e^{1,6,14}$					1	−1			
$e^{1,7,13}$						0	0		
$e^{1,8,12}$							−1	1	
$e^{1,9,11}$								1	−1
$e^{2,3,16}$	−1	−1		1					
$e^{2,4,15}$	1		1		−1				
$e^{2,5,14}$	0			0		0			
$e^{2,6,13}$	−1				−1		1		
$e^{2,7,12}$	1					1		−1	
$e^{2,8,11}$	0						0		0
$e^{2,9,10}$	−1							−1	−1
⋮									

(iii) $(r = 1)\ k = 19 = 6 \cdot 3 + 1$**.**

	$e^{1,18}$	$e^{2,17}$	$e^{3,16}$	$e^{4,15}$	$e^{5,14}$	$e^{6,13}$	$e^{7,12}$	$e^{8,11}$	$e^{9,10}$
$e^{1,2,16}$	1	0	1						
$e^{1,3,15}$	0		−1	−1					
$e^{1,4,14}$	−1			1	0				
$e^{1,5,13}$	1				0	1			
$e^{1,6,12}$	0					−1	−1		
$e^{1,7,11}$	−1						1	0	
$e^{1,8,10}$	①							0	1
$e^{2,3,14}$		1	0		1				
$e^{2,4,13}$		0		−1		−1			
$e^{2,5,12}$		−1			1		0		
$e^{2,6,11}$		1				0		1	
$e^{2,7,10}$		0					−1		−1
$e^{2,8,9}$		−1						1	0
⋮									

(iii) $(r = 2)$ $k = 20 = 6 \cdot 3 + 2$.

	$e^{1,19}$	$e^{2,18}$	$e^{3,17}$	$e^{4,16}$	$e^{5,15}$	$e^{6,14}$	$e^{7,13}$	$e^{8,12}$	$e^{9,11}$
$e^{1,2,17}$	0	1	1						
$e^{1,3,16}$	−1		0	−1					
$e^{1,4,15}$	1			−1	0				
$e^{1,5,14}$	0				1	1			
$e^{1,6,13}$	−1					0	−1		
$e^{1,7,12}$	1						−1	0	
$e^{1,8,11}$	0							1	1
$e^{1,9,10}$	$\textcircled{-1}$								0
$e^{2,3,15}$		0	1		1				
$e^{2,4,14}$		−1		0		−1			
$e^{2,5,13}$		1			−1		0		
$e^{2,6,12}$		0				1		1	
$e^{2,7,11}$		$\textcircled{-1}$					0		−1
$e^{2,8,10}$		1						−1	
⋮									

(iii) $(r = 4)$ $k = 22 = 6 \cdot 3 + 4$.

	$e^{1,21}$	$e^{2,20}$	$e^{3,19}$	$e^{4,18}$	$e^{5,17}$	$e^{6,16}$	$e^{7,15}$	$e^{8,14}$	$e^{9,13}$	$e^{10,12}$
$e^{1,2,19}$	1	0	1							
$e^{1,3,18}$	0		−1	−1						
$e^{1,4,17}$	−1			1	0					
$e^{1,5,16}$	1				0	1				
$e^{1,6,15}$	0					−1	−1			
$e^{1,7,14}$	−1						1	0		
$e^{1,8,13}$	1							0	1	
$e^{1,9,12}$	0								−1	−1
$e^{1,10,11}$	$\textcircled{-1}$									1
$e^{2,3,17}$		1	0		1					
$e^{2,4,16}$		0		−1		−1				
$e^{2,5,15}$		−1			1		0			
$e^{2,6,14}$		1				0		1		
$e^{2,7,13}$		0					−1		−1	
$e^{2,8,12}$		−1						1		0
$e^{2,9,11}$		$\textcircled{1}$							0	
⋮										

(iii) $(r = 5)$ $k = 23 = 6 \cdot 3 + 5$**.**

	$e^{1,22}$	$e^{2,21}$	$e^{3,20}$	$e^{4,19}$	$e^{5,18}$	$e^{6,17}$	$e^{7,16}$	$e^{8,15}$	$e^{9,14}$	$e^{10,13}$	$e^{11,12}$
$e^{1,2,20}$	0	1	1								
$e^{1,3,19}$	−1		0	−1							
$e^{1,4,18}$	1			−1	0						
$e^{1,5,17}$	0				1	1					
$e^{1,6,16}$	−1					0	1				
$e^{1,7,15}$	1						−1	0			
$e^{1,8,14}$	0							1	1		
$e^{1,9,13}$	−1								0	−1	
$e^{1,10,12}$	1									−1	0
$e^{2,3,18}$		0	1		1						
$e^{2,4,17}$		−1		0		1					
$e^{2,5,16}$		1			−1		0				
$e^{2,6,15}$		0				1		1			
$e^{2,7,14}$		−1					0		−1		
$e^{2,8,13}$		1						−1		0	
$e^{2,9,12}$		0							1		1
$e^{2,10,11}$		−1								0	1
⋮											

(iii) $(r = 0)$ $k = 24 = 6 \cdot 4 + 0$**.**

	$e^{1,23}$	$e^{2,22}$	$e^{3,21}$	$e^{4,20}$	$e^{5,19}$	$e^{6,18}$	$e^{7,17}$	$e^{8,16}$	$e^{9,15}$	$e^{10,14}$	$e^{11,13}$
$e^{1,2,21}$	-1	-1	1								
$e^{1,3,20}$	1		1	-1							
$e^{1,4,19}$	0			0	0						
$e^{1,5,18}$	-1				-1	1					
$e^{1,6,17}$	1					1	-1				
$e^{1,7,16}$	0						0	0			
$e^{1,8,15}$	-1							-1	1		
$e^{1,9,14}$	1								1	-1	
$e^{1,10,13}$	0									0	0
$e^{1,11,12}$	-1										-1
$e^{2,3,19}$		-1	-1		1						
$e^{2,4,18}$		1		1		-1					
$e^{2,5,17}$		0			0		0				
$e^{2,6,16}$		-1				-1		1			
$e^{2,7,15}$		1					1		-1		
$e^{2,8,14}$		0						0		0	
$e^{2,9,13}$		-1							-1		1
$e^{2,10,12}$		1								1	
$\vdots$											

(iii) $(r = 3)$ $k = 21 = 6 \cdot 3 + 3$**.**

Here the shaded rows indicate $\alpha_{1,k-1} = -\alpha_{s(k)-1,k-s(k)+1} = \alpha_{2,k-2}$ as in (21), and the single shaded entries indiate the terms $\alpha_{i+1,k-i-1}$ and $\alpha_{i+2,k-i-2}$ from (22).

	$e^{1,20}$	$e^{2,19}$	$e^{3,18}$	$e^{4,17}$	$e^{5,16}$	$e^{6,15}$	$e^{7,14}$	$e^{8,13}$	$e^{9,12}$	$e^{10,11}$
$e^{1,2,18}$	-1	-1	1							
$e^{1,3,17}$	1		1	-1						
$e^{1,4,16}$	0			0	0					
$e^{1,5,15}$	-1				-1	1				
$e^{1,6,14}$	1					1	-1			
$e^{1,7,13}$	0						0	0		
$e^{1,8,12}$	-1							-1	1	
$e^{1,9,11}$	1								1	-1
$e^{2,3,16}$		-1	-1		1					
$e^{2,4,15}$		1		1		-1				
$e^{2,5,14}$		0			0		0			
$e^{2,6,13}$		-1				-1		1		
$e^{2,7,12}$		1					1		-1	
$e^{2,8,11}$		0						0		0
$e^{2,9,10}$		-1							-1	-1
$\vdots$										