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REMARKS ON TOTALLY POSITIVE FLAG MANIFOLDS

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INTRODUCTION

0.1. We often identify an algebraic variety defined over \bf{R} with its set of \bf{R} -points. Let G be a split, connected, simply connected semisimple algebraic group of simply laced type defined over **R** with a fixed pinning $(B^+, B^-, x_i, y_i (i \in I))$ as in [L94]. Here B^+, B^- are opposed Borel subgroups of G defined over **R** with unipotent radicals U^+, U^- and $x_i : \mathbf{R} \to U^+, y_i : \mathbf{R} \to U^-$ are certain imbeddings; I is a finite set. Let $T = B^+ \cap B^-$. Let $G_{>0}, U_{>0}^+, U_{>0}^-, T_{>0}$ be the (open) sub-semigroups of G, U^+, U^-, T defined in [L94].

For any $(u, u') \in U_{>0}^- \times U_{>0}^-$ we have defined in [L94, 7.1] the open subset

$$
\mathcal{T}_{u,u'} = \{ t \in T_{>0}; tut^{-1}u'^{-1} \in U_{>0}^{-} \}
$$

of $T_{>0}$ and proved that it is nonempty. In this paper we will state a conjecture on the structure of $\mathcal{T}_{u,u'}$ (see 0.3), we prove some special cases of it and we derive some consequences of it.

0.2. We introduce some notation. For any $\lambda = (\lambda_i)_{i \in I} \in \mathbb{N}^I$ let V_λ be an irreducible rational representation of G (over C) whose highest weight is λ . We fix a highest weight vector e_{λ} of V_{λ} . Let β_{λ} be the canonical basis of V_{λ} that contains e_{λ} , see [L90]. Let e'_{λ} λ be the lowest weight vector in β_{λ} .

Let $j \in I$ and let $\lambda = \omega(j) \in \mathbb{N}^{I}$ be such that $\lambda_i = 1$ if $i = j$, $\lambda_i = 0$ if $i \neq j$ (a fundamental weight). If $u \in U^-$ we can write $ue_{\omega(j)}$ as an **R**-linear combination of vectors in $\beta_{\omega(j)}$; let $Z_j(u) \in \mathbf{R}$ be the coefficient of e'_ν $\mathcal{L}_{\omega(j)}$ in this linear combination. This defines a function $Z_j : U^- \to \mathbf{R}$ (in fact a morphism of real algebraic varieties). From [L90], [L94] it is known that $Z_j(U_{>0}^-)$ $_{>0}^{(-)}$) $\subset \mathbf{R}_{>0}$. Hence for $(u, u', t) \in U_{>0}^- \times U_{>0}^- \times T_{>0}$ such that $t \in \mathcal{T}_{u, u'}$,

$$
z(u,u',t)=(Z_j(tut^{-1}u'^{-1}))_{j\in I}\in {\bf R}^I_{>0}
$$

is defined.

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We define

 $\Theta: \{(u, u', t) \in U_{>0}^- \times U_{>0}^- \times T_{>0}; t \in \mathcal{T}_{u, u'}\} \to U_{>0}^- \times U_{>0}^- \times \mathbb{R}_{>0}^1$

by $(u, u', t) \mapsto (u, u', z(u, u', t)).$

We can now state:

Conjecture 0.3. Θ is a homeomorphism.

This will be verified in §1 in some examples.

0.4. Let β be the (real) flag manifold of G that is, the real algebraic variety whose points are the Borel subgroups of G defined over **R**. Following [L4,§8], let $\mathcal{B}_{>0}$ be the (open) subset

$$
\{u'B^+u'^{-1};u'\in U_{>0}^-\}=\{uB^-u^{-1};u\in U_{>0}^+\}
$$

of \mathcal{B} (the last equality is proved in [L94, 8.7]). According to [L94, 8.9],

(a) for any $g \in G_{>0}$ there is a unique $B \in G_{>0}$ such that $g \in B$;

moreover, the map $\zeta : G_{>0} \to \mathcal{B}_{>0}$ given by $g \mapsto B$ is continuous. According to [L21, 5.5(a)], the map ζ is surjective. Thus, the fibres of ζ (that is, the sets $B \cap G_{>0}$ for various $B \in \mathcal{B}_{>0}$ define a partition of $G_{>0}$ into non-empty closed subsets indexed by $\mathcal{B}_{>0}$.

We are interested in the study of the open set $B \cap G_{>0}$ of B (for any $B \in \mathcal{B}_{>0}$). The following result (conjectured in $[L21,\S5]$) shows (assuming 0.3) that this open set is homeomorphic to a product of copies of $\mathbf{R}_{>0}$.

Proposition 0.5. Assume that 0.3 holds for G. For any $B \in \mathcal{B}_{>0}$ there exists a canonical homeomorphism

$$
\sigma_B: B \cap G_{>0} \xrightarrow{\sim} U_{>0}^+ \times \mathbf{R}_{>0}^I.
$$

The proof is given in §2.

0.6. We now fix $J \subset I$. Let P_J^+ U_J^+ be the subgroup of G generated by B^+ and by $\{y_j(a); j \in J, a \in \mathbf{R}\}\$. Let \mathcal{P}^J be the set of subgroups of G that are G-conjugate to P_J^+ \mathcal{P}_J^+ . Following [L98] we define $\mathcal{P}_{>0}^J$ to be the set of subgroups $P \in \mathcal{P}_J^J$ such that $\gamma_P := \{B \in \mathcal{B}_{>0}; B \subset P\}$ is nonempty. The following result is a generalization of $0.4(a)$.

Proposition 0.7. Let $g \in G_{>0}$. There is a unique $P \in \mathcal{P}_{>0}^J$ such that $g \in P$.

The proof is given in 3.3.

0.8. From 0.7 we see that there is a well defined map $\zeta_J : G_{>0} \to \mathcal{P}_{>0}^J$ given by $g \mapsto P$ where $P \in \mathcal{P}_{>0}^J$ contains g. It generalizes the map $\zeta : G_{>0} \to \mathcal{B}_{>0}$ in 0.4. It is again continuous. It is also surjective (this follows from the surjectivity of $ζ$). Thus, the fibres of $ζ_J$ (that is, the sets $P ∩ G_{>0}$ for various $P ∈ P_{>0}^J$) define a partition of $G_{>0}$ into non-empty closed subsets indexed by $\mathcal{P}_{>0}^J$. Note that, if $P \in \mathcal{P}_{\geq 0}^J$, then $P \cap G_{>0} = \sqcup_{B \in \gamma_P} (B \cap G_{>0})$. (In other words, if $g \in P \cap G_{>0}$ and if $B \in \mathcal{B}_{\geq 0}$ is defined by $g \in B$ then $B \subset P$. Indeed, we have $B \subset P'$ for a unique $P' \in \mathcal{P}_{\geq 0}^J$ so that $g \in P', g \in P$ and then $P = P'$ by 0.7.) Using then 0.5 (under the assumption that 0.3 holds) we see that there is a well defined bijection

(a) $P \cap G_{>0} = \gamma_P \times U_{\geq 0}^+ \times \mathbf{R}_{>0}^I$ whose restriction to $B \cap G_{>0}$ (for any $B \in \gamma_P$) is given by $g \mapsto (B, \sigma_B(g))$. From the definitions we see that

(b) the bijection (a) is a homeomorphism.

In 3.4 we show that

(c) if $P \in \mathcal{P}_{\geq 0}^J$ then γ_P is homeomorphic to a product of copies of $\mathbf{R}_{>0}$. Combining (b) , (c) we obtain:

Proposition 0.9. Assume that 0.3 holds for G. For any $P \in \mathcal{P}_{>0}^J$, the intersection $P \cap G_{>0}$ is homeomorphic to a product of copies of $\mathbf{R}_{>0}$.

1. Examples

1.1. In this section we shall give some examples when 0.3 holds. We first assume that $G = SL_2(\mathbf{R})$ with the standard pinning. Let

$$
u = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in U_{>0}^-, \quad u' = \begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix} \in U_{>0}^-,
$$

so that $(a, a') \in \mathbb{R}^2_{\geq 0}$. Now $T_{\geq 0}$ is the set of all matrices

$$
\begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix}
$$

with $(r, s) \in \mathbf{R}_{>0}, rs = 1$; we can identify $T_{>0}$ with $\mathbf{R}_{>0}$ by $(r, s) \mapsto R = s/r$. Let t_R be the element of $T_{>0}$ corresponding to $R \in \mathbf{R}_{>0}$. If $t = t_R$, then

$$
tut^{-1}u'^{-1} = \begin{pmatrix} 1 & 0 \\ Ra - a' & 1 \end{pmatrix}
$$

so that

$$
\mathcal{T}_{u,u'} = \{ R \in \mathbf{R}; Ra - a' > 0 \}.
$$

Now $\tau : \mathcal{T}_{u,u'} \to \mathbf{R}_{>0}, R \mapsto Ra - a'$ is a homeomorphism $\mathcal{T}_{u,u'} \to \mathbf{R}_{>0}$. This shows that 0.3 holds in our case: the map $Z_i: U^- \to \mathbf{R}$ (for the unique $i \in I$) attaches to a matrix

$$
\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in U^-
$$

the number x.

4 G. LUSZTIG

1.2. In this subsection we assume that $G = SL_3(\mathbf{R})$ with the standard pinning. We shall prove 0.3 in this case. Let

$$
u = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \in U_{>0}^-, \quad u' = \begin{pmatrix} 1 & 0 & 0 \\ a' & 1 & 0 \\ c' & b' & 1 \end{pmatrix} \in U_{>0}^-,
$$

so that $(a, b, c, ab - c, a', b', c', a'b' - c') \in \mathbb{R}_{\geq 0}^8$. Now $T_{>0}$ is the set of all matrices

$$
\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & p \end{pmatrix}
$$

with $(r, s, p) \in \mathbf{R}_{>0},$ $rsp = 1$; we can identify $T_{>0}$ with $\mathbf{R}_{>0}^2$ by $(r, s, p) \mapsto (R, S) =$ $(s/r, p/s)$. Let $t_{R,S}$ be the element of $T_{>0}$ corresponding to $(R, S) \in \mathbb{R}^2_{>0}$. If $t = t_{R,S}$, then

$$
tut^{-1}u'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ Ra - a' & 1 & 0 \\ RSc - Sa'b + a'b' - c' & Sb - b' & 1 \end{pmatrix}
$$

so that

$$
\mathcal{T}_{u,u'} = \{ (R, S) \in \mathbf{R}^2; Ra - a' > 0, Sb - b' > 0, (Rc - a'b)S + a'b' - c' > 0, R(S(ab - c) - ab') + c' > 0 \}.
$$

(The last inequality is obtain by rewriting $(Ra-a')(Sb-b')-RSc+Sa'b-a'b'+c'$) 0.) Note that any (R, S) in the right hand side of (a) automatically satisfies $R > 0, S > 0$ since $Ra > a', Sb > b'$. We define $\tau : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$
\tau(R, S) = ((Rc - a'b)S + a'b' - c', R(S(ab - c) - ab') + c').
$$

Let $(A, B) \in \mathbb{R}_{\geq 0}^2$. We show that $\tau^{-1}(A, B)$ consists of two elements. Let $(R, S) \in$ $\tau^{-1}(A, B)$ that is

$$
RSc = a'bS - a'b' + c' + A, RS(ab - c) = ab'R - c' + B.
$$

We have

$$
(ab - c)(a'bS - a'b' + c' + A) = c(ab'R - c' + B)
$$

so that

(b)
$$
R = (ab - c)a'b(ab'c)^{-1}S + (-aa'bb' + a'b'c + abc' + (ab - c)A - cB)(ab'c)^{-1}
$$
.

Substituting this into $RSc = a'bS - a'b' + c' + A$ we obtain

$$
(ab - c)a'bS2 + (-2aa'bb' + a'b'c + abc' + (ab - c)A - cB)S + (a'b' - c' - A)ab' = 0.
$$

We set $\tilde{R} = R - a'/a$, $\tilde{S} = S - b'/b$. We obtain

$$
(ab - c)a'b(\tilde{S} + b'/b)^{2} + (-2aa'b' + a'b'c + abc' + (ab - c)A - cB)(\tilde{S} + b'/b) + (a'b' - c' - A)ab' = 0
$$

that is

(c)
$$
(ab - c)a'b\tilde{S}^2 + (abc' - a'b'c + (ab - c)A - cB)\tilde{S} - (A + B)cb'/b = 0;
$$

Since $\left(-\frac{(A+B)c^{b}}{\lambda}\right)/((ab-c)a^{\prime}b)^{-1} < 0$, the equation (c) for \tilde{S} has two roots, \tilde{S}_+ , \tilde{S}_- such that $\tilde{S}_+ \in \mathbb{R}_{>0}$, $\tilde{S}_- \in -\mathbb{R}_{>0}$. Thus we have

$$
\tilde{S} \in \{ \tilde{S}_+, \tilde{S}_- \}
$$

and

(d)
$$
\tilde{S}_+ + \tilde{S}_- = -\mu((ab - c)a'b)^{-1},
$$

where $\mu = abc' - a'b'c + (ab - c)A - cB$.

Note that

$$
\tilde{S}_{+} = (-\mu + \sqrt{\mu^2 + 4(ab - c)a'b'c(A + B)})/(2(ab - c)a'b),
$$

$$
\tilde{S}_{-} = (-\mu - \sqrt{\mu^2 + 4(ab - c)a'b'c(A + B)})/(2(ab - c)a'b),
$$

where $\sqrt{R}_{>0} \to \mathbf{R}_{>0}$ is the square root.

In the case where $\tilde{S} = \tilde{S}_+$ (resp. $\tilde{S} = \tilde{S}_-$) we set $\tilde{R} = \tilde{R}_+$ (resp. $\tilde{R} = \tilde{R}_-$). We can rewrite (b) as

$$
\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}\tilde{S}_{\pm}\mu(ab'c)^{-1}.
$$

Using (d) this becomes

$$
\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}\tilde{S}_{\pm} - ((ab - c)a'b)(ab'c)^{-1}(\tilde{S}_{+} + \tilde{S}_{-})
$$

that is

$$
\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}(\tilde{S}_{\pm} - \tilde{S}_{+} - \tilde{S}_{-}).
$$

Thus,

$$
\tilde{R}_{+} = -(ab - c)a'b(ab'c)^{-1}\tilde{S}_{-} \in \mathbf{R}_{>0}
$$

(since $\tilde{S}_- \in -\mathbf{R}_{>0}$,)

$$
\tilde{R}_{-} = -(ab - c)a'b(ab'c)^{-1}\tilde{S}_{+} \in -\mathbf{R}_{>0}
$$

(since $S_+ \in \mathbf{R}_{>0}$). We see that

$$
\tau^{-1}(A, B) \subset \{ (\tilde{R}_{+} + a'/a, \tilde{S}_{+} + b'/b), (\tilde{R}_{-} + a'/a, \tilde{S}_{-} + b'/b) \}
$$

The same proof shows also the reverse inclusion so that

$$
\tau^{-1}(A, B) = \{ (\tilde{R}_{+} + a'/a, \tilde{S}_{+} + b'/b), (\tilde{R}_{-} + a'/a, \tilde{S}_{-} + b'/b) \}
$$

Note that $(A, B) \mapsto (\tilde{R}_+ + a'/a, \tilde{S}_+ + b'/b)$ is a continuous map $\mathbb{R}^2_{\geq 0} \to \mathcal{T}_{u,u'}$ which is the inverse of the continuous map $\mathcal{T}_{u,u'} \to \mathbb{R}^2_{>0}$ defined by τ . This shows that 0.3 holds in our case: one of the two maps $Z_i: U^- \to \mathbf{R}$ $(i \in I)$ attaches to a matrix

$$
\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \in U^-
$$

the number z; the other attaches to the same matrix the number $xy - z$.

2. Proof of Proposition 0.5

2.1. In this section we prove Proposition 0.5. For $u' \in U_{\geq 0}^$ $z_{>0}^-, v \in U_{>0}^+$ we have $u'v \in G_{>0}$ hence we can define $(u'v)^+ \in U_{>0}^+$ t_1^+ , $t_1 \in T_{>0}$, $(u'v)^- \in U_{>0}^-$ by the equation $u'v = (u'v)^+ t_1(u'v)^-$. Note that the map $U_{>0}^- \times U_{>0}^+ \to U_{>0}^+ \times T_{>0}^- \times U_{>0}^ >0$, $(u', v) \mapsto ((u'v)^+, t_1, (u'v)^-)$ is continuous.

Lemma 2.2. Let $B \in \mathcal{B}_{>0}$. We write $B = u'B^+u'^{-1}$ with $u' \in U_{>0}^-$ uniquely determined. We have a homeomorphism

$$
\{(v,t)\in U_{>0}^+\times T_{>0}; t\in \mathcal{T}_{(u'v)^-,u'}\}\xrightarrow{\sim} B\cap G_{>0}
$$

given by $(v, t) \mapsto u'vtu'^{-1}$.

We have $B = \{u'vtu'^{-1}; v \in U^+, t \in T\}$. If such $u'vtu'^{-1}$ is in $G_{>0}$ then $u'vt \in G_{>0}u' \subset \tilde{G}_{>0}U_{>0}^- \subset G_{>0}$ (see [L94, 2.12]) so that $u'vt = u'_0$ $\int_0' v_0 t_0$ with $u'_0 \in U^-_{>0}$ $\zeta_0, v_0 \in U^+_{>0}$ $t_{>0}^{+}, t_0 \in T_{>0}$. It follows that $u' = u_0'$ $v_0, v = v_0, t = t_0$ so that $v \in U^+_{>0}$ $y_{>0}^{+}$, $t \in T_{>0}$. Next we have $u'v = (u'v)^+ t_1(u'v)^-$ as in 2.1) and

$$
u'vtu'^{-1} = (u'v)^{+}t_{1}(u'v)^{-}tu'^{-1} = (u'v)^{+}t_{1}t(t^{-1}(u'v)^{-}tu'^{-1}) \in U_{>0}^{+}T_{>0}U^{-}.
$$

Since $u'vtu'^{-1} \in G_{>0}$ we have also $u'vtu'^{-1} = u_2t_2u'_2$ with $u_2 \in U_{>0}^+$ $x_0^+, u_2' \in U_{>0}^ >0$ $t_2 \in T_{>0}$. It follows that $(u'v)^+ = u_2, t_1t = t_2, t^{-1}(u'v)^-tu'^{-1} = u'_2$ 2 . In particular, $t^{-1}(u'v)^{-}tu'^{-1} \in U_{>0}^{-}$ >0 .

Conversely, if $v \in U^+_{\geq 0}$ ⁺_{>0} and *t* ∈ *T*_{>0} are given such that $t^{-1}(u'v)$ ⁻ tu'^{-1} </sup> ∈ $U_{>0}^ >0$, then we have $u'vtu'^{-1} \in G_{>0}$. Indeed, as in 2.1 we have

$$
u'vtu'^{-1} = (u'v)^{+}t_{1}(u'v)^{-}tu'^{-1}
$$

= $(u'v)^{+}(t_{1}t)(t^{-1}(u'v)^{-}tu'^{-1}) \in U_{>0}^{+}T_{>0}U_{>0}^{-} = G_{>0}.$

The lemma follows.

2.3. We prove 0.5. Let $B \in \mathcal{B}_{>0}$. We write $B = u'B^+u'^{-1}$ with $u' \in U_{>0}^-$ uniquely determined. From 0.3 we obtain a homeomorphism

$$
\{(v, t) \in U_{>0}^+ \times T_{>0}; t \in \mathcal{T}_{(u'v)^-, u'}\} \to U_{>0}^+ \times \mathbf{R}_{>0}^1
$$

given by $(v, t) \mapsto (v, z((u'v)^{-}, u', t))$. Composing this with the inverse of the homeomorphism in Lemma 2.2, we obtain a homeomorphism of $B \cap G_{>0}$ with $U_{>0}^{+} \times \mathbf{R}_{>0}^{I}$ so that 0.5 holds.

3. Proof of Proposition 0.7 and of $0.8(c)$

3.1. Let W be the Weyl group of G; let $\{s_i; i \in I\}$ be the simple reflections in W. Let $w \mapsto |w|$ be the length function; we have $|s_i| = 1$ for $i \in I$. For any $J \subset I$ let W_J be the subgroup of W generated by $\{s_i; i \in J\}$ and let w_0^J be the unique element $w \in W_J$ with $|w|$ maximal. Let $w_0 = w_0^I$.

For $w \in W$ let $U^+(w)$ (resp. $U^-(w)$) be the subset of U^+ (resp. U^-) defined in [L94, 2.7] (resp. [L94, 2.9]).

In the remainder of this section we fix $J \subset I$. From the definitions we see that (a) $u \mapsto u P_J^+ u^{-1}$ is a bijection $U^-(w_0w_0^J) \xrightarrow{\sim} \mathcal{P}_{>0}^J$.

3.2. For any $\lambda = (\lambda_i)_{i \in I} \in \mathbb{N}^I$ let V_λ, β_λ be as in 0.2. Let $V_{\lambda,\mathbf{R}}$ be the **R**subspace of V_{λ} spanned by β_{λ} . Let **P** be the set of all lines in $V_{\lambda,\mathbf{R}}$. Note that G acts naturally on **P**. We shall assume that $\{i \in I; \lambda_i \neq 0\} = I - J$. Then for any $P \in \mathcal{P}^J$ there is a unique $L_P^{\lambda} \in \mathbf{P}$ such that the stabilizer of L_P^{λ} in G is equal to P; moreover,

(a) the map $P \mapsto L_P^{\lambda}$ from \mathcal{P}^{J} to **P** is injective.

A line in **P** is said to be in $P_{>0}$ if it is spanned by a linear combination of elements in β_{λ} with all coefficients being in $\mathbf{R}_{>0}$. We shall now assume in addition that λ is such that λ_i is sufficiently large for any $i \in I - J$ so that [L98, 3.4] is applicable; thus the following holds:

(b) For $P \in \mathcal{P}^J$ we have $P \in \mathcal{P}_{\geq 0}^J$ if and only if $L_P^{\lambda} \in \mathbf{P}_{>0}$. Now let $g \in G_{>0}$. From [L94, 5.2] we see that the following holds.

(c) There is a unique line $L_g \in \mathbf{P}_{>0}$ such that $gL_g = L_g$.

3.3. Let $g \in G_{>0}$. We prove Proposition 0.7.

By [L94, 8.9] there exists $B \in \mathcal{B}_{>0}$ such that $g \in B$. Let $P \in \mathcal{P}^J$ be such that $B \subset P$. We have $P \in \mathcal{P}_{\geq 0}^J, g \in P$. This proves the existence in 0.7. Assume now that $P' \in \mathcal{P}_{>0}^J, P'' \in \mathcal{P}_{>0}^J$ satisfy $g \in P', g \in P''$. In the setup of 3.2(b) we have $L_{P'}^{\lambda} \in \mathbf{P}_{>0}$, $L_{P''}^{\lambda} \in \mathbf{P}_{>0}$. Since $g \in P'$, the line $L_{P'}^{\lambda}$ is g-stable hence, with notation of 3.2(c), we have $L_{P'}^{\lambda} = L_g$. Similarly we have $L_{P''}^{\lambda} = L_g$. Thus we have $L_{P'}^{\lambda} = L_{P''}^{\lambda}$. Using 3.2(a) we deduce $P' = P''$. This proves the uniqueness in 0.7.

3.4. We now fix $P \in \mathcal{P}_{\geq 0}^J$. By 3.1(a) we have $P = uP_J^+u^{-1}$ for a well defined $u \in U^{-}(w_0 w_0^J).$

We show:

8 G. LUSZTIG

(a) $v \mapsto uvB^+v^{-1}u^{-1}$ is a bijection $U^-(w_0^J) \xrightarrow{\sim} \gamma_P$. If $v \in U^{-}(w_0^J)$ then $uv \in U^{-}(w_0) = U_{>0}^{-}$ hence $uvB^{+}v^{-1}u^{-1} \in \mathcal{B}_{>0}$; moreover we have $vB+v^{-1} \in P_J^+$ u^+ hence $uvB^+v^{-1}u^{-1} \subset uP_J^+u^{-1} = P$ so that $uvB^+v^{-1}u^{-1} \in$ γ_P . Thus (a) is a well defined map. Now let $B \in \gamma_P$. We have $B = u_1 B^+ u_1^{-1}$ 1 where $u_1 \in U^-(w_0)$ and $B \subset uP_J^+u^{-1}$ that is $u_1B^+u_1^{-1} \subset uP_J^+u^{-1}$. Now there is a unique $P' \in \mathcal{P}^J$ containing $u_1 B^+ u_1^{-1}$ $_1^{-1}$. Since $u_1 P_J^+$ $y_1^+ u_1^{-1}$ $_{1}^{-1}$ and $uP_J^+u^{-1}$ are such P', we must have $u_1 P_J^+$ $u_1^+ u_1^{-1} = u P_J^+ u^{-1}$. We have $u_1 = u_1' u_1''$ where $u_1' \in$ $U^-(w_0w_0^J)$, $u_1'' \in U^-(w_0^J)$; Hence $u_1P_J^+$ $u_1^{\dagger} u_1^{-1} = u_1' u_1'' P_J^+$ $y_1^+ u_1''$ $''^{-1}u_1'$ $y_1^{\prime -1} = u_1^{\prime} P_J^+$ $J^+u'_1$ $\frac{1}{1}$ so that $u'_1P_J^+$ $J^+u'_1$ $u_1^{1-1} = u P_J^+ u^{-1}$. Using this and 3.1(a) we deduce $u_1' = u$ so that $B = uu''_1 B^+ u''_1$ \int_{1}^{π} 1 u^{-1} . We see that the map (a) is surjective. The injectivity of the map (a) is immediate. This proves (a).

From the definitions we see that (a) is a homeomorphism. Hence $0.8(c)$ holds.

3.5. Let $P \in \mathcal{P}_{\geq 0}^J$ and let P_{red} be the reductive quotient of P. Define $u \in$ $U^-(w_0w_0^J)$ by $P = uP_J^+u^{-1}$ (see 3.1(a)). Now P_{red} has a natural pinning. (It is induced by the obvious pinning of the reductive quotient of P_J^+ U_J^+ using the isomorphism $P_J^+ \to P$ given by conjugation by u.) Hence we can define $\mathcal{B}_{P,>0}$ (an open subset of the real flag manifold of P_{red}) in terms of P_{red} with its pinning in the same way as $\mathcal{B}_{>0}$ was defined in terms of G with its pinning. (Although P_{red} is not necessarily semisimple, the same definitions can be applied.) We have a bijection (a) $\mathcal{B}_{P,>0} \xrightarrow{\sim} \gamma_P$

obtained by taking inverse image under the obvious map $P \rightarrow P_{red}$. This can be deduced from the proof in 3.4.

REFERENCES

- [L90] G.Lusztig, Canonical bases arising from quantized enveloping algebras, J.Amer.Math.Soc. 3 (1990), 447-498.
- [L94] G.Lusztig, Total positivity in reductive groups, Lie theory and geometry, Progr.in Math.123, Birkhäuser, Boston, 1994, pp. 531-568.
- [L98] G.Lusztig, Total positivity in partial flag manifolds, Represent.Th. 2 (1998), 70-78.
- [L21] G.Lusztig, Total positivity in Springer fibres, Quart.J.Math. **72** (2021), 31-49.

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