REMARKS ON TOTALLY POSITIVE FLAG MANIFOLDS

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INTRODUCTION

0.1. We often identify an algebraic variety defined over \mathbf{R} with its set of \mathbf{R} -points. Let G be a split, connected, simply connected semisimple algebraic group of simply laced type defined over \mathbf{R} with a fixed pinning $(B^+, B^-, x_i, y_i (i \in I))$ as in [L94]. Here B^+, B^- are opposed Borel subgroups of G defined over \mathbf{R} with unipotent radicals U^+, U^- and $x_i : \mathbf{R} \to U^+, y_i : \mathbf{R} \to U^-$ are certain imbeddings; I is a finite set. Let $T = B^+ \cap B^-$. Let $G_{>0}, U^+_{>0}, U^-_{>0}$, $T_{>0}$ be the (open) sub-semigroups of G, U^+, U^-, T defined in [L94].

For any $(u, u') \in U_{>0}^- \times U_{>0}^-$ we have defined in [L94, 7.1] the open subset

$$\mathcal{T}_{u,u'} = \{ t \in T_{>0}; tut^{-1}u'^{-1} \in U_{>0}^{-} \}$$

of $T_{>0}$ and proved that it is nonempty. In this paper we will state a conjecture on the structure of $\mathcal{T}_{u,u'}$ (see 0.3), we prove some special cases of it and we derive some consequences of it.

0.2. We introduce some notation. For any $\lambda = (\lambda_i)_{i \in I} \in \mathbf{N}^I$ let V_{λ} be an irreducible rational representation of G (over \mathbf{C}) whose highest weight is λ . We fix a highest weight vector e_{λ} of V_{λ} . Let β_{λ} be the canonical basis of V_{λ} that contains e_{λ} , see [L90]. Let e'_{λ} be the lowest weight vector in β_{λ} .

Let $j \in I$ and let $\lambda = \omega(j) \in \mathbf{N}^{I}$ be such that $\lambda_{i} = 1$ if i = j, $\lambda_{i} = 0$ if $i \neq j$ (a fundamental weight). If $u \in U^{-}$ we can write $ue_{\omega(j)}$ as an **R**-linear combination of vectors in $\beta_{\omega(j)}$; let $Z_{j}(u) \in \mathbf{R}$ be the coefficient of $e'_{\omega(j)}$ in this linear combination. This defines a function $Z_{j} : U^{-} \to \mathbf{R}$ (in fact a morphism of real algebraic varieties). From [L90], [L94] it is known that $Z_{j}(U_{>0}^{-}) \subset \mathbf{R}_{>0}$. Hence for $(u, u', t) \in U_{>0}^{-} \times U_{>0}^{-} \times T_{>0}$ such that $t \in \mathcal{T}_{u,u'}$,

$$z(u, u', t) = (Z_j(tut^{-1}u'^{-1}))_{j \in I} \in \mathbf{R}_{>0}^I$$

is defined.

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We define

 $\Theta: \{(u, u', t) \in U_{>0}^{-} \times U_{>0}^{-} \times T_{>0}; t \in \mathcal{T}_{u, u'}\} \to U_{>0}^{-} \times U_{>0}^{-} \times \mathbf{R}_{>0}^{I}$

by $(u, u', t) \mapsto (u, u', z(u, u', t))$.

We can now state:

Conjecture 0.3. Θ is a homeomorphism.

This will be verified in $\S1$ in some examples.

0.4. Let \mathcal{B} be the (real) flag manifold of G that is, the real algebraic variety whose points are the Borel subgroups of G defined over **R**. Following [L4,§8], let $\mathcal{B}_{>0}$ be the (open) subset

$$\{u'B^+u'^{-1}; u' \in U_{>0}^-\} = \{uB^-u^{-1}; u \in U_{>0}^+\}$$

of \mathcal{B} (the last equality is proved in [L94, 8.7]). According to [L94, 8.9],

(a) for any $g \in G_{>0}$ there is a unique $B \in G_{>0}$ such that $g \in B$;

moreover, the map $\zeta : G_{>0} \to \mathcal{B}_{>0}$ given by $g \mapsto B$ is continuous. According to [L21, 5.5(a)], the map ζ is surjective. Thus, the fibres of ζ (that is, the sets $B \cap G_{>0}$ for various $B \in \mathcal{B}_{>0}$) define a partition of $G_{>0}$ into non-empty closed subsets indexed by $\mathcal{B}_{>0}$.

We are interested in the study of the open set $B \cap G_{>0}$ of B (for any $B \in \mathcal{B}_{>0}$). The following result (conjectured in [L21,§5]) shows (assuming 0.3) that this open set is homeomorphic to a product of copies of $\mathbf{R}_{>0}$.

Proposition 0.5. Assume that 0.3 holds for G. For any $B \in \mathcal{B}_{>0}$ there exists a canonical homeomorphism

$$\sigma_B: B \cap G_{>0} \xrightarrow{\sim} U_{>0}^+ \times \mathbf{R}_{>0}^I.$$

The proof is given in $\S 2$.

0.6. We now fix $J \subset I$. Let P_J^+ be the subgroup of G generated by B^+ and by $\{y_j(a); j \in J, a \in \mathbf{R}\}$. Let \mathcal{P}^J be the set of subgroups of G that are G-conjugate to P_J^+ . Following [L98] we define $\mathcal{P}_{>0}^J$ to be the set of subgroups $P \in \mathcal{P}^J$ such that $\gamma_P := \{B \in \mathcal{B}_{>0}; B \subset P\}$ is nonempty. The following result is a generalization of 0.4(a).

Proposition 0.7. Let $g \in G_{>0}$. There is a unique $P \in \mathcal{P}_{>0}^J$ such that $g \in P$.

The proof is given in 3.3.

0.8. From 0.7 we see that there is a well defined map $\zeta_J : G_{>0} \to \mathcal{P}_{>0}^J$ given by $g \mapsto P$ where $P \in \mathcal{P}_{>0}^J$ contains g. It generalizes the map $\zeta : G_{>0} \to \mathcal{B}_{>0}$ in 0.4. It is again continuous. It is also surjective (this follows from the surjectivity of ζ). Thus, the fibres of ζ_J (that is, the sets $P \cap G_{>0}$ for various $P \in \mathcal{P}_{>0}^J$) define a partition of $G_{>0}$ into non-empty closed subsets indexed by $\mathcal{P}_{>0}^J$. Note that, if $P \in \mathcal{P}_{>0}^J$, then $P \cap G_{>0} = \sqcup_{B \in \gamma_P} (B \cap G_{>0})$. (In other words, if $g \in P \cap G_{>0}$ and if $B \in \mathcal{B}_{>0}$ is defined by $g \in B$ then $B \subset P$. Indeed, we have $B \subset P'$ for a unique $P' \in \mathcal{P}_{>0}^J$ so that $g \in P', g \in P$ and then P = P' by 0.7.) Using then 0.5 (under the assumption that 0.3 holds) we see that there is a well defined bijection

(a) $P \cap G_{>0} = \gamma_P \times U^+_{>0} \times \mathbf{R}^I_{>0}$ whose restriction to $B \cap G_{>0}$ (for any $B \in \gamma_P$) is given by $g \mapsto (B, \sigma_B(g))$. From the definitions we see that

(b) the bijection (a) is a homeomorphism.

In 3.4 we show that

(c) if $P \in \mathcal{P}_{>0}^J$ then γ_P is homeomorphic to a product of copies of $\mathbf{R}_{>0}$. Combining (b),(c) we obtain:

Proposition 0.9. Assume that 0.3 holds for G. For any $P \in \mathcal{P}_{>0}^J$, the intersection $P \cap G_{>0}$ is homeomorphic to a product of copies of $\mathbf{R}_{>0}$.

1. EXAMPLES

1.1. In this section we shall give some examples when 0.3 holds. We first assume that $G = SL_2(\mathbf{R})$ with the standard pinning. Let

$$u = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \in U_{>0}^{-}, \quad u' = \begin{pmatrix} 1 & 0 \\ a' & 1 \end{pmatrix} \in U_{>0}^{-},$$

so that $(a, a') \in \mathbf{R}^2_{>0}$. Now $T_{>0}$ is the set of all matrices

$$\left(\begin{array}{cc} r & 0\\ 0 & s \end{array}\right)$$

with $(r, s) \in \mathbf{R}_{>0}$, rs = 1; we can identify $T_{>0}$ with $\mathbf{R}_{>0}$ by $(r, s) \mapsto R = s/r$. Let t_R be the element of $T_{>0}$ corresponding to $R \in \mathbf{R}_{>0}$. If $t = t_R$, then

$$tut^{-1}u'^{-1} = \begin{pmatrix} 1 & 0\\ Ra - a' & 1 \end{pmatrix}$$

so that

$$\mathcal{T}_{u,u'} = \{ R \in \mathbf{R}; Ra - a' > 0 \}.$$

Now $\tau : \mathcal{T}_{u,u'} \to \mathbf{R}_{>0}, R \mapsto Ra - a'$ is a homeomorphism $\mathcal{T}_{u,u'} \to \mathbf{R}_{>0}$. This shows that 0.3 holds in our case: the map $Z_i : U^- \to \mathbf{R}$ (for the unique $i \in I$) attaches to a matrix

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in U^-$$

the number x.

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1.2. In this subsection we assume that $G = SL_3(\mathbf{R})$ with the standard pinning. We shall prove 0.3 in this case. Let

$$u = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \in U_{>0}^{-}, \quad u' = \begin{pmatrix} 1 & 0 & 0 \\ a' & 1 & 0 \\ c' & b' & 1 \end{pmatrix} \in U_{>0}^{-},$$

so that $(a, b, c, ab - c, a', b', c', a'b' - c') \in \mathbb{R}^8_{>0}$. Now $T_{>0}$ is the set of all matrices

$$\begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & p \end{pmatrix}$$

with $(r, s, p) \in \mathbf{R}_{>0}$, rsp = 1; we can identify $T_{>0}$ with $\mathbf{R}_{>0}^2$ by $(r, s, p) \mapsto (R, S) = (s/r, p/s)$. Let $t_{R,S}$ be the element of $T_{>0}$ corresponding to $(R, S) \in \mathbf{R}_{>0}^2$. If $t = t_{R,S}$, then

$$tut^{-1}u'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ Ra - a' & 1 & 0 \\ RSc - Sa'b + a'b' - c' & Sb - b' & 1 \end{pmatrix}$$

so that

(a)
$$\mathcal{T}_{u,u'} = \{ (R,S) \in \mathbf{R}^2; Ra - a' > 0, Sb - b' > 0, \\ (Rc - a'b)S + a'b' - c' > 0, R(S(ab - c) - ab') + c' > 0 \}.$$

(The last inequality is obtain by rewriting (Ra-a')(Sb-b')-RSc+Sa'b-a'b'+c' > 0.) Note that any (R, S) in the right hand side of (a) automatically satisfies R > 0, S > 0 since Ra > a', Sb > b'. We define $\tau : \mathbf{R}^2 \to \mathbf{R}^2$ by

$$\tau(R,S) = ((Rc - a'b)S + a'b' - c', R(S(ab - c) - ab') + c').$$

Let $(A, B) \in \mathbf{R}^2_{>0}$. We show that $\tau^{-1}(A, B)$ consists of two elements. Let $(R, S) \in \tau^{-1}(A, B)$ that is

$$RSc = a'bS - a'b' + c' + A, RS(ab - c) = ab'R - c' + B.$$

We have

$$(ab - c)(a'bS - a'b' + c' + A) = c(ab'R - c' + B)$$

so that

(b)
$$R = (ab-c)a'b(ab'c)^{-1}S + (-aa'bb'+a'b'c+abc'+(ab-c)A-cB)(ab'c)^{-1}.$$

Substituting this into RSc = a'bS - a'b' + c' + A we obtain

$$(ab-c)a'bS^{2} + (-2aa'bb' + a'b'c + abc' + (ab-c)A - cB)S + (a'b' - c' - A)ab' = 0.$$

We set $\tilde{R} = R - a'/a$, $\tilde{S} = S - b'/b$. We obtain

$$(ab - c)a'b(\tilde{S} + b'/b)^{2} + (-2aa'bb' + a'b'c + abc' + (ab - c)A - cB)(\tilde{S} + b'/b) + (a'b' - c' - A)ab' = 0$$

that is

(c)
$$(ab-c)a'b\tilde{S}^2 + (abc'-a'b'c+(ab-c)A-cB)\tilde{S} - (A+B)cb'/b = 0;$$

Since $(-(A+B)cb'/b)/((ab-c)a'b)^{-1} < 0$, the equation (c) for \tilde{S} has two roots, \tilde{S}_+, \tilde{S}_- such that $\tilde{S}_+ \in \mathbf{R}_{>0}, \tilde{S}_- \in -\mathbf{R}_{>0}$. Thus we have

$$\tilde{S} \in \{\tilde{S}_+, \tilde{S}_-\}$$

and

(d)
$$\tilde{S}_{+} + \tilde{S}_{-} = -\mu((ab - c)a'b)^{-1},$$

where $\mu = abc' - a'b'c + (ab - c)A - cB$.

Note that

$$\tilde{S}_{+} = (-\mu + \sqrt{\mu^{2} + 4(ab - c)a'b'c(A + B)})/(2(ab - c)a'b),$$

$$\tilde{S}_{-} = (-\mu - \sqrt{\mu^{2} + 4(ab - c)a'b'c(A + B)})/(2(ab - c)a'b),$$

where $\sqrt{:}\mathbf{R}_{>0} \to \mathbf{R}_{>0}$ is the square root.

In the case where $\tilde{S} = \tilde{S}_+$ (resp. $\tilde{S} = \tilde{S}_-$) we set $\tilde{R} = \tilde{R}_+$ (resp. $\tilde{R} = \tilde{R}_-$). We can rewrite (b) as

$$\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}\tilde{S}_{\pm}\mu(ab'c)^{-1}.$$

Using (d) this becomes

$$\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}\tilde{S}_{\pm} - ((ab - c)a'b)(ab'c)^{-1}(\tilde{S}_{+} + \tilde{S}_{-})$$

that is

$$\tilde{R}_{\pm} = (ab - c)a'b(ab'c)^{-1}(\tilde{S}_{\pm} - \tilde{S}_{+} - \tilde{S}_{-}).$$

Thus,

$$\tilde{R}_+ = -(ab-c)a'b(ab'c)^{-1}\tilde{S}_- \in \mathbf{R}_{>0}$$

(since $\tilde{S}_{-} \in -\mathbf{R}_{>0}$,)

$$\tilde{R}_{-} = -(ab-c)a'b(ab'c)^{-1}\tilde{S}_{+} \in -\mathbf{R}_{>0}$$

(since $\tilde{S}_+ \in \mathbf{R}_{>0}$). We see that

$$\tau^{-1}(A,B) \subset \{ (\tilde{R}_+ + a'/a, \tilde{S}_+ + b'/b), (\tilde{R}_- + a'/a, \tilde{S}_- + b'/b) \}$$

The same proof shows also the reverse inclusion so that

$$\tau^{-1}(A,B) = \{ (\tilde{R}_+ + a'/a, \tilde{S}_+ + b'/b), (\tilde{R}_- + a'/a, \tilde{S}_- + b'/b) \}$$

Note that $(A, B) \mapsto (\tilde{R}_+ + a'/a, \tilde{S}_+ + b'/b)$ is a continuous map $\mathbf{R}_{>0}^2 \to \mathcal{T}_{u,u'}$ which is the inverse of the continuous map $\mathcal{T}_{u,u'} \to \mathbf{R}_{>0}^2$ defined by τ . This shows that 0.3 holds in our case: one of the two maps $Z_i : U^- \to \mathbf{R}$ $(i \in I)$ attaches to a matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} \in U^-$$

the number z; the other attaches to the same matrix the number xy - z.

2. Proof of Proposition 0.5

2.1. In this section we prove Proposition 0.5. For $u' \in U_{>0}^-$, $v \in U_{>0}^+$ we have $u'v \in G_{>0}$ hence we can define $(u'v)^+ \in U_{>0}^+$, $t_1 \in T_{>0}$, $(u'v)^- \in U_{>0}^-$ by the equation $u'v = (u'v)^+t_1(u'v)^-$. Note that the map $U_{>0}^- \times U_{>0}^+ \to U_{>0}^+ \times T_{>0} \times U_{>0}^-$, $(u',v) \mapsto ((u'v)^+, t_1, (u'v)^-)$ is continuous.

Lemma 2.2. Let $B \in \mathcal{B}_{>0}$. We write $B = u'B^+u'^{-1}$ with $u' \in U^-_{>0}$ uniquely determined. We have a homeomorphism

$$\{(v,t) \in U_{>0}^+ \times T_{>0}; t \in \mathcal{T}_{(u'v)^-, u'}\} \xrightarrow{\sim} B \cap G_{>0}$$

given by $(v,t) \mapsto u'vtu'^{-1}$.

We have $B = \{u'vtu'^{-1}; v \in U^+, t \in T\}$. If such $u'vtu'^{-1}$ is in $G_{>0}$ then $u'vt \in G_{>0}u' \subset G_{>0}U_{>0}^- \subset G_{>0}$ (see [L94, 2.12]) so that $u'vt = u'_0v_0t_0$ with $u'_0 \in U_{>0}^-, v_0 \in U_{>0}^+, t_0 \in T_{>0}$. It follows that $u' = u'_0, v = v_0, t = t_0$ so that $v \in U_{>0}^+, t \in T_{>0}$. Next we have $u'v = (u'v)^+t_1(u'v)^-$ as in 2.1) and

$$u'vtu'^{-1} = (u'v)^{+}t_{1}(u'v)^{-}tu'^{-1} = (u'v)^{+}t_{1}t(t^{-1}(u'v)^{-}tu'^{-1}) \in U_{>0}^{+}T_{>0}U^{-}.$$

Since $u'vtu'^{-1} \in G_{>0}$ we have also $u'vtu'^{-1} = u_2t_2u'_2$ with $u_2 \in U^+_{>0}, u'_2 \in U^-_{>0}$, $t_2 \in T_{>0}$. It follows that $(u'v)^+ = u_2, t_1t = t_2, t^{-1}(u'v)^-tu'^{-1} = u'_2$. In particular, $t^{-1}(u'v)^-tu'^{-1} \in U^-_{>0}$.

Conversely, if $v \in U_{>0}^+$ and $t \in T_{>0}$ are given such that $t^{-1}(u'v)^- tu'^{-1} \in U_{>0}^-$, then we have $u'vtu'^{-1} \in G_{>0}$. Indeed, as in 2.1 we have

$$u'vtu'^{-1} = (u'v)^{+}t_{1}(u'v)^{-}tu'^{-1}$$

= $(u'v)^{+}(t_{1}t)(t^{-1}(u'v)^{-}tu'^{-1}) \in U_{>0}^{+}T_{>0}U_{>0}^{-} = G_{>0}$

The lemma follows.

2.3. We prove 0.5. Let $B \in \mathcal{B}_{>0}$. We write $B = u'B^+u'^{-1}$ with $u' \in U^-_{>0}$ uniquely determined. From 0.3 we obtain a homeomorphism

$$\{(v,t) \in U_{>0}^+ \times T_{>0}; t \in \mathcal{T}_{(u'v)^-,u'}\} \to U_{>0}^+ \times \mathbf{R}_{>0}^I$$

given by $(v,t) \mapsto (v, z((u'v)^-, u', t))$. Composing this with the inverse of the homeomorphism in Lemma 2.2, we obtain a homeomorphism of $B \cap G_{>0}$ with $U_{>0}^+ \times \mathbf{R}_{>0}^I$ so that 0.5 holds.

3. Proof of Proposition 0.7 and of 0.8(C)

3.1. Let W be the Weyl group of G; let $\{s_i; i \in I\}$ be the simple reflections in W. Let $w \mapsto |w|$ be the length function; we have $|s_i| = 1$ for $i \in I$. For any $J \subset I$ let W_J be the subgroup of W generated by $\{s_i; i \in J\}$ and let w_0^J be the unique element $w \in W_J$ with |w| maximal. Let $w_0 = w_0^J$.

For $w \in W$ let $U^+(w)$ (resp. $U^-(w)$) be the subset of U^+ (resp. U^-) defined in [L94, 2.7] (resp. [L94, 2.9]).

In the remainder of this section we fix $J \subset I$. From the definitions we see that (a) $u \mapsto uP_J^+ u^{-1}$ is a bijection $U^-(w_0 w_0^J) \xrightarrow{\sim} \mathcal{P}_{>0}^J$.

3.2. For any $\lambda = (\lambda_i)_{i \in I} \in \mathbf{N}^I$ let $V_{\lambda}, \beta_{\lambda}$ be as in 0.2. Let $V_{\lambda, \mathbf{R}}$ be the **R**-subspace of V_{λ} spanned by β_{λ} . Let **P** be the set of all lines in $V_{\lambda, \mathbf{R}}$. Note that G acts naturally on **P**. We shall assume that $\{i \in I; \lambda_i \neq 0\} = I - J$. Then for any $P \in \mathcal{P}^J$ there is a unique $L_P^{\lambda} \in \mathbf{P}$ such that the stabilizer of L_P^{λ} in G is equal to P; moreover,

(a) the map $P \mapsto L_P^{\lambda}$ from \mathcal{P}^J to **P** is injective.

A line in **P** is said to be in $\mathbf{P}_{>0}$ if it is spanned by a linear combination of elements in β_{λ} with all coefficients being in $\mathbf{R}_{>0}$. We shall now assume in addition that λ is such that λ_i is sufficiently large for any $i \in I - J$ so that [L98, 3.4] is applicable; thus the following holds:

(b) For $P \in \mathcal{P}^J$ we have $P \in \mathcal{P}^J_{>0}$ if and only if $L_P^{\lambda} \in \mathbf{P}_{>0}$. Now let $g \in G_{>0}$. From [L94, 5.2] we see that the following holds.

(c) There is a unique line $L_g \in \mathbf{P}_{>0}$ such that $gL_g = L_g$.

3.3. Let $g \in G_{>0}$. We prove Proposition 0.7.

By [L94, 8.9] there exists $B \in \mathcal{B}_{>0}$ such that $g \in B$. Let $P \in \mathcal{P}^J$ be such that $B \subset P$. We have $P \in \mathcal{P}_{>0}^J$, $g \in P$. This proves the existence in 0.7. Assume now that $P' \in \mathcal{P}_{>0}^J$, $P'' \in \mathcal{P}_{>0}^J$ satisfy $g \in P', g \in P''$. In the setup of 3.2(b) we have $L_{P'}^{\lambda} \in \mathbf{P}_{>0}$, $L_{P''}^{\lambda} \in \mathbf{P}_{>0}$. Since $g \in P'$, the line $L_{P'}^{\lambda}$ is g-stable hence, with notation of 3.2(c), we have $L_{P'}^{\lambda} = L_g$. Similarly we have $L_{P''}^{\lambda} = L_g$. Thus we have $L_{P'}^{\lambda} = L_{P''}^{\lambda}$. Using 3.2(a) we deduce P' = P''. This proves the uniqueness in 0.7.

3.4. We now fix $P \in \mathcal{P}_{>0}^J$. By 3.1(a) we have $P = uP_J^+ u^{-1}$ for a well defined $u \in U^-(w_0 w_0^J)$.

We show:

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(a) $v \mapsto uvB^+v^{-1}u^{-1}$ is a bijection $U^-(w_0^J) \xrightarrow{\sim} \gamma_P$. If $v \in U^-(w_0^J)$ then $uv \in U^-(w_0) = U_{>0}^-$ hence $uvB^+v^{-1}u^{-1} \in \mathcal{B}_{>0}$; moreover we have $vB^+v^{-1} \in P_J^+$ hence $uvB^+v^{-1}u^{-1} \subset uP_J^+u^{-1} = P$ so that $uvB^+v^{-1}u^{-1} \in \gamma_P$. Thus (a) is a well defined map. Now let $B \in \gamma_P$. We have $B = u_1B^+u_1^{-1}$ where $u_1 \in U^-(w_0)$ and $B \subset uP_J^+u^{-1}$ that is $u_1B^+u_1^{-1} \subset uP_J^+u^{-1}$. Now there is a unique $P' \in \mathcal{P}^J$ containing $u_1B^+u_1^{-1}$. Since $u_1P_J^+u_1^{-1}$ and $uP_J^+u^{-1}$ are such P', we must have $u_1P_J^+u_1^{-1} = uP_J^+u^{-1}$. We have $u_1 = u'_1u''_1$ where $u'_1 \in U^-(w_0w_0^J)$, $u''_1 \in U^-(w_0^J)$; Hence $u_1P_J^+u_1^{-1} = u'_1u''_1P_J^+u''_1^{-1} = u'_1P_J^+u'_1^{-1}$ so that $u'_1P_J^+u'_1^{-1} = uP_J^+u^{-1}$. Using this and 3.1(a) we deduce $u'_1 = u$ so that $B = uu''_1B^+u''_1^{-1}u^{-1}$. We see that the map (a) is surjective. The injectivity of the map (a) is immediate. This proves (a).

From the definitions we see that (a) is a homeomorphism. Hence 0.8(c) holds.

3.5. Let $P \in \mathcal{P}_{>0}^J$ and let P_{red} be the reductive quotient of P. Define $u \in U^-(w_0w_0^J)$ by $P = uP_J^+u^{-1}$ (see 3.1(a)). Now P_{red} has a natural pinning. (It is induced by the obvious pinning of the reductive quotient of P_J^+ using the isomorphism $P_J^+ \to P$ given by conjugation by u.) Hence we can define $\mathcal{B}_{P,>0}$ (an open subset of the real flag manifold of P_{red}) in terms of P_{red} with its pinning in the same way as $\mathcal{B}_{>0}$ was defined in terms of G with its pinning. (Although P_{red} is not necessarily semisimple, the same definitions can be applied.) We have a bijection

(a) $\mathcal{B}_{P,>0} \xrightarrow{\sim} \gamma_P$ obtained by taking inverse image under the obvious map $P \to P_{red}$. This can be deduced from the proof in 3.4.

References

- [L90] G.Lusztig, Canonical bases arising from quantized enveloping algebras, J.Amer.Math.Soc. 3 (1990), 447-498.
- [L94] G.Lusztig, Total positivity in reductive groups, Lie theory and geometry, Progr.in Math.123, Birkhäuser, Boston, 1994, pp. 531-568.
- [L98] G.Lusztig, Total positivity in partial flag manifolds, Represent. Th. 2 (1998), 70-78.
- [L21] G.Lusztig, Total positivity in Springer fibres, Quart.J.Math. 72 (2021), 31-49.

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