

Partial actions on quotient spaces and globalization

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Abstract

Given a partial action of a topological group G on a space X we determine properties \mathcal{P} which can be extended from X to its globalization. We treat the cases when \mathcal{P} is any of the following: Hausdorff, regular, metrizable, second countable and having invariant metric. Further, for a normal subgroup H we introduce and study a partial action of G/H on the orbit space X/\sim , applications to invariant metrics and inverse limits are presented.

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1 Introduction

Given an action $a : G \times Y \rightarrow Y$ of a group G on a space Y and an invariant subset X of Y (i.e. $a(g, x) \in X, x \in X, g \in G$), the restriction of a to $G \times X$ is an action of G over X . If X is not invariant, we get what is called a *partial action* on X , that is a collection of partial maps $\eta_g, g \in G$ on X satisfying $\eta_1 = \text{id}_X$ and $\eta_g \circ \eta_h \subseteq \eta_{gh}$, for each $g, h \in G$. The notion of partial group action appeared in the context of C^* -algebras in [8], there C^* -algebraic crossed products by partial automorphisms played an important role to analyze and characterize their internal structure. Since [8], partial actions have been spreading in several branches of mathematics, for a detailed account on partial actions the interested reader may consult [5] or [10]. A relevant question is if a partial action can be obtained by restriction of a corresponding collection of maps on some superspace. In the topological context, this is known as the globalization problem and was studied in [1] and independently in [13]. It was proven that for any partial action η of a topological group G on a topological space X there is a topological superspace Y of X and a continuous action μ of G on Y such that the restriction of μ to X is η . Such a space is called a globalization of X . It is also shown that there is a minimal globalization X_G called the enveloping space of X .

We shall mainly work with partial actions for which the partial maps have clopen domains, that is closed and open, this kind of partial actions were considered in [6] where the authors studied the ideal structure of the algebraic partial crossed product $\mathcal{L}_c(X) \rtimes G$ being $\mathcal{L}_c(X)$ the algebra consisting of all locally constant, compactly supported functions on X , while in [11] the authors showed that partial actions on the Cantor set by clopen subsets are exactly the

ones for which the enveloping space is Hausdorff, also in [3] partial actions with clopen domains were relevant to introduce and study topological entropy for a partial action of \mathbb{Z} on metric spaces, and in [12] the authors studied topological dynamics arising from partial actions on clopen subsets of a compact space.

Our work is organized in the following way: After the introduction, in Section 2 we present some notions, examples and results that will be useful during the work, especially Proposition 2.8 gives conditions for the enveloping space to be T_1 while Theorem 2.12 establish that the globalization of a partial action is actually an orbit space. At the beginning of Section 3 we treat the question if a structural property \mathcal{P} of a space X endowed with a partial action of a group G is inherited by the spaces X/\sim_G and X_G (see equations (2.6) and (2.2) for the proper definitions of X/\sim_G and X_G , respectively). To do so we first show in Lemma 3.1 that the quotient map π_G defined in (2.7) is perfect, this allows us to present in Theorem 3.2 sufficient conditions in which an affirmative answer holds for when \mathcal{P} is any properties of being Hausdorff, regular, metrizable and second countable. Second part of Section 3 deals with invariant metrics, there we give in Theorem 3.10 a condition for a space X with a partial action of a compact group so that it admits an invariant metric. At this point it is important to note that in the classical case of finding characterizations of G -spaces having invariant metric have been extensively studied, in particular it is known that if a space X with a global action admits an invariant metric, then the orbit space X/\sim_G is metrizable provided that is T_1 , however, this result does not hold for partial actions, where one needs to impose regularity conditions (see Remark 3.11 and Proposition 3.12, respectively). In Section 4 we take a partial action η of G on a space X , a normal subgroup H of G and we show in Theorem 4.1 how to construct a partial action $\eta_{G/H}$ of G/H on the orbit space X/\sim_H , moreover, in the same Theorem is shown that the orbit spaces $(X/\sim_H)/\sim_{G/H}$ and X/\sim_G are homeomorphic. The structure of the partial action $\eta_{G/H}$ as well as its globalization are presented in Theorem 4.2. As an application for the construction of $\eta_{G/H}$ we treat in Proposition 4.9 partial actions on inverse limits, where we provide suitable conditions for which a space X is G -equivalent to an inverse limit $\varprojlim X_i$, and such that the partial action on X satisfies a compatibility relation with the partial actions associated to X_i

Throughout the work, several examples are shown to clarify the notions and results.

2 Preliminaries

Let G be a group with identity element 1, X be a set, and $\eta : G \times X \rightarrow X$, $(g, x) \mapsto g \cdot x$ be a partially defined function, that is, a function whose domain, denoted by $G * X$, is contained in $G \times X$. We shall write $\exists g \cdot x$ to mean that (g, x) belongs to $G * X$. We say that η is a *partial action* of G on X if for each $g, h \in G$ and $x \in X$ the following assertions hold:

- (PA1) If $\exists g \cdot x$, then $\exists g^{-1} \cdot (g \cdot x)$ and $g^{-1} \cdot (g \cdot x) = x$,
- (PA2) If $\exists g \cdot (h \cdot x)$, then $\exists (gh) \cdot x$ and $g \cdot (h \cdot x) = (gh) \cdot x$,
- (PA3) $\exists 1 \cdot x$ and $1 \cdot x = x$.

For $g \in G$ set $X_g = \{x \in X \mid \exists g^{-1} \cdot x\}$. Then η induces a family of bijections $\{\eta_g : X_{g^{-1}} \ni x \mapsto g \cdot x \in X_g\}_{g \in G}$. We also denote this family by η . Notice that η acts (globally) on X if $\exists g \cdot x$, for all $(g, x) \in G \times X$, or equivalently, $X_g = X$, for any $g \in G$. The following result characterizes partial actions in terms of a family of bijections.

Proposition 2.1. [17, Lemma 1.2] A partial action η of G on X is a family $\eta = \{\eta_g : X_{g^{-1}} \rightarrow X_g\}_{g \in G}$, where $X_g \subseteq X$, $\eta_g : X_{g^{-1}} \rightarrow X_g$ is bijective, for all $g \in G$, and:

- (i) $X_1 = X$ and $\eta_1 = \text{id}_X$;

$$(ii) \eta_g(X_{g^{-1}} \cap X_h) = X_g \cap X_{gh};$$

$$(iii) \eta_g \eta_h: X_{h^{-1}} \cap X_{h^{-1}g^{-1}} \rightarrow X_g \cap X_{gh}, \text{ and } \eta_g \eta_h = \eta_{gh} \text{ in } X_{h^{-1}} \cap X_{h^{-1}g^{-1}};$$

for all $g, h \in G$.

Definition 2.2. Let G be a topological group and X be a topological space. A topological partial action of G on X is a partial action $\eta = \{\eta_g: X_{g^{-1}} \rightarrow X_g\}_{g \in G}$ on the underlying set X such that X_g is open, η_g is homeomorphism for any $g \in G$. Moreover we say that η is continuous if $\eta: G * X \rightarrow X$ is continuous, where $G \times X$ has the product topology and $G * X$ is endowed with the relative topology.

Throughout this paper G will denote a Hausdorff topological group, X a topological space and all partial actions will be topological.

Now we present an example of a continuous and topological partial action that will be useful in Section 3. We endow \mathbb{Z} with the p -adic topology \mathcal{T}_p , where p is a prime number. For the reader's convenience we recall its construction here. See [18, Example 1.18] for details. The family $\mathcal{V} = \{p^k \mathbb{Z}\}_{k \in \mathbb{Z}^+}$ satisfy the conditions given in [18, Theorem 1.13], then $\mathcal{B} = \{m + p^k \mathbb{Z} : m \in \mathbb{Z}, k \in \mathbb{Z}^+\}$ is a basis for the topology \mathcal{T}_p and $(\mathbb{Z}, +, \mathcal{T}_p)$ is a topological group.

Example 2.3. Let X be a disconnected topological space, $U \subseteq X$ be a proper clopen set, $f: U \rightarrow U$ be a homeomorphism and $n \in \mathbb{Z}$. We set $f^0 = \text{id}_U$, if $n \in \mathbb{Z}^+$ we write f^n as n -times the composition of f with itself and if $n < 0$, then $f^n = (f^{-1})^{-n}$. We define a partial action η of \mathbb{Z} on X by setting

$$\mathbb{Z} * X = (\mathbb{Z} \times U) \cup (\{0\} \times X) \text{ and } \eta: \mathbb{Z} * X \ni (n, a) \mapsto \begin{cases} f^n(a), & \text{if } a \in U, \\ a, & \text{if } n = 0 \text{ and } a \notin U. \end{cases} \in X. \quad (2.1)$$

Suppose there is p a prime number such that $f^p = \text{id}_U$, and consider \mathbb{Z} with the p -adic topology. Since U is open, then η is a topological partial action. To show that it is continuous take $(n, x) \in \mathbb{Z} * X$, and $V \subseteq X$ an open set such that $\eta(n, x) \in V$. There are two cases to consider.

Case 1: $x \in U$. Then $\eta(n, x) = f^n(x) \in V$. Since $V \cap U$ is open in U , there is $Z \subseteq U$ an open set such that $f^n(Z) \subseteq V \cap U$. First, suppose that p does not divide $|n|$. Then the open set $W = [(n + Z_1) \times Z] \cap (\mathbb{Z} * X) \subseteq \mathbb{Z} * X$ satisfies $\eta(W) \subseteq V$, because for $(t, y) \in W$ we have $y \in Z \subseteq U$ and there is $m \in \mathbb{Z}$ such that $t = n + pm$. Note that $t \neq 0$ since p does not divide $|n|$. Further since $y \in U$, we get $(n, y) \in \mathbb{Z} * X$, and $(pm, f^n(y)) \in \mathbb{Z} * X$. The fact $f^n(y) \in U$ gives

$$\eta(t, y) = f^t(y) = f^{pm}(f^n(y)) = f^n(y) \in V.$$

Now if p divides $|n|$ we let $i = \max\{k \in \mathbb{Z}^+ : p^k \text{ divides } n\}$. Consider the open set $W = [(n + Z_{i+1}) \times Z] \cap (\mathbb{Z} * X)$. Then for $(t, y) \in W$, by the maximality of i there is $m \in \mathbb{Z}$ such that $t = n + p^{i+1}m$, $y \in Z \subseteq U$ and $t \neq 0$. Since $y \in U$, we get

$$\eta(t, y) = f^{n+p^{i+1}m}(y) = f^{p^{i+1}m}(f^n(y)) = f^n(y) \in V,$$

and $\eta(W) \subseteq V$.

Case 2: $x \notin U$. By (2.1) we have $n = 0$ and $\eta(n, x) = x \in V$. Since U is closed, then $V \cap (X \setminus U)$ is an open subset of X containing x . Take $Z \subseteq X$ open such that $x \in Z \subseteq V \cap (X \setminus U)$ and let $W = (Z_1 \times Z) \cap (\mathbb{Z} * X)$. It is clear that $(n, x) = (0, x) \in W$. Further if $(t, y) \in W$, then $y \notin U$ and $t = 0$ from this we get $\eta(t, y) = \eta(0, y) = y \in V$, showing that η is continuous.

2.1 On the enveloping space

Partial actions can be induced from global ones as the following example shows.

Example 2.4. (Induced partial action) Let $u: G \times Y \rightarrow Y$ be a continuous action of G on a topological space Y and $X \subseteq Y$ an open set. For $g \in G$, set $X_g = X \cap u_g(X)$ and let $\eta_g = u_g \upharpoonright X_{g^{-1}}$. Then $\eta: G * X \ni (g, x) \mapsto \eta_g(x) \in X$ is a continuous and topological partial action of G on X . In this case we say that η is *induced* by u .

Remark 2.5. Given a continuous global action η of G on X , its induced partial action on an open (resp. closed) subset Y of X has open (resp. closed) domain in $G \times X$.

An important problem on partial actions is whether they can be induced by global actions. In the topological sense, this turns out to be affirmative and a proof was presented in [1, Theorem 1.1] and independently in [13, Section 3.1]. For the reader's convenience, we recall their construction. Let η be a partial action of G on X . Define an equivalence relation on $G \times X$ as follows:

$$(g, x)R(h, y) \iff x \in X_{g^{-1}h} \text{ and } \eta_{h^{-1}g}(x) = y, \quad (2.2)$$

and denote by $[g, x]$ the equivalence class of the pair (g, x) . Consider $X_G = (G \times X)/R$ with the quotient topology and the map

$$\mu: G \times X_G \ni (g, [h, x]) \mapsto [gh, x] \in X_G, \quad (2.3)$$

is a well defined action, and the map

$$\iota: X \ni x \mapsto [1, x] \in X_G, \quad (2.4)$$

is injective.

Definition 2.6. Let η be a partial action of G on X . The action μ defined in (2.3) is called the enveloping action of η and X_G is the enveloping space or globalization of X .

In the next result we summarize some basic results about the enveloping space and the enveloping action, see [1, Theorem 1.1], [13, Theorem 3.13] and [13, Proposition 3.9].

Proposition 2.7. Let η be a partial action of G on X . Then the following assertions hold.

- (i) The maps μ and ι are continuous.
- (ii) If η is continuous and $G * X$ is open in $G \times X$, then ι is an open map.
- (iii) The quotient map

$$q: G \times X \ni (g, x) \mapsto [g, x] \in X_G, \quad (2.5)$$

is continuous and open.

Now we provide conditions for X_G to be T_1 .

Proposition 2.8. Let η be a continuous partial action of G on X . Consider the following assertions.

- (i) $G * X$ is closed;
- (ii) For any $x \in X$ the set $G^x = \{g \in G \mid \exists g \cdot x\}$ is closed;
- (iii) X_G is T_1 .

Then (i) \Rightarrow (ii), and (ii) \Rightarrow (iii) provided that X is Hausdorff.

Proof. (i) \Rightarrow (ii) For a net $(g_\lambda)_{\lambda \in \Lambda}$ in G^x such that $\lim g_\lambda = g$, one has $(g_\lambda, x)_{\lambda \in \Lambda} \rightarrow (g, x) \in \overline{G * X} = G * X$, thus $g \in G^x$ and G^x is closed.

For the rest of the proof we assume that X is Hausdorff.

(ii) \Rightarrow (iii) Take $(g, x) \in G \times X$, and let q be the quotient map defined in (2.5), then

$$q^{-1}(\{[g, x]\}) = \bigcup_{l \in G} \{gl^{-1}\} \times \eta(\{x\} \cap X_{l^{-1}}) = \{(gl^{-1}, l \cdot x) \mid l \in G^x\}.$$

We prove that $q^{-1}(\{[g, x]\})$ is closed. For this let $(h, y) \in \overline{q^{-1}(\{[g, x]\})}$, then there exists a net $\{l_i\}_{i \in I}$ in G^x such that $(gl_i^{-1}, l_i \cdot x) \rightarrow (h, y)$, in particular $l_i \rightarrow h^{-1}g \in G^x$. Set $\eta^x : G^x \ni g \mapsto g \cdot x \in X$, using the fact that η^x is continuous one gets $l_i \cdot x \rightarrow (h^{-1}g) \cdot x$, and $y = (h^{-1}g) \cdot x$ because of the uniqueness of limits in Hausdorff spaces. From this we obtain

$$(h, y) = (g(h^{-1}g)^{-1}, (h^{-1}g) \cdot x) \in q^{-1}(\{[g, x]\}),$$

thus X_G is T_1 . □

Remark 2.9. With respect to Proposition 2.8 we have the next.

- The space X_G is T_1 when G is discrete and X is Hausdorff.
- Part (ii) \Rightarrow (i) does not necessarily holds. Indeed, for the partial action of $\mathbb{Z}_2 = \{1, -1\}$ on $X = [0, 1]$ presented in [1, Example 1.4.] that is $\alpha_1 = \text{id}_X$ and $\alpha_{-1} = \text{id}_V$, where $V = (0, 1]$. One has that \mathbb{Z}_2^x is closed for any $x \in [0, 1]$ while $\mathbb{Z}_2 * [0, 1] = \{(-1, 0)\} \cup (\mathbb{Z}_2 \times V)$ is not closed in $\mathbb{Z}_2 \times [0, 1]$.
- Also part (iii) \Rightarrow (ii) does not hold in general, for this let $G = GL(2; \mathbb{R})$ be the general linear group of degree 2 acting partially on \mathbb{R} as follows: For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$, set $\mathbb{R}_{g^{-1}} = \{x \in \mathbb{R} : cx + d \neq 0\}$ and $\eta_g : \mathbb{R}_{g^{-1}} \ni x \mapsto \frac{ax + b}{cx + d} \in \mathbb{R}_g$. There is a homeomorphism from \mathbb{R}_G to the space \mathbb{C} of complex numbers, then \mathbb{R}_G is Hausdorff but $G^0 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G : d \neq 0 \right\}$ is not closed in G .

Definition 2.10. Suppose that the spaces X and Y are equipped with partial actions η and ρ by G . A function $\epsilon : X \rightarrow Y$ is called a or G -map if for every $(g, x) \in G * X$ then $(g, \epsilon(x)) \in G * Y$ and $\epsilon \eta(g, x) = \rho(g, \epsilon(x))$. If moreover ϵ is a homeomorphism we say that X and Y are G -equivalent.

We have the next.

Proposition 2.11. The following assertions hold.

- (i) Let X and Y be two G -equivalent spaces. Then X_G and Y_G are homeomorphic, as well as $G * X$ and $G * Y$.
- (ii) Let $\beta : G \times Y \rightarrow Y$ be a continuous action of G on a space Y . Let $X \subseteq Y$ be an open set such that $G \cdot X = Y$ and $\eta : G * X \rightarrow X$ be the induced partial action of β on X (see Example 2.4). Then the spaces X_G and Y are G -equivalent.

Proof. Part (i) is clear, for (ii) let $i : G \times X \rightarrow G \times Y$ be the inclusion and $\alpha : X_G \ni [g, x] \mapsto \beta(g, x) \in Y$, then the following diagram

$$\begin{array}{ccc} G \times X & \xrightarrow{i} & G \times Y \\ q \downarrow & & \downarrow \beta \\ X_G & \xrightarrow{\alpha} & Y \end{array}$$

is commutative. Moreover, by [13, Proposition 3.5] the map α is a well defined bijection, moreover it is continuous because the map $\alpha \circ q$ is continuous. Also, since β is open the map α is a homeomorphism, finally the fact that it is a G -map is straightforward. \square

2.2 The orbit equivalence relation

Given a partial action η of G on X the *orbit equivalence relation* \sim_G on X is:

$$x \sim_G y \iff \exists g \in G^x \text{ such that } g \cdot x = y, \quad (2.6)$$

for each $x, y \in X$. The elements of X/\sim_G are the *orbits* $G^x \cdot x$ with $x \in X$ and X/\sim_G is endowed with the quotient topology. By [16, Lemma 3.2] the *induced quotient map* of η

$$\pi_G : X \ni x \mapsto G^x \cdot x \in X/\sim_G, \quad (2.7)$$

is continuous and open.

It is known that globalizations of topological spaces endowed with a partial action can be seen as orbit equivalence spaces. Indeed the following result was shown in [16, Theorem 3.3].

Theorem 2.12. Let η be a topological partial action of G on X , then the family $\hat{\eta} = \{\hat{\eta}_g : (G \times X)_{g^{-1}} \rightarrow (G \times X)_g\}_{g \in G}$, where $(G \times X)_g = G \times X_g$ and

$$\hat{\eta}_g : G \times X_{g^{-1}} \ni (h, x) \mapsto (hg^{-1}, \eta_g(x)) \in G \times X_g,$$

is a topological partial action of G on $G \times X$, and the enveloping space X_G of η is the space of orbits of $G \times X$ by $\hat{\eta}$.

Let η be a partial action of G on X , and H be a subgroup of G , then the family $\eta_H : \{\eta_h : X_{h^{-1}} \rightarrow X_h\}_{h \in H}$ is a partial action of H on X . The corresponding orbit equivalence relation of η_H is denoted by \sim_H .

For convenience, the orbits in the space X_G/\sim_H will be denoted by $H[g, x]$ for any $[g, x] \in X_G$. We finish this section with the next.

Lemma 2.13. Let η be a continuous partial action of G on X with $G * X$ open. Then for each subgroup H of G the map

$$\varphi : X/\sim_H \ni H^x \cdot x \mapsto H[1, x] \in X_G/\sim_H \quad (2.8)$$

is an embedding, that is continuous, open and injective.

Proof. First of all note that φ is well defined. In fact, let $x, y \in X$ be such that $x \sim_H y$ and take $h \in H^x$ with $\eta_h(x) = y$. Thus, $[1, y] \stackrel{(2.2)}{=} [h, x] \stackrel{(2.3)}{=} \mu_h([1, x])$ and $[1, y] \sim_H [1, x]$, then φ is well defined. It is easy to check that φ is injective. To prove that φ is continuous, consider $\pi_H : X \rightarrow X/\sim_H$ and $\Pi_H : X_G \rightarrow X_G/\sim_H$ the corresponding quotient maps. Since the map ι defined in (2.4) is continuous and $\varphi \circ \pi_H = \Pi_H \circ \iota$ we conclude that φ is continuous. It remains to check that φ is open, let $U \subseteq X/\sim_H$ be an open set, then $\varphi(U) = \Pi_H(\iota(\pi_H^{-1}(U)))$ is open because $\pi_H^{-1}(U)$ is open in X and the functions ι and Π_H are open thanks to Proposition 2.7 and [16, Lemma 3.2], respectively. Therefore φ is an open map. \square

3 Properties preserved by the enveloping space and invariant metrics

Recall that a continuous surjection $f : X \rightarrow Y$ is *perfect* if it is closed and $f^{-1}(\{y\})$ is compact for all $y \in Y$.

We proceed with the next.

Lemma 3.1. Let $\eta : G * X \rightarrow X$ be a continuous partial action such that $G * X$ is closed in $G \times X$ and G is compact, then the following assertions hold.

- (i) η is closed;
- (ii) The maps π_G and $\hat{\pi}_G$ are perfect, being $\hat{\pi}_G$ the corresponding quotient map of $\hat{\eta}$ in Theorem 2.12 .

Proof. (i). Let C be a nonempty closed subset of $G * X$ and $y \in \overline{\eta(C)}$, then there is a directed set Λ and a net $(g_\lambda, x_\lambda)_{\lambda \in \Lambda}$ in C such that $\lim g_\lambda \cdot x_\lambda = y$. Since G is compact, we can suppose that $\lim g_\lambda = g$, for some $g \in G$. Note that $(g_\lambda^{-1}, g_\lambda \cdot x_\lambda)_{\lambda \in \Lambda}$ is a net in $G * X$ and $\lim(g_\lambda^{-1}, g_\lambda \cdot x_\lambda) = (g^{-1}, y)$, then $(g^{-1}, y) \in G * X$ because this is a closed subset of $G \times X$. Now consider the net $(g_\lambda, x_\lambda)_{\lambda \in \Lambda} = (g_\lambda, g_\lambda^{-1} \cdot (g_\lambda \cdot x_\lambda))_{\lambda \in \Lambda}$ in C , then

$$(g, g^{-1} \cdot y) = \lim(g_\lambda, g_\lambda^{-1} \cdot (g_\lambda \cdot x_\lambda)) = \lim(g_\lambda, x_\lambda) \in C,$$

and $y = g \cdot (g^{-1} \cdot y) = \eta(g, g^{-1} \cdot y) \in \eta(C)$ which implies that η is closed.

(ii) The map π_G is closed because of (i) above and the equality $\pi_G^{-1}(\pi_G(F)) = \eta((G \times F) \cap (G * X))$, for any closed subset F of X . Hence to prove our assertion we need to check that $\pi_G^{-1}(\pi_G(x))$ is a compact for any $x \in X$. First, by Proposition 2.8 we have that G^x is a compact subset G , then $\pi_G^{-1}(\pi_G(x)) = G^x \cdot x = \eta(G^x \times \{x\})$ is a compact subset of X . To show that $\hat{\pi}_G$ is closed we have by [15, Proposition 2.6] that the map $\hat{\eta}$ is continuous, moreover from [15, Corollary 3.3] we get that $G * (G \times X)$ is closed in $G \times (G \times X)$ then the result follows. \square

Theorem 3.2. Let G be a compact group and $\eta : G * X \rightarrow X$ be a continuous partial action such that $G * X$ is closed in $G \times X$. Let \mathcal{P} be any of the properties: Hausdorff, regular, metrizable and second countable. Then the following statements hold.

- (i) If X is \mathcal{P} , then X/\sim_G is \mathcal{P} .
- (ii) If $G \times X$ is \mathcal{P} , then X_G is \mathcal{P} .

Proof. (i) This follows from item (ii) in Lemma 3.1 and [7, Theorem 5.2] while (ii) is a consequence of item (ii) in Lemma 3.1, item (i) above and the last assertion in Theorem 2.12. \square

Remark 3.3. We remark the following facts.

- (i) In general the assumption that $G * X$ is closed in $G \times X$ cannot be removed in part (ii) of Theorem 3.2. Indeed, for the Abadie's partial action of $\mathbb{Z}_2 = \{1, -1\}$ on $X = [0, 1]$ presented in Remark 2.9 we have by Proposition 2.8 that the space $X_{\mathbb{Z}_2}$ is T_1 but not Hausdorff.
- (ii) Also the fact that X_G is Hausdorff does not imply that G is compact, for instance in [11, Proposition 2.1] a characterization for X_G to be Hausdorff is presented in the case when G is countable and discrete.

We illustrate the previous theorem with some examples.

Example 3.4. Consider $X = \mathbb{R} \setminus \{0\}$ as a subspace of \mathbb{R} . A partial action of \mathbb{Z}_3 on X is defined as follows. Let $X_1 = (-\infty, 0)$ and $X_2 = (0, \infty)$, note that X_1 and X_2 are clopen subsets of X such that $X = X_1 \cup X_2$. Set $\eta_0 = \text{id}_X$, $\eta_2 : X_1 \ni x \mapsto -x \in X_2$ and $\eta_1 = \eta_2^{-1}$, moreover let

$$\mathbb{Z}_3 * X = (\{0\} \times X) \cup (\{1\} \times X_2) \cup (\{2\} \times X_1),$$

Then $\eta : \mathbb{Z}_3 * X \rightarrow X$, is a partial action of \mathbb{Z}_3 on X such that $\mathbb{Z}_3 * X$ is clopen in $\mathbb{Z}_3 \times X$ thus by Theorem 3.2 the enveloping space $X_{\mathbb{Z}_3}$ is metrizable.

Example 3.5. Let X be a disconnected space and $U \subseteq X$ be a non-empty clopen subset of X with $U \neq X$. Then $\eta : \mathbb{Z}_2 * X \rightarrow X$ is a partial action of \mathbb{Z}_2 on X where $\mathbb{Z}_2 * X = (\{0\} \times X) \cup (\{1\} \times U)$, and $\eta(1, u) = u$ for any $u \in U$. Since $\mathbb{Z}_2 * X$ is closed in $\mathbb{Z}_2 \times X$ we conclude that $X_{\mathbb{Z}_2}$ is metrizable.

In view of (ii) in Remark 3.3 we give the next.

Proposition 3.6. Let G be a compact group, X be a compact Hausdorff space and $\eta : G * X \rightarrow X$ be a partial action. If X_G is Hausdorff, then $G * X$ is closed.

Proof. Let $\{(g_\lambda, x_\lambda)\}_{\lambda \in \Lambda}$ be a net in $G * X$ such that $\lim(g_\lambda, x_\lambda) = (g, x) \in G \times X$. Since X_G is Hausdorff, we have by [1, Proposition 1.2] that the space $\text{Graph}(\eta) = \{(g, x, y) \in G \times X \times X : (g, x) \in G * X, g \cdot x = y\}$ is a closed subset of $G \times X \times X$, and thus compact. Therefore we may assume that $(g_\lambda, x_\lambda, g_\lambda \cdot x_\lambda)_{\lambda \in \Lambda}$ converges to $(g, x, p) \in \text{Graph}(\eta)$, for some $p \in X$. In particular, $(g, x) \in G * X$ and $G * X$ is closed. \square

Having at hand Proposition 3.6 one may ask if its assumptions imply that if the orbit space X/\sim_G is Hausdorff then $G * X$ is closed in $G \times X$. But this is not the case as Example 3.7 below shows.

Example 3.7. Consider again the partial action of \mathbb{Z}_2 on $X = [0, 1]$ given in [1, Example 1.4.]. We observed in Remark 2.9 that $\mathbb{Z}_2 * X$ is not closed in $\mathbb{Z}_2 \times X$. Moreover, since $\eta(1, x) = x$ for any $x \in (0, 1]$ we have $\pi_{\mathbb{Z}_2} : X \rightarrow X/\sim_{\mathbb{Z}_2}$ is injective and thus a homeomorphism and $X/\sim_{\mathbb{Z}_2}$ is Hausdorff.

3.1 Invariant metrics

Let $\eta : G * X \ni (g, x) \mapsto g \cdot x \in X$ be a partial action of G on the metric space (X, ρ) . We say that ρ is η -invariant if for any $g \in G$ and $x, y \in X_{g^{-1}}$, $\rho(g \cdot x, g \cdot y) = \rho(x, y)$.

Example 3.8. Let η be as in equation 2.1. Suppose that X is metric, U is a clopen subset of X and f is an isometry, then η is a topological and continuous partial action with invariant metric in any of the following cases.

- \mathbb{Z} is considered as a discrete space.
- \mathbb{Z} is endowed with the p -adic topology and $f^p = \text{id}_U$, for some prime number p .

In the context of hyperspaces endowed with partial actions we give the next.

Example 3.9. Let $\eta : G * X \ni (g, x) \mapsto g \cdot x \in X$ be a continuous partial action of G on a compact metric space (X, d) . Denote by 2^X the hyperspace of nonempty compact subsets of X endowed with the Hausdorff metric d_H , which is defined by the rule

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq N(B, \varepsilon) \text{ and } B \subseteq N(A, \varepsilon)\},$$

where $A, B \in 2^X$ and $N(A, \varepsilon) = \bigcup_{a \in A} B_d(a, \varepsilon)$. If $\{\eta_g\}_{g \in G}$ is the induced family of bijections by η , then follows by [14, Theorem 3.2] that $2^\eta : G * 2^X \ni (g, A) \mapsto \eta_g(A) \in 2^X$, is a continuous partial action of G in 2^X , being

$$G * 2^X = \{(g, A) \in G \times 2^X : (g, a) \in G * X \ (\forall a \in A)\}.$$

Suppose that d is η -invariant, we observe that d_H is 2^η -invariant. For this take $g \in G$ and $A, B \in 2^X$ for which $(g, A), (g, B) \in G * 2^X$. Let $\varepsilon > 0$ with $A \subseteq N(B, \varepsilon)$ and $B \subseteq N(A, \varepsilon)$. Now given $a \in A$ there exists $b \in B$ such that $a \in B_d(b, \varepsilon)$, then $d(g \cdot a, g \cdot b) = d(a, b) < \varepsilon$ and we

have proven that $\eta_g(A) \subseteq N(\eta_g(B), \varepsilon)$, in a similar way one shows that $\eta_g(B) \subseteq N(\eta_g(A), \varepsilon)$. Therefore, $d_H(\eta_g(A), \eta_g(B)) \leq \varepsilon$, and $d_H(\eta_g(A), \eta_g(B)) \leq d_H(A, B)$.

On the other hand, take $\varepsilon > 0$ with $\eta_g(A) \subseteq N(\eta_g(B), \varepsilon)$ and $\eta_g(B) \subseteq N(\eta_g(A), \varepsilon)$. For $a \in A$ choose $b \in B$ such that $g \cdot a \in B_d(g \cdot b, \varepsilon)$ then $d(a, b) = d(g \cdot a, g \cdot b) < \varepsilon$ and $A \subseteq N(B, \varepsilon)$, again one verifies $B \subseteq N(A, \varepsilon)$ which implies $d_H(A, B) \leq d_H(\eta_g(A), \eta_g(B))$ hence $d_H(A, B) = d_H(\eta_g(A), \eta_g(B))$, as desired.

It follows from [2, Proposition 5] that there is a compatible η -invariant metric for X provided that η is global and G is countably compact. Our next goal is to obtain a generalization of this result to the frame of partial actions.

Theorem 3.10. Let $\eta : G * X \rightarrow X$ be a partial action, then X and X_G are metrizable by a invariant metric under any of the following conditions:

- (i) G is countably compact and X_G is metrizable.
- (ii) G is compact and metric, X is metric and $G * X$ is closed.

Moreover if (i) holds and $X_G / \sim G$ is T_1 , then X / \sim_G is metrizable.

Proof. In both cases it is enough to prove that X_G has a compatible μ -invariant metric ρ . Indeed, since η is continuous we have by [13, Proposition 3.12] that the spaces X and $\iota(X)$ are homeomorphic, where ι is given by (2.4), thus one obtains an invariant metric for X by restricting ρ to $\iota(X)$. (i) Since the action μ of G on X_G given by (2.3) is continuous the result follows from [2, Proposition 5]. (ii) In this case the space $G \times X$ is metrizable, thus X_G is metrizable thanks to Theorem 3.2 and again the result follows from [2, Proposition 5]. To show the last assertion, we observe that X_G admits an invariant metric, then the result follows from [4, Theorem 2.16] and Lemma 2.13. \square

Remark 3.11. It is known that when G acts globally on a space X admitting an invariant metric, then the space X / \sim_G is metric provided that it is T_1 , however this not hold for partial actions. For a concrete example take the partial action given in Remark 2.9 and use Theorem 2.12 and Remark 3.3.

The following result tells us that one needs to impose the regularity condition on X / \sim_G .

Proposition 3.12. Let X be a separable second countable space endowed with a partial action of G , then the following assertions are equivalent.

- (i) X / \sim_G is metrizable;
- (ii) X / \sim_G is regular and T_1 .

Proof. Clearly (i) implies (ii). To see that (ii) implies (i), notice that X / \sim_G is separable and second countable, because the quotient map π_G is open. Therefore, by Urysohn's metrization Theorem, the space X / \sim_G is metrizable. \square

4 Partial actions on orbit spaces

Let η be a partial action of G on X and H be a normal subgroup of G . The idea now is to construct a partial action of G/H on X / \sim_H . If η is a global action, then G/H acts globally on X / \sim_H via

$$\eta_{gH}(H \cdot x) = H \cdot (g \cdot x), \quad (4.1)$$

for any $g \in G$ and $x \in X$.

For the case of partial action, we notice that mimicking the construction above does not yield to a partial action of G/H on X/\sim_H because it is not natural how to define the set $G/H * (X/\sim_H)$. Indeed the construction of such a partial action is essentially more laborious than the global one, as we shall see in the next.

Theorem 4.1. Let η be a partial action of G on X and H be a normal subgroup of G . Then there is a continuous partial action $\eta_{G/H}$ of G/H on X/\sim_H , such that the orbit spaces $(X/\sim_H)/\sim_{G/H}$ and X/\sim_G are homeomorphic.

Proof. Let μ be the globalization of η . Then μ is continuous and by (4.1) it induces a continuous action $\mu_{G/H}$ on X_G/\sim_H as follows:

$$\mu_{gH} : X_G/\sim_H \ni H[t, x] \mapsto H[gt, x] \in X_G/\sim_H,$$

for each $gH \in G/H$. Now let φ be defined by (2.8). By Example 2.4 and Lemma 2.13 the map $\mu_{G/H}$ induces a continuous partial action $\eta'_{G/H}$ of G/H on the open set $\text{Im}(\varphi)$ of X_G/\sim_H , where $\eta'_{G/H} = \{\eta'_{gH} : X_{g^{-1}H} \rightarrow X_{gH}\}_{gH \in G/H}$ and

$$X_{gH} = \mu_{gH}(\text{Im}(\varphi)) \cap \text{Im}(\varphi) \quad \text{and} \quad \eta'_{gH} = \mu_{gH} \upharpoonright X_{g^{-1}H}. \quad (4.2)$$

Let $\Omega := X/\sim_H$, then one obtains a partial action $\eta_{G/H}$ of G/H on Ω by setting

$$\Omega_{gH} = \varphi^{-1}(X_{gH}), g \in G \quad \text{and} \quad \eta_{gH} : \Omega_{g^{-1}H} \ni x \mapsto \varphi^{-1}(\eta'_{gH}(\varphi(x))) \in \Omega_{gH}. \quad (4.3)$$

Then

$$\eta_{gH}(x) = (\varphi^{-1} \circ \mu_{gH} \circ \varphi)(x), \quad (4.4)$$

for each $x \in \Omega_{g^{-1}H}$. The fact that $\eta_{G/H}$ is continuous is straightforward.

Let $\sim_{G/H}$ be the orbit equivalence relation in Ω induced by $\eta_{G/H}$. To finish the proof we show that the spaces $\Omega/\sim_{G/H}$ and X/\sim_G are homeomorphic. Consider the diagram:

$$\begin{array}{ccc} X & \xrightarrow{\pi_G} & X/\sim_G \\ \pi_H \downarrow & & \uparrow \psi \\ \Omega & \xrightarrow{\pi_{G/H}} & \Omega/\sim_{G/H} \end{array}$$

where ψ is made such that it commutes, that is

$$\psi(\pi_{G/H}(\pi_H(x))) = \pi_G(x), \quad (4.5)$$

for each $x \in X$. Let us prove that ψ is well defined. Take $z \in \Omega/\sim_{G/H}$ and $x, y \in X$ such that $\pi_{G/H}(\pi_H(x)) = \pi_{G/H}(\pi_H(y))$. Then there is $g \in G$ with

$$\pi_H(y) = \eta_{gH}(\pi_H(x)) \stackrel{(4.4)}{=} \varphi^{-1}(\mu_{gH}(\varphi(\pi_H(x)))) = \varphi^{-1}(H[g, x]),$$

which implies $H[g, x] = H[1, y]$ and there is $h \in H$ such that $[hg, x] = [1, y]$, thus $\eta_{hg}(x) = y$ and $\pi_G(x) = \pi_G(y)$, which shows that ψ is well defined. Moreover by its construction, the map ψ is continuous and surjective.

Let us prove that ψ is injective. Let $z_1, z_2 \in \Omega/\sim_{G/H}$ be such that $\psi(z_1) = \psi(z_2)$, and let $x, y \in X$ be such that $\pi_{G/H}(\pi_H(x)) = z_1$ and $\pi_{G/H}(\pi_H(y)) = z_2$. Since $\pi_G(x) = \pi_G(y)$, there is $g \in G^x$ satisfying $\eta_g(x) = y$. To prove that $z_1 = z_2$ we need to find $t \in G$ for which $\eta_{tH}(\pi_H(x)) = \pi_H(y)$. We claim that $\eta_{gH}(\pi_H(x)) = \pi_H(y)$. In fact, by (4.4) we get

$$\eta_{gH}(\pi_H(x)) = \varphi^{-1}(\mu_{gH}(\varphi(\pi_H(x)))) = \varphi^{-1}(H[g, x]),$$

and $\varphi(\pi_H(y)) = H[1, y] = H[g, x]$, then $\eta_{gH}(\pi_H(x)) = \pi_H(y)$ and ψ is injective. Finally let $U \subseteq \Omega/\sim_{G/H}$ be an open set. Since π_G is open, $\pi_G(\pi_H^{-1}(\pi_{G/H}^{-1}(U))) \subseteq X/\sim_G$ is open. Thus $\psi(U)$ is open and $\psi : \Omega/\sim_{G/H} \rightarrow X_G/\sim_G$ is a homeomorphism. \square

The following result describes explicitly the partial action $\eta_{G/H}$ and its globalization.

Theorem 4.2. Let η be a partial action of G on X , H be a normal subgroup of G and $\eta_{G/H}$ be the partial action of G/H on X/\sim_H defined by (4.4). Then the following assertions hold.

- (i) For $g \in G$ we have $(X/\sim_H)_{gH} = \{\pi_H(x) \mid \exists h \in H \text{ such that } (hg^{-1}, x) \in G * X\}$.
- (ii) The domain of $\eta_{G/H}$ is

$$G/H * X/\sim_H = \{(gH, \pi_H(x)) : (g, x) \in G \times X \wedge \exists h \in H \text{ such that } (hg, x) \in G * X\}.$$

- (iii) We have

$$\eta_{G/H} : G/H * X/\sim_H \ni (gH, \pi_H(x)) \mapsto \pi_H((hg) \cdot x) \in X/\sim_H, \quad (4.6)$$

where $h \in H$ is such that $(hg, x) \in G * X$.

- (iv) The globalization of $\eta_{G/H}$ is (G/H) -equivalent to X_G/\sim_H , where G/H acts on X_G/\sim_H , via $\mu_{G/H}$.

Proof. (i). Take $g \in G$ and $x \in X$ such that $\pi_H(x) \in (X/\sim_H)_{gH}$ then by (4.3) $\varphi(\pi_H(x)) = H[1, x] \in X_{gH}$ and (4.2) gives an element $y \in X$ such that $\mu_{gH}(H[1, y]) = H[1, x]$, that is, $H[g, y] = H[1, x]$ and $[h_0, x] = [g, y]$ for some $h_0 \in H$, therefore $(g^{-1}h_0, x) \in G * X$. Since H is normal in G we have $g^{-1}h_0 = hg^{-1}$ for some $h \in H$ and $(hg^{-1}, x) \in G * X$. Conversely if $x \in X$ verifies $(h_0g^{-1}, x) \in G * X$ for some $h_0 \in H$. Then $h_0g^{-1} = g^{-1}h$ for some $h \in H$ and we have $[h, x] = [g, y]$, where $y = (g^{-1}h) \cdot x$ and

$$\varphi(\pi_H(x)) = H[1, x] = H[1, (h^{-1}g) \cdot y] \stackrel{(2.2)}{=} H[h^{-1}g, y] = H[g, y] = \mu_{gH}(H[1, y]) \in \mu_{gH}(\text{Im}\varphi),$$

thus $\pi_H(x) \in (X/\sim_H)_{gH}$ thanks to equations (4.2) and (4.3).

(ii). This is a consequence of part (i) and the fact that $(gH, \pi_H(x)) \in G/H * X/\sim_H$ if and only if $\pi_H(x) \in (X/\sim_H)_{g^{-1}H}$.

(iii). For $(gH, \pi_H(x)) \in G/H * X/\sim_H$, there exists $h \in H$ such that $(hg, x) \in G * X$. Then $[hg, x] = [1, (hg) \cdot x]$ and $\varphi(\pi_H((hg) \cdot x)) = H[hg, x] = H[g, x]$. Then follows by (4.4) that

$$\eta_{G/H}(gH, \pi_H(x)) = \varphi^{-1}(H[g, x]) = \pi_H((hg) \cdot x),$$

as desired.

(iv). By Lemma 2.13 we know that $\text{Im}\varphi = \{H[1, x] \mid x \in X\}$, then $\mu_{G/H}[\text{Im}\varphi] = X_G/\sim_H$, thus by (ii) of Proposition 2.11 the spaces $(\text{Im}\varphi)_{G/H}$ and X_G/\sim_H are homeomorphic. Now we must show that the spaces $\text{Im}\varphi$ and X/\sim_H are G/H -equivalent, but by (i) in Lemma 2.11 it is enough to show that φ is a (G/H) -map, and this follows from (4.3). \square

Example 4.3. Consider the partial action $\eta : \mathbb{Z} * X \rightarrow X$ of Example 3.8 and let $m \in \mathbb{Z}^+$ be such that $f^m = \text{id}_U$, where m is the smallest positive integer with this property. If $H = m\mathbb{Z}$, then the induced quotient morphism π_H satisfies $\pi_H(x) = \{x\}$, for any $x \in X$, thus the spaces X and X/\sim_H are homeomorphic. Now we determine $\eta_{\mathbb{Z}/H}$. Take $(k + H, \pi_H(x)) \in \mathbb{Z}/H * X/\sim_H$, if $k \in H$, by (4.6) we get

$$\eta_{\mathbb{Z}/H}(k + H, \pi_H(x)) = \eta_{\mathbb{Z}/H}(H, \pi_H(x)) = \pi_H(x).$$

Suppose $k \notin H$. By (ii) of Theorem 4.2, there is $h \in H$ such that $(h+k, x) \in \mathbb{Z} * X$ and $\eta_{\mathbb{Z}/H}(k+H, \pi_H(x)) = \pi_H(\eta(h+k, x))$, thanks to (4.6). Since $(h+k, x) \in \mathbb{Z} * X$ and $k \notin H$ the equality (2.1) implies $x \in U$. Then, (h, x) and $(k+h, x)$ belong to $\mathbb{Z} * X$, which gives $\eta(h+k, x) = \eta(k+h, x) = f^{k+h}(x) = f^k(x)$. We have shown that if $k \notin H$ with $(k+H, \pi_H(x)) \in \mathbb{Z}/H * X / \sim_H$, one gets

$$\eta_{\mathbb{Z}/H}(k+H, \pi_H(x)) = \pi_H(\eta(h+k, x)) = \pi_H(f^k(x)) = \pi_H(\eta(k, x)).$$

Corollary 4.4. Let G be compact and Hausdorff group, H be a closed normal subgroup of G , and $\eta : G * X \rightarrow X$ be a continuous partial action on a compact Hausdorff space X . If $G * X$ is closed in $G \times X$, then $G/H * X / \sim_H$ is closed in $G/H \times X / \sim_H$.

Proof. Let η' be the partial action defined (4.2). By construction we get that $\eta_{G/H}$ and $\eta'_{G/H}$ are G/H -equivalent, and thus it is enough to show that $G/H * \text{Im}(\varphi)$ is closed in $G/H \times \text{Im}(\varphi)$. Having at hand Remark 2.5 we only need to see that $\text{Im}(\varphi)$ is closed in X_G / \sim_H . Now by (ii) in Theorem 3.2 the enveloping space X_G is Hausdorff and since H is compact then the first item of Theorem 3.2 implies that X_G / \sim_H is Hausdorff. Also X / \sim_H is compact which implies that φ is a closed map, then $\text{Im}(\varphi)$ is closed in X_G / \sim_H and $G/H * \text{Im}(\varphi)$ is closed in $G/H \times \text{Im}(\varphi)$ which finishes the proof. \square

The following is clear.

Lemma 4.5. Let G and H be topological groups and $\phi : G \rightarrow H$ be a group homomorphism. If $\{\eta_h : X_{h^{-1}} \rightarrow X_h\}_{h \in H}$ is a partial action of H on X , then the family $\{\eta_{\phi(g)} : U_{g^{-1}} \rightarrow U_g\}_{g \in G}$, where $U_g = X_{\phi(g)}$, $g \in G$, is a partial action of G on X such that

$$G * X = (\phi \times \text{id}_X)^{-1}(H * X) \text{ and } G * X \ni (g, x) \mapsto \eta(\phi(g), x) \in X. \quad (4.7)$$

Remark 4.6. Using $\eta_{G/H}$ and the canonical homomorphism $p_H : G \rightarrow G/H$, it follows by Theorem 4.1 and Lemma 4.5 that there is a partial action η^{p_H} of G on X / \sim_H which by (4.7) has domain

$$G * (X / \sim_H) = \{(g, \pi_H(x)) \mid g \in G, x \in X, (gH, \pi_H(x)) \in G/H * X / \sim_H\}, \quad (4.8)$$

and

$$\eta^{p_H}(g, \pi_H(x)) = \eta_{G/H}(gH, \pi_H(x)). \quad (4.9)$$

From now on we always consider G acting partially on X / \sim_H via η^{p_H} .

Let H_1, H_2 be subgroups of G such that $H_1 \subseteq H_2$. We define $\pi_{H_1, H_2} : X / \sim_{H_1} \rightarrow X / \sim_{H_2}$ as the only map such that

$$\pi_{H_2} = \pi_{H_1, H_2} \circ \pi_{H_1}, \quad (4.10)$$

in particular for a subgroup H of G the map $\pi_{H, H}$ is the identity on X / \sim_H .

Proposition 4.7. Let H, H_1 and H_2 be normal subgroups of G with $H_1 \subseteq H_2$. Then π_H and π_{H_1, H_2} are G -maps.

Proof. We first show that π_H is a G -map. Take $(g, x) \in G * X$, by (ii) of Theorem (4.2) the pair $(gH, \pi_H(x))$ belongs to $G/H * X / \sim_H$ and follows by (4.6) that $\pi_H(\eta(g, x)) = \eta_{G/H}(gH, \pi_H(x))$. Hence $(g, \pi_H(x)) \in G * X / \sim_H$ and $\eta^{p_H}(g, \pi_H(x)) = \pi_H(\eta(g, x))$ which shows that π_H is a G -map. Now we show that π_{H_1, H_2} is a G -map. Suppose $(g, \pi_{H_1}(x)) \in G * X / \sim_{H_1}$. We need to show that $(g, \pi_{H_2}(x)) \in G * X / \sim_{H_2}$ and $\pi_{H_1, H_2}(\eta_{G/H_1}(gH_1, \pi_{H_1}(x))) = \eta_{G/H_2}(gH_2, \pi_{H_2}(x))$. We have $(gH_1, \pi_{H_1}(x)) \in G/H_1 * X / \sim_{H_1}$ using (ii) of Theorem 4.2 there exists an $h \in H_1 \subseteq H_2$

such that $(hg, x) \in G * X$, thus $(gH_2, \pi_{H_2}(x)) \in G/H_2 * X / \sim_{H_2}$ and $(g, \pi_{H_2}(x)) \in G * X / \sim_{H_2}$. It follows from (4.6) that

$$\eta^{p_{H_1}}(g, \pi_{H_1}(x)) = \eta_{G/H_1}(gH_1, \pi_{H_1}(x)) = \pi_{H_1}(\eta(hg, x)),$$

in a similar way $\eta^{p_{H_2}}(g, \pi_{H_2}(x)) = \eta_{G/H_2}(gH_2, \pi_{H_2}(x)) = \pi_{H_2}(\eta(hg, x))$. Therefore

$$\pi_{H_1, H_2}(g \cdot \pi_{H_1}(x)) = \pi_{H_1, H_2}(\pi_{H_1}(hg \cdot x)) = \pi_{H_2}(hg \cdot x) = g \cdot \pi_{H_2}(x),$$

and we conclude that π_{H_1, H_2} is a G -map. \square

4.1 Inverse limits

As an application of Theorem 4.1 we extend [2, Theorem 9] to the context of partial actions. Suppose that G is a compact group, let (I, \leq) be a directed set and consider an inverse system $\{G_i; p_i^j; I\}$ in the category of topological groups such that $G = \varprojlim G_i$, where $\{p_i : G \rightarrow G_i\}_{i \in I}$ is the family of projections such that $p_i^j \circ p_j = p_i$ for $i, j \in I$ and $i \leq j$. Take $i \in I$, then $H_i = \ker(p_i) = p_i^{-1}(\{e_i\})$ is a closed normal subgroup of G thus is compact and $H_j \leq H_i$ for every $i, j \in I$ with $i \leq j$. Let η be a partial action of G on X , now for $i \in I$ the group H_i acts partially on X via restriction, setting $X_i = X / \sim_{H_i}$ we denote by $\pi_i^j = \pi_{H_j, H_i} : X_j \rightarrow X_i$, the G -map defined in (4.10) and $\pi_i = \pi_{H_i} : X \rightarrow X_i$, the orbit equivalence map.

We proceed with the next.

Lemma 4.8. Following the notations above consider $i, j \in I$ with $i \leq j$ let $\eta : G * X \rightarrow X$ be a continuous partial action with $G * X$ is closed, then the family $\{\pi_i : X \rightarrow X_i\}_{i \in I}$ separates points of closed sets in X .

Proof. The proof follows the lines of [2, Lemma 3], where it is shown that $\pi_i(x) \notin \pi_i(C)$ for any $x \in X$ and $C \subseteq X$ a closed subset such that $x \notin C$. On the other hand, the fact that $G * X$ is closed is used to guarantee that $H_i * X = (G * X) \cap (H_i \times X)$, is closed in $H_i \times X$, which implies that π_i is closed, for any $i \in I$. Then the family $\{\pi_i : X \rightarrow X_i\}_{i \in I}$ separates points of closed sets in X , as desired. \square

Assuming X Hausdorff and letting $i, j, k \in I$ be such that $i \leq j \leq k$. For $x \in X$, we have $\pi_i^k(H_k^x \cdot x) = (\pi_i^j \circ \pi_j^k)(H_k^x \cdot x)$, and $\mathcal{X} = \{X_i, \pi_i^j, I\}$ is an inverse system of spaces endowed with partial actions of G .

We finish this work with the next.

Proposition 4.9. Under the assumptions above, let $\mathcal{X} = \{\varphi_i : \varprojlim X_i \rightarrow X_i\}_{i \in I}$ be the family of projections associated to $\varprojlim X_i$, if X is Hausdorff and $G * X$ is closed in $G \times X$, then the following assertions hold.

- (i) There is a partial action θ of G on $\varprojlim X_i$ such that X is G -equivalent to $\varprojlim X_i$.
- (ii) For any $j \in I$ the diagram

$$\begin{array}{ccc} G * \varprojlim X_i & \xrightarrow{\theta} & \varprojlim X_i \\ \text{id}_G \times \varphi_j \downarrow & & \downarrow \varphi_j \\ G * X_j & \xrightarrow{\eta^{p_{H_j}}} & X_j \end{array}$$

commutes, where $\eta^{p_{H_j}}$ is the partial action of G on X_j given by (4.9).

Proof. (i) It is not difficult to see that the family $\Pi = \{\pi_i : X \rightarrow X_i\}_{i \in I}$ is compatible with \mathcal{X} then by the universal property of the inverse limit there exists a continuous map $\lambda : X \rightarrow \varprojlim X_i$, such that $\varphi_i \circ \lambda = \pi_i$, for any $i \in I$. We shall prove that λ is a homeomorphism. First, by Lemma 4.8, the family Π separates points of closed sets in X , further by (i) in Theorem 3.2 each orbit space X_i is T_2 , then the map λ is an embedding. Let $(x_i)_{i \in I} \in \varprojlim X_i$, since $H_i * X$ is closed in $H_i \times X$ and by Lemma 3.1 the map π_i is perfect, we have $A_i = \pi_i^{-1}(x_i)$ is a compact subset of X , for all $i \in I$. Now write $\mathcal{A} = \{A_i\}_{i \in I}$ and take $i, j \in I$ such that $i \leq j$. For $y \in A_j$ we have $\pi_i(y) = \pi_i^j(\pi_j(y)) = \pi_i^j(x_j) = x_i$, and $A_j \subseteq A_i$, from this one concludes that \mathcal{A} has the finite intersection property, therefore $\bigcap_{i \in I} A_i \neq \emptyset$. Finally if $y \in \bigcap_{i \in I} A_i$, then $\pi_i(y) = x_i$, that is $(x_i)_{i \in I} = \lambda(y)$, therefore $\varprojlim X_i = \lambda(X)$ and λ is a homeomorphism. To define a partial action of G on $\varprojlim X_i$ we set $G * \varprojlim X_i = \left\{ (g, x) \in G \times \varprojlim X_i \mid (g, \lambda^{-1}(x)) \in G * X \right\}$ and

$$\theta : G * \varprojlim X_i \ni (g, x) \mapsto \lambda(g \cdot \lambda^{-1}(x)) \in \varprojlim X_i,$$

thus λ is a G -map which shows the first item.

(ii) Take $j \in I$ we first check that the map $\text{id}_G \times \varphi_j$ is well defined, that is for $(g, x) \in G * \varprojlim X_i$ one has that $(g, x_j) \in G * X_j$, where $x = (x_i)_{i \in I}$. Indeed, if $(g, x) \in G * \varprojlim X_i$ we get that $(g, \lambda^{-1}(x)) \in G * X$ which by item (ii) in Theorem 4.2 implies $(gH_j, \pi_j(\lambda^{-1}(x))) \in G/H_j * X_j$ and thus $(g, x_j) = (g, \pi_j(\lambda^{-1}(x))) \in G * X_j$ thanks to (4.8), and $\text{id}_G \times \varphi_j$ is well defined. To check that the diagram commutes observe that

$$\eta^{pH_j} \circ (\text{id}_G \times \varphi_j)(g, x) = \eta_{G/H_j}(gH_j, \pi_j(\lambda^{-1}(x))) = \pi_j((hg) \cdot \lambda^{-1}(x)),$$

where by (ii) of Theorem 4.2 the element $h \in H_j$ is such that $(hg, \lambda^{-1}(x)) \in G * X$. Since $\lambda^{-1}(x) \in X_{g^{-1}h^{-1}} \cap X_{g^{-1}}$ we get by item (ii) of Proposition 2.1 that $g \cdot \lambda^{-1}(x) \in X_{h^{-1}}$ thus $(hg) \cdot \lambda^{-1}(x) = h \cdot (g \cdot \lambda^{-1}(x))$ and $\pi_j((hg) \cdot \lambda^{-1}(x)) = \pi_j(g \cdot \lambda^{-1}(x))$. On the other hand $\varphi_j \circ \theta(g, x) = \varphi_j \lambda(g \cdot \lambda^{-1}(x)) = \pi_j(g \cdot \lambda^{-1}(x))$. Then $\eta^{pH_j} \circ (\text{id}_G \times \varphi_j) = \varphi_j \circ \theta$ which ends the proof. \square

References

- [1] F. Abadie, Enveloping actions and Takai duality for partial actions. *Journal of Funct. Anal.* (2003) 197: 14-67.
- [2] S.A. Antonyan, Equivariant embeddings into G-AR's *Glasnik Matematički*. 22 (42) (1987) 503-533.
- [3] A. Baraviera, R. Exel, D. Gonçalves, F. B. Rodrigues and D. Royer, Entropy for partial actions of \mathbb{Z} , *Proc. Amer. Math. Soc.* 150 (2022), 1089-1103
- [4] S. De Neymet and R. Jiménez, Introducción a los grupos topológicos de transformaciones. Aportaciones matemáticas: Textos, 2005.
- [5] M. Dokuchaev, Recent developments around partial actions, *São Paulo J. Math. Sci.* (2019) **13** (1) 195-247.
- [6] M. Dokuchaev and R. Exel, The ideal structure of algebraic partial crossed products. *Proc. Lond. Math. Soc.* (3) 115 (2017), no. 1, 91-134.
- [7] J. Dugundji, Topology, *Allyn and Bacon, Inc.*, Boston, 1966.

- [8] R. Exel, Circle actions on C^* -algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequences, *J. Funct. Anal.* **122** (1994), (3), 361 - 401.
- [9] R. Exel, Partial actions of group and actions of inverse semigroups, *Proc. Am. Math. Soc.* **126** (12) (1998) 3481–3494.
- [10] R. Exel, Partial dynamical systems, Fell bundles and applications, Mathematical surveys and monographs; volume 224, Providence, Rhode Island: American Mathematical Society, 2017.
- [11] R. Exel, T. Giordano, and D. Gonçalves, Enveloping algebras of partial actions as groupoid C^* -algebras, *J. Operator Theory* **65** (2011) 197–210.
- [12] D. Gonçalves, J. Öinert and D. Royer, Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics. *J. Algebra* **420** (2014), 201–216.
- [13] J. Kellendonk and M. V. Lawson, Partial Actions of Groups, *International Journal of Algebra and Computation* **14** (2004) 87-114.
- [14] L. Martínez, H. Pinedo and E. Ramírez, Partial actions of groups on hyperspaces, *Appl. Gen. Topol.* **23** (2022), no. 2, 255–268.
- [15] L. Martínez, H. Pinedo and A. Villamizar, Partial actions of groups on profinite spaces *arXiv*: 2101.04779v3.
- [16] H. Pinedo and C. Uzcátegui, Polish globalization of Polish group partial actions, *Math. Log. Quart.* **63** 6 (2017) 481–490.
- [17] J.C. Quigg and I. Raeburn, Characterizations of crossed products by partial actions. *J. Operator Theory* (1997) **37**: 311-340.
- [18] M. Tkačenko, L. Villegas, C. Hernández, and O. Rendón, Grupos topológicos. *SEP*, 1997.