ASYMPTOTIC BEHAVIOR AND LIOUVILLE-TYPE THEOREMS FOR AXISYMMETRIC STATIONARY NAVIER-STOKES EQUATIONS OUTSIDE OF AN INFINITE CYLINDER WITH A PERIODIC BOUNDARY CONDITION

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Abstract. We study the asymptotic behavior of solutions to the steady Navier-Stokes equations outside of an infinite cylinder in $\mathbb{R}^3$. We assume that the flow is periodic in x_3-direction and has no swirl. This problem is closely related with two-dimensional exterior problem. Under a condition on the generalized finite Dirichlet integral, we give a pointwise decay estimate of the vorticity at the spatial infinity. Moreover, we prove a Liouville-type theorem only from the condition of the generalized finite Dirichlet integral.

1. Introduction

Let $D = \{(x_1, x_2) \in \mathbb{R}^2; \sqrt{x_1^2 + x_2^2} > r_0\}$ with some constant $r_0 > 0$, and let $S^1 = [-\pi, \pi]$. We consider the stationary Navier-Stokes equations

$$\begin{cases} (v \cdot \nabla)v + \nabla p = \Delta v, \\ \nabla \cdot v = 0, \\ v(x_1, x_2, x_3 + 2\pi) = v(x_1, x_2, x_3), \end{cases} \qquad x \in D \times S^1, \tag{1.1}$$

where $v = v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ denote the velocity vector field and the scalar pressure at the point $x = (x_1, x_2, x_3)$, respectively. The last condition in (1.1) means that the flow is periodic in x_3-direction.

The problem (1.1) is closely related with the two-dimensional exterior problem

$$\begin{cases} (v \cdot \nabla)v + \nabla p = \Delta v, \\ \nabla \cdot v = 0, \end{cases} \qquad x = (x_1, x_2) \in D, \quad \text{where } v = (v_1, v_2). \tag{1.2}$$

For (1.2), Gilbarg and Weinberger [6] and the subsequent studies by Amick [1], Korobkov, Pileckas, and Russo [9, 10] investigated the asymptotic behavior of the solution at the spatial infinity under the assumption of finite Dirichlet integral

$$\int_D |\nabla v(x)|^2 \, dx < \infty. \tag{1.3}$$

More precisely, they proved that there exists a constant vector $v_\infty \in \mathbb{R}^2$ such that

$$\lim_{r \to \infty} \sup_{\theta \in (0, 2\pi)} |v(r, \theta) - v_\infty| = 0,$$

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where $r=\sqrt{x_1^2+x_2^2}$ and $\theta=\tan^{-1}(x_2/x_1)$. Moreover, the asymptotic behavior

$$\omega(r,\theta)=o(r^{-3/4}),\quad \nabla v(r,\theta)=o(r^{-3/4}(\log r))$$

uniformly in $\theta\in(0,2\pi)$ as $r\to\infty$ was obtained. Furthermore, the following Liouville type theorem was also proved: let (v,p) be a smooth solution to (1.2) in $\mathbb{R}^2$ with the finite Dirichlet integral $\nabla v\in L^2(\mathbb{R}^2)$. Then, v and p are constant.

Recently, the authors [11] studied (1.2) under the condition of generalized finite Dirichlet integral

$$\int_D |\nabla v(x)|^q\,dx<\infty \tag{1.4}$$

with some $q\in(2,\infty)$. From the viewpoint of the decay of ∇v at the spatial infinity, the condition (1.4) is weaker than (1.3). In [11], the following asymptotic behavior was obtained:

$$\omega(r,\theta)=o\left(r^{-\left(1/q+1/q^2\right)}\right),\quad \nabla v(r,\theta)=o\left(r^{-\frac{2}{q}-\frac{1}{q^2}+\frac{1}{2}}\right)$$

uniformly in $\theta\in(0,2\pi)$ as $r\to\infty$. Moreover, when $D=\mathbb{R}^2$, the Liouville type theorem was proved.

It is a natural question to ask whether similar properties as above are valid in three-dimensional domains. For this purpose, we consider the axially symmetric solution. Let us use the cylindrical coordinates $r=\sqrt{x_1^2+x_2^2}$, $\theta=\tan^{-1}(x_2/x_1)$, $z=x_3$, and let $e_r=(x_1/r,x_2/r,0)$, $e_\theta=(-x_2/r,x_1/r,0)$, and $e_z=(0,0,1)$. Using such $\{e_r,e_\theta,e_z\}$ as an orthogonal basis in $\mathbb{R}^3$, we express the vector field $v=v(x)$ as

$$v(x)=v^r(r,\theta,z)e_r+v^\theta(r,\theta,z)e_\theta+v^z(r,\theta,z)e_z.$$

A vector field v and a function $p=p(r,\theta,z)$ are called axially symmetric if they are independent of θ. The Navier-Stokes equations in the axially symmetric case is written as follows:

$$\begin{cases}
\left(v^r\partial_r+v^z\partial_z\right)v^r-\dfrac{\left(v^\theta\right)^2}{r}+\partial_r p=\left(\partial_r^2+\dfrac{1}{r}\partial_r+\partial_z^2-\dfrac{1}{r^2}\right)v^r,\\
\left(v^r\partial_r+v^z\partial_z\right)v^\theta+\dfrac{v^r v^\theta}{r}=\left(\partial_r^2+\dfrac{1}{r}\partial_r+\partial_z^2-\dfrac{1}{r^2}\right)v^\theta,\\
\left(v^r\partial_r+v^z\partial_z\right)v^z+\partial_z p=\left(\partial_r^2+\dfrac{1}{r}\partial_r+\partial_z^2\right)v^z,\\
\partial_r v^r+\dfrac{v^r}{r}+\partial_z v^z=0.
\end{cases}\tag{1.5}$$

Moreover, the vorticity $\omega=\nabla\times v$ is expressed by $\omega=\omega^r e_r+\omega^\theta e_\theta+\omega^z e_z$, where

$$\omega^r=-\partial_z v^\theta,\quad \omega^\theta=\partial_z v^r-\partial_r v^z,\quad \omega^z=\frac{1}{r}\partial_r(rv^\theta).$$

For the axially symmetric Navier-Stokes equations (1.5) in an exterior $\mathcal{D}$ of an axisymmetric body in $\mathbb{R}^3$, under the assumptions $\nabla v\in L^2(\mathcal{D})$, $v=0$ on $\partial\mathcal{D}$, and

$\lim_{|x|\to 0} v(x) = 0$, Choe and Jin [5] obtained the asymptotic behavior of the solution

$$|v^r| + |v^z| = O\left(\left(\frac{\log r}{r}\right)^{\frac{1}{2}}\right), \quad |v^\theta| = O\left(\frac{(\log r)^{\frac{1}{8}}}{r^{\frac{3}{8}}}\right),$$
$$|\omega^\theta| = O\left(r^{-\frac{7}{8}}\right)$$

uniformly in z as $r \to \infty$. Later on, in the whole space case $\mathbb{R}^3$, Weng [18] and Carrillo, Pan, and Zhang [3] improved the above behavior to

$$|v| = O\left(\left(\frac{\log r}{r}\right)^{\frac{1}{2}}\right),$$
$$|\nabla v^r| + |\nabla v^z| = O\left(r^{-\left(\frac{9}{8}\right)^-}\right), \quad \left|\nabla v^\theta\right| = O\left(r^{-\left(\frac{67}{64}\right)^-}\right),$$
$$|\omega^r| + |\omega^z| = O\left(\frac{(\ln r)^{\frac{11}{8}}}{r^{\frac{9}{8}}}\right), \quad \left|\omega^\theta\right| = O\left(\frac{(\ln r)^{\frac{3}{4}}}{r^{\frac{5}{4}}}\right) \tag{1.6}$$

uniformly in z as $r \to \infty$, where a^- denotes arbitrary constant less than a. Recently, Li and Pan [15] studied the case of generalized finite Dirichlet integral $\nabla v \in L^q(\mathbb{R}^3)$ with $q \in (2, \infty)$, and obtained the asymptotic behavior

$$\left|\omega^\theta\right| = O\left(r^{-\left(\frac{1}{q}+\frac{3}{q^2}\right)^-}\right), \quad |\omega^r| + |\omega^z| = O\left(r^{-\left(\frac{1}{q}+\frac{1}{q^2}+\frac{3}{q^3}\right)^-}\right) \tag{1.7}$$

for $q \in [3, \infty)$, provided that $\sup |v(r_*, z)| \leq C$ holds for some $r_* > 0$;

$$\left|\omega^\theta(r,z)\right| = O\left(r^{-\left(\frac{2}{q}\right)^-}\right), \quad |\omega^r(r,z)| + |\omega^z(r,z)| = O\left(r^{-\left(\frac{1}{q}+\frac{2}{q^2}\right)^-}\right)$$

for $q \in (2, 3)$, provided that $v^z \to 0$ as $r \to \infty$.

The Liouville type theorem for (1.5) in $\mathbb{R}^3$ is still an open question, but partial results were given by Lei and Zhang [13] with the additional conditions that rv^θ is bounded and the stream function is a BMO function. Later on, Lei, Zhang, and Zhao [14] showed that if $rv^\theta \in L^p(\mathbb{R}^3)$ with some $1 \leq p < \infty$, or if $\lim_{r\to\infty} rv^\theta = 0$, then v must be a constant. On the other hand, Zhao [19] proved that if $\nabla v \in L^2(\mathbb{R}^3)$, $\lim_{|x|\to\infty} v(x) = 0$, and if $|v| \leq C(1+r)^{-(2/3)^+}$ or $|\omega| \leq C(1+r)^{-(5/3)^+}$ holds, then v must be zero.

Recently, Carrillo, Pan, Zhang and Zhao [4] studied the Liouville type theorem for axially symmetric problem in a periodic slab domain

$$\begin{cases} (v\cdot\nabla)v + \nabla p = \Delta v \quad \text{in} \quad \mathbb{R}^2 \times S^1, \\ \nabla \cdot v = 0, \\ v\left(x_1, x_2, x_3\right) = v\left(x_1, x_2, x_3 + 2\pi\right), \\ \lim\limits_{|x|\to\infty} v = 0. \end{cases}$$

They proved that, Under the condition of finite Dirichlet integral $\nabla v \in L^2(\mathbb{R}^2 \times S^1)$, $v \equiv 0$.

In this paper, to investigat more detailed properties of the solution, we further assume that the solution has no swirl, that is, $v^\theta \equiv 0$. In this case, the Navier-Stokes equations are written as

$$\begin{cases} (v^r\partial_r + v^z\partial_z)v^r + \partial_r p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2 - \dfrac{1}{r^2}\right) v^r, \\ (v^r\partial_r + v^z\partial_z)v^z + \partial_z p = \left(\partial_r^2 + \dfrac{1}{r}\partial_r + \partial_z^2\right) v^z, \\ \partial_r v^r + \dfrac{v^r}{r} + \partial_z v^z = 0, \end{cases} \tag{1.8}$$

Moreover, since $v^\theta = 0$, the vorticity ω satisfies $\omega^r = \omega^z = 0$, and the equation of ω^θ becomes

$$(v^r\partial_r + v^z\partial_z)\omega^\theta - \frac{v^r}{r}\omega^\theta = \left(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2}\right)\omega^\theta.$$

Introducing a new unknown function $\Omega = \omega^\theta/r$, we write the equation above to simpler form

$$-\left(\partial_r^2 + \partial_z^2 + \frac{3}{r}\partial_r\right)\Omega + (v^r\partial_r + v^z\partial_z)\,\Omega = 0. \tag{1.9}$$

Here, we remark that the fact $\omega^\theta(0,z) = 0$ (cf. Liu and Wang [16, Lemma 2.2]) guarantees continuity of $\Omega(r,z)$ as $r \to +0$. This equation has similar properties to those of two-dimensional vorticity equation such as maximum principle. When the domain is an outside of a cylinder $D \times \mathbb{R}$, the authors [12] investigated the asymptotic behavior of Ω and ω^θ. Under the assumptions

$$\int_{D\times\mathbb{R}} |\nabla v(x)|^q \, dx < \infty \tag{1.10}$$

with some $q \in [2,\infty)$ and $|v(r,z)| \le C(1+r)^k$ with some $k \in \mathbb{R}$, they obtained

$$|\Omega| = o\left(r^{-\left(1+\frac{3}{q}-\frac{1}{2q}\max\{0,1+k\}\right)}\right), \quad |\omega^\theta| = o\left(r^{-\left(\frac{3}{q}-\frac{1}{2q}\max\{0,1+k\}\right)}\right) \tag{1.11}$$

uniformly in z as $r \to \infty$.

Concerning the Liouville type theorem in the swirl free case, Koch, Nadirashvili, Seregin, and Sverak [8] proved that if v is a bounded weak solution to (1.8) in $\mathbb{R}^3$, then v is a constant. Lei, Zhang, and Zhao [14] proved that if $\lim_{r\to\infty}|\Omega| = 0$ uniformly in z, then $\omega^\theta = 0$ and v is harmonic. In particular, if v is sublinear, that is, $|v| = o(|x|)$ as $|x| \to \infty$, then v must be a constant. Also, Pan and Li [17] showed that if v satisfies

$$|v^r(r,z)| + |v^z(r,z)| \le C(r+|z|+c)^\alpha,$$
$$|v^r(r,z)| + |v^z(r,z) - v^z(0,z)| \le Cr$$

with some $\alpha < 1$ and $C > 0$, then v must be a constant. In [12], the authors obtained the Liouville type result under $\nabla v \in L^q(\mathbb{R}^3)$ with some $q \in [2,\infty)$ and

$$v^r(r,z) \ge -C(1+r)^k, \quad (\operatorname{sign} z)v^z(r,z) \ge -C(1+r)^k$$

with some $k \le q+1$. This condition does not require upper bounds of v^r and $(\operatorname{sign} z)v^z$.

In this paper, we study the equation (1.1), that is, the case of periodic slab domain. We put the following condition of finite generalized Dirichlet integral.

$$\int_{D\times S^1} |\nabla v(x)|^q \, dx < \infty \tag{1.12}$$

with some $q \in [2, \infty)$. We remark that, formally, the assumption (1.12) is weaker than (1.10), since no decay is required for z-direction in (1.12). Our main result on the asymptotic behavior of Ω and ω^θ is as follows:

Theorem 1.1. *Let* (v, p) *be a smooth axisymmetric solution of* (1.1) *with no swirl satisfying* (1.12) *with some* $q \in [2, \infty)$. *Then, we have*

$$\begin{cases} \lim\limits_{r\to\infty} r^{1+2/q-1/q^2} \sup\limits_{z\in[-\pi,\pi]} |\Omega(r,z)| = 0 & (q > 2), \\ \lim\limits_{r\to\infty} r^{3/2} \sup\limits_{z\in[-\pi,\pi]} |\Omega(r,z)| = 0 & (q = 2) \end{cases}$$

and

$$\begin{cases} \lim\limits_{r\to\infty} r^{2/q-1/q^2} \sup\limits_{z\in[-\pi,\pi]} |\omega^\theta(r,z)| = 0 & (q > 2), \\ \lim\limits_{r\to\infty} r^{1/2} \sup\limits_{z\in[-\pi,\pi]} |\omega^\theta(r,z)| = 0 & (q = 2). \end{cases}$$

Remark 1.1. *Compared to the results in the non periodic cases* (1.6), (1.7)*, and* (1.11)*, the decay rates in Theorem 1.1 are in general slightly worse. This seems to be due to the weaker assumption* (1.12) *and that we do not assume any assumption on the pointwise behavior of* v.

Moreover, by modifying the method of the proof of Theorem 1.1, we obtain the Liouville type theorem in $\mathbb{R}^2 \times S^1$:

Theorem 1.2. *Let* $D = \mathbb{R}^2$ *and let* (v, p) *be an axisymmetric smooth solution to* (1.1) *with no swirl. We assume* (1.12) *with some* $q \in [2, \infty)$. *Then,* v *must be a constant vector.*

Remark 1.2. (i) *We do not assume the boundedness or pointwise behavior of* v *or* Ω*, and hence, Theorem 1.2 is not included by those of* [4, 8, 14, 17].
(ii) *Applying Theorem 1.1 in* $D = \mathbb{R}^2$*, we have, in particular,* $\lim_{r\to\infty} |\Omega| = 0$ *uniformly in* z. *This enables us to make use of the result by Lei, Zhang, and Zhao* [14] *so that the same conclusion as Theorem 1.2 may be obtained. However, we shall give another independent proof of Theorem 1.2 in Section 3. The advantage of our proof is that the argument is simpler and does not rely on the maximum principle.*
(iii) *Recently, Bang, Gui, Wang and Xie* [2] *proved the Liouville-type theorem in* $\mathbb{R}^2 \times S^1$ *in such a way that any bounded solution* v *has the form* $v = (0, 0, c)$ *with some constant* $c \in \mathbb{R}$ *provided either* v^r *or* v^θ *is axisymmetric, or provided* $\lim\limits_{r\to\infty} rv^r = 0$.

2. Proof of Theorem 1.1

In what follows, we denote by C generic constants. In particular, $C = C(*, \ldots, *)$ denotes constants depending only on the quantities appearing in the parenthesis. We sometimes use the symbols $\mathrm{div}_{r,z}, \nabla_{r,z}$, and $\Delta_{r,z}$, which mean the differential operators defined by $\mathrm{div}_{r,z}(f_1, f_2)(r, z) = \partial_r f_1(r, z) + \partial_z f_2(r, z)$, $\nabla_{r,z} f(r, z) = (\partial_r f, \partial_z f)(r, z)$, and $\Delta_{r,z} f(r, z) = \partial_r^2 f(r, z) + \partial_z^2 f(r, z)$, respectively.

Since

$$|\nabla_x v(x)|^2 = |\nabla_{r,z} v|^2 + \frac{1}{r^2}|v^r(r,z)|^2$$

for the axisymmetric vector field v without swirl, we first note that the condition (1.12) implies

$$\infty > \int_{D\times S^1} |\nabla v(x)|^q \, dx \geq C \int_{D\times S^1} \left[|\nabla_{r,z} v(r,z)|^q + r^{-q} |v^r(r,z)|^q \right] r \, drdz,$$

and hence, ω^θ and Ω satisfy

$$\int_{-\pi}^{\pi}\int_{r_0}^{\infty} |\omega^\theta(r,z)|^q r \, drdz < \infty, \quad \int_{-\pi}^{\pi}\int_{r_0}^{\infty} r^{q+1} |\Omega(r,z)|^q \, drdz < \infty, \tag{2.1}$$

respectively.

2.1. **Preliminary estimates.** When $q > 2$. following the argument of Li and Pan [15], we prepare a pointwise growth estimate in slab domain.

Lemma 2.1. *Let $r_0 > 0$ and let $f = f(r,z) : [r_0, \infty) \times [-\pi, \pi] \to \mathbb{R}$ satisfy*

$$\int_{-\pi}^{\pi}\int_{r_0}^{\infty} |\nabla_{r,z} f(r,z)|^q r \, drdz < \infty$$

with some $q \in (2, \infty)$. Then, there exists $C > 0$ depending only on r_0, q such that

$$|f(r,z)| \leq C(1+r)^{1-2/q}.$$

Proof. Let $R > 4r_0$ be a parameter, and consider the transformation $r = R\rho$ and

$$\phi(\rho, z) = f(R\rho, z).$$

Since $2 < q < \infty$, it follows from Morrey's inequality (see,e.g., Gilbarg-Trudinger [7, Theorem 7.17]) that for $(\rho_1, z_1), (\rho_2, z_2) \in [1/4, 4] \times [-\pi, \pi]$,

$$|\phi(\rho_1, z_1) - \phi(\rho_2, z_2)| \leq C \left(\int_{-\pi}^{\pi}\int_{1/4}^{4} |\nabla_{\rho,z}\phi(\rho,z)|^q \, d\rho dz \right)^{1/q}$$

with some constant $C = C(q) > 0$. Since $R \geq 4r_0$ and $\nabla_{\rho,z}\phi(\rho,z) = (R\partial_r f, \partial_z f)(R\rho, z)$, we obtain

$$\begin{aligned} C \left(\int_{-\pi}^{\pi}\int_{1/4}^{4} |\nabla_{\rho,z}\phi(\rho,z)|^q \, d\rho dz \right)^{1/q} &\leq CR \left(\int_{-\pi}^{\pi}\int_{1/4}^{4} |\nabla_{r,z} f(R\rho,z)|^q \, d\rho dz \right)^{1/q} \\ &\leq CR^{1-1/q} \left(\int_{-\pi}^{\pi}\int_{R/4}^{4R} |\nabla_{r,z} f(r,z)|^q \, drdz \right)^{1/q} \\ &\leq CR^{1-2/q} \left(\int_{-\pi}^{\pi}\int_{R/4}^{4R} |\nabla_{r,z} f(r,z)|^q r \, drdz \right)^{1/q}. \end{aligned}$$

From this, putting $r_1 = R\rho_1$ and $r_2 = R\rho_2$, we have, for any $(r_1, z_1), (r_2, z_2) \in [-R/4, 4R] \times [-\pi, \pi]$,

$$|f(r_1, z_1) - f(r_2, z_2)| \leq CR^{1-2/q} \left(\int_{-\pi}^{\pi}\int_{r_0}^{\infty} |\nabla_{r,z} f(r,z)|^q r \, drdz \right)^{1/q},$$

where we note that the constant C is independent of R. Now, fix a point $r_* \in (r_0, \infty)$, and let $r > r_*, z \in S^1$. There exists a nonnegative integer n such that $2^n \le r/r_* \le 2^{n+1}$. Therefore, we have

$$\begin{aligned} |f(r,z) - f(r_*,z)| &\le \sum_{j=0}^{n-1} \left| f\left(\frac{r}{2^j}, z\right) - f\left(\frac{r}{2^{j+1}}, z\right)\right| + \left| f\left(\frac{r}{2^n}, z\right) - f(r_*, z)\right| \\ &\le C \sum_{j=0}^{\infty} \left(\frac{r}{2^j}\right)^{1-2/q} \\ &\le C r^{1-2/q}. \end{aligned}$$

Furthermore, it follows again from Morrey's inequality that

$$|f(r_*,z) - f(r_*,0)| \le C.$$

Now, by the above two estimates we have

$$|f(r,z) - f(r_*,0)| \le C(1+r)^{1-2/q}.$$

This completes the proof. □

For the case $q = 2$, we recall the following estimate proved by Gilbarg and Weinberger [6, Lemma 2.1].

Lemma 2.2. *Let $f = f(r,z) \in C^1((r_0,\infty) \times S^1)$ satisfy*

$$\int_{-\pi}^{\pi} \int_{r_0}^{\infty} |\partial_r f(r,z)|^2 r \, dr dz < \infty.$$

Then, we have

$$\lim_{r\to\infty} \frac{1}{\log r} \int_{-\pi}^{\pi} |f(r,z)|^2 \, dz = 0.$$

2.2. **L^q-energy estimates.**

Lemma 2.3. *Suppose that the assumptions of Theorem 1.1 holds. Let $r_1 > r_0$, and let $\alpha \in \mathbb{R}$ be*

$$\alpha = \begin{cases} q + 1 - 2/q & (q > 2), \\ 1 & (q = 2). \end{cases}$$

Then, we have

$$\int_{-\pi}^{\pi} \int_{r_1}^{\infty} r^{\alpha} |\Omega(r,z)|^{q-2} |\nabla \Omega(r,z)|^2 \, dr dz \le C \int_{-\pi}^{\pi} \int_{r_0}^{\infty} r^{q+1} |\Omega(r,z)|^q \, dr dz,$$

where $C = C(q, \alpha, r_0, r_1)$.

Proof. Let r_2 be $r_1 > r_2 > r_0$, and let $\xi_1(r), \xi_2(r) \in C^\infty((0,\infty))$ be nonnegative functions satisfying

$$\xi_1(r) = \begin{cases} 1 & (r \ge r_1), \\ 0 & (r_0 < r \le r_2), \end{cases} \qquad \xi_2(r) = \begin{cases} 1 & (0 \le r \le 1/2), \\ \text{decreasing} & (1/2 < r < 1), \\ 0 & (r \ge 1). \end{cases} \tag{2.2}$$

For a parameter $R > \max\{2r_0, 1\}$, we define a cut-off function

$$\eta_R(r) = \xi_1(r) \xi_2\left(\frac{r}{R}\right).$$

Then, we see that

$$\begin{aligned} &|\nabla_{r,z}\eta_R(r)| \le C(|\xi_1'(r)| + R^{-1}), \\ &|\Delta_{r,z}\eta_R(r)| \le C(|\xi_1''(r)| + R^{-1}|\xi_1'(r)| + R^{-2}). \end{aligned} \tag{2.3}$$

Let $h = h(\Omega)$ be a C^1 and piecewise C^2 function determined later. Based on the idea of [6, p.385], we consider the following identity.

$$\begin{aligned} &\mathrm{div}_{r,z}\left[(r^\alpha\eta_R)\nabla_{r,z}h(\Omega) - h(\Omega)\nabla_{r,z}(r^\alpha\eta_R) - (r^\alpha\eta_R)h(\Omega)v\right] \\ &= r^\alpha\eta_R h''(\Omega)|\nabla_{r,z}\Omega|^2 - h(\Omega)\left[\Delta_{r,z}(r^\alpha\eta_R) + v\cdot\nabla_{r,z}(r^\alpha\eta_R)\right] \\ &\quad + r^\alpha\eta_R h'(\Omega)\left[\Delta_{r,z}\Omega - v\cdot\nabla_{r,z}\Omega\right] - r^\alpha\eta_R h(\Omega)\,\mathrm{div}_{r,z}\,v. \end{aligned}$$

A straightforward computation shows

$$\begin{aligned} \Delta_{r,z}\left(r^\alpha\eta_R\right) &= \alpha(\alpha-1)r^{\alpha-2}\eta_R + 2\alpha r^{\alpha-1}\partial_r\eta_R + r^\alpha\partial_r^2\eta_R, \\ v\cdot\nabla_{r,z}\left(r^\alpha\eta_R\right) &= \alpha r^{\alpha-1}v^r\eta_R + r^\alpha v^r\partial_r\eta_R. \end{aligned} \tag{2.4}$$

Applying the equation (1.9), we calculate

$$\begin{aligned} &r^\alpha\eta_R h'(\Omega)\left[\Delta_{r,z}\Omega - v\cdot\nabla_{r,z}\Omega\right] \\ &= r^\alpha\eta_R h'(\Omega)\left(-\frac{3}{r}\partial_r\Omega\right) \\ &= -3\partial_r\left(r^{\alpha-1}\eta_R h(\Omega)\right) + 3\partial_r\left(r^{\alpha-1}\eta_R\right)h(\Omega), \\ &= -3\partial_r\left(r^{\alpha-1}\eta_R h(\Omega)\right) + 3(\alpha-1)r^{\alpha-2}\eta_R h(\Omega) + 3r^{\alpha-1}\partial_r\eta_R h(\Omega). \end{aligned} \tag{2.5}$$

Also, from the third line of (1.8), we obtain

$$-r^\alpha\eta_R h(\Omega)\,\mathrm{div}_{r,z}\,v = r^{\alpha-1}\eta_R h(\Omega)v^r. \tag{2.6}$$

Therefore, we see from (2.4), (2.5), and (2.6) that

$$\begin{aligned} &\mathrm{div}_{r,z}\left[(r^\alpha\eta_R)\nabla_{r,z}h(\Omega) - h(\Omega)\nabla_{r,z}(r^\alpha\eta_R) - (r^\alpha\eta_R)h(\Omega)v\right] \\ &\quad + 3\partial_r(r^{\alpha-1}\eta_R h(\Omega)) \\ &= r^\alpha\eta_R h''(\Omega)|\nabla_{r,z}\Omega|^2 - h(\Omega)\left[r^\alpha\partial_r^2\eta_R + (2\alpha-3)r^{\alpha-1}\partial_r\eta_R\right] \\ &\quad - h(\Omega)\left[r^\alpha v\partial_r\eta_R + (\alpha-1)r^{\alpha-1}v^r\eta_R\right] \\ &\quad - (\alpha-3)(\alpha-1)h(\Omega)r^{\alpha-2}\eta_R. \end{aligned} \tag{2.7}$$

Now, we divide the proof into two cases for $q > 2$ and for $q = 2$. First, when $q > 2$, we take

$$h(\Omega) = |\Omega|^q.$$

Integrating both sides of the above identity (2.7) over $D\times S^1$, we have by periodicity in z-direction that

$$\begin{aligned} &q(q-1)\int_{D\times S^1} r^\alpha\eta_R|\Omega(r,z)|^{q-2}|\nabla_{r,z}\Omega(r,z)|^2\,drdz \\ &= \int_{D\times S^1}|\Omega(r,z)|^q\left[r^\alpha\partial_r^2\eta_R + (2\alpha-3)r^{\alpha-1}\partial_r\eta_R\right]\,drdz \\ &\quad + \int_{D\times S^1}|\Omega(r,z)|^q\left[r^\alpha v\partial_r\eta_R + (\alpha-1)r^{\alpha-1}v^r\eta_R\right]\,drdz \\ &\quad + (\alpha-3)(\alpha-1)\int_{D\times S^1}|\Omega(r,z)|^q r^{\alpha-2}\eta_R\,drdz. \end{aligned}$$

Applying (2.3) and Lemma 2.1, we have by the property of the support of η_R and its derivatives that

$$\begin{aligned}
&\int_{D\times S^1} r^\alpha \eta_R |\Omega(r,z)|^{q-2}|\nabla_{r,z}\Omega(r,z)|^2\,drdz\\
&\le C\sum_{l=0}^{2} R^{-l}\int_{-\pi}^{\pi}\int_{\frac{R}{2}}^{R} r^{\alpha-2+l}|\Omega(r,z)|^q\,drdz\\
&\quad + CR^{-1}\int_{-\pi}^{\pi}\int_{\frac{R}{2}}^{R} r^{1-2/q+\alpha}|\Omega(r,z)|^q\,drdz\\
&\quad + C\int_{-\pi}^{\pi}\int_{r_0}^{R} (r^{-2/q+\alpha}+r^{\alpha-2})|\Omega(r,z)|^q\,drdz\\
&\quad + C\int_{-\pi}^{\pi}\int_{r_2}^{r_1} |\Omega(r,z)|^q\,drdz\\
&\le C(q,\alpha,r_0,r_1)\int_{D\times S^1} r^{-2/q+\alpha}|\Omega(r,z)|^q\,drdz.
\end{aligned}$$

From the assumption $\alpha = q+1+2/q$, we obtain

$$\int_{D\times S^1} r^\alpha \eta_R |\Omega(r,z)|^{q-2}|\nabla_{r,z}\Omega(r,z)|^2\,drdz \le C\int_{D\times S^1} r^{q+1}|\Omega(r,z)|^q\,drdz.$$

Finally, noting $\eta_R = 1$ on $[r_1, R/2]$ and letting $R\to\infty$, we conclude that

$$\int_{D_1\times S^1} r^\alpha |\Omega(r,z)|^{q-2}|\nabla_{r,z}\Omega(r,z)|^2\,drdz \le C\int_{D\times S^1} r^{q+1}|\Omega(r,z)|^q\,drdz.$$

Next, we consider the case $q=2$. In this case, we take $\alpha = 1$ and

$$h(\Omega) = \begin{cases} |\Omega|^2 & (|\Omega|\le A),\\ A(2|\Omega|-A) & (|\Omega|\ge A). \end{cases}$$

with a positive constant A. Then, $h(\Omega)$ is C^1 and piecewise C^2, and $|h(\Omega)| \le \min\{|\Omega|^2, 2A|\Omega|\}$ holds. In this case, the identity (2.7) has the form

$$\begin{aligned}
&\mathrm{div}_{r,z}\left[(r\eta_R)\nabla_{r,z}h(\Omega) - h(\Omega)\nabla_{r,z}(r\eta_R) - (r\eta_R)h(\Omega)v\right] + 3\partial_r(\eta_R h(\Omega))\\
&= r\eta_R h''(\Omega)|\nabla_{r,z}\Omega|^2 - h(\Omega)\left[r\partial_r^2\eta_R - \partial_r\eta_R\right]\\
&\quad - h(\Omega) r v^r \partial_r\eta_R.
\end{aligned}$$

Integrating both sides of the identity above over $D\times S^1$, we have

$$\begin{aligned}
2\int_{\substack{D\times S^1\\ |\Omega|\le A}} r\eta_R|\nabla_{r,z}\Omega|^2\,drdz = &\int_{D\times S^1} h(\Omega)\left[r\partial_r^2\eta_R - \partial_r\eta_R\right]\,drdz\\
&+\int_{D\times S^1} h(\Omega) r v^r\partial_r\eta_R\,drdz.
\end{aligned}\tag{2.8}$$

The first term of the right-hand side can be estimated in the same way as before. For the second term, we put

$$\overline{v^r} = \frac{1}{2\pi}\int_{-\pi}^{\pi} v^r(r,z)\,dz.$$

Then, we have

$$\begin{aligned}\int_{D\times S^1} h(\Omega) r v^r \partial_r \eta_R \, drdz &= \int_{D\times S^1} h(\Omega) r (v^r - \overline{v^r}) \partial_r \eta_R \, drdz \\ &\quad + \overline{v^r} \int_{D\times S^1} h(\Omega) r \partial_r \eta_R \, drdz \\ &=: I + II.\end{aligned}$$

For the term I, we use (2.3) and $|h(\Omega)| \le \min\{|\Omega|^2, 2A|\Omega|\}$, and we apply the Schwarz and the Wirtinger inequality to obtain

$$\begin{aligned}|I| &\le CAR^{-2} \left(\int_{-\pi}^{\pi} \int_{\frac{R}{2}}^{R} r^3 |\Omega|^2 \, drdz \right)^{1/2} \left(\int_{\frac{R}{2}}^{R} r \left(\int_{-\pi}^{\pi} |\partial_z v^r|^2 \, dz \right) dr \right)^{1/2} \\ &\quad + C \int_{-\pi}^{\pi} \int_{r_2}^{r_1} |\Omega(r,z)|^2 \, drdz \\ &\le CAR^{-2} \left(\int_{-\pi}^{\pi} \int_{\frac{R}{2}}^{R} r^3 |\Omega|^2 \, drdz \right)^{1/2} \|\nabla v\|_{L^2} \\ &\quad + C \int_{-\pi}^{\pi} \int_{r_2}^{r_1} |\Omega(r,z)|^2 \, drdz.\end{aligned}$$

For the term II, we use (2.3) and $|h(\Omega)| \le |\Omega|^2$, and we apply Lemma 2.2 to deduce

$$|II| \le C(\log R)^{1/2} R^{-3} \int_{-\pi}^{\pi} \int_{\frac{R}{2}}^{R} r^3 |\Omega|^2 \, drdz + C \int_{-\pi}^{\pi} \int_{r_2}^{r_1} |\Omega(r,z)|^2 \, drdz$$

Applying these estimates to (2.8), we conclude

$$\begin{aligned}\int_{\substack{D\times S^1 \\ |\Omega|\le A}} r\eta_R |\nabla_{r,z} \Omega|^2 \, drdz &\le CAR^{-2} \left(\int_{-\pi}^{\pi} \int_{\frac{R}{2}}^{R} r^3 |\Omega|^2 \, drdz \right)^{1/2} \|\nabla v\|_{L^2} \\ &\quad + C(\log R)^{1/2} R^{-3} \int_{-\pi}^{\pi} \int_{\frac{R}{2}}^{R} r^3 |\Omega|^2 \, drdz \\ &\quad + C \int_{-\pi}^{\pi} \int_{r_2}^{r_1} |\Omega(r,z)|^2 \, drdz.\end{aligned}$$

Letting $R \to \infty$ yields

$$\int_{\substack{D_1\times S^1 \\ |\Omega|\le A}} r |\nabla_{r,z} \Omega|^2 \, drdz \le C \int_{-\pi}^{\pi} \int_{r_2}^{r_1} |\Omega(r,z)|^2 \, drdz.$$

Since the right-hand side is independent of A, we obtain

$$\int_{-\pi}^{\pi} \int_{r_1}^{\infty} r |\nabla_{r,z} \Omega|^2 \, drdz \le C \int_{-\pi}^{\pi} \int_{r_0}^{\infty} r^3 |\Omega(r,z)|^2 \, drdz$$

This completes the proof of Lemma 2.3. □

Remark 2.1. *In the case $q = 2$, we have to take $\alpha = 1$, otherwise the term*

$$(\alpha - 1) \int_{D\times S^1} |\Omega(r,z)|^2 r^{\alpha-1} v^r \eta_R \, drdz$$

remains in the identity (2.8). *When $\alpha \ne 1$, the above proof does not seem to work well.*

2.3. **Pointwise behavior via maximum principle.** From (2.1) and Lemma 2.3, we have

$$\int_{-\pi}^{\pi}\int_{r_1}^{\infty} r^{q+1}|\Omega(r,z)|^q\,drdz<\infty,\qquad \int_{-\pi}^{\pi}\int_{r_1}^{\infty} r^{\alpha}|\Omega(r,z)|^{q-2}|\nabla\Omega(r,z)|^2\,drdz<\infty$$

with

$$\alpha=\begin{cases} q+1-2/q & (q>2),\\ 1 & (q=2).\end{cases}$$

The following proposition shows that the above bounds with the maximum principle yield a pointwise behavior of Ω as $r\to\infty$.

Proposition 2.4. *Let $r_1>0$, and let $f=f(r,z)\in C^1((r_1,\infty)\times S^1)$ be 2π-periodic in z and satisfy*

$$\int_{-\pi}^{\pi}\int_{r_1}^{\infty} r^{\alpha}|f(r,z)|^{q-2}|\nabla_{r,z}f(r,z)|^2\,drdz+\int_{-\pi}^{\pi}\int_{r_1}^{\infty} r^{q+1}|f(r,z)|^q\,drdz<\infty$$

with some $q\in[2,\infty)$ and $\alpha\in\mathbb{R}$. Moreover, we assume that f satisfies the maximum principle in $(r_1,\infty)\times S^1$, that is, for every $\rho_1,\rho_2\in(r_1,\infty)$ with $\rho_1<\rho_2$, the function f restricted in $[\rho_1,\rho_2]\times S^1$ attains its maximum at the boundary $r=\rho_1$ or $r=\rho_2$. Then, we have

$$\lim_{r\to\infty} r^{\beta/q}\sup_{-\pi\le z\le\pi}|f(r,z)|=0$$

with $\beta=\min\{q+2,\frac{\alpha+q+3}{2}\}$.

Proof. For $n\in\mathbb{N}$ satisfying $2^n>r_1$, by the assumption and the Schwarz inequality, we have

$$\begin{aligned}\infty &> \int_{2^n}^{2^{n+1}}\int_{-\pi}^{\pi}|f|^{q-2}\left(r^{q+1}|f|^2+r^{\alpha}|\nabla_{r,z}f|^2\right)dzdr\\ &\ge \int_{2^n}^{2^{n+1}}\int_{-\pi}^{\pi}|f|^{q-2}\left(r^{q+1}|f|^2+r^{\frac{\alpha+q+1}{2}}|f||\nabla_{r,z}f|\right)dzdr\\ &= \int_{2^n}^{2^{n+1}}\frac{dr}{r}\int_{-\pi}^{\pi}|f|^{q-2}\left(r^{q+2}|f|^2+r^{\frac{\alpha+q+3}{2}}|f||\nabla_{r,z}f|\right)dz.\end{aligned}$$

By the mean value theorem for integration, there exists a sequence $r_n\in[2^n,2^{n+1}]$ such that the right-hand side of the above equals to

$$\log 2\int_{-\pi}^{\pi}|f(r_n,z)|^{q-2}\left(r_n^{q+2}|f(r_n,z)|^2+r_n^{\frac{\alpha+q+3}{2}}|f(r_n,z)||\nabla_{r,z}f(r_n,z)|\right)dz.$$

By the fundamental theorem of calculus, for $z_1,z_2\in S^1$ we have with $\beta=\min\{q+2,\frac{\alpha+q+3}{2}\}$ that

$$\begin{aligned}r_n^{\beta}|f(r_n,z_1)|^q-r_n^{\beta}|f(r_n,z_2)|^q &= r_n^{\beta}\left|\int_{z_2}^{z_1}\partial_z|f(r_n,z)|^q\,dz\right|\\ &\le Cr_n^{\beta}\int_{-\pi}^{\pi}|f(r_n,z)|^{q-1}|\nabla_{r,z}f(r_n,z)|\,dz.\end{aligned}$$

Integrating both sides of the above inequality on $[-\pi, \pi]$ with respect to z_2, we deduce

$$r_n^\beta |f(r_n, z_1)|^q \le C r_n^\beta \int_{-\pi}^{\pi} |f(r_n, z)|^q \, dz + C r_n^\beta \int_{-\pi}^{\pi} |f(r_n, z)|^{q-1} |\nabla_{r,z} f(r_n, z)| \, dz.$$

Noting $\beta = \min\{q+2, \frac{\alpha+q+3}{2}\}$ and by the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} r_n^\beta \sup_{-\pi\le z\le\pi} |f(r_n, z)|^q = 0.$$

For $r \in (r_1, \infty)$, we take $n \in \mathbb{N}$ so that $r \in [r_n, r_{n+1}]$ and apply the maximum principle to obtain

$$\sup_{[r_n, r_{n+1}]\times S^1} r^{\beta/q} |f(r,z)| \le C \max\left\{ \sup_{-\pi\le z\le\pi} r_n^{\beta/q} |f(r_n, z)|, \sup_{-\pi\le z\le\pi} r_{n+1}^{\beta/q} |f(r_{n+1}, z)| \right\} \to 0 \quad (n\to\infty).$$

This implies

$$\lim_{r\to\infty} r^{\beta/q} \sup_{-\pi\le z\le\pi} |f(r,z)| = 0,$$

and the proof is complete. □

2.4. **Proof of Theorem 1.1.** By the assumptions on Theorem 1.1 and the result of Lemma 2.3, we can apply Proposition 2.4 with

$$\alpha = \begin{cases} q + 1 - 2/q & (q > 2), \\ 1 & (q = 2). \end{cases}$$

In this case, we have

$$\beta = \min\left\{q+2, \frac{\alpha+q+3}{2}\right\} = \begin{cases} q + 2 - \dfrac{1}{q} & (q > 2), \\ 3 & (q = 2). \end{cases}$$

Therefore, we obtain

$$\begin{cases} \lim_{r\to\infty} r^{1+2/q-1/q^2} \sup_{z\in[-\pi,\pi]} |\Omega(r,z)| = 0 & (q > 2), \\ \lim_{r\to\infty} r^{3/2} \sup_{z\in[-\pi,\pi]} |\Omega(r,z)| = 0 & (q = 2). \end{cases}$$

Concerning the estimate of ω^θ, we have by the relation $\omega^\theta = r\Omega$ that

$$\begin{cases} \lim_{r\to\infty} r^{2/q-1/q^2} \sup_{z\in[-\pi,\pi]} |\omega^\theta(r,z)| = 0 & (q > 2), \\ \lim_{r\to\infty} r^{1/2} \sup_{z\in[-\pi,\pi]} |\omega^\theta(r,z)| = 0 & (q = 2). \end{cases}$$

This completes the proof of Theorem 1.1.

3. Proof of Theorem 1.2

We first remark that the maximum principle for Ω cannot be directly applicable due to the presence of z-axis. Therefore, we repeat a similar argument of the proof of Lemma 2.3 with $r_0 = 0$. We first note that, corresponding to the property (2.1), we have

$$\int_{-\pi}^{\pi}\int_0^{\infty} r^{q+1}|\Omega|^q \, drdz < \infty. \tag{3.1}$$

Let us first take $\alpha \in (1,3)$, and let $h(\Omega)$ be a nonnegative, C^1, and piecewise C^2 function such that h'' is locally bounded. We use the cut-off function ξ_2 in (2.2), and for $R > 0$ to define

$$\eta_R(r,z) = \xi_2\left(\frac{r}{R}\right).$$

In the same way to the proof of Lemma 2.3, we start with the same identity as (2.7):

$$\begin{aligned}
&\mathrm{div}_{r,z}\left[(r^\alpha\eta_R)\nabla_{r,z}h(\Omega) - h(\Omega)\nabla_{r,z}(r^\alpha\eta_R) - (r^\alpha\eta_R)h(\Omega)v\right] \\
&\quad + 3\partial_r(r^{\alpha-1}\eta_R h(\Omega)) \\
&= r^\alpha\eta_R h''(\Omega)|\nabla_{r,z}\Omega|^2 - h(\Omega)\left[r^\alpha\partial_r^2\eta_R + (2\alpha-3)r^{\alpha-1}\partial_r\eta_R\right] \\
&\quad - h(\Omega)\left[r^\alpha v\partial_r\eta_R + (\alpha-1)r^{\alpha-1}v^r\eta_R\right] \\
&\quad - (\alpha-3)(\alpha-1)h(\Omega)r^{\alpha-2}\eta_R.
\end{aligned}$$

Now, we integrate both sides of the above identity over $[0,R]\times S^1$. Since $\alpha > 1$, we have

$$\left(r^\alpha\eta_R\nabla_{r,z}h(\Omega) - h(\Omega)\nabla_{r,z}\left(r^\alpha\eta_R\right) - \left(r^\alpha\eta_R\right)h\left(\Omega\right)v + 3r^{\alpha-1}\eta_R h(\Omega)\right)\Big|_{r=0} = 0.$$

From this and periodicity in z-direction, we deduce

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_0^R r^\alpha\eta_R h''(\Omega)|\nabla_{r,z}\Omega(r,z)|^2\,drdz \\
&= \int_{-\pi}^{\pi}\int_0^R h(\Omega)\left[r^\alpha\partial_r^2\eta_R + (2\alpha-3)r^{\alpha-1}\partial_r\eta_R\right]\,drdz \\
&\quad + \int_{-\pi}^{\pi}\int_0^R h(\Omega)\left[r^\alpha v^r\partial_r\eta_R + (\alpha-1)r^{\alpha-1}v^r\eta_R\right]\,drdz \\
&\quad + (\alpha-3)(\alpha-1)\int_{-\pi}^{\pi}\int_0^R r^{\alpha-2}h(\Omega)\eta_R\,drdz.
\end{aligned}$$

Since $h(\Omega)$ is nonnegative and $\alpha \in (1,3)$, the last term can be dropped, and we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi}\int_0^R r^\alpha\eta_R h''(\Omega)|\nabla_{r,z}\Omega(r,z)|^2\,drdz \\
&\le \int_{-\pi}^{\pi}\int_0^R h(\Omega)\left[r^\alpha\partial_r^2\eta_R + (2\alpha-3)r^{\alpha-1}\partial_r\eta_R\right]\,drdz \\
&\quad + \int_{-\pi}^{\pi}\int_0^R h(\Omega)\left[r^\alpha v^r\partial_r\eta_R + (\alpha-1)r^{\alpha-1}v^r\eta_R\right]\,drdz.
\end{aligned}$$

Moreover, since h'' is locally bounded, by letting $\alpha \to 1+0$ in the above estimate, we have

$$\begin{aligned} &\int_{-\pi}^{\pi}\int_0^R r\eta_R h''(\Omega)|\nabla_{r,z}\Omega(r,z)|^2\,drdz \\ &\le \int_{-\pi}^{\pi}\int_0^R h(\Omega)\left[r\partial_r^2\eta_R - \partial_r\eta_R\right]\,drdz \\ &\quad + \int_{-\pi}^{\pi}\int_0^R h(\Omega) r v^r \partial_r\eta_R\,drdz. \end{aligned} \tag{3.2}$$

Now, we estimate the right-hand side. For the case $q > 2$, we take $h(\Omega) = |\Omega|^q$. Then, by using (2.3) and Lemma 2.1, the right-hand side of (3.2) is estimated by

$$C\int_{-\pi}^{\pi}\int_{R/2}^R |\Omega(r,z)|^q \left(rR^{-2} + R^{-1} + r(1+r)^{1-2/q}R^{-1}\right)\,drdz.$$

Using the condition (3.1), we easily see that the above integral tends to 0 as $R \to \infty$. Namely, letting $R \to \infty$ in (3.2), we have

$$\int_{-\pi}^{\pi}\int_0^{\infty} r|\Omega(r,z)|^{q-2}|\nabla_{r,z}\Omega(r,z)|^2\,drdz = 0.$$

This implies $\nabla|\Omega|^{q/2} = 0$ and Ω is constant. By the bound (3.1), we have $\Omega = 0$, and so is ω^θ.

Next, when $q = 2$, we take

$$h(\Omega) = \begin{cases} |\Omega|^2 & (|\Omega| \le A), \\ A(2|\Omega| - A) & (|\Omega| \ge A), \end{cases}$$

with $A > 0$, and perform the completely same argument as in the proof of Lemma 2.3. Hence, we can obtain

$$\begin{aligned} \int_{\substack{(0,\infty)\times S^1 \\ |\Omega|\le A}} r\eta_R|\nabla_{r,z}\Omega|^2\,drdz \le CAR^{-2}\left(\int_{-\pi}^{\pi}\int_{\frac{R}{2}}^R r^3|\Omega|^2\,drdz\right)^{1/2}\|\nabla v\|_{L^2} \\ + C(\log R)^{1/2}R^{-3}\int_{-\pi}^{\pi}\int_{\frac{R}{2}}^R r^3|\Omega|^2\,drdz. \end{aligned}$$

Letting first $R \to \infty$, and then $A \to \infty$, we have

$$\int_{-\pi}^{\pi}\int_0^{\infty} r|\nabla_{r,z}\Omega(r,z)|^2\,drdz = 0,$$

which also implies $\nabla\Omega = 0$. Thus, in the same way as before, we see that $\omega^\theta = 0$.

Finally, the formula

$$0 = \nabla \times (\nabla \times v) = \nabla(\nabla\cdot v) - \Delta v$$

and $\nabla\cdot v = 0$ lead to $\Delta v = 0$. Therefore, the assumption $\nabla v \in L^q(\mathbb{R}^2 \times S^1)$ and the periodicity of v with respect to z show that v must be constant. This completes the proof. □

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