Minimal Binary Linear Codes from Vectorial Boolean Functions

Yanjun Li, Jie Peng^{*}, Haibin Kan, and Lijing Zheng

Abstract

Recently, much progress has been made to construct minimal linear codes due to their preference in secret sharing schemes and secure two-party computation. In this paper, we put forward a new method to construct minimal linear codes by using vectorial Boolean functions. Firstly, we give a necessary and sufficient condition for a generic class of linear codes from vectorial Boolean functions to be minimal. Based on that, we derive some new three-weight minimal linear codes and determine their weight distributions. Secondly, we obtain a necessary and sufficient condition for another generic class of linear codes from vectorial Boolean functions to be minimal and to be violated the AB condition. As a result, we get three infinite families of minimal linear codes violating the AB condition. To the best of our knowledge, this is the first time that minimal liner codes are constructed from vectorial Boolean functions. Compared with other known ones, in general the minimal liner codes obtained in this paper have higher dimensions.

Index Terms: Minimal linear code, vectorial Boolean function, three-weight code, AB condition, secret sharing.

1 Introduction

Linear codes, especially those with few weights, paly an important role in consumer electronics, communication and data storage system. Recently, a special kind of linear codes, called minimal linear codes, has attracted great attention and research interest because of their significant applications in secret sharing schemes [7, 45] and secure two-party computation [8, 16]. They are those linear codes $C \subseteq \mathbb{F}_p^n$, in which every codeword is minimal, that is, every codeword $c \in C$ only covers *ac* for any $a \in \mathbb{F}_p$, but no other codewords in *C*.

In 1998, a sufficient condition for judging whether a linear code is minimal was presented by Ashikhmin and Barg (see Lemma 2 and Remark 1 of this paper), which is called the AB condition. In the literature, several classes of minimal linear codes satisfying the AB condition are obtained by studying linear codes with few weights (see for instance [16, 17, 19, 25, 26, 28, 29, 33, 34, 40, 44, 49]). Although the AB condition is not necessary, there is not any infinite family of minimal linear codes violating the AB condition before 2018 until Chang and Hyun's break through in [9], where the first infinite family of such codes was discovered amongst the linear code of the form

$$C_f = \left\{ c(\mu, \nu) = \left(\mu f(x) + \nu \cdot x \right)_{x \in \mathbb{F}_2^{n*}} : \mu \in \mathbb{F}_2, \nu \in \mathbb{F}_2^n \right\},\tag{1}$$

where f is a Boolean function on \mathbb{F}_2^n . Then a necessary and sufficient condition for the code C_f to be minimal was given by Ding et al. in [14], and thus three new infinite families of minimal

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linear codes violating the AB condition were found. Later the same authors also extended their results to the case of finite fields of characteristic 3, and obtained an infinite family of linear codes with $\frac{w_{\min}}{w_{\max}} \leq \frac{2}{3}$ in [18]. Then Bartoli and Bonini [2] further extended the results of [14] and [18] to odd characteristic cases, and found an infinite family of minimal linear codes violating the AB condition. Thereafter, more families of minimal linear codes violating the AB condition were constructed by investigating the linear code of the form (1). For instance, some minimal linear codes violating the AB condition were obtained from cutting blocking sets [3, 32]; from partial difference sets [38]; from characteristic functions [30]; from weakly regular plateaued/bent functions [31, 43]; from Partial Spreads [41]; from Maiorana-McFarland functions [42, 47]; and from the direct sum of Boolean functions [48], etc.

In the literature, there is another general form to construct minimal linear codes violating the AB condition, which is given by

$$C_D = \{ (d_1 \cdot x, d_2 \cdot x, \dots, d_s \cdot x) : x \in \mathbb{F}_p^n \},$$

$$(2)$$

where p is a prime and $D := \{d_1, d_2, \ldots, d_s\} \subseteq \mathbb{F}_p^n$ is called the definition set of C_D . Some conditions for C_D to be minimal have been obtained in [24, 35]. Some related minimal linear codes violating the AB condition of this form can be found in [23, 24, 35, 46].

This paper concentrates on the construction of minimal linear codes by using vectorial Boolean functions. Explicitly, we consider the minimality of binary linear codes of the form

$$C_F = \left\{ c(\mu, \nu) = \left(\mu \cdot F(x) + \nu \cdot x \right)_{x \in \mathbb{F}_2^{n*}} : \mu \in \mathbb{F}_2^m, \nu \in \mathbb{F}_2^n \right\},\tag{3}$$

where F is any vectorial Boolean (n, m)-function. Note that some linear codes have been obtained from certain vectorial Boolean functions in the literature. For instance, the parameters of the linear code

$$C_F = \left\{ c(\mu, \nu, \lambda) = \left(\mu \cdot F(x) + \nu \cdot x + \lambda \right)_{x \in \mathbb{F}_2^n} : \mu \in \mathbb{F}_2^m, \nu \in \mathbb{F}_2^n, \lambda \in \mathbb{F}_2 \right\}$$
(4)

were determined in [6] and [39], where F is an (n, m)-function. In (2), let $D = \{x \in \mathbb{F}_{2^n}^* : \operatorname{Tr}_l^n(F(x)) = 0\}$, the authors of [10] obtained three classes of binary linear codes with few weights, and determined their weight distributions, where F is a power function on \mathbb{F}_{2^n} . Modifying the linear code in (2) as

$$C_{D_{f_{\lambda}}} = \left\{ \left(\operatorname{Tr}_{1}^{m}(xd_{1}) + \operatorname{Tr}_{1}^{s}(yF(d_{1})), \dots, \operatorname{Tr}_{1}^{m}(xd_{n_{f_{\lambda}}}) + \operatorname{Tr}_{1}^{s}(yF(d_{n_{f_{\lambda}}})) \right) : x \in \mathbb{F}_{2^{m}}, y \in \mathbb{F}_{2^{s}} \right\},\$$

the authors of [36] obtained several classes of linear codes with few weights and determine their weight distributions by choosing some suitable (m, s)-functions F, where $\lambda \in \mathbb{F}_{2^s}^*$, $f_{\lambda}(x) = \operatorname{Tr}_1^s(\lambda F(x))$, $n_{f_{\lambda}}$ is the Hamming weight of f_{λ} and $D_{f_{\lambda}}$ is the support of f_{λ} . In [15], Ding et al. used vectorial bent functions for a construction of a two-parameter family of binary linear codes that do not satisfy the conditions of the Assmus-Mattson theorem. However, it is harder to construct minimal linear codes of the form (3) than that of the form (1), since if C_F in (3) is minimal, then for any nonzero Boolean function $f = \mu \cdot F$, the linear code C_f in (1) is always minimal. To the best of our knowledge, there has not been any infinite families of minimal linear codes constructed from vectorial Boolean functions up to now.

In this paper, we first give a necessary and sufficient condition such that the linear code C_F in (3) is minimal. Based on that, we obtain some three-weight minimal linear codes from vectorial plateaued functions and determine their weight distributions. Then, by rewriting the (n, m)-function F as F = (f, G) (where f is a Boolean function on \mathbb{F}_2^n and G is an (n, m - 1)-function), we give another necessary and sufficient condition such that C_F in (3) is minimal and violates the AB condition. This condition enables us to derive three infinite families of minimal linear codes violating the AB condition. Moreover, we find that the dimensions of our minimal linear codes violating the AB condition are also better than almost all known ones.

Form	Length	Dimension	Minimal distance	Method	References
	$2^n - 1$	n+1	$2^{ A } + 2^{ B } - 2^{ A \cap B } - 1$		[9, Prop 3]
	$2^n - 1$	n+1	$s(2^t - 1), n = 2t, s \le 2^{t-2}$		[14, Thm 18]
	$2^n - 1$	n+1	$2^{n-1} - 2^{\frac{n-1}{2}-1}(\frac{n+1}{2}-1)$		[14, Thm 23]
	$2^n - 1$	n+1	$\frac{\sum_{j=1}^{k} \binom{n}{j}, 2 \le k \le \lfloor \frac{n-3}{2} \rfloor}{2^{t-1}(s+1), n=s+t}$		[14, Thm 31]
	$2^n - 1$	n+1	$2^{t-1}(s+1), n = s+t$		[42, Thm 4.1]
	$2^n - 1$	n+1	$2^{t-1}(2^k-1)d, n = 2k+t$	Boolean	[42, Thm 4.2]
	2^n	n+1	$2^{n-1} - 2^{\lambda-1}(\kappa - 1), \ n = \lambda + \kappa$	functions	[47, Thm 2(i)]
	2^n	n+1	$2^{n-1} - 2^{\lambda-1}(\kappa - 3), \ n = \lambda + \kappa$		[47, Thm 2(ii)]
(1)	2^n	n+1	$2^{n-1} - 2^{\lambda-1}(2^{\kappa} - \kappa - 3)$		[47, Thm 3]
	2^n	n+1	$2^{n-1} - 2^{\lambda}(\kappa + 1)$		[47, Thm 4]
	2^n	n+2	$\frac{2^{n-1} - 2^{\lambda}(\kappa+1)}{2^{n-1} - 2^{\lambda}(\kappa+1)}$		[47, Thm 5]
	2^n	$n + \frac{s}{2} + 2$	$2^{\frac{r-1}{2}+s}(\frac{r+1}{2}+1), \ n=r+s$		[47, Thm 7]
	$2^n - 1$	n+1	$2^{\frac{n-1}{2}-1}(\frac{n+1}{2}+1) - 1$		[30, Thm 3.5]
	$2^n - 1$	n+1	$2^{\frac{n-1}{2}-1}(\frac{n+1}{2}+1) - 1$		[30, Thm 3.5]
	$2^n - 1$	n+1	-		[30]
	$2^n - 1$	n+1	-	Blocking	[3]
	2^n	n+1		sets	[32]
	$2^n - 1$	n+1	$2^{n-1} + \theta_2$	P-D. sets	[38]
	N_1	n	d_1		[23, Thm 3.7]
	N_2	n	d_2	Boolean	[23, Thm 3.14]
	N_3	n	d_3	functions	[23, Thm 3.20]
(-)	N_4	n	d_4		[23, Thm 3.26]
(2)	$\binom{n}{2}$	n-1	n-1		[46, Thm 3]
	$\binom{n}{2} + 1$	n-1	n	-	[46, Thm 4]
	$\binom{n}{1} + \binom{n}{2}$	n	n		[46, Thm 5]
	$\binom{n}{1} + \binom{n}{2} + 1$	n	n+1		[46, Thm 6]
	$2^n - 1$	n+2	$2^{\frac{n}{2}} - 2$		Thm 5
	077 4	n + m (2 <	$2n-2$, $2^{n}+1$	Vectorial	
(3)	$2^{n} - 1$	$m \le \frac{n}{2} + 1)$	$2^{n-2} + 2^{\frac{n}{2}+1}$	Boolean	Thm 6
	07 1	-	$2n-2+2^{n}+1$	functions	
	$\frac{2^n - 1}{2^n - 1}$	2n+1	$2^{n-2} + 2^{\frac{n}{2}+1}$		Thm 7

Table 1: Known minimal linear codes over \mathbb{F}_2 violating the AB condition

the values of N_i and d_i , i = 1, 2, 3, 4, are referred to [23]

The parameters of known minimal linear codes over \mathbb{F}_2 violating the AB condition are listed in Table 1, from which the advantage of dimensions of our codes can be seen clearly.

The rest of this paper is organized as follows. In Section 2, we fix some notations used in this paper, and introduce some basic knowledge on Boolean functions and linear codes. In Section 3, we define a generic construction of linear codes from vectorial Boolean functions, and fully characterize its minimality and parameters. Based on that, we give some 3-weight minimal linear codes with complete weight distribution in Section 4. In Section 5, we present a necessary and sufficient condition for a generic class of linear codes to be minimal and to be violated the AB condition. Based on that, we construct several minimal linear codes violating the AB condition in Section 6. Finally, we conclude the paper in Section 7.

2 Preliminaries

Throughout this paper, n and m are two positive integers. For any set E, let $E^* = E \setminus \{0\}$ and let |E| be the cardinality of E. Let \mathbb{F}_{2^n} be the finite field of 2^n elements, and \mathbb{F}_2^n be the n-dimensional linear space over \mathbb{F}_2 . When fix a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ of \mathbb{F}_{2^n} over \mathbb{F}_2 , there is a natural one-to-one correspondence between \mathbb{F}_{2^n} and \mathbb{F}_2^n , that is, for every $a \in \mathbb{F}_{2^n}$, there is a unique vector $(a_1, a_2, \ldots, a_n) \in \mathbb{F}_2^n$, such that $a = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$. Thus in this paper, \mathbb{F}_{2^n} and \mathbb{F}_2^n are often treated as the same. The trace function $\operatorname{Tr}_r^n : \mathbb{F}_{2^n} \to \mathbb{F}_{2^r}$, where r|n, is defined as

$$\operatorname{Tr}_{r}^{n}(x) = x + x^{2^{r}} + x^{2^{2r}} + \dots + x^{2^{n-r}}, \quad \forall \ x \in \mathbb{F}_{2^{n}}.$$

For r = 1, Tr_1^n is also called the *absolute trace function*. It is well known that the trace function satisfies $\operatorname{Tr}_r^n(x) = \operatorname{Tr}_r^n(x^{2^r})$, $\operatorname{Tr}_r^n(x+y) = \operatorname{Tr}_r^n(x) + \operatorname{Tr}_r^n(y)$ (\mathbb{F}_{2^r} -linear) and $\operatorname{Tr}_r^n(x) = \operatorname{Tr}_r^k(\operatorname{Tr}_k^n(x))$ (transitivity property) for any r|k and k|n.

2.1 Vectorial Boolean function and Walsh-Hadamard transform

A mapping F from \mathbb{F}_2^n to \mathbb{F}_2^m is called an (n, m)-function, and F is called a *Boolean function* on \mathbb{F}_2^n if m = 1. For an (n, m)-function F and a vector $\mu \in \mathbb{F}_2^{m*}$, the Boolean function $\mu \cdot F(x)$ is said to be the *component* of F with respect to μ . The *Walsh-Hadamard transform* of $\mu \cdot F$ at $\nu \in \mathbb{F}_2^n$ is defined as

$$W_{\mu \cdot F(x)}(\nu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\mu \cdot F(x) + \nu \cdot x},$$

where $\nu \cdot x$ is the usual product of ν and x in \mathbb{F}_2^n . In \mathbb{F}_{2^n} , one has $\nu \cdot x = \operatorname{Tr}_1^n(\nu x)$. For convenience, $W_{\mu \cdot F(x)}(\nu)$ is also denoted by $W_F(\mu, \nu)$ in this paper. The *nonlinearity* of F is defined as

$$NL(F) = 2^{n-1} - \frac{1}{2} \max_{\mu \in \mathbb{F}_2^{m*}, \nu \in \mathbb{F}_2^n} |W_F(\mu, \nu)|.$$

Let f be a Boolean function on \mathbb{F}_2^n . Then it is easy to check that

$$\sum_{\nu \in \mathbb{F}_2^n} W_f^2(\nu) = 2^{2n},\tag{5}$$

which is the well-known Parseval formula. It indicates that $\max_{\nu \in \mathbb{F}_2^n} |W_f(\nu)| \geq 2^{n/2}$. The equality occurs if and only if n is even and $W_f(\nu) = \pm 2^{n/2}$ for any $\nu \in \mathbb{F}_2^n$. In this case, f is called a *bent function*. It is well known that every bent function f satisfies that $W_f(\nu) = 2^{n/2}(-1)^{f^*(\nu)}$ for any $\nu \in \mathbb{F}_2^n$, where f^* is called the *dual* of f.

According to Parseval formula, we can obtain the following result.

Lemma 1. Let f be a Boolean function on \mathbb{F}_2^n . Then $W_f(\nu) \leq 0$ (or ≥ 0) for any $\nu \in \mathbb{F}_2^n$ if and only if f is affine with $f(\mathbf{0_n}) = 1$ (or 0), where $\mathbf{0_n}$ is the zero vector of \mathbb{F}_2^n .

Proof. Since the proof of the case " ≥ 0 " is similar as that of " ≤ 0 ", we only prove the latter case here.

Let f be affine with $f(\mathbf{0_n}) = 1$, that is, $f(x) = \nu_0 \cdot x + 1$ for some vector $\nu_0 \in \mathbb{F}_2^n$. Then it is easily obtained by definition that $W_f(\nu)$ is equal to -2^n if $\nu = \nu_0$, and equal to 0 otherwise.

Conversely, suppose that $W_f(\nu) \leq 0$ for any $\nu \in \mathbb{F}_2^n$. Then by the relation

$$\sum_{\nu \in \mathbb{F}_2^n} W_f(\nu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)} \sum_{\nu \in \mathbb{F}_{2^n}} (-1)^{\nu \cdot x} = 2^n (-1)^{f(\mathbf{0_n})},\tag{6}$$

one has $\sum_{\nu \in \mathbb{F}_2^n} W_f(\nu) = -2^n$ and $f(\mathbf{0_n}) = 1$. Combining Parseval formula, it holds

$$\left(\sum_{\nu\in\mathbb{F}_2^n} W_f(\nu)\right)^2 = \sum_{\nu\in\mathbb{F}_2^n} W_f^2(\nu),$$

which implies that $W_f(\mu)W_f(\nu) = 0$ for any distinct vectors $\mu, \nu \in \mathbb{F}_2^n$. Equivalently, there is a unique $\nu_0 \in \mathbb{F}_2^n$ such that $W_f(\nu_0) \neq 0$. Hence $W_f(\nu)$ equals -2^n if $\nu = \nu_0$, and equals 0 otherwise. Therefore, $f(x) = \nu_0 \cdot x + 1$. It is well known [5] that every (n, m)-function F can be uniquely represented as a polynomial of the form

$$F(x) = \sum_{I \subseteq \{1,2,\dots,n\}} a_I \left(\prod_{i \in I} x_i\right), \ a_I \in \mathbb{F}_2^m,\tag{7}$$

which is usually called the algebraic normal form (ANF) of F. The algebraic degree of F, denoted by deg(F), is defined as the maximum cardinality of I such that $a_I \neq \mathbf{0}_{\mathbf{m}}$ in its ANF. An (n, m)-function F is called affine if deg $(F) \leq 1$.

Definition 1. A Boolean function f on \mathbb{F}_2^n is called plateaued if its Walsh-Hadamard transform satisfies that $W_f(\nu) \in \{0, \pm \Lambda\}$ for any $\nu \in \mathbb{F}_2^n$, where $\Lambda = 2^{(n+\lambda)/2}$ for some $0 \le \lambda \le n$ with the same parity of n is called the amplitude of f. In particular, when n is even and $\Lambda = 2^{n/2}$, f is clearly a bent function.

Definition 2. An (n,m)-function F is called vectorial plateaued if all its components are plateaued functions. In particular, F is called vectorial bent if n is even and F is a vectorial plateaued function with single amplitude $2^{n/2}$; and F is called almost bent (AB) if n is odd and F is a vectorial plateaued function with single amplitude $2^{(n+1)/2}$.

2.2 Minimal linear codes

For a vector $c = (c_1, c_2, \ldots, c_n) \in \mathbb{F}_2^n$, the set $\{1 \leq i \leq n : c_i = 1\}$ (denoted by $\operatorname{suppt}(c)$) is called the *support* of c, whose cardinality (usually denoted by $\operatorname{wt}(c)$) is called the (Hamming) weight of c, i.e., $\operatorname{wt}(c) = |\operatorname{suppt}(c)| = \sum_{i=1}^n c_i$. Any vector $c^{(1)} \in \mathbb{F}_2^n$ is said to be covered by $c^{(2)} \in \mathbb{F}_2^n$, denoted by $c^{(1)} \leq c^{(2)}$, if $\operatorname{suppt}(c^{(1)}) \subseteq \operatorname{suppt}(c^{(2)})$. The (Hamming) distance of $c^{(1)}, c^{(2)} \in \mathbb{F}_2^n$, denoted by $d(c^{(1)}, c^{(2)})$, is defined as the weight of $c^{(1)} + c^{(2)}$.

Any code $C \subseteq \mathbb{F}_2^n$ is said to be an [n, k, d] linear code if it is a k-dimensional linear subspace of \mathbb{F}_2^n with minimal distance $d = \min\{d(c^{(1)}, c^{(2)}) : c^{(1)}, c^{(2)} \in C^*, c^{(1)} \neq c^{(2)}\}$, where n, k and d are called the *length*, *dimension and minimal distance* of C, respectively. Since C is a linear subspace, d is in fact the minimal weight of all nonzero codewords in C. The *weight enumerator* of C is defined by a polynomial $1 + A_1 z + A_2 z^2 + \cdots + A_n z^n$, where each coefficient A_i denotes the number of codewords in C whose weight is equal to i. The code C is said to be a *t*-weight code if the number of nonzero coefficients A_i in its weight enumerator is equal to t.

For a code C over \mathbb{F}_2 , a codeword $c \in C$ is called *minimal* if in C it is covered by itself only. A linear code C is called minimal if every nonzero codeword in C is minimal. The following result is a sufficient condition for a linear code to be minimal.

Lemma 2. [1] Let $C \subseteq \mathbb{F}_2^n$ be a linear code. Let w_{\max} and w_{\min} be the maximum and minimum nonzero Hamming weights in C, respectively. If

$$\frac{w_{\min}}{w_{\max}} > \frac{1}{2},\tag{8}$$

then C is minimal.

Remark 1. If C is a linear code over \mathbb{F}_p , where p is a prime, then (8) in Lemma 2 becomes that $\frac{w_{\min}}{w_{\max}} > \frac{p-1}{p}$.

The condition (8) is called the *AB condition* in this paper. AB condition is a sufficient but not necessary condition for a linear code to be minimal. Until now, many infinite families of minimal linear codes violating the AB condition have been found, see for instance [3, 9, 14, 23, 30, 32, 46, 47, 48]. The following result gives a necessary and sufficient condition for a binary linear code to be minimal.

Theorem 1. [14] Let $C \subseteq \mathbb{F}_2^n$ be a binary linear code. Then C is minimal if and only if

$$\operatorname{wt}(c^{(1)} + c^{(2)}) \neq \operatorname{wt}(c^{(2)}) - \operatorname{wt}(c^{(1)}),$$
(9)

for any $c^{(1)}, c^{(2)}, c^{(1)} + c^{(2)} \in C^*$.

3 A generic construction of minimal binary linear codes

Recently, many minimal binary linear codes have been obtained from the linear code of the form (1):

$$C_f = \left\{ c(\mu, \nu) = \left(\mu f(x) + \nu \cdot x \right)_{x \in \mathbb{F}_2^{n*}} : \mu \in \mathbb{F}_2, \nu \in \mathbb{F}_2^n \right\}$$

by choosing some suitable Boolean functions f on \mathbb{F}_{2^n} (see e.g., [3, 13, 14, 18, 30, 31, 42]). In this section, we consider this linear code for the case f being a vectorial Boolean function.

Construction 1. Let F be an (n,m)-function. Define a linear code as in (3):

$$C_F = \left\{ c(\mu, \nu) = \left(\mu \cdot F(x) + \nu \cdot x \right)_{x \in \mathbb{F}_2^{n*}} : \mu \in \mathbb{F}_2^m, \nu \in \mathbb{F}_2^n \right\}.$$

In fact, a more general linear code defined as

$$C_F = \left\{ c(\mu, \nu, \lambda) = \left(\mu \cdot F(x) + \nu \cdot x + \lambda \right)_{x \in \mathbb{F}_2^n} : \mu \in \mathbb{F}_2^m, \nu \in \mathbb{F}_2^n, \lambda \in \mathbb{F}_2 \right\}$$
(10)

has been considered in [6] and [39], where it was proved to be a $[2^n, n + m + 1, NL(F)]$ binary linear code when $NL(F) \neq 0$ (see [6, Proposition 6] and [39, Lemma 10]), where NL(F) is the nonlinearity of F. But this code is clearly not minimal, since $c(\mathbf{0_m}, \mathbf{0_n}, 1) \in C_F$ with $wt(c(\mathbf{0_m}, \mathbf{0_n}, 1)) = 2^n$. Hence, in this section, we give a further study on the binary linear code C_F in Construction 1 to find some minimal codes. We first give an observation on the parameters of C_F in the following subsection.

3.1 The parameters of C_F

It is noticed that, the parameters (including the length, dimension and minimal distance) of the linear code C_F in (10) have been obtained in [6] and [39], respectively. The code C_F in Construction 1 (i.e., the code in (3)) is obtained by modifying (10) slightly. So the parameters of the former can be obtained similarly as that of the later. For completeness, we also present the parameters of C_F in Construction 1 in this subsection. The techniques used here are similar to that of [3, 14, 39].

Proposition 1. Let F be an (n,m)-function such that $\mu \cdot F(x)$ is not linear for any $\mu \in \mathbb{F}_2^{m*}$. Then the binary linear code C_F generated by Construction 1 has length $2^n - 1$ and dimension n + m.

Proof. Clearly, the length of C_F is $2^n - 1$.

It is noted that every codeword $c(\mu,\nu) = (\mu \cdot F(x) + \nu \cdot x)_{x \in \mathbb{F}_2^{n^*}}$ in C_F can be linearly represented by $c(\mu^{(i)}, \mathbf{0_n}) = (\mu^{(i)} \cdot F(x))_{x \in \mathbb{F}_2^{n^*}}, i = 1, 2, ..., m$, and $c(\mathbf{0_m}, \nu^{(j)}) = (\nu^{(j)} \cdot x)_{x \in \mathbb{F}_2^{n^*}}, j = 1, 2, ..., n$, where $\{\mu^{(1)}, \mu^{(2)}, ..., \mu^{(m)}\}$ is a basis of \mathbb{F}_2^m over \mathbb{F}_2 , and $\{\nu^{(1)}, \nu^{(2)}, ..., \nu^{(n)}\}$ is a basis of \mathbb{F}_2^n over \mathbb{F}_2 . Below, we prove that these codewords are linearly independent over \mathbb{F}_2 .

Assume that there are some $a_i, b_j \in \mathbb{F}_2$, i = 1, 2, ..., m, j = 1, 2, ..., n, such that

$$\sum_{i=1}^{m} a_i c(\mu^{(i)}, \mathbf{0_n}) + \sum_{j=1}^{n} b_j c(\mathbf{0_m}, \nu^{(j)}) = \mathbf{0}.$$

Then we have

$$\mu' \cdot F(x) + \nu' \cdot x = 0, \quad \forall \ x \in \mathbb{F}_2^{n*}, \tag{11}$$

where $\mu' = a_1 \mu^{(1)} + a_2 \mu^{(2)} + \dots + a_m \mu^{(m)}$ and $\nu' = b_1 \nu^{(1)} + b_2 \nu^{(2)} + \dots + b_n \nu^{(n)}$. Firstly, we claim that $\mu' = \mathbf{0}_{\mathbf{m}}$. Otherwise, (11) is impossible since $\mu' \cdot F(x)$ is not linear by the assumption. This also implies that $\nu' = \mathbf{0}_{\mathbf{n}}$, since the cardinality of $\{x \in \mathbb{F}_2^{n*} : \nu' \cdot x = 1\}$ is 2^{n-1} when $\nu' \neq \mathbf{0}_{\mathbf{n}}$. Hence, we obtain that $\mu' = \mathbf{0}_{\mathbf{m}}$ and $\nu' = \mathbf{0}_{\mathbf{n}}$, which implies that $a_i = 0$ for any $1 \leq i \leq m$, and $b_i = 0$ for any $1 \leq j \leq n$. This completes the proof. We now characterize the weight distribution of the binary linear code C_F given in Construction 1.

Proposition 2. Let F be an (n, m)-function with $F(\mathbf{0_n}) = \mathbf{0_m}$, and let C_F be the binary linear code generated by Construction 1. Then for any $c(\mu, \nu) \in C_F$, it holds that

wt(c(
$$\mu, \nu$$
)) =

$$\begin{cases}
0, & \text{if } \mu = \mathbf{0_m}, \nu = \mathbf{0_n}; \\
2^{n-1}, & \text{if } \mu = \mathbf{0_m}, \nu \in \mathbb{F}_2^{n*}; \\
2^{n-1} - \frac{1}{2}W_F(\mu, \nu), & \text{otherwise.}
\end{cases}$$

Proof. By definition, for any $(\mu, \nu) \in \mathbb{F}_2^m \times \mathbb{F}_2^n$, the weight of $c(\mu, \nu) \in C_F$ is given by

$$wt(c(\mu,\nu)) = \sum_{x \in \mathbb{F}_2^{n*}} (\mu \cdot F(x) + \nu \cdot x) = \sum_{x \in \mathbb{F}_2^n} (\mu \cdot F(x) + \nu \cdot x).$$
(12)

Since $\mu \cdot F(x) + \nu \cdot x$ is a Boolean function, we have $\mu \cdot F(x) + \nu \cdot x = \frac{1 - (-1)^{\mu \cdot F(x) + \nu \cdot x}}{2}$. Hence (12) becomes that

wt
$$(c(\mu,\nu)) = 2^{n-1} - \frac{1}{2} \sum_{x \in \mathbb{F}_2^n} (-1)^{\mu \cdot F(x) + \nu \cdot x},$$

which is equal to 0 if $\mu = \mathbf{0}_{\mathbf{m}}, \nu = \mathbf{0}_{\mathbf{n}}$; equal to 2^{n-1} if $\mu = \mathbf{0}_{\mathbf{m}}, \nu \neq \mathbf{0}_{\mathbf{n}}$; and equal to $2^{n-1} - \frac{1}{2}W_F(\mu,\nu)$ otherwise.

Remark 2. When F is not affine, Lemma 1 implies that $\max_{\mu \in \mathbb{F}_2^{m*}, \nu \in \mathbb{F}_2^n} W_F(\mu, \nu) > 0$. Thus, by Proposition 2, we obtain that the minimal distance d of the binary linear code C_F in Construction 1 satisfies that

$$d = 2^{n-1} - \frac{1}{2} \max_{\mu \in \mathbb{F}_2^{m*}, \nu \in \mathbb{F}_2^n} |W_F(\mu, \nu)| = NL(F).$$

Remark 3. When F is a vectorial plateaued function with single amplitude, the linear code C_F in Construction 1 is a 3-weight linear code by Proposition 2.

Remark 4. Let F be an (n,m)-function such that $\mu \cdot F(x)$ is not affine for some $\mu \in \mathbb{F}_2^{m*}$. Then, similarly as the proof of [32, Proposition 3.2], one can show that the linear code C_F in Construction 1 does not satisfy the AB condition if and only if

$$2 \max_{\mu \in \mathbb{F}_{2}^{m*}, \nu \in \mathbb{F}_{2}^{n}} W_{F}(\mu, \nu) - \min_{\mu \in \mathbb{F}_{2}^{m*}, \nu \in \mathbb{F}_{2}^{n}} W_{F}(\mu, \nu) \ge 2^{n}.$$
 (13)

Reformulating the above results to the finite field version, we have the following result.

Proposition 3. Let F be a function from \mathbb{F}_{2^n} to \mathbb{F}_{2^m} such that F(0) = 0, and $\operatorname{Tr}_1^m(\mu F(x))$ is not affine for any $\mu \in \mathbb{F}_{2^m}^*$. Let C_F be the binary linear code defined as

$$C_F = \left\{ c(\mu, \nu) = \left(\operatorname{Tr}_1^m(\mu F(x)) + \operatorname{Tr}_1^n(\nu x) \right)_{x \in \mathbb{F}_{2^n}^*} : (\mu, \nu) \in \mathbb{F}_{2^m} \times \mathbb{F}_{2^n} \right\}.$$
 (14)

Then C_F is a linear code with parameters $[2^n - 1, n + m, NL(F)]$.

3.2 Minimality of C_F

In this subsection, we give some conditions for the binary linear code C_F in Construction 1 to be minimal. The main result is given as follows.

Theorem 2. Let F be an (n,m)-function such that $F(\mathbf{0_n}) = \mathbf{0_m}$ and $\mu \cdot F$ is not affine for any $\mu \in \mathbb{F}_2^{m*}$. Let C_F be the linear code generated by Construction 1. Then C_F is minimal if and only if both the following two conditions are satisfied:

(1) For any $\mu \in \mathbb{F}_2^{m*}$ and any $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$, it holds that

$$W_F(\mu,\nu) \pm W_F(\mu,\nu') \neq 2^n;$$
 (15)

(2) For any $\mu, \mu' \in \mathbb{F}_2^{m*}$ with $\mu \neq \mu'$, and any $\nu, \nu' \in \mathbb{F}_2^n$, it holds that

$$W_F(\mu,\nu) + W_F(\mu',\nu') - W_F(\mu+\mu',\nu+\nu') \neq 2^n.$$
(16)

Proof. By the definition of C_F and the assumption that $\mu \cdot F$ is not affine for any $\mu \in \mathbb{F}_2^{m*}$, it is easily seen that $c(\mu, \nu) = \mathbf{0}$ if and only if $(\mu, \nu) = (\mathbf{0_m}, \mathbf{0_n})$, and $c(\mu, \nu) + c(\mu', \nu') = c(\mu + \mu', \nu + \nu')$ for any $(\mu, \nu), (\mu', \nu') \in \mathbb{F}_2^m \times \mathbb{F}_2^n$. So by theorem 1, we need to show that

$$\operatorname{wt}(c(\mu+\mu',\nu+\nu')) \neq \operatorname{wt}(c(\mu,\nu)) - \operatorname{wt}(c(\mu',\nu')),$$
(17)

for any $(\mu, \nu), (\mu', \nu'), (\mu + \mu', \nu + \nu') \in \mathbb{F}_2^m \times \mathbb{F}_2^n \setminus \{(\mathbf{0_m}, \mathbf{0_n})\}$. According to whether μ and μ' are nonzero, our proof can be divided into the following five cases.

Case 1. $\mu = \mu' = \mathbf{0}_{\mathbf{m}}$ and $\nu, \nu', \nu + \nu' \in \mathbb{F}_2^{n*}$. In this case, (17) obviously holds as $\operatorname{wt}(c(\mu, \nu)) = \operatorname{wt}(c(\mu', \nu')) = \operatorname{wt}(c(\mu + \mu', \nu + \nu')) = 2^{n-1}$ by Proposition 2.

Case 2. $\mu = \mathbf{0}_{\mathbf{m}}, \mu' \neq \mathbf{0}_{\mathbf{m}}$ and $\nu \neq \mathbf{0}_{\mathbf{n}}$. In this case, $\operatorname{wt}(c(\mu,\nu)) = 2^{n-1}$, $\operatorname{wt}(c(\mu',\nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu',\nu')$, and $\operatorname{wt}(c(\mu+\mu',\nu+\nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu',\nu+\nu')$ by Proposition 2. Hence, (17) holds if and only if

$$W_F(\mu',\nu') + W_F(\mu',\nu+\nu') \neq 2^n$$

Case 3. $\mu \neq \mathbf{0}_{\mathbf{m}}, \mu' = \mathbf{0}_{\mathbf{m}}$ and $\nu' \neq \mathbf{0}_{\mathbf{n}}$. In this case, $\operatorname{wt}(c(\mu,\nu)) = 2^{n-1} - \frac{1}{2}W_F(\mu,\nu)$, $\operatorname{wt}(c(\mu',\nu')) = 2^{n-1}, \operatorname{wt}(c(\mu+\mu',\nu+\nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu,\nu+\nu')$ by Proposition 2. Hence, (17) holds if and only if

$$W_F(\mu,\nu+\nu') - W_F(\mu,\nu) \neq 2^n$$

Case 4. $\mu = \mu' \neq \mathbf{0}_{\mathbf{m}}$ and $\nu \neq \nu'$. In this case, $\operatorname{wt}(c(\mu, \nu)) = 2^{n-1} - \frac{1}{2}W_F(\mu, \nu)$, $\operatorname{wt}(c(\mu', \nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu', \nu')$, $\operatorname{wt}(c(\mu + \mu', \nu + \nu')) = 2^{n-1}$ by Proposition 2. Hence, (17) holds if and only if

$$W_F(\mu, \nu') - W_F(\mu, \nu) \neq 2^n.$$

Case 5. $\mu \neq \mathbf{0_m}, \mu' \neq \mathbf{0_m}$ and $\mu + \mu' \neq \mathbf{0_m}$. In this case, $\operatorname{wt}(c(\mu, \nu)) = 2^{n-1} - \frac{1}{2}W_F(\mu, \nu),$ wt $(c(\mu', \nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu', \nu'),$ wt $(c(\mu + \mu', \nu + \nu')) = 2^{n-1} - \frac{1}{2}W_F(\mu + \mu', \nu + \nu')$ by Proposition 2. Hence, (17) holds if and only if

$$W_F(\mu',\nu') - W_F(\mu,\nu) + W_F(\mu+\mu',\nu+\nu') \neq 2^n.$$

The result follows then from the above five cases.

Remark 5. When m = 1, that is, F is a Boolean function, Theorem 2 reduces to [14, Theorem 15].

4 Three-weight minimal linear codes from vectorial plateaued functions

In this section, we present some three-weight minimal linear codes from certain vectorial plateaued functions. The main result is given as follows.

Theorem 3. Let n > 4. Let F be an (n,m)-function satisfying the following two conditions:

- (1) $F(\mathbf{0_n}) = \mathbf{0_m};$
- (2) $\max_{\mu \in \mathbb{F}_2^{m*}, \nu \in \mathbb{F}_2^n} |W_F(\mu, \nu)| = 2^{\frac{n+\lambda}{2}}$ such that $0 \le \lambda \le n-4$.

Then the code C_F generated by Construction 1 is a minimal binary linear code with parameters $[2^n - 1, n + m, 2^{n-1} - 2^{\frac{n+\lambda}{2}-1}].$

Proof. Firstly, C_F is obviously a binary linear code with parameters $[2^n - 1, n + m, 2^{n-1} - 2^{\frac{n+\lambda}{2}-1}]$ by Proposition 1 and Remark 2, since $NL(F) = 2^{n-1} - 2^{\frac{n+\lambda}{2}-1} \neq 0$ (which means that F has no affine components).

Then, we only need to show that C_F is minimal, which can be done by verifying the two conditions of Theorem 2. In fact, both conditions are obvious, since for any $\mu \in \mathbb{F}_2^{m*}$ and any distinct $\nu, \nu' \in \mathbb{F}_2^n$, one has

$$|W_F(\mu,\nu) \pm W_F(\mu,\nu')| \le 2^{\frac{n+\lambda}{2}+1},$$

which is strictly less than 2^n when $\lambda \leq n-4$; and for any distinct $\mu, \mu' \in \mathbb{F}_2^{m*}$, and any $\nu, \nu' \in \mathbb{F}_2^n$, one has

$$|W_F(\mu,\nu) + W_F(\mu',\nu') - W_F(\mu+\mu',\nu+\mu')| \le 3 \times 2^{\frac{n+\lambda}{2}},$$

which is also strictly less than 2^n when $\lambda \leq n-4$.

Applying Theorem 3 to vectorial plateaued functions with single amplitude, we obtain the following result.

Corollary 1. Let n > 4 and $0 \le \lambda \le n-4$. Let F be a vectorial plateaued (n, m)-function such that $F(\mathbf{0_n}) = \mathbf{0_m}$ and F has the single amplitude $2^{\frac{n+\lambda}{2}}$. Then the code C_F defined in Construction 1 is a three-weight minimal linear code with parameters $[2^n - 1, n+m, 2^{n-1} - 2^{\frac{n+\lambda}{2}-1}]$. Moreover, the weight distribution of C_F is given in Table 2.

Table 2: Weight distribution of C_F in Corollary 1

Weight	Frequency
0	1
2^{n-1}	$2^{n} - 1 + (2^{m} - 1)(2^{n} - 2^{n-\lambda})$
$2^{n-1} + 2^{\frac{n+\lambda}{2}-1}$	$(2^m - 1)(2^{n-\lambda-1} - 2^{\frac{n-\lambda}{2}-1})$
$2^{n-1} - 2^{\frac{n+\lambda}{2}-1}$	$(2^m - 1)(2^{n-\lambda-1} + 2^{\frac{n-\lambda}{2}-1})$

Proof. The first part of this corollary is clear by Theorem 3 and Remark 3. Below we calculate the weight distribution of C_F .

Since $W_F(\mu,\nu) \in \{0,\pm 2^{\frac{n+\lambda}{2}}\}$ for any $(\mu,\nu) \in \mathbb{F}_2^{m*} \times \mathbb{F}_2^n$, we have that

wt
$$(\mu, \nu) \in T := \{0, 2^{n-1}, 2^{n-1} + 2^{\frac{n+\lambda}{2}-1}, 2^{n-1} - 2^{\frac{n+\lambda}{2}-1}\}$$

for any $(\mu, \nu) \in \mathbb{F}_2^m \times \mathbb{F}_2^n$ by proposition 2. To determine the weight distribution of C_F , we need to determine the frequency of every element in T, which are denoted by $\mathbf{f}(0)$, $\mathbf{f}(2^{n-1})$, $\mathbf{f}(2^{n-1}+2^{\frac{n+\lambda}{2}-1})$ and $\mathbf{f}(2^{n-1}-2^{\frac{n+\lambda}{2}-1})$, respectively.

For every fixed $\mu \in \mathbb{F}_2^{m*}$, let $S_F := \{\nu \in \mathbb{F}_2^n : W_F(\mu, \nu) \neq 0\}$. Then by Parseval formula $\sum_{\nu \in \mathbb{F}_2^n} W_F^2(\mu, \nu) = 2^{2n}$, it is easily obtained that $|S_F| = 2^{n-\lambda}$. Hence $|\overline{S_F}| = 2^n - 2^{n-\lambda}$, where $\overline{S_F} = \mathbb{F}_2^n \backslash S_F$. Let

$$S_F^{(+)} := \{\nu \in \mathbb{F}_2^n : W_F(\mu, \nu) > 0\} \text{ and } S_F^{(-)} := \{\nu \in \mathbb{F}_2^n : W_F(\mu, \nu) < 0\},\$$

then we have

$$|S_F^{(+)}| + |S_F^{(-)}| = 2^{n-\lambda}.$$
(18)

In addition, by (6), we get

$$\sum_{\nu \in \mathbb{F}_2^n} W_F(\mu, \nu) = 2^n (-1)^{\mu \cdot F(\mathbf{0_n})} = 2^n.$$

Consequently, we have

$$2^{\frac{n+\lambda}{2}} \left(|S_F^{(+)}| - |S_F^{(-)}| \right) = 2^n.$$
(19)

Combining (18) and (19), we obtain that $|S_F^{(+)}| = \frac{1}{2}(2^{n-\lambda}+2^{\frac{n-\lambda}{2}})$ and $|S_F^{(-)}| = \frac{1}{2}(2^{n-\lambda}-2^{\frac{n-\lambda}{2}})$. Therefore, by proposition 2, we have

$$\mathbf{f}(0) = 1, \quad \mathbf{f}(2^{n-1}) = 2^n - 1 + (2^m - 1)|\overline{S_F}| = 2^n - 1 + (2^m - 1)(2^n - 2^{n-\lambda}),$$

$$\mathbf{f}(2^{n-1} + 2^{\frac{n+\lambda}{2}-1}) = (2^m - 1)|S_F^{(-)}| = (2^m - 1)(2^{n-\lambda-1} - 2^{\frac{n-\lambda}{2}-1}), \text{ and}$$

$$\mathbf{f}(2^{n-1} - 2^{\frac{n+\lambda}{2}-1}) = (2^m - 1)|S_F^{(+)}| = (2^m - 1)(2^{n-\lambda-1} + 2^{\frac{n-\lambda}{2}-1}).$$

This completes the proof.

Applying Corollary 1 to vectorial bent functions and AB functions, respectively, we obtain the following two results.

Corollary 2. Let n > 4 be an even integer and let $m \leq \frac{n}{2}$. Let F be a vectorial bent (n, m)function with $F(\mathbf{0_n}) = \mathbf{0_m}$. Then the code C_F generated by Construction 1 is a three-weight
minimal linear code with parameters $[2^n - 1, n + m, 2^{n-1} - 2^{\frac{n}{2}-1}]$. Moreover, the weight distribution of C_F is given in Table 3.

Weight	Frequency
0	1
2^{n-1}	$2^{n} - 1$
$2^{n-1} + 2^{\frac{n}{2}-1}$	$(2^m - 1)(2^{n-1} - 2^{\frac{n}{2}-1})$
$2^{n-1} - 2^{\frac{n}{2}-1}$	$(2^m - 1)(2^{n-1} + 2^{\frac{n}{2}-1})$

Table 3: Weight distribution of C_F in Corollary 2

Remark 6. For any even integer n > 4, the minimal distance of the linear codes C_F in Corollary 2 reaches the optimal amongst the linear codes in Construction 1, since their nonlinearity is the highest.

Corollary 3. Let n > 4 be an odd positive integer. Let F be an AB function with $F(\mathbf{0_n}) = \mathbf{0_n}$. Then the code C_F generated by Construction 1 is a three-weight minimal linear code with parameters $[2^n - 1, 2n, 2^{n-1} - 2^{\frac{n+1}{2}-1}]$. Moreover, the weight distribution of C_F is given in Table 4.

Table 4: Weight distribution of C_F in Corollary 3

Weight	Frequency
0	1
2^{n-1}	$2^n - 1 + (2^n - 1)2^{n-1}$
$2^{n-1} + 2^{\frac{n+1}{2}-1}$	$(2^n - 1)(2^{n-2} - 2^{\frac{n-1}{2}-1})$
$2^{n-1} - 2^{\frac{n+1}{2}-1}$	$(2^n - 1)(2^{n-2} + 2^{\frac{n-1}{2}-1})$

The following numerical data is consistent with the conclusions of Corollaries 2 and 3, respectively.

Example 1. Let n = 6 and m = 3. Then the code C_F in Corollary 2 is a three-weight minimal linear code with parameters [63,9,28] and weight enumerator

$$1 + 252z^{28} + 63z^{32} + 196z^{36}.$$

Example 2. Let n = 7. Then the code C_F in Corollary 3 is a three-weight minimal linear code with parameters [127, 14, 56] and weight enumerator

$$1 + 4572z^{56} + 8255z^{64} + 3556z^{72}$$
.

Note that [14, Examples 20, 24, 33] are also examples of minimal linear codes for n = 6 and n = 7. According to the values of n, we list the parameters of linear codes in Examples 1, 2, and [14, Examples 20, 24, 33] in Tables 5 and 6.

Table 5: The parameters of linear codes in Example 1 and [14, Example 20]

n=6	Length	Dimension	Minimal distance	Weights
Example 1	63	9	28	3-weight
[14, Example 20]	63	7	14	4-weight

Table 6: The parameters of linear codes in Example 2 and [14, Examples 24, 33]

n=7	Length	Dimension	Minimal distance	Weights
Example 2	127	14	56	3-weight
[14, Example 24]	127	8	52	6-weight
[14, Example 33]	127	8	28	6-weight

From Tables 5 and 6, it is clear that the dimension, minimal distance and weights of the minimal linear codes in Examples 1, 2 are better than that of the minimal linear codes in [14, Examples 20, 24, 33].

It is noted that $w_{\min} = 2^{n-1} - 2^{\frac{n+\lambda}{2}-1}$ and $w_{\max} = 2^{n-1} + 2^{\frac{n+\lambda}{2}-1}$ in Theorem 3, which satisfy $\frac{w_{\min}}{w_{\max}} > \frac{1}{2}$ when $\lambda \leq n-4$. Hence, the minimal linear codes obtained in this section satisfy the AB condition.

Linear codes with few weights play a significant role in data storage systems, communication systems and consumer electronics. Many papers are devoted to constructing such linear codes. At the end of this section, we collect the parameters of some known three-weight linear codes over \mathbb{F}_2 in Table 7.

From Table 7, it is easily seen that the dimensions and minimal distances of the linear codes from Corollaries 1, 2 and 3 are better than many of the others.

5 A generic construction of minimal linear codes violating the AB condition

In this section, we present a generic construction of minimal linear codes violating the AB condition, which is given as follows.

Construction 2. Let f be a Boolean function on \mathbb{F}_2^n with $f(\mathbf{0_n}) = 0$, and the following two conditions are satisfied:

- (a) $W_f(\nu) \pm W_f(\nu') \neq 2^n$ for any $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$;
- (**b**) $2 \max_{\nu \in \mathbb{F}_2^n} W_f(\nu) \min_{\nu \in \mathbb{F}_2^n} W_f(\nu) \ge 2^n$.

Let G be an (n, m-1)-function with $G(\mathbf{0_n}) = \mathbf{0_{m-1}}$, such that $\mu_1 f + \tilde{\mu} \cdot G$ is not affine for any nonzero $\mu = (\mu_1, \tilde{\mu}) \in \mathbb{F}_2 \times \mathbb{F}_2^{m-1}$, and the linear code C_G defined by

$$C_G = \left\{ c(\tilde{\mu}, \nu) = \left(\tilde{\mu} \cdot G(x) + \nu \cdot x \right)_{x \in \mathbb{F}_2^{n*}} : (\tilde{\mu}, \nu) \in \mathbb{F}_2^{m-1} \times \mathbb{F}_2^n \right\}$$
(20)

	r		
Length	Dimension	Minimal distance	References
$2^n - 1$	n+1	$2^{n-1} - 2^{\frac{n}{2}-1}$ or $2^{n-1} - 2^{\frac{n}{2}-1} - 1$	[14, Remark 19]
$2^{n-1} - 1$	n	$2^{n-2} - 2^{\frac{n-3}{2}}$	[12, Theorem 7]
n_f (see (19) of [12])	m	$\frac{n_f - 2^{(m-1)/2}}{2}$	[12, Corollary 11]
n_f (see (22) of [12])	m	$\frac{\frac{2}{n_f - 2^{m-1-r_f/2}}}{2}$	[12, Theorem 14]
$2^n - 1$	n+1	$2^{n-1} + \theta_2$, where $\theta_2 < 0$, see [38, Lemma 6]	[38, Remark 17]
$2^n - 1$	n+1	$2^{n-1} - 1$ or $2^{n-1} - 2$	[9, Remarks 1, 2]
2^{n-1}	n	$2^{n-2} - 2^{\frac{n-2}{2}}$ or $2^{n-2} - 2^{\frac{n-3}{2}}$	[13, Theorems 9, 21]
$2^n - 3 \times 2^{\frac{n-1}{2}} + 1$	n	$2^{n-1} - 3 \times 2^{\frac{n-3}{2}}$	[37, Theorem 7]
2^n	n+1	$2^{n-1} - 2^{\frac{s}{2}-1}2^{\frac{r+1}{2}}$, where $n = r+s$	[48]
$2^n - 1$	n+1	$2^{n-1} - 2^{\frac{n}{2}-1}$	[28]
$2^n - 1$	n+1	$2^{n-1} - 2^{\frac{n+s}{2}-1}$	[29, Theorem 1]
$2^{n-1} - 2^{\frac{n}{2}-1} - 1$	n	$2^{n-2} - (3l-1)2^{\frac{n}{2}-2}$, where $l < \frac{2^{n/2}+1}{3}$	[26, Theorem 1 (i)]
$2^{2l} - 1, \ n = 3l$	n	$2^{2l-1} - 2^{l-1}$	[10, Theorem 1]
$2^{n} - 1$	n+m	$2^{n-1} - 2^{\frac{n+\lambda}{2}-1}$	Corollary 1
$2^n - 1$	n+m	$2^{n-1} - 2^{\frac{n}{2}-1}$	Corollary 2
$2^n - 1$	2n	$2^{n-1} - 2^{\frac{n+1}{2}-1}$	Corollary 3

Table 7: The parameters of some known three-weight linear codes over \mathbb{F}_2

is minimal. Let F be an (n,m)-function given by

$$F(x) = (f(x), G(x)),$$
 (21)

and let C_F be the linear code generated by Construction 1.

In the following, we study the conditions for the linear code C_F in Construction 2 to be minimal and to be violated the AB condition. To this end, we need first to calculate the Walsh-Hadamard transform of F.

Proposition 4. Let F be the (n,m)-function defined in Construction 2. Then for any nonzero $\mu = (\mu_1, \tilde{\mu}) \in \mathbb{F}_2 \times \mathbb{F}_2^{m-1}$ and any $\nu \in \mathbb{F}_2^n$, it holds that

$$W_F(\mu,\nu) = \begin{cases} W_G(\tilde{\mu},\nu), & \text{if } \mu_1 = 0, \ \tilde{\mu} \neq \mathbf{0_{m-1}}; \\ W_f(\nu), & \text{if } \mu_1 = 1, \ \tilde{\mu} = \mathbf{0_{m-1}}; \\ W_{A_{\tilde{\mu}}}(\nu), & \text{if } \mu_1 = 1, \ \tilde{\mu} \neq \mathbf{0_{m-1}}, \end{cases}$$

where $A_{\tilde{\mu}}(x) = f(x) + \tilde{\mu} \cdot G(x)$.

Proof. By the definition of Walsh-Hadamard transform, one has

$$W_F(\mu,\nu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\mu \cdot F(x) + \nu \cdot x} = \sum_{x \in \mathbb{F}_2^n} (-1)^{\mu_1 f(x) + \tilde{\mu} \cdot G(x) + \nu \cdot x}, \quad \forall \ (\mu,\nu) \in \mathbb{F}_2^{m*} \times \mathbb{F}_2^n,$$

from which the result follows clearly.

Since it is required in Construction 2 that

$$2\max_{\nu\in\mathbb{F}_2^n}W_f(\nu) - \min_{\nu\in\mathbb{F}_2^n}W_f(\nu) \ge 2^n$$

by Proposition 4 one immediately gets

$$2 \max_{\mu \in \mathbb{F}_{2}^{m*}, \nu \in \mathbb{F}_{2}^{n}} W_{F}(\mu, \nu) - \min_{\mu \in \mathbb{F}_{2}^{m*}, \nu \in \mathbb{F}_{2}^{n}} W_{F}(\mu, \nu) \ge 2^{n}.$$

Then by Remark 4, we arrive at the following result.

Proposition 5. Let C_F be the linear code generated by Construction 2. Then C_F does not satisfy the AB condition.

Thus, we only need to investigate the conditions under which the linear code C_F in Construction 2 is minimal. We give the following theorem.

Theorem 4. The linear code C_F generated by Construction 2 is minimal if and only if the following three conditions are fulfilled:

(1) For any $\tilde{\mu} \in \mathbb{F}_2^{m-1} \setminus \{\mathbf{0}_{m-1}\}$ and any $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$, it holds that $W_{A_{\tilde{\mu}}}(\nu) \pm W_{A_{\tilde{\mu}}}(\nu') \neq 2^n;$ (22)

(2) For any $\tilde{\mu} \in \mathbb{F}_2^{m-1} \setminus \{\mathbf{0_{m-1}}\}\ and\ any\ \nu, \nu' \in \mathbb{F}_2^n$, it holds that

$$(-1)^{i}W_{f}(\nu) + (-1)^{j}W_{G}(\tilde{\mu},\nu') + (-1)^{k}W_{A_{\tilde{\mu}}}(\nu+\nu') \neq 2^{n},$$
(23)

where $i, j, k \in \{0, 1\}$ such that exactly two of them are 0;

(3) For any $\tilde{\mu}, \tilde{\mu}' \in \mathbb{F}_2^{m-1} \setminus \{\mathbf{0}_{m-1}\}$ with $\tilde{\mu} \neq \tilde{\mu}'$, and any $\nu, \nu' \in \mathbb{F}_2^n$, it holds that

$$W_{A_{\tilde{\mu}}}(\nu) + (-1)^{i} W_{G}(\tilde{\mu}',\nu') + (-1)^{j} W_{A_{\tilde{\mu}+\tilde{\mu}'}}(\nu+\nu') \neq 2^{n},$$
(24)

where $i, j \in \{0, 1\}$ such that exactly one of them is 0.

Proof. To prove the result, we need to verify both the conditions of Theorem 2. For Condition (1) of Theorem 2, let $\mu \in \mathbb{F}_2^{m*}$ and let $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$, by Proposition 4 one gets

$$W_{F}(\mu,\nu) \pm W_{F}(\mu,\nu') = \begin{cases} W_{G}(\tilde{\mu},\nu) \pm W_{G}(\tilde{\mu},\nu'), & \text{if } \mu = (0,\tilde{\mu}), \\ W_{f}(\nu) \pm W_{f}(\nu'), & \text{if } \mu = (1,\mathbf{0_{m-1}}), \\ W_{A_{\tilde{\mu}}}(\nu) \pm W_{A_{\tilde{\mu}}}(\nu'), & \text{if } \mu = (1,\tilde{\mu}), \tilde{\mu} \neq \mathbf{0_{m-1}} \end{cases}$$

In each case one always has $W_F(\mu,\nu) \pm W_F(\mu,\nu') \neq 2^n$. In fact, the first case is because that C_G is minimal, the second case is due to Condition (a) of Construction 2, while the third case is the same as Condition (1) of this theorem.

Now we verify the second condition of Theorem 2. According to the relations $\mu = (\mu_1, \tilde{\mu})$ and $\mu' = (\mu'_1, \tilde{\mu}')$, where $\mu, \mu', \mu + \mu' \in \mathbb{F}_2^{m*}$, the discussions can be divided into 10 cases, which are listed in Table 8. Since the discussions of some cases are similar, without loss of generality, here we only discuss the cases 1, 2 and 4 in detail.

Case 1. $\mu_1 = 0, \tilde{\mu} \neq \mathbf{0_{m-1}}, \, \mu'_1 = 0, \tilde{\mu}' \neq \mathbf{0_{m-1}}, \, \tilde{\mu} \neq \tilde{\mu}'$. In this case, by Proposition 4 we have

$$W_F(\mu,\nu) - W_F(\mu',\nu') + W_F(\mu+\mu',\nu+\nu') = W_G(\tilde{\mu},\nu) - W_G(\tilde{\mu}',\nu') + W_G(\tilde{\mu}+\tilde{\mu}',\nu+\nu'),$$

which can never be equal to 2^n for any $\nu, \nu' \in \mathbb{F}_2^n$, as C_G is minimal.

Case 2. $\mu_1 = 0, \tilde{\mu} \neq \mathbf{0_{m-1}}, \mu'_1 = 1, \tilde{\mu}' = \mathbf{0_{m-1}}$. In this case, by Proposition 4 we have

$$W_F(\mu,\nu) - W_F(\mu',\nu') + W_F(\mu+\mu',\nu+\nu') = W_G(\tilde{\mu},\nu) - W_f(\nu') + W_{A_{\tilde{\mu}}}(\nu+\nu'),$$

which does not equal 2^n for any $\nu, \nu' \in \mathbb{F}_2^n$ if and only if

$$-W_f(\nu) + W_G(\tilde{\mu}, \nu') + W_{A_{\tilde{\mu}}}(\nu + \nu') \neq 2^n$$

for any $\nu, \nu' \in \mathbb{F}_2^n$. This corresponds to Condition (2) of this theorem with i = 1 and j = k = 0. Case 4. $\mu_1 = 0, \tilde{\mu} \neq \mathbf{0_{m-1}}, \mu'_1 = 1, \tilde{\mu}' \neq \mathbf{0_{m-1}}, \tilde{\mu} \neq \tilde{\mu}'$. In this case, one has

$$W_F(\mu,\nu) - W_F(\mu',\nu') + W_F(\mu+\mu',\nu+\nu') = W_G(\tilde{\mu},\nu) - W_{A_{\tilde{\mu}'}}(\nu') + W_{A_{\tilde{\mu}+\tilde{\mu}'}}(\nu+\nu'),$$

which does not equal 2^n for any $\nu, \nu' \in \mathbb{F}_2^n$ if and only Condition (3) of this theorem holds with i = 0 and j = 1, since in this case the roles of $\tilde{\mu}$, $\tilde{\mu}'$ and $\tilde{\mu} + \tilde{\mu}'$ are the same, and the roles of ν , ν' and $\nu + \nu'$ are the same, respectively.

Similarly, it is checked that Case 9 corresponds to Condition (2) of this theorem with i = 1, j = 0 and k = 0; Cases 5 and 7 correspond to Condition (2) with i = 0, j = 1 and k = 0; Cases 3 and 6 correspond to Condition (2) with i = 0, j = 0 and k = 1; and Cases 8 and 10 correspond to Condition (3) with i = 1 and j = 0.

The result then follows from the above discussions.

Cases	μ_1	$ ilde{\mu}$	μ'_1	$ ilde{\mu}'$	$\tilde{\mu} + \tilde{\mu}'$
1	0	$\neq \mathbf{0_{m-1}}$	0	$\neq \mathbf{0_{m-1}}$	$\neq \mathbf{0_{m-1}}$
2	0	$\neq \mathbf{0_{m-1}}$	1	0_{m-1}	$\neq \mathbf{0_{m-1}}$
3	0	$ eq \mathbf{0_{m-1}} onumber$	1	$ eq 0_{m-1} $	0_{m-1}
4	0	$ eq \mathbf{0_{m-1}} onumber$	1	$ eq 0_{m-1} $	$\neq \mathbf{0_{m-1}}$
5	1	0_{m-1}	0	$ eq \mathbf{0_{m-1}} onumber$	$\neq \mathbf{0_{m-1}}$
6	1	0_{m-1}	1	$ eq 0_{m-1} $	$\neq \mathbf{0_{m-1}}$
7	1	$\neq \mathbf{0_{m-1}}$	0	$\neq \mathbf{0_{m-1}}$	0_{m-1}
8	1	$ eq \mathbf{0_{m-1}} onumber$	0	$ eq 0_{m-1} $	$\neq \mathbf{0_{m-1}}$
9	1	$\neq \mathbf{0_{m-1}}$	1	0 _{m-1}	$\neq \mathbf{0_{m-1}}$
10	1	$ eq \mathbf{0_{m-1}}$	1	$ eq \mathbf{0_{m-1}} onumber$	$\neq \mathbf{0_{m-1}}$

Table 8: The cases of μ and μ' in the proof of Theorem 4

Remark 7. Note that for m = 2, all the three conditions of Theorem 4 are fulfilled if and only if the first two of them are fulfilled.

Remark 8. To find minimal linear codes from Construction 2, firstly one has to find a vectorial Boolean function G such that C_G is minimal, and then to find a Boolean function f satisfying Conditions (a) and (b) (In fact, according to [14, Theorem 15] and Remark 4, Conditions (a) and (b) just mean that the linear code C_f in (1) is minimal violating the AB condition). Finally, one has to verify the three conditions of Theorem 4. Therefore, compared with the construction of minimal linear codes violating the AB condition from Boolean functions, the method of this paper by using vectorial Boolean functions is more challenging.

6 Several minimal linear codes violating the AB condition

In this section, let n = 2t. Let $E_0, E_1, \ldots, E_{2^t}$ be t-dimensional linear subspaces of \mathbb{F}_2^n such that $E_i \cap E_j = \{\mathbf{0_n}\}$ for any $0 \le i < j \le 2^t$. Let $1_{E_i}(x)$ be the characteristic function of E_i , which is equal to 1 if $x \in E_i$, and is equal to 0 otherwise. Denote by $k = (k_0, k_1, \ldots, k_{t-1})_2$ the binary expansion of the integer $0 \le k \le 2^t - 1$, this is, $k = \sum_{i=0}^{t-1} k_i 2^i$. This representation is also regarded as a vector of \mathbb{F}_2^t for convenience. We give several minimal linear codes violating the AB condition by using Theorem 4. In the proof of our main result, we need the following lemma, which is inspired by [14, Lemma 16].

Lemma 3. Let $f(x) = \sum_{i=1}^{s} 1_{E_i}(x)$ for some $1 \le s \le 2^t$. Then the Walsh-Hadamard transform of f is given by

$$W_f(\nu) = \begin{cases} 2^n - 2s(2^t - 1) - 2f(\mathbf{0_n}), & \text{if } \nu = \mathbf{0_n}; \\ -2^{t+1} + 2s - 2f(\mathbf{0_n}), & \text{if } \nu \in \bigcup_{i=1}^s E_i^{\perp} \setminus \{\mathbf{0_n}\}; \\ 2s - 2f(\mathbf{0_n}), & \text{otherwise.} \end{cases}$$

Proof. By the relation $(-1)^{f(x)} = 1 - 2f(x)$, for any $\nu \in \mathbb{F}_2^n$, we have

$$W_f(\nu) = \sum_{x \in \mathbb{F}_2^n} (-1)^{\nu \cdot x} - 2 \sum_{x \in \mathbb{F}_2^n} f(x)(-1)^{\nu \cdot x}$$
$$= 2^n \delta_0(\nu) - 2 \left(f(\mathbf{0_n}) + \sum_{i=1}^s \left(\sum_{x \in E_i} (-1)^{\nu \cdot x} - 1 \right) \right),$$

where $\delta_0(x)$ equals 1 if $x = \mathbf{0}_{\mathbf{n}}$, and equals 0 otherwise. Then the result follows from the facts that $\sum_{x \in E_i} (-1)^{\nu \cdot x} = 2^t \mathbf{1}_{E_i^{\perp}}(\nu)$ with $E_i^{\perp} = \{x \in \mathbb{F}_2^n : \nu \cdot x = 0, \forall \nu \in E_i\}$, and that $E_i^{\perp} \cap E_j^{\perp} = \{\mathbf{0}_{\mathbf{n}}\}$ for any $1 \leq i < j \leq 2^t$.

6.1 The case n even and m = 2

In this subsection, we present an application of Theorem 4 for the case n even and m = 2, that is, for the case F = (f, g), where f and g are two Boolean functions on \mathbb{F}_2^n .

Theorem 5. Let n = 2t with $t \ge 3$. Let $f(x) = 1_{E_0}(x) + 1_{E_{2t}}(x)$ and

$$g_i(x) = \sum_{0 \le k = (k_0, k_1, \dots, k_{t-1})_2 \le 2^t - 1, k_i = 1} \mathbf{1}_{E_k}(x) + \mathbf{1}_{E_{2^t}}(x) + 1, \ \forall \ 0 \le i \le t - 1.$$
(25)

Let F_i be the (n, 2)-function given by $F_i(x) = (f(x), g_i(x))$, and let C_{F_i} be the linear code generated by Construction 1. Then C_{F_i} is a minimal linear code with parameters $[2^n - 1, n + 2, 2^{t+1} - 2]$ and weight distribution in Table 9. Moreover, C_{F_i} violates the AB condition.

Proof. By Lemma 3, we have

$$W_{f}(\nu) = \begin{cases} 2^{n} - 4(2^{t} - 1), & \text{if } \nu = \mathbf{0}_{\mathbf{n}}; \\ -2^{t+1} + 4, & \text{if } \nu \in (E_{0}^{\perp} \cup E_{2^{t}}^{\perp}) \setminus \{\mathbf{0}_{\mathbf{n}}\}; \\ 4, & \text{otherwise}, \end{cases}$$
(26)

which clearly satisfies Conditions (a) and (b) in Construction 2 when $t \ge 2$. For any $0 \le i \le t-1$, we also have

$$W_{g_i}(\nu) = \begin{cases} 2^t, & \text{if } \nu \in (\bigcup_{0 \le k \le 2^t - 1, k_i = 1} E_k^{\perp}) \cup E_{2^t}^{\perp}; \\ -2^t, & \text{otherwise}, \end{cases}$$
(27)

which is a bent function with $g_i(\mathbf{0_n}) = 0$. Hence C_{g_i} is a minimal linear code by Corollary 2. Consequently, f and every g_i satisfy the conditions of Construction 2. Therefore, each C_{F_i} violates the AB condition by Proposition 5.

Then by Remark 7, to show C_{F_i} is minimal for any $0 \le i \le t - 1$, we need to verify the first two conditions of Theorem 4. It is noted that

$$f(x) + g_i(x) = \sum_{0 \le k = (k_0, k_1, \dots, k_{t-1})_2 \le 2^t - 1, k_i = 1} 1_{E_k}(x) + 1_{E_0}(x) + 1.$$

Let $A_i(x) = f(x) + g_i(x)$. For any $\nu \in \mathbb{F}_2^n$, it holds

$$W_{A_i}(\nu) = \begin{cases} 2^t, & \text{if } \nu \in (\bigcup_{0 \le k \le 2^t - 1, k_i = 1} E_k^{\perp}) \cup E_0^{\perp}; \\ -2^t, & \text{otherwise.} \end{cases}$$
(28)

Hence, for any $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$, we have

$$W_{A_i}(\nu) \pm W_{A_i}(\nu') \in \{0, \pm 2^{t+1}\},\$$

which can never be equal to 2^n when $t \ge 2$. In other words, Condition (1) of Theorem 4 is satisfied. In addition, for any $\nu, \nu' \in \mathbb{F}_2^n$ and any $p, q, r \in \mathbb{F}_2$, it is easily checked that

$$(-1)^{p}W_{f}(\nu) + (-1)^{q}W_{g_{i}}(\nu') + (-1)^{r}W_{A_{i}}(\nu+\nu') \in \left\{ \pm (2^{n} - 4(2^{t} - 1)), \pm (2^{n} - 2^{t+1} + 4), \pm (2^{n} - 6 \times 2^{t} + 4), \pm (2^{t+1} - 4), \pm (2^{t+1} + 4), \pm (2^{t+2} - 4), \pm 4 \right\},$$

which also can never be equal to 2^n when $t \ge 2$, and thus Condition (2) of Theorem 4 is satisfied. Hence C_{F_i} is minimal. Moreover, according to Propositions 2 and 4, and Relations (26), (27) and (28), we can obtain the weight distribution of C_{F_i} , as listed in Table 9.

Weight	Frequency
0	1
2^{n-1}	$2^n - 1$
$2^{t+1} - 2$	1
$2^{n-1} + 2^t - 2$	$2^{t+1} - 2$
$2^{n-1}-2$	$2^n - 2^{t+1} + 1$
$2^{n-1} - 2^{t-1}$	$2^n + 2^t$
$2^{n-1} + 2^{t-1}$	$2^{n} - 2^{t}$

Table 9: Weight distribution of C_{F_i} in Theorem 5

Remark 9. Let $f(x) = 1_{E_{2^t-1}}(x) + 1_{E_{2^t}}(x)$, and

$$g_i(x) = \sum_{0 \le k = (k_0, k_1, \dots, k_{t-1})_2 \le 2^t - 1, k_i = 0} \mathbf{1}_{E_k}(x) + \mathbf{1}_{E_{2^t}}(x) + 1, \ \forall \ 0 \le i \le t - 1.$$
(29)

Then one can get a similar result as that of Theorem 5.

Example 3. Let n = 6. Then the linear code C_{F_i} in Theorem 5 is a minimal binary linear code with parameters [63, 8, 14] and weight enumerator

$$1 + z^{14} + 72z^{28} + 49z^{30} + 63z^{32} + 56z^{36} + 14z^{38}.$$

Clearly, $w_{\min}/w_{\max} = 14/38 < 1/2$.

Comparing the parameters of C_{F_i} in Example 3 with [14, Example 20], the dimension of C_{F_i} is larger than that of [14, Example 20].

It is noted that Theorem 5 cannot be generalized to the case m > 2 by letting $G = (g_0, g_1, \ldots, g_{r-1})$ with $2 \leq r \leq t$ (which is a vectorial bent function by Lemma 4), even the Boolean function f in Theorem 5 and the newly defined function G satisfy the conditions of Construction 2. Because in this case, one can check that though the conditions of Theorem 4 are satisfied when wt($\tilde{\mu}$) is odd, Condition (2) of Theorem 4 is not satisfied when wt($\tilde{\mu}$) is even. So we have to find another function f satisfying the conditions of Construction 2 for generalization, which is done in the next section.

6.2 The case n even and $2 < m \le t+1$

In this subsection, we present an application of Theorem 4 for the case n even and $2 < m \leq t+1$, that is, the case F = (f, G), where f is a Boolean function on \mathbb{F}_2^n , and G is an (n, r)-function with $2 \leq r = m - 1 \leq t$. To this end, we first give a class of vectorial bent functions, which is obtained by modifying the vectorial bent functions in [21, 22].

Lemma 4. Let $G = (g_0, g_1, \ldots, g_{r-1})$ be an (n, r)-function with $2 \le r \le t$, g_i is defined by (25) for any $0 \le i \le r-1$. Then G is a vectorial bent function.

Proof. For any $\tilde{\mu} = (\mu_0, \mu_1, \dots, \mu_{r-1}) \in \mathbb{F}_2^{r*}$, we have

$$\tilde{\mu} \cdot G(x) = \sum_{0 \le k \le 2^{t} - 1} \left(\sum_{0 \le i \le r - 1, k_{i} = 1} \mu_{i} \right) \mathbf{1}_{E_{k}}(x) + (\mathbf{1}_{E_{2^{t}}}(x) + 1) \sum_{i=0}^{r-1} \mu_{i}$$

$$= \sum_{0 \le k \le 2^{t} - 1, \tilde{\mu} \cdot \tilde{k} = 1} \mathbf{1}_{E_{k}}(x) + (\mathbf{1}_{E_{2^{t}}}(x) + 1) \sum_{i=0}^{r-1} \mu_{i}$$

$$= \begin{cases} \sum_{0 \le k \le 2^{t} - 1, \tilde{\mu} \cdot \tilde{k} = 1} \mathbf{1}_{E_{k}}(x) + \mathbf{1}_{E_{2^{t}}}(x) + 1, & \text{if wt}(\tilde{\mu}) \text{ is odd}; \\ \sum_{0 \le k \le 2^{t} - 1, \tilde{\mu} \cdot \tilde{k} = 1} \mathbf{1}_{E_{k}}(x), & \text{if wt}(\tilde{\mu}) \text{ is even}, \end{cases}$$
(30)

where $\tilde{k} = (k_0, k_1, \dots, k_{r-1})$. Since the cardinality of the set

$$\left\{ 0 \le k = (k_0, k_1, \dots, k_{t-1})_2 \le 2^t - 1 : \tilde{\mu} \cdot \tilde{k} = 1, \tilde{\mu} \in \mathbb{F}_2^{r*} \right\}$$

is equal to 2^{t-1} , by Lemma 3 we have:

1) When $wt(\tilde{\mu})$ is odd, it holds that

$$W_G(\tilde{\mu}, \nu) = \begin{cases} 2^t, & \text{if } \nu \in (\bigcup_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 1} E_k^{\perp}) \cup E_{2^t}^{\perp}; \\ -2^t, & \text{otherwise.} \end{cases}$$
(31)

2) When $wt(\tilde{\mu})$ is even, it holds that

$$W_G(\tilde{\mu}, \nu) = \begin{cases} -2^t, & \text{if } \nu \neq \mathbf{0_n}, \nu \in \bigcup_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 1} E_k^{\perp};\\ 2^t, & \text{otherwise.} \end{cases}$$
(32)

This completes the proof.

To give an infinite family of minimal binary linear codes violating the AB condition, we also need the following results.

Lemma 5. [4, 27] Let φ_1, φ_2 and φ_3 be three Boolean functions on \mathbb{F}_2^n . Let $f(x) = \varphi_1(x)\varphi_2(x) + \varphi_1(x)\varphi_3(x) + \varphi_2(x)\varphi_3(x)$. Then for any $\nu \in \mathbb{F}_2^n$, it holds that

$$W_f(\nu) = \frac{1}{2} \big(W_{\varphi_1}(\nu) + W_{\varphi_2}(\nu) + W_{\varphi_3}(\nu) - W_{\varphi_4}(\nu) \big),$$

where $\varphi_4 = \varphi_1 + \varphi_2 + \varphi_3$.

A generalized version of Lemma 5 is referred to [20].

Lemma 6. Let n = 2t with $t \ge 2$. Let $a, b \in \mathbb{F}_2^{n*}$ with $a \ne b$, and let E be a t-dimensional linear subspace of \mathbb{F}_2^n . Then the Boolean function f on \mathbb{F}_2^n defined by

$$f(x) = 1_E(x) + (a \cdot x)(b \cdot x) + 1$$
(33)

satisfies Condition (a) of Construction 2 if and only if $a, b, a + b \notin E^{\perp}$.

Proof. Let $\varphi(x) = 1_E(x) + 1$. By Lemma 3, one has

$$W_{\varphi}(\nu) = \begin{cases} -2^{n} + 2^{t+1}, & \text{if } \nu = \mathbf{0}_{\mathbf{n}};\\ 2^{t+1}, & \text{if } \nu \in E^{\perp} \setminus \{\mathbf{0}_{\mathbf{n}}\};\\ 0, & \text{otherwise.} \end{cases}$$
(34)

Note that $f(x) = \varphi(x) + (a \cdot x)(b \cdot x) = \varphi_1(x)\varphi_2(x) + \varphi_1(x)\varphi_3(x) + \varphi_2(x)\varphi_3(x)$, where $\varphi_1(x) = \varphi(x)$, $\varphi_2(x) = \varphi(x) + a \cdot x$ and $\varphi_3(x) = \varphi(x) + b \cdot x$. By Lemma 5, we have

$$W_f(\nu) = \frac{1}{2} \big(W_{\varphi}(\nu) + W_{\varphi}(\nu+a) + W_{\varphi}(\nu+b) - W_{\varphi}(\nu+a+b) \big), \ \forall \ \nu \in \mathbb{F}_2^n.$$
(35)

Now if $a, b, a + b \notin E^{\perp}$, we can deduce from (35) that

$$W_{f}(\nu) = \begin{cases} -2^{n-1} + 2^{t}, & \text{if } \nu \in \{\mathbf{0_{n}}, a, b\};\\ 2^{n-1} - 2^{t}, & \text{if } \nu = a + b;\\ 2^{t}, & \text{if } \nu \in (E^{\perp} \cup a + E^{\perp} \cup b + E^{\perp}) \setminus \{\mathbf{0_{n}}, a, b, a + b\};\\ -2^{t}, & \text{if } \nu \in (a + b + E^{\perp}) \setminus \{\mathbf{0_{n}}, a, b, a + b\};\\ 0, & \text{otherwise.} \end{cases}$$
(36)

Then it is easily checked that Condition (a) of Construction 2 is satisfied when $t \ge 2$.

Conversely, assume that $a, b \in E^{\perp} \setminus \{\mathbf{0}_n\}$ with $a \neq b$, we deduce from (35) that

$$W_f(\mathbf{0_n}) = W_f(a) = W_f(b) = -2^{n-1} + 2^{t+1}$$
 and $W_f(a+b) = 2^{n-1} + 2^{t+1}$,

from which we can get $W_f(a+b) - W_f(a) = 2^n$. This means that Condition (a) of Construction 2 is not satisfied for the case $a, b \in E^{\perp} \setminus \{\mathbf{0_n}\}$ with $a \neq b$. Similarly, one can show that Condition (a) is also not satisfied for the cases $\mathbf{0} \neq a \in E^{\perp}, b \notin E^{\perp}$ and $\mathbf{0} \neq b \in E^{\perp}, a \notin E^{\perp}$. This completes the proof.

Remark 10. When $t \ge 3$, it is easily checked that the function f in Lemma 6 also satisfies Condition (b) of Construction 2.

Lemma 7. Let φ be a bent function on \mathbb{F}_2^{2t} . Let $a, b \in \mathbb{F}_2^{2t}$, and $h(x) = \varphi(x) + (a \cdot x)(b \cdot x)$. Then for any $\nu \in \mathbb{F}_2^{2t}$, it holds that $W_h(\nu) \in \{0, \pm 2^t, \pm 2^{t+1}\}$.

Proof. Since φ is bent, by (35), for any $\nu \in \mathbb{F}_2^{2t}$, we have

$$W_h(\nu) = 2^{t-1} \left((-1)^{\varphi^*(\nu)} + (-1)^{\varphi^*(\nu+a)} + (-1)^{\varphi^*(\nu+b)} - (-1)^{\varphi^*(\nu+a+b)} \right),$$

where φ^* is the dual of φ . Note that

$$(-1)^{\varphi^*(\nu)} + (-1)^{\varphi^*(\nu+a)} + (-1)^{\varphi^*(\nu+b)} - (-1)^{\varphi^*(\nu+a+b)} \in \{0, \pm 2, \pm 4\}$$

for any $a, b \in \mathbb{F}_2^{2t}$. Therefore, $W_h(\nu) \in \{0, \pm 2^t, \pm 2^{t+1}\}$ for any $\nu \in \mathbb{F}_2^{2t}$.

Now we give an infinite family of minimal binary linear codes violating the AB condition.

Theorem 6. Let n = 2t with $t \ge 3$. Let f be a Boolean function on \mathbb{F}_2^n defined by

$$f(x) = 1_{E^{2^{t}}}(x) + (a \cdot x)(b \cdot x) + 1,$$

where $a, b, a + b \in E_0^{\perp} \setminus \{\mathbf{0_n}\}$. Let G be the (n, r)-bent function generated by Lemma 4 with $2 \leq r \leq t$. Let F = (f, G), and let C_F be the binary linear code generated by Construction 1. Then C_F is a minimal linear code with parameters $[2^n - 1, n + r + 1, 2^{n-2} + 2^{t-1}]$ violating the AB condition.

Proof. Since $a, b, a + b \in E_0^{\perp} \setminus \{\mathbf{0_n}\}$, one has $a, b, a + b \notin E_{2^t}^{\perp} \setminus \{\mathbf{0_n}\}$. By Lemma 6 and Remark 10, we obtain that f is a Boolean function satisfying Conditions (a) and (b) of Construction 2. In addition, as G is an (n, r)-bent function with $G(\mathbf{0_n}) = \mathbf{0_r}$, C_G defined by (20) is a minimal linear code by Corollary 2. Consequently, f and G satisfy the conditions of Construction 2, and thus C_F violates the AB condition by Proposition 5. Below we show that C_F is minimal, for which we need to verify the three conditions of Theorem 4.

Firstly, for any $\tilde{\mu} \in \mathbb{F}_2^{r*}$, we obtain from (30) that

$$f(x) + \tilde{\mu} \cdot G(x) = \begin{cases} \sum_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 1} 1_{E_k}(x) + (a \cdot x)(b \cdot x), & \text{if wt}(\tilde{\mu}) \text{ is odd}; \\ \sum_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 1} 1_{E_k}(x) + 1_{E_{2^t}}(x) + (a \cdot x)(b \cdot x) + 1, & \text{if wt}(\tilde{\mu}) \text{ is even.} \end{cases}$$

Let $A_{\tilde{\mu}}(x) = f(x) + \tilde{\mu} \cdot G(x)$. Then by (30), (31), (32) and Lemma 7, we obtain that

$$W_{A_{\tilde{\mu}}}(\nu) \in \{0, \pm 2^t, \pm 2^{t+1}\}, \quad \forall \ \nu \in \mathbb{F}_2^n.$$
 (37)

Hence, for any $\tilde{\mu} \in \mathbb{F}_2^{r^*}$, and any $\nu, \nu' \in \mathbb{F}_2^n$ with $\nu \neq \nu'$, we have

$$W_{A_{\tilde{\mu}}}(\nu) \pm W_{A_{\tilde{\mu}}}(\nu') \in \{0, \pm 2^{t}, \pm 2^{t+1}, \pm 3 \cdot 2^{t}, \pm 2^{t+2}\},\$$

which can never be equal to 2^n when $t \ge 3$. So Condition (1) of Theorem 4 is satisfied. In addition, for any $\tilde{\mu} \in \mathbb{F}_2^{r*}$, $\nu, \nu' \in \mathbb{F}_2^n$, and for any $i, j, k \in \mathbb{F}_2$, it is easy to check from (31), (32), (36) and (37) that

$$(-1)^{i}W_{f}(\nu) + (-1)^{j}W_{G}(\tilde{\mu},\nu') + (-1)^{k}W_{A_{\tilde{\mu}}}(\nu+\nu') \in \left\{0,\pm 2^{t},\pm 2^{t+1},\pm 3\cdot 2^{t},\pm 2^{t+2},\pm 2^{n-1},\pm 2^{n-1}\pm 2^{t},\pm 2^{n-1}\pm 2^{t+1},\pm (2^{n-1}-2^{t+1}),\pm (2^{n-1}-3\cdot 2^{t}),\pm (2^{n-1}-2^{t+2})\right\},$$

which also cannot be equal to 2^n when t > 2. Hence Condition (2) of Theorem 4 is satisfied. Moreover, for any $\tilde{\mu}, \tilde{\mu}' \in \mathbb{F}_2^{r*}$ with $\tilde{\mu} \neq \tilde{\mu}'$, any $\nu, \nu' \in \mathbb{F}_2^n$, and for any $i, j \in \mathbb{F}_2$, it is easy to check from (31), (32) and (37) that

$$W_{A_{\tilde{\mu}}}(\nu) + (-1)^{i} W_{G}(\tilde{\mu}',\nu') + (-1)^{j} W_{A_{\tilde{\mu}+\tilde{\mu}'}}(\nu+\nu') \in \left\{0,\pm 2^{t},\pm 2^{t+1},\pm 3\cdot 2^{t},\pm 2^{t+2},\pm 5\cdot 2^{t}\right\},$$

which can never be equal to 2^n when t > 2. Therefore, Condition (3) of Theorem 4 is also satisfied. Consequently, C_F is minimal by Theorem 4.

Finally, according to Propositions 1 and 4, Remark 2, Lemma 4, and Equations (36) and (37), one can obtain that C_F is a $[2^n - 1, n + r + 1, 2^{n-2} + 2^{t-1}]$ code. The proof is finished. \Box

Note that the weight distribution of C_F in Theorem 6 is not obtained. It seems difficult to get the exact numbers of $\nu \in \mathbb{F}_2^n$ in (37) such that $W_{A_{\tilde{\mu}}}(\nu) = 0, 2^t, -2^t, 2^{t+1}, -2^{t+1}$, respectively. If those numbers are obtained, then by Propositions 2 and 4, Relations (31), (32) and (36), the weight distribution of C_F can also be obtained.

Remark 11. In Lemma 4, let g_i be defined by (29) for any $0 \le i \le r-1$. Then for any $\tilde{\mu} \in \mathbb{F}_2^{r*}$, one has

$$\tilde{\mu} \cdot G(x) = \begin{cases} \sum_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 0} \mathbf{1}_{E_k}(x) + \mathbf{1}_{E_{2^t}}(x) + 1, & \text{if } \operatorname{wt}(\tilde{\mu}) \text{ is odd}; \\ \sum_{0 \le k \le 2^t - 1, \tilde{\mu} \cdot \tilde{k} = 1} \mathbf{1}_{E_k}(x), & \text{if } \operatorname{wt}(\tilde{\mu}) \text{ is even}, \end{cases}$$

which also implies that $G = (g_0, g_1, \ldots, g_{r-1})$ is vectorial bent. Applying this result to Theorem 6, one can obtain a similar result as that of Theorem 6.

6.3 The case n even and m = n + 1

In this subsection, let n = 2t, *i* be an positive integer and $\lambda = \gcd(n, i)$ such that $\frac{n}{\lambda}$ is odd. We present an application of Theorem 4 for the case m = n + 1, that is, the case F = (f, G), where f is a Boolean function on \mathbb{F}_{2^n} and G is an (n, n)-function. Here we choose $G(x) = x^{2^i+1}$ to be a Gold function. We need the following known result on Gold functions.

Lemma 8. [11] Let $\lambda = \gcd(n, i)$ such that $\frac{n}{\lambda}$ is odd. Then $G(x) = x^{2^i+1}$ is a vectorial plateaued (n, n)-function with single amplitude $2^{\frac{n+\lambda}{2}}$.

We also need the following result.

Lemma 9. Let n = 2t and $\lambda = \gcd(n, i)$ such that $\frac{n}{\lambda}$ is odd. Let $E = \mathbb{F}_{2^t}$. For any $\tilde{\mu} \in \mathbb{F}_{2^n}^*$, let

$$\varphi(x) = \operatorname{Tr}_{1}^{n}(\tilde{\mu}x^{2^{i}+1}) + 1_{E}(x).$$
(38)

Then for any $\nu \in \mathbb{F}_{2^n}$, it holds that

$$W_{\varphi}(\nu) \in \left\{0, -2^{t+1}, \pm 2^{\frac{n+\lambda}{2}}, \pm 2^{\frac{t+\lambda}{2}+1}, \pm 2^{\frac{n+\lambda}{2}} - 2^{t+1}, \pm 2^{\frac{n+\lambda}{2}} \pm 2^{\frac{t+\lambda}{2}+1}\right\}.$$

Proof. The Walsh-Hadamard transform of φ at $\nu \in \mathbb{F}_{2^n}$ is that

$$W_{\varphi}(\nu) = \sum_{x \in \mathbb{F}_{2^{n}}} (-1)^{\varphi(x) + \operatorname{Tr}_{1}^{n}(\nu x)}$$
$$= \sum_{x \in \mathbb{F}_{2^{n}}} (-1)^{\operatorname{Tr}_{1}^{n}(\tilde{\mu}x^{2^{i+1}} + \nu x)} - 2 \sum_{x \in \mathbb{F}_{2^{t}}} (-1)^{\operatorname{Tr}_{1}^{t}\left((\tilde{\mu} + \tilde{\mu}^{2^{t}})x^{2^{i+1}} + (\nu + \nu^{2^{t}})x\right)}.$$

Observe that

$$\sum_{x \in \mathbb{F}_{2^t}} (-1)^{\operatorname{Tr}_1^t \left((\tilde{\mu} + \tilde{\mu}^{2^t}) x^{2^i + 1} + (\nu + \nu^{2^t}) x \right)} = \begin{cases} 2^t, & \text{if } \tilde{\mu} \in \mathbb{F}_{2^t}^*, \nu \in \mathbb{F}_{2^t}; \\ 0, & \text{if } \tilde{\mu} \in \mathbb{F}_{2^t}^*, \nu \notin \mathbb{F}_{2^t}. \end{cases}$$

Then by Lemma 8, we obtain that

$$\sum_{x \in \mathbb{F}_{2^n}} (-1)^{\operatorname{Tr}_1^n(\tilde{\mu}x^{2^{i+1}} + \nu x)} \in \left\{ 0, \pm 2^{\frac{n+\lambda}{2}} \right\}$$

and

$$\sum_{x \in \mathbb{F}_{2^t}} (-1)^{\operatorname{Tr}_1^t \left((\tilde{\mu} + \tilde{\mu}^{2^t}) x^{2^i + 1} + (\nu + \nu^{2^t}) x \right)} \in \left\{ 0, \pm 2^{\frac{t + \frac{\lambda}{2}}{2}}, 2^t \right\},$$

since $\lambda = 2 \operatorname{gcd}(t, i)$ and $\frac{t}{\operatorname{gcd}(t, i)} = \frac{n}{\lambda}$ as $\frac{n}{\lambda}$ is odd. The result follows then from the above two relations.

Now we present our main result of this subsection.

2

Theorem 7. Let n = 2t > 8, $2 \le i \le n-1$ and $\lambda = \gcd(n,i)$ such that $\frac{n}{\lambda}$ is odd. Let $E = \mathbb{F}_{2^t}$ and $a, b \in \mathbb{F}_{2^n} \setminus E$ such that $a + b \notin E$. Let $G(x) = x^{2^i+1}$, $f(x) = 1_E(x) + \operatorname{Tr}_1^n(ax)\operatorname{Tr}_1^n(bx) + 1$ and F = (f,G). Let C_F be the linear code given by (14). Then C_F is a minimal linear code violating the AB condition with parameters $[2^n - 1, 2n + 1, 2^{n-2} + 2^{t-1}]$.

Proof. Note that $E^{\perp} = \{x \in \mathbb{F}_{2^n} : \operatorname{Tr}_1^n(xy) = 0, \forall y \in E\} = \mathbb{F}_{2^t} = E$. Then by Lemma 6 and Remark 10, f is a Boolean function satisfying Conditions (a) and (b) of Construction 2. By Lemma 8 and Corollary 1, it is also clear that C_G is a minimal linear code. Therefore, f and Gsatisfy the conditions of Construction 2, and hence C_F violates the AB condition by Proposition 5. Below we show that C_F is minimal, for which we need to verify the three conditions of Theorem 4.

For any $\tilde{\mu} \in \mathbb{F}_{2^n}^*$, let

$$A_{\tilde{\mu}}(x) = f(x) + \operatorname{Tr}_{1}^{n}(\tilde{\mu}G(x)) = \varphi(x) + \operatorname{Tr}_{1}^{n}(ax)\operatorname{Tr}_{1}^{n}(bx) + 1,$$

where φ is the function in (38). Then by Lemmas 5 and 9, and Relation (35), we have

$$\max_{(\tilde{\mu},\nu)\in\mathbb{F}_{2n}^*\times\mathbb{F}_{2n}}|W_{A_{\tilde{\mu}}}(\nu)| \le 2^{\frac{n+\lambda}{2}+1}+2^{t+2},\tag{39}$$

since $\lambda \leq t$ (recall that $\lambda = \gcd(n, i), 2 \leq i \leq n-1$). Hence, for any $\tilde{\mu} \in \mathbb{F}_{2^n}^*$ and any $\nu, \nu' \in \mathbb{F}_{2^n}$ with $\nu \neq \nu'$, we have

$$W_{A_{\tilde{\mu}}}(\nu) \pm W_{A_{\tilde{\mu}}}(\nu') \le 2^{\frac{n+\lambda}{2}+2} + 2^{t+3}$$

which is strictly less than 2^n when t > 4. For any $\tilde{\mu} \in \mathbb{F}_{2^n}^*$, any $\nu, \nu' \in \mathbb{F}_{2^n}$ and for any $p, q, r \in \mathbb{F}_2$, by Lemma 8 and Relations (36) and (39), we have

$$(-1)^{p}W_{f}(\nu) + (-1)^{q}W_{G}(\tilde{\mu},\nu') + (-1)^{r}W_{A_{\tilde{\mu}}}(\nu+\nu') \le (2^{n-1}-2^{t}) + 2^{\frac{n+\lambda}{2}} + (2^{\frac{n+\lambda}{2}+1}+2^{t+2}),$$

which is also strictly less than 2^n when t > 4. For any distinct $\tilde{\mu}, \tilde{\mu}' \in \mathbb{F}_{2^n}^*$, any $\nu, \nu' \in \mathbb{F}_{2^n}$, and for any $p, q \in \mathbb{F}_2$, by Lemma 8 and Relation (39), we have

$$W_{A_{\tilde{\mu}}}(\nu) + (-1)^{p} W_{G}(\tilde{\mu}',\nu') + (-1)^{q} W_{A_{\tilde{\mu}+\tilde{\mu}'}}(\nu+\nu') \le (2^{\frac{n+\lambda}{2}+2}+2^{t+3}) + 2^{\frac{n+\lambda}{2}},$$

which is also strictly less than 2^n when t > 4. Therefore, C_F is minimal by Theorem 4.

Finally, according to Propositions 1 and 4, Remark 2, Lemma 8, and Relations (36) and (39), one can obtain that C_F is a $[2^n - 1, 2n + 1, 2^{n-2} + 2^{t-1}]$ code. This completes the proof. \Box

Note that the weight distribution of C_F in Theorem 7 is not obtained. We invite the interested readers to attack it.

Note also that Lemma 8 holds for any positive integer n. Thus, a natural question is that, whether Theorem 7 can be generalized to the case n odd. In this case, the function f in Theorem 7 cannot be used any more. So one has to find another function f satisfying Conditions (**a**) and (**b**) of Construction 2, such that the Walsh-Hadamard transform of $A_{\tilde{\mu}}(x) = f(x) + \operatorname{Tr}_{1}^{n}(\tilde{\mu}G(x))$ can be evaluated for any $\tilde{\mu} \in \mathbb{F}_{2^{n}}^{*}$. It seems very difficult to find such a function, and we leave it for further research.

Remark 12. As three applications of Theorem 4, Theorems 5, 6 and 7 give minimal binary linear codes C_F violating the AB condition for any $2 \le m \le \frac{n}{2} + 1$ and m = n + 1 when n is even. It would be interesting if someone can apply Theorem 4 to other cases, including other values m and the case n odd.

7 Concluding remarks

The main contributions of this paper are presented as follows:

- A generic construction of minimal linear codes from vectorial Boolean functions (Construction 1).
- A necessary and sufficient condition for the linear codes in Construction 1 to be minimal (Theorem 2).
- New three-weight minimal binary linear codes with complete weight distribution (Corollaries 1, 2 and 3).
- A generic construction of minimal linear codes violating the AB condition (Construction 2).
- A necessary and sufficient condition for the linear codes in Construction 2 to be minimal (Theorem 4).
- Three infinite families of minimal linear codes violating the AB condition (Theorems 5, 6 and 7).

As remarked in Conclusion of [14], to find an infinite family of minimal linear codes violating the AB condition is a hard problem in general. This paper focus on constructing minimal linear codes from vectorial Boolean functions. To the best of our knowledge, this is the first time that minimal linear codes are constructed from vectorial Boolean functions. Compared with known minimal linear codes, the dimensions of the minimal codes in this paper are in general better.

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References

- A. Ashikhmin and A. Barg, "Minimal vectors in linear codes," *IEEE Trans. Inf. Theory*, vol. 44, no. 5, pp. 2010-2017, Sep. 1998. 5
- [2] D. Bartoli and M. Bonini, "Minimal linear codes in odd characteristic," *IEEE Trans. Inf. Theory*, vol. 65, no. 7, pp. 4152-4155, Jul. 2019. 2
- [3] M. Bonini and M. Borello, "Minimal linear codes arising from blocking sets," J. Algebr. Combinatorics, vol. 53, pp. 327-341, 2021. 2, 3, 5, 6
- [4] C. Carlet, "On bent and highly nonlinear balanced/resilient functions and their algebraic immunities," in *Proc. AAECC*, vol. 3857. Berlin, Germany: Springer, 2006, pp. 1-28. 17
- [5] C. Carlet, "Vectorial Boolean Functions for Cryptography," Chapter of the monography Boolean Models and Methods in *Mathematics, Computer Science, and Engineering*, Y. Crama and P. Hammer, (eds), Cambridge University Press, pp. 398-469, 2010. 5
- [6] C. Carlet and C. Ding, "Nonlinearities of S-boxes," *Finite Fields Appl.*, vol. 13, no. 1, pp. 121-135, 2007. 2, 6
- [7] C. Carlet, C. Ding, and J. Yuan, "Linear codes from perfect nonlinear mappings and their secret sharing schemes," *IEEE Trans. Inf. Theory*, vol. 51, no. 6, pp. 2089-2102, Jun. 2005.
- [8] H. Chabanne, G. Cohen, and A. Patey, "Towards secure two-party computation from the wire-tap channel," in *Proc. ICISC*, in Lecture Notes in Computer Science, vol. 8565, H.-S. Lee and D.-G. Han, Eds. Berlin, Germany: Springer-Verlag, 2014, pp. 34-46. 1
- [9] S. Chang and J. Y. Hyun, "Linear codes from simplicial complexes," Des., Codes Cryptogr., vol. 86, no. 10, pp. 2167-2181, Oct. 2018. 1, 3, 5, 12
- [10] Y. Chen, X. Zeng, L. Zhang, and B. Xiao, "Linear codes from vectorial Boolean power functions," *Finite Fields Appl.*, vol. 67, 2020, Art. no. 101726. 2, 12
- [11] R. S. Coulter, "On the evaluation of a class of Weil sums in characteristic 2," New Zealand J Math., vol. 28, no. 2, pp. 171-184, 1999.
- [12] C. Ding, "Linear codes from some 2-designs," *IEEE Trans. Inf. Theory*, vol. 60, no. 6, pp. 3265-3275, Jun. 2015. 12
- [13] C. Ding, "A construction of binary linear codes from Boolean functions," Discrete Math., vol. 339, no. 9, pp. 2288-2303, 2016. 6, 12
- [14] C. Ding, Z. Heng, and Z. Zhou, "Minimal binary linear codes," *IEEE Trans. Inf. Theory*, vol. 64, no. 10, pp. 6536-6545, Oct. 2018. 1, 2, 3, 5, 6, 8, 11, 12, 14, 16, 21
- [15] C. Ding, A. Munemasa, and V. D. Tonchev, "Bent vectorial functions, codes and designs," *IEEE Trans. Inf. Theory*, vol. 65, no. 11, pp. 7533-7541, 2019.
- [16] C. Ding and J. Yuan, "Covering and secret sharing with linear codes," in *Discrete Mathematics and Theoretical Computer Science*, vol. 2731. Berlin, Germany: Springer-Verlag, 2003, pp. 11-25.
- [17] K. Ding and C. Ding, "A class of two-weight and three-weight codes and their applications in secret sharing," *IEEE Trans. Inf. Theory*, vol. 61, no. 11, pp. 5835-5842, Nov. 2015. 1
- [18] Z. Heng, C. Ding, and Z. Zhou, "Minimal linear codes over finite fields," *Finite Fields Appl.*, vol. 54, pp. 176-196, Nov. 2018. 2, 6

- [19] Z. Heng and Q. Yue, "Two classes of two-weight linear codes," *Finite Fields Appl.*, vol. 38, pp. 72-92, 2016.
- [20] Y. Li, H. Kan, S. Mesnager, J. Peng, C.H. Tan, and L. Zheng, "Generic constructions of (Boolean and vectorial) bent functions and their consequences," *IEEE Trans. Inf. Theory*, vol. 68, no. 4, pp. 2735-2751, 2022. 17
- [21] Y. Li, J. Peng, and C.H. Tan, "An answer to an open problem of Mesnager on bent functions," Inf. Process. Lett., vol. 161, 2020, Art. no. 105974. 16
- [22] Y. Li, J. Peng, C.H. Tan, H. Kan, and L. Zheng, "Further constructions of bent functions and their duals," *IET Inf. Secur.*, vol. 15, pp. 87-97, 2021. 16
- [23] X. Li and Q. Yue, "Four classes of minimal binary linear codes with $w_{\min}/w_{\max} < 1/2$ derived from Boolean functions," *Des. Codes Cryptogr.*, vol. 88, pp. 257-271, 2020. 2, 3, 5
- [24] W. Lu, X. Wu, and X. Cao, "The parameters of minimal linear codes," *Finite Fields Appl.*, vol. 71, Mar. 2021, Art. no. 101799. 2
- [25] G. Luo and X. Cao, "Five classes of optimal two-weight linear codes," Cryptogr. Commun., vol. 10, no. 6, pp. 1119-1135, 2018. 1
- [26] G. Luo, X. Cao, S. Xu, and J. Mi, "Binary linear codes with two or three weights from niho exponents," *Cryptogr. Commun.*, vol. 10, no. 2, pp. 301-318, 2018. 1, 12
- [27] S. Mesnager, "Several new infinite families of bent functions and their duals," *IEEE Trans. Inf. Theory*, vol. 60, no. 7, pp. 4397-4407, 2014. 17
- [28] S. Mesnager, "Linear codes with few weights from weakly regular bent functions based on a generic construction," *Cryptogr. Commun.*, vol. 9, no. 1, pp. 71-84, Jan. 2017. 1, 12
- [29] S. Mesnager, F. Ozbudak, and A. Sınak, "Linear codes from weakly regular plateaued functions and their secret sharing schemes," *Des., Codes Cryptogr.*, vol. 87, pp. 463-480, 2019. 1, 12
- [30] S. Mesnager, Y. Qi, H. Ru, and C. Tang, "Minimal linear codes from characteristic functions," *IEEE Trans. Inf. Theory*, vol. 66, no. 9, pp. 5404-5413, Sep. 2020. 2, 3, 5, 6
- [31] S. Mesnager and A. Sınak, "Several classes of minimal linear codes with few weights from weakly regular plateaued functions," *IEEE Trans. Inf. Theory*, vol. 66, no. 4, pp. 2296-2310, 2020. 2, 6
- [32] E. Pasalic, R. Rodríguez, F. Zhang, and Y. Wei, "Several classes of minimal binary linear codes violating the Aschikhmin-Barg bound," *Cryptogr. Commun.*, vol. 13, pp. 637-659, 2021. 2, 3, 5, 7
- [33] M. Shi, Y. Guan, and P. Solé, "Two new families of two-weight codes," IEEE Trans. Inf. Theory, vol. 63, no. 10, pp. 6240-6246, 2017. 1
- [34] C. Tang, N. Li, Y. Qi, Z. Zhou, and T. Helleseth, "Linear codes with two or three weights from weakly regular bent functions," *IEEE Trans. Inf. Theory*, vol. 62, no. 3, pp. 1166-1176, Mar. 2016. 1
- [35] C. Tang, Y. Qiu, Q. Liao, and Z. Zhou, "Full Characterization of Minimal Linear Codes as Cutting Blocking Sets," *IEEE Trans. Inf. Theory*, vol. 67, no. 6, pp. 3690-3700, 2021.
- [36] D. Tang, C. Carlet, and Z. Zhou, "Binary linear codes from vectorial Boolean functions and their weight distribution," *Discrete Math.*, vol. 340, no. 12, pp. 3055-3072, 2017. 2
- [37] D. Tang and X. Li, "A note on the minimal binary linear code," Cryptogr. Commun., vol. 12, pp. 375-388, 2020. 12

- [38] R. Tao, T. Feng, and W. Li, "A construction of minimal linear codes from partial difference sets," *IEEE Trans. Inf. Theory*, vol. 67, no. 6, pp. 3724-3734, 2021. 2, 3, 12
- [39] T. Wadayama, T. Hada, K. Wakasugi, and M. Kasahara, "Upper and lower bounds on maximum nonlinearity of n-input m-output Boolean function," Des., Codes Cryptogr., vol. 23, no. 1, pp. 23-33, 2001. 2, 6
- [40] Q. Wang, K. Ding, and R. Xue, "Binary linear sodes with two weights," *IEEE Commun. Lett.*, vol. 19, no. 7, pp. 1097-1100, 2015. 1
- [41] G. Xu and L. Qu, "Three classes of minimal linear codes over the finite fields of odd characteristic," *IEEE Trans. Inf. Theory*, vol. 65, no. 11, pp. 7067-7078, Nov. 2019. 2
- [42] G. Xu, L. Qu and X. Cao, "Minimal linear codes from Maiorana-McFarland functions," *Finite Fields Appl.*, vol. 65, 2020, Art. no. 101688. 2, 3, 6
- [43] G. Xu, L. Qu, and G. Luo, "Minimal linear codes from weakly regular bent functions," *Cryptogr. Commun.*, vol. 14, pp. 415-431, 2022. 2
- [44] S. Yang and Z. Yao, "Complete weight enumerators of a family of three-weight linear codes," Des. Codes Cryptogr., vol. 82, no. 3, pp. 663-674, 2017. 1
- [45] J. Yuan and C. Ding, "Secret sharing schemes from three classes of linear codes," IEEE Trans. Inf. Theory, vol. 52, no. 1, pp. 206-212, Jan. 2006. 1
- [46] W. Zhang, H. Yan, and H. Wei, "Four families of minimal binary linear codes with $w_{\min}/w_{\max} \leq 1/2$," Appl. Algebra Eng. Commun. Comput., vol. 30, pp. 75-184, 2019. 2, 3, 5
- [47] F. Zhang, E. Pasalic, R. Rodríguez, and Y. Wei, "Wide minimal binary linear codes from the general Maiorana-McFarland class," *Des. Codes Cryptogr.*, vol. 89, pp. 1485-1507, 2021.
 2, 3, 5
- [48] F. Zhang, E. Pasalic, R. Rodríguez, and Y. Wei, "Minimal binary linear codes: a general framework based on bent concatenation," *Des. Codes Cryptogr.*, vol. 90, pp. 1289-1318, 2022. 2, 5, 12
- [49] Z. Zhou, N. Li, C. Fan, and T. Helleseth, "Linear codes with two or three weights from quadratic bent functions," *Des. Codes Cryptogr.*, vol. 81, no. 2, pp. 283-295, 2016.