

ON A GENERALIZATION OF BUSEMANN'S INTERSECTION INEQUALITY

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ABSTRACT. Busemann's intersection inequality gives an upper bound for the volume of the intersection body of a star body in terms of the volume of the body itself. Koldobsky, Paouris, and Zymonopoulou asked if there is a similar result for k -intersection bodies. We solve this problem for star bodies that are close to the Euclidean ball in the Banach-Mazur distance. We also improve a bound obtained by Koldobsky, Paouris, and Zymonopoulou for general star bodies in the case when k is proportional to the dimension.

1. INTRODUCTION

We say that a set K in \mathbb{R}^n is *star-shaped* if for every $x \in K$ the closed line segment connecting x to the origin lies in K . A compact set K in \mathbb{R}^n is called a *star body* if it is star-shaped and its *radial function* defined by

$$\rho_K(\xi) = \max\{a \geq 0 : a\xi \in K\}, \quad \xi \in S^{n-1},$$

is positive and continuous. Geometrically, $\rho_K(\xi)$ is the distance from the origin to the point on the boundary in the direction of ξ .

Let K and L be star bodies in \mathbb{R}^n . Following Lutwak [L], we say that L is the *intersection body* of K if

$$\rho_L(\xi) = \text{vol}_{n-1}(K \cap \xi^\perp),$$

for every $\xi \in S^{n-1}$.

For any star body K its intersection body always exists and we will denote it by $I(K)$. The well-known Busemann intersection inequality asserts that

$$\text{vol}_n(I(K)) \leq \frac{\kappa_{n-1}^n}{\kappa_n^{n-2}} \text{vol}_n(K)^{n-1}$$

with equality if and only if K is a centered ellipsoid; see, e.g., [Ga, p. 373]. Here and below,

$$\kappa_p = \frac{\pi^{\frac{p}{2}}}{\Gamma\left(1 + \frac{p}{2}\right)},$$

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which equals the volume of the unit Euclidean ball in \mathbb{R}^p when p is a positive integer.

Koldobsky introduced a generalization of the notion of intersection body; see [K, p. 75]. Let K and L be origin-symmetric star bodies in \mathbb{R}^n and k be an integer, $1 \leq k \leq n-1$. We say that L is the k -intersection body of K if

$$\text{vol}_k(L \cap H) = \text{vol}_{n-k}(K \cap H^\perp)$$

for every k -dimensional subspace H of \mathbb{R}^n .

For a given origin-symmetric star body K its k -intersection body may not exist, but when it does, we will denote it by $I_k(K)$.

Koldobsky, Paouris, and Zymonopoulou [KPZ] asked whether an analogue of Busemann's intersection inequality holds for k -intersection bodies. Namely, if K is an origin-symmetric star body in \mathbb{R}^n whose k -intersection body exists and such that $\text{vol}_n(K) = \text{vol}_n(B_2^n)$, is it true that

$$\text{vol}_n(I_k(K)) \leq \text{vol}_n(I_k(B_2^n)) \quad (1)$$

They proved that

$$\left(\frac{\text{vol}_n(I_k(K))}{\text{vol}_n(I_k(B_2^n))} \right)^{1/n} \leq c \min\{\log n, k \log k\}, \quad (2)$$

for some absolute constant c , under the assumption that $I_k(K)$ is a convex body.

We will slightly modify and extend their conjecture. First of all, let us write (1) somewhat differently. If L is the k -intersection body of K , and we do not put any restrictions on the volume of K , then (1) is equivalent to

$$(\text{vol}_n(L))^k \leq C_{n,k} (\text{vol}_n(K))^{n-k}, \quad (3)$$

where $C_{n,k}$ is an appropriate constant so that the latter inequality becomes equality in the case of centered balls.

It follows directly from the definition that if L is the k -intersection body of K , then K is the $(n-k)$ -intersection body of L . This means that if inequality (3) is true for $0 < k < n/2$, then the reversed inequality should be true for $n/2 < k < n$.

Further, in terms of the Fourier transform the condition that L is the k -intersection body K can be written as follows:

$$\|\theta\|_L^{-k} = \frac{k}{(2\pi)^k(n-k)} \left(\|\cdot\|_K^{-n+k} \right)^\wedge(\theta), \quad \theta \in S^{n-1}; \quad (4)$$

see [K, Theorem 4.6]. Here $\|\cdot\|_K$ denotes the *Minkowski functional* of K and is defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad x \in \mathbb{R}^n.$$

Note that relation (4) allows us to write (3) as follows:

$$\int_{S^{n-1}} \left[(\|\cdot\|_K^{-n+k})^\wedge(\theta) \right]^{n/k} d\theta \leq c_{n,k} (\text{vol}_n(K))^{\frac{n}{k}-1}, \quad (5)$$

where $c_{n,k}$ is a constant that turns (5) into equality for centered balls. The exact value of this constant will be computed later.

As one can see, (5) makes sense even if k is not an integer. Thus we will write (5) for a larger set of values by using a real number p instead of k . Let us also note that the assumption that the Fourier transform of $\|\cdot\|_K^{-n+k}$ is a positive continuous function is very restrictive. So this assumption will be dropped.

Summarizing all of the above remarks, let us now state the question in the following form.

Question 1. *Let $0 < p < n$ be a real number and K be an origin-symmetric star body in \mathbb{R}^n . Are the following inequalities true?*

$$\int_{S^{n-1}} \left| (\|\cdot\|_K^{-n+p})^\wedge(\theta) \right|^{n/p} d\theta \leq c_{n,p} (\text{vol}_n(K))^{\frac{n}{p}-1}, \quad \text{if } p < n/2, \quad (6)$$

and

$$\int_{S^{n-1}} \left| (\|\cdot\|_K^{-n+p})^\wedge(\theta) \right|^{n/p} d\theta \geq c_{n,p} (\text{vol}_n(K))^{\frac{n}{p}-1}, \quad \text{if } p > n/2, \quad (7)$$

with equality if and only if K is a centered ellipsoid.

The case $p = n/2$ is omitted above since these inequalities become equalities for every origin-symmetric star body K . This is just an application of the spherical version of Parseval's formula (see [K, p.66]):

$$\int_{S^{n-1}} \left[(\|\cdot\|_K^{-n/2})^\wedge(\theta) \right]^2 d\theta = (2\pi)^n \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta = n(2\pi)^n \text{vol}_n(K).$$

It is interesting to note that the conjectured inequalities (6) and (7) have connections to other known inequalities. First of all, let us repeat that the case $p = 1$ corresponds to the Busemann intersection inequality. Thus Question 1 has an affirmative answer for $p = 1$ and $p = n - 1$. Let us now look at the case when $p < 1$ and p is not an even integer. For these values of p the Fourier transform of $\|\cdot\|_K^{-n+p}$ can be expressed as follows:

$$(\|\cdot\|_K^{-n+p})^\wedge(\theta) = \frac{\pi(n-p)}{2\Gamma(1-p)\sin(\pi p/2)} \int_K |\langle x, \theta \rangle|^{-p} dx; \quad (8)$$

see [K, Corollary 3.15].

The reader may recognize such integrals: they appear, for example, in the definition of polar q -centroid bodies (see [LZ] for $q \geq 1$ and [YY] for $-1 < q < 1$). Let $K \subset \mathbb{R}^n$ be a star body and $q > -1$, $q \neq 0$. The polar q -centroid body of K is the star body $\Gamma_q^* K$ given by

$$\|x\|_{\Gamma_q^* K} = \left(\frac{1}{\text{vol}_n(K)} \int_K |\langle x, y \rangle|^q dy \right)^{\frac{1}{q}}, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Note that in [LZ] the normalization is different.

Lutwak and Zhang [LZ] have shown that if $K \subset \mathbb{R}^n$ is a star body and $q \geq 1$, then

$$\text{vol}_n(K)\text{vol}_n(\Gamma_q^*K) \leq \text{vol}_n(B_2^n)\text{vol}_n(\Gamma_q^*B_2^n), \quad (9)$$

with equality if and only if K is an ellipsoid centered at the origin.

By virtue of formula (8) one can check that, in the case when q is not an even integer, the Lutwak-Zhang inequality is equivalent to

$$\int_{S^{n-1}} \left| (\|\cdot\|_K^{-n-q})^\wedge(\theta) \right|^{-n/q} d\theta \leq |c_{n,-q}| (\text{vol}_n(K))^{-\frac{n}{q}-1}.$$

Thus, Question 1 can also be viewed as an extension of the Lutwak-Zhang inequality.

In this paper we show that Question 1 has an affirmative answer when the body K is sufficiently close to the Euclidean ball in the Banach-Mazur distance. For general star bodies we will obtain an improvement of inequality (2) when $k > cn/\log^2 n$.

2. PRELIMINARIES

Two of the main tools used in this paper are the Fourier transform of distributions and spherical harmonics. The reader is referred to the books [K] and [Gr] for detailed discussions of such techniques. We will just briefly mention some important facts. Let f be a continuous function on the sphere S^{n-1} and consider its homogeneous extension to $\mathbb{R}^n \setminus \{0\}$ of degree $-n+p$, where $0 < p < n$. We can think of $|x|_2^{-n+p}f(x/|x|_2)$ as a distribution acting on test functions by integration and therefore we can define its Fourier transform in the distributional sense. If $f \in C^\infty(S^{n-1})$, then the Fourier transform of $|x|_2^{-n+p}f(x/|x|_2)$ is equal to a homogeneous of degree $-p$ function, that is infinitely smooth on $\mathbb{R}^n \setminus \{0\}$; see [K, Lemma 3.16]. Thus we can introduce a linear operator $I_p : C^\infty(S^{n-1}) \rightarrow C^\infty(S^{n-1})$, that maps a function f to the function $I_p f$ equal to the restriction to the sphere of the Fourier transform of $\frac{\Gamma(\frac{n-p}{2})}{2^p \pi^{\frac{n}{2}} \Gamma(\frac{p}{2})} |x|_2^{-n+p} f(x/|x|_2)$. The coefficient in front of the latter function is chosen in such a way that $I_p(1) = 1$, as will be shown later.

When $0 < p < 1$, I_p has the following integral representation, which is just (8) with an appropriate normalization.

$$\begin{aligned} I_p f(\theta) &= \frac{\Gamma(\frac{n-p}{2})}{2^p \pi^{\frac{n}{2}} \Gamma(\frac{p}{2})} \frac{\pi}{2\Gamma(1-p) \sin(\pi p/2)} \int_{S^{n-1}} |\langle x, \theta \rangle|^{-p} f(x) dx \\ &= \frac{\sqrt{\pi} \Gamma(\frac{n-p}{2})}{\Gamma(\frac{n}{2}) \Gamma(\frac{1-p}{2})} \int_{S^{n-1}} |\langle x, \theta \rangle|^{-p} f(x) d\sigma(x), \end{aligned} \quad (10)$$

where σ is the rotationally invariant probability measure on the sphere. To compute the coefficient in front of the above integral we used [K, Lemma 2.18].

For a function f on the sphere, let $\sum_{m=0}^{\infty} H_m$ denote its spherical harmonic expansion, where each H_m is a spherical harmonic of degree m in dimension n . If f is even, then it has only harmonics of even degrees in its expansion: $\sum_{m \geq 0, m \text{ even}} H_m$. Furthermore, $I_p f$ is also even and its spherical harmonic expansion is given by

$$\sum_{\substack{m \geq 0, \\ m \text{ even}}} \lambda_m(n, p) H_m,$$

where

$$\lambda_m(n, p) = \frac{\Gamma\left(\frac{n-p}{2}\right) \Gamma\left(\frac{m+p}{2}\right)}{\Gamma\left(\frac{p}{2}\right) \Gamma\left(\frac{m+n-p}{2}\right)};$$

see, e.g., [GY] (note that the normalization of I_p is different there).

Note that $\lambda_0(n, p) = 1$ for all $p \in (0, n)$, and thus $I_p(1) = 1$. Another important observation is that $|\lambda_m(n, p)| > |\lambda_{m+2}(n, p)|$ for all even $m \geq 0$ when $0 < p < n/2$ (and the inequality gets reversed when $n/2 < p < n$). In particular, for $0 < p < n/2$, we have

$$\|I_p f\|_2^2 = \sum_{\substack{m \geq 0, \\ m \text{ even}}} \lambda_m^2(n, p) \|H_m\|_2^2 \leq \|f\|_2^2.$$

Hence, I_p (when $0 < p < n/2$) is well defined as a linear operator from $L_{\text{even}}^2(S^{n-1})$ to $L_{\text{even}}^2(S^{n-1})$ with the operator norm equal to 1.

Here and below we denote by $\|f\|_q$ the $L^q(S^{n-1})$ -norm of f :

$$\|f\|_q = \left(\int_{S^{n-1}} |f(\theta)|^q d\sigma(\theta) \right)^{1/q}.$$

Let us finally remark that in terms of the operator I_p the conjectured inequality (6) can be written as follows.

Question 1 (Reformulated). *Let K be an origin-symmetric star body in \mathbb{R}^n and $0 < p < n/2$. Is it true that*

$$\int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) \leq (\kappa_n)^{1-\frac{n}{p}} (\text{vol}_n(K))^{\frac{n}{p}-1}, \quad (11)$$

with equality if and only if K is a centered ellipsoid?

Below we will not discuss inequality (7), since, at least in the language of k -intersection bodies, the cases $0 < k < n/2$ and $n/2 < k < n$ are the same.

3. MAIN RESULTS

Following the discussion in the previous section, we will now show that I_p can be extended to a bounded linear operator on a larger space of functions.

Theorem 2. *Let $0 < p < n/2$. I_p is a bounded linear operator from $L_{\text{even}}^{n/(n-p)}(S^{n-1})$ to $L_{\text{even}}^{n/p}(S^{n-1})$. Moreover,*

$$\|I_p\|_{L_{\text{even}}^{n/(n-p)}(S^{n-1}) \rightarrow L_{\text{even}}^{n/p}(S^{n-1})} \leq \left(\frac{n}{2p} \right)^{\frac{p}{2}}.$$

Proof. We will follow the Stein interpolation theorem; see [S] or [SW, Chapter V]. First observe that for every $f, g \in C^\infty(S^{n-1})$ the mapping

$$p \mapsto \int_{S^{n-1}} I_p f(x) g(x) d\sigma(x)$$

is analytic in the strip $0 < \Re(p) < n/2$, and continuous up to the boundary of the strip. In addition,

$$\left| \int_{S^{n-1}} I_p f(x) g(x) d\sigma(x) \right| \leq \|I_p f\|_2 \|g\|_2 \leq \|f\|_2 \|g\|_2.$$

We will now investigate the behavior of I_p on the boundary of the strip. In order to be consistent with the notation in Stein's theorem we will write the strip in the form $0 \leq \Re(z) \leq 1$, where $z = \frac{2}{n}p$. Thus, the boundary of the strip consists of two lines $z = is$ and $z = 1 + is$, where $s \in \mathbb{R}$. Therefore, when $\Re(z) = 1$, i.e., $p = \frac{n}{2}(1 + is)$, we get

$$\lambda_m \left(n, \frac{n}{2}(1 + is) \right) = \frac{\Gamma\left(\frac{n}{4} - \frac{is}{4}\right) \Gamma\left(\frac{m}{2} + \frac{n}{4} + \frac{is}{4}\right)}{\Gamma\left(\frac{n}{4} + \frac{is}{4}\right) \Gamma\left(\frac{m}{2} + \frac{n}{4} - \frac{is}{4}\right)}.$$

Since $\Gamma(a + ib)$ is the complex conjugate of $\Gamma(a - ib)$, we see that $|\lambda_m(n, \frac{n}{2}(1 + is))| = 1$ for all real s , and so

$$\|I_{\frac{n}{2}(1+is)} f\|_2 = \|f\|_2.$$

Now consider the case $\Re(z) = 0$. Extending (10) to $p = \frac{n}{2}is$, $s \in \mathbb{R}$, we get

$$\|I_{\frac{n}{2}is} f\|_\infty \leq \frac{\sqrt{\pi} |\Gamma\left(\frac{n-isn/2}{2}\right)|}{\Gamma\left(\frac{n}{2}\right) |\Gamma\left(\frac{1-isn/2}{2}\right)|} \int_{S^{n-1}} |f(x)| d\sigma(x).$$

Let $A(s)$ be the coefficient in front of the latter integral. To estimate $A(s)$ we will distinguish two cases according to the parity of n . If n is odd, then

$$\begin{aligned} A(s) &= \frac{\sqrt{\pi} |\Gamma\left(\frac{n-isn/2}{2}\right)|}{\Gamma\left(\frac{n}{2}\right) |\Gamma\left(\frac{1-isn/2}{2}\right)|} \\ &= \frac{\sqrt{\pi} \left| \frac{n-2-isn/2}{2} \right| \cdot \left| \frac{n-4-isn/2}{2} \right| \cdots \left| \frac{1-isn/2}{2} \right| \cdot \left| \Gamma\left(\frac{1-isn/2}{2}\right) \right|}{\frac{n-2}{2} \cdot \frac{n-4}{2} \cdots \frac{1}{2} \sqrt{\pi} \cdot |\Gamma\left(\frac{1-isn/2}{2}\right)|} \\ &= \left| \frac{n-2-isn/2}{n-2} \right| \cdot \left| \frac{n-4-isn/2}{n-4} \right| \cdots \left| \frac{1-isn/2}{1} \right| \\ &= \prod_{k=0}^{(n-3)/2} \left(1 + \frac{s^2 n^2}{4(2k+1)^2} \right)^{1/2} \leq \prod_{k=0}^{\infty} \left(1 + \frac{s^2 n^2}{4(2k+1)^2} \right)^{1/2} \\ &= \left(\cosh \frac{\pi s n}{4} \right)^{1/2} \leq e^{\frac{\pi |s| n}{8}}. \end{aligned}$$

Above we used a representation of \cosh as an infinite product. See e.g., [C, VII, §5-6] for details on the Weierstrass factorization theorem.

If n is even, the argument is similar, but we will additionally need the following formulas:

$$|\Gamma(1 + is)| = \left(\frac{\pi s}{\sinh \pi s} \right)^{1/2}, \quad \left| \Gamma \left(\frac{1}{2} + is \right) \right| = \left(\frac{\pi}{\cosh \pi s} \right)^{1/2},$$

which can be obtained from the Euler reflection formula; see [AAR, p. 9 and p. 22] for some details. The above formulas imply

$$\frac{|\Gamma \left(\frac{2-isn/2}{2} \right)|}{|\Gamma \left(\frac{1-isn/2}{2} \right)|} = \left(\frac{sn}{4} \coth \frac{\pi sn}{4} \right)^{1/2}.$$

Then we have, for even n ,

$$\begin{aligned} A(s) &= \frac{\sqrt{\pi} |\Gamma \left(\frac{n-isn/2}{2} \right)|}{\Gamma \left(\frac{n}{2} \right) |\Gamma \left(\frac{1-isn/2}{2} \right)|} \\ &= \frac{\sqrt{\pi} \left| \frac{n-2-isn/2}{2} \right| \cdot \left| \frac{n-4-isn/2}{2} \right| \cdots \left| \frac{2-isn/2}{2} \right| \cdot |\Gamma \left(\frac{2-isn/2}{2} \right)|}{\frac{n-2}{2} \cdot \frac{n-4}{2} \cdots \frac{2}{2} \cdot |\Gamma \left(\frac{1-isn/2}{2} \right)|} \\ &= \sqrt{\pi} \left| \frac{n-2-isn/2}{n-2} \right| \cdot \left| \frac{n-4-isn/2}{n-4} \right| \cdots \left| \frac{2-isn/2}{2} \right| \left(\frac{sn}{4} \coth \frac{\pi sn}{4} \right)^{1/2} \\ &= \sqrt{\pi} \left(\frac{sn}{4} \coth \frac{\pi sn}{4} \right)^{1/2} \prod_{k=1}^{(n-2)/2} \left(1 + \frac{n^2 s^2}{4(2k)^2} \right)^{1/2} \\ &\leq \sqrt{\pi} \left(\frac{sn}{4} \coth \frac{\pi sn}{4} \right)^{1/2} \prod_{k=1}^{\infty} \left(1 + \frac{n^2 s^2}{4(2k)^2} \right)^{1/2} \\ &= \sqrt{\pi} \left(\frac{sn}{4} \coth \frac{\pi sn}{4} \right)^{1/2} \left(\frac{4 \sinh \frac{\pi sn}{4}}{\pi sn} \right)^{1/2} \\ &= \left(\cosh \frac{\pi sn}{4} \right)^{1/2} \leq e^{\frac{\pi |s|n}{8}}. \end{aligned}$$

Therefore, regardless of the parity of n we have $A(s) \leq e^{\frac{\pi |s|n}{8}}$. Denoting $A_0(s) = e^{\frac{\pi |s|n}{8}}$ and $A_1(s) = 1$, we obtain

$$\|I_{\frac{n}{2}is} f\|_{\infty} \leq A_0(s) \|f\|_1$$

and

$$\|I_{\frac{n}{2}(1+is)} f\|_2 \leq A_1(s) \|f\|_2$$

for all $s \in \mathbb{R}$.

The Stein interpolation theorem now implies that for $0 < p < n/2$ we have

$$\|I_p f\|_{n/p} \leq C(n, p) \|f\|_{n/(n-p)}, \quad (12)$$

for some constant $C(n, p)$.

Instead of using Stein's bound for $C(n, p)$, we will proceed as follows. Consider the function $F(z) = -\frac{n}{4}\Re(z \log z)$, which is clearly harmonic in the strip $0 < \Re(z) < 1$. Now let us compute $F(z)$ on the boundary of the strip. When $z = is$, we get $\log(is) = \log|s| + \frac{i\pi}{2}$ if $s > 0$ and $\log(is) = \log|s| - \frac{i\pi}{2}$ if $s < 0$. Therefore,

$$F(is) = -\frac{n}{4}\Re\left(is\left(\log|s| + \operatorname{sgn}(s)\frac{i\pi}{2}\right)\right) = \frac{\pi n}{8}|s|.$$

Observe that $F(is) = \log A_0(s)$.

When $z = 1 + is$, we get

$$\begin{aligned} F(1 + is) &= -\frac{n}{4}\Re\left((1 + is)\left(\log\sqrt{1 + s^2} + i \arctan s\right)\right) \\ &= -\frac{n}{8}\log(1 + s^2) + \frac{n}{4}s \arctan s. \end{aligned}$$

The derivative of the latter function is negative when $s < 0$ and positive when $s > 0$. Therefore the function achieves its minimum at zero, and thus $F(1 + is) \geq 0 = \log A_1(s)$ for all real s .

Summarizing, on the boundary of the strip the following holds: $\log A_0(s) = F(is)$ and $\log A_1(s) \leq F(1 + is)$, for all $s \in \mathbb{R}$. We can now conclude that for all $p \in (0, \frac{n}{2})$ we have

$$\log C(n, p) \leq F\left(\frac{2p}{n}\right) = -\frac{p}{2}\log\left(\frac{2p}{n}\right).$$

That is,

$$C(n, p) \leq \left(\frac{n}{2p}\right)^{\frac{p}{2}},$$

which together with (12) yields the result. \square

We will now show that (11) holds up to a multiplicative constant (depending on n and p) for all origin-symmetric star bodies.

Theorem 3. *For every $0 < p < n/2$ and every origin-symmetric star body K in \mathbb{R}^n we have*

$$\int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) \leq \left(\frac{n}{2p}\right)^{\frac{n}{2}} (\kappa_n)^{1-\frac{n}{p}} (\operatorname{vol}_n(K))^{\frac{n}{p}-1}.$$

Proof. This is a direct application of Theorem 2 to the function $f = \|\cdot\|_K^{-n+p}$.

$$\begin{aligned} & \left(\int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) \right)^{p/n} \\ & \leq \left(\frac{n}{2p} \right)^{\frac{p}{2}} \left(\int_{S^{n-1}} \left(\|\theta\|_K^{-n+p} \right)^{n/(n-p)} d\sigma(\theta) \right)^{(n-p)/n} \\ & = \left(\frac{n}{2p} \right)^{\frac{p}{2}} (\kappa_n^{-1} \text{vol}_n(K))^{(n-p)/n}. \end{aligned}$$

Raising both sides to the power n/p , we get the result. □

Remark. Observe that the result above improves inequality (2) when $k > c \frac{n}{\log^2 n}$. Indeed, let $0 < k < n/2$ and let K be an origin-symmetric star body in \mathbb{R}^n whose k -intersection body exists. If we assume that $\text{vol}_n(K) = \text{vol}_n(B_2^n)$, then Theorem 3 yields

$$\left(\frac{\text{vol}_n(I_k(K))}{\text{vol}_n(I_k(B_2^n))} \right)^{1/n} \leq \sqrt{\frac{n}{2k}}.$$

Note that we do not require K or $I_k(K)$ to be convex.

Before solving a local version of Question 1 we will prove the following lemma pertaining to the equality case in Question 1.

Lemma 4. *The conjectured inequality (11) becomes equality if K is a centered ellipsoid.*

Proof. First, let us show that inequality (11) is invariant under invertible linear transformations. Indeed, let $T \in GL_n(\mathbb{R})$. Applying the transformation T to the right-hand side of inequality (11) yields a factor of $|\det T|^{n/p-1}$. Now let us show that the same happens to the left-hand side of (11).

Consider the origin-symmetric star-shaped set L defined by the formula

$$\|\theta\|_L^{-p} = \left| \left(\|\cdot\|_K^{-n+p} \right)^\wedge(\theta) \right|, \quad \theta \in S^{n-1}.$$

Using the connection between the Fourier transform and linear transformations, we get

$$\begin{aligned}
& \int_{S^{n-1}} \left| (\|\cdot\|_{TK}^{-n+p})^\wedge(\theta) \right|^{n/p} d\theta = \int_{S^{n-1}} \left| (\|T^{-1}(\cdot)\|_K^{-n+p})^\wedge(\theta) \right|^{n/p} d\theta \\
& = |\det T|^{n/p} \int_{S^{n-1}} \left| (\|\cdot\|_K^{-n+p})^\wedge(T^t\theta) \right|^{n/p} d\theta \\
& = |\det T|^{n/p} \int_{S^{n-1}} \|T^t\theta\|_L^{-n} d\theta = |\det T|^{n/p} \int_{S^{n-1}} \|\theta\|_{T^{-t}L}^{-n} d\theta \\
& = |\det T|^{n/p} n \operatorname{vol}_n(T^{-t}L) = |\det T|^{n/p-1} n \operatorname{vol}_n(L) \\
& = |\det T|^{n/p-1} \int_{S^{n-1}} \left| (\|\cdot\|_K^{-n+p})^\wedge(\theta) \right|^{n/p} d\theta,
\end{aligned}$$

where we used T^t and T^{-t} to denote the transpose and the inverse of the transpose of T correspondingly. The calculations above remain valid if the Fourier transform is replaced by the operator I_p since they are equal up to a constant multiple. Thus, the linear invariance of (11) follows.

It remains to show that (11) turns into equality when K is a unit ball. Indeed, we have

$$\int_{S^{n-1}} \left| I_p(\|\cdot\|_2^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) = 1 = (\kappa_n)^{1-\frac{n}{p}} (\operatorname{vol}_n(B_2^n))^{\frac{n}{p}-1}.$$

□

We will now prove that a local version of Question 1 has a positive answer.

Theorem 5. *Let $0 < p < n/2$ and let K be an origin-symmetric star body in \mathbb{R}^n . If K is sufficiently close to the Euclidean ball in the Banach-Mazur distance, then*

$$\int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) \leq (\kappa_n)^{1-\frac{n}{p}} (\operatorname{vol}_n(K))^{\frac{n}{p}-1}, \quad (13)$$

with equality if and only if K is an ellipsoid centered at the origin.

Proof. Since K is close to the Euclidean ball in the Banach-Mazur distance, we can apply an appropriate linear transformation and assume that K is close to B_2^n in the Hausdorff distance, that is,

$$(1 - \varepsilon)B_2^n \subset K \subset (1 + \varepsilon)B_2^n \quad (14)$$

for some small $\varepsilon > 0$.

Applying another linear transformation we can put K in isotropic position, i.e., a position for which the following holds:

$$\int_K x_i x_j dx = \lambda \delta_{ij},$$

for some positive constant λ and all $1 \leq i, j \leq n$; see [BGVV, Section 2.3.2] for details.

After putting K in isotropic position, one can check that it is still close to B_2^n in the Hausdorff distance; see [ANRY, Section 4]. So we can assume that (14) holds (with a different ε).

Let us write

$$\|x\|_K^{-n+p} = H_0(1 + \varphi(x)), \quad x \in S^{n-1}, \quad (15)$$

where H_0 is a constant (the harmonic of order zero in the spherical harmonic expansion of $\|x\|_K^{-n+p}$) and $\int_{S^{n-1}} \varphi(x) d\sigma(x) = 0$.

Note that (14) implies

$$(1 + \varepsilon)^{-n+p} \leq \|x\|_K^{-n+p} \leq (1 - \varepsilon)^{-n+p}$$

for all $x \in S^{n-1}$. Therefore,

$$(1 + \varepsilon)^{-n+p} \leq H_0 \leq (1 - \varepsilon)^{-n+p}.$$

Since H_0 is close to one, if we dilate K by a factor of $(H_0)^{1/(n-p)}$, K will still be close to B_2^n in the Hausdorff metric. So from now on we will assume that

$$\|x\|_K^{-n+p} = 1 + \varphi(x), \quad (16)$$

where $\max_{x \in S^{n-1}} |\varphi(x)| < \varepsilon$ and $\int_{S^{n-1}} \varphi(x) d\sigma(x) = 0$.

Let

$$\sum_{\substack{m \geq 2, \\ m \text{ even}}} H_m$$

be the spherical harmonic expansion of φ .

Recall that K is in isotropic position. We will show that this implies that

$$\|H_2\|_2 \leq C\varepsilon \|\varphi\|_2. \quad (17)$$

Indeed, let $H(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$ be a harmonic quadratic polynomial on \mathbb{R}^n . Observe that we necessarily have $\sum_{i=1}^n a_{ii} = 0$.

Therefore,

$$\begin{aligned} \int_{S^{n-1}} \|x\|_K^{-n-2} H(x) dx &= (n+2) \int_K H(x) dx \\ &= (n+2) \sum_{i,j=1}^n a_{ij} \int_K x_i x_j dx = (n+2) \sum_{i,j=1}^n a_{ij} \lambda \delta_{ij} = 0. \end{aligned}$$

Thus $\|x\|_K^{-n-2}$ has no second order harmonic in its spherical harmonic expansion.

Raising (16) to the power $(n+2)/(n-p)$ and using the Taylor expansion, we get

$$\left| \|x\|_K^{-n-2} - 1 - \frac{n+2}{n-p} \varphi(x) \right| \leq C\varepsilon |\varphi(x)|.$$

Taking the L^2 -norms of both sides and keeping only the second order harmonic in the left-hand side, we obtain (17).

We will now compute the left-hand side of (13). To this end, applying I_p to both sides of (16) and raising to the power n/p , we get

$$\left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} = |1 + I_p \varphi(\theta)|^{n/p}, \quad (18)$$

for all $\theta \in S^{n-1}$.

To expand the right-hand side of the latter equality, we will need the following observation. For a fixed $\alpha > 2$ let $\zeta = \zeta_\alpha$ be the function defined by $\zeta(t) = |1 + t|^\alpha - 1 - \alpha t - \frac{\alpha(\alpha-1)}{2}t^2$, for $t \in \mathbb{R}$. We claim that

$$|\zeta(t)| \leq \begin{cases} D|t|^\alpha, & \text{if } 2 < \alpha \leq 3, \\ D(|t|^3 + |t|^\alpha), & \text{if } 3 \leq \alpha, \end{cases} \quad (19)$$

where D is a constant (depending on α). To prove the claim, observe that $\zeta(t)$ divided by the right-hand side of (19) is a continuous function of $t \in \mathbb{R} \setminus \{0\}$ with finite limits when $t \rightarrow 0$ and $t \rightarrow \pm\infty$.

Thus,

$$\begin{aligned} \int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) &= \int_{S^{n-1}} \left| 1 + I_p\varphi(\theta) \right|^{n/p} d\sigma(\theta) \\ &= \int_{S^{n-1}} \left(1 + \frac{n}{p} I_p\varphi(\theta) + \frac{n(n-p)}{2p^2} (I_p\varphi(\theta))^2 + \zeta(I_p\varphi(\theta)) \right) d\sigma(\theta). \end{aligned}$$

Since φ has no spherical harmonic of degree zero, neither does $I_p\varphi$. That is,

$$\int_{S^{n-1}} I_p\varphi(\theta) d\sigma(\theta) = 0.$$

Denoting

$$R_1 = \int_{S^{n-1}} \zeta(I_p\varphi(\theta)) d\sigma(\theta),$$

we get

$$\int_{S^{n-1}} \left| I_p(\|\cdot\|_K^{-n+p})(\theta) \right|^{n/p} d\sigma(\theta) = 1 + \frac{n(n-p)}{2p^2} \|I_p\varphi\|_2^2 + R_1. \quad (20)$$

We will now show that $R_1 = o(\|\varphi\|_2^2)$. By Theorem 2 there is a constant C (depending on n and p) such that

$$\|I_p\varphi\|_{n/p} \leq C\|\varphi\|_2.$$

When $2 < n/p \leq 3$, by (19) we have

$$|R_1| \leq D \int_{S^{n-1}} |I_p\varphi(\theta)|^{n/p} d\sigma(\theta) = D \|I_p\varphi\|_{n/p}^{n/p} \leq C^{n/p} D \|\varphi\|_2^{n/p} = o(\|\varphi\|_2^2).$$

When $3 \leq n/p$, (19) yields

$$\begin{aligned} |R_1| &\leq D \left(\int_{S^{n-1}} |I_p\varphi(\theta)|^{n/p} d\sigma(\theta) + \int_{S^{n-1}} |I_p\varphi(\theta)|^3 d\sigma(\theta) \right) \\ &= D \left(\|I_p\varphi\|_{n/p}^{n/p} + \|I_p\varphi\|_{n/p}^3 \right) \\ &\leq D \left(C^{n/p} \|\varphi\|_2^{n/p} + C^3 \|\varphi\|_2^3 \right) = o(\|\varphi\|_2^2). \end{aligned}$$

Thus, in both cases, $R_1 = o(\|\varphi\|_2^2)$.

We will now compute the right-hand side of (13). Using (16) we get

$$\|x\|_K^{-n} = (1 + \varphi(x))^{n/(n-p)} = 1 + \frac{n}{n-p} \varphi(x) + \frac{1}{2} \frac{np}{(n-p)^2} \varphi^2(x) + \eta(x),$$

where

$$|\eta| \leq c\varepsilon\varphi^2,$$

for some constant c .

Using that the integral of φ over the sphere vanishes, we get

$$\begin{aligned} \text{vol}_n(K) &= \kappa_n \int_{S^{n-1}} \|x\|_K^{-n} d\sigma(x) \\ &= \kappa_n \left(1 + \frac{np}{2(n-p)^2} \int_{S^{n-1}} \varphi^2(x) d\sigma(x) + R_2 \right), \end{aligned}$$

where $R_2 = o(\|\varphi\|_2^2)$.

Hence,

$$(\kappa_n)^{1-\frac{n}{p}} \text{vol}_n(K)^{\frac{n}{p}-1} = 1 + \frac{n}{2(n-p)} \|\varphi\|_2^2 + R_2, \quad (21)$$

where R_2 is different from that above, but is still of order $o(\|\varphi\|_2^2)$.

Let us now compare (20) and (21). Since R_1 and R_2 are of order $o(\|\varphi\|_2^2)$, to finish the proof we need to show that

$$\frac{n(n-p)}{2p^2} \|I_p\varphi\|_2^2 \leq \frac{n}{2(n-p)} \|\varphi\|_2^2 + o(\|\varphi\|_2^2), \quad (22)$$

provided $\|\varphi\|_2$ is sufficiently small.

Indeed,

$$\begin{aligned} \frac{n(n-p)}{2p^2} \|I_p\varphi\|_2^2 &= \frac{n(n-p)}{2p^2} \left(\lambda_2^2(n,p) \|H_2\|_2^2 + \sum_{\substack{m \geq 4, \\ m \text{ even}}} \lambda_m^2(n,p) \|H_m\|_2^2 \right) \\ &\leq \frac{n(n-p)}{2p^2} \left(\lambda_2^2(n,p) \|H_2\|_2^2 + \lambda_4^2(n,p) \sum_{\substack{m \geq 4, \\ m \text{ even}}} \|H_m\|_2^2 \right) \\ &= \frac{n(n-p)}{2p^2} \left((\lambda_2^2(n,p) - \lambda_4^2(n,p)) \|H_2\|_2^2 + \lambda_4^2(n,p) \sum_{\substack{m \geq 2, \\ m \text{ even}}} \|H_m\|_2^2 \right) \\ &= o(\|\varphi\|_2^2) + \frac{n(n-p)}{2p^2} \lambda_4^2(n,p) \|\varphi\|_2^2 \\ &= o(\|\varphi\|_2^2) + \frac{n}{2(n-p)} \frac{(p+2)^2}{(n-p+2)^2} \|\varphi\|_2^2. \end{aligned}$$

Since $\frac{(p+2)^2}{(n-p+2)^2} < 1$, (22) follows. \square

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