

# CATEGORY $\mathcal{O}$ FOR THE LIE ALGEBRA OF VECTOR FIELDS ON THE LINE

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ABSTRACT. Let  $\mathfrak{W}$  be the Lie algebra of vector fields on the line. Via computing extensions between all simple modules in the category  $\mathcal{O}$ , we give the block decomposition of  $\mathcal{O}$ , and show that the representation type of each block of  $\mathcal{O}$  is wild using the Ext-quiver. Each block of  $\mathcal{O}$  has infinite simple objects. This result is very different from that of  $\mathcal{O}$  for complex semisimple Lie algebras. To find a connection between  $\mathcal{O}$  and the module category over some associative algebra, we define a subalgebra  $H_1$  of  $U(\mathfrak{b})$ . We give an exact functor from  $\mathcal{O}$  to the category  $\Omega$  of finite dimensional modules over  $H_1$ . We also construct new simple  $\mathfrak{W}$ -modules from Weyl modules and modules over the Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{W}$ .

**Keywords:** Category  $\mathcal{O}$ , block, Ext-quiver, wild, Whittaker module.

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## 1. INTRODUCTION

The category  $\mathcal{O}$  for complex semisimple Lie algebras was introduced by Joseph Bernstein, Israel Gelfand and Sergei Gelfand in the early 1970s, see [2], and it includes all highest weight modules. This category is very important in the representation theory. For more details on category  $\mathcal{O}$ , one can see the monograph [12].

The category  $\mathcal{O}$  can be defined for any Lie algebra with a triangular decomposition, see the book [14]. For a Lie algebra  $\mathfrak{g}$  with a triangular decomposition  $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ , there always exists an anti-involution  $\sigma$  of  $\mathfrak{g}$  such that  $\sigma(\mathfrak{g}^+) = \mathfrak{g}^-$  and  $\sigma|_{\mathfrak{h}} = \text{id}_{\mathfrak{h}}$ . For example, the finite dimensional simple Lie algebras, the Virasoro algebra, the affine Kac-Moody algebras, the Heisenberg Lie algebras are all Lie algebras with triangular decompositions, see [14]. For the Kac-Moody algebras and the Virasoro algebra, categories  $\mathcal{O}$  were studied in [8, 4] and references therein. For these algebras, the Hom-spaces between Verma modules determine the block decomposition of the category  $\mathcal{O}$  to a great extent.

For the Lie algebra  $\mathfrak{W} = \mathfrak{W}^- \oplus \mathfrak{h} \oplus \mathfrak{W}^+$  of vector fields on the line,  $\mathfrak{W}^-$  is one dimensional,  $\mathfrak{W}^+$  is infinite dimensional. So  $\mathfrak{W}$  is not a Lie algebra with a triangular decomposition in the sense of [14]. Although we can also define the category  $\mathcal{O}$  for  $\mathfrak{W}$  similar as that of complex semisimple Lie algebras, however

several properties for  $\mathcal{O}$  in [14] dose not hold for  $\mathfrak{W}$ . For example, the embeddings between Verma modules has little impact on the block decomposition of  $\mathcal{O}$ . Our initial motivation of the present paper was to explore the differences between the category  $\mathcal{O}$  of  $\mathfrak{W}$  and the categories  $\mathcal{O}$  of semi-simple Lie algebras. In this paper, through giving extensions between all simple modules in  $\mathcal{O}$ , we obtain the block decomposition of the category  $\mathcal{O}$  for  $\mathfrak{W}$ , and study the representation type of each block of  $\mathcal{O}$ . We also detect the relation between  $\mathcal{O}$  and the module category for some associative algebra, and construct simple  $\mathfrak{W}$ -modules from modules over the Weyl algebra and modules over the Borel subalgebra  $\mathfrak{b}$  of  $\mathfrak{W}$ .

The paper is organized as follows. In Section 2, we introduce the category  $\mathcal{O}$  for the Lie algebra  $\mathfrak{W}$ . In Section 3, we first study the Verma modules and recall extensions on the  $\mathfrak{W}$ -modules  $F_\lambda$  of Feigin and Fuchs defined in [9]. Then using these extensions and the duality between  $F_\lambda$  and the Verma module  $\Delta(\lambda)$ , we can give all nontrivial extensions between Verma modules in  $\mathcal{O}$ . Consequently, we obtain  $\text{Ext}_{\mathcal{O}}^1(M, N)$  for all simple modules  $M, N \in \mathcal{O}$ , see Theorem 3.11. It should be mentioned that extensions between simple modules for the finite dimensional Witt algebra  $W(1, 1)$  over an algebraically closed field of characteristic  $p > 3$  were determined in [3]. Furthermore we give the block decomposition  $\mathcal{O} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{O}_{[\lambda]}$ , and show that each block  $\mathcal{O}_{[\lambda]}$  is wild by studying a sub-quiver of its Ext-quiver, see Theorem 3.14. Let  $H_1 = \{u \in U(\mathfrak{b}) \mid u(d_{-1} - 1) \subset (d_{-1} - 1)U(\mathfrak{W})\}$  which is a subalgebra of  $U(\mathfrak{b})$ . Moreover  $H_1$  is isomorphic to the endomorphism algebra of an induced right  $U(\mathfrak{W})$ -module  $Q'_1$  which is the universal Whittaker module defined in [21]. In subsection 3.5, we construct a functor  $\Gamma$  from  $\mathcal{O}$  to the category  $\Omega_1$  of finite dimensional  $H_1$ -modules. We show that  $\Gamma$  is an exact functor. At the end of Section 3, we also conjecture that some non-integral block  $\mathcal{O}_{[\lambda]}$  may be equivalent to some subcategory of  $\Omega_1$ . In Section 4, we construct new simple tensor  $\mathfrak{W}$ -modules  $T(P, V)$  from modules  $P$  over the Weyl algebra and  $\mathfrak{b}$ -modules  $V$ . The isomorphism criterion for  $T(P, V)$  is also given.

## 2. PRELIMINARIES

In this paper, we denote by  $\mathbb{Z}$ ,  $\mathbb{Z}_{>0}$ ,  $\mathbb{Z}_{\geq 0}$  and  $\mathbb{C}$  the sets of integers, positive integers, nonnegative integers and complex numbers, respectively. All vector spaces and Lie algebras are over  $\mathbb{C}$ . For a Lie algebra  $\mathfrak{g}$  we denote by  $U(\mathfrak{g})$  its universal enveloping algebra. We write  $\otimes$  for  $\otimes_{\mathbb{C}}$ .

**2.1. Witt algebra.** Let  $A = \mathbb{C}[x]$  be the polynomial algebra and  $\mathfrak{W}$  the derivation Lie algebra of  $A$ , i.e.,  $\mathfrak{W} = \text{Der}_{\mathbb{C}}A$ . The Lie algebra  $\mathfrak{W}$  is called the Lie algebra of vector fields on the line, or the Witt algebra of rank one. Denote  $\partial = \frac{\partial}{\partial x}$  and  $d_i = x^{i+1}\partial$ , for any  $i \in \mathbb{Z}_{\geq -1}$ . Then  $\{d_i \mid i \in \mathbb{Z}_{\geq -1}\}$  is a basis of  $\mathfrak{W}$ . We can write the Lie bracket in  $\mathfrak{W}$  as follows:

$$[d_i, d_j] = (j - i)d_{i+j}, \text{ for all } i, j \in \mathbb{Z}_{\geq -1}.$$

Note that the subspace  $\mathfrak{h} = \mathbb{C}d_0$  is a Cartan subalgebra of  $\mathfrak{W}$ , i.e., a maximal abelian subalgebra that is diagonalizable on  $\mathfrak{W}$  with respect to the adjoint action. Let  $\mathfrak{W}^+ = \text{span}\{d_i \mid i \in \mathbb{Z}_{>0}\}$  and  $\mathfrak{W}^- = \mathbb{C}d_{-1}$ . Then  $\mathfrak{W} = \mathfrak{W}^- \oplus \mathfrak{h} \oplus \mathfrak{W}^+$  is a decomposition of  $\mathfrak{W}$ , and the Lie subalgebra  $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{W}^+$  is called a Borel subalgebra of  $\mathfrak{W}$ .

One can see that  $\Phi = \{\varepsilon_{-1}, \varepsilon_1, \varepsilon_2, \dots\}$  is the root system of  $\mathfrak{W}$ , where  $\varepsilon_i \in \mathfrak{h}^*$  such that  $\varepsilon_i(d_0) = i$ ,  $i \in \mathbb{Z}_{\geq -1}$ . The subalgebra  $\mathbb{C}d_{-1} \oplus \mathbb{C}d_0 \oplus \mathbb{C}d_1 \cong \mathfrak{sl}_2$ , and  $z = -d_1d_{-1} + d_0^2 - d_0$  is its Casimir element, i.e.,  $z$  is a central element in  $U(\mathfrak{sl}_2)$ .

**Definition 2.1.** *A left  $U(\mathfrak{W})$ -module  $M$  is called a weight module if  $d_0$  acts diagonally on  $M$ , i.e.,*

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda,$$

where  $M_\lambda := \{v \in M \mid d_0v = \lambda v\}$ . For a weight module  $M$ , denote

$$\text{Supp}(M) := \{\lambda \in \mathbb{C} \mid M_\lambda \neq 0\}.$$

If  $M$  is a simple weight  $\mathfrak{W}$ -module, then  $\text{Supp}(M) \subset \lambda + \mathbb{Z}$  for some  $\lambda \in \mathbb{C}$ . For a  $\lambda \in \text{Supp}(M)$ , a nonzero vector  $v \in M_\lambda$  is called a maximal vector if  $\mathfrak{W}^+v = 0$ . A weight module is called a highest weight module if it is generated by a maximal weight vector.

We use  $U(\mathfrak{W})\text{-Mod}$  to denote the category of all left  $U(\mathfrak{W})$ -modules.

**2.2. Category  $\mathcal{O}$ .** Next we introduce the category  $\mathcal{O}$  for  $\mathfrak{W}$ .

**Definition 2.2.** *The category  $\mathcal{O}$  for  $\mathfrak{W}$  is a full subcategory of  $U(\mathfrak{W})\text{-Mod}$  whose objects are  $\mathfrak{W}$ -modules  $M$  satisfying the following axioms:*

- (a)  *$M$  is a finitely generated  $U(\mathfrak{W})$ -module;*
- (b)  *$M$  is a weight module;*
- (c)  *$M$  is locally  $\mathfrak{W}^+$ -finite: for each  $v \in M$ , the subspace  $U(\mathfrak{W}^+)v$  is finite dimensional.*

Let  $M$  be a module in  $\mathcal{O}$ . By (a) and (c) in Definition 2.2, we can assume that  $M$  is generated by a finite dimensional  $U(\mathfrak{W}^+)$ -module  $N$ . By induction on the dimension of  $N$ , we can show that  $M$  has the following property.

**Lemma 2.3.** *Any module  $M$  in  $\mathcal{O}$  has a finite filtration of submodules as follows:*

$$0 = M_0 \subset M_1 \subset \dots \subset M_m = M,$$

where each factor  $M_j/M_{j-1}$  for  $1 \leq j \leq m$  is a highest weight module.

So highest weight modules are basic constituents of  $\mathcal{O}$ .

3. BLOCK DECOMPOSITION OF  $\mathcal{O}$ 

In this section, we study extensions between Verma modules and simple modules in  $\mathcal{O}$ . Using the Ext-quiver, we show that each block of  $\mathcal{O}$  has wild representation type. We also construct an exact functor from  $\mathcal{O}$  to the category  $\Omega_1$  of finite dimensional modules over  $H_1$ .

**3.1. The Verma modules.** For a  $\lambda \in \mathbb{C}$ , denote by  $\mathbb{C}_\lambda$  the one-dimensional  $\mathfrak{b}$ -module with the generator  $v_\lambda$  and the action given by

$$\mathfrak{W}^+ v_\lambda = 0, \quad d_0 v_\lambda = \lambda v_\lambda.$$

The Verma module over  $\mathfrak{W}$  is defined as follows:

$$\Delta(\lambda) := \text{Ind}_{\mathfrak{b}}^{\mathfrak{W}} \mathbb{C}_\lambda \cong U(\mathfrak{W}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda.$$

The module  $\Delta(\lambda)$  has the unique simple quotient module  $L(\lambda)$ . By Lemma 2.3, the modules  $L(\lambda)$  for  $\lambda \in \mathbb{C}$  provide a complete set of irreducible modules in category  $\mathcal{O}$ .

**Lemma 3.1.** (1)  $zv = \lambda(\lambda + 1)v$  for all  $v \in \Delta(\lambda)$ .  
(2) The module  $\Delta(\lambda)$  is simple if and only if  $\lambda \neq 0$ . So  $\Delta(\lambda) = L(\lambda)$  for  $\lambda \neq 0$ .  
(3) The module  $\Delta(0)$  is a uniserial module whose structure can be described by the following exact sequence:

$$(3.1) \quad 0 \rightarrow \Delta(-1) \xrightarrow{\alpha} \Delta(0) \xrightarrow{\beta} L(0) \rightarrow 0.$$

*Proof.* (1) The proof follows from  $[z, d_{-1}] = 0$ ,  $\Delta(\lambda) = \mathbb{C}[d_{-1}]v_\lambda$  and  $zv_\lambda = \lambda(\lambda + 1)v_\lambda$ .

(2) For any  $i \in \mathbb{Z}_{\geq 0}$ , denote  $v_{\lambda-i} := d_{-1}^i \cdot v_\lambda$ . We can deduce that  $d_0 \cdot v_{\lambda-i} = (\lambda - i)v_{\lambda-i}$ . Since  $\Delta(\lambda)$  is generated by  $v_\lambda$ ,  $\Delta(\lambda)$  is reducible if and only if there is an  $i \in \mathbb{Z}_{>0}$  such that  $d_1 v_{\lambda-i} = d_2 v_{\lambda-i} = 0$ .

We can compute

$$\begin{aligned} d_1 \cdot v_{\lambda-i} &= d_1 \cdot d_{-1}^i \cdot v_\lambda \\ &= ([d_1, d_{-1}^i] + d_{-1}^i d_1) \cdot v_\lambda \\ &= \sum_{t=0}^{i-1} d_{-1}^t [d_1, d_{-1}] d_{-1}^{i-t-1} \cdot v_\lambda \\ &= -2 \sum_{t=0}^{i-1} d_{-1}^t d_0 d_{-1}^{i-t-1} \cdot v_\lambda \\ &= -2 \sum_{t=0}^{i-1} (\lambda - i + t + 1) v_{\lambda-i+1} \\ &= i(i-1-2\lambda) v_{\lambda-i+1}. \end{aligned}$$

Similarly,

$$d_2 \cdot v_{\lambda-i} = i(i-1)(3\lambda-i+2)v_{\lambda-i+2}.$$

Consider  $d_1 \cdot v_{\lambda-i} = 0$  and  $d_2 \cdot v_{\lambda-i} = 0, i \in \mathbb{Z}_{>0}$ , we have the equations

$$\begin{cases} i(i-1-2\lambda) = 0, \\ i(i-1)(3\lambda-i+2) = 0. \end{cases}$$

The solution of this equation is  $\lambda = 0, i = 1$ . Moreover when  $\lambda = 0$ , the submodule generated by  $v_{-1}$  is a proper submodule which is isomorphic to  $\Delta(-1)$ . Thus  $\Delta(\lambda)$  is simple if and only if  $\lambda \neq 0$ .

(3) By (2), the submodule  $N$  generated by  $v_{-1}$  of  $\Delta(0)$  is simple and  $\Delta(0)/N \cong L(0)$ . So  $N$  is the unique nontrivial submodule of  $\Delta(0)$ . Hence  $\Delta(0)$  is a uniserial module.  $\square$

**Remark 3.2.** *If we denote  $e_{\lambda-i} := \frac{(-1)^i}{i!} d_{-1}^i \cdot v_\lambda$ , for any  $i \in \mathbb{Z}_{\geq 0}$ , then  $\{e_{\lambda-i} : i \in \mathbb{Z}_{\geq 0}\}$  is also a basis of  $\Delta(\lambda)$  such that the action of  $\mathfrak{W}$  on  $\Delta(\lambda)$  is defined as follows:*

$$(3.2) \quad d_k e_{\lambda-i} = ((k+1)\lambda + k - i) e_{\lambda-i+k}, \quad \forall k \in \mathbb{Z}_{\geq -1},$$

where  $e_{\lambda-i+k} = 0$  when  $k - i > 0$ .

**Corollary 3.3.** *Any module  $M$  in  $\mathcal{O}$  has finite composition length.*

*Proof.* By lemma 2.3,  $M$  has a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M,$$

such that each factor  $M_j/M_{j-1}$  for  $1 \leq j \leq m$  is a highest weight module. By Lemma 3.1, every Verma module has finite composition length, so does any highest weight module. Consequently the composition length of  $M$  is finite.  $\square$

**Remark 3.4.** *In [7], the authors defined another category  $\mathcal{O}'$  for  $\mathfrak{W}$ . Any module  $M$  in  $\mathcal{O}'$  is locally finite over  $\mathbb{C}d_{-1} \oplus \mathbb{C}d_0$  rather than  $\mathfrak{W}^+$ . Similar as  $\Delta(\lambda)$ , define the  $\mathfrak{W}$ -module:*

$$\Delta'(\lambda) := U(\mathfrak{W}) \otimes_{U(\mathbb{C}d_{-1} \oplus \mathbb{C}d_0)} \mathbb{C}'_\lambda,$$

where  $\mathbb{C}'_\lambda = \mathbb{C}v'_\lambda$  is the  $U(\mathbb{C}d_{-1} \oplus \mathbb{C}d_0)$ -module defined by  $d_{-1}v'_\lambda = 0, d_0v'_\lambda = \lambda v'_\lambda$ . Since any simple weight module over  $\mathfrak{W}$  has one dimensional weight spaces, see [19],  $\Delta'(\lambda)$  does not has finite composition length. So  $\mathcal{O}'$  does not satisfy Corollary 3.3.

**3.2. Extension between Verma modules.** Recall that for  $\mathfrak{W}$ -modules  $M, N \in \mathcal{O}$ , the first cohomology space  $\text{Ext}_{U(\mathfrak{W})}^1(M, N)$  classifies the short exact sequences:  $0 \rightarrow N \xrightarrow{\alpha} K \xrightarrow{\beta} M \rightarrow 0$ , also called the extension of  $N$  by  $M$ . Generally  $K$  may not lie in  $\mathcal{O}$ . We are only interested in that  $K \in \mathcal{O}$ , i.e.,  $K$  needs to be a weight module. So  $\text{Ext}_{\mathcal{O}}^1(M, N) \subset \text{Ext}_{U(\mathfrak{W})}^1(M, N)$ . Note that  $\mathcal{O}$  is closed under weight module extensions. That is  $M, N \in \mathcal{O}$  and  $K$  is a weight module, then  $K \in \mathcal{O}$ . In this subsection, we will give all extensions between Verma modules in  $\mathcal{O}$ .

**Lemma 3.5.** *Let  $\lambda, \mu \in \mathbb{C}$ .*

- (1) *If  $\lambda - \mu \in \mathbb{Z}_{\geq 0}$  and  $M$  is a highest weight module with the highest weight  $\mu$ , then  $\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), M) = 0$ ;*
- (2)  *$\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), \Delta(\lambda)) = 0$ .*

*Proof.* (1) Suppose that

$$(3.3) \quad 0 \rightarrow M \xrightarrow{\alpha} N \xrightarrow{\beta} \Delta(\lambda) \rightarrow 0$$

is a short exact sequence in  $\mathcal{O}$ , where  $N$  is a weight module. As  $\lambda - \mu \in \mathbb{Z}_{\geq 0}$ ,  $\text{Supp}(N) = \text{Supp}(M) \cup \text{Supp}(\Delta(\lambda))$ , so  $\lambda$  is a maximum weight in  $N$ . Recall that  $\Delta(\lambda)$  is generated by the highest weight vector  $v_\lambda$ . Since  $\beta$  is surjective, there exists weight vector  $0 \neq v \in N_\lambda$ , such that  $\beta(v) = v_\lambda$ . Moreover,  $v$  must be a maximal vector. Otherwise, there exists  $d_i \in \mathfrak{W}^+, 0 \neq d_i \cdot v \in N_{\lambda+i}$  such that  $\lambda + i \in \text{Supp}(N)$ , which contradicts to the maximality of  $\lambda$ . The map such that  $v_\lambda \mapsto v$  can be extended to a  $U(\mathfrak{W})$ -module homomorphism  $\beta' : \Delta(\lambda) \rightarrow N$ , and  $\beta\beta' = 1_{\Delta(\lambda)}$ . So the exact sequence (3.3) is split and hence  $\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), M) = 0$ .

(2) is an immediate corollary of (1).  $\square$

Let us recall the  $\mathfrak{W}$ -modules  $F_\lambda$  of Feigin and Fuchs defined in [9], with  $\lambda \in \mathbb{C}$ . The module  $F_\lambda$  has a basis  $\{f_j \mid j \in \mathbb{Z}_{\geq 0}\}$  with the  $\mathfrak{W}$ -action defined by

$$d_i f_j = (j - (i + 1)\lambda) f_{i+j},$$

where  $i \in \mathbb{Z}_{\geq -1}, j \in \mathbb{Z}_{\geq 0}$ . These modules are shown to be restricted dualities of Verma modules. For a weight  $\mathfrak{W}$ -module  $V = \bigoplus_\lambda V_\lambda$ , the restricted duality  $\mathfrak{W}$ -module  $V^* = \bigoplus_\lambda \text{Hom}_{\mathbb{C}}(V_\lambda, \mathbb{C})$  is defined by the natural action:

$$(d_i \phi)(v) = \phi(-d_i v),$$

for all  $i \in \mathbb{Z}_{\geq -1}, \phi \in V^*, v \in V$ . By the universal property of  $\Delta(\lambda)$ , one can check that  $F_\lambda^* \cong \Delta(\lambda)$ . Feigin and Fuchs gave the classification of the extensions of  $F_\mu$  by the modules  $F_\lambda$ .

**Proposition 3.6.** [9] *Suppose that  $\lambda, \mu \in \mathbb{C}$ . Then*

$$\text{Ext}_{U(\mathfrak{g})}^1(F_\lambda, F_\mu) = \begin{cases} \mathbb{C}, & \text{if } \lambda - \mu = 0, 2, 3, 4; \\ \mathbb{C} \oplus \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -5) \text{ or } (4, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = \left(\frac{5 \pm \sqrt{19}}{2}, \frac{-7 \pm \sqrt{19}}{2}\right); \\ 0, & \text{otherwise.} \end{cases}$$

Moreover all nontrivial extensions of  $F_\mu$  by  $F_\lambda$  were listed in the table 1 on page 207 of [9]. We recall these extensions in a slightly different form as follows.

(1) The unique non-split extension  $E(F_\lambda, F_\lambda)$  of  $F_\lambda$  by itself has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$(3.4) \quad \begin{aligned} d_i f_j &= (j - (i + 1)\lambda) f_{i+j}, \\ d_i f'_j &= (j - (i + 1)\lambda) f'_{i+j} + (i + 1) f_{i+j}. \end{aligned}$$

(2) There are two non-split extensions:  $E(F_0, F_{-1})$ ,  $E'(F_0, F_{-1})$  of  $F_{-1}$  by  $F_0$ . The module  $E(F_0, F_{-1})$  has a a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j + i + 1) f_{i+j}, \\ d_i f'_j &= j f'_{i+j} + (i + 1) j f_{i+j-1}. \end{aligned}$$

The module  $E'(F_0, F_{-1})$  has a a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j + i + 1) f_{i+j}, \\ d_i f'_j &= j f'_{i+j} + (i + 1) i f_{i+j-1}. \end{aligned}$$

(3) The unique non-split extension  $E(F_\lambda, F_{\lambda-2})$  of  $F_{\lambda-2}$  by  $F_\lambda$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j - (i + 1)(\lambda - 2)) f_{i+j}, \\ d_i f'_j &= (j - (i + 1)\lambda) f'_{i+j} + ((i + 1)i(i - 1) + 2(i + 1)ij) f_{i+j-2}. \end{aligned}$$

(4) The unique non-split extension  $E(F_\lambda, F_{\lambda-3})$  of  $F_{\lambda-3}$  by  $F_\lambda$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j - (i + 1)(\lambda - 3)) f_{i+j}, \\ d_i f'_j &= (j - (i + 1)\lambda) f'_{i+j} + ((i + 1)i(i - 1)j + (i + 1)ij(j - 1)) f_{i+j-3}. \end{aligned}$$

(5) The unique non-split extension  $E(F_\lambda, F_{\lambda-4})$  of  $F_{\lambda-4}$  by  $F_\lambda$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j - (i + 1)(\lambda - 4)) f_{i+j}, \\ d_i f'_j &= (j - (i + 1)\lambda) f'_{i+j} + \left( \frac{(i + 1)!}{(i - 4)!} \lambda + \frac{(i + 1)! j}{(i - 3)!} \right. \\ &\quad \left. - 6(i + 1)i(i - 1)j(j - 1) - 4(i + 1)ij(j - 1)(j - 2) \right) f_{i+j-4}. \end{aligned}$$

(6) The unique non-split extension  $E(F_0, F_{-5})$  of  $F_{-5}$  by  $F_0$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j + 5(i + 1))f_{i+j}, \\ d_i f'_j &= j f'_{i+j} + \left( 2 \frac{(i+1)!j}{(i-4)!} - 5 \frac{(i+1)!j(j-1)}{(i-3)!} \right. \\ &\quad \left. + 10(i+1)i(i-1)j(j-1)(j-2) + 5(i+1)ij(j-1)(j-2)(j-3) \right) f_{i+j-5}. \end{aligned}$$

(7) The unique non-split extension  $E(F_4, F_{-1})$  of  $F_{-1}$  by  $F_4$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j + i + 1)f_{i+j}, \\ d_i f'_j &= (j - 4(i + 1))f'_{i+j} + \left( 12 \frac{(i+1)!}{(i-5)!} + 22 \frac{(i+1)!j}{(i-4)!} + 5 \frac{(i+1)!j(j-1)}{(i-3)!} \right. \\ &\quad \left. - 10(i+1)i(i-1)j(j-1)(j-2) - 5(i+1)ij(j-1)(j-2)(j-3) \right) f_{i+j-5}. \end{aligned}$$

(8) When  $(\lambda, \mu) = (\frac{5 \pm \sqrt{19}}{2}, \frac{-7 \pm \sqrt{19}}{2})$ , the unique non-split extension  $E(F_\lambda, F_\mu)$  of  $F_\mu$  by  $F_\lambda$  has a basis  $\{f_j, f'_j \mid j \in \mathbb{Z}_{\geq 0}\}$  such that

$$\begin{aligned} d_i f_j &= (j - (i + 1)\mu)f_{i+j}, \\ d_i f'_j &= (j - (i + 1)\lambda)f'_{i+j} + \left( \frac{(i+1)!(22 \pm 5\sqrt{19})}{(i-6)!4} - \frac{(i+1)!j(31 \pm 7\sqrt{19})}{(i-5)!2} \right. \\ &\quad - \frac{(i+1)!j(j-1)(25 \pm 7\sqrt{19})}{(i-4)!2} - \frac{(i+1)!j(j-1)(j-2)5}{(i-3)!} \\ &\quad \left. + 5(i+1)i(i-1)j(j-1)(j-2)(j-3) + \frac{j!2(i+1)i}{(j-5)!} \right) f_{i+j-6}. \end{aligned}$$

In the above formulas,  $f_j = f'_j = 0$  if  $j < 0$ . By the isomorphism  $F_\lambda^* \cong \Delta(\lambda)$ , we obtain all nontrivial extensions between Verma modules.

**Proposition 3.7.** *Suppose that  $\lambda, \mu \in \mathbb{C}$ . Then*

$$\text{Ext}_{\mathcal{O}}^1(\Delta(\mu), \Delta(\lambda)) = \begin{cases} \mathbb{C}, & \text{if } \lambda - \mu = 2, 3, 4; \\ \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1), (0, -5) \text{ or } (4, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = (\frac{5 \pm \sqrt{19}}{2}, \frac{-7 \pm \sqrt{19}}{2}); \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* By  $\dim \text{Ext}_{\mathcal{O}}^1(\Delta(\mu), \Delta(\lambda)) \leq \dim \text{Ext}_{U(\mathfrak{g})}^1(\Delta(\mu), \Delta(\lambda))$  and  $F_\lambda^* \cong \Delta(\lambda)$ , we have

$$\dim \text{Ext}_{\mathcal{O}}^1(\Delta(\mu), \Delta(\lambda)) \leq \begin{cases} 1, & \text{if } \lambda - \mu = 0, 2, 3, 4; \\ 2, & \text{if } (\lambda, \mu) = (0, -1); \\ 1, & \text{if } (\lambda, \mu) = (0, -5) \text{ or } (4, -1); \\ 1, & \text{if } (\lambda, \mu) = (\frac{5 \pm \sqrt{19}}{2}, \frac{-7 \pm \sqrt{19}}{2}). \end{cases}$$



If  $E(F_\lambda, F_\lambda)$  is a weight module, then there are nonzero  $a_j \in \mathbb{C}$  such that  $d_0(f'_j + a_j f_j) = (j - \lambda)(f'_j + a_j f_j)$  for almost all  $j$ . However on the other side,  $d_0(f'_j + a_j f_j) = (j - \lambda)f'_j + f_j + a_j(j - \lambda)f_j$  by (3.4), which is a contradiction. So  $E(F_\lambda, F_\lambda)$  is not a weight module, and hence  $\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), \Delta(\lambda)) = 0$ . In fact, by (2) in Lemma 3.5, we can also see that  $\text{Ext}_{\mathcal{O}}^1(\Delta(\lambda), \Delta(\lambda)) = 0$ .

Similarly  $E(F_0, F_{-1})$  is not a weight module. By the action of  $d_0$  on  $f'_j$ , we can see that  $E(F_0, F_{-1})$ ,  $E(F_\lambda, F_{\lambda-2})$ ,  $E(F_\lambda, F_{\lambda-3})$ ,  $E(F_\lambda, F_{\lambda-4})$ ,  $E(F_0, F_{-5})$ ,  $E(F_4, F_{-1})$ , and  $E(F_{\frac{5 \pm \sqrt{19}}{2}}, F_{\frac{-7 \pm \sqrt{19}}{2}})$  are weight modules. So these modules are also no-split extensions between Verma modules in  $\mathcal{O}$ . Then we can complete the proof.  $\square$

**3.3. Extensions between simple modules.** In this subsection, we compute  $\text{Ext}_{\mathcal{O}}^1(M, N)$  for all simple modules  $M, N \in \mathcal{O}$ .

**Lemma 3.8.** *If  $\lambda - \mu \notin \mathbb{Z}$ , then  $\text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\mu)) = 0$ .*

*Proof.* If  $M \in \mathcal{O}$  such that  $L(\mu) \subset M$  and  $M/L(\mu) \cong L(\lambda)$ , then  $\text{Supp}(M) = \text{Supp}(L(\mu)) \cup \text{Supp}(L(\lambda))$ . Since  $\lambda - \mu \notin \mathbb{Z}$ ,  $\text{Supp}(L(\mu)) \cap \text{Supp}(L(\lambda)) = \emptyset$ . So  $M \cong L(\mu) \oplus L(\lambda)$ .  $\square$

**Lemma 3.9.** (1) *For all  $\lambda \in \mathbb{C}$ ,  $\dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\lambda)) = 0$ .*

(2) *We have  $\dim \text{Ext}_{\mathcal{O}}^1(L(0), L(-1)) = 1$ . That is, if*

$$(3.5) \quad 0 \rightarrow \Delta(-1) \rightarrow M \rightarrow L(0) \rightarrow 0,$$

*is a non-split exact sequence of  $\mathfrak{W}$ -modules in  $\mathcal{O}$ , then  $M \cong \Delta(0)$ .*

(3)  *$\dim \text{Ext}_{\mathcal{O}}^1(L(0), L(\lambda)) = 0$  for all  $\lambda \in \mathbb{C} \setminus \{-1\}$ .*

*Proof.* (1) When  $\lambda \neq 0$ , thanks to (1) in Lemma 3.5 and  $L(\lambda) = \Delta(\lambda)$ , we obtain  $\text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\lambda)) = 0$ . It is enough to prove that  $\text{Ext}_{\mathcal{O}}^1(L(0), L(0)) = 0$ .

Consider the short exact sequence

$$0 \rightarrow \Delta(-1) \rightarrow \Delta(0) \rightarrow L(0) \rightarrow 0,$$

where  $\Delta(-1)$  is the unique nonzero submodule of  $\Delta(0)$ . We can get a long exact sequence by using the functor  $\text{Hom}_{\mathcal{O}}(-, L(0))$ :

$$\cdots \rightarrow \text{Hom}_{\mathcal{O}}(\Delta(-1), L(0)) \rightarrow \text{Ext}_{\mathcal{O}}^1(L(0), L(0)) \rightarrow \text{Ext}_{\mathcal{O}}^1(\Delta(0), L(0)) \rightarrow \cdots.$$

According to (1) in Lemma 3.5, and the fact that  $L(0)$  is not a composition factor of  $\Delta(-1)$ , we have  $\text{Ext}_{\mathcal{O}}^1(\Delta(0), L(0)) = 0$ ,  $\text{Hom}_{\mathcal{O}}(\Delta(-1), L(0)) = 0$ , whence  $\text{Ext}_{\mathcal{O}}^1(L(0), L(0)) = 0$ . Thus  $\text{Ext}_{\mathcal{O}}^1(L(\lambda), L(\lambda)) = 0$  for any  $\lambda \in \mathbb{C}$ .

(2) Consider the short exact sequence

$$(3.6) \quad 0 \rightarrow \Delta(-1) \rightarrow \Delta(0) \rightarrow L(0) \rightarrow 0,$$

where  $\Delta(-1)$  is the maximal submodule of codimension 1 of  $\Delta(0)$ . According to Lemma 3.5, we have  $\text{Ext}_{\mathcal{O}}^1(\Delta(0), L(-1)) = 0$ . Applying  $\text{Hom}_{\mathcal{O}}(-, L(-1))$  to

(3.6), from  $\text{Hom}_{\mathcal{O}}(\Delta(0), L(-1)) = 0$ , we can get

$$0 \rightarrow \text{Hom}_{\mathcal{O}}(\Delta(-1), L(-1)) \rightarrow \text{Ext}_{\mathcal{O}}^1(L(0), L(-1)) \rightarrow \text{Ext}_{\mathcal{O}}^1(\Delta(0), L(-1)) \rightarrow \cdots$$

Thus  $\text{Ext}_{\mathcal{O}}^1(L(0), L(-1)) \cong \text{Hom}_{\mathcal{O}}(\Delta(-1), L(-1)) \cong \mathbb{C}$ .

(3) By (1) and (2), we can assume that  $\lambda \neq 0, -1$ . Let  $M$  be a non-split extension of  $L(\lambda)$  by  $L(0)$  in  $\mathcal{O}$ . We can suppose that  $L(\lambda) \subset M$  and  $M/L(\lambda) = L(0)$ . Then  $1 \leq \dim M_0 \leq 2$ . There must exist  $e'_0 \in M_0 \setminus L(\lambda)_0$  such that  $d_1 e'_0 = 0$ . Thus  $d_1 d_{-1} e'_0 = 0$ . As  $z d_{-1} e'_0 = \lambda(\lambda + 1) d_{-1} e'_0 = d_{-1} z e'_0 = 0$  and  $\lambda \neq 0, -1$ , we deduce that  $d_{-1} e'_0 = 0$ . Since  $M$  is indecomposable,  $d_2 e'_0$  is a nonzero element in  $L(\lambda)$ . So  $[d_{-1}, d_2] e'_0 = 0$  implies that  $d_{-1} d_2 e'_0 = 0$ , contradicting with that  $d_{-1}$  acts injectively on  $L(\lambda)$ .  $\square$

Next we use the relative Lie algebra cohomology to compute  $\dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(0))$ . For two weight  $\mathfrak{W}$ -modules  $M, N$ ,

$$\begin{aligned} \text{Ext}_{\mathfrak{W}, \mathfrak{h}}^1(M, N) &\cong H^1(\mathfrak{W}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(M, N)) \\ &\cong C^1(\mathfrak{W}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(M, N)) / B^1(\mathfrak{W}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(M, N)), \end{aligned}$$

where the set of 1-cocycles  $C^1(\mathfrak{W}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(M, N))$  is the subspace of all  $c \in \text{Hom}_{\mathfrak{h}}(\mathfrak{W}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(M, N))$  such that

$$(3.7) \quad c(\mathfrak{h}) = 0, \quad c([g_1, g_2]) = [g_1, c(g_2)] - [g_2, c(g_1)],$$

for all  $g_1, g_2 \in \mathfrak{W}$ , where  $[g, \psi] \in \text{Hom}_{\mathbb{C}}(M, N)$  such that

$$[g, \psi](v) = g\psi(v) - \psi(gv),$$

for  $g \in \mathfrak{W}, \psi \in \text{Hom}_{\mathbb{C}}(M, N), v \in M$ . A 1-cocycle  $c$  is a coboundary if there is a  $\psi \in \text{Hom}_{\mathfrak{h}}(M, N)$  such that  $c(g) = [g, \psi]$  for any  $g \in \mathfrak{W}$ .

**Lemma 3.10.**  $\dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(0)) = \begin{cases} 1, & \text{if } \lambda = -1, -2; \\ 0, & \text{if } \lambda \neq -1, -2. \end{cases}$

*Proof.* By (1) in Lemma 3.9, we can assume  $\lambda \neq 0$ . So  $L(\lambda) = \Delta(\lambda) = U(\mathfrak{W})v_{\lambda} = \mathbb{C}[d_{-1}]v_{\lambda}$ . Suppose that  $L(0) = \mathbb{C}v_0$

According to (3.1.2) in [17], we have

$$\begin{aligned} \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(0)) &\cong \text{Ext}_{(\mathfrak{W}, \mathfrak{h})}^1(\Delta(\lambda), L(0)) \\ &\cong \text{Ext}_{(\mathfrak{W}, \mathfrak{h})}^1(U(\mathfrak{W}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda}, L(0)) \\ &\cong \text{Ext}_{(\mathfrak{b}, \mathfrak{h})}^1(\mathbb{C}_{\lambda}, L(0)) \\ &\cong H^1(\mathfrak{b}, \mathfrak{h}, \text{Hom}_{\mathbb{C}}(\mathbb{C}_{\lambda}, L(0))), \end{aligned}$$

where  $\mathbb{C}_{\lambda} = \mathbb{C}v_{\lambda}$  is the one dimensional  $\mathfrak{b}$ -module.

For  $\omega \in C^1(\mathfrak{b}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(\mathbb{C}_{\lambda}, L(0)))$ ,  $k, j \in \mathbb{Z}_{\geq 0}$ , we have

$$(3.8) \quad (j - k)\omega(d_{k+j}) = [d_k, \omega(d_j)] - [d_j, \omega(d_k)].$$

Taking  $k = 0$ , by  $\omega(d_0) = 0$ , we have  $j\omega(d_j) = [d_0, \omega(d_j)]$ . After multiplying a suitable scalar, we can assume that  $\omega(d_j)(v_\lambda) = \delta_{\lambda+j,0}v_0$ . If  $\lambda \in \mathbb{Z}_{\leq -3}$ , then  $\omega(d_1) = \omega(d_2) = \omega(d_{-1}) = 0$ , hence  $\omega = 0$  and  $\dim \text{Ext}_{\mathcal{O}}^1(L(\lambda), L(0)) = 0$ . If  $\lambda = -2$ , then  $\omega(d_2)(v_{-2}) = v_0$ ,  $\omega(d_j) = 0$  for any  $j \neq 2$ . So from  $B^1(\mathfrak{b}, \mathfrak{h}; \text{Hom}_{\mathbb{C}}(\mathbb{C}_\lambda, L(0))) = 0$ , we have  $\dim \text{Ext}_{\mathcal{O}}^1(L(-2), L(0)) = 1$ . Similarly  $\dim \text{Ext}_{\mathcal{O}}^1(L(-1), L(0)) = 1$ .  $\square$

We can summarize the results on extensions of simple modules as follows:

**Theorem 3.11.** *Suppose that  $\lambda, \mu \in \mathbb{C}$ . Then*

$$\text{Ext}_{\mathcal{O}}^1(L(\mu), L(\lambda)) = \begin{cases} \mathbb{C}, & \text{if } \lambda - \mu = 2, 3, 4, \lambda\mu \neq 0; \\ \mathbb{C}, & \text{if } (\lambda, \mu) = (0, -1), (0, -2), (-1, 0) \text{ or } (4, -1); \\ \mathbb{C}, & \text{if } (\lambda, \mu) = \left(\frac{5+\sqrt{19}}{2}, \frac{-7+\sqrt{19}}{2}\right); \\ 0, & \text{otherwise.} \end{cases}$$

It should be mentioned that extensions between simple modules for the finite dimensional Witt algebra  $W(1, 1)$  over an algebraically closed field of characteristic  $p > 3$  were determined in [3].

**3.4. Block decomposition of  $\mathcal{O}$ .** We first recall the notion of blocks of an abelian category  $\mathcal{C}$ . We assume that any object of  $\mathcal{C}$  has finite composition length. We introduce an equivalence relation on the set of isomorphism classes of simple objects of  $\mathcal{C}$  as follows: two simple objects  $V, V'$  are equivalent if there exists a sequence  $V = V_1, V_2, \dots, V_r = V'$  of simple objects satisfying  $\text{Ext}_{\mathcal{C}}^1(V_i, V_{i+1}) \neq 0$  or  $\text{Ext}_{\mathcal{C}}^1(V_{i+1}, V_i) \neq 0$  for all  $i$ . Then for each equivalence class  $\chi$ , we denote by  $\mathcal{C}_\chi$  the full subcategory of  $\mathcal{C}$  consisting of objects whose all composition factors belong to  $\chi$ . Each  $\mathcal{C}_\chi$  is called a block of  $\mathcal{C}$  and

$$(3.9) \quad \mathcal{C} = \bigoplus_{\chi} \mathcal{C}_\chi.$$

Moreover, each  $\mathcal{C}_\chi$  cannot be decomposed into a direct sum of two nontrivial abelian full subcategories. The decomposition in (3.9) is called the block decomposition of the category  $\mathcal{C}$ .

For any  $\lambda \in \mathbb{Z}$ , let  $\mathcal{O}_{[\lambda]}$  be the full subcategory of  $\mathcal{O}$  consisting of modules  $M$  such that  $\text{Supp} M \subset \lambda + \mathbb{Z}$ .

**Proposition 3.12.** *We have the block decomposition  $\mathcal{O} = \bigoplus_{\lambda \in \mathbb{C}/\mathbb{Z}} \mathcal{O}_{[\lambda]}$ , each  $\mathcal{O}_{[\lambda]}$  is indecomposable. The set  $\{L(\lambda + n) \mid n \in \mathbb{Z}\}$  is the set of all simple modules in  $\mathcal{O}_{[\lambda]}$ .*

*Proof.* This result follows from Theorem 3.11.  $\square$

Let  $\mathbb{C}\langle x_1, x_2 \rangle$  be the free associative algebra over  $\mathbb{C}$  in two variables  $x_1, x_2$ . Recall that an abelian category  $\mathcal{C}$  is wild if there exists an exact functor from the category of finite dimensional representations of the algebra  $\mathbb{C}\langle x_1, x_2 \rangle$  to  $\mathcal{C}$

which preserves indecomposability and takes non-isomorphic modules to non-isomorphic ones, see Definition 2 in [18]. The following Lemma is useful for the study of representations of infinite dimensional algebras. For its proof, one can see Proposition 2.1 in [11]

**Lemma 3.13.** *Let  $\mathcal{C}$  be an abelian category. If the Ext-quiver of  $\mathcal{C}$  contains a finite subquiver  $Q$  whose underlying unoriented graph is neither a Dynkin nor an affine diagram such that two arrows in  $Q$  can not be concatenated, then  $\mathcal{C}$  is wild.*

**Theorem 3.14.** *For any  $\lambda \in \mathbb{C}$ , the block  $\mathcal{O}_{[\lambda]}$  is wild.*

*Proof.* By Theorem 3.11, the Ext-quiver of every block  $\mathcal{O}_{[\lambda]}$  contains the following subquiver:

$$\begin{array}{ccccc}
 & & L(\mu) & & \\
 & \nearrow & & \nwarrow & \\
 L(\mu - 3) & & & & L(\mu - 4) \longrightarrow L(\mu - 2) \\
 & \searrow & & \swarrow & \\
 & & L(\mu - 1) & & 
 \end{array}$$

where  $\mu \in \lambda + \mathbb{Z}$  such that  $\mu(\mu - 1)(\mu - 2)(\mu - 3)(\mu - 4) \neq 0$ . This subquiver is neither a Dynkin nor an affine diagram. So the block  $\mathcal{O}_{[\lambda]}$  is wild by Lemma 3.13.  $\square$

**Remark 3.15.** *We can compare the category  $\mathcal{O}$  of  $\mathfrak{W}$  with the category  $\mathcal{O}_{\mathfrak{sl}_2}$  of  $\mathfrak{sl}_2$ . Each non-regular block of  $\mathcal{O}_{\mathfrak{sl}_2}$  is semi-simple. Every regular block of  $\mathcal{O}_{\mathfrak{sl}_2}$  is equivalent to the category of finite dimensional representations over  $\mathbb{C}$  of the following quiver*

$$\bullet \begin{array}{c} \xrightarrow{a} \\ \xleftarrow{b} \end{array} \bullet, \quad ab = 0.$$

*So every block of  $\mathcal{O}_{\mathfrak{sl}_2}$  is not wild. It should be mentioned that representation types of all blocks of  $\mathcal{O}$  for complex simple Lie algebras were independently obtained in [6] and [10].*

**3.5. Relation between  $\mathcal{O}$  and a subalgebra  $H_1$  of  $U(\mathfrak{b})$ .** In order to find the connection between  $\mathcal{O}$  and finite dimensional modules over some associative algebra, we define

$$H_1 = \{u \in U(\mathfrak{b}) \mid u(d_{-1} - 1) \subset (d_{-1} - 1)U(\mathfrak{W})\},$$

which is a subalgebra of  $U(\mathfrak{b})$ .

Let  $N_1 = \mathbb{C}v_1$  be the one dimensional right  $\mathbb{C}[d_{-1}]$ -module such that  $v_1 \cdot d_{-1} = v_1$ . Let  $Q'_1$  the induced right  $U(\mathfrak{W})$ -module from the right  $\mathbb{C}[d_{-1}]$ -module  $N_1$ ,

and

$$(3.10) \quad H'_1 = \text{End}_{\mathfrak{W}}(Q'_1).$$

**Lemma 3.16.** *The algebra  $H_1$  is isomorphic to the algebra  $H'_1$ .*

*Proof.* For any  $u \in H_1$ , we define a  $\psi_u \in H'_1 = \text{End}_{\mathfrak{W}}(Q'_1)$  such that  $\psi_u(v_1) = v_1u$ . Then one can check that the linear map

$$\psi : H_1 \rightarrow H'_1, u \mapsto \psi_u,$$

is an algebra isomorphism. □

We define a functor  $\Gamma$  from  $\mathcal{O}$  to the category of finite dimensional  $H_1$ -modules. For any module  $M \in \mathcal{O}$ , let

$$\Gamma(M) = M/(d_{-1} - 1)M.$$

Clearly  $\Gamma(M)$  is an  $H_1$ -module. The following result is immediate.

**Lemma 3.17.**  $\dim \Gamma(L(\lambda)) = \begin{cases} 1, & \text{if } \lambda \neq 0, \\ 0, & \text{if } \lambda = 0. \end{cases}$

Let  $\Omega$  be the category of finite dimensional  $H_1$ -modules.

**Theorem 3.18.** *The functor  $\Gamma : \mathcal{O} \rightarrow \Omega$  is exact.*

*Proof.* Suppose that

$$(3.11) \quad 0 \rightarrow N \xrightarrow{\alpha} M \xrightarrow{\beta} L \rightarrow 0,$$

is an exact sequence in  $\mathcal{O}$ . We will show that

$$(3.12) \quad 0 \rightarrow \Gamma(N) \xrightarrow{\Gamma(\alpha)} \Gamma(M) \xrightarrow{\Gamma(\beta)} \Gamma(L) \rightarrow 0,$$

is exact.

From the surjectivity of  $\beta$  and  $\Gamma(\beta)\Gamma(\alpha) = 0$ , we see that  $\Gamma(\beta)$  is surjective and  $\text{Im}\Gamma(\alpha) \subset \text{Ker}\Gamma(\beta)$ . We need to show that  $\Gamma(\alpha)$  is injective and  $\text{Ker}\Gamma(\beta) \subset \text{Im}\Gamma(\alpha)$ .

For  $n + (d_{-1} - 1)N \in \text{Ker}\Gamma(\alpha)$ , there exists some  $m \in M$  such that  $\alpha(n) = (d_{-1} - 1)m$ . Then  $(d_{-1} - 1)\beta(m) = \beta\alpha(n) = 0$ . Since  $L$  is a weight module,  $d_{-1} - 1$  acts injectively on  $L$ . Thus  $\beta(m) = 0$  and there is an  $n_1 \in N$  such that  $m = \alpha(n_1)$ . Consequently,  $\alpha(n) = (d_{-1} - 1)\alpha(n_1)$ . Then the injectivity of  $\alpha$  implies that  $n = (d_{-1} - 1)n_1$ , i.e.,  $n \in (d_{-1} - 1)N$ . So  $\Gamma(\alpha)$  is injective.

For  $m + (d_{-1} - 1)M \in \text{Ker}\Gamma(\beta)$ , we have that  $\beta(m) = (d_{-1} - 1)l$  for some  $l \in L$ . As  $\beta$  is surjective,  $\beta(m) = (d_{-1} - 1)\beta(m')$  for some  $m' \in M$ , i.e.,  $m - (d_{-1} - 1)m' \in \text{ker}\beta = \text{Im}\alpha$ . So  $m - (d_{-1} - 1)m' = \alpha(n)$  for some  $n \in N$ , hence  $\text{Ker}\Gamma(\beta) \subset \text{Im}\Gamma(\alpha)$ . Therefore  $\Gamma : \mathcal{O} \rightarrow \Omega$  is exact. □

Denote the restriction of  $\Gamma$  to  $\mathcal{O}_{[\lambda]}$  by  $\Gamma^{[\lambda]}$ , and by  $\Omega_{[\lambda]}$  the subcategory of  $\Omega$  consisting of the  $H_1$ -modules isomorphic to  $\Gamma^{[\lambda]}(M)$  for  $M \in \mathcal{O}_{[\lambda]}$ . Finally we give a conjecture on  $\mathcal{O}_{[\lambda]}$ .

**Conjecture 3.1.** *There is a  $\lambda \notin \mathbb{Z}$  such that  $\mathcal{O}_{[\lambda]}$  is equivalent to  $\Omega_{[\lambda]}$ .*

#### 4. TENSOR $\mathfrak{W}$ -MODULES FROM $\mathfrak{b}$ -MODULES AND WEYL MODULES

Let  $\mathfrak{D}$  be the Weyl algebra of rank one, that is,  $\mathfrak{D}$  is the associative algebra over  $\mathbb{C}$  generated by  $x, \partial$  subject to the relation  $[\partial, x] = 1$ . In this section, we construct simple  $\mathfrak{W}$ -modules from  $\mathfrak{D}$ -modules and  $\mathfrak{b}$ -modules.

**4.1. Tensor module  $T(P, V)$ .** The following interesting algebra homomorphism was given in [20] and [5] independently, which plays an important role in the classification of simple weight modules for  $\mathfrak{W}_n$ , see [20, 16].

**Lemma 4.1.** *There is an algebra monomorphism  $\phi$  from  $U(\mathfrak{W})$  to  $\mathfrak{D} \otimes U(\mathfrak{b})$  such that*

$$\begin{aligned} d_{-1} &\mapsto \partial \otimes 1, \\ d_m &\mapsto x^{m+1} \partial \otimes 1 + \sum_{r=0}^m \binom{m+1}{r+1} x^{m-r} \otimes d_r, \quad m \in \mathbb{Z}_{\geq 0}. \end{aligned}$$

By Lemma 4.1, for any  $\mathfrak{D}$ -module  $P$ , any  $\mathfrak{b}$ -module  $V$ , the tensor product  $P \otimes V$  can be made to be a  $\mathfrak{W}$ -module denoted by  $T(P, V)$ .

**Remark 4.2.** *Let  $S = \mathbb{C}[x^{\pm 1}]/\mathbb{C}[x]$  which is a simple  $\mathfrak{D}$ -module. Then we have a functor  $T(S, -)$  from the category  $\mathfrak{b}\text{-mod}$  of finite dimensional  $\mathfrak{b}$ -modules to the category  $\mathcal{O}$ . We originally intended to use the functor  $T(S, -)$  to study  $\mathcal{O}$ . However, in view of extensions between simple modules in  $\mathcal{O}$ ,  $T(S, V)$  may be decomposable for some indecomposable  $\mathfrak{b}$ -module  $V$ . Nevertheless, we can still use the bifunctor  $T(-, -)$  to construct simple  $\mathfrak{W}$ -modules.*

In the following theorem, we will give the simplicity of  $T(P, V)$  under some natural conditions.

**Theorem 4.3.** *Suppose that  $P$  is a simple  $\mathfrak{D}$ -module, and  $V$  is a simple  $\mathfrak{b}$ -module such that there is an  $l \in \mathbb{Z}_{>0}$  satisfying*

- (1)  $d_l$  acts injectively on  $V$ ;
- (2)  $d_i V = 0$  for any  $i > l$ ,

*then  $T(P, V)$  is a simple  $\mathfrak{W}$ -module.*

*Proof.* Let  $N$  be a nonzero submodule of  $T(P, V)$ . Suppose that  $u = \sum_{n=0}^q p_n \otimes v_n$  is a nonzero element in  $N$ , where  $p_0, \dots, p_q$  are linearly independent.

**Claim 1.** For any  $X \in \mathfrak{D}$ , we have that  $\sum_{n=0}^q X p_n \otimes d_l^2 v_n \in N$ .

For any  $k$  with  $k \geq 2l$  and any  $m$  with  $-1 \leq m \leq 2l + 1$ , we can compute that

$$\begin{aligned}
d_{k-m}d_m(p \otimes v) &= d_{k-m}\left(d_m p \otimes v + \sum_{r=0}^m \binom{m+1}{r+1} x^{m-r} p \otimes d_r v\right) \\
&= (d_{k-m}d_m p) \otimes v + \sum_{r=0}^m \binom{m+1}{r+1} (d_{k-m} x^{m-r} p) \otimes d_r v \\
&\quad + \sum_{s=0}^{k-m} \binom{k-m+1}{s+1} x^{k-m-s} d_m p \otimes d_s v \\
&\quad + \sum_{s=0}^{k-m} \sum_{r=0}^m \binom{k-m+1}{s+1} \binom{m+1}{r+1} x^{k-r-s} p \otimes d_s d_r v \\
&= m^{2l+2} x^{k-2l} p \otimes \frac{d_l^2 v}{((l+1)!)^2} + g(m),
\end{aligned}$$

where  $g(m)$  is the term with degree of  $m$  smaller than  $2l + 2$ . Consider the coefficient of  $m^{2l+2}$  in  $d_{k-m}d_m u$ . By letting  $m = -1, 0, 1, \dots, 2l + 1$ , using the Vandermonde matrix, we deduce that  $\sum_{n=0}^q x^i p_n \otimes d_l^2 v_n \in N$ , for all  $i \in \mathbb{Z}_{\geq 0}$ . From the action of  $d_{-1}$  on  $N$ , we see that  $\sum_{n=0}^q \partial^j x^i p_n \otimes d_l^2 v_n \in N$  for all  $i, j \in \mathbb{Z}_{\geq 0}$ . So  $\sum_{n=0}^q X p_n \otimes d_l^2 v_n \in N$  for any  $X \in \mathfrak{D}$ . Then Claim 1 follows.

**Claim 2.**  $P \otimes d_l^2 v_n \subset N$ , for any  $0 \leq n \leq q$ .

Since  $P$  is a simple  $\mathfrak{D}$ -module, by the Jacobson density theorem, for any  $p \in P$ , there is a  $X_n$  such that

$$X_n p_i = \delta_{n,i} p, \quad i = 0, \dots, q.$$

By Claim 1, we obtain that  $P \otimes d_l^2 v_n \subset N$ , for any  $0 \leq n \leq q$ . Claim 2 follows.

**Claim 3.**  $N = T(P, V)$ . Hence  $T(P, V)$  is simple.

Let  $V_1 = \{v \in V \mid P \otimes v \subset N\}$ . By Claim 2 and that  $d_l$  acts injectively on  $V$ ,  $V_1 \neq 0$ . For any  $v \in V_1, p \in P$ , taking  $m = 0, 1, \dots, l$  in

$$d_m(p \otimes v) = d_m p \otimes v + \sum_{r=0}^m \binom{m+1}{r+1} x^{m-r} p \otimes d_r v,$$

we can see that  $p \otimes d_0 v, p \otimes d_1 v, \dots, p \otimes d_l v \in N$ . So  $V_1$  is a nonzero  $\mathfrak{b}$ -submodule of  $V$ . The simplicity of  $V$  forces that  $V_1 = V$ . Then Claim 3 follows.  $\square$

**4.2. Isomorphism criterion for  $T(P, V)$ .** Next we give the following isomorphism criterion for  $T(P, V)$ .

**Theorem 4.4.** *Suppose that  $P, P'$  are simple  $\mathfrak{D}$ -modules,  $V, V'$  are simple  $\mathfrak{b}$ -modules such that there are  $l, s \in \mathbb{Z}_{>0}$  satisfying  $d_l$  (resp.  $d_s$ ) acts injectively on*

$V$  (resp.  $V'$ ) and  $d_i V = 0$  (resp.  $d_i V' = 0$ ) for any  $i > l$  (resp.  $i > s$ ). Then  $T(P, V) \cong T(P', V')$  if and only if  $P \cong P'$ ,  $l = s$  and  $V \cong V'$ .

*Proof.* The sufficiency is obvious. Now suppose that

$$\psi : T(P, V) \rightarrow T(P', V')$$

is an isomorphism of  $\mathfrak{W}$ -modules. Let  $p \otimes v$  be a nonzero element in  $T(P, V)$ . Write

$$\psi(p \otimes v) = \sum_{n=0}^q p'_n \otimes v'_n \in T(P', V'),$$

where  $p'_0, \dots, p'_q$  are linearly independent. Similar to the proof of Claim 1 of Theorem 4.3, comparing the the highest degree of  $m$  on both sides of

$$\psi(d_{k-m} d_m(p \otimes v)) = d_{k-m} d_m \psi(p \otimes v),$$

we have that  $l = s$  and

$$\psi(Xp \otimes v) = \sum_{n=0}^q Xp_n \otimes d_l^2 v_n, \quad \forall X \in \mathfrak{D}.$$

By the Jacobson density theorem, there exists  $Y \in \mathfrak{D}$  such that  $Yp_i = \delta_{i0} p_0$ . Then

$$\psi(Yp \otimes v) = p_0 \otimes d_l^2 v_n.$$

Replacing  $Yp$  by  $p$ ,  $p_0$  by  $p'$  and  $d_l^2 v_n$  by  $v'$ , we have

$$\psi(Xp \otimes v) = Xp' \otimes v', \quad \forall X \in \mathfrak{D}.$$

Consequently  $\psi_1 : P \rightarrow P'$  satisfying  $\psi_1(Xp) = Xp'$  is a well-defined map. Since  $P, P'$  are simple  $\mathfrak{D}$ -modules,

$$\text{Ann}_{\mathfrak{D}}(p) = \text{Ann}_{\mathfrak{D}}(p'), \quad \text{and } P \cong \mathfrak{D}/\text{Ann}_{\mathfrak{D}}(p) \cong P'.$$

Thus  $\psi_1$  is a  $\mathfrak{D}$ -module isomorphism and

$$\psi(p \otimes v) = \psi_1(p) \otimes v', \quad \forall p \in P.$$

Then from  $\psi(d_m(p \otimes v)) = d_m \psi(p \otimes v)$ , we obtain that

$$\psi(p \otimes d_r v) = \psi_1(p) \otimes d_r v', \quad \forall p \in P, r \in \mathbb{Z}_{\geq 0}.$$

So

$$\psi(p \otimes yv) = \psi_1(p) \otimes yv', \quad \forall p \in P, \forall y \in U(\mathfrak{b}).$$

Therefore we have  $\text{Ann}_{U(\mathfrak{b})}(v) = \text{Ann}_{U(\mathfrak{b})}(v')$ . The simplicity of  $V$  and  $V'$  implies that  $V \cong U(\mathfrak{b})/\text{Ann}_{U(\mathfrak{b})}(v) \cong V'$ . □



**Remark 4.5.** For each  $r > 0$ , denote the quotient algebra  $\mathfrak{b}/\langle d_{r+i} : i > 0 \rangle$  by  $\mathfrak{a}_r$ . From Theorem 4.3, we know that, to obtain new simple  $\mathfrak{W}$ -modules  $T(P, V)$ , it is enough to construct infinite dimensional simple modules  $V$  over  $\mathfrak{a}_r$  for  $r > 0$  such that the action of  $d_r$  on  $V$  is injective. Simple modules over  $\mathfrak{a}_1$  and  $\mathfrak{a}_2$  were classified in [15].

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