ON ORIENTED m-SEMIREGULAR REPRESENTATIONS OF FINITE GROUPS

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Abstract. A finite group G admits an *oriented regular representation* if there exists a Cayley digraph of G such that it has no digons and its automorphism group is isomorphic to G. Let m be a positive integer. In this paper, we extend the notion of oriented regular representations to oriented m-semiregular representations using m-Cayley digraphs. Given a finite group G, an *m-Cayley digraph* of G is a digraph that has a group of automorphisms isomorphic to G acting semiregularly on the vertex set with m orbits. We say that a finite group G admits an *oriented m-semiregular representation* if there exists a regular m-Cayley digraph of G such that it has no digons and G is isomorphic to its automorphism group. In this paper, we classify finite groups admitting an oriented m-semiregular representation for each positive integer m.

Keywords: Regular group, semiregular group, regular representation, m-Cayley digraph, ORR, OmSR.

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1. Introduction

All groups and digraphs considered in this paper are finite. For a digraph Γ, we denote by $V(\Gamma)$ and $A(\Gamma)$ the set of vertices and the set of arcs respectively, where $V(\Gamma) \neq \emptyset$ and $A(\Gamma) \subseteq V(\Gamma) \times V(\Gamma)$. A digraph Γ is called a *graph* if the binary relation $A(\Gamma)$ is symmetric, that is, $A(\Gamma) = \{(y, x) \mid (x, y) \in A(\Gamma)\}$. A digraph is called *regular* if each vertex has the same in- and out-valency. Let Γ be a digraph. An automorphism of Γ is a permutation σ of $V(\Gamma)$ fixing $A(\Gamma)$ setwise, that is, $(x^\sigma, y^\sigma) \in A(\Gamma)$ if and only if $(x, y) \in A(\Gamma)$. The full automorphism group of Γ is denoted by $\mathrm{Aut}(\Gamma)$.

Let G be a group with a subset $R \subseteq G \setminus \{1\}$. A *Cayley digraph* $\Gamma = \mathrm{Cay}(G, R)$ is the digraph with $V(\Gamma) = G$ and $A(\Gamma) = \{(g, rg) \mid g \in G, r \in R\}$. In particular, Γ is a Cayley graph if and only if $R = R^{-1}$. The right regular representation of G gives rise to an embedding of G into $\mathrm{Aut}(\Gamma)$, and thus we identify G with its image under this permutation representation. We say that a group G admits a *(di)graphical regular representation* (GRR or DRR for short) if there exists a Cayley (di)graph Γ of G satisfying $G \cong \mathrm{Aut}(\Gamma)$. Babai [1] proved that except Q_8, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ and $\mathbb{Z}_3^2$, each finite group admits a DRR. It is clear that if a group G admits a GRR then G admits a DRR. However, the converse is not true. Many researchers proposed and investigated various kinds of generalizations of the classifications of DRRs and GRRs. Despite the classification of groups admitting a DRR by Babai [1], the classification of groups admitting a GRR needs considerably more work. For more results along this way, we refer the reader to [7, 10, 11, 18, 19]. A *tournament* is a digraph Γ such that for any two distinct vertices $x, y \in V(\Gamma)$, exactly one of (x, y) and (y, x) is in $A(\Gamma)$. Note that Cayley digraph $\mathrm{Cay}(G, R)$ is a tournament if and only if $R \cap R^{-1} = \emptyset$ and $R \cup R^{-1} = G \setminus \{1\}$, where 1 is the identity of G. It is known that if $\mathrm{Cay}(G, R)$ is a tournament then the order of G is odd. We say that a group G admits a *tournament regular representation* (TRR for short) if there exists a Cayley digraph Γ of G such that Γ is a tournament satisfying $G \cong \mathrm{Aut}(\Gamma)$. Babai and Imrich [2] proved that except $\mathbb{Z}_3^2$, each group of odd order admits a TRR. An *oriented Cayley digraph* is a Cayley digraph $\mathrm{Cay}(G, R)$ satisfying $R \cap R^{-1} = \emptyset$ (i.e., it has no digons.). We say that a group G admits an *oriented regular representation* (ORR for short) if there exists a Cayley digraph $\Gamma = \mathrm{Cay}(G, R)$ of G such that $R \cap R^{-1} = \emptyset$ and $G \cong \mathrm{Aut}(\Gamma)$ both hold. It is clear that if a group G admits a TRR then it admits an ORR. In 2017, Morris and Spiga [15] proved that each finite non-solvable group admits an ORR. Spiga [20] proved that generalized dihedral groups do not admit ORRs. Morris and Spiga [16] completely classified the finite groups admitting ORRs. For more results, we refer the reader to [8, 13, 14, 17, 21, 22, 23].

Let G be a permutation group on a set Ω and let $\omega \in \Omega$. Denote by G_ω the stabilizer of ω in G (i.e., the subgroup of G fixing ω.). We say that G is *semiregular* on Ω if $G_\omega = 1$ for every $\omega \in \Omega$, and *regular* if it is semiregular and transitive. For a positive integer m, an *m-Cayley (di)graph* of a group G is defined as a (di)graph which has a semiregular group of automorphisms isomorphic to G with m orbits on its vertex set. Note that 1-Cayley (di)graphs are the usual Cayley (di)graphs. For $m = 2$, 2-Cayley (di)graphs are called *bi-Cayley (di)graphs* (For more concrete definition, see Section 2.). We say that a group G admits a *(di)graphical m-semiregular representation* (GmSR or DmSR for short) if there exists a regular m-Cayley (di)graph Γ of G satisfying $G \cong \mathrm{Aut}(\Gamma)$. In particular, a G1SR (D1SR, respectively) is the usual GRR (DRR, respectively). In [4], Du, Feng and Spiga classified finite groups admitting a GmSR or a DmSR for each positive integer m.

Now we consider n-partite digraphs. We say that a group G admits an *n-partite digraphical representation* (n-PDR for short) if there exists a regular n-partite digraph Γ such that $G \cong \mathrm{Aut}(\Gamma)$, $\mathrm{Aut}(\Gamma)$ acts semiregularly on $V(\Gamma)$ and the orbits of $\mathrm{Aut}(\Gamma)$ on $V(\Gamma)$ form a partition into n parts giving the structure of Γ. It is known that each n-PDR is also a DnSR, but the converse is not true. Recently, Du, Feng and Spiga [5, 6] classified the finite groups admitting an n-PDR for each positive integer n.

In this paper, we extend the notion of oriented regular representations to oriented m-semiregular representations using m-Cayley digraphs. We say that for a positive integer m, a group G admits an *oriented m-semigular representation* (OmSR for short) if there exists a regular oriented m-Cayley digraph Γ of G satisfying $G \cong \mathrm{Aut}(\Gamma)$. In particular, an O1SR is the usual ORR. Finite groups admitting ORRs are completely classified (see Proposition 2.2, [16]).

Now we introduce the main result of this paper. In Theorem 1.1, we classify the finite groups admitting an oriented m-semiregular representation ($m \geq 1$) as follows.

Theorem 1.1. *Let m be a positive integer and let G be a finite group. Then G satisfies one of the following:*

(1) *G admits an OmSR;*
(2) *$m = 1$, and G is isomorphic to either a generalized dihedral group of order greater than 2 or one of the 11 exceptional groups listed in* Table 2.1*;*
(3) *$m = 2$, and G is isomorphic to either $\mathbb{Z}_1$, $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$;*
(4) *$3 \leq m \leq 6$, and G is isomorphic to $\mathbb{Z}_1$.*

This paper is organized as follows. In Section 2, we set up notations and review known results which will be used in this paper. In Section 3, we prove Theorem 1.1. We first show in Lemma 3.1 that any finite group G of order at least 3 admitting an ORR must admit an OmSR for any integer $m \geq 2$. To consider finite groups which do not admit ORRs, we mainly consider elementary abelian 2-groups and non-abelian generalized dihedral groups. First, we consider when elementary abelian 2-groups admit an OmSR for positive integer m. To show this, we divide into two cases: groups of order at most 16 (Lemma 3.4); groups of order greater than 16 (Lemma 3.6). On the other hand, we prove in Lemma 3.7 that finite non-abelian generalized dihedral groups always admit an OmSR for any integer $m \geq 2$. Using these results, we classify the finite groups admitting an oriented m-semiregular representation for each positive integer m (Theorem 1.1).

2. ORRs, m-Cayley digraphs and notations

In this section, we set up notations and review several results which will be used in this paper.

As usual, for a positive integer n and a prime p, we denote by $\mathbb{Z}_n$, $\mathbb{Z}_p^n$, Q_8 and D_n the cyclic group of order n, the elementary abelian group of order p^n, the quoternion group of order 8 and the dihedral group of order $2n$, respectively. For an abelian group H, the *generalized dihedral group* over H is the semidirect product of H with the cyclic group of order two, with the non-identity element acting as the inverse map on H. In the following remark, we note some properties of generalized dihedral groups which will be used later.

Remark 2.1. *Let G be the generalized dihedral group over an abelian group H. Then we have the following.*

(1) *The exponent of G is the least common multiple of 2 and the exponent of H.*
(2) *G is abelian if and only if G is an elementary abelian 2-group.*

To state finite groups admitting ORRs, we first give the following table which shows 11 exceptional groups admitting no ORRs (see [16]).

Group	Comments
$Q_8,\ \mathbb{Z}_4 \times \mathbb{Z}_2,\ \mathbb{Z}_4 \times \mathbb{Z}_2^2,\ \mathbb{Z}_4 \times \mathbb{Z}_2^3,\ \mathbb{Z}_4 \times \mathbb{Z}_2^4,\ \mathbb{Z}_3^2,\ \mathbb{Z}_3 \times \mathbb{Z}_2^3$	
$H_1 := \langle x, y \mid x^4 = y^4 = (xy)^2 = (xy^{-1})^2 = 1 \rangle$	order 16
$H_2 := \langle x, y, z \mid x^4 = y^4 = z^4 = (yx)^2 = (yx^{-1})^2 = (yz)^2 = (yz^{-1})^2 = 1,\ x^2 = y^2 = z^2, x^z = x^{-1} \rangle$	order 16
$H_3 := \langle x, y, z \mid x^4 = y^4 = z^4 = (xy)^2 = (xy^{-1})^2 = (xz)^2 = (xz^{-1})^2 = (yz)^2 = (yz^{-1})^2 = x^2y^2z^2 = 1 \rangle$	order 32
$D_4 \circ D_4$ (the central product of D_4 and D_4)	order 32

Table 2.1. The exceptional groups admitting no ORRs

The following is the classification of finite groups admitting ORRs.

Proposition 2.2. [16, Theorem 1.2] *Let G be a finite group. Then G admits an ORR, except*

(1) *G is a generalized dihedral group of order greater than 2;*
(2) *G is isomorphic to one of the 11 exceptional groups given in* Table 2.1.

Let $m \geq 1$ be an integer. Recall that an *m-Cayley digraph* of a finite group G is defined as a digraph which has a semiregular group of automorphisms isomorphic to G with m orbits on its vertex set. Now, we consider more concrete definition for m-Cayley digraphs. For each $i \in \mathbb{Z}_m$, put $G_i := \{g_i \mid g \in G\}$. For any subset $H \subseteq G$ and $i \in \mathbb{Z}_m$, we also denote $H_i := \{h_i \mid h \in H\}$. Similar to Cayley digraphs, an m-Cayley digraph Γ can be viewed as the digraph

$$\mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$$

with the vertex set and the arc set:

$$\bigcup_{i \in \mathbb{Z}_m} G_i \text{ and } \bigcup_{i,j \in \mathbb{Z}_m} \{(g_i, (tg)_j) \mid t \in T_{i,j}\}$$

respectively, where $T_{i,j}$ $(i, j \in \mathbb{Z}_m)$ are subsets of G satisfying $1 \notin T_{i,i}$. In particular, the above digraph $\mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ is also an m-Cayley digraph of a group G (with respect to $T_{i,j}$ $(i, j \in \mathbb{Z}_m)$). Note here that an m-Cayley digraph $\mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ is an oriented m-Cayley digraph if and only if $T_{i,j} \cap T_{j,i}^{-1} = \emptyset$ for all $i, j \in \mathbb{Z}_m$. Without loss of generality, let $\Gamma = \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ be an m-Cayley (di)graph of a group G with respect to $T_{i,j}$ $(i, j \in \mathbb{Z}_m)$. For any element $g \in G$, the right multiplication $R(g)$ (i.e., mapping each vertex $x_i \in G_i$ to $(xg)_i \in G_i$ for all $i \in \mathbb{Z}_m$) is an automorphism of Γ, and $R(G)$ is a semiregular group of automorphisms of G with G_i as orbits, where

$$R(G) := \{R(g) \mid g \in G\}.$$

In particular, 2-Cayley (di)graph $\mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_2)$ is called a *bi-Cayley (di)graph* and it is also denoted by

$$\mathrm{BiCay}(G, R, L, S, T)$$

where $R = T_{0,0}$, $L = T_{1,1}$, $S = T_{0,1}$ and $T = T_{1,0}$.

The following proposition gives a result of the automorphisms between bi-Cayley digraph $\mathrm{BiCay}(G, \emptyset, \emptyset, S, T)$ and Cayley digraphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G, T)$.

Proposition 2.3. [5, Lemma 3.2] *Let G be a finite group and let σ be a permutation of G. Denote by σ' the permutation of $G_0 \cup G_1$ mapping g_i to g_i^σ for each $g \in G$ and $i \in \mathbb{Z}_2$. Then the following are equivalent.*

(1) $\sigma \in \mathrm{Aut}(\mathrm{Cay}(G, S)) \cap \mathrm{Aut}(\mathrm{Cay}(G, T))$
(2) $\sigma' \in \mathrm{Aut}(\mathrm{BiCay}(G, \emptyset, \emptyset, S, T))$

To end this section, we set up some notations for this paper.

Notation 2.4. (1) *Let Γ be a (di)graph. For any subset X of the vertex set $V(\Gamma)$, denote by $\Gamma[X]$ the induced sub-(di)graph by X in Γ. We simply write $[X]$ for $\Gamma[X]$ if there are no confusions.*

(2) *Let Γ be a digraph. For a vertex $x \in V(\Gamma)$, denote by $\Gamma^+(x)$ and $\Gamma^-(x)$ the out- and in-neighbors of x in Γ, respectively. For a subset $X \subseteq V(\Gamma)$, define*

$$\Gamma^+(X) := \bigcup_{x \in X} \Gamma^+(x) \text{ and } \Gamma^-(X) := \bigcup_{x \in X} \Gamma^-(x).$$

(3) *Let Γ be a digraph and let X, Y be subsets of $V(\Gamma)$. Recall that $A([X])$ is the set of arcs of the induced sub-digraph $[X]$ by X in Γ (i.e., the set of arcs of Γ in $X \times X$). Denote by $A(X, Y)$ the set of arcs of Γ in $X \times Y$. If $X = Y$ then we simply write $A(X)$ instead of $A(X, X)$. If $X = \{x\}$ or $Y = \{y\}$ then we simply write*

$$A(x, Y) := A(\{x\}, Y), \ A(X, y) := A(X, \{y\}) \text{ and } A(x, y) := A(\{x\}, \{y\})$$

if there are no confusions.

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. We first consider the case when given group admits an ORR.

Lemma 3.1. *Let G be a finite group of order at least 3. If G admits an ORR, then G admits an OmSR for any integer $m \geq 2$.*

Proof. Assume that G admits an ORR. That is, there exists a subset $R \subseteq G$ such that $\mathrm{Cay}(G, R)$ is an ORR of G. If $R = \emptyset$ then $\mathrm{Cay}(G, R)$ is the union of $|G|$ isolated vertices. Since $\mathrm{Cay}(G, R)$ is an ORR, G is $\mathbb{Z}_1$ or $\mathbb{Z}_2$ which is impossible as $|G| \geq 3$. Thus

$$R \neq \emptyset.$$

Let $\Sigma := \mathrm{Cay}(G, R)$, $\Phi := \mathrm{Cay}(G, R^{-1})$ and $E_H := \{(x, y, z) \mid x, y, z \in H, xy = z\}$ for $H \in \{R, R^{-1}\}$. As $(x, y, z) \in E_R$ if and only if $(y^{-1}, x^{-1}, z^{-1}) \in E_{R^{-1}}$, we have $|E_R| = |E_{R^{-1}}|$. Put $k := |E_R| = |E_{R^{-1}}|$. As $|A([\Sigma^+(1)])| = |E_R|$ and $|A([\Phi^+(1)])| = |E_{R^{-1}}|$ hold (see Notation 2.4), we obtain

$$|A([\Sigma^+(1)])| = |A([\Phi^+(1)])| = k \tag{1}$$

where 1 is the identity of G. As $R \neq \emptyset$, take $a \in R$. Note that

$$R \cap R^{-1} = \emptyset, \quad a \in R, \quad a^{-1} \notin R, \quad o(a) \geq 3, \quad 1 \notin R \cup R^{-1} \tag{2}$$

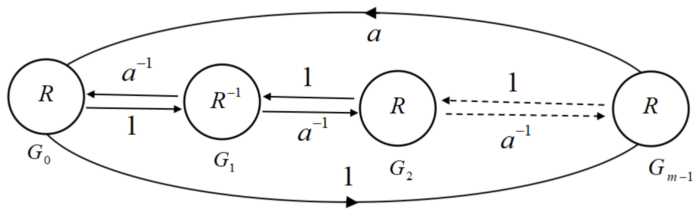

FIGURE 3.1. The digraph Γ_m for $m \geq 3$

all hold, where $o(a)$ is the order of a in G. Now, we divide the proof into two cases: $m = 2$ (Case 1) and $m \geq 3$ (Case 2).

Case 1: Show that G admits an O2SR.
Proof of Case 1. Let $\Gamma_2 := \mathrm{BiCay}(G, R, R^{-1}, \{1\}, \{a^{-1}\})$ and $\mathcal{A} := \mathrm{Aut}(\Gamma_2)$. Then Γ_2 is an oriented 2-Cayley digraph of G with valency $|R| + 1$ satisfying $\Gamma_2^+(1_0) = \{r_0, 1_1 \mid r \in R\}, \Gamma_2^+(1_1) = \{r_1^{-1}, a_0^{-1} \mid r \in R\}$ and $\Gamma_2^-(1_1) = \{r_1, 1_0 \mid r \in R\}$.
We first show that $\mathcal{A}$ is not vertex-transitive by considering $|A([\Gamma_2^+(1_0)])|$ and $|A([\Gamma_2^+(1_1)])|$. As $[\Gamma_2^+(1_0) \cap G_0] \cong [\Sigma^+(1)]$, we need to count the number of arcs between r_0 and 1_1 for $r \in R$. Since

$$\Gamma_2^+(1_1) \cap \Gamma_2^+(1_0) = \emptyset \ \text{ and } \ \Gamma_2^-(1_1) \cap \Gamma_2^+(1_0) = \emptyset$$

holds (see (2)), 1_1 is an isolated vertex in $[\Gamma_2^+(1_0)]$. Hence $|A([\Gamma_2^+(1_0)])| = |A([\Sigma^+(1)])| = k$ follows by (1). On the other hand, $|A([\Gamma_2^+(1_1)])| \geq |A([\Phi^+(1)])| + 1 = k + 1$ holds since (a_0^{-1}, a_1^{-1}) is an arc in $[\Gamma_2^+(1_1)]$ by $T_{0,1} = \{1\}$. Therefore $[\Gamma_2^+(1_0)] \ncong [\Gamma_2^+(1_1)]$ and $\mathcal{A}$ is not vertex-transitive. Recall that $R(G)$ is a group of automorphisms of Γ_2 with two orbits G_0 and G_1, and $\mathcal{A}$ also has two orbits G_0 and G_1 satisfying $\mathcal{A} = R(G)\mathcal{A}_{1_0}$. Since $\mathcal{A}_{1_0}$ fixes G_0 setwise and $[G_0] \cong \Sigma$ is an ORR, $\mathcal{A}_{1_0}$ fixes G_0 pointwise. Moreover, $\mathcal{A}_{1_0}$ fixes G_1 pointwise as $T_{0,1} = \{1\}$. Therefore, $\mathcal{A}_{1_0} = 1$ and $\mathcal{A} = R(G)\mathcal{A}_{1_0} = R(G)$. This shows that Γ_2 is an O2SR of G. This completes the proof of Case 1. □

Case 2: For $m \geq 3$, show that G admits an OmSR.
Proof of Case 2. For $m \geq 3$, define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\begin{array}{lll} T_{0,1} = \{1\}, & T_{1,0} = \{a^{-1}\},\ T_{1,1} = R^{-1},\ T_{m-1,0} = \{a\}\ ; & \\ T_{i,i-1} = \{1\}, & T_{i,i} = R & \text{for } i \neq 1\ ; \\ T_{i,i+1} = \{a^{-1}\} & & \text{for } i \neq 0, m-1\ ; \\ T_{i,j} = \emptyset & & \text{for } i \neq j, j \pm 1. \end{array}$$

Let $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$. Then Γ_m is an oriented m-Cayley digraph of G with valency $|R| + 2$. To complete the proof for Case 2, we need the following claim that counts the number of arcs in the induced digraph $[\Gamma_m^+(1_i)]$ for $i \in \mathbb{Z}_m$.

Claim:

(i) $|A([\Gamma_3^+(1_0)])| = k + 2$, $|A([\Gamma_m^+(1_0)])| = k + 1$ $(m \geq 4)$.
(ii) $|A([\Gamma_3^+(1_1)])| = k + 3$, $|A([\Gamma_m^+(1_1)])| = k + 2$ $(m \geq 4)$.
(iii) $|A([\Gamma_m^+(1_{m-1})])| = k + 1$ $(m \geq 3)$.
(iv) $|A([\Gamma_m^+(1_i)])| = k$ $(2 \leq i \leq m-2)$.

Proof of Claim. We first need to consider the set of out- and in- neighbors of 1_i, a_i and a_i^{-1} in Γ_m for $i \in \mathbb{Z}_m$. With the aid of FIGURE 3.1, we find the following:

$$\begin{array}{ll} \Gamma_m^+(1_0) = \{r_0, 1_1, 1_{m-1} \mid r \in R\}, & \Gamma_m^-(1_0) = \{r_0^{-1}, a_1, a_{m-1}^{-1} \mid r \in R\}, \\ \Gamma_m^+(1_1) = \{r_1^{-1}, a_2^{-1}, a_0^{-1} \mid r \in R\}, & \Gamma_m^-(1_1) = \{r_1, 1_2, 1_0 \mid r \in R\}, \\ \Gamma_m^+(1_{m-1}) = \{r_{m-1}, a_0, 1_{m-2} \mid r \in R\}, & \\ \Gamma_m^+(1_i) = \{r_i, a_{i+1}^{-1}, 1_{i-1} \mid r \in R\}\ (2 \leq i \leq m-2), & \Gamma_m^-(1_i) = \{r_i^{-1}, 1_{i+1}, a_{i-1} \mid r \in R\}\ (i \neq 0, 1), \\ \Gamma_m^+(a_0^{-1}) = \{(ra^{-1})_0, a_1^{-1}, a_{m-1}^{-1} \mid r \in R\}, & \Gamma_m^-(a_0^{-1}) = \{(r^{-1}a^{-1})_0, 1_1, a_{m-1}^{-2} \mid r \in R\}, \\ \Gamma_m^+(a_{m-1}^{-1}) = \{(ra^{-1})_{m-1}, 1_0, a_{m-2}^{-1} \mid r \in R\}, & \\ \Gamma_m^+(a_{i+1}^{-1}) = \{(ra^{-1})_{i+1}, a_{i+2}^{-2}, a_i^{-1} \mid r \in R\}\ (1 \leq i \leq m-3), & \Gamma_m^-(a_{i+1}^{-1}) = \{(r^{-1}a^{-1})_{i+1}, a_{i+2}^{-1}, 1_i \mid r \in R\}\ (i \neq 0, m-1), \\ \Gamma_m^+(a_0) = \{(ra)_0, a_1, a_{m-1} \mid r \in R\}, & \Gamma_m^-(a_0) = \{(r^{-1}a)_0, a_1^2, 1_{m-1} \mid r \in R\}. \end{array}$$

The above sets enable us to prove the claim as follows.
Proof of (i). It is easy to find

$$\begin{array}{llll} \Gamma_3^+(1_1) \cap \Gamma_3^+(1_0) = \emptyset, & \Gamma_3^+(1_2) \cap \Gamma_3^+(1_0) = \{a_0, 1_1\}, & \Gamma_3^-(1_1) \cap \Gamma_3^+(1_0) = \{1_2\}, & \Gamma_3^-(1_2) \cap \Gamma_3^+(1_0) = \emptyset\ ; \\ \Gamma_m^+(1_1) \cap \Gamma_m^+(1_0) = \emptyset, & \Gamma_m^+(1_{m-1}) \cap \Gamma_m^+(1_0) = \{a_0\}, & \Gamma_m^-(1_1) \cap \Gamma_m^+(1_0) = \emptyset, & \Gamma_m^-(1_{m-1}) \cap \Gamma_m^+(1_0) = \emptyset\ (m \geq 4). \end{array}$$

As $|A([\Gamma_m^+(1_0)])| = |A(R_0)| + |A(1_1, R_0 \cup \{1_{m-1}\})| + |A(1_{m-1}, R_0 \cup \{1_1\})| - |A(1_1, 1_{m-1})|$ holds (see Notation 2.4), the result (i) now follows by

$$\begin{aligned} |A(R_0)| &= |[\Gamma_m^+(1_0) \cap G_0]| = |[\Sigma^+(1)]| = k \ ; \\ |A(1_1, R_0 \cup \{1_{m-1}\})| &= |\Gamma_m^+(1_1) \cap \Gamma_m^+(1_0)| + |\Gamma_m^-(1_1) \cap \Gamma_m^+(1_0)| \ ; \\ |A(1_{m-1}, R_0 \cup \{1_0\})| &= |\Gamma_m^+(1_{m-1}) \cap \Gamma_m^+(1_0)| + |\Gamma_m^-(1_{m-1}) \cap \Gamma_m^+(1_0)| \ ; \\ |A(1_1, 1_2)| &= 1 \ ; \\ |A(1_1, 1_{m-1})| &= 0 \end{aligned}$$

for $m \geq 4$.

Proof of (ii). It is easy to find

$$\begin{array}{llll} \Gamma_3^+(a_0^{-1}) \cap \Gamma_3^+(1_1) = \{a_1^{-1}, a_2^{-1}\}, & \Gamma_3^-(a_0^{-1}) \cap \Gamma_3^+(1_1) = \emptyset, & \Gamma_3^+(a_2^{-1}) \cap \Gamma_3^+(1_1) = \{a_1^{-1}\}, & \Gamma_3^-(a_2^{-1}) \cap \Gamma_3^+(1_1) = \{a_0^{-1}\} \ ; \\ \Gamma_m^+(a_0^{-1}) \cap \Gamma_m^+(1_1) = \{a_1^{-1}\}, & \Gamma_m^-(a_0^{-1}) \cap \Gamma_m^+(1_1) = \emptyset, & \Gamma_m^+(a_2^{-1}) \cap \Gamma_m^+(1_1) = \{a_1^{-1}\}, & \Gamma_m^-(a_2^{-1}) \cap \Gamma_m^+(1_1) = \emptyset \ (m \geq 4). \end{array}$$

The result (ii) follows by

$$\begin{aligned} |A([\Gamma_m^+(1_1)])| = \ & k - |A(a_0^{-1}, a_2^{-1})| + \\ & |\Gamma_m^+(a_0^{-1}) \cap \Gamma_m^+(1_1)| + |\Gamma_m^-(a_0^{-1}) \cap \Gamma_m^+(1_1)| + |\Gamma_m^+(a_2^{-1}) \cap \Gamma_m^+(1_1)| + |\Gamma_m^-(a_2^{-1}) \cap \Gamma_m^+(1_1)| \end{aligned}$$

and $|A(a_0^{-1}, a_2^{-1})| = 1$ for $m = 3$, and $|A(a_0^{-1}, a_2^{-1})| = 0$ for $m \geq 4$.

Proof of (iii). For $m \geq 3$, it is easy to find

$$\Gamma_m^+(a_0) \cap \Gamma_m^+(1_{m-1}) = \{a_{m-1}\},\ \Gamma_m^-(a_0) \cap \Gamma_m^+(1_{m-1}) = \emptyset,\ \Gamma_m^+(1_{m-2}) \cap \Gamma_m^+(1_{m-1}) = \emptyset,\ \Gamma_m^-(1_{m-2}) \cap \Gamma_m^+(1_{m-1}) = \emptyset.$$

As $|A(a_0, 1_{m-2})| = 0$ holds by FIGURE 3.1, we obtain (iii):

$$\begin{aligned} |A([\Gamma_m^+(1_{m-1})])| &= k - |A(a_0, 1_{m-2})| + |\Gamma_m^+(a_0) \cap \Gamma_m^+(1_{m-1})| + |\Gamma_m^-(a_0) \cap \Gamma_m^+(1_{m-1})| \\ &\quad + |\Gamma_m^+(1_{m-2}) \cap \Gamma_m^+(1_{m-1})| + |\Gamma_m^-(1_{m-2}) \cap \Gamma_m^+(1_{m-1})| \\ &= k + 1. \end{aligned}$$

Proof of (iv). Let $m \geq 4$. Since there are no arcs between 1_{i-1} and a_{i+1}^{-1} $(2 \leq i \leq m-2)$, we obtain

$$|A(1_{i-1}, a_{i+1}^{-1})| = 0 \ (2 \leq i \leq m-2). \tag{3}$$

We divide the proof of (iv) into three cases $i = 2$, $i = m-2$ and $3 \leq i \leq m-3$.
First, we consider $3 \leq i \leq m-3$. Then $m \geq 6$ and

$$\Gamma_m^+(1_{i-1}) \cap \Gamma_m^+(1_i) = \emptyset,\ \Gamma_m^-(1_{i-1}) \cap \Gamma_m^+(1_i) = \emptyset,\ \Gamma_m^+(a_{i+1}^{-1}) \cap \Gamma_m^+(1_i) = \emptyset,\ \Gamma_m^-(a_{i+1}^{-1}) \cap \Gamma_m^+(1_i) = \emptyset.$$

Using (3), we have

$$\begin{aligned} |A([\Gamma_m^+(1_i)])| &= k - |A(1_{i-1}, a_{i+1}^{-1})| + |\Gamma_m^+(1_{i-1}) \cap \Gamma_m^+(1_i)| + |\Gamma_m^-(1_{i-1}) \cap \Gamma_m^+(1_i)| + |\Gamma_m^+(a_{i+1}^{-1}) \cap \Gamma_m^+(1_i)| + |\Gamma_m^-(a_{i+1}^{-1}) \cap \Gamma_m^+(1_i)| \\ &= k \end{aligned}$$

and this shows (iv) for $3 \leq i \leq m-3$.
Now we consider $i = 2$. By (3) and

$$\Gamma_m^+(1_1) \cap \Gamma_m^+(1_2) = \emptyset,\ \Gamma_m^-(1_1) \cap \Gamma_m^+(1_2) = \emptyset,\ \Gamma_m^+(a_3^{-1}) \cap \Gamma_m^+(1_2) = \emptyset,\ \Gamma_m^-(a_3^{-1}) \cap \Gamma_m^+(1_2) = \emptyset,$$

we obtain

$$\begin{aligned} |A([\Gamma_m^+(1_2)])| &= k - |A(1_1, a_3^{-1})| + |\Gamma_m^+(1_1) \cap \Gamma_m^+(1_2)| + |\Gamma_m^-(1_1) \cap \Gamma_m^+(1_2)| + |\Gamma_m^+(a_3^{-1}) \cap \Gamma_m^+(1_2)| + |\Gamma_m^-(a_3^{-1}) \cap \Gamma_m^+(1_2)| \\ &= k \end{aligned}$$

and this shows (iv) for $i = 2$.
Finally, let $i = m-2$. Note that $m \geq 5$ follows by $i = m-2 \neq 2$ and $m \geq 4$. By (3) and

$$\Gamma_m^+(1_{m-3}) \cap \Gamma_m^+(1_{m-2}) = \emptyset,\ \Gamma_m^-(1_{m-3}) \cap \Gamma_m^+(1_{m-2}) = \emptyset,\ \Gamma_m^+(a_{m-1}^{-1}) \cap \Gamma_m^+(1_{m-2}) = \emptyset,\ \Gamma_m^-(a_{m-1}^{-1}) \cap \Gamma_m^+(1_{m-2}) = \emptyset,$$

we obtain (iv):

$$\begin{aligned} |A([\Gamma_m^+(1_{m-2})])| &= k - |A(1_{m-3}, a_{m-1}^{-1})| + |\Gamma_m^+(1_{m-3}) \cap \Gamma_m^+(1_{m-2})| + |\Gamma_m^-(1_{m-3}) \cap \Gamma_m^+(1_{m-2})| \\ &\quad + |\Gamma_m^+(a_{m-1}^{-1}) \cap \Gamma_m^+(1_{m-2})| + |\Gamma_m^-(a_{m-1}^{-1}) \cap \Gamma_m^+(1_{m-2})| \\ &= k. \end{aligned}$$

This completes the proof of the claim. □

Now we are ready to complete the proof of Case 2 in Lemma 3.1 by using the above claim. Let $\mathcal{A} := \mathrm{Aut}(\Gamma_m)$. As $|A([\Gamma_m^+(1_1)])| \neq |A([\Gamma_m^+(1_i)])|$ $(i \neq 1)$ by the claim, we have $[\Gamma_m^+(1_1)] \ncong [\Gamma_m^+(1_i)]$ $(i \neq 1)$ and thus $\mathcal{A}$ fixes G_1 setwise. By

$|A([\Gamma_m^+(1_0)])| \neq |A([\Gamma_m^+(1_2)])|$ (see the above claim), $\mathcal{A}$ fixes G_i $(i = 0, 2)$ setwise. By $|T_{2,3}| = 1$ and $|T_{2,i}| = 0$ $(i \neq 1, 2, 3)$, $\mathcal{A}$ fixes G_3 setwise. Similary, we obtain that $\mathcal{A}$ fixes G_i $(i \in \mathbb{Z}_m)$ setwise. Recall that $R(G)$ is a group of automorphisms of Γ_m with m orbits G_i $(i \in \mathbb{Z}_m)$. Then $\mathcal{A}$ also has m orbits satisfying $\mathcal{A} = R(G)\mathcal{A}_{1_0}$. Since $\mathcal{A}_{1_0}$ fixes G_i setwise and $[G_0 \cup G_1] \cong \Gamma_2$ is an O2SR, $\mathcal{A}_{1_0}$ fixes G_0 and G_1 pointwise. As $T_{0,m-1} = \{1\}$, $\mathcal{A}_{1_0}$ fixes G_{m-1} pointwise. Similarly, we find that $\mathcal{A}_{1_0}$ fixes G_i $(2 \leq i \leq m-2)$ pointwise and hence $\mathcal{A}_{1_0} = 1$. Thus, $\mathcal{A} = R(G)\mathcal{A}_{1_0} = R(G)$ holds and therefore Γ_m is an OmSR of G. This completes the proof of Case 2 in Lemma 3.1. □

Now, we consider elementary abelian 2-groups in two cases: of order at most 16 (Lemma 3.4); of order greater than 16 (Lemma 3.6). We need the following notation and remark on oriented digraphs with valency two.

Notation 3.2. *Let Δ be an oriented digraph with valency two. Define $\mathcal{C}(\Delta)$ by the set of oriented 3-cycles in Δ and let $V(\mathcal{C}(\Delta))$ be the set of vertices in $\mathcal{C}(\Delta)$. For a subset $S \subseteq V(\Delta)$, define $\mathcal{C}_S(\Delta)$ by the set of cycles in $\mathcal{C}(\Delta)$ containing at least one vertex of S. For notational simplicity, put $\mathcal{C}_s(\Delta) := \mathcal{C}_{\{s\}}(\Delta)$. For a subset $W \subseteq \mathcal{C}(\Delta)$, define $N_\Delta(W)$ by the set of vertices and their neighbors of each cycle in W. Put $N_\Delta(C) := N_\Delta(\{C\})$. If there are no confusions, we may omit (Δ) for each notation; $\mathcal{C}(\Delta)$, $V(\mathcal{C}(\Delta))$, $\mathcal{C}_S(\Delta)$, $\mathcal{C}_s(\Delta)$ and $N_\Delta(W)$.*

The following simple observation on oriented digraphs with valency two will be used in the proof of Lemma 3.4.

Remark 3.3. *Let Δ be an oriented digraph with valency two and let $\sigma \in \mathrm{Aut}(\Delta)$. Suppose that σ fixes a vertex $v \in V(\Delta)$.*

1. *If σ fixes an out-neighbor and an in-neighbor of v, then σ fixes every neighbors of v in Δ.*
2. *Suppose that σ fixes an oriented cycle $C \in \mathcal{C}(\Delta)$ pointwise.*
   (i) Then σ fixes $N_\Delta(C)$ pointwise. Moreover, if $s \in C$ then σ fixes every vertex of $N_\Delta(\mathcal{C}_s)$ in Δ.
   (ii) Let $S \subseteq V(\Delta)$. If $C \cap S \neq \emptyset$ and cycles in $\mathcal{C}_S$ are connected to one another, then σ fixes $N_\Delta(\mathcal{C}_S)$ pointwise.

Lemma 3.4. *Let $m \geq 1$ be an integer. Each group $\mathbb{Z}_2^n$ $(0 \leq n \leq 4)$ satisfies the following.*

1. *$\mathbb{Z}_1$ does not admit OmSRs if and only if $2 \leq m \leq 6$.*
2. *$\mathbb{Z}_2$ does not admit OmSRs if and only if $m = 2$.*
3. *For each integer $2 \leq n \leq 4$, $\mathbb{Z}_2^n$ does not admit OmSRs if and only if $m = 1$ or 2.*

Proof. Let $G := \mathbb{Z}_2^n$ $(0 \leq n \leq 4)$. By Remark 2.1 and Proposition 2.2, G does not admit O1SRs if and only if $G = \mathbb{Z}_2^2, \mathbb{Z}_2^3$ or $\mathbb{Z}_2^4$. For the rest of the proof, we assume $m \geq 2$. We first show that

(4) if $(m, G) \in \{(2, \mathbb{Z}_1), (3, \mathbb{Z}_1), (4, \mathbb{Z}_1), (5, \mathbb{Z}_1), (6, \mathbb{Z}_1), (2, \mathbb{Z}_2), (2, \mathbb{Z}_2^2), (2, \mathbb{Z}_2^3), (2, \mathbb{Z}_2^4)\}$ then G does not admit OmSRs.

Since any vertex-transitive digraph cannot be OmSR $(m \geq 2)$, digraphs with valency 0, 1 cannot be OmSR $(m \geq 2)$. Moreover, digraphs of order less than 5 cannot be OmSR. Hence $\mathbb{Z}_1$ does not admit OmSRs for $2 \leq m \leq 4$, and $\mathbb{Z}_2$ does not admit O2SRs. With the aid of Magma [3], we can find that G does not admit OmSRs for each $(m, G) \in \{(5, \mathbb{Z}_1), (6, \mathbb{Z}_1), (2, \mathbb{Z}_2^2), (2, \mathbb{Z}_2^3), (2, \mathbb{Z}_2^4)\}$. This shows statement (4).

To complete the proof, we need to show that if G satisfies $G = \mathbb{Z}_1$ with $m \geq 7$ or $G = \mathbb{Z}_2^n$ $(1 \leq n \leq 4)$ with $m \geq 3$, then G admits an OmSR. We divide the rest of the proof into five cases.

Case 1: Show that $G = \mathbb{Z}_1$ admits an OmSR for any $m \geq 7$.
Proof of Case 1. For a given m, we will define a subset $T_{i,j} \subseteq G = \mathbb{Z}_1$ $(i, j \in \mathbb{Z}_m)$ and $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$. Let $\mathcal{A} := \mathrm{Aut}(\Gamma_m)$, $\mathcal{C} := \mathcal{C}(\Gamma_m)$ and $\mathcal{C}_S := \mathcal{C}_S(\Gamma_m)$ for a subset $S \subseteq V(\Gamma_m)$ (see Notation 3.2).
For each $7 \leq m \leq 11$, define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\begin{array}{ll} T_{0,3} = T_{1,4} = T_{2,1} = T_{m-1,2} = T_{j,j+2} = \{1\} & \text{for } 3 \leq j \leq m-2; \\ T_{i,i+1} = \{1\} & \text{for } i \in \mathbb{Z}_m; \\ T_{i,j} = \emptyset & \text{otherwise.} \end{array}$$

Then Γ_m is an oriented m-Cayley digraph of G with valency two. With the aid of Magma [3], we find that Γ_m is an OmSR of G for each $7 \leq m \leq 11$.

On the other hand, consider Case 1 for $m \geq 12$. For each even integer $m \geq 12$, define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\{1\} = T_{0,1} = T_{1,2} = T_{2,3} = T_{3,4} = T_{4,5} = T_{5,6} = T_{6,0}\ ; \tag{5}$$

$$\begin{aligned} \{1\} &= T_{9,10} = T_{10,12} = \cdots = T_{i,i+2} = \cdots = T_{m-4,m-2} = T_{m-2,m-1} \\ &= T_{m-1,m-3} = \cdots = T_{i,i-2} = \cdots = T_{13,11} = T_{11,1} = T_{1,9}\ ; \end{aligned} \tag{6}$$

$$\begin{aligned} \{1\} &= T_{0,4} = T_{4,2} = T_{2,6} = T_{6,7} = T_{7,8} = T_{8,3} = T_{3,5} = T_{5,7} = T_{7,m-2} = T_{m-2,8} = T_{8,m-1} \\ &= T_{m-1,m-4} = T_{m-4,m-3} = \cdots = T_{i,i-3} = T_{i-3,i-2} = \cdots = T_{13,10} = T_{10,11} = T_{11,9} = T_{9,0}\ ; \end{aligned} \tag{7}$$

$$\emptyset = T_{i,j} \ \text{otherwise}\ .$$

Then Γ_m is an oriented m-Cayley digraph of G with valency two and Γ_m consists of three oriented cycles which correspond to (5)-(7), respectively (see the left one of FIGURE 3.2).

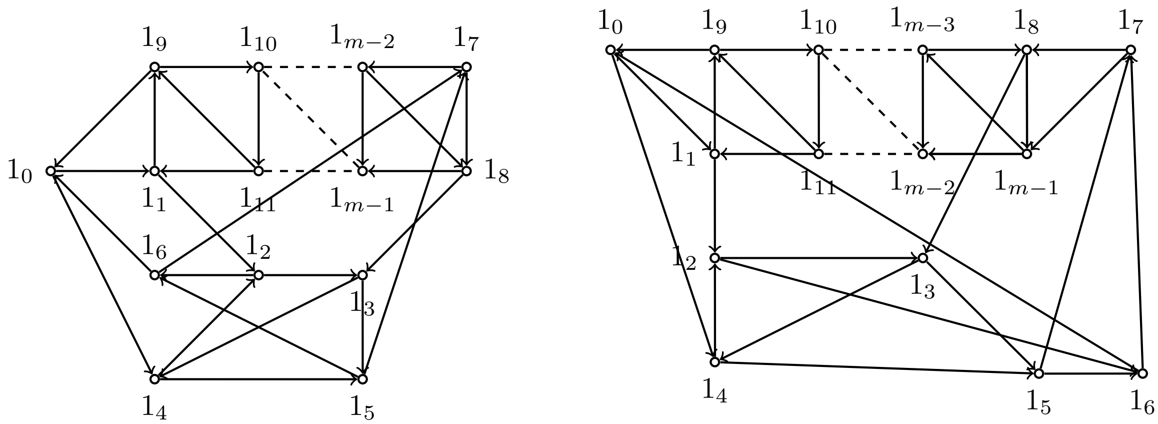

FIGURE 3.2. OmSR of $\mathbb{Z}_1$ for $m \geq 12$

We first show $\mathcal{A}_{1_0} = 1$ by using FIGURE 3.2 and Remark 3.3. By $|\mathcal{C}_{1_0}| = 1$ and $\mathcal{C}_{1_0} = [\{1_0, 1_1, 1_9\}]$, $\mathcal{A}_{1_0}$ fixes cycle $\mathcal{C}_{1_0}$. By Remark 3.3 with $(\sigma, s, S) = (\mathcal{A}_{1_0}, 1_9, \{1_9, 1_{10}, 1_{12}, \cdots, 1_{m-1}\})$, $\mathcal{A}_{1_0}$ fixes $N(\mathcal{C}_{\{1_9,1_{10},1_{12},\cdots,1_{m-1}\}})$ pointwise. On the other hand, we find that $|\mathcal{C}_{1_4}| = 1$ and two cycles $\mathcal{C}_{1_0}$ and $\mathcal{C}_{1_4} = [\{1_4, 1_2, 1_3\}]$ are adjacent. This implies that $\mathcal{A}_{1_0}$ fixes $\mathcal{C}_{1_4}$ pointwise, and also $N(\mathcal{C}_{\{1_9,1_{10},1_{12},\cdots,1_{m-1}\}} \cup \mathcal{C}_{1_4})$. Now $\mathcal{A}_{1_0} = 1$ follows as $N(\mathcal{C}_{\{1_9,1_{10},1_{12},\cdots,1_{m-1}\}} \cup \mathcal{C}_{1_4}) = V(\Gamma_m)$. By the left one of FIGURE 3.2, it is easy to find that $\{1_5, 1_6, 1_7, 1_8\} = V(\Gamma_m) \setminus V(\mathcal{C})$ and $1_5 \in N(\mathcal{C})$ both hold. This implies that $\mathcal{A}$ fixes 1_5, and $\mathcal{A}$ fixes directed path $(1_5, 1_6, 1_7)$ pointwise because the induced digraph $[\{1_5, 1_6, 1_7, 1_8\}]$ is the union of directed path $(1_5, 1_6, 1_7, 1_8)$ and arc $(1_5, 1_7)$. Since 1_0 is a neighbor of 1_6, $\mathcal{A}$ fixes 1_0 by Remark 3.3. As $\mathcal{A}_{1_0} = 1$, $\mathcal{A}$ fixes each vertex of Γ_m. Now Case 1 holds for even integer $m \geq 12$.

Finally, we consider Case 1 for odd integer $m > 12$. Define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\{1\} = T_{0,1} = T_{1,2} = T_{2,3} = T_{3,4} = T_{4,5} = T_{5,6} = T_{6,0}\ ; \tag{8}$$

$$\begin{aligned}\{1\} &= T_{9,10} = T_{10,12} = \cdots = T_{i,i+2} = \cdots = T_{m-5,m-3} = T_{m-3,8} = T_{8,m-1} = T_{m-1,m-2} \\ &= T_{m-2,m-4} = \cdots = T_{i,i-2} = \cdots = T_{13,11} = T_{11,1} = T_{1,9}\ ;\end{aligned} \tag{9}$$

$$\begin{aligned}\{1\} &= T_{0,4} = T_{4,2} = T_{2,6} = T_{6,7} = T_{7,8} = T_{8,3} = T_{3,5} = T_{5,7} = T_{7,m-1} = T_{m-1,m-3} = T_{m-3,m-2} = T_{m-2,m-5} \\ &= \cdots = T_{i,i+1} = T_{i+1,i-2} = \cdots = T_{12,13} = T_{13,10} = T_{10,11} = T_{11,9} = T_{9,0}\ ;\end{aligned} \tag{10}$$

$$\emptyset = T_{i,j} \ \text{ otherwise}\ .$$

Then Γ_m is an oriented m-Cayley digraph of G with valency two, and Γ_m consists of three oriented cycles which correspond to (8)-(10), respectively (see the right one of FIGURE 3.2). With $S := \{1_9, 1_{10}, 1_{12} \cdots 1_{m-3}\}$, we can obtain $\mathcal{A}_{1_0} = 1$ in the same way as Case 1 for even integer $m \geq 12$. By the right one of FIGURE 3.2, it is easy to find that $\{1_5, 1_6, 1_7\} = V(\Gamma_m) \setminus V(\mathcal{C})$ and $1_5 \in N(\mathcal{C})$ both hold. This implies that $\mathcal{A}$ fixes 1_5, and $\mathcal{A}$ fixes directed path $(1_5, 1_6, 1_7)$ pointwise because induced digraph $[\{1_5, 1_6, 1_7\}]$ is the union of directed path $(1_5, 1_6, 1_7)$ and arc $(1_5, 1_7)$. Since 1_0 is a neighbor of 1_6, $\mathcal{A}$ fixes 1_0 by Remark 3.3. As $\mathcal{A}_{1_0} = 1$, $\mathcal{A}$ fixes each vertex of Γ_m. Now Case 1 holds for odd integer $m > 12$. This completes the proof of Case 1. □

For the remaining cases we need to use the following statement (11) for new graph Δ from Γ_m, where Γ_m is the graph defined in Case 1. For a positive integer $m \geq 12$ and a subset $\mathbb{E} \subseteq \{(1_6, 1_0), (1_1, 1_2), (1_8, 1_3), (1_4, 1_5)\}$, let Δ be a connected oriented digraph with valency two containing $\Gamma_m - \mathbb{E}$ as a subgraph, where $\Gamma_m - \mathbb{E}$ is the subgraph of Γ_m by deleting all arcs in $\mathbb{E}$. Note that the graph $\Gamma_m - \mathbb{E}$ is connected and it contains every oriented 3-cycles in $\mathcal{C}(\Gamma_m) \cup \mathcal{C}_{1_4}(\Gamma_m)$ and arc $(1_0, 1_4)$, where $\mathcal{C}_{1_4}(\Gamma_m) = [\{1_4, 1_2, 1_3\}]$. For each $e \in \mathbb{E}$, we denote by $v(e)$ the set of two vertices incident to e and let $\mathrm{Aut}(\Delta)_{(v(e))}$ be the subgroup of $\mathrm{Aut}(\Delta)$ fixing $v(e)$ pointwise. We now show that the following holds.

$$\text{Each group in } \{\mathrm{Aut}(\Delta)_{1_0}, \mathrm{Aut}(\Delta)_{1_5}, \mathrm{Aut}(\Delta)_{(v(e))} \mid e \in \mathbb{E}\} \text{ fixes each vertex of } \Gamma_m. \tag{11}$$

First, we can show that $\mathrm{Aut}(\Delta)_{1_0}$ fixes each vertex of Γ_m in the same way with Case 1. Let $e \in \mathbb{E}$. If $e = (1_6, 1_0)$ then $\mathrm{Aut}(\Delta)_{(v(e))}$ fixes 1_0. If $e \neq (1_6, 1_0)$, then e and $\mathcal{C}_{1_4}(\Gamma_m)$ has a common vertex. By Remark 3.3, we obtain that $\mathrm{Aut}(\Delta)_{(v(e))}$ fixes $\mathcal{C}_{1_4}(\Gamma_m) = [\{1_4, 1_2, 1_3\}]$ pointwise, and also fixes each neighbors of 1_4. Since 1_0 is a neighbor of 1_4, $\mathrm{Aut}(\Delta)_{(v(e))}$ fixes 1_0 and hence $\mathrm{Aut}(\Delta)_{(v(e))}$ fixes each vertex of Γ_m. Now consider $\mathrm{Aut}(\Delta)_{1_5}$. Since two out-neighbors of 1_5 in Δ are 1_6 and 1_7, and $(1_6, 1_7)$ is an arc, we obtain that $\mathrm{Aut}(\Delta)_{1_5}$ fixes cycle $[\{1_5, 1_6, 1_7\}]$ pointwise. As 1_2 is adjacent to 1_6, $\mathrm{Aut}(\Delta)_{1_5}$ fixes 1_2 by Remark 3.3. Thus, $\mathrm{Aut}(\Delta)_{1_5}$ fixes cycle $\mathcal{C}_{1_4}(\Gamma_m) = [\{1_4, 1_2, 1_3\}]$ pointwise, and thus it fixes each vertex of Γ_m. Now

(11) follows and we are ready to use it.

For the remaining cases Case 2–Case 5, we will define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ for each m and G, and let $\Sigma_m := \mathrm{Cay}(G, T_{i,j} : i,j \in \mathbb{Z}_m)$ and $\mathcal{A} := \mathrm{Aut}(\Sigma_m)$. Then we can check that each Σ_m is an oriented m-Cayley digraph of G with valency two.

Case 2: Show that $G = \langle x \rangle = \mathbb{Z}_2$ admits an OmSR for any $m \geq 3$.
Proof of Case 2. For each $3 \leq m \leq 11$, define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ as follows. Let $T_{i,j} = \emptyset$ $(i,j \in \mathbb{Z}_m)$ except

for $m = 3$ $\quad T_{0,1} = T_{0,2} = T_{2,0} = \{1\},\ T_{1,0} = T_{1,2} = T_{2,1} = \{x\}$;
for $m = 4$ $\quad T_{0,1} = T_{0,2} = T_{3,0} = T_{3,1} = \{1\},\ T_{1,0} = T_{1,2} = \{x\},\ T_{2,3} = \{1, x\}$;
for $m = 5$ $\quad T_{0,1} = T_{0,3} = T_{2,0} = T_{2,4} = T_{3,1} = T_{3,4} = T_{4,3} = T_{4,2} = \{1\},\ T_{1,0} = T_{1,2} = \{x\}$;
for $m = 6$ $\quad T_{0,1} = T_{0,3} = T_{2,0} = T_{2,5} = T_{3,1} = T_{3,4} = T_{4,3} = T_{5,4} = \{1\},\ T_{1,0} = T_{1,2} = T_{4,5} = T_{5,2} = \{x\}$;
for $7 \leq m \leq 11$ $\quad T_{0,1} = T_{0,3} = T_{2,0} = T_{2,5} = T_{3,1} = T_{3,4} = T_{4,3} = T_{5,6} = T_{i,i+1} = T_{m-1,4} = \{1\}$,
$T_{1,0} = T_{1,2} = T_{4,m-1} = T_{5,2} = T_{i,i-1} = T_{m-1,m-2} = \{x\}$ $(6 \leq i \leq m-2)$.

With the aid of Magma [3], we find that Σ_m is an OmSR of G for each $3 \leq m \leq 11$.

For $m \geq 12$, take $T_{i,j}$ $(i,j \in \mathbb{Z}_m)$ the same as Case 1 except $T_{6,0} = \{x\}$. Note that we denote by Γ_m (Σ_m, respectively) the m-Cayley digraph defined in Case 1 (Case 2-Case 5, respectively). Clearly, Σ_m has two layers: vertex sets $\{1_i \mid i \in \mathbb{Z}_m\}$ and $\{x_i \mid i \in \mathbb{Z}_m\}$. They are called 1-*layer* and x-*layer of* Σ_m, respectively (this notation i-*layer* will be used in Case 2-Case 5). For each $g \in \{1, x\}$, the induced subgraph of g-layer is isomorphic to the subgraph $\Gamma_m - \{(1_6, 1_0)\}$ of Γ_m. By (11), $\mathcal{A}_{1_0}$ fixes 1-layer pointwise. Since 1-layer has only two neighbors in x-layer (say, x_0 and x_6), $\mathcal{A}_{1_0}$ fixes x_0 and x_6, respectively. As $\mathcal{A}_{(\{x_0,x_6\})}$ fixes each vertex of x-layer, $\mathcal{A}_{1_0}$ fixes every vertex of Σ_m (i.e., $\mathcal{A}_{1_0} = 1$). We also obtain $\mathcal{A}_{1_5} = \mathcal{A}_{1_0} = 1$ by (11). In view of Figure 3.2, we can check that for even $m \geq 12$, the vertices not on any oriented 3-cycle of Σ_m are the vertices in $G_5 \cup G_6 \cup G_7 \cup G_8$, and only G_5 has two in-neighbors on an oriented 3-cycle; for odd $m > 12$, the vertices not on any oriented 3-cycle of Σ_m are the vertices in $G_5 \cup G_6 \cup G_7$, and only G_5 has two in-neighbors on an oriented 3-cycle. Thus, $\mathcal{A}$ fixes G_5 setwise. Since $R(G)$ is transitive on G_5, we have $\mathcal{A} = R(G)\mathcal{A}_{1_5} = R(G)$. Hence Σ_m is an OmSR of G. This completes the proof of Case 2. □

Case 3: Show that $G = \langle x, y \rangle = \mathbb{Z}_2^2$ admits an OmSR for any $m \geq 3$.
Proof of Case 3. For each $3 \leq m \leq 11$, define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ as follows. Let $T_{i,j} = \emptyset$ $(i,j \in \mathbb{Z}_m)$ except

for $m = 3$ $\quad T_{0,1} = T_{1,2} = \{1\},\ T_{2,1} = \{x\},\ T_{1,0} = T_{2,0} = \{y\},\ T_{0,2} = \{xy\}$;
for $m = 4$ $\quad T_{0,1} = T_{1,3} = T_{3,2} = \{1\},\ T_{2,3} = T_{3,1} = \{x\},\ T_{1,0} = T_{2,0} = \{y\},\ T_{0,2} = \{xy\}$;
for $5 \leq m \leq 11$ $\quad T_{0,1} = T_{1,3} = T_{3,4} = T_{i,i+1} = T_{m-1,2} = \{1\},\ T_{1,0} = T_{2,0} = \{y\},\ T_{0,2} = \{xy\}$,
$T_{2,m-1} = T_{3,1} = T_{i,i-1} = T_{m-1,m-2} = \{x\}$ $(4 \leq i \leq m-2)$.

With the aid of Magma [3], we find that Σ_m is an OmSR of G for each $3 \leq m \leq 11$.
For $m \geq 12$, define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ the same as Case 1 except

$$T_{6,0} = \{x\} \text{ and } T_{1,2} = \{y\}.$$

Then Σ_m has four layers. The induced subgraph of each layer is isomorphic to the induced subgraph of 1-layer, which is exactly the subgraph $\Gamma_m - \{(1_6, 1_0), (1_1, 1_2)\}$ of Γ_m. By (11), $\mathcal{A}_{1_0}$ fixes 1-layer pointwise, and also it fixes each vertex of $\{x_0, x_6, y_1, y_2\}$. Again by (11), $\mathcal{A}_{(\{x_0,y_6\})}$ fixes x-layer pointwise; $\mathcal{A}_{(\{y_1,y_2\})}$ fixes y-layer pointwise; $\mathcal{A}_{(\{y_1,y_2\})}$ fixes $(xy)_0$ and $(xy)_6$ because they are adjacent to y_6 and y_0, respectively. Therefore $\mathcal{A}_{(\{(xy)_0,(xy)_6\})}$ fixes xy-layer pointwise, and hence $\mathcal{A}_{1_0}$ fixes every vertex of Σ_m (i.e., $\mathcal{A}_{1_0} = 1$). Thus, $\mathcal{A}_{1_5} = \mathcal{A}_{1_0} = 1$. In the same way as Case 2, we can show that $\mathcal{A}$ fixes G_5 setwise, and Σ_m is an OmSR of G. This completes the proof of Case 3. □

Case 4: Show that $G = \langle x, y, z \rangle = \mathbb{Z}_2^3$ admits an OmSR for any $m \geq 3$.
Proof of Case 4. For each $3 \leq m \leq 11$, define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ as follows. Let $T_{i,j} = \emptyset$ $(i,j \in \mathbb{Z}_m)$ except

for $m = 3$ $\quad T_{0,1} = T_{1,2} = \{1\},\ T_{2,1} = \{x\},\ T_{2,0} = \{y\},\ T_{1,0} = T_{0,2} = \{z\}$;
for $m = 4$ $\quad T_{0,1} = T_{1,3} = T_{3,2} = \{1\},\ \ T_{2,3} = T_{3,1} = \{x\},\ \ T_{2,0} = \{y\},\ T_{0,2} = \{z\}$;
for $5 \leq m \leq 11$ $\quad T_{0,1} = T_{1,3} = T_{3,4} = T_{i,i+1} = T_{m-1,2} = \{1\},\ T_{2,0} = \{y\},\ T_{1,0} = T_{0,2} = \{z\}$,
$T_{2,m-1} = T_{3,1} = T_{i,i-1} = T_{m-1,m-2} = \{x\}$ $(4 \leq i \leq m-2)$.

With the aid of Magma [3], we find that Σ_m is an OmSR of G for each $3 \leq m \leq 11$.
For $m \geq 12$, define a subset $T_{i,j} \subseteq G$ $(i,j \in \mathbb{Z}_m)$ the same as Case 1 except

$$T_{6,0} = \{x\},\ T_{1,2} = \{y\},\ T_{8,3} = \{z\}.$$

In the same way as Case 3, we can show that Σ_m is an OmSR of G. This completes the proof of Case 4. □

Case 5: Show that $G=\langle x,y,z,w\rangle=\mathbb{Z}_2^4$ admits an OmSR for any $m\geq 3$.

Proof of Case 5. For each $3\leq m\leq 11$, define a subset $T_{i,j}\subseteq G$ $(i,j\in\mathbb{Z}_m)$ as follows. Let $T_{i,j}=\emptyset$ $(i,j\in\mathbb{Z}_m)$ except

for $m=3$ $\quad T_{0,1}=\{1,y,xy\},\ T_{1,2}=\{1,z,w\},\ T_{2,0}=\{y,w,xw\}$;
for $4\leq m\leq 11$ $\quad T_{0,1}=\{1,y,xy\},\ T_{1,2}=\{1,z,w\},\ T_{m-1,0}=\{y,w,xw\},\ T_{i,i+1}=\{x,y,w\}\ (2\leq i\leq m-2)$.

With the aid of Magma [3], we find that Σ_m is an OmSR of G for each $3\leq m\leq 11$.
For $m\geq 12$, define a subset $T_{i,j}\subseteq G$ $(i,j\in\mathbb{Z}_m)$ the same as Case 1 except

$$T_{6,0}=\{x\},\ T_{1,2}=\{y\},\ T_{8,3}=\{z\},\ T_{4,5}=\{w\}.$$

In the same way as Case 3, we can show that Σ_m is an OmSR of G. This completes the proof of Case 5. Now Lemma 3.4 follows. □

Remark 3.5. *The oriented m-Cayley digraphs constructed in Lemma 3.4 have valency two. In particular, $\mathbb{Z}_1$ admits an OmSR with valency two if and only if $m\geq 7$, and $\mathbb{Z}_2^n$ $(1\leq n\leq 4)$ admits an OmSR with valency two if and only if $m\geq 3$.*

Lemma 3.6. *For any integers $n\geq 5$ and $m\geq 2$, $\mathbb{Z}_2^n$ admits an OmSR.*

Proof. For a given integer $n\geq 5$, let $G:=\mathbb{Z}_2^n=\langle x_1,x_2,\cdots,x_n\rangle$, $x:=x_1x_2\cdots x_n$ and $\overline{x_i}:=xx_i$ $(1\leq i\leq n)$. Take subsets $S,R,T\subseteq G$ as follows:

$$S=\{1,\,x_i\mid 1\leq i\leq n\},\ R=\{x,\,\overline{x_i}\mid 1\leq i\leq n\},\ T=\{x_1x_2x_{n-2}x_{n-1},\,x_1x_2x_{n-1}x_n,\,x_ix_{i+1}\mid 1\leq i\leq n-1\}.$$

It is easy to see that $R=xS$ and $|R^2|=|S^2|=|\{1,\,x_i,\,x_ix_j\mid 1\leq i,j\leq n\}|=1+n+\frac{n(n-1)}{2}$ hold. Since set $\{x_ix_{i+1}x_{i+2}\mid 1\leq i\leq n-2\}$ appears twice in ST where

$$ST=T\cup\{x_i,\,x_kx_jx_{j+1}\mid 1\leq i,j,k\leq n,\,j\neq n,\,k\neq j,j+1\}\cup\{x_1x_2x_ix_{n-2}x_{n-1},\,x_1x_2x_jx_{n-1}x_n\mid 3\leq i,j\leq n-2,i\neq n-2\},$$

we obtain

$$|ST|=n+1+n+(n-1)(n-3)+(n-5)+(n-4)-(n-2)=n^2-n-3.$$

By $x_1x_2\in ST$, $x_1x_2\notin SR$, $\overline{x_1}\in SR$ and $\overline{x_1}\notin ST$, we have the following :

$$|R^2|=|S^2|=|SR|,\ |RT|=|ST|>|R^2|,\ ST\not\subseteq SR,\ SR\not\subseteq ST. \tag{12}$$

We divide the proof into two cases: $m=2$ (Case 1) and $m\geq 3$ (Case 2).

Case 1: Show that G admits an O2SR.

Proof of Case 1. Let $\Gamma_2:=\mathrm{BiCay}(G,\emptyset,\emptyset,S,T)$ and $\mathcal{A}:=\mathrm{Aut}(\Gamma_2)$. Then Γ_2 is an oriented 2-Cayley digraph of G with valency $n+1$. If $n=5$, then we find that Γ_2 is an O2SR of G by using Magma [3]. Let $n\geq 6$. Note that

$$\Gamma_2^+(1_0)=S_1=\{1_1,\,(x_i)_1\mid 1\leq i\leq n\},\ x_iT=\{x_ix_jx_{j+1},\,x_ix_1x_2x_{n-2}x_{n-1},\,x_ix_1x_2x_{n-1}x_n\mid 1\leq j<n\}\ (1\leq i\leq n)$$

all hold. Since $R(G)\in\mathcal{A}$ and G is abelian, we have

$$\Gamma_2^+(1_1)=T_0\ \text{ and }\Gamma_2^+((x_i)_1)=(x_iT)_0\ (1\leq i\leq n).$$

It follows by $T\cap x_iT=\emptyset$ $(1\leq i\leq n)$ and $x_iT\cap x_{i+2}T=\{x_{i+1},\,x_ix_{i+1}x_{i+2}\}$ $(1\leq i\leq n-2)$ that

$$\Gamma_2^+((x_i)_1)\cap\Gamma_2^+(1_1)=\emptyset\ (1\leq i\leq n)\text{ and }\Gamma_2^+((x_i)_1)\cap\Gamma_2^+((x_{i+2})_1)=\{x_{i+1},\,x_ix_{i+1}x_{i+2}\}_0\ (1\leq i\leq n-2). \tag{13}$$

This shows that 1_1 is the only vertex in $\Gamma_2^+(1_0)$ that has no common out-neighbor with any other vertex in $\Gamma_2^+(1_0)$, and $(x_i)_1$ $(1\leq i\leq n)$ has at least two common out-neighbors with some vertex in $\Gamma_2^+(1_0)$. Hence $\mathcal{A}_{1_0}$ fixes 1_1 and thus $\mathcal{A}_{1_0}\subseteq\mathcal{A}_{1_1}$. Similarly, the out-neighbors of 1_1 can be written as

$$\Gamma_2^+(1_1)=T_0=\{(x_1x_2x_{n-2}x_{n-1})_0,\,(x_1x_2x_{n-1}x_n)_0,(x_jx_{j+1})_0\mid 1\leq j\leq n-1\}.$$

We also find $\Gamma_2^+(t_0)=(tS)_1$ $(t_0\in T_0)$ and the following:

$$\begin{aligned}x_jx_{j+1}S\cap x_{j+1}x_{j+2}S &=\{x_{j+1},\,x_jx_{j+1}x_{j+2}\}\ (1\leq j\leq n-2),\\ x_1x_2x_{n-2}x_{n-1}S\cap x_1x_2S &=\{x_1x_2x_{n-2},\,x_1x_2x_{n-1}\},\\ x_1x_2x_{n-1}x_nS\cap x_{n-1}x_nS &=\{x_1x_{n-1}x_n,\,x_2x_{n-1}x_n\}.\end{aligned}$$

This shows that each vertex in $\Gamma_2^+(1_1)$ has at least two common out-neighbors with some vertex in $\Gamma_2^+(1_1)$. As Γ_2 cannot be vertex transitive by (13), $\mathcal{A}$ fixes G_0 and G_1 setwise. As $R(G)\subseteq\mathcal{A}$, $\mathcal{A}$ has exactly two orbits G_0 and G_1 and hence $\mathcal{A}=R(G)\mathcal{A}_{1_0}=R(G)\mathcal{A}_{1_1}$. Since $R(G)$ is semiregular and $\mathcal{A}_{1_0}\subseteq\mathcal{A}_{1_1}$, $\mathcal{A}_{1_1}=\mathcal{A}_{1_1}\cap\mathcal{A}=\mathcal{A}_{1_1}\cap R(G)\mathcal{A}_{1_0}=\mathcal{A}_{1_0}$ holds and so $\mathcal{A}_{g_0}=\mathcal{A}_{g_1}$ $(g\in G)$. Let $\sigma\in\mathcal{A}$. Suppose $\{g_0,g_1\}^\sigma\cap\{g_0,g_1\}\neq\emptyset$. Then it follows by $\mathcal{A}_{g_0}=\mathcal{A}_{g_1}$ and $\mathcal{A}$ fixes G_i $(i=0,1)$ that

$$g_0^\sigma=g_0\text{ and }g_1^\sigma=g_1$$

both hold. This means that $\{\{g_0,g_1\}\mid g\in G\}$ is a block system of $\mathcal{A}$ on $V(\Gamma_2)$, and hence there is a permutation $\overline{\sigma}$ of G satisfying $(g^{\overline{\sigma}})_0=(g_0)^\sigma$ and $(g^{\overline{\sigma}})_1=(g_1)^\sigma$ for each $g\in G$. By Proposition 2.3, $\overline{\sigma}\in\mathrm{Aut}(\mathrm{Cay}(G,T))$. Since $\mathrm{Cay}(G,T)$ is a GRR of G by [9, pp.654], we obtain $\overline{\sigma}\in R(G)$. This implies $\sigma\in R(G)$ and hence $\mathcal{A}=R(G)$. Therefore, Γ_2 is an O2SR of G. This completes the proof of Case 1. □

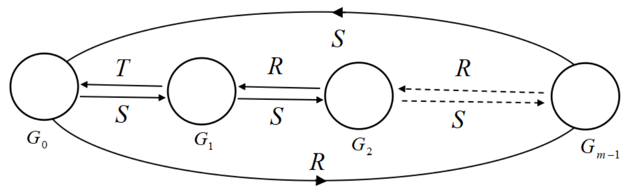

FIGURE 3.3. The digraph Γ_m for $m \geq 3$

Case 2: Show that G admits an OmSR for any $m \geq 3$.
Proof of Case 2. Let $m \geq 3$ and take a subset $T_{i,j} \subseteq G$ ($i, j \in \mathbb{Z}_m$) as follows:

$$T_{1,0} = T,\ T_{0,1} = T_{i,i+1} = S,\ T_{i,i-1} = R,\ T_{i,j} = \emptyset \text{ for } i \neq 1,\ j \neq i \pm 1.$$

Let $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ and $\mathcal{A} := \mathrm{Aut}(\Gamma_m)$. Then Γ_m is an oriented m-Cayley digraph of G with valency $2n+2$ (See FIGURE 3.3). It is easy to see that $\Gamma_m^+(1_1) = S_2 \cup T_0$ and $\Gamma_m^+(1_i) = S_{i+1} \cup R_{i-1}$ for each $i \neq 1$. Recall that $\Gamma_m^+(\Gamma_m^+(1_i))$ is the set of out-neighbors of $\Gamma_m^+(1_i)$ (See Notation 2.4). Since G is abelian, we obtain

$$\begin{aligned}
\Gamma_m^+(\Gamma_m^+(1_0)) &= \Gamma_m^+(S_1 \cup R_{m-1}) = (S^2)_2 \cup (ST)_0 \cup (SR)_0 \cup (R^2)_{m-2}\ ;\\
\Gamma_m^+(\Gamma_m^+(1_1)) &= \Gamma_m^+(S_2 \cup T_0) = (S^2)_3 \cup (SR)_1 \cup (ST)_1 \cup (TR)_{m-1}\ ;\\
\Gamma_m^+(\Gamma_m^+(1_2)) &= \Gamma_m^+(S_3 \cup R_1) = (S^2)_4 \cup (SR)_2 \cup (RT)_0\ ;\\
\Gamma_m^+(\Gamma_m^+(1_i)) &= \Gamma_m^+(S_{i+1} \cup R_{i-1}) = (S^2)_{i+2} \cup (SR)_i \cup (R^2)_{i-2}\ (3 \leq i \leq m-1).
\end{aligned}$$

By (12), it is easy to find the following:

$$\begin{aligned}
&|\Gamma_m^+(\Gamma_m^+(1_0))| > |\Gamma_m^+(\Gamma_m^+(1_i))|, \quad |\Gamma_m^+(\Gamma_m^+(1_1))| > |\Gamma_m^+(\Gamma_m^+(1_i))|\ (3 \leq i \leq m-1)\ ;\\
&|\Gamma_m^+(\Gamma_m^+(1_1))| > |\Gamma_m^+(\Gamma_m^+(1_2))|, \quad |\Gamma_m^+(\Gamma_m^+(1_0))| > |\Gamma_m^+(\Gamma_m^+(1_2))|.
\end{aligned}$$

So $\mathcal{A}$ fixes $G_0 \cup G_1$ setwise. Since $[G_0 \cup G_1] \cong \Gamma_2$ is an O2SR of G, $\mathcal{A}$ fixes $G_0 \cup G_1$ pointwise. Since all out-neighbors of G_1 are in $G_0 \cup G_2$, $\mathcal{A}$ fixes G_2 setwise. Take $g, h \in G$ with $g \neq h$. Then

$$\Gamma_m^-(g_2) \cap G_1 = (gS)_1 = \{g, x_i g \mid 1 \leq i \leq n\}_1 \text{ and } \Gamma_m^-(h_2) \cap G_1 = (hS)_1 = \{h, x_i h \mid 1 \leq i \leq n\}_1.$$

As $gS \neq hS$, $\mathcal{A}$ fixes G_2 pointwise. In the same way, we can show that $\mathcal{A}$ fixes G_i pointwise ($3 \leq i \leq m-1$). Therefore, $\mathcal{A}_{1_1} = 1$ and thus $\mathcal{A} = R(G)\mathcal{A}_{1_1} = R(G)$ (i.e., Γ_m is an OmSR of G). This completes the proof for Case 2 of Lemma 3.6. □

Now we consider finite non-abelian generalized dihedral groups.

Lemma 3.7. *Let G be a finite non-abelian generalized dihedral group. Then G admits an OmSR for any integer $m \geq 2$.*

Proof. In view of Remark 2.1, take an abelian group H of exponent greater than 2 and an element $b \in G$ satisfying $G = \langle H, b\rangle$, $o(b) = 2$ and $bhb = h^{-1}$ for each $h \in H$, where $o(b)$ is the order of b in G. Now, we divide the proof into two cases.

Case 1: H admits an ORR.
Proof of Case 1. Let R be a subset of H such that $\Sigma := \mathrm{Cay}(H, R)$ is an ORR of H. If $R = \emptyset$, then H is $\mathbb{Z}_1$ or $\mathbb{Z}_2$ and thus G is $\mathbb{Z}_2$ or $\mathbb{Z}_2^2$. This is impossible as G is non-abelian. Thus, $R \neq \emptyset$ and take $a \in R$. Since $b \notin H$, note that

(14) $$o(a) \geq 3,\quad a \in R,\quad a \notin R^{-1},\quad 1 \notin R,\quad b \notin R \cup R^{-1},\ H = \langle R\rangle,\ R \cap R^{-1} = \emptyset$$

all hold. Put $k := |A([\Sigma^+(1)])|$. Now, we divide Case 1 into two subcases: $m = 2$ (Subcase 1.1) and $m \geq 3$ (Subcase 1.2).

Subcase 1.1: Show that G admits an O2SR.
Let $\Gamma_2 := \mathrm{BiCay}(G, R, R^{-1}, \{1, a\}, \{a, b\})$ with $T_{0,1} = \{1, a\}$ and $T_{1,0} = \{a, b\}$. Let $\mathcal{A} := \mathrm{Aut}(\Gamma_2)$. Then Γ_2 is an oriented 2-Cayley digraph of G with valency $|R| + 2$. In the same way as Lemma 3.1, we first show that $\mathcal{A}$ is not vertex-transitive by counting $|A([\Gamma_2^+(1_0)])|$ and $|A([\Gamma_2^+(1_1)])|$. In the beginning, we find the following:

$$\begin{aligned}
&\Gamma_2^+(1_0) = \{r_0, 1_1, a_1 \mid r \in R\}, && \Gamma_2^-(1_0) = \{r_0^{-1}, b_1, a_1^{-1} \mid r \in R\},\\
&\Gamma_2^+(1_1) = \{r_1, a_0, b_0 \mid r \in R\}, && \Gamma_2^-(1_1) = \{r_1^{-1}, a_0^{-1}, 1_0 \mid r \in R\},\\
&\Gamma_2^+(a_1) = \{(ra)_1, a_0^2, (ba)_0 \mid r \in R\}, && \Gamma_2^-(a_1) = \{(r^{-1}a)_1, 1_0, a_0 \mid r \in R\},\\
&\Gamma_2^+(a_0) = \{(ra)_0, a_1, a_1^2 \mid r \in R\}, && \Gamma_2^-(a_0) = \{(r^{-1}a)_0, (ba)_1, 1_1 \mid r \in R\},\\
&\Gamma_2^+(b_0) = \{(rb)_0, b_1, (ab)_1 \mid r \in R\}, && \Gamma_2^-(b_0) = \{(r^{-1}b)_0, 1_1, (a^{-1}b)_1 \mid r \in R\}.
\end{aligned}$$

Using the above sets, (14) and $bh \in H$ (for $h \in H$), it is easy to find the following:

$$\begin{aligned}
&\Gamma_2^+(1_1) \cap \Gamma_2^+(1_0) = \{a_0, a_1\}, && \Gamma_2^-(1_1) \cap \Gamma_2^+(1_0) = \emptyset, && \Gamma_2^-(a_1) \cap \Gamma_2^+(1_0) = \{1_1, a_0\},\\
&\Gamma_2^+(a_1) \cap \Gamma_2^+(1_0) = \{a_0^2\} \text{ if } a^2 \in R, && \Gamma_2^+(a_1) \cap \Gamma_2^+(1_0) = \emptyset \text{ if } a^2 \notin R, &&\\
&\Gamma_2^+(b_0) \cap \Gamma_2^+(1_1) = \emptyset, && \Gamma_2^-(b_0) \cap \Gamma_2^+(1_1) = \emptyset, && \Gamma_2^-(a_0) \cap \Gamma_2^+(1_1) = \emptyset,\\
&\Gamma_2^+(a_0) \cap \Gamma_2^+(1_1) = \{a_1, a_1^2\} \text{ if } a^2 \in R, && \Gamma_2^+(a_0) \cap \Gamma_2^+(1_1) = \{a_1\} \text{ if } a^2 \notin R. &&
\end{aligned}$$

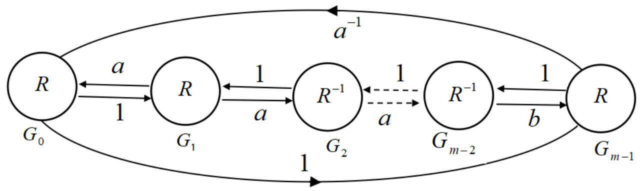

FIGURE 3.4. The digraph Γ_m for $m \geq 3$

As $[\Gamma_2^+(1_0) \cap G_0] \cong [\Sigma^+(1)]$, we obtain

$$|A([\Gamma_2^+(1_0)])| \;=\; k - |A(1_1, a_1)| + |\Gamma_2^+(1_1) \cap \Gamma_2^+(1_0)| + |\Gamma_2^-(1_1) \cap \Gamma_2^+(1_0)| + |\Gamma_2^+(a_1) \cap \Gamma_2^+(1_0)| + |\Gamma_2^-(a_1) \cap \Gamma_2^+(1_0)|.$$

As $(1_1, a_1)$ is an arc in Γ_2, $|A(1_1, a_1)| = 1$ and thus

$$A|[\Gamma_2^+(1_0)]| = k + 4 \text{ if } a^2 \in R \text{ and } A|[\Gamma_2^+(1_0)]| = k + 3 \text{ if } a^2 \notin R \tag{15}$$

hold. As $[\Gamma_2^+(1_1) \cap G_1] \cong [\Sigma^+(1)]$, we obtain

$$|A([\Gamma_2^+(1_1)])| \;=\; k - |A(a_0, b_0)| + |\Gamma_2^+(a_0) \cap \Gamma_2^+(1_1)| + |\Gamma_2^-(a_0) \cap \Gamma_2^+(1_1)| + |\Gamma_2^+(b_0) \cap \Gamma_2^+(1_1)| + |\Gamma_2^-(b_0) \cap \Gamma_2^+(1_1)|.$$

As $|A(a_0, b_0)| = 0$, we can find

$$A|[\Gamma_2^+(1_1)]| = k + 2 \text{ if } a^2 \in R \text{ and } A|[\Gamma_2^+(1_1)]| = k + 1 \quad \text{ if } a^2 \notin R. \tag{16}$$

By (15)–(16), $[\Gamma_2^+(1_0)] \not\cong [\Gamma_2^+(1_1)]$ holds and thus Γ_2 is not vertex-transitive. Recall that $R(G)$ is a group of automorphisms of Γ_2 with two orbits G_0 and G_1. Then $\mathcal{A}$ also has two orbits G_0 and G_1 satisfying $\mathcal{A} = R(G)\mathcal{A}_{1_1}$. Note that $G_i = H_i \cup (Hb)_i$ $(i = 0, 1)$ and $[H_i] \cong [(Hb)_i] \cong \Sigma$. Since $\mathcal{A}_{1_1}$ fixes G_1 setwise and $[H_1]$ is an ORR of H, $\mathcal{A}_{1_1}$ fixes H_1 pointwise. Since $[\Gamma_2^-(1_1)] = ((a^{-1})_0, 1_0)$ is an arc, $\mathcal{A}_{1_1}$ fixes 1_0. Since $[H_0]$ is an ORR, $\mathcal{A}_{1_1}$ fixes H_0 pointwise. On the other hand, it follows by $[\Gamma_2^+(1_1)] = (a_0, (ab)_0)$ that $\mathcal{A}_{1_1}$ fixes $(ab)_0$ and thus it fixes $(Hb)_0$ pointwise. By $[\Gamma_2^-(b_0)] = ((ab)_1, a_1)$, $\mathcal{A}_{1_1}$ fixes $(Hb)_1$ pointwise. This shows $\mathcal{A}_{1_1} = 1$ and thus $\mathcal{A} = R(G)\mathcal{A}_{1_1} = R(G)$. Therefore, Γ_2 is an O2SR of G. This completes the proof for Subcase 1.1.

Subcase 1.2: For $m \geq 3$, show that G admits an OmSR.
For $m \geq 3$, define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\begin{array}{ll}
T_{0,0} = T_{1,1} = T_{m-1,m-1} = R, \quad T_{1,0} = \{a\}, T_{0,1} = \{1\}, T_{m-2,m-1} = \{b\}, T_{m-1,0} = \{a^{-1}\}\ ; & \\
T_{i,i} = R^{-1} & \text{for } i \neq 0, 1, m-1; \\
T_{i,i-1} = \{1\} & \text{for } i \neq 1; \\
T_{i,i+1} = \{a\} & \text{for } i \neq 0, m-2, m-1; \\
T_{i,j} = \emptyset & \text{for } i \neq j, j \pm 1.
\end{array}$$

Let $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$. Then Γ_m is an oriented m-Cayley digraph of G with valency $|R| + 2$ (See FIGURE 3.4). To complete the proof for Subcase 1.2, we need the following claim that counts the number of arcs in the induced digraph $[\Gamma_m^+(1_i)]$ for $i \in \mathbb{Z}_m$.

Claim:
(i) $|A([\Gamma_3^+(1_0)])| = k + 2$, $|A([\Gamma_m^+(1_0)])| = k + 1$ $(m \geq 4)$.
(ii) $|A([\Gamma_3^+(1_1)])| = k + 1$, $|A([\Gamma_m^+(1_1)])| = k + 2$ $(m \geq 4)$.
(iii) $|A([\Gamma_m^+(1_{m-1})])| = k$ $(m \geq 3)$.
(iv) $|A([\Gamma_m^+(1_{m-2})])| = k$ $(m \geq 4)$.
(v) $|A([\Gamma_m^+(1_i)])| = k$ $(2 \leq i \leq m - 3)$.

Proof of Claim. With the aid of FIGURE 3.4, we find the following:

$\Gamma_m^+(1_0) = \{r_0, 1_1, 1_{m-1} \mid r \in R\}$,
$\Gamma_3^+(1_1) = \{r_1, b_2, a_0 \mid r \in R\}$,
$\Gamma_m^+(1_1) = \{r_1, a_2, a_0 \mid r \in R\}$ $(m \geq 4)$,
$\Gamma_m^+(1_{m-2}) = \{r_{m-2}^{-1}, b_{m-1}, 1_{m-3} \mid r \in R\}$ $(m \geq 4)$,
$\Gamma_m^+(1_{m-1}) = \{r_{m-1}, a_0^{-1}, 1_{m-2} \mid r \in R\}$,
$\Gamma_m^+(1_i) = \{r_i^{-1}, a_{i+1}, 1_{i-1} \mid r \in R\}$ $(2 \leq i \leq m - 3)$,
$\Gamma_m^+(a_0) = \{(ra)_0, a_1, a_2 \mid r \in R\}$,
$\Gamma_m^+(a_{m-2}) = \{(r^{-1}a)_{m-2}, (ba)_{m-1}, a_{m-3} \mid r \in R\}$ $(m \geq 4)$,
$\Gamma_m^+(a_{i+1}) = \{(r^{-1}a)_{i+1}, a_{i+2}^2, a_i \mid r \in R\}$ $(1 \leq i \leq m - 4)$,
$\Gamma_m^+(a_0^{-1}) = \{(ra^{-1})_0, a_1^{-1}, a_{m-1}^{-1} \mid r \in R\}$,
$\Gamma_m^+(b_{m-1}) = \{(rb)_{m-1}, (a^{-1}b)_0, b_{m-2} \mid r \in R\}$,

$\Gamma_m^-(1_0) = \{r_0^{-1}, a_1^{-1}, a_{m-1} \mid r \in R\}$,
$\Gamma_m^-(1_1) = \{r_1^{-1}, 1_2, 1_0 \mid r \in R\}$,
$\Gamma_m^-(1_{m-1}) = \{r_{m-1}^{-1}, 1_0, b_{m-2} \mid r \in R\}$,
$\Gamma_m^-(1_i) = \{r_i, 1_{i+1}, a_{i-1}^{-1} \mid r \in R\}$ $(2 \leq i \leq m - 2)$,
$\Gamma_m^-(a_0) = \{(r^{-1}a)_0, 1_1, a_2^2 \mid r \in R\}$,
$\Gamma_m^-(a_{i+1}) = \{(ra)_{i+1}, a_{i+2}, 1_i \mid r \in R\}$ $(1 \leq i \leq m - 3)$,
$\Gamma_m^-(a_0^{-1}) = \{(r^{-1}a^{-1})_0, a_1^{-2}, 1_{m-1} \mid r \in R\}$,
$\Gamma_m^-(b_{m-1}) = \{(r^{-1}b)_{m-1}, b_0, 1_{m-2} \mid r \in R\}$.

The above sets enable us to prove the claim as follows.

Proof of (i). It is easy to find by the above sets and (14) that

$$\Gamma_3^+(1_1)\cap\Gamma_3^+(1_0)=\{a_0\},\quad \Gamma_3^+(1_2)\cap\Gamma_3^+(1_0)=\{1_1\},\quad \Gamma_3^-(1_1)\cap\Gamma_3^+(1_0)=\{1_2\},\quad \Gamma_3^-(1_2)\cap\Gamma_3^+(1_0)=\emptyset\ ;$$
$$\Gamma_m^+(1_1)\cap\Gamma_m^+(1_0)=\{a_0\},\quad \Gamma_m^+(1_{m-1})\cap\Gamma_m^+(1_0)=\emptyset,\quad \Gamma_m^-(1_1)\cap\Gamma_m^+(1_0)=\emptyset,\quad \Gamma_m^-(1_{m-1})\cap\Gamma_m^+(1_0)=\emptyset\ (m\geq 4).$$

As $[\Gamma_m^+(1_0)\cap G_0]\cong[\Sigma^+(1)]$, we obtain

$$|A([\Gamma_m^+(1_0)])| = k-|A(1_1,1_{m-1})|+ |\Gamma_m^+(1_1)\cap\Gamma_m^+(1_0)|+|\Gamma_m^-(1_1)\cap\Gamma_m^+(1_0)|+|\Gamma_m^+(1_{m-1})\cap\Gamma_m^+(1_0)|+|\Gamma_m^-(1_{m-1})\cap\Gamma_m^+(1_0)|.$$

The result (i) follows by $|A(1_1,1_2)|=1$ and $|A(1_1,1_{m-1})|=0$ $(m\geq 4)$ (See FIGURE 3.4).

Proof of (ii). By (14), we have

$$\Gamma_3^+(a_0)\cap\Gamma_3^+(1_1)=\{a_1\},\quad \Gamma_3^+(b_2)\cap\Gamma_3^+(1_1)=\emptyset,\quad \Gamma_3^-(a_0)\cap\Gamma_3^+(1_1)=\emptyset,\quad \Gamma_3^-(b_2)\cap\Gamma_3^+(1_1)=\emptyset\ ;$$
$$\Gamma_m^+(a_0)\cap\Gamma_m^+(1_1)=\{a_1\},\quad \Gamma_m^+(a_2)\cap\Gamma_m^+(1_1)=\{a_1\},\quad \Gamma_m^-(a_0)\cap\Gamma_m^+(1_1)=\emptyset,\quad \Gamma_m^-(a_2)\cap\Gamma_m^+(1_1)=\emptyset\ (m\geq 4).$$

As $[\Gamma_3^+(1_1)\cap G_1]\cong[\Sigma^+(1)]$, we have

$$|A([\Gamma_3^+(1_1)])| = k-|A(b_2,a_0)|+|\Gamma_3^+(b_2)\cap\Gamma_3^+(1_1)|+|\Gamma_3^-(b_2)\cap\Gamma_3^+(1_1)|+|\Gamma_3^+(a_0)\cap\Gamma_3^+(1_1)|+|\Gamma_3^-(a_0)\cap\Gamma_3^+(1_1)|.$$

This shows $A|[\Gamma_3^+(1_1)]|=k+1$ by $|A(a_0,b_2)|=0$ (see FIGURE 3.4).
On the other hand, consider $m\geq 4$. As $[\Gamma_m^+(1_1)\cap G_1]\cong[\Sigma^+(1)]$, we also obtain

$$|A([\Gamma_m^+(1_1)])| = k-|A(a_2,a_0)|+|\Gamma_m^+(a_2)\cap\Gamma_m^+(1_1)|+|\Gamma_m^-(a_2)\cap\Gamma_m^+(1_1)|+|\Gamma_m^+(a_0)\cap\Gamma_m^+(1_1)|+|\Gamma_m^-(a_0)\cap\Gamma_m^+(1_1)|.$$

Now Claim (ii) follows by $|A(a_0,a_2)|=0$ $(m\geq 4)$ (see FIGURE 3.4).

Proof of (iii). The result (iii) follows by $|A(a_0^{-1},1_{m-2})|=0$ $(m\geq 3)$ and

$$\Gamma_m^+(a_0^{-1})\cap\Gamma_m^+(1_{m-1})=\emptyset,\ \Gamma_m^+(1_{m-2})\cap\Gamma_m^+(1_{m-1})=\emptyset,\ \Gamma_m^-(a_0)\cap\Gamma_m^+(1_{m-1})=\emptyset,\ \Gamma_m^-(1_{m-2})\cap\Gamma_m^+(1_{m-1})=\emptyset.$$

Proof of (iv). The result (iv) follows by $|A(b_{m-1},1_{m-3})|=0$ $(m\geq 4)$ and

$$\Gamma_m^+(b_{m-1})\cap\Gamma_m^+(1_{m-2})=\emptyset,\ \Gamma_m^+(1_{m-3})\cap\Gamma_m^+(1_{m-1})=\emptyset,\ \Gamma_m^-(b_{m-1})\cap\Gamma_m^+(1_{m-2})=\emptyset,\ \Gamma_m^-(1_{m-3})\cap\Gamma_m^+(1_{m-1})=\emptyset.$$

Proof of (v). Let $m\geq 5$. For each $2\leq i\leq m-3$, we find

$$\Gamma_m^+(a_{i+1})\cap\Gamma_m^+(1_i)=\emptyset,\ \Gamma_m^+(1_{i-1})\cap\Gamma_m^+(1_i)=\emptyset,\ \Gamma_m^-(a_{i+1})\cap\Gamma_m^+(1_i)=\emptyset,\ \Gamma_m^-(1_{i-1})\cap\Gamma_m^+(1_i)=\emptyset.$$

The result (v) follows by $|A(a_{i+1},1_{i-1})|=0$ (FIGURE 3.4). This completes the proof of the claim. □

Now we are ready to complete the proof of Subcase 1.2 in Lemma 3.7 by using the above claim. Let $\mathcal{A}:=\mathrm{Aut}(\Gamma_m)$. By $|A([\Gamma_m^+(1_0)])|\neq|A([\Gamma_m^+(1_i)])|$ $(i\neq 0)$ and $|A([\Gamma_m^+(1_1)])|\neq|A([\Gamma_m^+(1_j)])|$ $(j\neq 1)$ (see the above claim), we obtain

$$[\Gamma_m^+(1_0)]\ncong[\Gamma_m^+(1_i)]\ (i\neq 0)\text{ and }[\Gamma_m^+(1_1)]\ncong[\Gamma_m^+(1_j)]\ (j\neq 1).$$

Thus $\mathcal{A}$ fixes G_0 and G_1 setwise. By $|T_{1,2}|=1$ and $T_{1,i}=\emptyset$ $(3\leq i\leq m-1)$, $\mathcal{A}$ fixes G_2 setwise. Similarly, we obtain that $\mathcal{A}$ fixes G_i setwise for each $i\in\mathbb{Z}_m$. Recall that $R(G)$ is a group of automorphisms of Γ_m with m orbits G_i $(i\in\mathbb{Z}_m)$. Then $\mathcal{A}$ also has m orbits satisfying $\mathcal{A}=R(G)\mathcal{A}_{1_0}$. To prove Γ_m is an OmSR, we only need to show $\mathcal{A}_{1_0}=1$. Let $\sigma\in\mathcal{A}_{1_0}$. Then σ fixes G_i setwise, where $G_i=H_i\cup(Hb)_i$ $(i\in\mathbb{Z}_m)$. Since $[H_0]\cong\Sigma$ is an ORR of H, σ fixes H_0 pointwise. By $T_{0,1}=\{1\}$, $\mathcal{A}_{1_0}$ fixes H_1 pointwise. By $T_{j,j+1}=\{a\}$ $(1\leq j\leq m-3)$, $\mathcal{A}_{1_0}$ fixes H_i $(2\leq i\leq m-2)$ pointwise. As $T_{m-2,m-1}=\{b\}$, $\mathcal{A}_{1_0}$ fixes H_{m-1} pointwise. On the other hand, it follows by $T_{m-2,m-1}=\{b\}$ that $\mathcal{A}_{1_0}$ fixes $(Hb)_{m-1}$ pointwise. By $T_{m-1,0}=\{a^{-1}\}$, $\mathcal{A}_{1_0}$ fixes $(Hb)_0$ pointwise. We can also obtain by $T_{j,j+1}=\{a\}$ $(1\leq j\leq m-3)$ that $\mathcal{A}_{1_0}$ fixes $(Hb)_i$ $(1\leq i\leq m-2)$ pointwise. As $G=H\cup Hb$, we conclude that $\mathcal{A}_{1_0}$ fixes G_i $(i\in\mathbb{Z}_m)$ pointwise (i.e., $\mathcal{A}_{1_0}=1$). Therefore Subcase 1.2 follows. This completes the proof of Case 1. □

Case 2: H does not admit ORRs.

Proof of Case 2. Recall that H is an abelian group and G is a non-abelian generalized dihedral group over H. By Remark 2.1 and Proposition 2.2, $H=\mathbb{Z}_4\times\mathbb{Z}_2$, $\mathbb{Z}_3^2$, $\mathbb{Z}_4\times\mathbb{Z}_2^2$, $\mathbb{Z}_3\times\mathbb{Z}_2^3$, $\mathbb{Z}_4\times\mathbb{Z}_2^3$ or $\mathbb{Z}_4\times\mathbb{Z}_2^4$. For each group H, take two subsets $R,L\subseteq H$ as follows:

(1) $R=\{x,xb\}$ and $L=\{x,x^{-1}y\}$ for $H=\mathbb{Z}_4\times\mathbb{Z}_2=\langle x,y\mid x^4=y^2=1,xy=yx\rangle$;
(2) $R=\{x,xb\}$ and $L=\{x,x^{-1}y\}$ for $H=\mathbb{Z}_3^2=\langle x,y\mid x^3=y^3=1,xy=yx\rangle$;
(3) $R=\{x,xy\}$ and $L=\{x,x^{-1}yz\}$ for $H=\mathbb{Z}_4\times\mathbb{Z}_2^2=\langle x,y,z\mid x^4=y^2=z^2=1,xy=yx,xz=zx,yz=zy\rangle$;
(4) $R=\{x,xy,xz,xw\}$ and $L=\{x,xz,xzw,x^{-1}yzw\}$
for $H=\mathbb{Z}_3\times\mathbb{Z}_2^3=\langle x,y,z,w\mid x^3=y^2=z^2=w^2=1,xy=yx,xz=zx,yz=zy,xw=wx,yw=wy,zw=wz\rangle$;

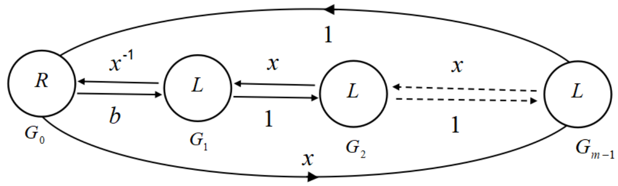

FIGURE 3.5. The digraph Γ_m for $m \geq 2$

(5) $R = \{x, xy, xz, xw\}$ and $L = \{x, xz, xzw, x^{-1}yzw\}$
for $H = \mathbb{Z}_4 \times \mathbb{Z}_2^3 = \langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1, xy = yx, xz = zx, yz = zy, xw = wx, yw = wy, zw = wz \rangle$;

(6) $R = \{x, xy, xz, xw, xu\}$ and $L = \{x, xz, x^{-1}zw, xyzw, x^{-1}zwu\}$
for $H = \mathbb{Z}_4 \times \mathbb{Z}_2^4 = \langle x, y, z, w, u \mid x^4 = y^2 = z^2 = w^2 = u^2 = 1, xy = yx, xz = zx, yz = zy, xw = wx, yw = wy, zw = wz, xu = ux, yu = uy, zu = uz, wu = uw \rangle$.

Note that for each case, R and L satisfy

$$|R| = |L| \text{ and } R \cap R^{-1} = L \cap L^{-1} = \emptyset.$$

For each case in (1)-(6) and for each $m \geq 2$, we define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\begin{array}{lll} T_{0,0} = R, & T_{1,0} = \{x^{-1}\}, & T_{0,1} = \{b\} \ ; \\ T_{i,i} = L, & T_{i,i+1} = \{1\} & \text{for } i \neq 0 \ ; \\ & T_{i,i-1} = \{x\} & \text{for } i \neq 1 \ ; \\ & T_{i,j} = \emptyset & \text{for } i \neq j, j \pm 1. \end{array}$$

Let $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ and $\mathcal{A} := \mathrm{Aut}(\Gamma_m)$. Then Γ_m is an oriented m-Cayley digraph of G, where the valency of Γ_m is $|R| + 1$ for $m = 2$ and $|R| + 2$ for $m \geq 3$ (See FIGURE 3.5). With the aid of MAGMA [3], we can find that Γ_m is an OmSR of G for $m = 2, 3$. In the rest of the proof, we will show that Γ_m is an OmSR of G for all $m \geq 4$. Let $\Sigma := \mathrm{Cay}(G, R)$ and $\Phi := \mathrm{Cay}(G, L)$. It is easy to see that $|A([\Sigma^+(1)])| = |A([\Phi^+(1)])| = 0$ holds for each case. By the definition of Γ_m and $R(g) \in \mathcal{A}$ for $g \in G$, we find the following:

$$\begin{array}{ll}
\Gamma_m^+(1_0) = \{r_0, b_1, x_{m-1} \mid r \in R\}, & \Gamma_m^-(1_0) = \{r_0^{-1}, x_1, 1_{m-1} \mid r \in R\}, \\
\Gamma_m^+(1_1) = \{l_1, 1_2, x_0^{-1} \mid l \in L\}, & \Gamma_m^-(1_1) = \{l_1^{-1}, x_2^{-1}, b_0 \mid l \in L\}, \\
\Gamma_m^+(1_i) = \{l_i, 1_{i+1}, x_{i-1} \mid l \in L\} \ (2 \leq i \leq m-1), & \Gamma_m^-(1_i) = \{l_i^{-1}, x_{i+1}^{-1}, 1_{i-1} \mid l \in L\} \ (2 \leq i \leq m-1), \\
\Gamma_m^+(b_1) = \{(lb)_1, b_2, (x^{-1}b)_0 \mid l \in L\}, & \Gamma_m^-(b_1) = \{(l^{-1}b)_1, (x^{-1}b)_2, 1_0 \mid l \in L\}, \\
\Gamma_m^+(x_1) = \{(lx)_1, x_2, 1_0 \mid l \in L\}, & \Gamma_m^-(x_1) = \{(l^{-1}x)_1, 1_2, (bx)_0 \mid l \in L\}, \\
\Gamma_m^+(x_{i-1}) = \{(lx)_{i-1}, x_i, x_{i-2}^2 \mid l \in L\} \ (3 \leq i \leq m-1), & \Gamma_m^-(x_{i-1}) = \{(l^{-1}x)_{i-1}, 1_i, x_{i-2} \mid l \in L\} \ (3 \leq i \leq m-1), \\
\Gamma_m^+(x_{m-1}) = \{(lx)_{m-1}, x_0, x_{m-2}^2 \mid l \in L\}, & \Gamma_m^-(x_{m-1}) = \{(l^{-1}x)_{m-1}, 1_0, x_{m-2} \mid l \in L\}, \\
\Gamma_m^+(x_0^{-1}) = \{(rx^{-1})_0, (bx^{-1})_1, 1_{m-1} \mid r \in R\}, & \Gamma_m^-(x_0^{-1}) = \{(r^{-1}x^{-1})_0, 1_1, x_{m-1}^{-1} \mid r \in R\}.
\end{array}$$

Using the above sets and $x \in R \cap L$, $b \notin R \cup L$, $1 \notin R$ and $1 \notin L$, it is easy to find the following:

$$\begin{array}{ll}
\Gamma_m^+(b_1) \cap \Gamma_m^+(1_0) = \emptyset, & \Gamma_m^-(b_1) \cap \Gamma_m^+(1_0) = \emptyset, \\
\Gamma_m^+(x_{m-1}) \cap \Gamma_m^+(1_0) = \{x_0\}, & \Gamma_m^-(x_{m-1}) \cap \Gamma_m^+(1_0) = \emptyset, \\
\Gamma_m^+(1_2) \cap \Gamma_m^+(1_1) = \{x_1\}, & \Gamma_m^-(1_2) \cap \Gamma_m^+(1_1) = \emptyset, \\
\Gamma_m^+(x_0^{-1}) \cap \Gamma_m^+(1_1) = \emptyset, & \Gamma_m^-(x_0^{-1}) \cap \Gamma_m^+(1_1) = \emptyset, \\
\Gamma_m^+(1_3) \cap \Gamma_m^+(1_2) = \{x_2\}, & \Gamma_m^-(1_3) \cap \Gamma_m^+(1_2) = \emptyset, \\
\Gamma_m^+(x_1) \cap \Gamma_m^+(1_2) = \{x_2\}, & \Gamma_m^-(x_1) \cap \Gamma_m^+(1_2) = \emptyset, \\
\Gamma_m^+(1_{i+1}) \cap \Gamma_m^+(1_i) = \{x_i\} \ (3 \leq i \leq m-1), & \Gamma_m^-(1_{i+1}) \cap \Gamma_m^+(1_i) = \emptyset \ (3 \leq i \leq m-1), \\
\Gamma_m^+(x_{i-1}) \cap \Gamma_m^+(1_i) = \{x_i\} \ (3 \leq i \leq m-1), & \Gamma_m^-(x_{i-1}) \cap \Gamma_m^+(1_i) = \emptyset \ (3 \leq i \leq m-1).
\end{array}$$

In the same way as Lemma 3.1, we first show

$$|A([\Gamma_m^+(1_i)])| = \begin{cases} 1 & \text{for } i = 0, 1 \\ 2 & \text{for } i \neq 0, 1 \end{cases} . \tag{17}$$

Let $i = 0$. As $|A(R_0)| = |[\Gamma_m^+(1_0) \cap G_0]| = |A([\Sigma^+(1)])| = 0$ and $|A(b_1, x_{m-1})| = 0$, we find (17) for $i = 0$ by

$$|A([\Gamma_m^+(1_0)])| = 0 - |A(b_1, x_{m-1})| + |\Gamma_m^+(b_1) \cap \Gamma_m^+(1_0)| + |\Gamma_m^-(b_1) \cap \Gamma_m^+(1_0)| + |\Gamma_m^+(x_{m-1}) \cap \Gamma_m^+(1_0)| + |\Gamma_m^-(x_{m-1}) \cap \Gamma_m^+(1_0)|.$$

Let $i = 1$. As $|A(L_1)| = |[\Gamma_m^+(1_1) \cap G_1]| = |A([\Phi^+(1)])| = 0$ and $|A(1_2, x_0^{-1})| = 0$, we find (17) for $i = 1$ by

$$|A([\Gamma_m^+(1_1)])| = 0 - |A(1_2, x_0^{-1})| + |\Gamma_m^+(1_2) \cap \Gamma_m^+(1_1)| + |\Gamma_m^-(1_2) \cap \Gamma_m^+(1_1)| + |\Gamma_m^+(x_0^{-1}) \cap \Gamma_m^+(1_1)| + |\Gamma_m^-(x_0^{-1}) \cap \Gamma_m^+(1_1)|.$$

Finally, we consider $i \neq 0, 1$. Since there are no arcs between 1_{i+1} and x_{i-1}, we have $|A(1_{i+1}, x_{i-1})| = 0$ $(i \neq 0, 1)$. Now (17) for $i \neq 0, 1$ follows immediately.

Since $[\Gamma_m^+(1_0)] \ncong [\Gamma_m^+(1_i)]$ and $[\Gamma_m^+(1_1)] \ncong [\Gamma_m^+(1_i)]$ $(i \neq 0, 1)$ all hold by (17), $\mathcal{A}$ fixes $G_0 \cup G_1$ setwise. Since $[G_0 \cup G_1] \cong \Gamma_2$

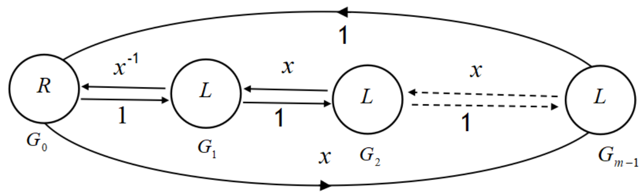

FIGURE 3.6. The digraph Γ_m for $m \geq 2$

is an O2SR, $\mathcal{A}$ fixes G_i $(i = 0, 1)$ setwise. By $T_{1,2} = \{1\}$ and $T_{1,i} = \emptyset$ $(i \neq 0, 1, 2)$, $\mathcal{A}$ fixes G_2 setwise. Similarly, $\mathcal{A}$ fixes G_i setwise for each $i \in \mathbb{Z}_m$. Since $[G_0 \cup G_1] \cong \Gamma_2$ is an O2SR, $\mathcal{A}_{1_0}$ fixes G_i $(i \in \mathbb{Z}_m)$ pointwise and so $\mathcal{A}_{1_0} = 1$. Hence $\mathcal{A} = R(G)\mathcal{A}_{1_0} = R(G)$ and so Γ_m is an OmSR of G. This completes the proof for Case 2 of Lemma 3.7. □

Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let G be a finite group and $m \geq 1$ be an integer. Without loss of generality, we may assume that G does not satisfy Theorem 1.1(1), that is, G does not admit OmSRs. If $m = 1$ then Theorem 1.1(2) follows by Proposition 2.2. Now assume $m \geq 2$ and G does not admit OmSRs. To complete the proof, we consider two cases, G admits an ORR or not.

If G admits an ORR, then G is $\mathbb{Z}_1$ or $\mathbb{Z}_2$ by Lemma 3.1. Hence, we obtain that G is either $\mathbb{Z}_1$ with $3 \leq m \leq 6$ (i.e., Theorem 1.1(4)) or $\mathbb{Z}_1, \mathbb{Z}_2$ with $m = 2$ by Lemma 3.4.

If G does not admit ORRs, then it follows by Proposition 2.2 that G is either a generalized dihedral group of order greater than 2 or one of the 11 exceptional groups given in TABLE 2.1. We first consider that G is a generalized dihedral group of order greater than 2. Then G is an elementary abelian 2-group by Remark 2.1 and Lemma 3.7. Moreover, G has order at most $2^4 = 16$ by Lemma 3.6. Now, Theorem 1.1(3) follows by Lemma 3.4. To complete the proof, we now show that none of the 11 exceptional groups in TABLE 2.1 does not admit OmSRs for any $m \geq 2$. For each group G in TABLE 2.1, take two subsets $R, L \subseteq G$ as follows:

(1) $R = \{x, xy\}$ and $L = \{x, x^{-1}y\}$ for $G = \mathbb{Z}_4 \times \mathbb{Z}_2 = \langle x, y \mid x^4 = y^2 = 1, xy = yx\rangle$;
(2) $R = \{x, xy\}$ and $L = \{x, x^{-1}y\}$ for $G = Q_8 = \langle x, y \mid x^4 = y^4 = 1, x^2 = y^2, x^y = x^{-1}\rangle$;
(3) $R = \{x, xy\}$ and $L = \{x, x^{-1}y\}$ for $G = \mathbb{Z}_3^2 = \langle x, y \mid x^3 = y^3 = 1, xy = yx\rangle$;
(4) $R = \{x, xy, xz\}$ and $L = \{x, x^{-1}y, x^{-1}yz\}$
for $G = \mathbb{Z}_4 \times \mathbb{Z}_2^2 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, xy = yx, xz = zx,\ yz = zy\rangle$;
(5) $R = \{x, xy, xz, xw\}$ and $L = \{x, x^{-1}y, x^{-1}yz, x^{-1}yzw\}$
for $G = \mathbb{Z}_3 \times \mathbb{Z}_2^3 = \langle x, y, z, w \mid x^3 = y^2 = z^2 = w^2 = 1, xy = yx, xz = zx, yz = zy, xw = wx, yw = wy, zw = wz\rangle$;
(6) $R = \{x, xy, xz, xw\}$ and $L = \{x, x^{-1}y, x^{-1}yz, x^{-1}yzw\}$
for $G = \mathbb{Z}_4 \times \mathbb{Z}_2^3 = \langle x, y, z, w \mid x^4 = y^2 = z^2 = w^2 = 1, xy = yx, xz = zx, yz = zy, xw = wx, yw = wy, zw = wz\rangle$;
(7) $R = \{x, xy, xz, xw, xu\}$ and $L = \{x, x^{-1}y, x^{-1}yz, x^{-1}yzw, x^{-1}yzwu\}$
for $G = \mathbb{Z}_4 \times \mathbb{Z}_2^4 = \langle x, y, z, w, u \mid x^4 = y^2 = z^2 = w^2 = u^2 = 1, xy = yx, xz = zx, yz = zy, xw = wx, yw = wy, zw = wz, xu = ux, yu = uy, zu = uz, wu = uw\rangle$;
(8) $R = \{x, xy\}$ and $L = \{x, x^{-1}y\}$ for $G = H_1$;
(9) $R = \{x, xy, xz\}$ and $L = \{x, x^{-1}y, x^{-1}yz\}$ for $G = H_2$;
(10) $R = \{x, xy, xz\}$ and $L = \{x, x^{-1}y, x^{-1}yz\}$ for $G = H_3$;
(11) $R = \{x, xy, xz, xw\}$ and $L = \{x, x^{-1}y, x^{-1}yz, x^{-1}yzw\}$
for $G = D_4 \circ D_4 = \langle x, y \mid x^4 = y^2 = 1, x^y = x^{-1}\rangle \circ \langle z, w \mid z^4 = w^2, z^w = z^{-1}\rangle$.

Note that for each case, R and L satisfy

$$|R| = |L| \text{ and } R \cap R^{-1} = L \cap L^{-1} = \emptyset.$$

For a given G in (1)-(11) and for each $m \geq 2$, we define a subset $T_{i,j} \subseteq G$ $(i, j \in \mathbb{Z}_m)$ as follows:

$$\begin{array}{lll} T_{0,0} = R, & T_{1,0} = \{x^{-1}\}\ ; & \\ T_{i,i} = L, & T_{i,i-1} = \{x\} & \text{for } i \neq 1\ ; \\ T_{i,i+1} = \{1\} & & \text{for } i \in \mathbb{Z}_m\ ; \\ T_{i,j} = \emptyset & & \text{for } i \neq j, j \pm 1. \end{array}$$

Let $\Gamma_m := \mathrm{Cay}(G, T_{i,j} : i, j \in \mathbb{Z}_m)$ and $\mathcal{A} := \mathrm{Aut}(\Gamma_m)$. Then Γ_m is an oriented m-Cayley digraph of G, where the valency of Γ_m is $|R| + 1$ for $m = 2$ and $|R| + 2$ for $m \geq 3$ (See FIGURE 3.6).

With the aid of MAGMA [3], we can find that Γ_m is an OmSR of G for $m = 2, 3$. In the rest of the proof, we will show that Γ_m is an OmSR of G for all $m \geq 4$. Let $\Sigma := \mathrm{Cay}(G, R)$ and $\Phi := \mathrm{Cay}(G, L)$. Both Σ and Φ are oriented Cayley digraphs of G satisfying $|A([\Sigma^+(1)])| = |A([\Phi^+(1)])| = 0$. By the definition of Γ_m and $R(g) \in \mathcal{A}$ for $g \in G$, we find the following:

$$\begin{array}{ll}
\Gamma^+_m(1_0) = \{r_0, 1_1, x_{m-1} \mid r \in R\}, & \Gamma^-_m(1_0) = \{r_0^{-1}, x_1, 1_{m-1} \mid r \in R\}, \\
\Gamma^+_m(1_1) = \{l_1, 1_2, x_0^{-1} \mid l \in L\}, & \\
\Gamma^+_m(1_i) = \{l_i, 1_{i+1}, x_{i-1} \mid l \in L\} \ (2 \le i \le m-1), & \Gamma^-_m(1_i) = \{l_i^{-1}, x_{i+1}^{-1}, 1_{i-1} \mid l \in L\} \ (1 \le i \le m-1), \\
\Gamma^+_m(x_1) = \{(lx)_1, x_2, 1_0 \mid l \in L\}, & \Gamma^-_m(x_1) = \{(l^{-1}x)_1, 1_2, x_0 \mid l \in L\}, \\
\Gamma^+_m(x_{i-1}) = \{(lx)_{i-1}, x_i, x_{i-2}^2 \mid l \in L\} \ (3 \le i \le m-1), & \Gamma^-_m(x_{i-1}) = \{(l^{-1}x)_{i-1}, 1_i, x_{i-2} \mid l \in L\} \ (3 \le i \le m-1), \\
\Gamma^+_m(x_0^{-1}) = \{(rx^{-1})_0, x_1^{-1}, 1_{m-1} \mid r \in R\}, & \Gamma^-_m(x_0^{-1}) = \{(r^{-1}x^{-1})_0, 1_1, x_{m-1}^{-1} \mid r \in R\}.
\end{array}$$

Using $x \in R \cap L$, $x^{-1} \notin R \cup L$, $1 \notin R \cup L \cup R^{-1} \cup L^{-1}$ and the above sets, it is easy to find the following:

$$\begin{array}{ll}
\Gamma^+_m(1_1) \cap \Gamma^+_m(1_0) = \emptyset, & \Gamma^-_m(1_1) \cap \Gamma^+_m(1_0) = \emptyset, \\
\Gamma^+_m(x_{m-1}) \cap \Gamma^+_m(1_0) = \{x_0\}, & \Gamma^-_m(x_{m-1}) \cap \Gamma^+_m(1_0) = \emptyset, \\
\Gamma^+_m(x_0^{-1}) \cap \Gamma^+_m(1_1) = \emptyset, & \Gamma^-_m(x_0^{-1}) \cap \Gamma^+_m(1_1) = \emptyset, \\
\Gamma^+_m(1_{i+1}) \cap \Gamma^+_m(1_i) = \{x_i\} \ (i \ne 0), & \Gamma^-_m(1_{i+1}) \cap \Gamma^+_m(1_i) = \emptyset \ (i \ne 0), \\
\Gamma^+_m(x_{i-1}) \cap \Gamma^+_m(1_i) = \{x_i\} \ (i \ne 0, 1), & \Gamma^-_m(x_{i-1}) \cap \Gamma^+_m(1_i) = \emptyset \ (i \ne 0, 1).
\end{array}$$

Note that $|A(1_1, x_{m-1})| = |A(1_2, x_0^{-1})| = |A(1_{i+1}, x_{i-1})| = 0$ holds for $2 \le i \le m-1$. In the same way of claims in Lemma 3.1 and Lemma 3.7, we can obtain

$$|A([\Gamma^+_m(1_0)])| = 1, \ |A([\Gamma^+_m(1_1)])| = 1, \ |A([\Gamma^+_m(1_i)])| = 2 \ (i \ne 0, 1).$$

Since $\mathcal{A}$ fixes $G_0 \cup G_1$ setwise and $[G_0 \cup G_1] \cong \Gamma_2$ is an O2SR, $\mathcal{A}$ fixes G_i $(i = 0, 1)$ setwise. By $T_{1,2} = \{1\}$ and $T_{1,i} = \emptyset$ $(i \ne 0, 1, 2)$, $\mathcal{A}$ fixes G_2 setwise. Similarly, $\mathcal{A}$ fixes G_i setwise for each $i \in \mathbb{Z}_m$. Since Γ_2 is an O2SR, $\mathcal{A}_{1_0}$ fixes G_i $(i \in \mathbb{Z}_m)$ pointwise and so $\mathcal{A}_{1_0} = 1$. Thus, $\mathcal{A} = R(G)\mathcal{A}_{1_0} = R(G)$ and hence Γ_m is an OmSR of G. Therefore, none of the 11 exceptional groups in Table 2.1 does not admit OmSRs for any $m \ge 2$. This completes the proof of Themrem 1.1. □

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References

[1] L. Babai, Finite digraphs with given regular automorphism groups, Period. Math. Hungar. 11 (1980), 257–270.
[2] L. Babai, W. Imrich, Tournaments with given regular group, Aequationes Math. 19 (1979), 232–244.
[3] W. Bosma, C. Cannon, C. Playoust, The MAGMA algebra system I: The user language, J. Symbolic Comput. 24 (1997), 235–265.
[4] J. -L. Du, Y. -Q. Feng, P. Spiga, A classification of the graphical m-semiregular representations of finite groups, J. Combin. Theory Ser. A 171 (2020), 105174.
[5] J. -L. Du, Y. -Q. Feng, P. Spiga, On Haar digraphical representations of groups, Discrete Math. 343 (2020), 112032.
[6] J. -L. Du, Y. -Q. Feng, P. Spiga, On n-partite digraphical representations of finite groups, J. Combin. Theory Ser. A 189 (2022), 105606.
[7] C.D. Godsil, GRR's for non-solvable groups, in Algebraic Methods in Graph theory (Proc. Conf. Szeged 1978 L. Lovász and V. T. Sos, eds), Coll. Math. Soc. J. Bolyai 25, North-Holland, Amsterdam, 1981, pp.221–239.
[8] A. Hujdurović, K. Kutnar, D. Marušič, On normality of n-Cayley graphs, Appl. Math. Comput. 332 (2018), 469–476.
[9] W. Imrich, Graphs with transitive abelian automorphism group, Coll. Soc. Janos Bolyai 4 (1969), 651–656.
[10] W. Imrich, Graphical regular representations of groups odd order, in: Combinatorics, Coll. Math. Soc. János. Bolayi 18 (1976), 611–621.
[11] W. Imrich, M.E. Watkins, On graphical regular representations of cyclic extensions of groups, Pac. J. Math. 55 (1974), 461–477.
[12] I. Kovács, A. Malnič, D. Marušič, Š. Miklavič, One-matching bi-Cayley graphs over abelian groups, European J. Combin. 30 (2009), 602–616.
[13] K. Kutnar, D. Marušič, S. Miklavič, P. Šparl, Strongly regular tri-Cayley graphs, European J. Combin. 30 (2009), 822–832.
[14] P.-H. Leemann, M. Salle, Cayley graphs with few automorphisms, J. Algebraic Combin. 53 (2021), 1117–1146.
[15] J. Morris, P. Spiga, Every finite non-solvable group admits an oriented regular representation, J. Combin. Theory Ser. B 126 (2017), 198–234.
[16] J. Morris, P. Spiga, Classification of finite groups that admit an oriented regular representation, Bull. Lond. Math. Soc. 50 (2018), 811-831.
[17] J. Morris, P. Spiga, Asymptotic enumeration of Cayley digraphs, Israel J. Math. 242 (2021), 401-459.
[18] L. A. Nowitz, M. E. Watkins, Graphical regular representations of non-abelain groups, I, Canad. J. Math. 24 (1972), 994–1008.
[19] L. A. Nowitz, M. E. Watkins, Graphical regular representations of non-abelain groups, II, Canad. J. Math. 24 (1972), 1009–1018.
[20] P. Spiga, Finite groups admitting an oriented regular representation, J. Combin. Theory Ser. A 153 (2018), 76–97.
[21] P. Spiga, Cubic graphical regular representations of finite non-abelian simple groups, Commu. Algebra 46 (2018), 2440–2450.
[22] G. Verret, B. Xia, Oriented regular representations of out-valency two for finite simple groups, Ars Math. Contemp. 22 (2022), ♯P1.07.
[23] B. Xia, On cubic graphical regular representations of finite simple groups, J. Combin. Theory Ser. B 141 (2020), 1–30.

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