

Minimum L_1 -norm Estimation for Fractional Ornstein-Uhlenbeck Process Driven by a Gaussian Process

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Abstract: We investigate the asymptotic properties of the minimum L_1 -norm estimator of the drift parameter for fractional Ornstein-Uhlenbeck type process driven by a general Gaussian process.

1 Introduction

Diffusion processes and diffusion type processes satisfying stochastic differential equations driven by Wiener processes are used for stochastic modeling in a wide variety of sciences such as population genetics, economic processes, signal processing as well as for modeling sunspot activity and more recently in mathematical finance. Statistical inference for diffusion type processes satisfying stochastic differential equations driven by Wiener processes have been studied earlier and a comprehensive survey of various methods is given in Prakasa Rao (1999). There has been a recent interest to study similar problems for stochastic processes driven by a fractional Brownian motion to model processes involving long range dependence (cf. Prakasa Rao (2010)). Le Breton (1998) studied parameter estimation and filtering in a simple linear model driven by a fractional Brownian motion. Kleptsyna and Le Breton (2002) studied parameter estimation problems for fractional Ornstein-Uhlenbeck process. The fractional Ornstein-Uhlenbeck process is a fractional analogue of the Ornstein-Uhlenbeck process, that is, a continuous time first order autoregressive process $X = \{X_t, t \geq 0\}$ which is the solution of a one-dimensional homogeneous linear stochastic differential equation driven by a fractional Brownian motion (fBm) $W^H = \{W_t^H, t \geq 0\}$ with Hurst parameter H . Such a process is the unique Gaussian process satisfying the linear integral equation

$$(1. 1) \quad X_t = x_0 + \theta \int_0^t X_s ds + \sigma W_t^H, t \geq 0.$$

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They investigate the problem of estimation of the parameters θ and σ^2 based on the observation $\{X_s, 0 \leq s \leq T\}$ and study the asymptotic behaviour of these estimators as $T \rightarrow \infty$.

In spite of the fact that maximum likelihood estimators (MLE) are consistent and asymptotically normal and also asymptotically efficient in general, they have some short comings at the same time. Their calculation is often cumbersome as the expression for MLE involve stochastic integrals at times which need good approximations for computational purposes. Further more MLE are not robust in the sense that a slight perturbation in the noise component will change the properties of MLE substantially. In order to circumvent such problems, the minimum distance approach is proposed. Properties of the minimum distance estimators (MDE) were discussed in Millar (1984) in a general frame work. Kutoyants and Pilibossian (2000) studied the problem of minimum L_1 -norm estimation for the Ornstein-Uhlenbeck process. Prakasa Rao (2004) investigated the problem of minimum L_1 -norm estimation for the fractional Ornstein-Uhlenbeck process driven by a fractional Brownian motion.

Our aim in this paper is to obtain the minimum L_1 -norm estimator of the drift parameter of a Ornstein-Uhlenbeck process driven by general Gaussian processes and investigate the asymptotic properties of such estimators. El Machkouri et al. (2015), Chen and Zhou (2020) and Lu (2022) study parameter estimation for an Ornstein-Uhlenbeck process driven by a general Gaussian process.

2 Minimum L_1 -norm Estimation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a stochastic basis satisfying the usual conditions and the processes discussed in the following are (\mathcal{F}_t) -adapted. Further the natural filtration of a process is understood as the P -completion of the filtration generated by this process. We consider centered a Gaussian process $G \equiv \{G_t, 0 \leq t \leq 1\}$.

Let us consider a stochastic process $\{X_t, t \in [0, 1]\}$ defined by the stochastic integral equation

$$(2. 1) \quad X_t = x_0 + \theta \int_0^t X(s)ds + \varepsilon G_t, 0 \leq t \leq 1,$$

where θ is an unknown drift parameters respectively. For convenience, we write the above integral equation in the form of a stochastic differential equation

$$(2. 2) \quad dX_t = \theta X(t)dt + \varepsilon dG_t, X_0 = x_0, 0 \leq t \leq 1,$$

driven by the Gaussian process G . The class of Gaussian processes G includes fractional Brownian motion, sub-fractional Brownian motion and bifractional Brownian motion.

We now consider the problem of estimation of the parameter θ based on the observation of fractional Ornstein-Uhlenbeck process $X = \{X_t, 0 \leq t \leq 1\}$ satisfying the stochastic differential equation

$$(2.3) \quad dX_t = \theta X(t)dt + \varepsilon dG_t, X_0 = x_0, 0 \leq t \leq 1$$

where $\theta \in \Theta \subset R$ and study its asymptotic properties as $\varepsilon \rightarrow 0$.

Let $x_t(\theta)$ be the solution of the above differential equation with $\varepsilon = 0$. It is obvious that

$$(2.4) \quad x_t(\theta) = x_0 e^{\theta t}, 0 \leq t \leq 1.$$

Let

$$(2.5) \quad S_1(\theta) = \int_0^1 |X_t - x_t(\theta)| dt$$

We define θ_ε^* to be a *minimum L_1 -norm estimator* if there exists a measurable selection θ_ε^* such that

$$(2.6) \quad S_1(\theta_\varepsilon^*) = \inf_{\theta \in \Theta} S_1(\theta).$$

Conditions for the existence of a measurable selection are given in Lemma 3.1.2 in Prakasa Rao (1987). We assume that there exists a measurable selection θ_ε^* satisfying the above equation. An alternate way of defining the estimator θ_ε^* is by the relation

$$(2.7) \quad \theta_\varepsilon^* = \arg \inf_{\theta \in \Theta} \int_0^1 |X_t - x_t(\theta)| dt.$$

We will now present some maximal inequalities for Gaussian processes out of which one will be used in the sequel.

Some Maximal Inequalities for Gaussian processes:

Let $G_1^* = \sup_{0 \leq t \leq 1} |G_t|$. If G is a fractional Brownian motion or a sub-fractional Brownian motion, maximal inequalities are known and are reviewed in Prakasa Rao (2014, 2017, 2020) following self-similarity for such processes. Maximal inequalities for general Gaussian processes are surveyed in Li and Shao (2001). We now present some maximal inequalities for general Gaussian processes G . Let

$$d_G(s, t) = [E(G_t - G_s)^2]^{1/2}, 0 \leq s, t \leq 1,$$

$$\rho(\varepsilon) = \sup_{t, s \in [0, 1], |s-t| \leq \varepsilon} d_G(s, t)$$

and

$$Q(\delta) = \int_0^\infty \rho(\delta e^{-y^2}) dy.$$

Suppose that the function $\rho(\cdot)$ is strictly increasing. Let $\sigma^2 = \sup_{0 \leq t \leq 1} E(G_t^2)$ and $Q^{-1}(\cdot)$ denote the inverse of the function $Q(\cdot)$. The following result is due to Berman (1985).

Theorem 2.1: Under the conditions stated above, given a Gaussian process G defined on the interval $[0, 1]$, there exists a constant C such that

$$P\left(\sup_{0 \leq t \leq 1} |G_t| > \varepsilon\right) \leq C(Q^{-1}(1/\varepsilon))^{-1} \exp\left(-\frac{\varepsilon^2}{2\sigma^2}\right).$$

for all $\varepsilon > 0$.

Consider a Gaussian process G on the interval $[0, 1]$ and let $D = \sup_{0 \leq s, t \leq 1} d_G(s, t)$. Let $N(\varepsilon, d_G, 1)$ denote the minimum number of closed intervals of length $2\varepsilon > 0$, needed to cover the interval $[0, 1]$ with mid-points in $[0, 1]$ in the semi-metric d_G . We say that the Gaussian process G is pairwise non-degenerate if for every $\varepsilon > 0$ there exist closed intervals of length $2\varepsilon > 0$, in the semi-metric d_G needed to cover the Interval $[0, 1]$. If the Gaussian process is pairwise non-degenerate, then the semi-metric d_G is a metric. The following result is due to Marcus and Rosen (2006).

Theorem 2.2: Suppose G is a Gaussian process which is pairwise non-degenerate and continuous in mean square. If

$$\int_0^D \sqrt{\log N(\varepsilon, d_G, 1)} d\varepsilon < \infty,$$

then the process G has continuous sample paths and there exists a positive constant C such that

$$E\left[\sup_{0 \leq s \leq 1} |G(s)|\right] \leq C \int_0^{D/2} \sqrt{\log N(\varepsilon, d_G, 1)} d\varepsilon < \infty.$$

The following result is due to Borovkov et al. (2017).

Theorem 2.3: Suppose G is a Gaussian process on the interval $[0, 1]$ which is pairwise non-degenerate and there exists constants $0 \leq C_1 \leq C_2$ and $H \in (0, 1)$ such that

$$C_1|t - s|^H \leq d_G(s, t) \leq C_2|t - s|^H, t, s \in [0, 1].$$

Then

$$\frac{C_1}{5\sqrt{H}} \leq E\left[\sup_{0 \leq t \leq 1} G_t\right] \leq \frac{(16.3)C_2}{\sqrt{H}}.$$

Another useful maximal inequality for Gaussian process is the following result due to Nourdin (2012).

Theorem 2.4: Suppose G is a centered and continuous Gaussian process on the interval $[0, 1]$. Let $\sigma^2 = \sup_{0 \leq t \leq 1} E[G_t^2]$. Suppose that $0 < \sigma^2 < \infty$. Then $m = E[\sup_{0 \leq t \leq 1} G_t]$ is finite and for all $x > m$,

$$P(\sup_{0 \leq t \leq 1} G_t \geq x) \leq \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right).$$

From the fact that G is centered Gaussian process, it follows that

$$P(\sup_{0 \leq t \leq 1} |G_t| \geq x) \leq 2 \exp\left(-\frac{(x - m)^2}{2\sigma^2}\right)$$

under the conditions stated in Theorem 2.4.

3 Consistency of the estimator:

Let θ_0 denote the true parameter. For any $\delta > 0$, define

$$(3. 1) \quad g(\delta) = \inf_{|\theta - \theta_0| > \delta} \int_0^1 |x_t(\theta) - x_t(\theta_0)| dt.$$

Note that $g(\delta) > 0$ for any $\delta > 0$.

Theorem 3.1: Suppose G is a centered and continuous Gaussian process on the interval $[0, 1]$. Let $\sigma^2 = \sup_{0 \leq t \leq 1} E[G_t^2]$ and $m = E[\sup_{0 \leq t \leq 1} G_t]$. Then there exists a positive constant C such that for every $\delta > 0$,

$$P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} = O(e^{-C[g(\delta)]^2 \varepsilon^{-2}}).$$

Proof: Let $\|\cdot\|$ denote the L_1 -norm. Then

$$\begin{aligned} (3. 2) \quad P_{\theta_0}^{(\varepsilon)}\{|\theta_\varepsilon^* - \theta_0| > \delta\} &= P_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta - \theta_0| \leq \delta} \|X - x(\theta)\| > \inf_{|\theta - \theta_0| > \delta} \|X - x(\theta)\|\right\} \\ &\leq P_{\theta_0}^{(\varepsilon)}\left\{\inf_{|\theta - \theta_0| \leq \delta} (\|X - x(\theta_0)\| + \|x(\theta) - x(\theta_0)\|) \right. \\ &\quad \left. > \inf_{|\theta - \theta_0| > \delta} (\|x(\theta) - x(\theta_0)\| - \|X - x(\theta_0)\|)\right\} \\ &= P_{\theta_0}^{(\varepsilon)}\left\{2\|X - x(\theta_0)\| > \inf_{|\theta - \theta_0| > \delta} \|x(\theta) - x(\theta_0)\|\right\} \\ &= P_{\theta_0}^{(\varepsilon)}\left\{\|X - x(\theta_0)\| > \frac{1}{2}g(\delta)\right\}. \end{aligned}$$

Since the process X satisfies the stochastic differential equation (2.1), it follows that

$$(3. 3) \quad \begin{aligned} X_t - x_t(\theta_0) &= x_0 + \theta_0 \int_0^t X_s ds + \varepsilon G_t - x_t(\theta_0) \\ &= \theta_0 \int_0^t (X_s - x_s(\theta_0)) ds + \varepsilon G_t \end{aligned}$$

since $x_t(\theta) = x_0 e^{\theta t}$. Let $U_t = X_t - x_t(\theta_0)$. Then it follows from the equation given above that

$$(3. 4) \quad U_t = \theta_0 \int_0^t U_s ds + \varepsilon G_t.$$

Let $V_t = |U_t| = |X_t - x_t(\theta_0)|$. The relation given above implies that

$$(3. 5) \quad V_t = |X_t - x_t(\theta_0)| \leq |\theta_0| \int_0^t V_s ds + \varepsilon |G_t|.$$

Applying the Gronwall-Bellman Lemma, it follows that

$$(3. 6) \quad \sup_{0 \leq t \leq 1} |V_t| \leq \varepsilon e^{|\theta_0|} \sup_{0 \leq t \leq 1} |G_t|.$$

Hence

$$(3. 7) \quad \begin{aligned} P_{\theta_0}^{(\varepsilon)} [\|X - x(\theta_0)\| > \frac{1}{2} g(\delta)] &\leq P[\sup_{0 \leq t \leq 1} |G_t| > \frac{e^{-|\theta_0|} g(\delta)}{2\varepsilon}] \\ &= P[G_1^* > \frac{e^{-|\theta_0|} g(\delta)}{2\varepsilon}]. \end{aligned}$$

Let $m = E[\sup_{0 \leq t \leq 1} G(t)]$. Applying the maximal inequalities for Gaussian processes given in Theorem 2.4, we get that, for fixed $\delta > 0$, we can choose ε sufficiently small so that $\frac{e^{-|\theta_0|} g(\delta)}{2\varepsilon} > m$. For such ε ,

$$(3. 8) \quad \begin{aligned} P_{\theta_0}^{(\varepsilon)} [|\theta_\varepsilon^* - \theta_0| > \delta] &\leq 2 \exp\left(-\frac{((e^{-|\theta_0|} g(\delta)/2\varepsilon) - m)^2}{2\sigma^2}\right) \\ &= O(e^{-C[g(\delta)]^2 \varepsilon^{-2}}) \end{aligned}$$

for some positive constant C independent of ε .

Remarks: As a consequence of the result obtained above, it follows that

$$P_{\theta_0}^{(\varepsilon)} \{ |\theta_\varepsilon^* - \theta_0| > \delta \} \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

for every $\delta > 0$. Hence the minimum norm L_1 -estimator θ_ε^* is weakly consistent for estimating the parameter θ_0 .

4 Asymptotic distribution of the estimator:

We will now study the asymptotic distribution if any of the estimator θ_ε^* after suitable scaling.

It can be checked that

$$(4. 1) \quad X_t = e^{\theta_0 t} \left\{ x_0 + \int_0^t e^{-\theta_0 s} \varepsilon dG_s \right\}$$

or equivalently

$$(4. 2) \quad X_t - x_t(\theta_0) = \varepsilon e^{\theta_0 t} \int_0^t e^{-\theta_0 s} dG_s.$$

Let

$$(4. 3) \quad Y_t = e^{\theta_0 t} \int_0^t e^{-\theta_0 s} dG_s.$$

Note that $\{Y_t, 0 \leq t \leq 1\}$ is a Gaussian process and can be interpreted as the "derivative" of the process $\{X_t, 0 \leq t \leq 1\}$ with respect to ε . We obtain that, P -a.s.,

$$(4. 4) \quad Y_t e^{-\theta_0 t} = \int_0^t e^{-\theta_0 s} dG_s$$

The integral with respect to the process G is interpreted as Young integral (cf. El Machkouri et al. (2015)). In particular it follows that the random variable $Y_t e^{-\theta_0 t}$ and hence Y_t has the normal distribution with mean zero and further more, for any $0 \leq t, s \leq 1$,

$$(4. 5) \quad \begin{aligned} Cov(Y_t, Y_s) &= e^{\theta_0 t + \theta_0 s} E \left[\int_0^t e^{-\theta_0 u} dG_u \int_0^s e^{-\theta_0 v} dG_v \right] \\ &= e^{\theta_0 t + \theta_0 s} \int_0^t \int_0^s e^{-\theta_0(u+v)} dudv \\ &= R(t, s) \text{ (say)}. \end{aligned}$$

In particular

$$(4. 6) \quad Var(Y_t) = R(t, t).$$

Observe that $\{Y_t, 0 \leq t \leq 1\}$ is a zero mean Gaussian process with $Cov(Y_t, Y_s) = R(t, s)$. Let

$$(4. 7) \quad \zeta = \arg \inf_{-\infty < u < \infty} \int_0^1 |Y_t - utx_0 e^{\theta_0 t}| dt.$$

Theorem 4.1: As $\varepsilon \rightarrow 0$, the random variable $\varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ converges in probability to a random variable whose probability distribution is the same as that of ζ under P_{θ_0} .

Proof: Let $x'_t(\theta) = x_0 t e^{\theta t}$ and let

$$(4. 8) \quad Z_\varepsilon(u) = \|Y - \varepsilon^{-1}(x(\theta_0 + \varepsilon u) - x(\theta_0))\|$$

and

$$(4.9) \quad Z_0(u) = \|Y - ux'(\theta_0)\|.$$

Further more, let

$$(4.10) \quad A_\varepsilon = \{\omega : |\theta_\varepsilon^* - \theta_0| < \delta_\varepsilon\}, \delta_\varepsilon = \varepsilon^\tau, \tau \in (\frac{1}{2}, 1), L_\varepsilon = \varepsilon^{\tau-1}.$$

Observe that the random variable $u_\varepsilon^* = \varepsilon^{-1}(\theta_\varepsilon^* - \theta_0)$ satisfies the equation

$$(4.11) \quad Z_\varepsilon(u_\varepsilon^*) = \inf_{|u| < L_\varepsilon} Z_\varepsilon(u), \omega \in A_\varepsilon.$$

Define

$$(4.12) \quad \zeta_\varepsilon = \arg \inf_{|u| < L_\varepsilon} Z_0(u).$$

Observe that, with probability one,

$$(4.13) \quad \begin{aligned} \sup_{|u| < L_\varepsilon} |Z_\varepsilon(u) - Z_0(u)| &= \| \|Y - ux'(\theta_0) - \frac{1}{2}\varepsilon u^2 x''(\tilde{\theta})\| - \|Y - ux'(\theta_0)\| \| \\ &\leq \frac{\varepsilon}{2} L_\varepsilon^2 \sup_{|\theta - \theta_0| < \delta_\varepsilon} \int_0^T |x''(\theta)| dt \\ &\leq C \varepsilon^{2\tau-1}. \end{aligned}$$

Here $\tilde{\theta} = \theta_0 + \alpha(\theta - \theta_0)$ for some $\alpha \in (0, 1)$. Note that the last term in the above inequality tends to zero as $\varepsilon \rightarrow 0$. Further more the process $\{Z_0(u), -\infty < u < \infty\}$ has a unique minimum u^* with probability one. This follows from the arguments given in Theorem 2 of Kutoyants and Pilibossian (1994). In addition, we can choose the interval $[-L, L]$ such that

$$(4.14) \quad P_{\theta_0}^{(\varepsilon)} \{u_\varepsilon^* \in (-L, L)\} \geq 1 - \beta(g(L))^{-1}$$

and

$$(4.15) \quad P\{u^* \in (-L, L)\} \geq 1 - \beta(g(L))^{-1}$$

where $\beta > 0$. Note that $g(L)$ increases as L increases. The processes $Z_\varepsilon(u), u \in [-L, L]$ and $Z_0(u), u \in [-L, L]$ satisfy the Lipschitz conditions and $Z_\varepsilon(u)$ converges uniformly to $Z_0(u)$ over $u \in [-L, L]$. Hence the minimizer of $Z_\varepsilon(\cdot)$ converges to the minimizer of $Z_0(u)$. This completes the proof.

Remarks : We have seen earlier that the process $\{Y_t, 0 \leq t \leq T\}$ is a zero mean Gaussian process with the covariance function $Cov(Y_t, Y_s) = R(t, s)$ for $0, t, s \leq 1$. Recall that

$$(4.16) \quad \zeta = \arg \inf_{-\infty < u < \infty} \int_0^1 |Y_t - utx_0 e^{\theta_0 t}| dt.$$

It is not clear what the distribution of the random variable ζ is. It depends on the Gaussian process G . Observe that for every u , the integrand in the above integral is the absolute value of a Gaussian process $\{J_t, 0 \leq t \leq 1\}$ with the mean function $E(J_t) = -utx_0e^{\theta_0 t}$ and the covariance function $Cov(J_t, J_s) = R(t, s)$ for $0 \leq s, t \leq 1$. It is easy to extend the results to any Gaussian process defined on any interval $[0, T]$ for any fixed $T > 0$.

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References

- Berman, S.M. (1985) An asymptotic bound for the tail of the distribution of the maximum of a Gaussian process, *Ann. Inst. H. Poincare Probab. Statist.*, **21**, 47-57.
- Borovkov, K., Mishura, Y., and Novikov, A. (2017) Bounds for expected maximum of Gaussian processes and their discrete approximations, *Stochastics*, **89**, 21-37.
- Chen, Y., and Zhou, H. (2020) Parameter estimation for an Ornstein-Uhlenbeck process driven by a general Gaussian noise, arxiv:2002.09641v1 [math.PR] 22 Feb 2020.
- El Machkouri, M., Es-Sebaiy, K., and Ouknine, Y. (2015) Parameter estimation for the non-ergodic Ornstein-Uhlenbeck processes driven by Gaussian process, arXiv:1507.00802v1 [math.PR] 3 July 2015.
- Kleptsyna, M.L. and Le Breton, A. (2002) Statistical analysis of the fractional Ornstein-Uhlenbeck type process, *Statist. Infer. Stoch. Proces.* **5**, 229-248.
- Kutoyants, Yu. and Pilibossian, P. (1994) On minimum L_1 -norm estimate of the parameter of the Ornstein-Uhlenbeck process, *Statist. Probab. Lett.*, **20**, 117-123.
- Le Breton, A. (1998) Filtering and parameter estimation in a simple linear model driven by a fractional Brownian motion, *Statist. Probab. Lett.* **38**, 263-274.
- Lu, Y. (2022) Parameter estimation of non-ergodic Ornstein-Uhlenbeck processes driven by general Gaussian processes, arXiv:2207.13355v1 [math.ST] 27 Jul 2022.
- Marcus, M.B. and Rosen, J. (2006) *Markov processes, Gaussian processes and Local Times*, Cambridge Studies in Advanced Mathematics, Vol. 100, Cambridge University Press, Cambridge.

- Millar, P.W. (1984) A general approach to the optimality of the minimum distance estimators, *Trans. Amer. Math. Soc.* **286**, 249-272.
- Nourdin, I. (2012) *Selected Aspects of Fractional Brownian Motion*, Bocconi and Springer Series, Bocconi University Press, Milan.
- Prakasa Rao, B.L.S. (1987) *Asymptotic Theory of Statistical Inference*, Wiley, New York.
- Prakasa Rao, B.L.S. (1999) *Statistical Inference for Diffusion Type Processes*, Arnold, London and Oxford University Press, New York.
- Prakasa Rao, B.L.S. (2004) Minimum L_1 -norm estimation for fractional Ornstein-Uhlenbeck type process, *Theor. Probability and Math. Statist.*, **71** (2004) 160-168.
- Prakasa Rao, B.L.S. (2010) *Statistical Inference for Fractional Diffusion Processes*, Wiley, London.
- Prakasa Rao, B.L.S. (2014) Maximal inequalities for fractional Brownian motion: An overview, *Stochastic Analysis and Applications*, **32**, 450-479.
- Prakasa Rao, B.L.S. (2017) On some maximal and integral inequalities for sub-fractional Brownian motion, *Stochastic Analysis and Applications*, **35**, 279-287.
- Prakasa Rao, B.L.S. (2020) More on maximal inequalities for sub-fractional Brownian motion, *Stochastic Analysis and Applications*, **38**, 238-247.

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