

PINCHUK SCALING METHOD ON DOMAINS WITH NON-COMPACT AUTOMORPHISM GROUPS

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ABSTRACT. In this paper, we characterize weakly pseudoconvex domains of finite type in \mathbb{C}^n in terms of the boundary behavior of automorphism orbits by using the scaling method.

1. INTRODUCTION

Let Ω be a domain in \mathbb{C}^n . The set of all automorphisms of Ω , denoted by $\text{Aut}(\Omega)$, makes a group under composition and this group is also a topological group with the topology of uniform convergence on compact sets of Ω . By a classical theorem of H. Cartan (see [Na71]), for a bounded domain Ω in \mathbb{C}^n , it follows that $\text{Aut}(\Omega)$ is non-compact if and only if there exist a point $a \in \Omega$, a point $p \in \partial\Omega$, and automorphisms $\varphi_j \in \text{Aut}(\Omega)$ such that $\varphi_j(a) \rightarrow p$ as $j \rightarrow \infty$. Such a point p is called a *boundary orbit accumulation point*. The local geometry of the boundary orbit accumulation point in turn gives global information about the characterization of domains. In particular, Greene and Krantz [GK93] posed a conjecture that for a smoothly bounded pseudoconvex domain admitting a non-compact automorphism group, the boundary orbit accumulation point is of finite type in the sense of D'Angelo [D'A82]. (In this paper, the finiteness of type is understood in the sense of D'Angelo.) The interested reader is referred to the recent papers [IK99, KN15, Kr21] for this conjecture.

In this paper, we study the problem of characterizing domains in \mathbb{C}^n with non-compact automorphism groups. The main results around this problem are due to B. Wong and J. P. Rosay [Wo77, Ro79], E. Bedford and S. Pinchuk [BP89, BP91, BP95, BP99], K.-T. Kim [Ki90], F. Berteloot [Be94, Be03], A. Isaev and S. Krantz [IK97], Do Duc Thai and the first author [DN09], the first and third authors [NN20]. Almost all previous work requires the finiteness of type and either the strong pseudoconvexity (or even convexity), or pseudoconvexity only in dimension 2. In contrast to these results, we provide a new characterization of weakly pseudoconvex domains of finite type in terms of the boundary behavior of automorphism orbits by using the scaling method, introduced by S. Pinchuk [Pi81].

The scaling method may be briefly described as follows. Let Ω be a domain in \mathbb{C}^{n+1} and $\{\varphi_j(a)\}$ be a sequence of automorphism orbits converging to a boundary point ξ_0 . Let us fix a small neighborhood U_0 of ξ_0 . By using the reasonable composition, say T_j , of polynomial automorphisms of \mathbb{C}^{n+1} , including translations and dilations, the sequence of domains $D_j := T_j(U_0 \cap \Omega)$ converges normally to a model M_P , given by

$$M_P := \{(z, w) \in \mathbb{C}^n \times \mathbb{C} : \text{Re}(w) + P(z, \bar{z}) < 0\},$$

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where P is a real-valued polynomial on \mathbb{C}^n . When P is a homogeneous plurisubharmonic polynomial, M_P is called a *local homogeneous model* at ξ_0 .

The sequence $F_j := T_j \circ \varphi_j$ is in turn called a *Pinchuk scaling sequence*. The most difficulty is to prove that the Pinchuk scaling sequence $\{F_j\}$ is normal, i.e., there exists a subsequence of $\{F_j\}$ that converges uniformly on compacta from Ω to M_P . We first assume, temporarily, that Ω is a bounded domain in \mathbb{C}^n . Then the approach of Bedford and Pinchuk conveniently splits into two steps. In the first step, E. Bedford and S. Pinchuk [BP89, BP91, BP95, BP99] considered, alternatively, the convergence of the backward scaling sequence $\{F_j^{-1}\}$. Thanks to the boundedness of Ω , the Montel theorem ensures that this sequence $\{F_j^{-1}\}$ contains a convergent subsequence. Next, that limit, say G , was showed to be one-one from M_P into Ω by using the uniform estimates of the Kobayashi metric (cf. [Cat89] for $n = 1$), or the Sibony metric (cf. [Sib81] for corank one domains) on a family of the scaling domains $\{D_j\}$. In addition, the existence of a plurisubharmonic exhaustion function for Ω (cf. [DF77]) yields the holomorphic map G is surjective. In the second step, they treated a one-dimensional subgroup $\{h_t\}_{t \in \mathbb{R}} \subset \text{Aut}(\Omega)$ defined by $h_t(z) = G(G(z) + (0', it))$, $z \in \Omega$. This subgroup is parabolic in the sense that $h_t(z)$ tends to some boundary point $p \in \partial\Omega$ as $t \rightarrow \pm\infty$ for any $z \in \Omega$. A careful analysis of the holomorphic vector field $H(z) := \frac{d}{dt} \big|_{t=0} h_t(z)$, defined on Ω and tangent to $\partial\Omega$ (by [Fe74] each $h_t, t \in \mathbb{R}$, extends smoothly to the boundary), at the parabolic fixed point ξ_0 shows that the polynomial P is a weighted homogeneous polynomial such as $P(z_1, \bar{z}_1) = c|z_1|^{2m}$ for $n = 1$ and $P(z, \bar{z}) = c|z_1|^{2m} + |z_2|^2 + \dots + |z_n|^2$ for corank one domains, where c is a positive constant.

Let us emphasize that the above-mentioned method does not work for unbounded domains. Therefore, an alternative approach is to prove directly the normality of $\{F_j\}$ and then the tautness of Ω indicates that $\{F_j^{-1}\}$ is also normal. Then, [GK87, Lemma 4.1](see also [DN09, Prop. 2.1]) guarantees that the limit of $\{F_j\}$ is a biholomorphism from Ω onto M_P . The tautness of Ω easily follows from the existence of a plurisubharmonic peak function at ξ_0 (cf. [Be94, Prop. 2.1]). Therefore, our work boils down to verify the normality of $\{F_j\}$.

In 1991, S. Pinchuk [Pi91] himself considered the case that ξ_0 is strongly pseudoconvex. Thanks to the locally convexifiability of Ω near ξ_0 , S. Pinchuk proved that $\{F_j\}$ is normal. Therefore, Ω is biholomorphically equivalent to the unit ball \mathbb{B}^n , which gives a local version of the Wong-Rosay theorem. Similar result was achieved by A. M. Efimov [Ef95] for unbounded strongly pseudoconvex domains in \mathbb{C}^n . In addition, for convex domains in \mathbb{C}^n the normality of our Pinchuk scaling sequence can instead be easily established (cf. [BP95, Ga97, Ni09]). However, the Frankel scaling method given in [Fra89] can be applied for convenience (cf. [Ki90, Ki01, Zi17, Jo18]).

For unbounded weakly pseudoconvex domains of finite type in \mathbb{C}^2 , F. Berteloot [Be94, Be03, Be06] obtained a significant progress by using the properties of polydiscs constructed by D. Catlin (see [Cat89]) and the corresponding estimate of Kobayashi metric near ξ_0 to show the normality of the Pinchuk scaling sequence. Hence, Ω is biholomorphically equivalent to some local homogeneous model M_P . This result was generalized by Do Duc Thai and the first author [DN09] for corank one domains in \mathbb{C}^n .

Recently, the first and third authors [NN20] investigated a pseudoconvex domain $\Omega \subset \mathbb{C}^n$ which is of finite Catlin's multitype near $\xi_0 \in \partial\Omega$. When M_P is a homogeneous model of finite type, there exists a plurisubharmonic peak function for M_P at the origin. Therefore, the attraction property of analytic discs yields the normality of the Pinchuk

scaling sequence. Consequently, Ω is biholomorphically equivalent to M_P provided that the automorphism orbit $\{\varphi_j(a)\}$ converges Λ -nontangentially to ξ_0 (cf. Remark 3.1 and [NN20, Theorem 1.1]).

One observes that the local homogeneous model M_P depends heavily on the boundary behavior of $\{\varphi_j(a)\}$. More precisely, the boundary behavior of $\{\varphi_j(a)\}$ suggests some choice of dilations. The purpose of this paper is to give a characterization of local homogeneous models when the automorphism orbit $\{\varphi_j(a)\}$ accumulates at ξ_0 “very” tangentially to $\partial\Omega$ - the remaining possibility (cf. Definitions 3.1 and 4.1).

The first aim of this paper is to prove the following theorem, which says that only the unit ball admits an automorphism orbit accumulating at ξ_0 uniformly Λ -tangentially to $\partial\Omega$ (cf. Definition 3.1).

Theorem 1.1. *Let Ω be a pseudoconvex domain in \mathbb{C}^{n+1} with C^∞ -smooth boundary $\partial\Omega$. Let $\xi_0 \in \partial\Omega$ be strongly h -extendible with Catlin’s finite multitype $(2m_1, \dots, 2m_n, 1)$ and let $\Lambda = (1/2m_1, \dots, 1/2m_n)$ (see Definition 3.2). Suppose that there exists a sequence $\{\varphi_j\} \subset \text{Aut}(\Omega)$ such that $\{\varphi_j(a)\}$ converges uniformly Λ -tangentially to ξ_0 for some $a \in \Omega$ (see Definition 3.1). Then Ω is biholomorphically equivalent to the unit ball \mathbb{B}^{n+1} .*

Remark 1.1. The uniform Λ -tangential convergence of $\{\varphi_j(a)\}$ allows us to choose a suitable sequence of dilations (cf. see Equation (3) in Section 3) so that our model is an analytic ellipsoid that is biholomorphically equivalent to \mathbb{B}^{n+1} . However, Example 3.3 in Section 3 points out that without this uniform Λ -tangential convergence, there still exists an alternative sequence of dilations to get such a model. In addition, the explicit description for the automorphism group of the Thullen domains or the finite multitype models (cf. [NNTK19]) demonstrates that any sequence of automorphism orbit converges Λ -nontangentially to some boundary orbit accumulation point. Therefore, it seems reasonable to expect that may drop the requirement of the uniformity of Λ -tangentially convergences (see Remark 3.3).

Now we turn to pseudoconvex Levi corank one domains in \mathbb{C}^{n+1} which includes pseudoconvex domains of finite type in \mathbb{C}^2 . Then, the point ξ_0 is h -extendible (cf. [Yu95]). In addition, if $n = 1$ and ξ_0 is strongly h -extendible, then Ω is biholomorphically equivalent to \mathbb{B}^2 by Theorem 1.1. However, without the strongly h -extendibility, the notion of spherically $\frac{1}{2m}$ -tangential convergence is necessary to determine if Ω is biholomorphically equivalent to \mathbb{B}^{n+1} (cf. Definition 4.1).

The second aim of this paper is to prove the following theorem.

Theorem 1.2. *Let Ω be a pseudoconvex domain in \mathbb{C}^{n+1} with C^∞ -smooth boundary $\partial\Omega$. Suppose that ξ_0 is a boundary point of Ω of D’Angelo finite type such that the Levi form has corank at most 1 at ξ_0 and there exists a sequence $\{\varphi_j\} \subset \text{Aut}(\Omega)$ such that $\varphi_j(a)$ converges spherically $\frac{1}{2m}$ -tangentially to ξ_0 for some $a \in \Omega$ (cf. Definition 4.1). Then Ω is biholomorphically equivalent to the unit ball \mathbb{B}^{n+1} .*

Now let $\Omega \subset \mathbb{C}^2$ be a pseudoconvex domain of finite type near $\xi_0 \in \partial\Omega$ with the type $2m$. Then the notion of uniformly $(\frac{1}{2m})$ -tangential convergence (cf. Definition 3.1) reduces to that of $(\frac{1}{2m})$ -tangential convergence. Moreover, the notion of spherically $\frac{1}{2m}$ -tangential convergence is exactly Definition 4.1. Therefore, Theorem 1.2 yields the following corollary.

Corollary 1.3. *Let Ω be a pseudoconvex domain in \mathbb{C}^2 with C^∞ -smooth boundary $\partial\Omega$. Suppose that $\xi_0 \in \partial\Omega$ is of finite type $2m$. Suppose that there exists a sequence $\{\varphi_j\} \subset$*

$\text{Aut}(\Omega)$ such that $\varphi_j(a)$ converges spherically $\frac{1}{2m}$ -tangentially to ξ_0 for some $a \in \Omega$ (see Definition 4.1). Then Ω is biholomorphically equivalent to the unit ball \mathbb{B}^2 .

In the case that $\{\varphi_j(a)\}$ does not converge spherically $\frac{1}{2m}$ -tangentially to ξ_0 for some $a \in \Omega$, we prove the following proposition, in which our model may be defined by a homogeneous polynomial of degree larger than 2.

Proposition 1.4. *Let Ω be a pseudoconvex domain in \mathbb{C}^2 with C^∞ -smooth boundary $\partial\Omega$. Suppose that $\xi_0 \in \partial\Omega$ is of finite type $2m$. Suppose that there exist a number $2 \leq \nu \leq m$, $a \in \Omega$ and a sequence $f_j \subset \text{Aut}(\Omega)$ such that $f_j(a)$ converges spherically $\frac{1}{2m}$ -tangentially of order 2ν to ξ_0 (see Definition 5.1). Then Ω is biholomorphically equivalent to a model of the form*

$$M_Q := \{(z, w) \in \mathbb{C}^2 : \text{Re}(w) + Q(z) < 0\},$$

where Q is a homogeneous polynomial of degree 2ν which is not harmonic.

Remark 1.2. We note that Example 5.1 in Section 5 illustrates that if the sequence of automorphism orbits does not converge spherically $\frac{1}{2m}$ -tangentially to a boundary point, then our domain Ω is not biholomorphically equivalent to the unit ball \mathbb{B}^2 but to M_Q with $\deg(Q) = 4$.

The organization of this paper is as follows: In Sections 2, we recall some basic definitions and results needed later. In Section 3, we present the notion of Λ -tangential convergence and give a proof of Theorem 1.1. Next, the notion of spherical $\frac{1}{2m}$ -tangential convergence and the proof of Theorem 1.2 are introduced in Section 4. Finally, the proof of Proposition 1.4 is given in Section 5.

2. PRELIMINARIES

First of all, we recall the following definition (see [GK87, Kr21], or [DN09]).

Definition 2.1. Let $\{\Omega_i\}_{i=1}^\infty$ be a sequence of domains in \mathbb{C}^n . The sequence $\{\Omega_i\}_{i=1}^\infty$ is said to *converge normally* to a domain $\Omega_0 \subset \mathbb{C}^n$ if the following two conditions hold:

- (i) If a compact set K is contained in the interior (i.e., the largest open subset) of $\bigcap_{j \geq m} \Omega_j$ for some positive integer m , then $K \subset \Omega_0$.
- (ii) If a compact subset $K' \subset \Omega_0$, then there exists a constant $m > 0$ such that $K' \subset \bigcap_{j \geq m} \Omega_j$.

In addition, when a sequence of map $\varphi_j: \Omega_j \rightarrow \mathbb{C}^m$ converges uniformly on compact sets to a map $\varphi: \Omega \rightarrow \mathbb{C}^m$ then we shall say that φ_j *converges normally* to φ .

Next, let us recall the following definition (cf. [Yu95]).

Definition 2.2. Let $\Lambda = (\lambda_1, \dots, \lambda_n)$ be a fixed n -tuple of positive numbers and $\mu > 0$. We denote by $\mathcal{O}(\mu, \Lambda)$ the set of smooth functions f defined near the origin of \mathbb{C}^n such that

$$D^\alpha \bar{D}^\beta f(0) = 0 \text{ whenever } \sum_{j=1}^n (\alpha_j + \beta_j) \lambda_j \leq \mu.$$

If $n = 1$ and $\Lambda = (1)$ then we use $\mathcal{O}(\mu)$ to denote the functions vanishing to order at least μ at the origin. Here and in what follows, D^α and \bar{D}^β denote the partial differential operators

$$\frac{\partial^{|\alpha|}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \text{ and } \frac{\partial^{|\beta|}}{\partial \bar{z}_1^{\beta_1} \cdots \partial \bar{z}_n^{\beta_n}},$$

respectively. Furthermore, \lesssim and \gtrsim denote inequality up to a positive constant. Moreover, we will use \approx for the combination of \lesssim and \gtrsim .

Finally, in order to give proofs of Theorem 1.1 and Theorem 1.3, let us recall the following proposition that is the main ingredient in our argument (see [NN20, Prop. 4.3]).

Proposition 2.1 ([NN20]). *Let ω be a domain in \mathbb{C}^k , $a \in \omega$ and $\sigma_j: \omega \rightarrow \Omega_j$ be a sequence of holomorphic mappings such that $\{\sigma_j(a)\} \in M_P$. If M_P is of finite type, then $\{\sigma_j\}$ contains a subsequence that converges locally uniformly to a holomorphic map $\sigma: \omega \rightarrow M_P$.*

3. THE BEHAVIOUR OF AUTOMORPHISM ORBITS ACCUMULATING AT A BOUNDARY POINT OF AN h -EXTENDIBLE DOMAIN IN \mathbb{C}^n

3.1. Λ -tangential convergence. Throughout this subsection, let Ω be a domain in \mathbb{C}^n and assume that $\xi_0 \in \partial\Omega$ is an h -extendible boundary point (cf. [Yu95, DH94]). Let ρ be a local defining function for Ω near ξ_0 and let the multitype $\mathcal{M}(\xi_0) = (2m_1, \dots, 2m_n, 1)$ be finite (see [Cat84]). (Note that because of the pseudoconvexity of Ω , the integers $2m_1, \dots, 2m_n$ are all even.) Let us denote by $\Lambda = (1/2m_1, \dots, 1/2m_n)$. By the definition of multitype, there are distinguished coordinates $(z, w) = (z_1, \dots, z_n, w)$ such that $\xi_0 = 0$ and $\rho(z, w)$ can be expanded near 0 as follows:

$$\rho(z, w) = \operatorname{Re}(w) + P(z) + Q(z, w),$$

where P is a Λ -homogeneous plurisubharmonic polynomial that contains no pluriharmonic monomials, Q is smooth and satisfies

$$|Q(z, w)| \leq C \left(|w| + \sum_{j=1}^n |z_j|^{2m_j} \right)^\gamma,$$

for some constant $\gamma > 1$ and $C > 0$. In what follows, $\operatorname{dist}(z, \partial\Omega)$ denotes the Euclidean distance from z to $\partial\Omega$.

Definition 3.1. We say that a sequence $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ with $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn})$, converges uniformly Λ -tangentially to ξ_0 if the following conditions hold:

- (a) $|\operatorname{Im}(\beta_j)| \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$;
- (b) $|\operatorname{dist}(\eta_j, \partial\Omega)| = o(|\alpha_{jk}|^{2m_k})$ for $1 \leq k \leq n$;
- (c) $|\alpha_{j1}|^{2m_1} \approx |\alpha_{j2}|^{2m_2} \approx \dots \approx |\alpha_{jn}|^{2m_n}$.

Remark 3.1. It is well-known that $\{\eta_j\} \subset \Omega$ converges nontangentially to ξ_0 if $|\operatorname{Im}(\beta_j)| \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$ and $|\alpha_{jk}| \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$ for every $1 \leq k \leq n$. Nevertheless, such sequence converges Λ -nontangentially to ξ_0 if $|\operatorname{Im}(\beta_j)| \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$ and $|\alpha_{jk}|^{2m_k} \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$ for every $1 \leq k \leq n$ (cf. [NN20]).

Denote by

$$\sigma(z) := \sum_{k=1}^n |z_k|^{2m_k}.$$

Definition 3.2. We say that a boundary point $\xi_0 \in \partial\Omega$ is *strongly h -extendible* if there exists $\delta > 0$ such that $P(z) - \delta\sigma(z)$ is plurisubharmonic, i.e. $dd^c P \geq \delta dd^c \sigma$.

Remark 3.2. The notion of strongly h -extendibility is exactly that M_P is *homogeneous finite diagonal type* given in [He92, He16]. Let $\xi_0 \in \partial\Omega$ be strongly h -extendible. Then by $dd^c P \gtrsim dd^c \sigma$, it follows that $\xi_0 \in \partial\Omega$ is in fact h -extendible and

$$\begin{aligned} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l}(\alpha) w_j \bar{w}_l &\gtrsim \sum_{k,l=1}^n \frac{\partial^2 \sigma}{\partial z_k \partial \bar{z}_l}(\alpha) w_j \bar{w}_l \\ &\gtrsim m_1^2 |\alpha_1|^{2m_1-2} |w_1|^2 + \cdots + m_n^2 |\alpha_n|^{2m_n-2} |w_n|^2 \end{aligned}$$

for all $\alpha, w \in \mathbb{C}^n$.

In the sequel, we will assume that $\xi_0 \in \partial\Omega$ is a strongly h -extendible point and let $\{\epsilon_j\} \subset \mathbb{R}^+$ be a given sequence. Then we define the sequence $\tau_j = (\tau_{j1}, \dots, \tau_{jn})$, associated to $\{\epsilon_j\}$, as follows:

$$\tau_{jk} := |\alpha_k| \cdot \left(\frac{\epsilon_j}{|\alpha_{jk}|^{2m_k}} \right)^{1/2}, \quad j \geq 1, 1 \leq k \leq n.$$

A simple calculation shows that $\tau_{jk}^{2m_k} = \epsilon_j \cdot \left(\frac{\epsilon_j}{|\alpha_{jk}|^{2m_k}} \right)^{m_k-1} \lesssim \epsilon_j$. Hence, we get the following estimates

$$(1) \quad \epsilon_j^{1/2} \lesssim \tau_{jk} \lesssim \epsilon_j^{1/2m_k}.$$

In what follows, we assign weights $\frac{1}{2m_1}, \dots, \frac{1}{2m_n}, 1$ to the variables z_1, \dots, z_n, w , respectively and denote by $wt(K) := \sum_{j=1}^n \frac{k_j}{2m_j}$ the weight of an n -tuple $K = (k_1, \dots, k_n) \in \mathbb{Z}_{\geq 0}^n$. We note that $wt(K + L) = wt(K) + wt(L)$ for any $K, L \in \mathbb{Z}_{\geq 0}^n$.

In order to prove Theorem 1.1, we need the following lemmas. First of all, from (1) one easily obtains the following lemma.

Lemma 3.1. *Let $f(z, w)$ be a C^∞ -smooth function defined in a neighborhood of the origin in \mathbb{C}^{n+1} vanishing to weight order greater than 1 at the origin. Then*

$$f(\tau_{j1}z_1, \dots, \tau_{jn}z_n, \epsilon_j w) = o(\epsilon_j).$$

For monomials with weight order ≤ 1 , we have the following lemmas.

Lemma 3.2. *Let $p, q \in \mathbb{N}^n$ be two multi-indices. Then, for all polynomials P one has*

$$\epsilon_j^{-1} |D^p \bar{D}^q P(\alpha_j) \tau_j^{p+q}| \rightarrow 0$$

for $|p| + |q| > 2$. In addition, if $|p| = |q| = 1$, then

$$\epsilon_j^{-1} |D^p \bar{D}^q P(\alpha_j) \tau_j^{p+q}| \lesssim 1.$$

Moreover, if $P(z) - \delta\sigma(z)$ is plurisubharmonic for some $\delta > 0$, then

$$\epsilon_j^{-1} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l}(\alpha_j) \tau_{jk} \tau_{jl} w_k \bar{w}_l \gtrsim m_1^2 |w_1|^2 + \cdots + m_n^2 |w_n|^2.$$

Proof. It suffices to prove the lemma for $P(z) = z^K \bar{z}^L$ and $K \geq p, L \geq q$. Then we have

$$\begin{aligned} \epsilon_j^{-1} |D^p \bar{D}^q P(\alpha_j) \tau_j^{p+q}| &= \epsilon_j^{-1} |\alpha_{j1}|^{k_1+l_1} \left(\frac{\tau_{j1}}{|\alpha_{j1}|} \right)^{p_1+q_1} \cdots |\alpha_{jn}|^{k_n+l_n} \left(\frac{\tau_{jn}}{|\alpha_{jn}|} \right)^{p_n+q_n} \\ &= \left(\frac{|\alpha_{j1}|^{2m_1}}{\epsilon_j} \right)^{\frac{k_1+l_1}{2m_1} - \frac{p_1+q_1}{2}} \cdots \left(\frac{|\alpha_{jn}|^{2m_n}}{\epsilon_j} \right)^{\frac{k_n+l_n}{2m_n} - \frac{p_n+q_n}{2}} \\ &\approx \left(\frac{|\alpha_{j1}|^{2m_1}}{\epsilon_j} \right)^{\sum_{s=1}^n \frac{k_s+l_s}{2m_s} - \frac{p_j+q_j}{2}} = \left(\frac{|\alpha_{j1}|^{2m_1}}{\epsilon_j} \right)^{1 - \frac{|p|+|q|}{2}}. \end{aligned}$$

Therefore, we get

$$\epsilon_j^{-1} |D^p \bar{D}^q P(\alpha_j) \tau_j^{p+q}| \rightarrow 0$$

as $j \rightarrow \infty$ for $|p| + |q| > 2$ and

$$\epsilon_j^{-1} |D^p \bar{D}^q P(\alpha_j) \tau_j^{p+q}| \lesssim 1$$

for $|p| + |q| = 2$. Finally, by Remark 3.2 one obtains that

$$\begin{aligned} \epsilon_j^{-1} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l}(\alpha_j) \tau_{jk} \tau_{jl} w_k \bar{w}_l &\gtrsim \epsilon_j^{-1} \sum_{k,l=1}^n \frac{\partial^2 \sigma}{\partial z_k \partial \bar{z}_l}(\alpha_j) \tau_{jk} \tau_{jl} w_j \bar{w}_l \\ &\gtrsim \epsilon_j^{-1} (m_1^2 |\alpha_1|^{2m_1-2} \tau_{j1}^2 |w_1|^2 + \cdots + m_n^2 |\alpha_n|^{2m_n-2} \tau_{jn}^2 |w_n|^2) \\ &\gtrsim m_1^2 |w_1|^2 + \cdots + m_n^2 |w_n|^2 \end{aligned}$$

for every $w \in \mathbb{C}^n$. □

In the same fashion we have the following lemma.

Lemma 3.3. *Let $Q(z)$ be a polynomial in $z \in \mathbb{C}^n$ such that $Q \in \mathcal{O}(1, \Lambda)$. Then we have*

$$\epsilon_j^{-1} |D^p \bar{D}^q Q(\alpha_j) \tau_j^{p+q}| \rightarrow 0$$

as $j \rightarrow \infty$ for $|p| + |q| \geq 2$.

Proof. As in the proof of Lemma 3.2, it suffices to consider $Q(z) = z^K \bar{z}^L$ and $K \geq p, L \geq q$ with $d := wt(K + L) > 1$. Then following the proof of Lemma 3.2, one has

$$\epsilon_j^{-1} |D^p \bar{D}^q Q(\alpha_j) \tau_j^{p+q}| \approx \left(\frac{|\alpha_{j1}|^{2m_1}}{\epsilon_j} \right)^{d - \frac{|p|+|q|}{2}}.$$

Therefore, we conclude that $\epsilon_j^{-1} |D^p \bar{D}^q Q(\alpha_j) \tau_j^{p+q}| \rightarrow 0$ as $j \rightarrow \infty$ for $|p| + |q| \geq 2$, as desired. □

3.2. Proof of Theorem 1.1. Let Ω and $\xi_0 \in \partial\Omega$ be as in the statement of Theorem 1.1. Let $\mathcal{M}(\xi_0) = (2m_1, \dots, 2m_n, 1)$ be the finite multitype of Ω at ξ_0 and denote by $\Lambda = (1/2m_1, \dots, 1/2m_n)$. As in Subsection 3.1, one can find local coordinates $(\tilde{z}, \tilde{w}) = (\tilde{z}_1, \dots, \tilde{z}_n, \tilde{w})$ near ξ_0 such that $\xi_0 = 0$ and the local defining function $\rho(\tilde{z}, \tilde{w})$ for Ω can be expanded near 0 as follows:

$$\rho(\tilde{z}, \tilde{w}) = \text{Re}(\tilde{w}) + P(\tilde{z}) + Q(\tilde{z}, \tilde{w}),$$

where P is a Λ -homogeneous plurisubharmonic polynomial that contains no pluriharmonic monomials, Q is smooth and satisfies

$$|Q(\tilde{z}, \tilde{w})| \leq C \left(|\tilde{w}| + \sum_{j=1}^n |\tilde{z}_j|^{2m_j} \right)^\gamma,$$

for some constant $\gamma > 1$ and $C > 0$.

By hypothesis of Theorem 1.1, there exist a sequence $\{\varphi_j\} \subset \text{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\eta_j := \varphi_j(a)$ converges uniformly Λ -tangentially to ξ_0 . Let us write $\eta_j = (\alpha_j, \beta_j) = (\alpha_{j1}, \dots, \alpha_{jn}, \beta_j)$. Then one has

- (a) $|\text{Im}(\beta_j)| \lesssim |\text{dist}(\eta_j, \partial\Omega)|$;
- (b) $|\text{dist}(\eta_j, \partial\Omega)| = o(|\alpha_{jk}|^{2m_k})$ for $1 \leq k \leq n$.
- (c) $|\alpha_{j1}|^{2m_1} \approx |\alpha_{j2}|^{2m_2} \approx \dots \approx |\alpha_{jn}|^{2m_n}$.

By following the proofs of Lemmas 4.10, 4.11 in [Yu95], after a change of variables

$$\begin{cases} z := \tilde{z}; \\ w := \tilde{w} + b_1(\tilde{z})\tilde{w} + b_2(\tilde{z})\tilde{w}^2 + b_3(\tilde{z}), \end{cases}$$

where b_1, b_2, b_3 are holomorphic functions of \tilde{z} satisfying $b_k = O(|\tilde{z}|^2)$, $k = 1, 2, 3$, there are local holomorphic coordinates (z, w) in which $\xi_0 = 0$ and Ω can be described near 0 as follows:

$$\Omega = \{(z, w) \in \mathbb{C}^{n+1} : \rho(z, w) = \text{Re}(w) + P(z) + R_1(z) + R_2(\text{Im}w) + (\text{Im}w)R(z) < 0\},$$

where $R_1 \in \mathcal{O}(1, \Lambda)$, $R \in \mathcal{O}(1/2, \Lambda)$, and $R_2 \in \mathcal{O}(2)$. We would like to emphasize that in the new coordinates the sequence $\{\eta_j\}$ still has the properties (a), (b), and (c).

Let U_0 be a fixed small neighborhood of $\xi_0 = 0$. Then for any sequence $\{\eta_j = (\alpha_j, \beta_j)\}$ of points converging uniformly Λ -tangentially to the origin in $U_0 \cap \{\rho < 0\} =: U_0^-$, we associate with a sequence of points $\eta'_j = (\alpha_j, a_j + \epsilon_j + ib_j)$, where $\epsilon_j > 0$ and $\beta_j = a_j + ib_j$, such that $\eta'_j = (\alpha_j, \beta'_j)$ with $\beta'_j = a_j + \epsilon_j + ib_j$ is in the hypersurface $\{\rho = 0\}$ for every $j \in \mathbb{N}^*$. We note that $\epsilon_j \approx \text{dist}(\eta_j, \partial\Omega)$

Before we begin the scaling procedure, we make several changes of coordinates. Firstly, let us consider the sequences of translations $L_{\eta'_j} : \mathbb{C}^n \rightarrow \mathbb{C}^n$, defined by

$$L_{\eta'_j}(z, w) := (z, w) - \eta'_j = (z - \alpha_j, w - \beta'_j).$$

Then, under the change of variables $(\tilde{z}, \tilde{w}) := L_{\eta'_j}(z, w)$, i.e.,

$$\begin{cases} w - \beta'_j = \tilde{w}; \\ z_k - \alpha_{jk} = \tilde{z}_k, \quad k = 1, \dots, n, \end{cases}$$

one observes that $L_{\eta'_j}(\alpha_j, \beta_j) = (0', -\epsilon_j)$ for every $j \in \mathbb{N}^*$.

Now let us write $w = u + iv$, $v = b_j + (v - b_j) = b_j + \text{Im}(\tilde{w})$, and $z = \alpha_j + (z - \alpha_j) = \alpha_j + \tilde{z}$. Since $R_2 \in \mathcal{O}(2)$ and $R \in \mathcal{O}(1/2, \Lambda)$, by using Taylor's theorem, we have

$$\begin{aligned} R_2(v) &= R_2(b_j) + R_2'(b_j)(v - b_j) + o(|v - b_j|) = R_2(b_j) + R_2'(b_j)\text{Im}(\tilde{w}) + o(|\text{Im}(\tilde{w})|); \\ vR(z) &= (b_j + (v - b_j))R(\alpha_j + \tilde{z}) \\ &= (b_j + \text{Im}(\tilde{w})) \left(R(\alpha_j) + 2\text{Re} \sum_{1 \leq |p| \leq 2} \frac{D^p R}{p!}(\alpha_j)(\tilde{z})^p + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R}{\partial \tilde{z}_k \partial \bar{\tilde{z}}_l}(\alpha_j) \tilde{z}_k \bar{\tilde{z}}_l + o(|\tilde{z}|^2) \right) \\ &= b_j R(\alpha_j) + b_j \left(2\text{Re} \sum_{1 \leq |p| \leq 2} \frac{D^p R}{p!}(\alpha_j)(\tilde{z})^p + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R}{\partial \tilde{z}_k \partial \bar{\tilde{z}}_l}(\alpha_j) \tilde{z}_k \bar{\tilde{z}}_l \right) \\ &\quad + o(|\tilde{z}|^2) + o(|\text{Im}(\tilde{w})|). \end{aligned}$$

Hence, using again Taylor's theorem we see that the hypersurface $L_{\eta_j}'(\{\rho = 0\})$ is defined by an equation of the form

$$\begin{aligned} (2) \quad \rho \left(L_{\eta_j}^{-1}(\tilde{z}, \tilde{w}) \right) &= \text{Re}(\tilde{w}) + R_2'(b_j)\text{Im}(\tilde{w}) + R(\alpha_j)\text{Im}(\tilde{w}) + o(|\text{Im}(\tilde{w})|) \\ &\quad + 2\text{Re} \sum_{1 \leq |p| \leq 2} \frac{D^p P}{p!}(\alpha_j)(\tilde{z})^p + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial \tilde{z}_k \partial \bar{\tilde{z}}_l}(\alpha_j) \tilde{z}_k \bar{\tilde{z}}_l \\ &\quad + 2\text{Re} \sum_{1 \leq |p| \leq 2} \frac{D^p R_1}{p!}(\alpha_j)(\tilde{z})^p + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R_1}{\partial \tilde{z}_k \partial \bar{\tilde{z}}_l}(\alpha_j) \tilde{z}_k \bar{\tilde{z}}_l \\ &\quad + b_j \left(2\text{Re} \sum_{1 \leq |p| \leq 2} \frac{D^p R}{p!}(\alpha_j)(\tilde{z})^p + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R}{\partial \tilde{z}_k \partial \bar{\tilde{z}}_l}(\alpha_j) \tilde{z}_k \bar{\tilde{z}}_l \right) + o(|\tilde{z}|^2) = 0. \end{aligned}$$

Next, to remove the pluriharmonic terms in (2) let us define a sequence $\{Q_j\}$ of automorphisms of \mathbb{C}^{n+1} by

$$\begin{cases} w := \tilde{w} + (R_2'(b_j) + R(\alpha_j))i\tilde{w} + 2 \sum_{1 \leq |p| \leq 2} \frac{D^p P}{p!}(\alpha_j)(\tilde{z})^p + 2 \sum_{1 \leq |p| \leq 2} \frac{D^p R_1}{p!}(\alpha_j)(\tilde{z})^p \\ \quad + b_j \sum_{1 \leq |p| \leq 2} \frac{D^p R}{p!}(\alpha_j); \\ z_k := \tilde{z}_k, \quad k = 1, \dots, n. \end{cases}$$

Then the composite $Q_j \circ L_{\eta_j}' \in \text{Aut}(\mathbb{C}^n)$ and satisfies that

$$Q_j \circ L_{\eta_j}'(\alpha_j, \beta_j) = (0, \dots, 0, -\epsilon_j - i(R_2'(b_j) + R(\alpha_j))\epsilon_j)$$

for every $j \in \mathbb{N}^*$. Moreover, the hypersurface $Q_j \circ L_{\eta_j}'(\{\rho = 0\})$ is given by an equation of the form

$$\begin{aligned} \rho \left(L_{\eta_j}^{-1} \circ Q_j^{-1}(z, w) \right) &= \text{Re}(w) + o(|\text{Im}(w)|) + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial z_k \partial \bar{z}_l}(\alpha_j) z_k \bar{z}_l \\ &\quad + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R_1}{\partial z_k \partial \bar{z}_l}(\alpha_j) z_k \bar{z}_l + \frac{b_j}{2} \sum_{k,l=1}^n \frac{\partial^2 R}{\partial z_k \partial \bar{z}_l}(\alpha_j) z_k \bar{z}_l + o(|z|^2) = 0. \end{aligned}$$

Finally, let us recall the following notation

$$\tau_{jk} := |\alpha_{jk}| \cdot \left(\frac{\epsilon_j}{|\alpha_{jk}|^{2m_k}} \right)^{1/2}, \quad 1 \leq k \leq n$$

and we define an anisotropic dilation $\Delta_j: \mathbb{C}^n \rightarrow \mathbb{C}^n$ by settings:

$$(3) \quad \Delta_j(z, w) := \Delta_{\eta_j}^{\epsilon_j}(z_1, \dots, z_n, w) = \left(\frac{z_1}{\tau_{j1}}, \dots, \frac{z_n}{\tau_{jn}}, \frac{w}{\epsilon_j} \right).$$

Then it follows that $\Delta_j \circ Q_j \circ L_{\eta_j}'(\alpha_j, \beta_j) = (0, \dots, 0, -1 - i(R_2'(b_j) + R(\alpha_j))) \rightarrow (0', -1)$ as $j \rightarrow \infty$. Furthermore, the hypersurface $\Delta_j \circ Q_j \circ L_{\eta_j}'(\{\rho = 0\})$ is now defined by an equation of the form

$$(4) \quad \begin{aligned} & \epsilon_j^{-1} \rho \left(L_{\eta_j}^{-1} \circ Q_j^{-1} \circ (\Delta_j)^{-1}(\tilde{z}, \tilde{w}) \right) \\ &= \operatorname{Re}(\tilde{w}) + \epsilon_j^{-1} o(\epsilon_j |\operatorname{Im}(\tilde{w})|) + \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 P}{\partial \tilde{z}_k \partial \tilde{z}_l}(\alpha_j) \epsilon_j^{-1} \tau_{jk} \tau_{jl} \tilde{z}_k \bar{\tilde{z}}_l \\ &+ \frac{1}{2} \sum_{k,l=1}^n \frac{\partial^2 R_1}{\partial \tilde{z}_k \partial \tilde{z}_l}(\alpha_j) \epsilon_j^{-1} \tau_{jk} \tau_{jl} \tilde{z}_k \bar{\tilde{z}}_l + \frac{\epsilon_j^{-1} b_j}{2} \sum_{k,l=1}^n \frac{\partial^2 R}{\partial \tilde{z}_k \partial \tilde{z}_l}(\alpha_j) \tau_{jk} \tau_{jl} \tilde{z}_k \bar{\tilde{z}}_l + \dots = 0, \end{aligned}$$

where the dots denote remainder terms. Note that by Lemma 3.1 the terms with weight order greater than one must converge uniformly on compacta of \mathbb{C}^{n+1} to 0. Hence, we consider only the convergence of monomials from (4) with weight order ≤ 1 .

Since the sequence $\{\eta_j := \varphi_j(a)\}$ converges uniformly Λ -tangentially to $\xi_0 = (0', 0)$, it follows that

$$\frac{|\alpha_{j1}|^{2m_1}}{\epsilon_j} \approx \frac{|\alpha_{j2}|^{2m_2}}{\epsilon_j} \approx \dots \approx \frac{|\alpha_{jn}|^{2m_n}}{\epsilon_j}.$$

Thus Lemma 3.2 yields

$$\epsilon_j^{-1} \left| \frac{D^p \bar{D}^q P}{p!q!}(\alpha_j) \tau_j^{p+q} \right| \rightarrow 0$$

as $j \rightarrow \infty$ for $|p| + |q| > 2$ and, after taking a subsequence if necessary, we may assume that

$$a_{kl} := \frac{1}{2} \lim_{j \rightarrow \infty} \frac{\partial^2 P}{\partial \tilde{z}_k \partial \tilde{z}_l}(\alpha_j) \epsilon_j^{-1} \tau_{jk} \tau_{jl}, \quad 1 \leq k, l \leq n.$$

Moreover, by Lemma 3.3 we also have

$$\epsilon_j^{-1} \left| \frac{D^p \bar{D}^q R_1}{p!q!}(\alpha_j) \tau_j^{p+q} \right| \rightarrow 0$$

as $j \rightarrow \infty$ for $|p| + |q| \geq 2$. In addition, since $|\epsilon_j^{-1} b_j| \lesssim 1$, we obtain that

$$\epsilon_j^{-1} b_j \left| \frac{D^p \bar{D}^q R}{p!q!}(\alpha_j) \tau_j^{p+q} \right| \rightarrow 0$$

as $j \rightarrow \infty$ for $|p| + |q| \geq 1$. Therefore, after taking a subsequence if necessary, we may assume that sequence of defining functions given in (4) converges uniformly on

compacta of \mathbb{C}^{n+1} to $\hat{\rho}(\tilde{z}, \tilde{w}) := \operatorname{Re}(\tilde{w}) + H(\tilde{z})$. Consequently, the sequence of domains $\Omega_j := \Delta_j \circ Q_j \circ L_{\eta_j}(U_0^-)$ converges normally to the following model

$$M_H := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^n \times \mathbb{C} : \hat{\rho}(\tilde{z}, \tilde{w}) := \operatorname{Re}(\tilde{w}) + H(\tilde{z}) < 0\},$$

where

$$H(\tilde{z}) = \sum_{k,l=1}^n a_{kl} \tilde{z}_k \bar{\tilde{z}}_l.$$

Note that M_H is a limit of a sequence of the pseudoconvex domains $\Delta_j \circ Q_j \circ L_{\eta_j}(U_0^-)$. Hence, M_H is also pseudoconvex, and thus H is plurisubharmonic. In addition, it follows directly from Lemma 3.2 that H is positive definite. Therefore, M_H is biholomorphically equivalent to unit ball \mathbb{B}^{n+1} (cf. [Gra75, Prop. 2]).

For simplicity, let us denote by $T_j := \Delta_j \circ Q_j \circ L_{\eta_j}$ and $\sigma_j := T_j \circ \varphi_j : \varphi_j^{-1}(U_0^-) \rightarrow \Omega_j$. Then $T_j(\eta_j) = (0', -1 - i(R'_2(b_j) + R(\alpha_j)))$ and $\{\sigma_j\}$ is a sequence of biholomorphic mappings satisfying

$$\sigma_j(a) = b_j := (0', -1 - i(R'_2(b_j) + R(\alpha_j))) \rightarrow b := (0', -1).$$

as $j \rightarrow \infty$. Thus, by Proposition 2.1, after passing to a subsequence, we may assume that σ_j converges locally uniformly to a holomorphic map $\sigma : \Omega \rightarrow M_H$ which satisfies $\sigma(a) = b$.

On the other hand, since Ω is taut (cf. [DN09, Prop. 2.2]), the sequence $\sigma_j^{-1} : \Omega_j \rightarrow \varphi_j^{-1}(U_0^-) \subset \Omega$ is also normal. Since $\sigma_j^{-1}(b_j) = a \in \Omega$ with $b_j \rightarrow b \in \Omega$ as $j \rightarrow \infty$, we may also assume, after switching a subsequence, that σ_j^{-1} converges locally uniformly to a holomorphic map $\sigma^* : M_H \rightarrow \Omega$. It then follows from [DN09, Prop. 2.1] that Ω is biholomorphically equivalent to M_H . Hence, Ω is biholomorphically equivalent to \mathbb{B}^{n+1} , and thus the proof of Theorem 1.1 is finally complete. \square

3.3. Example. Denote by $E_{1,2,4}$ the domain in \mathbb{C}^3 , given by

$$E_{1,2,4} := \{(z_1, z_2, w) \in \mathbb{C}^3 : \operatorname{Re}(w) + |z_1|^4 + |z_1|^2|z_2|^4 + |z_2|^8 < 0\}.$$

Denote by $P(z) = |z_1|^4 + |z_1|^2|z_2|^4 + |z_2|^8$ and $\sigma(z) = |z_1|^4 + |z_2|^8$. Then a computation shows that

$$\begin{aligned} dd^c P(z) &= (4|z_1|^2 + |z_2|^4)dz_1d\bar{z}_1 + 2\bar{z}_1z_2|z_2|^2dz_1d\bar{z}_2 + 2z_1\bar{z}_2|z_2|^2d\bar{z}_1dz_2 \\ &\quad + (16|z_2|^6 + 4|z_1|^2|z_2|^2)dz_2d\bar{z}_2 \\ &= 4|z_1|^2dz_1d\bar{z}_1 + 16|z_2|^6dz_2d\bar{z}_2 + |z_2|^2|z_2d\bar{z}_1 + 2\bar{z}_1dz_2|^2 \\ &\geq dd^c \sigma(z). \end{aligned}$$

Therefore, the origin is strongly h -extendible with multitype $(4, 8, 1)$ and thus the weight Λ is now given by $\Lambda := (\frac{1}{4}, \frac{1}{8})$.

Now we consider the sequence $\{(\frac{1}{j^{1/4}}, \frac{1}{j^{3/8}}, -\frac{1}{j} - \frac{2}{j^2} - \frac{1}{j^3})\}$ that converges Λ -tangentially but not uniformly to $(0', 0)$. We are going to show that $\{\eta_j\}$ is not a sequence of automorphism orbits, that is, there do not exist a sequence $\{f_j\} \subset \operatorname{Aut}(E_{1,2,4})$ and $a \in E_{1,2,4}$ such that $f_j(a) = (\frac{1}{j^{1/4}}, \frac{1}{j^{3/8}}, -\frac{1}{j} - \frac{2}{j^2} - \frac{1}{j^3})$ for all $n \in \mathbb{N}^*$. Assume for the sake of seeing a contradiction that $\{f_j\}$ and a exist. Then, although we cannot apply a scaling given in the proof of Theorem 1.1, an alternative scaling can be introduced as follows. Indeed, let $\rho(z_1, z_2, w) = \operatorname{Re}(w) + |z_1|^4 + |z_1|^2|z_2|^4 + |z_2|^8$ and let $\eta_j = (\frac{1}{j^{1/4}}, \frac{1}{j^{3/8}}, -\frac{1}{j} - \frac{2}{j^2} - \frac{1}{j^3})$ for every $j \in \mathbb{N}^*$. We see that $\rho(\eta_j) = -\frac{1}{j^2} \approx -\operatorname{dist}(\eta_j, \partial E_{1,2,4})$.

Set $\epsilon_j = |\rho(\eta_j)| = \frac{1}{j^2}$. In addition, we consider a change of variables $(\tilde{z}, \tilde{w}) := L_j(z, w)$, i.e.,

$$\begin{cases} w = \tilde{w}; \\ z_1 - \frac{1}{j^{1/4}} = \tilde{z}_1; \\ z_2 - \frac{1}{j^{3/8}} = \tilde{z}_2. \end{cases}$$

Then, a direct calculation shows that

$$\begin{aligned} \rho \circ L_j^{-1}(\tilde{w}, \tilde{z}_1, \tilde{z}_2) &= \operatorname{Re}(w) + \left| \frac{1}{j^{1/4}} + \tilde{z}_1 \right|^4 + \left| \frac{1}{j^{1/4}} + \tilde{z}_1 \right|^2 \left| \frac{1}{j^{3/8}} + \tilde{z}_2 \right|^4 + \left| \frac{1}{j^{3/8}} + \tilde{z}_2 \right|^8 \\ &= \operatorname{Re}(w) + \frac{1}{j} + \frac{4}{j^{3/4}} \operatorname{Re}(\tilde{z}_1) + \frac{2}{j^{1/2}} |\tilde{z}_1|^2 + \frac{1}{j^{1/2}} (2\operatorname{Re}(\tilde{z}_1))^2 + \frac{4}{j^{1/4}} |\tilde{z}_1|^2 \operatorname{Re}(\tilde{z}_1) + |\tilde{z}_1|^4 \\ &\quad + \left(\frac{1}{j^{1/2}} + \frac{2}{j^{1/4}} \operatorname{Re}(\tilde{z}_1) + |\tilde{z}_1|^2 \right) \times \\ &\quad \left(\frac{1}{j^{3/2}} + \frac{4}{j^{9/8}} \operatorname{Re}(\tilde{z}_2) + \frac{2}{j^{3/4}} |\tilde{z}_2|^2 + \frac{1}{j^{3/4}} (2\operatorname{Re}(\tilde{z}_2))^2 + \frac{4}{j^{3/8}} |\tilde{z}_2|^2 \operatorname{Re}(\tilde{z}_2) + |\tilde{z}_2|^4 \right) + \left| \frac{1}{j^{3/8}} + \tilde{z}_2 \right|^8. \end{aligned}$$

To define an anisotropic dilation, let us denote by $\tau_{1j} := \tau_1(\eta_j) = \frac{1}{2j^{3/4}}$, $\tau_{2j} := \tau_2(\eta_j) = \frac{1}{j^{3/8}}$ for all $j \in \mathbb{N}^*$. Now let us introduce a sequence of polynomial automorphisms ϕ_{η_j} of \mathbb{C}^n ($j \in \mathbb{N}^*$), given by

$$\begin{aligned} &\phi_{\eta_j}^{-1}(\tilde{z}_1, \tilde{z}_2, \tilde{w}) \\ &= \left(\frac{1}{j^{1/4}} + \tau_{1j} \tilde{z}_1, \frac{1}{j^{3/8}} + \tau_{2j} \tilde{z}_2, -\frac{1}{j} - \frac{1}{j^2} - \frac{1}{j^3} + \epsilon_j \tilde{w} - \frac{4}{j^{3/4}} \tau_{1j} \tilde{z}_1 - \frac{2}{j^{1/2}} (\tau_{1j})^2 \tilde{z}_1^2 \right). \end{aligned}$$

Therefore, for each $j \in \mathbb{N}^*$ the hypersurface $\phi_{\eta_j}(\{\rho = 0\})$ is then defined by

$$\begin{aligned} &\epsilon_j^{-1} \rho \circ \phi_{\eta_j}^{-1}(\tilde{z}_1, \tilde{z}_2, \tilde{w}) \\ &= \epsilon_j^{-1} \rho \left(\frac{1}{j^{1/4}} + \tau_{1j} \tilde{z}_1, \frac{1}{j^{3/8}} + \tau_{2j} \tilde{z}_2, -\frac{1}{j} - \frac{1}{j^2} - \frac{1}{j^3} + \epsilon_j \tilde{w} - \frac{4}{j^{3/4}} \tau_{1j} \tilde{z}_1 - \frac{2}{j^{1/2}} (\tau_{1j})^2 \tilde{z}_1^2 \right) \\ &= \operatorname{Re}(\tilde{w}) + |\tilde{z}_1|^2 + \frac{1}{16j} |\tilde{z}_1|^4 + \frac{1}{2j^{1/4}} |\tilde{z}_1|^2 \operatorname{Re}(\tilde{z}_1) + (|\tilde{z}_2 + 1|^4 - 1) + O\left(\frac{1}{j^{1/2}}\right) + O\left(\frac{1}{j}\right) = 0. \end{aligned}$$

Hence, the sequence of domains $\Omega_j := \phi_{\eta_j}(E_{1,2,4})$ converges normally to the following model

$$M_{1,2} := \{(\tilde{z}_1, \tilde{z}_2, \tilde{w}) \in \mathbb{C}^3 : \operatorname{Re}(\tilde{w}) + |\tilde{z}_1|^2 + (|\tilde{z}_2 + 1|^4 - 1) < 0\}.$$

Finally, by the same argument as in the proof of Theorem 1.1 we conclude that $E_{1,2,2}$ is biholomorphically equivalent to $M_{1,2}$ that is biholomorphically equivalent to

$$\{(z_1, z_2, w) \in \mathbb{C}^3 : \operatorname{Re}(w) + |z_1|^2 + |z_2|^4 < 0\}.$$

It is absurd by [CP01, Main Theorem].

Remark 3.3. We consider a sequence $\left\{ \left(\frac{1}{j^{1/4}}, \eta_{j2}, -P\left(\frac{1}{j^{1/4}}, \eta_{j2}\right) - \frac{1}{j^2} \right) \right\}$ that converges Λ -tangentially but not uniformly to $(0', 0)$. Then, following the argument as in Example 3.3, we may assume that $\Omega_j := \phi_{\eta_j}(E_{1,2,4})$ converges normally to the following model

$$\left\{ (\tilde{z}_1, \tilde{z}_2, \tilde{w}) \in \mathbb{C}^3 : \operatorname{Re}(\tilde{w}) + |\tilde{z}_1|^2 + \tilde{P}(\tilde{z}_2) < 0 \right\},$$

where

- i) $\tilde{P}(\tilde{z}_2) = |\tilde{z}_2 + \alpha|^4 - |\alpha|^4$ for some $\alpha \in \mathbb{C}$ if $|\eta_{j2}| \approx \frac{1}{j^{3/8}}$;
- ii) $\tilde{P}(\tilde{z}_2) = |\tilde{z}_2|^4$ if $|\eta_{j2}| = o\left(\frac{1}{j^{3/8}}\right)$;
- iii) $\tilde{P}(\tilde{z}_2) = |\tilde{z}_2|^2$ if $\frac{1}{j^{3/8}} = o(|\eta_{j2}|)$.

This indicates that our model depends deeply on the behavior of the orbit $\{\eta_j\} \subset \Omega$.

4. THE BEHAVIOUR OF AUTOMORPHISM ORBITS ACCUMULATING AT A BOUNDARY POINT OF A PSEUDOCONVEX LEVI CORANK ONE DOMAINS IN \mathbb{C}^{n+1}

In this section, we are going to give a proof of Theorem 1.2. To do that, let Ω be a domain in \mathbb{C}^{n+1} such that $\partial\Omega$ is pseudoconvex of finite type and has corank one near ξ_0 .

4.1. Spherical $\frac{1}{2m}$ -tangential convergence. In what follows, let us write $z = (z_1, \dots, z_n)$ and $z^* = (0, z_2, \dots, z_n)$. Let $2m$ be the D'Angelo type of $\partial\Omega$ at ξ_0 . Without loss of generality, we may assume that ξ_0 and the rank of Levi form at ξ_0 is exactly $n - 1$. Let ρ be a smooth defining function for Ω . After an appropriate change of coordinates (cf. [BP91, Cho94]), we can find coordinate functions z_1, \dots, z_n, w defined on a neighborhood U_0 of ξ_0 such that $\xi_0 = 0$ and

$$\begin{aligned} \rho(z) = & \operatorname{Re}(w) + P(z_1, \bar{z}_1) + \sum_{\alpha=2}^n |z_\alpha|^2 + \sum_{\alpha=2}^n \operatorname{Re}(Q^\alpha(z_1, \bar{z}_1)z_\alpha) \\ & + O(|w| |(z, w)| + |z^*|^2 |z| + |z^*|^2 |z_1|^{m+1} + |z_1|^{2m+1}), \end{aligned}$$

where $P(z_1, \bar{z}_1), Q^\alpha(z_1, \bar{z}_1)$ ($2 \leq \alpha \leq n$) are homogeneous subharmonic real-valued polynomials of degree $2m$ and m , respectively, containing no harmonic terms.

Definition 4.1. We say that a sequence $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ with $\alpha_j = (\alpha_{j1}, \dots, \alpha_{jn})$, converges spherically $\frac{1}{2m}$ -tangentially to ξ_0 if

- (a) $|\operatorname{Im}(\beta_j)| \lesssim |\operatorname{dist}(\eta_j, \partial\Omega)|$;
- (b) $|\operatorname{dist}(\eta_j, \partial\Omega)| = o(|\alpha_{j1}|^{2m})$;
- (c) $\Delta P(\alpha_{j1}) \gtrsim |\alpha_{j1}|^{2m-2}$.

Remark 4.1. Let Ω be a pseudoconvex domain in \mathbb{C}^2 . Suppose that $\xi_0 \in \partial\Omega$ is of D'Angelo finite type, say, $\tau(\partial\Omega, \xi_0) = 2m$. It is known that Ω is h -extendible at ξ_0 . Let $\{\epsilon_j\} \subset \mathbb{R}^+$ be a sequence such that $\eta'_j := (\alpha_j, \beta_j + \epsilon_j)$ is in the hypersurface $\{\rho = 0\}$ for every $j \in \mathbb{N}^*$. Then the condition (c) simply says that Ω is strongly pseudoconvex at η'_j for every $j \in \mathbb{N}^*$. Consequently, the condition (c) is clearly satisfied if the model $M_P := \{(z, w) \in \mathbb{C}^2 : \operatorname{Re}(w) + P(z_1) < 0\}$ is a WB-domain.

4.2. Homogeneous subharmonic polynomials. Let us write $P(z) = \sum_{j=1}^{2m-1} a_j z^j \bar{z}^{2m-j}$.

Writing $z = |z|e^{i\theta}$, one defines $g(\theta)$ by

$$P(z) = |z|^{2m} g(\theta).$$

Then we have

$$\Delta P(z) = |z|^{2m-2} ((2m)^2 g(\theta) + g_{\theta\theta}(\theta)) \geq 0.$$

(See cf. [BF78].)

Given a sequence $\{\epsilon_j\} \subset \mathbb{R}^+$, we associate the sequence $\{\tau_j\}$ given by

$$\tau_j := \tau(\alpha_j, \epsilon_j) = |\alpha_j| \cdot \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{1/2}.$$

To give a proof of Theorem 1.2, it suffices to check the following lemma that is similar to Lemma 3.2.

Lemma 4.1. *We have*

$$\left| \frac{\partial^k P}{\partial z^l \partial \bar{z}^{k-l}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^k \lesssim \left(\frac{|\alpha_j|^{2m}}{\epsilon_j} \right)^{1-\frac{k}{2}}, \quad k \geq 3.$$

In addition, if $k = 2$ and $j = 1$, then

$$\left| \frac{\partial^2 P}{\partial z \partial \bar{z}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^2 = (2m)^2 g(\theta_j) + g_{\theta\theta}(\theta_j).$$

Proof. It suffices to prove the lemma for $P(z) = z^s \bar{z}^{2m-s}$ for $1 \leq s \leq 2m - 1$. Indeed, a simple computation shows that

$$\frac{\partial^k P}{\partial z^l \partial \bar{z}^{k-l}}(z) = \begin{cases} z^{s-l} \bar{z}^{2m-s-(k-l)} & \text{if } l \leq s, k-l \leq 2m-s \\ 0 & \text{if otherwise} \end{cases}$$

and

$$\begin{aligned} \left| \frac{\partial^k P}{\partial z^l \partial \bar{z}^{k-l}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^k &\lesssim |\alpha_j|^{2m-k} \epsilon_j^{-1} \tau_j^k = |\alpha_j|^{2m} \epsilon_j^{-1} \left(\frac{\tau_j}{|\alpha_j|} \right)^k \\ &\lesssim \frac{|\alpha_j|^{2m}}{\epsilon_j} \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{k/2} = \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{k/2-1}. \end{aligned}$$

Therefore, we get

$$\left| \frac{\partial^k P}{\partial z^l \partial \bar{z}^{k-l}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^k \lesssim \left(\frac{|\alpha_j|^{2m}}{\epsilon_j} \right)^{1-\frac{k}{2}}$$

for $k > 2$. Furthermore, in the case that $k = 2, l = 1$ one has

$$\begin{aligned} \left| \frac{\partial^2 P}{\partial z \partial \bar{z}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^2 &= |\alpha_j|^{2m-2} ((2m)^2 g(\theta_j) + g_{\theta\theta}(\theta_j)) |\alpha_j|^2 \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right) \\ &= (2m)^2 g(\theta_j) + g_{\theta\theta}(\theta_j). \end{aligned}$$

□

4.3. Proof of Theorem 1.2. Throughout this section, the domain Ω and the boundary point $\xi_0 \in \partial\Omega$ are assumed to satisfy the hypothesis of Theorem 1.2. Let $2m$ be the D'Angelo type of $\partial\Omega$ at ξ_0 . Without loss of generality, we may assume that $\xi_0 = 0 \in \mathbb{C}^n$ and the rank of Levi form at ξ_0 is exactly $n - 1$. Let ρ be a smooth defining function for Ω . After an appropriate change of coordinates (cf. [BP91, Cho94]), we can find the coordinate functions z_1, \dots, z_n, w defined on a neighborhood U_0 of ξ_0 such that $\xi_0 = 0$ and

$$\begin{aligned} \rho(z, w) &= \operatorname{Re}(w) + P(z_1, \bar{z}_1) + \sum_{\alpha=2}^n |z_\alpha|^2 + \sum_{\alpha=2}^n \operatorname{Re}(Q^\alpha(z_1, \bar{z}_1) z_\alpha) \\ &\quad + O(|w|(z, w) + |z^*|^2 |z| + |z^*|^2 |z_1|^{m+1} + |z_1|^{2m+1}), \end{aligned}$$

where $P(z_1, \bar{z}_1), Q^\alpha(z_1, \bar{z}_1)$ ($2 \leq \alpha \leq n$) are homogeneous subharmonic real-valued polynomials of degree $2m$ and m , respectively, containing no harmonic terms.

By hypothesis of Theorem 1.2, there exist a sequence $\{\varphi_j\} \subset \text{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\eta_j := \varphi_j(a)$ converges spherically $\frac{1}{2m}$ -tangentially to ξ_0 . Let us write $\eta_j = (\alpha_j, \beta_j) = (\alpha_j, \beta_j)$. Then one has

- (a) $|\text{Im}(\beta_j)| \lesssim |\text{dist}(\eta_j, \partial\Omega)|$;
- (b) $|\text{dist}(\eta_j, \partial\Omega)| = o(|\alpha_{j1}|^{2m})$;
- (c) $\Delta P(\alpha_{j1}) \gtrsim |\alpha_{j1}|^{2m-2}$.

Let us fix a small neighborhood U_0 of the origin. For any sequence $\{\eta_j = (\alpha_j, \beta_j)\}$ of points converging $\frac{1}{2m}$ -tangentially to the origin in $U_0 \cap \{\rho < 0\} =: U_0^-$, we associate with a sequence of points $\eta'_j = (\alpha_j, a_j + \epsilon_j + ib_j)$, where $\epsilon_j > 0$ and $\beta_j = a_j + ib_j$, such that $\eta'_j = (\alpha_j, \beta'_j)$ is in the hypersurface $\{\rho = 0\}$ for every $j \in \mathbb{N}^*$. We note that $\epsilon_j \approx \text{dist}(\eta_j, \partial\Omega)$.

By [Cho94, Proposition 2.2] (see also [DN09, Proposition 3.1]), for each point η'_j , there exists a biholomorphism $\Phi_{\eta'_j}$ of \mathbb{C}^{n+1} , $(z, w) = \Phi_{\eta'_j}^{-1}(\tilde{z}, \tilde{w})$, such that

$$(5) \quad \begin{aligned} \rho(\Phi_{\eta'_j}^{-1}(\tilde{z}, \tilde{w})) &= \text{Re}(\tilde{w}) + \sum_{\substack{k+l \leq 2m \\ k, l > 0}} a_{k,l}(\eta'_j) \tilde{z}_1^k \bar{\tilde{z}}_1^l \\ &+ \sum_{\alpha=2}^{n-1} |\tilde{z}_\alpha|^2 + \sum_{\alpha=2}^n \sum_{\substack{k+l \leq m \\ k, l > 0}} \text{Re}[(b_{k,l}^\alpha(\eta'_j)) \tilde{z}_1^k \bar{\tilde{z}}_1^l] \tilde{z}_\alpha \\ &+ O(|\tilde{w}| |(\tilde{z}, \tilde{w})| + |\tilde{z}^*|^2 |\tilde{w}| + |\tilde{z}^*|^2 |\tilde{z}_1|^{m+1} + |\tilde{z}_1|^{2m+1}), \end{aligned}$$

where $\tilde{z}^* = (0, \tilde{z}_2, \dots, \tilde{z}_n)$.

Now let us define

$$\tau_{j1} := |\alpha_j| \cdot \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{1/2}, \tau_{j2} = \dots = \tau_{jn} = \epsilon_j^{1/2}, j \geq 1.$$

This implies that

$$\epsilon_j^{1/2} \lesssim \tau(\eta_j, \epsilon_j) \lesssim \epsilon_j^{1/(2m)}.$$

To finish the scaling procedure, we define an anisotropic dilation Δ_j by

$$\Delta_j(z, w) = \left(\frac{z_1}{\tau_{j1}}, \frac{z_2}{\tau_{j2}}, \dots, \frac{z_n}{\tau_{jn}}, \frac{w}{\epsilon_j} \right), j \in \mathbb{N}^*.$$

This yields $\Delta_j \circ \Phi_{\eta'_j}(\eta_j) = (0', -1 + \gamma_j)$ for some sequence $\{\gamma_j\} \subset \mathbb{C}$, depending on $\{\Phi_{\eta'_j}\}$, that converges to 0 as $j \rightarrow \infty$. Furthermore, for each $j \in \mathbb{N}^*$, if we set $\rho_j(z, w) = \epsilon_j^{-1} \rho \circ \Phi_{\eta'_j}^{-1} \circ (\Delta_j)^{-1}(z, w)$, then (5) implies that

$$\rho_j(z, w) = \text{Re}(w) + P_{\eta'_j}(z_1, \bar{z}_1) + \sum_{\alpha=2}^n |z_\alpha|^2 + \sum_{\alpha=2}^n \text{Re}(Q_{\eta'_j}^\alpha(z_1, \bar{z}_1) z_\alpha) + O(\tau(\eta'_j, \epsilon_j)),$$

where

$$P_{\eta'_j}(z_1, \bar{z}_1) := \sum_{\substack{k,l \leq 2m \\ k,l > 0}} a_{k,l}(\eta'_j) \epsilon_j^{-1} \tau(\eta'_j, \epsilon_j)^{k+l} z_1^k \bar{z}_1^l,$$

$$Q_{\eta'_j}^\alpha(z_1, \bar{z}_1) := \sum_{\substack{k+l \leq m \\ k,l > 0}} b_{k,l}^\alpha(\eta'_j) \epsilon_j^{-1/2} \tau(\eta'_j, \epsilon_j)^{k+l} z_1^k \bar{z}_1^l.$$

Note that the sequence $\{\eta_j := \varphi_j(a)\}$ converges spherically $\frac{1}{2m}$ -tangentially to $\xi_0 = (0', 0)$. Then $\frac{|\alpha_{j1}|^{2m}}{\epsilon_j} \rightarrow +\infty$ as $j \rightarrow \infty$. In addition, we have, for $1 \leq k, l$ with $k+l \leq 2m$, that

$$a_{k,l}(\eta'_j) = \frac{\partial^{k+l} \rho}{\partial \bar{z}_1^k \partial \bar{z}_1^l}(0', 0) \approx \frac{\partial^{k+l} P}{\partial z_1^k \partial \bar{z}_1^l}(\alpha_{j1}).$$

Therefore, by Lemma 4.1 we get $a_{k,l}(\eta'_j) \epsilon_j^{-1} \tau(\eta'_j, \epsilon_j)^{k+l} \rightarrow 0$ as $j \rightarrow \infty$ for $k, l > 0$ with $2 < k+l \leq 2m$. In addition, by the condition (c), without loss of generality, we may assume that the limit $a := \lim_{j \rightarrow \infty} \frac{1}{2} \frac{\partial^2 P}{\partial z_1 \partial \bar{z}_1}(\alpha_{j1}) \epsilon_j^{-1} \tau_{j1}^2 > 0$ exists, and thus we conclude that $\{P_{\eta'_j}(z_1, \bar{z}_1)\}$ converges uniformly on compacta to $a|z_1|^2$.

For the sequences $\{Q_{\eta'_j}^\alpha(z_1, \bar{z}_1)\}$ ($2 \leq \alpha \leq n$), by [Cho94, Lemma 2.4] it follows that

$$|Q_{\eta'_j}^\alpha(z_1, \bar{z}_1)| \leq \tau(\eta'_j, \epsilon_j)^{\frac{1}{10}}, \quad j \geq 1,$$

for all $\alpha = 2, \dots, n$ and $|z_1| \leq 1$. Consequently, $\{Q_{\eta'_j}^\alpha\}$ converge uniformly on every compact subset of \mathbb{C} to 0. Therefore, after taking a subsequence if necessary, we may assume that the sequence $\{\hat{\rho}_j\}$ converges to the following function

$$\hat{\rho}(z, w) := \operatorname{Re}(w) + a|z_1|^2 + |z_2|^2 + \dots + |z_n|^2,$$

where $a = \frac{1}{2} \frac{\partial^2 P}{\partial z_1 \partial \bar{z}_1}(\alpha_{j1}) \epsilon_j^{-1} \tau_{j1}^2 > 0$. Hence, after taking a subsequence if necessary, we may assume that the sequence of domains $\Omega_j := \Delta_j \circ \Phi_{\eta'_j}(U_0^-)$ converges normally to the Siegel half-space

$$M_{|z|^2} := \{(z, w) \in \mathbb{C}^{n+1} : \hat{\rho}(z, w) = \operatorname{Re}(w) + a|z_1|^2 + |z_2|^2 + \dots + |z_n|^2 < 0\},$$

which is clearly biholomorphically equivalent to the unit ball \mathbb{B}^{n+1} (by using the Cayley transform). Moreover, by the same argument as in the proof of Theorem 1.1 we conclude that Ω is biholomorphically equivalent to $M_{|z|^2}$, or Ω is biholomorphically equivalent to \mathbb{B}^{n+1} . Hence, the proof of Theorem 1.2 is finally complete. \square

5. THE BEHAVIOUR OF AUTOMORPHISM ORBITS ACCUMULATING AT A BOUNDARY POINT OF A DOMAIN IN \mathbb{C}^2

In this section, we shall restrict the discussion to domains in \mathbb{C}^2 . More precisely, let Ω be a pseudoconvex domain of finite type near $\xi_0 \in \partial\Omega$ with the type $\tau(\partial\Omega, \xi_0) = 2m$. Then the notion of uniformly $(\frac{1}{2m})$ -tangential convergence given in Section 3 is just the that of $(\frac{1}{2m})$ -tangential convergence. In addition, the notion of spherically $\frac{1}{2m}$ -tangential convergence is exactly given in Section 4. Therefore, Corollary 1.3 follows directly from Theorem 1.2 and Lemma 4.1. In this situation our model is biholomorphically equivalent to the unit ball \mathbb{B}^2 .

In the sequel, we consider the case that the condition c) in Definition 4.1 is violated, i.e. $\frac{\Delta P(\alpha_j)}{|\alpha_j|^{2m-2}} \rightarrow 0$ as $j \rightarrow \infty$ for some sequence $\{\alpha_j\}$ converging to the origin in \mathbb{C} (see Definition 5.1 below). Then our model may be defined by a homogeneous polynomial of degree larger than 2.

5.1. Spherically $\frac{1}{2m}$ -tangentially convergence of higher order. Let ρ be a local defining function for Ω near ξ_0 and let the D'Angelo type $\tau(\partial\Omega, \xi_0) = 2m$ be finite. As in the proof of Theorem 1.1, we may assume that there are local holomorphic coordinates (z, w) in which $\xi_0 = 0$ and Ω can be described near 0 as follows:

$$\Omega = \{\rho(z, w) = \operatorname{Re}(w) + P(z) + R_1(z) + R_2(\operatorname{Im}w) + (\operatorname{Im}w)R(z) < 0\}.$$

Here P is homogeneous subharmonic real-valued polynomial of degree $2m$ containing no harmonic terms, $R_1 \in \mathcal{O}(1, \Lambda)$, $R \in \mathcal{O}(1/2, \Lambda)$ with $\Lambda = (\frac{1}{2m})$, and $R_2 \in \mathcal{O}(2)$.

Now let us write

$$P(z) = \sum_{l=1}^{2m-1} a_l z^l \bar{z}^{2m-l},$$

where $a_l = \overline{a_{l'}}$ if $l + l' = 2m$. Furthermore, in the polar coordinates $z = |z|e^{i\theta}$, for $l + l' \leq 2m$, we have

$$(6) \quad \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(z) = |z|^{2m-l-l'} g_{l,l'}(\theta); \quad \frac{\partial^{l+l'} R_1}{\partial z^l \partial \bar{z}^{l'}}(z) = |z|^{2m-l-l'} h_{l,l'}(|z|, \theta),$$

where $g_{l,l'}$ is a trigonometric polynomial of degree $2m - l - l'$ and $h_{l,l'}(|z|, \theta)$ is a \mathcal{C}^∞ -smooth function defined on \mathbb{C} with $h_{l,l'}(|z|, \theta) = O(|z|)$.

In analogy to Corollary 1.3, we discuss a situation where our domain is biholomorphically equivalent to some model defined by a homogeneous polynomial of degree larger than 2. To this purpose, the following variant of Definition 4.1 seems to be necessary.

Definition 5.1. A sequence $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ is said to converge spherically $\frac{1}{2m}$ -tangentially of order 2ν ($2 \leq \nu \leq m$) to $\xi_0 \in \partial\Omega$ if the following conditions hold:

- (i) $|\operatorname{Im}(\beta_j)| \leq \operatorname{dist}(\eta_j, \partial\Omega)$;
- (ii) $\operatorname{dist}(\eta_j, \partial\Omega) = o(|\alpha_j|^{2m})$;
- (iii) If $l + l' < 2\nu$, then

$$\lim_{j \rightarrow \infty} \left(\frac{\operatorname{dist}(\eta_j, \partial\Omega)}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}-1} (g_{l,l'}(\theta_j) + h_{l,l'}(|\alpha_j|, \theta_j)) = 0,$$

where $\theta_j := \arg(\alpha_j)$;

- (iv) There exists l_0, l'_0 with $l_0 + l'_0 = 2\nu$, $\max(l_0, l'_0) \geq 1$ such that

$$\liminf_{j \rightarrow \infty} |g_{l_0, l'_0}(\theta_j)| > 0.$$

We now assume that $\{\eta_j = (\alpha_j, \beta_j)\} \subset \Omega$ converges spherically $\frac{1}{2m}$ -tangentially of order 2ν ($2 \leq \nu \leq m$) to $\xi_0 \in \partial\Omega$. Then, for a given sequence $\{\epsilon_j\} \subset \mathbb{R}^+$ converging to 0, we define

$$\tau_j := |\alpha_j| \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{1}{2\nu}}, \quad j \geq 1.$$

With this notation, by Definition 5.1 we have the following lemma.

Lemma 5.1. *For any integers l, l' with $l, l' \geq 1$, we have*

- (a) $\lim_{j \rightarrow \infty} \frac{\partial^{l+l'}(P + R_1)}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} = 0$ for $l + l' < 2\nu$.
- (b) $\lim_{j \rightarrow \infty} \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} = 0$ for $l + l' > 2\nu$.
- (c) $\lim_{j \rightarrow \infty} \frac{\partial^{l+l'} R_1}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} = 0$ for $l + l' \geq 2\nu$.
- (d) $\left| \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^{l+l'} \lesssim 1$ for $l + l' = 2\nu$ and

$$\liminf_{j \rightarrow \infty} \left| \frac{\partial^{2\nu} P}{\partial z^{l_0} \partial \bar{z}^{2\nu-l_0}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^{2\nu} = \liminf_{j \rightarrow \infty} g_{l_0, 2\nu-l_0}(\theta_j) > 0.$$

Proof. Indeed, by a direct computation using (6) we obtain

$$\begin{aligned} \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} &= |\alpha_j|^{2m-l-l'} g_{l,l'}(\theta_j) \epsilon_j^{-1} |\alpha_j|^{l+l'} \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}} \\ &= \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}-1} g_{l,l'}(\theta_j) \end{aligned}$$

for $l, l' \geq 1$. Similarly, one also has

$$\frac{\partial^{l+l'} R_1}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} = \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}-1} h_{l,l'}(|\alpha_j|, \theta_j)$$

for $l, l' \geq 1$. Thus the assertion (a) follows directly from the condition (iii).

In the same fashion, for $l + l' \geq 2\nu$ we have

$$(7) \quad \begin{aligned} \left| \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^{l+l'} &\lesssim |\alpha_j|^{2m-l-l'} \epsilon_j^{-1} |\alpha_j|^{l+l'} \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}} \lesssim \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}-1}; \\ \left| \frac{\partial^{l+l'} R_1}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^{l+l'} &\lesssim |\alpha_j|^{2m+1-l-l'} \epsilon_j^{-1} |\alpha_j|^{l+l'} \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}} \lesssim |\alpha_j| \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{l+l'}{2\nu}-1}. \end{aligned}$$

This easily implies (c). Moreover, if $l + l' > 2\nu$, then we get

$$\lim_{j \rightarrow \infty} \frac{\partial^{l+l'} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} = 0,$$

and hence (b) follows.

Finally, it follows from (7) and the condition (iii) that we observe that every sequence $\left\{ \frac{\partial^{2\nu} P}{\partial z^l \partial \bar{z}^{l'}}(\alpha_j) \epsilon_j^{-1} \tau_j^{l+l'} \right\}_{j \geq 1}$ is bounded if $l + l' = 2\nu$, whereas

$$\liminf_{j \rightarrow \infty} \left| \frac{\partial^{2\nu} P}{\partial z^{l_0} \partial \bar{z}^{2\nu-l_0}}(\alpha_j) \right| \epsilon_j^{-1} \tau_j^{2\nu} = \liminf_{j \rightarrow \infty} g_{l_0, 2\nu-l_0}(\theta_j) > 0.$$

□

5.2. Proof of Proposition 1.4. This subsection is devoted to a proof of Proposition 1.4. Throughout this section, the domain Ω and the boundary point $\xi_0 \in \partial\Omega$ are assumed to satisfy the hypothesis of Proposition 1.4. Let ρ be a local defining function for Ω near ξ_0 and let the D'Angelo type $\tau(\partial\Omega, p) = 2m$ be finite. As in the proof of Theorem

1.1, we may assume that there are local holomorphic coordinates (z, w) in which $\xi_0 = 0$ and Ω can be described near 0 as follows:

$$\Omega = \{\rho(z, w) = \operatorname{Re}(w) + P(z) + R_1(z) + R_2(\operatorname{Im}w) + (\operatorname{Im}w)R(z) < 0\}.$$

Here P is homogeneous subharmonic real-valued polynomial of degree $2m$ containing no harmonic terms, $R_1 \in \mathcal{O}(1, \Lambda)$, $R \in \mathcal{O}(1/2, \Lambda)$ with $\Lambda = (\frac{1}{2m})$, and $R_2 \in \mathcal{O}(2)$.

By hypothesis of Proposition 1.4, there exist a sequence $\{\varphi_j\} \subset \operatorname{Aut}(\Omega)$ and a point $a \in \Omega$ such that $\eta_j := \varphi_j(a)$ converges spherically $\frac{1}{2m}$ -tangentially of order 2ν ($2 \leq k \leq m$) to ξ_0 .

Let us fix a small neighborhood U_0 of the origin. For any sequence $\{\eta_j = (\alpha_j, \beta_j)\}$ of points converging $\frac{1}{2m}$ -tangentially to the origin in $U_0 \cap \{\rho < 0\} =: U_0^-$, we associate with a sequence of points $\eta'_j = (\alpha_j, a_j + \epsilon_j + ib_j)$, where $\epsilon_j > 0$ and $\beta_j = a_j + ib_j$, such that $\eta'_j = (\alpha_j, \beta'_j)$ is in the hypersurface $\{\rho = 0\}$ for every $j \in \mathbb{N}^*$. We note that $\epsilon_j \approx \operatorname{dist}(\eta_j, \partial\Omega)$.

As in the proof of Theorem 1.1, let us consider the sequences of translations $L_{\eta'_j}$ and polynomial automorphisms Q_j of \mathbb{C}^2 ($j \in \mathbb{N}^*$), defined respectively by

$$L_{\eta'_j}(z, w) := (z, w) - \eta'_j = (z - \alpha_j, w - \beta'_j)$$

and

$$\begin{cases} w & := \tilde{w} + (R'_2(b_j) + R(\alpha_j))i\tilde{w} + 2 \sum_{k=1}^{2m} \frac{1}{k!} \frac{\partial^k P}{\partial z^k} \tilde{w}^k(\alpha_j) \\ & + 2 \sum_{k=1}^{2m} \frac{1}{k!} \frac{\partial^k R_1}{\partial z^k} \tilde{w}^k(\alpha_j) + 2b_j \sum_{k=1}^{2m} \frac{1}{k!} \frac{\partial^k R(\alpha_j)}{\partial z^k} \tilde{w}^k; \\ z & := \tilde{z}. \end{cases}$$

Then one sees that $Q_j \circ L_{\eta'_j}(\alpha_j, \beta_j) = (0, -\epsilon_j - i(R'_2(b_j) + R(\alpha_j))\epsilon_j)$ for every $j \in \mathbb{N}^*$. Moreover, the hypersurface $Q_j \circ L_{\eta'_j}(\{\rho = 0\})$ is defined by an equation of the form

$$\begin{aligned} \rho \left(L_{\eta'_j}^{-1} \circ Q_j^{-1}(z, w) \right) &= \operatorname{Re}(w) + o(|\operatorname{Im}(w)|) + \sum_{\substack{k+l \leq 2m \\ k, l > 0}} \frac{1}{k!l!} \frac{\partial^{k+l} P}{\partial z^k \partial \bar{z}^l}(\alpha_j) z^k \bar{z}^l \\ &+ \sum_{\substack{k+l \leq 2m \\ k, l > 0}} \frac{1}{k!l!} \frac{\partial^{k+l} R_1}{\partial z^k \partial \bar{z}^l}(\alpha_j) z^k \bar{z}^l + b_j \sum_{\substack{k+l \leq 2m \\ k, l > 0}} \frac{1}{k!l!} \frac{\partial^{k+l} R}{\partial z^k \partial \bar{z}^l}(\alpha_j) z^k \bar{z}^l + \dots = 0, \end{aligned}$$

where the dots denote remainder terms.

Now let us recall that

$$\tau_j := |\alpha_j| \cdot \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{1/2\nu}.$$

and then we define an anisotropic dilation Δ_j by

$$\Delta_j(z, w) := \left(\frac{z}{\tau_j}, \frac{w}{\epsilon_j} \right), \quad j \in \mathbb{N}^*.$$

This yields $\Delta_j \circ Q_j \circ L_{\eta'_j}(\alpha_j, \beta_j) = (0, -1 - i(R'_2(b_j) + R(\alpha_j))) \rightarrow (0, -1)$ as $j \rightarrow \infty$. By the definition of τ_j , a simple calculation shows that $\tau_j^{2m} = \epsilon_j \cdot \left(\frac{\epsilon_j}{|\alpha_j|^{2m}} \right)^{\frac{m}{\nu}-1} \lesssim \epsilon_j$. Hence, we get the following estimates

$$(8) \quad \epsilon_j^{1/2} \lesssim \tau_j \lesssim \epsilon_j^{1/2m}.$$

Furthermore, the hypersurface $\Delta_j \circ Q_j \circ L_{\eta'_j} (\{\rho = 0\})$ is defined by an equation of the form

$$(9) \quad \begin{aligned} \epsilon_j^{-1} \rho \left(L_{\eta_j}^{-1} \circ Q_j^{-1} \circ (\Delta_j)^{-1} (\tilde{z}, \tilde{w}) \right) &= \operatorname{Re}(\tilde{w}) + \sum_{\substack{k+l \leq 2m \\ k, l > 0}} \frac{1}{k!l!} \frac{\partial^{k+l}(P + R_1)}{\partial \tilde{z}^k \partial \tilde{z}^l} (\alpha_j) \epsilon_j^{-1} \tau_j^{k+l} \tilde{z}^k \tilde{z}^l \\ &+ \epsilon_j^{-1} b_j \sum_{\substack{k+l \leq 2m \\ k, l > 0}} \frac{1}{k!l!} \frac{\partial^{k+l} R}{\partial \tilde{z}^k \partial \tilde{z}^l} (\alpha_j) \epsilon_j^{-1} \tau_j^{k+l} \tilde{z}^k \tilde{z}^l + \epsilon_j^{-1} o(\epsilon_j |\operatorname{Im}(\tilde{w})|) + O(\tau(\eta_j, \epsilon_j)) = 0, \end{aligned}$$

where (8) yields the terms with weight greater than one equal $O(\tau(\eta_j, \epsilon_j))$. Hence, we consider only the convergence of monomials from (9) with weight order ≤ 1 .

Now thanks to Lemma 5.1, we have

$$\frac{\partial^{k+l}(P + R_1)}{\partial \tilde{z}^k \partial \tilde{z}^l} (\alpha_j) \epsilon_j^{-1} \tau_j^{k+l} \rightarrow 0$$

as $j \rightarrow \infty$ for $k + l \neq 2\nu$. For the case that $k + l = 2\nu$,

$$\frac{\partial^{2\nu} R_1}{\partial \tilde{z}^k \partial \tilde{z}^{2\nu-k}} (\alpha_j) \epsilon_j^{-1} \tau_j^{2\nu} \rightarrow 0$$

as $j \rightarrow \infty$. Moreover, since $|\epsilon_j^{-1} b_j| \lesssim 1$, we obtain that

$$\epsilon_j^{-1} b_j \frac{\partial^{k+l} R}{\partial \tilde{z}^k \partial \tilde{z}^l} (\alpha_j) \epsilon_j^{-1} \tau_j^{k+l} \rightarrow 0$$

as $j \rightarrow \infty$ for any $k, l \geq 1$. In addition, since every sequence $\left\{ \frac{\partial^{2\nu} P}{\partial \tilde{z}^k \partial \tilde{z}^{2\nu-k}} (\alpha_j) \epsilon_j^{-1} \tau_j^{2\nu} \right\}_{j \geq 1}$ is bounded if $k + l = 2\nu$, hence after taking a subsequence if necessary we may assume that

$$a_{k, 2\nu-k} := \frac{1}{k!(2\nu-k)!} \lim_{j \rightarrow \infty} \frac{\partial^{2\nu} P}{\partial \tilde{z}^k \partial \tilde{z}^{2\nu-k}} (\alpha_j) \epsilon_j^{-1} \tau_j^{2\nu}, 1 \leq k, l \leq n.$$

Therefore, after taking a subsequence if necessary, we may achieve that sequence of defining functions given in (9) converges uniformly on compacta of \mathbb{C}^2 to $\hat{\rho} := \operatorname{Re}(\tilde{w}) + Q(\tilde{z})$. Consequently, the sequence of domains $\Omega_j := \Delta_j \circ Q_j \circ L_{\eta'_j} (U_0^-)$ converges normally to the following model

$$M_Q := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^2 : \hat{\rho}(\tilde{z}, \tilde{w}) := \operatorname{Re}(\tilde{w}) + Q(\tilde{z}) < 0\},$$

where

$$Q(\tilde{z}) = \sum_{k=1}^{2\nu-1} a_{k, 2\nu-k} \tilde{z}^k \tilde{z}^{2\nu-k}.$$

Note that since M_Q is pseudoconvex, we infer that Q is subharmonic. Moreover, by the condition (iii), we have $a_{l_0, 2\nu-l_0} \neq 0$ for some $1 \leq l_0 \leq 2\nu - 1$, and hence the polynomial Q is not harmonic. Furthermore, by the same argument as in the proof of Theorem 1.1 we conclude that Ω is biholomorphically equivalent to M_Q . Hence, the proof of Proposition 1.4 is eventually complete. \square

5.3. The Kohn-Nirenberg type domains. The Kohn-Nirenberg domain Ω_{KN} is given firstly in [KN73] by

$$\Omega_{KN} := \left\{ (z, w) \in \mathbb{C}^2 : \operatorname{Re}(w) + |z|^8 + \frac{15}{7}|z|^2 \operatorname{Re}(z^6) < 0 \right\}$$

This domain is a weakly pseudoconvex domain of finite type having no supporting function at the origin. In addition, Ω_{KN} cannot be biholomorphically equivalent to a bounded domain in \mathbb{C}^2 with real analytic boundary. Indeed, if this would be the case, then [BP99] tells us that Ω_{KN} is biholomorphically equivalent to the ellipsoid

$$\{(z, w) \in \mathbb{C}^2 : |w|^2 + |z|^8 < 1\},$$

which is biholomorphically equivalent to the model $\{(z, w) \in \mathbb{C}^2 : \operatorname{Re}(w) + |z|^8 < 1\}$. This leads to a contradiction by [CP01, Main Theorem].

Furthermore, Ω_{KN} is a *WB*-domain and thus it satisfies the hypothesis of Theorem 1.3. Therefore, Theorem 1.3 shows that if a sequence of automorphism orbits $\{\eta_j = \varphi_j(a)\}$, $\{\varphi_j\} \subset \operatorname{Aut}(\Omega_{KN})$, converges to the origin, then it must converge $\frac{1}{8}$ -nontangentially to the origin.

The condition that η'_j is strongly pseudoconvex ensures that a limit of the scaling domains (the model M_H) is just biholomorphically equivalent to the unit ball \mathbb{B}^2 . If this condition is not satisfied such as $\Delta P(\alpha_j) = 0$ for all j , then the model now becomes $M_Q = \{(z, w) \in \mathbb{C}^2 : \operatorname{Re}(w) + Q(z) < 0\}$ for some homogeneous subharmonic polynomial Q with $\deg(Q) \geq 4$. The following example will demonstrate these phenomena. It also describes a situation when Proposition 1.4 may occur.

Example 5.1. Let $\tilde{\Omega}_{KN}$ be a domain in \mathbb{C}^2 defined by

$$\tilde{\Omega}_{KN} := \left\{ (z, w) \in \mathbb{C}^2 : \operatorname{Re}(w) + |z|^8 - \frac{16}{7}|z|^2 \operatorname{Re}(z^6) < 0 \right\}.$$

We see that the sequence $\left\{ \left(\frac{1}{\sqrt[8]{j}}, \frac{9}{7j} - \frac{1}{j^2} \right) \right\}$ converges $\frac{1}{8}$ -tangentially but not spherically $\frac{1}{8}$ -tangentially to $(0, 0)$.

Let $\rho(z, w) = \operatorname{Re}(w) + |z|^8 - \frac{16}{7}|z|^2 \operatorname{Re}(z^6)$ and let $\eta_j = \left(\frac{1}{\sqrt[8]{j}}, \frac{9}{7j} - \frac{1}{j^2} \right)$ for every $j \in \mathbb{N}^*$. This implies that $\rho(\eta_j) = \frac{9}{7j} - \frac{1}{j^2} - \frac{9}{7j} = -\frac{1}{j^2} \approx -\operatorname{dist}(\eta_j, \partial \tilde{\Omega}_{KN})$. Set $\epsilon_j = |\rho(\eta_j)| = \frac{1}{j^2}$.

Then, a computation shows that

$$\begin{aligned}
\rho(z, w) &= \operatorname{Re}(w) + \left| \left(z - \frac{1}{j^{1/8}} \right) + \frac{1}{j^{1/8}} \right|^8 - \frac{16}{7} \left| \left(z - \frac{1}{j^{1/8}} \right) + \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(\left(z - \frac{1}{j^{1/8}} \right) + \frac{1}{j^{1/8}} \right)^6 \\
&= \operatorname{Re}(w) + \frac{1}{j^{8/8}} + \frac{8}{j^{7/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right) + \frac{16}{j^{6/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 + \frac{12}{j^{6/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 \\
&+ \frac{8}{j^{5/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^3 + \frac{48}{j^{5/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right) \\
&+ \frac{2}{j^{4/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^4 + \frac{32}{j^4} \left| z - \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 + \frac{36}{j^{4/8}} \left| z - \frac{1}{j^{1/8}} \right|^4 \\
&- \frac{16}{7} \left(\frac{1}{j^{8/8}} + \frac{8}{j^{7/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right) + \frac{21}{j^{6/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 + \frac{7}{j^{6/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 \right) \\
&- \frac{16}{7} \left(\frac{35}{j^{5/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^3 + \frac{21}{j^{5/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right) \right) \\
&- \frac{16}{7} \left(\frac{21}{j^{4/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^4 + \frac{35}{j^{4/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 \right) + O \left(\frac{1}{j^{3/8}} \left| z - \frac{1}{j^{1/8}} \right|^5 \right) \\
&= \operatorname{Re}(w) - \frac{9}{7j^{8/8}} - \frac{72}{7j^{7/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right) - \frac{36}{j^{6/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 - \frac{72}{j^{5/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^3 \\
&- \frac{46}{j^{4/8}} \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^4 - \frac{48}{j^{4/8}} \left| z - \frac{1}{j^{1/8}} \right|^2 \operatorname{Re} \left(z - \frac{1}{j^{1/8}} \right)^2 \\
&+ \frac{36}{j^{4/8}} \left| z - \frac{1}{j^{1/8}} \right|^4 + O \left(\frac{1}{j^{3/8}} \left| z - \frac{1}{j^{1/8}} \right|^5 \right).
\end{aligned}$$

To define an anisotropic dilation, let us denote by $\tau_j := \tau(\eta_j) = \frac{1}{j^{3/8}}$ for all $j \in \mathbb{N}^*$. Now let us introduce a sequence of polynomial automorphisms $\phi_{\eta_j}^{-1}$ of \mathbb{C}^2 , given by

$$\begin{cases} z = \frac{1}{\sqrt[8]{j}} + \tau_j \tilde{z} \\ w = \epsilon_j \tilde{w} + \frac{9}{7j} + \frac{72}{7j^{7/8}} \tau_j \tilde{z} + \frac{36}{j^{6/8}} \tau_j^2 \tilde{z}^2 + \frac{72}{j^{5/8}} \tau_j^3 \tilde{z}^3 + \frac{46}{j^{4/8}} \tau_j^4 \tilde{z}^4. \end{cases}$$

Therefore, we have

$$\epsilon_j^{-1} \rho \circ \phi_{\eta_j}^{-1}(\tilde{z}, \tilde{w}) = \operatorname{Re}(\tilde{w}) + 36|\tilde{z}|^4 - 48|\tilde{z}|^2 \operatorname{Re}(\tilde{z}^2) + O\left(\frac{1}{j^{1/4}}\right).$$

We now show that there do not exist a sequence $\{f_j\} \subset \operatorname{Aut}(\tilde{\Omega}_{KN})$ and $a \in \tilde{\Omega}_{KN}$ such that $\eta_j = f_j(a) \rightarrow (0, 0) \in \partial\tilde{\Omega}_{KN}$ as $n \rightarrow \infty$. Indeed, suppose otherwise that there exist such a sequence $\{f_j\}$ and such a point $a \in \tilde{\Omega}_{KN}$. Then by the same argument as in the proof of Theorem 1.3, $\tilde{\Omega}_{KN}$ is biholomorphically equivalent to the following domain

$$D := \{(\tilde{z}, \tilde{w}) \in \mathbb{C}^2 : \operatorname{Re}(\tilde{w}) + 36|\tilde{z}|^4 - 48|\tilde{z}|^2 \operatorname{Re}(\tilde{z}^2) < 0\}.$$

However, since the D'Angelo type of ∂D is always less than or is equal to 4, it follows that D is not biholomorphically equivalent to $\tilde{\Omega}_{KN}$ (cf. [CP01, Main Theorem]). It is impossible.

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