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λ-TD ALGEBRAS, GENERALIZED SHUFFLE PRODUCTS AND LEFT COUNITAL HOPF ALGEBRAS

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Abstract. The theory of operated algebras has played a pivotal role in mathematics and physics. In this paper, we introduce a λ-TD algebra that appropriately includes both the Rota-Baxter algebra and the TD-algebra. The explicit construction of free commutative λ-TD algebra on a commutative algebra is obtained by generalized shuffle products, called λ-TD shuffle products. We then show that the free commutative λ-TD algebra possesses a left counital bialgera structure by means of a suitable 1-cocycle condition. Furthermore, the classical result that every connected filtered bialgebra is a Hopf algebra, is extended to the context of left counital bialgebras. Given this result, we finally prove that the left counital bialgebra on the free commutative λ-TD algebra is connected and filtered, and thus is a left counital Hopf algebra.

Contents

1. Introduction

An algebra with (one or more) linear operators, first appeared in [22] in the 1960s, is vital in the recent developments in a wide range of areas. The notion of an operated algebra (that is, an associative algebra with only one linear operator) was proposed by Guo for constructing the free Rota-Baxter algebra [18]. A **Rota-Baxter algebra of weight** λ (also called a λ**-Rota-Baxter algebra**) is an associative algebra R equipped with a linear operator $P : R \to R$ satisfying the **Rota-Baxter equation**

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy), \text{ for all } x, y \in R. \tag{1}$$

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Then P is called a **Rota-Baxter operator of weight** λ or a λ**-Rota-Baxter operator**, where λ is a constant. Derived from the work of Baxter on probability study [5], the Rota-Baxter operator is intimately connected with the classical Yang-Baxter equation [4], number theory [21], combinatorics [18, 30, 31, 33], and most conspicuously, renormalization in quantum field theory based on the Hopf algebra framework of Connes and Kreimer [10, 11].

On the other hand, the direct connection between Rota-Baxter algebras and dendriform algebras was first provided by Aguiar [2], who proved that every Rota-Baxter algebra of weight 0 naturally gives a dendriform algebra. Likewise, Ebrahimi-Fard and Guo showed that every Rota-Baxter algebra of non-zero weight λ carries a tridendriform algebra structure [9]. In order to offer another way to produce the tridendriform algebra, the TD operator, which can be formally viewed as an analog of the Rota-Baxter operator, was invented by Leroux [24]. A **TD operator** $P : R \to R$ is a linear operator satisfying the **TD equation**

$$P(x)P(y) = P(xP(y) + P(x)y - xP(1)y), \quad \text{for all } x, y \in R. \tag{2}$$

Note that when λ takes $-P(1)$ in Eq. (1), the Rota-Baxter opeator becomes a TD operator. From this viewpoint, the TD operator can be considered as a special class of Rota-Baxter operators. Indeed, the TD operator belongs to the category of Rota-Baxter type operator [36], which was proposed for solving the Rota's problem of classifying all linear operators on an associative algebra.

In recent years, classical operators, such as differential operators, Rota-Baxter operators and TD operators, are generalized in diverse ways for developing the various algebraic structures and phenomena [20, 36]. For instance, The λ-different operator was introduced by Guo and Keigher in [20] for uniformly studying the algebraic structure with both a differential operator and a difference operator, and for the same reason the λ-differential Rota-Baxter operator was discovered. Subsequently, in [1], the concept of **Rota-Baxter Nijenhuis TD operaotrs** or **RBNTD operators** was represented as a combination of Rota-Baxter operator, Nijenhuis operator and TD operator, giving rise to a RBNTD-dendriform algebra and a five-part splitting of associativity. The Rota-Baxter Nijenhuis TD operator is defined by the **Rota-Baxter Nijenhuis TD equation**

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy - P(xy) - xP(1)y), \qquad \text{for all } x, y \in R. \tag{3}$$

Lately, Zhou and Guo [38] introduced the concept of a **Rota-Baxter TD operator**, given by the **Rota-Baxter TD equation**

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy - xP(1)y - xyP(1)), \qquad \text{for all } x, y \in R. \tag{4}$$

As a consequence, a Rota-Baxter TD operator also gives a five-part splitting of the associativity and induces a quinquedendriform algebra structure. One can see at once that every TD operator contains the Rota-Baxter operator by taking $P(1) = 0$. But if we require $\lambda = 0$, the Rota-Baxter TD operator is not a TD operator in general. In this work, we will mainly develop a more appropriate fusion of a Rota-Baxter operator and a TD operator, called a λ**-TD operator** or a **TD operator of weight** λ (See Definition 2.1).

It is well-known that one of the most meaningful examples of Hopf algebras for applications in mathematical physics is the Connes-Kreimer Hopf algebra of rooted forests [6, 7], whose coproduct satisfies the 1-cocycle property. Lately, Hopf algebraic structures on the free non-commutative Rota-Baxter algebra of decorated rooted forests has been achieved by the same way [34]. Furthermore, it is worth mentioning that the explicit constructions of free non-commutative TD algebras and free non-commutative Rota-Baxter TD algebras were also accomplished by using the rooted trees [38, 39]. Based on the construction of shuffle product Hopf algebras, a Hopf algebra structure was established on the free commutative (modified) Rota-Baxter algebra by means of various generalized shuffle products [3, 8, 19, 35]. Motivated by this, in [15, 37], the Hopf algebraic structure on free commutative and non-commutative Nijenhuis algebras was considered spontaneously. However, it turns out that this method can not produce a genuine Hopf algebra again, only one with a left-sided counit and right-sided antipode. Such Hopf-type algebra is called a **left counital Hopf algebra** (See Definition 4.4) to distinguish between this Hopf algebra and the usual Hopf algebra. Intriguingly, algebra structures associated with it have already occurred in the study of quantum group [16, 29] and combinatorics [13, 14]. See [25, 32] for other variants of Hopf algebra under more weaker conditions.

Thanks to Rota-Baxter operators, TD operators and Nijenhuis operators [23, 26] sharing analogous properties and similar applications, it is reasonable to speculate that the free λ-TD algebras should possess a weakened form of Hopf algebra structure. In this paper, we primarily aim to equip the free commutative λ-TD algebra with a left counital Hopf algebra structure.

The paper is organized as follows. First of all, in Section 2, we give the concept of λ-TD algebras and then provide some general properties of λ-TD algebras in parallel to that of λ-Rota-Baxter algebras [17]. Then we combine the quasi-shuffle product and the left-shift shuffle product [12] together, thus yielding a λ-TD shuffle product. This allows us to construct the free commutative λ-TD algebra on a commutative algebra. In Section 3, we recall the concepts of left counital coalgebra and left counital bialgebra. Then applying a proper 1-cocycle property gives a coproduct on the free commutative λ-TD algebra. Afterwards, a left counit is also defined on it. Thus the free commutative λ-TD algebra possesses a left counital bialgebra. Finally in Section 4, we first prove that every connected filtered left counital operated bialgebra is a Hopf algebra. We then show that the aforementioned left counital bialgebra on the free commutative λ-TD algebra satisfies the connectedness and has an increasing filtration, and thus leads to a left counital Hopf algebra.

Convention. In this paper, all algebras are taken to be unitary commutative over a unitary commutative ring $\mathbf{k}$ unless otherwise specified. Also linear maps and tensor products are taken over $\mathbf{k}$.

2. Free commutative λ-TD algebras on a commutative algebra

In this section, we first present a more opportune combination of Rota-Baxter algebras and TD algebras, which can be regarded as one of Rota-Baxter type algebras. Then the general properties of λ-TD lagebas are developed. The construction of free commutative λ-TD algebras will be given by the λ-TD shuffle product, as a generalization of shuffle product [12].

2.1. General properties of λ-TD algebras.

Definition 2.1. Let $\lambda \in \mathbf{k}$. A **λ-TD algebra** is an algebra R equipped with a linear operator P, called a **λ-TD operator**, satisfying the **λ-TD equation**:

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy - xP(1)y), \quad \text{for all } x, y \in R. \tag{5}$$

Formally, every λ-TD-operator can be obtained from a Rota-Baxter operator by adding the last term $-P(xP(1)y)$ on the right hand side of Eq. (2) to the right hand side of Eq. (1). Note that every TD operator is a 0-TD operator. From this viewpoint, a λ-TD operator can be viewed as a natural generalization of TD-operator. Furthermore, we have

Proposition 2.2. *Let P be a linear operator on an algebra R. Let $\lambda \in \mathbf{k}$ be given.*

(a) *If $P(1) = 0$, then P is a λ-TD operator if and only if P is a Rota-Baxter operator of weight λ.*
(b) *If $P(1) = \lambda$, then P is a λ-TD operator if and only if P is a Rota-Baxter operator of weight 0.*
(c) *If $P(1) = 2\lambda$, then P is a λ-TD operator if and only if P is a Rota-Baxter operator of weight $-\lambda$.*

Proof. Items (a), (b) and (c) follow from Eqs. (1) and (5). □

By Item (c) and [38, Proposition 2.2], if a λ-TD operator P satisfies $P(1) = 2\lambda$, then P is a λ-RBTD operator. By [17, Proposition 1.1.12], a λ-Rota-Baxter operator P leads to another λ-Rota-Baxter operator $-\lambda\mathrm{id} - P$. However, it is not necessarily true that if P is a λ-TD operator, so is $-\lambda\mathrm{id} - P$. From the λ-TD equation, we obtain

Proposition 2.3. *Let P is a linear operator on a $\mathbf{k}$-algebra R. Then P is a λ-TD operator if and only if $-P$ is a $-\lambda$-TD operator.*

Definition 2.4. Let P be a linear operator on R. Then P is called a **λ-modified TD operator** if P satisfies the **λ-modified TD equation**

$$P(x)P(y) = P(xP(y) + P(x)y + \lambda xy) - xP(1)y. \tag{6}$$

In this case, we call (R, P) a **λ-modified TD algebra**.

Every Rota-Baxter algebra contains naturally a double structure, which is intimately related to the splitting of associativity in algebras such as Loday type algebras, including dendriform algebras and tridendriform algebras [17]. To explore this structure on a λ-TD algebra (R, P), we define another operation $*_\lambda$ on R, given by

$$x *_\lambda y = xP(y) + P(x)y + \lambda xy - xP(1)y. \tag{7}$$

We can prove that $(R, *_\lambda, P)$ is not a λ-TD algebra by a direct calculation, that is, a λ-TD algebra does not have the double structure in general. But if the λ-TD operator P satisfies $P^2 = P$, then $(R, *_\lambda, P)$ is a λ-TD algebra. Furthermore, we obtain the following observation.

Proposition 2.5. *Let (R, P) be a λ-TD algebra. Then*

(a) *The pair $(R, *_\lambda)$ is a nonunitary associative algebra;*
(b) *The triple $(R, *_\lambda, P)$ is a λ-modified TD algebra.*

Proof. (a) follows from [36, Proposition 2.37].
(b) By Eqs. (5) and (7), we obtain

$$\begin{aligned} P(x) *_\lambda P(y) &= P(x)P^2(y) + P^2(x)P(y) + \lambda P(x)P(y) - P(x)P(1)P(y) \\ &= P(x)P^2(y) + P^2(x)P(y) + \lambda P(x)P(y) - (P^2(x) + \lambda P(x))P(y) \\ &= P(x)P^2(y) \\ &= P(P(x)P(y)). \end{aligned}$$

On the other hand, by Item (a) and Eqs. (5) and (7), we get

$$\begin{aligned} & P\Big(x *_\lambda P(y) + P(x) *_\lambda y + \lambda x *_\lambda y\Big) - x *_\lambda P(1) *_\lambda y \\ =\ & P\Big(P(x)P(y) + P(x)P(y)\Big) + \lambda P(x)P(y) - (P^2(x)P(y) + \lambda P(x)P(y)) \end{aligned}$$

$$\begin{aligned} &= P(P(x)P(y) + P(x)P(y)) - P(P(x)P(y)) \\ &= P(P(x)P(y)). \end{aligned}$$

This gives

$$P(x) *_\lambda P(y) = P\Big(x *_\lambda P(y) + P(x) *_\lambda y + \lambda x *_\lambda y\Big) - x *_\lambda P(1) *_\lambda y,$$

proving Item (b). □

2.2. **The construction of free commutative λ-TD algebras.** The explicit construction of free commutative TD algebras on a commutative algebra A was carried out in [12] by using generalized shuffle products. This section will investigate the construction of the free commutative λ-TD algebra on A by another generalized shuffle product. We first give the notion of the free commutative λ-TD algebra on a commutative algebra.

Definition 2.6. Let A be a commutative algebra. A free commutative λ-TD algebra on A is a commutative λ-TD algebra $F_{\mathrm{L}}(A)$ with a λ-TD operator P_L and an algebra homomorphism $j_A : A \to F_{\mathrm{L}}(A)$ such that, for any commutative λ-TD algebra (R, P) and any algebra homomorphism $f : A \to R$, there is a unique λ-TD algebra homomorphism $\bar{f} : F_{\mathrm{L}}(A) \to R$ such that $f = \bar{f} \circ j_A$, that is, the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\ j_A\ } & F_{\mathrm{L}}(A) \\ & \searrow^{f} & \downarrow{\scriptstyle \bar{f}} \\ & & R \end{array}$$

commutes.

For a given unital commutative algebra A with unit 1_A, the free commutative Rota-Baxter algebra on A is given by the quasi-shuffle or mixable shuffle product in [17], and the free commutative TD algebra on A is given by the left-shift shuffle product in [12].

Let

$$\text{Ш}^+(A) := \bigoplus_{n\geq 0} A^{\otimes n}.$$

Here $A^{\otimes n}$ is the n-th tensor power of A with the convention that $A^{\otimes 0} = \mathbf{k}$. We next generalize the quasi-shuffle product and left-shift shuffle product by combining them together. For $\mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \cdots \otimes b_n \in A^{\otimes n}$ with $m, n \geq 0$, denote $\mathfrak{a}' = a_2 \otimes \cdots \otimes a_m$ if $m \geq 1$ and $\mathfrak{b}' = b_2 \otimes \cdots \otimes b_n$ if $n \geq 1$, so that $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Define a binary operation ш_r on $\text{Ш}^+(A)$ as follows. If $m = 0$ or $n = 0$, that is, $\mathfrak{a} = c \in \mathbf{k}$ or $\mathfrak{b} = c \in \mathbf{k}$, we define $\mathfrak{a}\text{ш}_r\mathfrak{b}$ to be the scalar product: $\mathfrak{a}\text{ш}_r\mathfrak{b} = c\mathfrak{b}$ or $\mathfrak{a}\text{ш}_r\mathfrak{b} = c\mathfrak{a}$. If $m \geq 1$ and $n \geq 1$, we define

$$\mathfrak{a}\text{ш}_r\mathfrak{b} = a_1 \otimes (\mathfrak{a}'\text{ш}_r\mathfrak{b}) + b_1 \otimes (\mathfrak{a}\text{ш}_r\mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}'\text{ш}_r\mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}'\text{ш}_r 1_A)\text{ш}_r\mathfrak{b}'). \tag{8}$$

Then we extend the product of two pure tensors to a binary operation on $\text{Ш}^+(A)$ by bilinearity, called the **λ-TD shuffle product**.

Example 2.1. Let $\mathfrak{a} = a_1$ and $\mathfrak{b} = b_1 \otimes b_2$. Then

$$\begin{aligned} \mathfrak{a}\text{ш}_r\mathfrak{b} &= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes (a_1\text{ш}_r b_2) + \lambda a_1 b_1 \otimes b_2 - a_1 b_1 \otimes (1_A\text{ш}_r b_2) \\ &= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes (a_1 \otimes b_2 + b_2 \otimes a_1 + \lambda a_1 b_2 - a_1 b_2 \otimes 1_A) \\ &\quad + \lambda a_1 b_1 \otimes b_2 - a_1 b_1 \otimes (1_A \otimes b_2 + b_2 \otimes 1_A + \lambda b_2 - b_2 \otimes 1_A) \\ &= a_1 \otimes b_1 \otimes b_2 + b_1 \otimes a_1 \otimes b_2 + b_1 \otimes b_2 \otimes a_1 + \lambda b_1 \otimes a_1 b_2 \\ &\quad - b_1 \otimes a_1 b_2 \otimes 1_A - a_1 b_1 \otimes 1_A \otimes b_2. \end{aligned}$$

We next give some properties of λ-TD shuffle product $ш_r$ for proving that it satisfies the commutativity and associativity.

Lemma 2.7. Let $\mathfrak{a} = a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$. Then

$$1_A ш_r \mathfrak{a} = \mathfrak{a} ш_r 1_A = \begin{cases} c1_A, & \text{if } \mathfrak{a} = c \in A^{\otimes 0}; \\ 1_A \otimes \mathfrak{a} + \lambda \mathfrak{a}. & \text{if } \mathfrak{a} \in A^{\otimes n} \text{ for } n \geq 1. \end{cases} \tag{9}$$

Proof. Let $\mathfrak{a} \in A^{\otimes n}$. For $n = 0$, let $\mathfrak{a} = c \in \mathbf{k}$. Then $1_A ш_r \mathfrak{a} = c1_A = \mathfrak{a} ш_r 1_A$ by the definition of $ш_r$. For $n \geq 1$, we let $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, where $\mathfrak{a}' = a_2 \otimes \cdots \otimes a_n$. Then by Eq. (8)

$$\begin{aligned} 1_A ш_r \mathfrak{a} &= 1_A ш_r (a_1 \otimes \mathfrak{a}') \\ &= 1_A \otimes a_1 \otimes \mathfrak{a}' + a_1 \otimes (1_A ш_r \mathfrak{a}') + \lambda a_1 \otimes \mathfrak{a}' - a_1 \otimes (1_A ш_r \mathfrak{a}') \\ &= 1_A \otimes \mathfrak{a} + \lambda \mathfrak{a}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathfrak{a} ш_r 1_A &= (a_1 \otimes \mathfrak{a}') ш_r 1_A \\ &= a_1 \otimes (\mathfrak{a}' ш_r 1_A) + 1_A \otimes a_1 \otimes \mathfrak{a}' + \lambda a_1 \otimes \mathfrak{a}' - a_1 \otimes (\mathfrak{a}' ш_r 1_A) \\ &= 1_A \otimes \mathfrak{a} + \lambda \mathfrak{a}. \end{aligned}$$

Thus Eq. (9) follows. □

Lemma 2.8. Let $\mathfrak{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes b_2 \otimes \cdots \otimes b_n \in A^{\otimes n}$ for $m, n \geq 1$. Then

$$(1_A \otimes \mathfrak{a}) ш_r \mathfrak{b} = \mathfrak{a} ш_r (1_A \otimes \mathfrak{b}). \tag{10}$$

Proof. We prove the claim by induction on $m + n \geq 2$. For $m + n = 2$, we have $m = n = 1$, and so

$$\begin{aligned} (1_A \otimes a_1) ш_r b_1 &= 1_A \otimes (a_1 ш_r b_1) + b_1 \otimes 1_A \otimes a_1 + \lambda b_1 \otimes a_1 - b_1 \otimes (a_1 ш_r 1_A) \quad \text{(by Eq. (8))} \\ &= 1_A \otimes (a_1 ш_r b_1) + b_1 \otimes 1_A \otimes a_1 + \lambda b_1 \otimes a_1 - b_1 \otimes (1_A \otimes a_1 + \lambda a_1) \quad \text{(by Eq. (9))} \\ &= 1_A \otimes (a_1 ш_r b_1). \end{aligned}$$

Likewise,

$$\begin{aligned} a_1 ш_r (1_A \otimes b_1) &= a_1 \otimes 1_A \otimes b_1 + 1_A \otimes (a_1 ш_r b_1) + \lambda a_1 \otimes b_1 - a_1 \otimes (1_A ш_r b_1) \quad \text{(by Eq. (8))} \\ &= a_1 \otimes 1_A \otimes b_1 + 1_A \otimes (a_1 ш_r b_1) + \lambda a_1 \otimes b_1 - a_1 \otimes (1_A \otimes b_1 + \lambda b_1) \quad \text{(by Eq. (9))} \\ &= 1_A \otimes (a_1 ш_r b_1). \end{aligned}$$

Thus Eq. (10) follows. Assume that the claim holds for $m + n \leq k$ with $k \geq 2$. Consider $m + n = k + 1$. Let $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. By Eqs. (8) and (9), we get

$$\begin{aligned} (1_A \otimes \mathfrak{a}) ш_r \mathfrak{b} &= 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}) + b_1 \otimes ((1_A \otimes \mathfrak{a}) ш_r \mathfrak{b}') + \lambda b_1 \otimes (\mathfrak{a} ш_r \mathfrak{b}') - b_1 \otimes ((\mathfrak{a} ш_r 1_A) ш_r \mathfrak{b}') \\ &= 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}) + b_1 \otimes ((1_A \otimes \mathfrak{a}) ш_r \mathfrak{b}') + \lambda b_1 \otimes (\mathfrak{a} ш_r \mathfrak{b}') - b_1 \otimes ((1_A \otimes \mathfrak{a} + \lambda \mathfrak{a}) ш_r \mathfrak{b}') \\ &= 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}). \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathfrak{a} ш_r (1_A \otimes \mathfrak{b}) &= a_1 \otimes (\mathfrak{a}' ш_r (1_A \otimes \mathfrak{b})) + 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}) + \lambda a_1 \otimes (\mathfrak{a}' ш_r \mathfrak{b}) - a_1 \otimes ((\mathfrak{a}' ш_r 1_A) ш_r \mathfrak{b}) \\ &= a_1 \otimes (\mathfrak{a}' ш_r (1_A \otimes \mathfrak{b})) + 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}) + \lambda a_1 \otimes (\mathfrak{a}' ш_r \mathfrak{b}) - a_1 \otimes ((1_A \otimes \mathfrak{a}' + \lambda \mathfrak{a}') ш_r \mathfrak{b}) \\ &= a_1 \otimes (\mathfrak{a}' ш_r (1_A \otimes \mathfrak{b})) + 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}) - a_1 \otimes ((1_A \otimes \mathfrak{a}') ш_r \mathfrak{b}) \\ &= 1_A \otimes (\mathfrak{a} ш_r \mathfrak{b}). \quad \text{(by the induction hypothesis)} \end{aligned}$$

Then Eq. (10) holds. Induction on $m + n$ completes the proof of the claim. □

From the proof of the above lemma, we also obtain

$$(1_A \otimes \mathfrak{a})ш_r\mathfrak{b} = 1_A \otimes (\mathfrak{a}ш_r\mathfrak{b}), \quad \text{for all } \mathfrak{a} \in A^{\otimes m}, \mathfrak{b} \in A^{\otimes n}. \tag{11}$$

Lemma 2.9. The λ-TD shuffle product $ш_r$ on $Ш^+(A)$ is commutative.

Proof. It suffices to prove

$$\mathfrak{a}\, ш_r\, \mathfrak{b} = \mathfrak{b}\, ш_r\, \mathfrak{a}, \tag{12}$$

for all pure tensors $\mathfrak{a} := a_1 \otimes a_2 \otimes \cdots \otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} := b_1 \otimes b_2 \otimes \cdots \otimes b_n \in A^{\otimes n}$ with $m, n \geq 0$. Use induction on $m + n \geq 0$. If $m = 0$, or $n = 0$, then $\mathfrak{a} = c \in \mathbf{k}$, or $\mathfrak{b} = c \in \mathbf{k}$, and so $\mathfrak{a}ш_r\mathfrak{b} = \mathfrak{b}ш_r\mathfrak{a}$ by the definition of $ш_r$. If $m \geq 1$ and $n \geq 1$, we let $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ with $\mathfrak{a}' \in A^{\otimes(m-1)}$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ with $\mathfrak{b}' \in A^{\otimes(n-1)}$. Assume that Eq. (12) holds for $m + n \leq k$. Consider $m + n = k + 1$. Then by Eq. (8), we get

$$\mathfrak{a}\, ш_r\mathfrak{b} = a_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b}) + b_1 \otimes (\mathfrak{a}ш_r\mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b}') - a_1 b_1 \otimes ((\mathfrak{a}'ш_r 1_A)ш_r\mathfrak{b}').$$

By Eq. (9), we obtain $\mathfrak{a}'ш_r 1_A = 1_A \otimes \mathfrak{a}' + \lambda\mathfrak{a}'$, and then using the induction hypothesis, we have

$$\mathfrak{a}\, ш_r\mathfrak{b} = a_1 \otimes (\mathfrak{b}ш_r\mathfrak{a}') + b_1 \otimes (\mathfrak{b}'ш_r\mathfrak{a}) - a_1 b_1 \otimes (\mathfrak{b}'ш_r(1_A \otimes \mathfrak{a}')).$$

By Eq. (8) again, we get

$$\mathfrak{b}\, ш_r\mathfrak{a} = b_1 \otimes (\mathfrak{b}'ш_r\mathfrak{a}) + a_1 \otimes (\mathfrak{b}ш_r\mathfrak{a}') + \lambda b_1 a_1 \otimes (\mathfrak{b}'ш_r\mathfrak{a}') - b_1 a_1 \otimes ((\mathfrak{b}'ш_r 1_A)ш_r\mathfrak{a}').$$

Applying Eq. (9) gives $\mathfrak{b}'ш_r 1_A = 1_A \otimes \mathfrak{b}' + \lambda\mathfrak{b}'$, and so

$$\mathfrak{b}\, ш_r\mathfrak{a} = b_1 \otimes (\mathfrak{b}'ш_r\mathfrak{a}) + a_1 \otimes (\mathfrak{b}ш_r\mathfrak{a}') - b_1 a_1 \otimes ((1_A \otimes \mathfrak{b}')ш_r\mathfrak{a}').$$

Then Eq. (12) follows from the commutativity of A and Eq. (10). This completes the induction and the proof of the lemma. □

Lemma 2.10. The λ-TD shuffle product $ш_r$ on $Ш^+(A)$ is associative.

Proof. To show that the associativity of $ш_r$, we need only prove

$$(\mathfrak{a}ш_r\mathfrak{b})ш_r\mathfrak{c} = \mathfrak{a}ш_r(\mathfrak{b}ш_r\mathfrak{c}), \tag{13}$$

for all pure tensors $\mathfrak{a} \in A^{\otimes m}$, $\mathfrak{b} \in A^{\otimes n}$, $\mathfrak{c} \in A^{\otimes \ell}$ with $m, n, \ell \geq 0$. Use induction on $s := m+n+\ell \geq 0$. If one of m, n, ℓ is 0, then Eq. (13) is true by the definition of $ш_r$. This proves Eq. (13) for $0 \leq s \leq 2$. Assume that Eq. (13) holds for $s \leq k$ with $k \geq 2$, and consider $s = m+n+\ell = k+1$ with $m, n, \ell \geq 1$. Denote $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$, $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$, and $\mathfrak{c} = c_1 \otimes \mathfrak{c}'$ with $\mathfrak{a}' \in A^{\otimes(m-1)}$, $\mathfrak{b}' \in A^{\otimes(n-1)}$, $\mathfrak{c}' \in A^{\otimes(\ell-1)}$. Then we have

$$\begin{aligned}
(\mathfrak{a}ш_r\mathfrak{b})ш_r\mathfrak{c} &= \Big(a_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b})\Big)ш_r\mathfrak{c} + \Big(b_1 \otimes (\mathfrak{a}ш_r\mathfrak{b}')\Big)ш_r\mathfrak{c} \\
&\quad + \Big(\lambda a_1 b_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b}')\Big)ш_r\mathfrak{c} - \Big(a_1 b_1 \otimes \Big((\mathfrak{a}'ш_r 1_A)ш_r\mathfrak{b}'\Big)\Big)ш_r\mathfrak{c} \quad \text{(by Eq. (8))} \\
&= a_1 \otimes \Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r\mathfrak{c}\Big) + c_1 \otimes \Big(\Big(a_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b})\Big)ш_r\mathfrak{c}'\Big) + \lambda a_1 c_1 \otimes \Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r\mathfrak{c}'\Big) \\
&\quad - a_1 c_1 \otimes \Big(\Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r 1_A\Big)ш_r\mathfrak{c}'\Big) + b_1 \otimes \Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r\mathfrak{c}\Big) + c_1 \otimes \Big(\Big(b_1 \otimes (\mathfrak{a}ш_r\mathfrak{b}')\Big)ш_r\mathfrak{c}'\Big) \\
&\quad + \lambda b_1 c_1 \otimes \Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r\mathfrak{c}'\Big) - b_1 c_1 \otimes \Big(\Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r 1_A\Big)ш_r\mathfrak{c}'\Big) + \lambda a_1 b_1 \otimes \Big((\mathfrak{a}'ш_r\mathfrak{b}')ш_r\mathfrak{c}\Big) \\
&\quad + c_1 \otimes \Big(\Big(\lambda a_1 b_1 \otimes (\mathfrak{a}'ш_r\mathfrak{b}')\Big)ш_r\mathfrak{c}'\Big) + \lambda^2 a_1 b_1 c_1 \otimes \Big((\mathfrak{a}'ш_r\mathfrak{b}')ш_r\mathfrak{c}'\Big)
\end{aligned}$$

$$
\begin{aligned}
&-\lambda a_1b_1c_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)-a_1b_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r\mathfrak{c}\Big)\\
&-c_1\otimes\Big(\big(a_1b_1\otimes((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}')\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)-\lambda a_1b_1c_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)\\
&+a_1b_1c_1\otimes\Big(\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r 1_A\Big)\sqcup\!\sqcup_r\mathfrak{c}'\Big).
\end{aligned}
$$

Combining the second, sixth, tenth and fourteenth terms and by Eq. (8), we obtain

$$
\begin{aligned}
(\mathfrak{a}\sqcup\!\sqcup_r\mathfrak{b})\sqcup\!\sqcup_r\mathfrak{c} \;=\;& a_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b})\sqcup\!\sqcup_r\mathfrak{c}\big)+c_1\otimes\Big((\mathfrak{a}\sqcup\!\sqcup_r\mathfrak{b})\sqcup\!\sqcup_r\mathfrak{c}'\Big)+\lambda a_1c_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b})\sqcup\!\sqcup_r\mathfrak{c}'\big)\\
&-a_1c_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b})\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)+b_1\otimes\big((\mathfrak{a}\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r\mathfrak{c}\big)+\lambda b_1c_1\otimes\big((\mathfrak{a}\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r\mathfrak{c}'\big)\\
&-b_1c_1\otimes\Big(\big((\mathfrak{a}\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)+\lambda a_1b_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r\mathfrak{c}\big)+\lambda^2a_1b_1c_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r\mathfrak{c}'\big)\\
&-\lambda a_1b_1c_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r\mathfrak{b}')\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)-a_1b_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r\mathfrak{c}\Big)\\
&-\lambda a_1b_1c_1\otimes\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r\mathfrak{c}'\Big)+a_1b_1c_1\otimes\Big(\Big(\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{b}'\big)\sqcup\!\sqcup_r 1_A\Big)\sqcup\!\sqcup_r\mathfrak{c}'\Big).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}) \;=\;& \mathfrak{a}\sqcup\!\sqcup_r\big(b_1\otimes(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)+\mathfrak{a}\sqcup\!\sqcup_r\big(c_1\otimes(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)\\
&+\mathfrak{a}\sqcup\!\sqcup_r\big(\lambda b_1c_1\otimes(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}')\big)-\mathfrak{a}\sqcup\!\sqcup_r\Big(b_1c_1\otimes\big((\mathfrak{b}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\quad\text{(by Eq. (8))}\\
=\;& a_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\big(b_1\otimes(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)\Big)+b_1\otimes\big(\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)+\lambda a_1b_1\otimes\big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)\\
&-a_1b_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)+a_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\big(c_1\otimes(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)\Big)+c_1\otimes\big(\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)\\
&+\lambda a_1c_1\otimes\big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)-a_1c_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)\\
&+a_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\big(\lambda b_1c_1\otimes(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}')\big)\Big)+\lambda b_1c_1\otimes\Big(\mathfrak{a}\sqcup\!\sqcup_r\big(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\\
&+\lambda^2a_1b_1c_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\big(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)-\lambda a_1b_1c_1\otimes\Big(\big(\mathfrak{a}'\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\big(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\\
&-a_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\Big(b_1c_1\otimes\big((\mathfrak{b}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\Big)-b_1c_1\otimes\Big(\mathfrak{a}\sqcup\!\sqcup_r\big((\mathfrak{b}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\\
&-\lambda a_1b_1c_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r\big((\mathfrak{b}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)+a_1b_1c_1\otimes\Big(\big(\mathfrak{a}'\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\big((\mathfrak{b}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)
\end{aligned}
$$

Adding the first, fifth, ninth and thirteenth terms and then using Eq. (8) again, we have

$$
\begin{aligned}
\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}) \;=\;& a_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c})\Big)+b_1\otimes\big(\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)+\lambda a_1b_1\otimes\big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)\\
&-a_1b_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c})\big)+c_1\otimes\big(\mathfrak{a}\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)+\lambda a_1c_1\otimes\big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)\\
&-a_1c_1\otimes\big((\mathfrak{a}'\sqcup\!\sqcup_r 1_A)\sqcup\!\sqcup_r(\mathfrak{b}\sqcup\!\sqcup_r\mathfrak{c}')\big)+\lambda b_1c_1\otimes\Big(\mathfrak{a}\sqcup\!\sqcup_r\big(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)\\
&+\lambda^2a_1b_1c_1\otimes\Big(\mathfrak{a}'\sqcup\!\sqcup_r(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}')\Big)-\lambda a_1b_1c_1\otimes\Big(\big(\mathfrak{a}'\sqcup\!\sqcup_r 1_A\big)\sqcup\!\sqcup_r\big(\mathfrak{b}'\sqcup\!\sqcup_r\mathfrak{c}'\big)\Big)
\end{aligned}
$$

$$-b_1c_1\otimes\Big(\mathfrak{a}ш_r\Big((\mathfrak{b}'ш_r 1_A)ш_r\mathfrak{c}'\Big)\Big)-\lambda a_1b_1c_1\otimes\Big(\mathfrak{a}'ш_r\Big((\mathfrak{b}'ш_r 1_A)ш_r\mathfrak{c}'\Big)\Big)$$
$$+a_1b_1c_1\otimes\Big(\Big(\mathfrak{a}'ш_r 1_A\Big)ш_r\Big((\mathfrak{b}'ш_r 1_A)ш_r\mathfrak{c}'\Big)\Big).$$

Applying Eq. (9) and the induction hypothesis to the seventh term

$$-a_1c_1\otimes\Big((\mathfrak{a}'ш_r 1_A)ш_r(\mathfrak{b}ш_r\mathfrak{c}')\Big)$$

gives

$$-a_1c_1\otimes\Big(((\mathfrak{a}'ш_r\mathfrak{b})ш_r 1_A)ш_r\mathfrak{c}'\Big).$$

Then by the induction hypothesis, we obtain

$$\begin{aligned}\mathfrak{a}ш_r(\mathfrak{b}ш_r\mathfrak{c}) \;=\;& a_1\otimes\Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r\mathfrak{c}\Big)+b_1\otimes\Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r\mathfrak{c}\Big)+\lambda a_1b_1\otimes\Big((\mathfrak{a}'ш_r\mathfrak{b}')ш_r\mathfrak{c}\Big)\\ &-a_1b_1\otimes\Big(\Big((\mathfrak{a}'ш_r 1_A)ш_r\mathfrak{b}'\Big)ш_r\mathfrak{c}\Big)+c_1\otimes\Big((\mathfrak{a}ш_r\mathfrak{b})ш_r\mathfrak{c}'\Big)+\lambda a_1c_1\otimes\Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r\mathfrak{c}'\Big)\\ &-a_1c_1\otimes\Big(\Big((\mathfrak{a}'ш_r\mathfrak{b})ш_r 1_A\Big)ш_r\mathfrak{c}'\Big)+\lambda b_1c_1\otimes\Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r\mathfrak{c}'\Big)\\ &+\lambda^2a_1b_1c_1\otimes\Big((\mathfrak{a}'ш_r\mathfrak{b}')ш_r\mathfrak{c}'\Big)-\lambda a_1b_1c_1\otimes\Big(\Big(\Big(\mathfrak{a}'ш_r 1_A\Big)ш_r\mathfrak{b}'\Big)ш_r\mathfrak{c}'\Big)\\ &-b_1c_1\otimes\Big(\Big((\mathfrak{a}ш_r\mathfrak{b}')ш_r 1_A\Big)ш_r\mathfrak{c}'\Big)-\lambda a_1b_1c_1\otimes\Big(\Big((\mathfrak{a}'ш_r\mathfrak{b}')ш_r 1_A\Big)ш_r\mathfrak{c}'\Big)\\ &+a_1b_1c_1\otimes\Big(\Big(\Big((\mathfrak{a}'ш_r 1_A)ш_r\mathfrak{b}'\Big)ш_r 1_A\Big)ш_r\mathfrak{c}'\Big).\end{aligned}$$

Then the i-th term in the expansion of $(\mathfrak{a}ш_r\mathfrak{b})ш_r\mathfrak{c}$ matches with the $\sigma(i)$-th term in the expansion of $\mathfrak{a}ш_r(\mathfrak{b}ш_r\mathfrak{c})$, where the permutation $\sigma\in\Sigma_{13}$ is

$$\left(\begin{array}{c} i\\ \sigma(i)\end{array}\right)=\left(\begin{array}{ccccccccccccc}1&2&3&4&5&6&7&8&9&10&11&12&13\\ 1&5&6&7&2&8&11&3&9&12&4&10&13\end{array}\right).$$

This completes the proof. □

Proposition 2.11. *The triple* $(Ш^+(A), ш_r, 1_{\mathbf{k}})$ *forms a unitary commutative algebra.*

Proof. This follows from Lemma 2.9 and Lemma 2.10. □

We next construct the free object of the category of λ-TD algebras on a commutative algebra A. Let

$$Ш(A) := A\otimes Ш^+(A) = (A\otimes\mathbf{k})\oplus A^{\otimes 2}\oplus\cdots(\cong\bigoplus_{n\geq 1}A^{\otimes n}). \tag{14}$$

Here $A^{\otimes n}$ is the n-th tensor power of A.

We first recall the definition of the **right-shift operator** P_λ on $Ш(A)$. Let $\mathfrak{a} := a_0\otimes\mathfrak{a}'\in Ш(A)$ for $\mathfrak{a}'\in A^{\otimes n}$ and all $n\geq 0$. If $n=0$, we let $\mathfrak{a}'=c\in\mathbf{k}(=A^{\otimes 0})$. Define

$$P_\lambda : Ш(A)\to Ш(A),\quad \mathfrak{a}\mapsto 1_A\otimes\mathfrak{a},\ n\geq 1\quad\text{and}\quad \mathfrak{a}\mapsto 1_A\otimes ca_0,\ n=0. \tag{15}$$

We next define a multiplication $\diamond_\lambda$ on $Ш(A)$ as follows. For this purpose, we just need to define the product of two pure tensors and then to extend by bilinearity. For $\mathfrak{a} := a_0\otimes\mathfrak{a}'\in A\otimes A^{\otimes m}$ and $\mathfrak{b} := b_0\otimes\mathfrak{b}'\in A\otimes A^{\otimes n}$, we define

$$\mathfrak{a}\diamond_\lambda\mathfrak{b} = a_0b_0\otimes(\mathfrak{a}'ш_r\mathfrak{b}'), \tag{16}$$

where $ш_r$ is the λ-TD shuffle product defined in Eq. (8). Then the associativity and commutativity of $\diamond_\lambda$ follows from that of the multiplication in A and $ш_r$ in $Ш^+(A)$.

Alternatively, let $Ш(A) = \bigoplus_{n\geq 1} A^{\otimes n}$. For $\mathfrak{a} = a_1\otimes\cdots\otimes a_m \in A^{\otimes m}$ and $\mathfrak{b} = b_1\otimes\cdots\otimes b_n \in A^{\otimes n}, m, n \geq 1$, denote $\mathfrak{a}' = a_2 \otimes \cdots \otimes a_m$ if $m \geq 2$ and $\mathfrak{b}' = b_2 \otimes \cdots \otimes b_n$ if $n \geq 2$, so that $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$. Then $\diamond_\lambda$ on $Ш(A)$ can also be defined by the following recursion.

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = \begin{cases} a_1 b_1, & m = n = 1, \\ a_1 b_1 \otimes \mathfrak{b}', & m = 1, n \geq 2, \\ a_1 b_1 \otimes \mathfrak{a}', & m \geq 2, n = 1, \\ a_1 b_1 \otimes \Big(\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}') + (1_A \otimes \mathfrak{a}') \diamond_\lambda \mathfrak{b}' & \\ +\lambda \mathfrak{a}' \diamond_\lambda \mathfrak{b}' - \Big(\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)\Big) \diamond_\lambda \mathfrak{b}'\Big), & m, n \geq 2. \end{cases} \tag{17}$$

Let

$$j_A : A \to Ш(A),\ a \mapsto a,$$

be the natural embedding. Then

$$j_A(ab) = ab = a \diamond_\lambda b = j_A(a) \diamond_\lambda j_A(b), \text{ for all } a, b \in A.$$

So j_A is an algebra homomorphism.

Theorem 2.12. *Let A be a commutative algebra. Let $Ш(A), P_\lambda$, $\diamond_\lambda$ and j_A be defined as above. Then*

(a) *The triple* $(Ш(A), \diamond_\lambda, P_\lambda)$ *is a commutative* λ-TD algebra;
(b) The quadruple $(Ш(A), \diamond_\lambda, P_\lambda, j_A)$ is the free commutative λ-TD algebra on A.

Proof. (a)Let $\mathfrak{a}, \mathfrak{b} \in Ш(A)$. Then by Eq. (17), we have

$$\begin{aligned} & P_\lambda(\mathfrak{a}) \diamond_\lambda P_\lambda(\mathfrak{b}) \\ = \ & (1_A \otimes \mathfrak{a}) \diamond_\lambda (1_A \otimes \mathfrak{b}) \\ = \ & 1_A \otimes \Big(\mathfrak{a} \diamond_\lambda (1_A \otimes \mathfrak{b})\Big) + 1_A \otimes \Big((1_A \otimes \mathfrak{a}) \diamond_\lambda \mathfrak{b}\Big) + \lambda 1_A \otimes (\mathfrak{a} \diamond_\lambda \mathfrak{b}) - 1_A \otimes \Big(\Big(\mathfrak{a} \diamond_\lambda P_\lambda(1_A)\Big) \diamond_\lambda \mathfrak{b}\Big) \\ = \ & P_\lambda\Big(\mathfrak{a} \diamond_\lambda P_\lambda(\mathfrak{b})\Big) + P_\lambda\Big(P_\lambda(\mathfrak{a}) \diamond_\lambda \mathfrak{b}\Big) + P_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b}) - P_\lambda\Big(\Big(\mathfrak{a} \diamond_\lambda P_\lambda(1_A)\Big) \diamond_\lambda \mathfrak{b}\Big). \end{aligned}$$

Thus P_λ is a λ-TD operator on $Ш(A)$, and so $(Ш(A), \diamond_\lambda, P_\lambda)$ forms a commutative λ-TD algebra.

(b) We now show that $(Ш(A), \diamond_\lambda, P_\lambda, j_A)$ is a free commutative λ-TD algebra, that is, $Ш(A)$ with j_A satisfies the universal property in Definition 2.6. Let (R, P) be a commutative λ-TD algebra and let $f : A \to R$ be an algebra homomorphism. For any pure tensor $\mathfrak{a} = a_1\otimes a_2\otimes\cdots\otimes a_m \in A^{\otimes m}$, we apply the induction on m to define a λ-TD algebra homomorphism $\bar{f} : Ш(A) \to R$. If $m = 1$, we define $\bar{f}(\mathfrak{a}) = f(\mathfrak{a})$. Then $\bar{f}(1_A) = f(1_A) = 1_R$, the unit of R. Assume that $\bar{f}(\mathfrak{a})$ has been defined for $m \leq k$ with $k \geq 1$. Consider $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes(k+1)}$ for $\mathfrak{a}' \in A^{\otimes k}$. Note that

$$\mathfrak{a} = a_1 \diamond_\lambda (1_A \otimes \mathfrak{a}') = a_1 \diamond_\lambda P_\lambda(\mathfrak{a}'). \tag{18}$$

Then define

$$\bar{f}(\mathfrak{a}) = f(a_1)P(\bar{f}(\mathfrak{a}')), \tag{19}$$

where $\bar{f}(\mathfrak{a}')$ is well-defined by the induction hypothesis. The uniqueness of $\bar{f}$ follows from the definition of $\bar{f}$.

Next we will verify that $\bar{f}$ is a λ-TD algebra homomorphism. By Eq. (19), we obtain

$$\bar{f}(P_\lambda(\mathfrak{a})) = \bar{f}(1_A \otimes \mathfrak{a}) = f(1_A)P(\bar{f}(\mathfrak{a})) = P(\bar{f}(\mathfrak{a})).$$

This gives

$$\bar{f} \circ P_\lambda = P \circ \bar{f}. \tag{20}$$

So it suffices to verify that $\bar{f}$ satisfies

$$\bar{f}(\mathfrak{a} \diamond_\lambda \mathfrak{b}) = \bar{f}(\mathfrak{a})\bar{f}(\mathfrak{b}), \quad \forall \mathfrak{a} \in A^{\otimes m},\ \mathfrak{b} \in A^{\otimes n}. \tag{21}$$

We will carry out the verification by induction on $m + n \geq 2$. If $m + n = 2$, then $m = n = 1$, and so $\mathfrak{a}, \mathfrak{b} \in A$. By Eq. (19), we have

$$\bar{f}(\mathfrak{a} \diamond_\lambda \mathfrak{b}) = \bar{f}(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a}\mathfrak{b}) = f(\mathfrak{a})f(\mathfrak{b}) = \bar{f}(\mathfrak{a})\bar{f}(\mathfrak{b}).$$

Assume that Eq. (21) holds for $m + n \leq k$. Let $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes m}$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}' \in A^{\otimes n}$ with $m + n = k + 1$. Then

$$\begin{aligned}
\bar{f}(\mathfrak{a} \diamond_\lambda \mathfrak{b}) &= \bar{f}\Big((a_1 \otimes \mathfrak{a}') \diamond_\lambda (b_1 \otimes \mathfrak{b}')\Big)\\
&= \bar{f}\Big(\big(a_1 \diamond_\lambda P_\lambda(\mathfrak{a}')\big) \diamond_\lambda \big(b_1 \diamond_\lambda P_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by Eq. (18))}\\
&= \bar{f}\Big((a_1 b_1) \diamond_\lambda \big(P_\lambda(\mathfrak{a}') \diamond_\lambda P_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by the commutativity of } \diamond_\lambda)\\
&= \bar{f}\Big((a_1 b_1) \diamond_\lambda P_\lambda\Big(\mathfrak{a}' \diamond_\lambda P_\lambda(\mathfrak{b}') + P_\lambda(\mathfrak{a}') \diamond_\lambda \mathfrak{b}'\\
&\quad + \lambda \mathfrak{a}' \diamond_\lambda \mathfrak{b}' - (\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}'\Big)\Big) \quad \text{(by } P_\lambda \text{ being a } \lambda\text{-TD operator)}\\
&= f(a_1 b_1) P\Big(\bar{f}\Big(\mathfrak{a}' \diamond_\lambda P_\lambda(\mathfrak{b}') + P_\lambda(\mathfrak{a}') \diamond_\lambda \mathfrak{b}'\\
&\quad + \lambda \mathfrak{a}' \diamond_\lambda \mathfrak{b}' - (\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}'\Big)\Big) \quad \text{(by Eq. (19))}\\
&= f(a_1 b_1) P\Big(\bar{f}(\mathfrak{a}')\bar{f}(P_\lambda(\mathfrak{b}')) + \bar{f}(P_\lambda(\mathfrak{a}'))\bar{f}(\mathfrak{b}')\\
&\quad + \lambda \bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}') - \Big(\bar{f}(\mathfrak{a}')\bar{f}(P_\lambda(1_A))\Big)\bar{f}(\mathfrak{b}')\Big) \quad \text{(by the induction hypothesis)}\\
&= f(a_1) f(b_1) P\Big(\bar{f}(\mathfrak{a}')P(\bar{f}(\mathfrak{b}')) + P(\bar{f}(\mathfrak{a}'))\bar{f}(\mathfrak{b}')\\
&\quad + \lambda \bar{f}(\mathfrak{a}')\bar{f}(\mathfrak{b}') - \bar{f}(\mathfrak{a}')P(\bar{f}(1_A))\bar{f}(\mathfrak{b}')\Big) \quad \text{(by Eq. (20))}\\
&= f(a_1) f(b_1) P\Big(\bar{f}(\mathfrak{a}')\Big) P\Big(\bar{f}(\mathfrak{b}')\Big) \quad \text{(by } \bar{f}(1_A) = 1_R \text{ and } P \text{ being a } \lambda\text{-TD operator)}\\
&= \Big(f(a_1) P\Big(\bar{f}(\mathfrak{a}')\Big)\Big)\Big(f(b_1) P\Big(\bar{f}(\mathfrak{b}')\Big)\Big) \quad \text{(by the commutativity of } A)\\
&= \bar{f}(\mathfrak{a})\bar{f}(\mathfrak{b}). \quad \text{(by Eq. (19))}
\end{aligned}$$

This completes the induction, and so the proof of Theorem 2.12. □

3. The cocycle bialgebra structure on free commutative λ-TD algebras

In this section, the free commutative λ-TD algebra $Ш(A)$ obtained in Theorem 2.12 will be equipped with a bialgebra structure, under the assumption that the generating algebra A is a bialgebra. So we let $A := (A, m_A, \mu_A, \Delta_A, \varepsilon_A)$ be a bialgebra. To achieve our goal, the first step in this process is to construct a comultiplication on the free commutative λ-TD algebra $Ш(A) := (Ш(A), \diamond_\lambda, P_\lambda)$ in terms of a suitable 1-cocycle property $\Delta P = (\mathrm{id} \otimes P)\Delta$, which was used to construct left counital Hopf algebras on free Nijenhuis algebras [15, 37] and on bi-decorated planar rooted forests [28]. Afterward, a left counit on $Ш(A)$ is given.

3.1. **Comultiplication by cocycle condition.** Let us first recall the definition of a left counital cocycle bialgebra.

Definition 3.1. [15, 37, 28]

(a) A **left counital coalgebra** is a triple (C, Δ, ε), where C is a $\mathbf{k}$-module, the comultiplication $\Delta : C \to C \otimes C$ is coassociative and the counit $\varepsilon : C \to \mathbf{k}$ is left counital, that is, $(\varepsilon \otimes \mathrm{id})\Delta = \beta_\ell$, where $\beta_\ell : C \to \mathbf{k} \otimes C$, given by $c \mapsto 1 \otimes c$, is a bijection.

(b) A **left counital operated bialgebra** is a sextuple $(H, m, \mu, \Delta, \varepsilon, P)$, where the quadruple (H, m, μ, P) is an operated algebra and the triple (H, Δ, ε) is a left counital coalgebra such that $\Delta : H \to H \otimes H$ and $\varepsilon : H \to \mathbf{k}$ are algebra homomorphisms;

(c) A left counintal operated bialgebra $(H, m, \mu, \Delta, \varepsilon, P)$ that satisfies the 1-cocycle property $\Delta P = (\mathrm{id} \otimes P)\Delta$ is called a **left counital cocycle bialgebra**.

In order to distinguish the multiplication in $Ш(A)$ and in $Ш(A) \otimes Ш(A)$, we denote by $\bullet$ the multiplication in $Ш(A) \otimes Ш(A)$.

Let $A := (A, m_A, \mu_A, \Delta_A, \varepsilon_A)$ be a bialgebra. Now we begin with the construction of the comultiplicaiton $\Delta_\lambda : Ш(A) \to Ш(A) \otimes Ш(A)$. For this, it suffices to define $\Delta_\lambda(\mathfrak{a})$ for $\mathfrak{a} := a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$ with $n \geq 1$, and then to extend by linearity. Use induction on n, starting with $n = 1$, that is, $\mathfrak{a} = a_1 \in A$. Then define $\Delta_\lambda(\mathfrak{a}) := \Delta_A(a_1)$ to be the coproduct Δ_A on A, giving

$$\Delta_\lambda(1_A) = 1_A \otimes 1_A. \tag{22}$$

Assume that $\Delta_\lambda(\mathfrak{a})$ has been defined for n. Consider $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes(n+1)}$ with $\mathfrak{a}' := a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$. By Eq. (18), we have

$$a_1 \otimes \mathfrak{a}' = a_1 \diamond_\lambda P_\lambda(\mathfrak{a}'). \tag{23}$$

By the 1-cocycle property, we first define

$$\Delta_\lambda(P_\lambda(\mathfrak{a}')) = (\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}'). \tag{24}$$

Then define

$$\Delta_\lambda(a_1 \otimes \mathfrak{a}') = \Delta_\lambda(a_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')\Big), \tag{25}$$

where $\Delta_\lambda(\mathfrak{a}')$ in Eq. (25) is well-defined by the induction hypothesis. So $\Delta_\lambda(\mathfrak{a})$ is well-defined.

Next, the counit ε_λ on $Ш(A)$ will be given in terms of the counit ε_A of A. Let $\mathfrak{a} = a_1 \otimes a_2 \otimes \cdots \otimes a_n \in A^{\otimes n}$ with $n \geq 1$. Define

$$\varepsilon_\lambda : Ш(A) \to \mathbf{k},\ \mathfrak{a} \mapsto \varepsilon_\lambda(\mathfrak{a}) = \begin{cases} \varepsilon_A(a_1), & \text{if } n = 1; \\ 0, & \text{if } n \geq 2. \end{cases} \tag{26}$$

Then extending by linearity, this map induces a linear map from $Ш(A)$ to $\mathbf{k}$. By ε_A being an algebra homomorphism, we obtain $\varepsilon_\lambda(1_A) = \varepsilon_A(1_A) = 1_{\mathbf{k}}$.

Lemma 3.2. Let $m, n \geq 1$ and let $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$ be pure tensors. Then

$$(\mathrm{id} \otimes \Delta_\lambda)(\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) = (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\mathrm{id} \otimes \Delta_\lambda)(\mathfrak{a} \otimes \mathfrak{b}). \tag{27}$$

and

$$(\Delta_\lambda \otimes \mathrm{id})(\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) = (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\Delta_\lambda \otimes \mathrm{id})(\mathfrak{a} \otimes \mathfrak{b}). \tag{28}$$

Proof. By Eq. (24), we obtain

$$\begin{aligned}
(\mathrm{id} \otimes \Delta_\lambda)(\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) &= (\mathrm{id} \otimes \Delta_\lambda P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) \\
&= \mathfrak{a} \otimes (\Delta_\lambda P_\lambda(\mathfrak{b})) \\
&= \mathfrak{a} \otimes \big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{b})\big) \\
&= (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\mathrm{id} \otimes \Delta_\lambda)(\mathfrak{a} \otimes \mathfrak{b}).
\end{aligned}$$

Thus Eq. (27) holds, and Eq. (28) can be done by straightforward computation. □

Lemma 3.3. Let $Ш(A) \otimes Ш(A)$, $\bullet$ and Δ_λ be as above. Then the triple $(Ш(A) \otimes Ш(A), \bullet, \mathrm{id} \otimes P_\lambda)$ forms a λ-TD algebra.

Proof. We only need to show that $\mathrm{id} \otimes P_\lambda$ satisfies Eq. (5). For all $\mathfrak{a} \otimes \mathfrak{b}, \mathfrak{c} \otimes \mathfrak{d} \in Ш(A) \otimes Ш(A)$, we have

$$\begin{aligned}
&(\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathrm{id} \otimes P_\lambda)(\mathfrak{c} \otimes \mathfrak{d}) \\
= \ &(\mathfrak{a} \diamond_\lambda \mathfrak{c}) \otimes ((P_\lambda(\mathfrak{b}) \diamond_\lambda P_\lambda(\mathfrak{d})) \\
= \ &(\mathfrak{a} \diamond_\lambda \mathfrak{c}) \otimes P_\lambda\Big(\mathfrak{b} \diamond_\lambda P_\lambda(\mathfrak{d}) + P_\lambda(\mathfrak{b}) \diamond_\lambda \mathfrak{d} + \lambda \mathfrak{b} \diamond_\lambda \mathfrak{d} - \mathfrak{b} \diamond_\lambda P_\lambda(1_A) \diamond_\lambda \mathfrak{d}\Big) \\
= \ &(\mathrm{id} \otimes P_\lambda)\Big((\mathfrak{a} \diamond_\lambda \mathfrak{c}) \otimes \Big(\mathfrak{b} \diamond_\lambda P_\lambda(\mathfrak{d}) + P_\lambda(\mathfrak{b}) \diamond_\lambda \mathfrak{d} + \lambda \mathfrak{b} \diamond_\lambda \mathfrak{d} - (\mathfrak{b} \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{d}\Big)\Big) \\
= \ &(\mathrm{id} \otimes P_\lambda)\Big((\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathrm{id} \otimes P_\lambda)(\mathfrak{c} \otimes \mathfrak{d}) + (\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) \\
&+ \lambda(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) - \Big(\mathfrak{a} \otimes (\mathfrak{b} \diamond_\lambda P_\lambda(1_A))\Big) \bullet (\mathfrak{c} \otimes \mathfrak{d})\Big) \\
= \ &(\mathrm{id} \otimes P_\lambda)\Big((\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathrm{id} \otimes P_\lambda)(\mathfrak{c} \otimes \mathfrak{d}) + (\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) \\
&+ \lambda(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) - \Big((\mathfrak{a} \otimes \mathfrak{b}) \bullet (1_A \otimes P_\lambda(1_A))\Big) \bullet (\mathfrak{c} \otimes \mathfrak{d})\Big) \\
= \ &(\mathrm{id} \otimes P_\lambda)\Big((\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathrm{id} \otimes P_\lambda)(\mathfrak{c} \otimes \mathfrak{d}) + (\mathrm{id} \otimes P_\lambda)(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) \\
&+ \lambda(\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathfrak{c} \otimes \mathfrak{d}) - (\mathfrak{a} \otimes \mathfrak{b}) \bullet (\mathrm{id} \otimes P_\lambda)(1_A \otimes 1_A) \bullet (\mathfrak{c} \otimes \mathfrak{d})\Big).
\end{aligned}$$

□

3.2. **The compatibilities of Δ_λ and ε_λ.** We are now going to show that Δ_λ and ε_λ as defined above are compatible with the multiplications.

Proposition 3.4. *The comultiplication* $\Delta_\lambda : Ш(A) \to Ш(A) \otimes Ш(A)$ *is an algebra homomorphism.*

Proof. It suffices to verify that for pure tensors $\mathfrak{a} \in A^{\otimes m}$ and $\mathfrak{b} \in A^{\otimes n}$ with $m, n \geq 1$,

$$\Delta_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b}) = \Delta_\lambda(\mathfrak{a}) \bullet \Delta_\lambda(\mathfrak{b}). \tag{29}$$

We prove Eq. (29) by induction on $m + n$. If $m + n = 2$, then $m = n = 1$, and so $\mathfrak{a}, \mathfrak{b} \in A$. By the definitions of $\diamond_\lambda$ and Δ_λ, together with Δ_A being an algebra homomorphism, Eq. (29) holds.

Suppose that Eq. (29) is true for $m+n \le k$. Let $m+n = k+1 \ge 3$. This leads to either $m \ge 2$ or $n \ge 2$. We just show that Eq. (29) holds for the case $m \ge 2$ and $n \ge 2$. The others are similar. When $m \ge 2$ and $n \ge 2$, denote $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ with $\mathfrak{a}' \in A^{\otimes(m-1)}$ and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ with $\mathfrak{b}' \in A^{\otimes(n-1)}$. On the one hand,

$$
\begin{aligned}
& \Delta_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b}) \\
= \; & \Delta_\lambda\Big((a_1 \otimes \mathfrak{a}') \diamond_\lambda (b_1 \otimes \mathfrak{b}')\Big) \\
= \; & \Delta_\lambda\Big((a_1 \diamond_\lambda P_\lambda(\mathfrak{a}')) \diamond_\lambda \big(b_1 \diamond_\lambda P_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by Eq. (23))} \\
= \; & \Delta_\lambda\Big(a_1 b_1 \diamond_\lambda \big(P_\lambda(\mathfrak{a}') \diamond_\lambda P_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by the definition of } \diamond_\lambda\text{)} \\
= \; & \Delta_\lambda\Big(a_1 b_1 \diamond_\lambda P_\lambda\Big(\mathfrak{a}' \diamond_\lambda P_\lambda(\mathfrak{b}') + P_\lambda(\mathfrak{a}') \diamond_\lambda \mathfrak{b}' + \lambda \mathfrak{a}' \diamond_\lambda \mathfrak{b}' - \big(\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)\big) \diamond_\lambda \mathfrak{b}'\Big)\Big) \\
& \qquad\qquad\qquad\qquad \text{(by } P_\lambda \text{ being a } \lambda\text{-TD operator)} \\
= \; & \Delta_\lambda\Big(a_1 b_1 \otimes \Big(\mathfrak{a}' \diamond_\lambda P_\lambda(\mathfrak{b}') + P_\lambda(\mathfrak{a}') \diamond_\lambda \mathfrak{b}' + \lambda \mathfrak{a}' \diamond_\lambda \mathfrak{b}' - \big(\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)\big) \diamond_\lambda \mathfrak{b}'\Big)\Big) \quad \text{(by Eq. (17))} \\
= \; & \Delta_\lambda(a_1 b_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda\big(\mathfrak{a}' \diamond_\lambda P_\lambda(\mathfrak{b}')\big) + (\mathrm{id} \otimes P_\lambda)\Delta_\lambda\big(P_\lambda(\mathfrak{a}') \diamond_\lambda \mathfrak{b}'\big) \\
& + \lambda(\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}' \diamond_\lambda \mathfrak{b}') - (\mathrm{id} \otimes P_\lambda)\Delta_\lambda\big((\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}'\big)\Big) \quad \text{(by Eq. (25))} \\
= \; & \Delta_\lambda(a_1 b_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\big(\Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(P_\lambda(\mathfrak{b}'))\big) + (\mathrm{id} \otimes P_\lambda)\big(\Delta_\lambda(P_\lambda(\mathfrak{a}')) \bullet \Delta_\lambda(\mathfrak{b}')\big) \\
& + \lambda(\mathrm{id} \otimes P_\lambda)\big(\Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(\mathfrak{b}')\big) - (\mathrm{id} \otimes P_\lambda)\Big(\big(\Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(P_\lambda(1_A))\big) \bullet \Delta_\lambda(\mathfrak{b}')\Big)\Big) \\
& \qquad\qquad\qquad\qquad \text{(by the induction hypothesis)} \\
= \; & \Delta_\lambda(a_1 b_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\big(\Delta_\lambda(\mathfrak{a}') \bullet (\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{b}') + (\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(\mathfrak{b}') \\
& + \lambda \Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(\mathfrak{b}') - \Delta_\lambda(\mathfrak{a}') \bullet (\mathrm{id} \otimes P_\lambda)(1_A \otimes 1_A) \bullet \Delta_\lambda(\mathfrak{b}')\big)\Big). \quad \text{(by Eqs. (22) and (24))}
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
& \Delta_\lambda(\mathfrak{a}) \bullet \Delta_\lambda(\mathfrak{b}) \\
= \; & \Big(\Delta_\lambda(a_1) \bullet \big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')\big)\Big) \bullet \Big(\Delta_\lambda(b_1) \bullet \big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by Eq. (25))} \\
= \; & \big(\Delta_\lambda(a_1) \bullet \Delta_\lambda(b_1)\big) \bullet \Big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \bullet \big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{b}')\big)\Big) \quad \text{(by the commutativity of } \bullet\text{)} \\
= \; & \Delta_\lambda(a_1 b_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\big(\Delta_\lambda(\mathfrak{a}') \bullet (\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{b}') + (\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(\mathfrak{b}') \\
& + \lambda \Delta_\lambda(\mathfrak{a}') \bullet \Delta_\lambda(\mathfrak{b}') - \Delta_\lambda(\mathfrak{a}') \bullet (\mathrm{id} \otimes P_\lambda)(1_A \otimes 1_A) \bullet \Delta_\lambda(\mathfrak{b}')\big)\Big). \quad \text{(by Lemma 3.3)}
\end{aligned}
$$

Thus the terms of $\Delta_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b})$ agree with the terms of $\Delta_\lambda(\mathfrak{a}) \bullet \Delta_\lambda(\mathfrak{b})$, and so Eq. (29) holds. This completes the induction. □

From Proposition 3.4, we obtain

Corollary 3.5. Let $Ш(A) \otimes Ш(A)$ and Δ_λ be as above. Then the induced maps

$$\mathrm{id} \otimes \Delta_\lambda : Ш(A) \otimes Ш(A) \to Ш(A) \otimes \Big(Ш(A) \otimes Ш(A)\Big), \mathfrak{a} \otimes \mathfrak{b} \mapsto \mathfrak{a} \otimes \Delta_\lambda(\mathfrak{b})$$

and

$$\Delta_\lambda \otimes \mathrm{id} : Ш(A) \otimes Ш(A) \to \Big(Ш(A) \otimes Ш(A)\Big) \otimes Ш(A), \mathfrak{a} \otimes \mathfrak{b} \mapsto \Delta_\lambda(\mathfrak{a}) \otimes \mathfrak{b}$$

are algebra homomorphisms, respectively.

We next verify that $\varepsilon_\lambda : Ш(A) \to \mathbf{k}$ given by Eq. (26) is an algebra homomorphism.

Proposition 3.6. *The linear map ε_λ is an algebra homomorphism.*

Proof. By the definition of ε_λ, $\varepsilon_\lambda(1_A) = 1_k$. So we just prove that

$$\varepsilon_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b}) = \varepsilon_\lambda(\mathfrak{a})\varepsilon_\lambda(\mathfrak{b}) \tag{30}$$

for any pure tensors $\mathfrak{a} := a_1 \otimes \mathfrak{a}' \in A^{\otimes m}$ and $\mathfrak{b} := b_1 \otimes \mathfrak{b}' \in A^{\otimes n}$ with $m, n \geq 1$. If $m = n = 1$, then by Eq. (17), $\mathfrak{a} \diamond_\lambda \mathfrak{b} = a_1 b_1$, and so Eq. (30) follow from Eq. (26). If $m \geq 2$ or $n \geq 2$, then $\mathfrak{a} \diamond_\lambda \mathfrak{b} = a_1 b_1 \otimes (\mathfrak{a}' Ш_r \mathfrak{b}')$ by Eq. (16, and so $\mathfrak{a} \diamond_\lambda \mathfrak{b} \in \sum_{i\geq 2}^{m+n-1} A^{\otimes i}$. Then by Eq. (26) again,

$$\varepsilon_\lambda(\mathfrak{a} \diamond_\lambda \mathfrak{b}) = 0 = \varepsilon_\lambda(\mathfrak{a})\varepsilon_\lambda(\mathfrak{b}).$$

□

3.3. **The coassociativity of Δ_λ and the left counitality of ε_λ.** In the following, we will show that Δ_λ satisfies the coassociativity and ε_λ satisfies the left counitality.

Proposition 3.7. *The comultiplication Δ_λ is coassociative, that is,*

$$(\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda = (\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda. \tag{31}$$

Proof. Let $\mathfrak{a} := a_1 \otimes \mathfrak{a}' \in A^{\otimes k}$ with $k \geq 1$. Then we shall verify that

$$(\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(\mathfrak{a}) = (\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(\mathfrak{a}). \tag{32}$$

We now proceed by induction on n. For $k = 1$, we have $\mathfrak{a} = a_1 \in A$. Then by the definition of Δ_λ and the coassociativity of Δ_A, Eq. (32) holds.

Assume that $k \geq 1$ and Eq. (32) is true for all $\mathfrak{a} \in A^{\otimes k}$. Consider $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes (k+1)}$. Expanding the left hand side $(\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(\mathfrak{a})$ of Eq. (32) gives

$$\begin{aligned}
& (\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(\mathfrak{a}) \\
= \; & (\mathrm{id} \otimes \Delta_\lambda)\Big(\Delta_\lambda(a_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')\Big)\Big) \quad \text{(by Eq. (25))} \\
= \; & (\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(a_1) \bullet (\mathrm{id} \otimes \Delta_\lambda)(\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \quad \text{(by Corollary 3.5)} \\
= \; & (\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(a_1) \bullet (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(\mathfrak{a}') \quad \text{(by Eq. (27))} \\
= \; & (\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(a_1) \bullet (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(\mathfrak{a}'). \quad \text{(by the induction hypothesis)}
\end{aligned}$$

On the other hand, we obtain

$$\begin{aligned}
& (\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(\mathfrak{a}) \\
= \; & (\Delta_\lambda \otimes \mathrm{id})\Big(\Delta_\lambda(a_1) \bullet \Big((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')\Big)\Big) \quad \text{(by Eq. (25))} \\
= \; & (\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(a_1) \bullet (\Delta_\lambda \otimes \mathrm{id})(\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \quad \text{(by Corollary 3.5)} \\
= \; & (\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(a_1) \bullet (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(\mathfrak{a}') \quad \text{(by Eq. (28))} \\
= \; & (\mathrm{id} \otimes \Delta_\lambda)\Delta_\lambda(a_1) \bullet (\mathrm{id} \otimes \mathrm{id} \otimes P_\lambda)(\Delta_\lambda \otimes \mathrm{id})\Delta_\lambda(\mathfrak{a}'). \quad \text{(by the coassociativity of } \Delta_A)
\end{aligned}$$

Then the expansion of $(\mathrm{id}\otimes\Delta_\lambda)\Delta_\lambda(\mathfrak{a})$ matches up with the expansion of $(\Delta_\lambda\otimes\mathrm{id})\Delta_\lambda(\mathfrak{a})$. This completes the induction, and thus proving Eq. (32). □

Proposition 3.8. *The linear map ε_λ satisfies the left counitality, that is,*

$$(\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda=\beta_\ell, \tag{33}$$

where $\beta_\ell:Ш(A)\to\mathbf{k}\otimes Ш(A)$ is given by $\mathfrak{a}\mapsto 1\otimes\mathfrak{a}$ for $\mathfrak{a}\in A^{\otimes k}$ and for $k\geq 1$.

Proof. It suffices to verify that

$$(\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda(\mathfrak{a})=\beta_\ell(\mathfrak{a}). \tag{34}$$

for every pure tensor $\mathfrak{a}\in A^{\otimes k}$. We do this by applying the induction on $k\geq 1$. If $k=1$, then $\mathfrak{a}\in A$, and so Eq. 34 follows from the left counitality of ε_A.

Assume $k>1$ and consider $\mathfrak{a}:=a_1\otimes\mathfrak{a}'\in A^{\otimes(k+1)}$. Then

$$\begin{aligned}
(\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda(\mathfrak{a}) &= (\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda\Big(a_1\diamond_\lambda P_\lambda(\mathfrak{a}')\Big)\\
&= (\varepsilon_\lambda\otimes\mathrm{id})\Big(\Delta_\lambda(a_1)\bullet\Delta_\lambda(P_\lambda(\mathfrak{a}'))\Big) \quad \text{(by Eq. (25))}\\
&= \Big((\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda(a_1)\Big)\Big((\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda\Big(P_\lambda(\mathfrak{a}')\Big)\Big) \quad \text{(by Proposition 3.6)}\\
&= \beta_\ell(a_1)(\varepsilon_\lambda\otimes\mathrm{id})(\mathrm{id}\otimes P_\lambda)\Delta_\lambda(\mathfrak{a}') \quad \text{(by Eq. (24))}\\
&= \beta_\ell(a_1)(\mathrm{id}\otimes P_\lambda)(\varepsilon_\lambda\otimes\mathrm{id})\Delta_\lambda(\mathfrak{a}')\\
&= \beta_\ell(a_1)(\mathrm{id}\otimes P_\lambda)\beta_\ell(\mathfrak{a}') \quad \text{(by the induction hypothesis)}\\
&= \beta_\ell(a_1)\beta_\ell(P_\lambda(\mathfrak{a}'))\\
&= \beta_\ell(a_1\diamond_\lambda P_\lambda(\mathfrak{a}')) \quad \text{(by } \beta_\ell \text{ being an algebra isomorphism)}\\
&= \beta_\ell(\mathfrak{a}).
\end{aligned}$$

This completes the induction and the proof of Eq. (34). □

However, ε_λ does not satisfy the right counitality. For example, we first define an algebra isomorphism $\beta_r:Ш(A)\to Ш(A)\otimes\mathbf{k}$, given by $\mathfrak{a}\mapsto\mathfrak{a}\otimes 1$, for all $\mathfrak{a}\in A^{\otimes k}$. Let $\mathfrak{a}=P_\lambda(x)$, where $x\in A$. By using Sweedler's notation: $\Delta_A(x)=\sum x'\otimes x''$, we get

$$\begin{aligned}
(\mathrm{id}\otimes\varepsilon_\lambda)\Delta_\lambda(P_\lambda(x)) &= (\mathrm{id}\otimes\varepsilon_\lambda)(\mathrm{id}\otimes P_\lambda)\Delta_A(x) \quad \text{(by Eq. (24))}\\
&= (\mathrm{id}\otimes\varepsilon_\lambda)(\mathrm{id}\otimes P_\lambda)\Big(\sum x'\otimes x''\Big)\\
&= \sum x'\otimes\varepsilon_\lambda\Big(P_\lambda(x'')\Big)\\
&= 0 \quad \text{(by Eq. (26))}\\
&\neq \beta_r(P_\lambda(x)).
\end{aligned}$$

Lastly, we state the main theorem of this section. It follows that there exists a linear map $\mu_\lambda:\mathbf{k}\to Ш(A)$, given by

$$c\mapsto c1_A,\quad c\in\mathbf{k}.$$

Then we can verify that μ_λ is a unit for $(Ш(A),\diamond_\lambda)$. According to our previous results, we obtain

Theorem 3.9. *The sextuple $Ш(A):=(Ш(A),\diamond_\lambda,\mu_\lambda,\Delta_\lambda,\varepsilon_\lambda,P_\lambda)$ is a left counital cocycle bialgebra.*

Proof. By Theorem 2.12, the quadruple $(Ш(A), \diamond_\lambda, \mu_\lambda, P_\lambda)$ is a commutative λ-TD algebra. Furthermore, the triple $(Ш(A), \Delta_\lambda, \varepsilon_\lambda)$ is a left counital coalgebra by Proposition 3.7 and Proposition 3.8. Finally, by Proposition 3.4 and Proposition 3.6, the sextuple $(Ш(A), \diamond_\lambda, \mu_\lambda, \Delta_\lambda, \varepsilon_\lambda, P_\lambda)$ is a left counital cocycle bialgebra. □

4. The left counital Hopf algebra structure on free commutative λ-TD algebras

This section will equip the free commutative λ-TD algebra $(Ш(A), \diamond_\lambda, \mu_\lambda, \Delta_\lambda, \varepsilon_\lambda)$ with a left counital Hopf algebra structure.

Definition 4.1. [17, 27]

(a) A left counital operated bialgebra $H := (H, m, \mu, \Delta, \varepsilon, P)$ is called **filtered** if there exists an increasing filtration H^n for $n \geq 0$ such that

$$\bigcup_{n\geq 0} H^n = H; \quad H^p H^q \subseteq H^{p+q}; \quad \Delta(H^n) \subseteq H^0 \otimes H^n + \sum_{\substack{p+q=n \\ p>0, q>0}} H^p \otimes H^q. \tag{35}$$

(b) A filtered left counital operated bialgebra H is **connected** if $H^0 = \mathrm{im}\mu (= \mathbf{k}1_H)$.

Lemma 4.2. Let $\mathbf{k}$ be a field. Let H be a connected filtered left counital operated bialgebra and let $e = \mu\varepsilon$. Then

$$H = \mathrm{im} u \oplus \ker \varepsilon.$$

Proof. By $\varepsilon : H \to \mathbf{k}$ being an algebra homomorphism, we obtain $\varepsilon\mu = \mathrm{id}_{\mathbf{k}}$. Then $e^2 = \mu(\varepsilon\mu)\varepsilon = e$, and so

$$H = \mathrm{im} e \oplus \ker e.$$

By $e = \mu\varepsilon$, we get $\mathrm{im} e \subseteq \mathrm{im}\mu$. If $x \in \mathrm{im}\mu$, then $\mu(c) = x$ for some $c \in \mathbf{k}$, and so $x = c\mu(1_{\mathbf{k}}) = c\mu(\varepsilon(1_H)) = e(c1_H) \in \mathrm{im} e$. Thus $\mathrm{im} e = \mathrm{im}\mu$. By $e = \mu\varepsilon$ again, $\ker \varepsilon \subseteq \ker e$. Let $z \in \ker e$. Then

$$e(z) = \mu(\varepsilon(z)) = \varepsilon(z)\mu(1_k) = \varepsilon(z)1_H = 0.$$

This gives $\varepsilon(z) = 0$, and then $\ker e \subseteq \ker \varepsilon$, yielding $\ker e = \ker \varepsilon$. Thus

$$H = \mathrm{im} u \oplus \ker \varepsilon.$$

□

By Lemma 4.2 and the connectedness of H, we obtain

$$H = \mathbf{k}1_H \oplus \ker \varepsilon. \tag{36}$$

Lemma 4.3. Let $\mathbf{k}$ be a field. Let H be a connected filtered left counital operated bialgebra.

(a) Let $\hat{H}^n := H^n \cap \ker \varepsilon$ for $n > 0$. Then

$$\hat{H}^n \subseteq \hat{H}^{n+1} \tag{37}$$

and

$$H^n = H^0 \oplus \hat{H}^n. \tag{38}$$

(b) Let $p, q > 0$. Then

$$H^p \otimes H^q \subseteq H^0 \otimes H^q + \hat{H}^p \otimes H^0 + \hat{H}^p \otimes \hat{H}^q. \tag{39}$$

(c) For $x \in \hat{H}^n$ with $n > 0$, we have

$$\Delta(x) = 1 \otimes x + \tilde{\Delta}(x), \quad \text{where } \tilde{\Delta}(x) \in (\ker\varepsilon \otimes H^0 + \ker\varepsilon \otimes \ker\varepsilon).$$

Proof. (a) For all $x \in H^n$, we get $x = \varepsilon(x)1_H + x - \varepsilon(x)1_H$. Since $\varepsilon(x - \varepsilon(x)1_H) = \varepsilon(x) - \varepsilon(x) = 0$ and $x - \varepsilon(x)1_H \in H^n + H^0 \subseteq H^n$, we have $H^n = H^0 + \hat{H}^n$. For every $y \in H^0 \cap \hat{H}^n$, we have $y = c1_H$ for some $c \in \mathbf{k}$ by the connectedness of H and $\varepsilon(y) = 0$. This leads to $0 = \varepsilon(y) = \varepsilon(c1_H) = c$. Thus $y = 0$, proving Eq. (38).

(b)Firstly, Eq. (37) follows from the increasing filtration $H^n \subseteq H^{n+1}$. Secondly, by Eq. (38), we obtain

$$\begin{aligned} H^p \otimes H^q &= (H^0 \oplus \hat{H}^p) \otimes (H^0 \oplus \hat{H}^q) \\ &\subseteq H^0 \otimes H^0 + H^0 \otimes \hat{H}^q + \hat{H}^p \otimes H^0 + \hat{H}^p \otimes \hat{H}^q \\ &\subseteq H^0 \otimes H^q + \hat{H}^p \otimes H^0 + \hat{H}^p \otimes \hat{H}^q \quad (\text{by } \hat{H}^q \subseteq H^q \text{ and } H^0 \subseteq H^q) \end{aligned}$$

(c) Let $n > 0$. By Eq. (35), we obtain

$$\begin{aligned} \Delta(H^n) &\subseteq H^0 \otimes H^n + \sum_{\substack{p+q=n \\ p>0,q>0}} H^p \otimes H^q \\ &\subseteq H^0 \otimes H^n + \sum_{\substack{p+q=n \\ p>0,q>0}} H^0 \otimes H^q + \hat{H}^p \otimes H^0 + \hat{H}^p \otimes \hat{H}^q \quad (\text{by Eq. (39)}) \\ &\subseteq H^0 \otimes H^n + \sum_{\substack{p+q=n \\ p>0,q>0}} H^0 \otimes H^q + \sum_{\substack{p+q=n \\ p>0,q>0}} \hat{H}^p \otimes H^0 + \sum_{\substack{p+q=n \\ p>0,q>0}} \hat{H}^p \otimes \hat{H}^q \\ &\subseteq H^0 \otimes H^n + H^0 \otimes H^{n-1} + \hat{H}^{n-1} \otimes H^0 + \sum_{\substack{p+q=n \\ p>0,q>0}} \hat{H}^p \otimes \hat{H}^q \quad (\text{by Eq. (37)}) \\ &\subseteq H^0 \otimes H^n + \hat{H}^{n-1} \otimes H^0 + \sum_{\substack{p+q=n \\ p>0,q>0}} \hat{H}^p \otimes \hat{H}^q \\ &\subseteq H^0 \otimes H^n + \ker\varepsilon \otimes H^0 + \ker\varepsilon \otimes \ker\varepsilon. \end{aligned}$$

Then for all $x \in \hat{H}^n$ for $n > 0$, we can write

$$\Delta(x) = 1 \otimes u + \tilde{\Delta}(x),$$

where $u \in H^n$ and $\tilde{\Delta}(x) \in (\ker\varepsilon \otimes H^0 + \ker\varepsilon \otimes \ker\varepsilon)$. Then by the left counitality of ε given by Definition 3.1,

$$\begin{aligned} x &= \beta^{-1}(\varepsilon \otimes \mathrm{id})\Delta(x) \\ &= \beta^{-1}(\varepsilon \otimes \mathrm{id})(1 \otimes u + \tilde{\Delta}(x)) \\ &= \beta^{-1}(\varepsilon(1) \otimes u + (\varepsilon \otimes \mathrm{id})\tilde{\Delta}(x)) \\ &= u. \quad (\text{by } \tilde{\Delta}(x) \in (\ker\varepsilon \otimes H^0 + \ker\varepsilon \otimes \ker\varepsilon)) \end{aligned}$$

This yields

$$\Delta(x) = 1 \otimes x + \tilde{\Delta}(x),$$

where $\tilde{\Delta}(x) \in (\ker\varepsilon \otimes H^0 + \ker\varepsilon \otimes \ker\varepsilon)$. ◻

From the above proof of Item (c), we also obtain

$$\tilde{\Delta}(x) \in \sum_{\substack{p+q=n \\ p>0, q>0}} H^p \otimes H^q. \tag{40}$$

Definition 4.4. Let $H := (H, m, \mu, \Delta, \varepsilon, P)$ be a left counital operated bialgebra.

(a) A linear map $S : H \to H$ is said to be an **right antipode** if S is a right inverse of id_H under the convolution product $*$, that is

$$\mathrm{id}_H * S = e.$$

(b) A left counital operated bialgebra H with a right antipode is called a **left counital Hopf algebra**.

The following fact is parallel to [27, Corollary II. 3.2].

Proposition 4.5. *A connected filtered left counital operated bialgebra is a left counital Hopf algebra. The right antipode is recursively defined by*

$$S(1_H) = 1_H, \quad S(x) = -\sum_x x' S(x''), \ x \in \ker \varepsilon, \tag{41}$$

using Sweedler's notation $\tilde{\Delta}(x) = \sum_x x' \otimes x''$

Proof. Verify directly that the linear map S defined in Eq. (41) satisfies the equation $\mathrm{id} * S = e$. By Δ being an algebra homomorphism, we get $\Delta(1_H) = 1_H \otimes 1_H$. The formula $e = \mu\varepsilon$ gives

$$e(1_H) = \mu(\varepsilon(1_H)) = \mu(1_{\mathbf{k}}) = 1_H.$$

Then

$$(\mathrm{id} * S)(1_H) = m(\mathrm{id} \otimes S)\Delta(1_H) = S(1_H) \Rightarrow (\mathrm{id} * S)(1_H) = 1_H = e(1_H).$$

Let $x \in \ker \varepsilon$. Then by Lemma 4.3 Item (c)),

$$(\mathrm{id} * S)(x) = m(\mathrm{id} \otimes S)\Delta(x) = m(\mathrm{id} \otimes S)(1 \otimes x + \tilde{\Delta}(x)) = S(x) + \sum_x x' S(x''),$$

where $\tilde{\Delta}(x) = \sum_x x' \otimes x'' \in \sum_{\substack{p+q=n \\ p>0, q>0}} H^p \otimes H^q$ follows immediately from Eq. (40). By Eq. (41), we obtain

$$S(x) + \sum_x x' S(x'') = 0.$$

This gives

$$(\mathrm{id} * S)(x) = 0 = e(x), \ x \in \ker \varepsilon.$$

□

Theorem 4.6. *Let* $A = \bigcup_{n \geq 0} A^n$ *is a connected filtered left counital bialgebra. Let* $Ш(A) = (Ш(A), \diamond_\lambda, \mu_\lambda, \Delta_\lambda, \varepsilon_\lambda, P_\lambda)$ *be as in Theorem 3.9. Then* $Ш(A)$ *is a left counital Hopf algebra.*

Proof. According to Proposition 4.5, we only need to verify that $Ш(A)$ is a connected filtered left counital operated bialgebra. For this reason, we denote the degree of a by

$$\deg(a) := \min\{k \in \mathbb{N} \,|\, a \in A^k\}, \quad \forall a \in A.$$

For any $m \geq 1$ and any pure tensor $0 \neq \mathfrak{a} = a_1 \otimes \cdots \otimes a_m \in A^{\otimes m}$, we set

$$\deg(\mathfrak{a}) := \deg(a_1) + \cdots + \deg(a_m) + m - 1. \tag{42}$$

For simplicity, we write $\Lambda := Ш(A) = \oplus_{n\geq 1} A^{\otimes n}$ and denote by Λ^k the linear span of pure tensors $\mathfrak{a} \in \Lambda$ with $\deg(\mathfrak{a}) \leq k$. Then we get a increasing filtration $\Lambda^k \subseteq \Lambda^{k+1}$ for all $k \geq 0$ and $\Lambda^0 = A^0 = \mathbf{k}1_A$ by the connectedness of A. Let $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes(n+1)}$ with $\mathfrak{a}' \in A^{\otimes n}$. Then by Eq. (42) we obtain

$$\deg(\mathfrak{a}) = \deg(a_1) + \deg(\mathfrak{a}') + 1. \tag{43}$$

Furthermore, if $\mathfrak{a} \in \Lambda^r$ for $r \geq 1$, then

$$\mathfrak{a}' \in \Lambda^{r-\deg(a_1)-1}. \tag{44}$$

We next show that the increasing filtration Λ^k satisfies that for all $p, q \geq 0$

$$\Lambda^q \diamond_\lambda \Lambda^p \subseteq \Lambda^{p+q} \tag{45}$$

and

$$\Delta_\lambda(\Lambda^k) \subseteq \Lambda^0 \otimes \Lambda^k + \sum_{\substack{p+q=k\\ p>0, q>0}} \Lambda^p \otimes \Lambda^q. \tag{46}$$

Now use induction on $p + q \geq 0$ to verify Eq. (45). For this it suffices to prove $\mathfrak{a} \diamond_\lambda \mathfrak{b} \in \Lambda^{p+q}$ for all pure tensors $\mathfrak{a} \in \Lambda^p$ and $\mathfrak{b} \in \Lambda^q$. When $p + q = 0$, we have $\mathfrak{a}, \mathfrak{b} \in \Lambda^0$, and so $\mathfrak{a} \diamond_\lambda \mathfrak{b} \in \Lambda^0$ by $\Lambda^0 = \mathbf{k}1_A$ and Eq. (17). Assume that Eq. (45) holds for $p + q \leq n$. Let $p + q = n + 1$. If $p = 0$ or $q = 0$, then $\mathfrak{a} \in \Lambda^0$ or $\mathfrak{b} \in \Lambda^0$, proving Eq. (45) by Eq. (17) again. Hence we suppose that $p, q \geq 1$. If $\mathfrak{a} \in A$ or $\mathfrak{b} \in A$, then $\deg(\mathfrak{a} \diamond_\lambda \mathfrak{b}) \leq \deg(\mathfrak{a}) + \deg(\mathfrak{b})$ by Eq. (17) and the connectedness of A, and so Eq. (45) holds. Thus we only consider $\mathfrak{a} \in A^{\otimes \ell}$ and $\mathfrak{b} \in A^{\otimes m}$ for $\ell, m \geq 2$. Write $\mathfrak{a} = a_1 \otimes \mathfrak{a}'$ with $\mathfrak{a}' = a_2 \otimes \cdots \otimes a_\ell$, and $\mathfrak{b} = b_1 \otimes \mathfrak{b}'$ with $\mathfrak{b}' = b_2 \otimes \cdots \otimes b_m$. By Eq. (17) again, we obtain

$$\mathfrak{a} \diamond_\lambda \mathfrak{b} = a_1 b_1 \otimes (\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}')) + a_1 b_1 \otimes ((1_A \otimes \mathfrak{a}') \diamond_\lambda \mathfrak{b}') + \lambda a_1 b_1 \otimes (\mathfrak{a}' \diamond_\lambda \mathfrak{b}') - a_1 b_1 \otimes \Big((\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}' \Big). \tag{47}$$

By Eq. (44), $\mathfrak{a}' \in \Lambda^{p-\deg(a_1)-1}$ and $\mathfrak{b}' \in \Lambda^{q-\deg(b_1)-1}$. Furthermore, by Eq. (43) and $\deg(1_A) = 0$ because $\Lambda^0 = \mathbf{k}1_A$, we have

$$\deg(1_A \otimes \mathfrak{a}') = \deg(1_A) + \deg(\mathfrak{a}') + 1 = \deg(\mathfrak{a}') + 1 \Rightarrow 1_A \otimes \mathfrak{a}' \in \Lambda^{p-\deg(a_1)}$$

and

$$\deg(1_A \otimes \mathfrak{b}') = \deg(1_A) + \deg(\mathfrak{b}') + 1 = \deg(\mathfrak{b}') + 1 \Rightarrow 1_A \otimes \mathfrak{b}' \in \Lambda^{q-\deg(b_1)}.$$

Since $p - \deg(a_1) - 1 + q - \deg(b_1) = p + q - \deg(a_1) - \deg(b_1) - 1 < p + q$, we have $\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}') \in \Lambda^{p+q-\deg(a_1)-\deg(b_1)-1}$ by the induction hypothesis. Thus

$$\begin{aligned} \deg(a_1 b_1 \otimes (\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}'))) &= \deg(a_1 b_1) + \deg(\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}')) + 1 \\ &\leq \deg(a_1) + \deg(b_1) + p + q - \deg(a_1) - \deg(b_1) - 1 + 1 \\ &= p + q. \end{aligned}$$

This gives $a_1 b_1 \otimes (\mathfrak{a}' \diamond_\lambda (1_A \otimes \mathfrak{b}')) \in \Lambda^{p+q}$. Similarly, $a_1 b_1 \otimes ((1_A \otimes \mathfrak{a}') \diamond_\lambda \mathfrak{b}') \in \Lambda^{p+q}$ and $\lambda a_1 b_1 \otimes (\mathfrak{a}' \diamond_\lambda \mathfrak{b}') \in \Lambda^{p+q-1}$. For the fourth term on the right-hand side of Eq. (47), by $\deg(P(1_A)) = \deg(1_A) + \deg(1_A) + 1 = 1$ and the induction hypothesis, we obtain

$$\mathfrak{a}' \diamond_\lambda P(1_A) \in \Lambda^{p-\deg(a_1)},$$

and thus using the induction hypothesis yields $((\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}') \in \Lambda^{p+q-\deg(a_1)-\deg(b_1)-1}$, thereby proving

$$a_1 b_1 \otimes \Big((\mathfrak{a}' \diamond_\lambda P_\lambda(1_A)) \diamond_\lambda \mathfrak{b}' \Big) \in \Lambda^{p+q}.$$

Hence all terms on the right-hand side of Eq. (47) are in Λ^{p+q}, yielding $\mathfrak{a} \diamond_\lambda \mathfrak{b} \in \Lambda^{p+q}$.

Finally, it remains to prove Eq. (46). The proof proceeds by induction on $k \geq 0$, with the case $k = 0$ is true, because $\Lambda^0 = \mathbf{k}1_A$ and $\Delta_\lambda(1_A) = 1_A \otimes 1_A$. Assume that $k \geq 0$ and Eq. (46) holds for all pure tensors $\mathfrak{a} \in \Lambda^k$. Consider $\mathfrak{a} \in \Lambda^{k+1}$. If $\mathfrak{a} \in A (= \cup_{n\geq 0} A^n)$, then by $\deg(\mathfrak{a}) \leq k+1$, we get $\mathfrak{a} \in A^{k+1}$. Since A is a connected filtered left counital bialgebra and $A^n \subseteq \Lambda^n$ for all $n \geq 0$, we have

$$\Delta_\lambda(\mathfrak{a}) = \Delta_A(\mathfrak{a}) \in A^0 \otimes A^{k+1} + \sum_{\substack{p+q=k+1\\ p>0,q>0}} A^p \otimes A^q \subseteq \Lambda^0 \otimes \Lambda^{k+1} + \sum_{\substack{p+q=k+1\\ p>0,q>0}} \Lambda^p \otimes \Lambda^q.$$

We then suppose that $\mathfrak{a} = a_1 \otimes \mathfrak{a}' \in A^{\otimes \ell+1}$ with $\mathfrak{a}' \in A^\ell$ for $\ell \geq 1$. Then

$$\begin{aligned} \Delta_\lambda(\mathfrak{a}) &= \Delta_\lambda(a_1 \otimes \mathfrak{a}') \\ &= \Delta_\lambda(a_1) \bullet ((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')) \quad \text{(by Eq. (25))} \\ &= \Delta_A(a_1) \bullet ((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')). \end{aligned}$$

By $\mathfrak{a} \in \Lambda^{k+1}$ and Eq. (44), we get $\mathfrak{a}' \in \Lambda^{k+1-\deg(a_1)-1} = \Lambda^{k-\deg(a_1)}$. Then applying the induction hypothesis gives

$$\Delta_\lambda(\mathfrak{a}') \in \Lambda^0 \otimes \Lambda^{k-\deg(a_1)} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_2>0,q_2>0}} \Lambda^{p_2} \otimes \Lambda^{q_2}.$$

Thus

$$\begin{aligned} \Delta_\lambda(\mathfrak{a}) &= \Delta_A(a_1) \bullet ((\mathrm{id} \otimes P_\lambda)\Delta_\lambda(\mathfrak{a}')) \\ &\in \Big(\Lambda^0 \otimes \Lambda^{\deg(a_1)} + \sum_{\substack{p_1+q_1=\deg(a_1)\\ p_1>0,q_1>0}} \Lambda^{p_1} \otimes \Lambda^{q_1}\Big) \\ &\quad \bullet \Big(\mathrm{id} \otimes P_\lambda\Big)\Big(\Lambda^0 \otimes \Lambda^{k-\deg(a_1)} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_2>0,q_2>0}} \Lambda^{p_2} \otimes \Lambda^{q_2}\Big) \\ &\subseteq \Big(\Lambda^0 \otimes \Lambda^{\deg(a_1)} + \sum_{\substack{p_1+q_1=\deg(a_1)\\ p_1>0,q_1>0}} \Lambda^{p_1} \otimes \Lambda^{q_1}\Big) \\ &\quad \bullet \Big(\Lambda^0 \otimes \Lambda^{k-\deg(a_1)+1} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_2>0,q_2>0}} \Lambda^{p_2} \otimes \Lambda^{q_2+1}\Big) \\ &\subseteq \Lambda^0 \otimes \Lambda^{k+1} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_2>0,q_2>0}} \Lambda^{p_2} \otimes \Lambda^{\deg(a_1)+q_2+1} \\ &\quad + \sum_{\substack{p_1+q_1=\deg(a_1)\\ p_1>0,q_1>0}} \Lambda^{p_1} \otimes \Lambda^{q_1+k-\deg(a_1)+1} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_1+q_1=\deg(a_1)\\ p_1>0,q_1>0,p_2>0,q_2>0}} \Lambda^{p_1+p_2} \otimes \Lambda^{q_1+q_2+1} \\ &\subseteq \Lambda^0 \otimes \Lambda^{k+1} + \sum_{\substack{p_2+q_2=k-\deg(a_1)\\ p_1+q_1=\deg(a_1)\\ p_1\geq 0,q_1>0,p_2\geq 0,q_2>0}} \Lambda^{p_1+p_2} \otimes \Lambda^{q_1+q_2+1} \quad (p_1^2+p_2^2 \neq 0) \\ &\subseteq \Lambda^0 \otimes \Lambda^{k+1} + \sum_{\substack{p+q=k+1\\ p>0,q>0}} \Lambda^p \otimes \Lambda^q \quad (p := p_1+p_2,\ q := q_1+q_2+1). \end{aligned}$$

This completes the induction and thus proves Eq. (46). □

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