Asymptotic behaviour of the finite blow-up points solutions of the fast diffusion equation

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Abstract

Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $i_0 \in \mathbb{Z}^+$, $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $a_1, a_2, \dots, a_{i_0} \in \Omega$, $\widehat{\Omega} = \Omega \setminus \{a_1, a_2, \dots, a_{i_0}\}$, $0 \leq f \in L^\infty(\partial\Omega)$ and $0 \leq u_0 \in L^p_{loc}(\widehat{\Omega})$ for some constant $p > \frac{n(1-m)}{2}$ which satisfies $\lambda_i |x - a_i|^{-\gamma_i} \leq u_0(x) \leq \lambda_i' |x - a_i|^{-\gamma_i'}$ $\forall 0 < |x - a_i| < \delta$, $i = 1, \dots, i_0$ where $\delta > 0$, $\lambda_i' \geq \lambda_i > 0$ and $\frac{2}{1-m} < \gamma_i \leq \gamma_i' < \frac{n-2}{m}$ $\forall i = 1, 2, \dots, i_0$ are constants. We will prove the asymptotic behaviour of the finite blow-up points solution u of $u_t = \Delta u^m$ in $\widehat{\Omega} \times (0, \infty)$, $u(a_i, t) = \infty$ $\forall i = 1, \dots, i_0, t > 0$, $u(x, 0) = u_0(x)$ in $\widehat{\Omega}$ and $u = f$ on $\partial\Omega \times (0, \infty)$, as $t \to \infty$. We will construct finite blow-up points solution in bounded cylindrical domain with appropriate lateral boundary value such that the finite blow-up points solution oscillates between two given harmonic functions as $t \to \infty$. We will also prove the existence of the minimal solution of $u_t = \Delta u^m$ in $\widehat{\Omega} \times (0, \infty)$, $u(x, 0) = u_0(x)$ in $\widehat{\Omega}$, $u(a_i, t) = \infty \quad \forall t > 0, i = 1, 2 \dots, i_0$ and $u = \infty$ on $\partial\Omega \times (0, \infty)$.

Keywords: finite blow-up points solutions, fast diffusion equation, asymptotic behaviour, blow-up at boundary

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1 Introduction

Recently there is a lot of study on the properties of the fast diffusion equation,

$$u_t = \Delta u^m \tag{1.1}$$

with $0 < m < 1$ by P. Daskalopoulos, M. Fila, S.Y. Hsu, K.M. Hui, T. Jin, S. Kim, Y.C. Kwong, P. Macková, M. del Pino, M. Sáez, N. Sesum, J. Takahashi, J.L. Vazquez, H. Yamamoto, E. Yanagida, M. Winkler and J. Xiong, etc. [DS1], [DS2], [FMTY], [H1], [H2], [HK1], [HK2], [Hs], [JX], [K], [PS], [TY], [VW1], [VW2]. The equation (1.1) arises in many physical models and in geometry. When $m > 1$, (1.1) is called the porous medium equation which arises in the modelling of gases passing through porous media and oil passing through sand [Ar]. (1.1) also arises as the diffusive limit for the generalized Carleman kinetic equation [CL], [GS], and as the large time asymptotic limit of the solution of the free boundary compressible Euler equation with damping [LZ]. When $m = 1$, it is the heat equation. When the dimension $n \geq 3$ and $m = \frac{n-2}{n+2}$, (1.1) arises in the study of the Yamabe flow [DS1], [DS2], [PS].

Various fundamental results in $\mathbb{R}^n$ for the equation (1.1) are obtained recently by A. Friedman and S. Kamin [FrK], and M. Bonfonte, J. Dolbeault, G. Grillo, M. del Pino and J.L. Vazquez [BBDGV], [BGV1], [BV1], [BV2], [CaV], [PD], [V1]. Results for the equation (1.1) in bounded domains are also obtained by D.G. Aronson and L.A. Peletier [ArP], B.E.J. Dahlberg and C.E. Kenig [DaK], E. Dibenedetto and Y.C. Kwong [DiK], E. Feireisl and F. Simondon [FeS], M. Bonfonte, G. Grillo and J.L. Vazquez [BGV2], [V2]. We refer the readers to the books by P. Daskalopoulos and C.E. Kenig [DK] and J.L. Vazquez [V3], [V4], for some recent results on the equation (1.1).

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $i_0 \in \mathbb{Z}^+$, $a_1, a_2, \dots, a_{i_0} \in \Omega$ and

$$\widehat{\Omega} = \Omega \setminus \{a_1, a_2, \dots, a_{i_0}\}. \tag{1.2}$$

Let $\delta_0 = \frac{1}{3}\min_{1\leq i,j\leq i_0}\left(\mathbf{dist}(a_i, \partial\Omega), \left|a_i - a_j\right|\right)$. For any $\delta > 0$, let

$$\Omega_\delta = \Omega \setminus \left(\cup_{i=1}^{i_0} B_\delta(a_i)\right) \quad \text{and} \quad D_\delta = \{x \in \Omega_\delta : \text{dist}\,(x, \partial\Omega) > \delta\}. \tag{1.3}$$

For any $j \in \mathbb{Z}^+$, let

$$E_j = \{x \in \widehat{\Omega} : \text{dist}\,(x, \partial\Omega) > 1/j\} \tag{1.4}$$

and $j_0 \in \mathbb{Z}^+$ be such that $j_0 > 1/\delta_0$.

When $n \geq 3$ and $\frac{n-2}{n} < m < 1$, existence of positive smooth solution of the Cauchy problem,

$$\begin{cases} u_t = \Delta u^m & \text{in } \mathbb{R}^n \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^n \end{cases} \tag{1.5}$$

for any $0 \leq u_0 \in L^1_{loc}(\mathbb{R}^n)$, $u_0 \not\equiv 0$, was proved by M.A. Herrero and M. Pierre in [HP]. This implies that the diffusion for the solution of (1.5) when $m < 1$ must be very fast so that for any $t > 0$ the solution $u(x, t)$ is positive everywhere even though the initial value u_0 may only have compact support in $\mathbb{R}^n$.

When $m > 1$, the Barenblatt solution

$$B(x, t) = t^{-\alpha}\left(C_1 - \frac{\alpha(m-1)|x|^2}{2mnt^{\alpha/n}}\right)_+^{\frac{1}{m-1}}, \quad x \in \mathbb{R}^n, t > 0$$

where $\alpha = \frac{n}{n(m-1)+2}$ and $C_1 > 0$ is any constant is a solution of (1.1) in $\mathbb{R}^n \times (0,\infty)$. Note that the Barenblatt solution $B(x,t)$ has compact support for any $t > 0$ and the diffusion for the solution of (1.1) when $m > 1$ is slow. Hence the behaviour of the solution of (1.1) is very different for $m < 1$ and $m > 1$.

Uniqueness of solutions of (1.5) for the case $0 < m < 1$ and $n \geq 1$ was also proved in [HP]. This result was later extended by G. Grillo, M. Muratori and F. Punzo [GMP], to the uniqueness of the strong solution of the equation

$$\begin{cases} u_t = \Delta u^m & \text{in } M \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } M \end{cases}$$

on a Riemannian manifold M whose Ricci curvatures satisfies some lower bound condition. Note that in the uniqueness theorems of [HP] and [GMP] the solutions considered are strong solutions. That is the solution u satisfies

$$u_t \in L^1_{loc}(\mathbb{R}^n \times (0,\infty)) \quad \text{in [HP]}$$

and

$$u_t \in L^1_{loc}(M \times (0,\infty)) \quad \text{in [GMP].}$$

However in the comparison results (Theorem 3.4 and Theorem 3.5) that we will prove in this paper the subsolutions and the supersolutions that we consider are $C^{2,1}(\mathbb{R}^n \times (0,T))$ functions and the condition $u_t \in L^1_{loc}(\mathbb{R}^n \times (0,T))$ is automatically satisfied.

Asymptotic behaviour of the solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega \times (0,T) \\ u = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases}$$

which vanishes at time $T > 0$ where $0 < m < 1$, $n \geq 3$ and $u_0 \geq 0$ is a function on Ω was studied by G. Akagi [A1], [A2], J.G. Berryman and C.J. Holland [BH], B. Choi , R.J. Mccann and C. Seis [CMS], etc. Let $\mu_0 > 0$, $\mu_0 \leq f_1 \in C^1(\overline{\Omega})$ and $\mu_0 \leq v_0 \in C^2(\overline{\Omega} \setminus \{a_1\})$ satisfies

$$\lambda_1 |x - a_1|^{-\gamma_1} \leq v_0(x) \leq \lambda_1' |x - a_1|^{-\gamma_1'} \quad \forall \Omega \setminus \{a_1\}$$

for some constants $\lambda_1' \geq \lambda_1 > 0$, $\gamma_1' \geq \gamma_1 > \frac{2}{1-m}$. When $n \geq 3$ and $0 < m \leq \frac{n-2}{n}$, existence and asymptotic large time behaviour of the Dirichlet blow-up solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } (\Omega \setminus \{a_1\}) \times (0,\infty) \\ u = f_1 & \text{on } \partial\Omega \times (0,\infty) \\ u(a_1,t) = \infty & \forall t > 0 \\ u(x,0) = v_0(x) & \text{in } \Omega \setminus \{a_1\} \end{cases}$$

has been proved by J.L. Vazquez and M. Winkler in [VW1], [VW2]. When $n \geq 3$ and $0 < m < \frac{n-2}{n}$, existence of finite blow-up points solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = f & \text{on } \partial\Omega \times (0,\infty) \\ u(a_i,t) = \infty & \forall t > 0, i = 1,2\dots,i_0 \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{1.6}$$

for any $0 \leq f \in L^\infty(\partial\Omega \times (0,\infty))$ and $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1,\dots,a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ which satisfies

$$u_0(x) \geq \lambda_i |x-a_i|^{-\gamma_i} \quad \forall |x-a_i| < \delta_1, i = 1,\dots,i_0$$

for some constants $0 < \delta_1 < \delta_0$, $\lambda_i > 0$ and $\gamma_i > \frac{2}{1-m}$ for any $i = 1,\dots,i_0$, has been proved by K.M. Hui and S. Kim [HK2]. When $n \geq 3$, $0 < m < \frac{n-2}{n}$ and f, u_0, also satisfy $f \geq \mu_0$ and $u_0 \geq \mu_0$ for some constant $\mu_0 > 0$ and

$$u_0(x) \leq \lambda_i' |x-a_i|^{-\gamma_i'} \quad \forall |x-a_i| < \delta_1, i = 1,\dots,i_0$$

for some constants $\lambda_i' \geq \lambda_i > 0$, $\gamma_i' \geq \gamma_i > \frac{2}{1-m}$, $i = 1,\dots,i_0$, asymptotic large time behaviour of the finite blow-up points solution of (1.6) has been proved by K.M. Hui and S. Kim in [HK2] and [H2].

When $n \geq 3$ and $0 < m \leq \frac{n-2}{n}$, existence of finite blow-up points solutions of (1.1) in bounded cylindrical domains was also proved by K.M. Hui and Sunghoon Kim in [HK1] using a different method when the initial value u_0 satisfies

$$u_0(x) \approx |x-a_i|^{-\gamma_i} \quad \text{for } x \approx a_i \quad \forall i = 1,2,\dots,i_0$$

for some constants $\gamma_i > \max\left(\frac{n}{2m}, \frac{n-2}{m}\right)$, $i = 1,2,\dots,i_0$.

Outline of our results:

- We improve the existence theorems of [HK2] (Theorem 1.1 and Theorem 1.2 of [HK2]) to the existence of unique maximal solutions of (1.6) (Theorem 1.1 and Theorem 1.2).
- We extend the comparison theorems of [H2] (Theorem 1.1 and Theorem 1.2 of [H2]) by removing the requirement that the boundary values and the initial values must be larger than some positive constant (Theorem 3.4 and Theorem 3.5).
- We extend the asymptotic large time behaviour of the finite blow-up points solutions results of [HK2] and [H2] by removing the requirement that the boundary value f and the initial value u_0 must be larger than some positive constant (Theorem 1.3 and Theorem 1.4). More precisely we prove the asymptotic large time behaviour of the finite blow-up points solution of (1.6) (Theorem 1.3 and Theorem 1.4) for any $0 \leq f \in L^\infty(\partial\Omega \times (0,\infty))$ and $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1,\dots,a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ which satisfies

$$\frac{\lambda_i}{|x-a_i|^{\gamma_i}} \leq u_0(x) \leq \frac{\lambda_i'}{|x-a_i|^{\gamma_i'}} \qquad \forall 0 < |x-a_i| < \delta_1, i = 1,\cdots,i_0 \tag{1.7}$$

for some constants $0<\delta_1<\delta_0$, $\lambda_1,\cdots,\lambda_{i_0},\lambda_1',\cdots,\lambda_{i_0}'\in\mathbb{R}^+$ and

$$\frac{2}{1-m}<\gamma_i\le\gamma_i'<\frac{n-2}{m}\quad\forall i=1,2,\dots,i_0. \tag{1.8}$$

- In the paper [H2] K.M. Hui constructed a solution of (1.6) which oscillates between some fixed positive number and infinity as $t\to\infty$. A natural question to ask is whether there exist solutions of (1.6) which oscillate between some functions on Ω. We answer this question in the affirmative. We will construct (Theorem 1.5) a solution of (1.6) with appropriate lateral boundary value such that the solution of (1.6) will oscillate between two given harmonic functions as $t\to\infty$.

- We will prove the existence of minimal finite blow-up points solutions of (1.1) in bounded cylindrical domains (Theorem 1.6) which also blow-up everywhere on the lateral boundary of the domain. Asymptotic large time behaviour of such solution is also prove in Theorem 1.6.

More precisely we obtain the following results. The first four theorems are extensions of Theorem 2.3, Theorem 2.4 of [H2] and Theorem 1.5 of [HK2].

Theorem 1.1. *Let $n\ge 3$, $0<m<\frac{n-2}{n}$, $0<\delta_1<\min(1,\delta_0)$, $0\le f\in C^3(\partial\Omega\times(0,\infty))\cap L^\infty(\partial\Omega\times(0,\infty))$ and $0\le u_0\in L^p_{loc}(\overline{\Omega}\setminus\{a_1,\cdots,a_{i_0}\})$ for some constant $p>\frac{n(1-m)}{2}$ be such that (1.7) holds for some constants*

$$\gamma_i'\ge\gamma_i>\frac{2}{1-m}\quad\forall i=1,2,\dots,i_0 \tag{1.9}$$

and $\lambda_1,\cdots,\lambda_{i_0},\lambda_1',\cdots,\lambda_{i_0}'\in\mathbb{R}^+$. Let $\widehat{\Omega}$ be given by (1.2). Then there exists a unique maximal solution u of (1.6) such that for any constants $T>0$ and $\delta_2\in(0,\delta_1)$ there exist constants $C_1=C_1(T)>0$, $C_2=C_2(T)>0$, depending only on $\lambda_1,\cdots,\lambda_{i_0},\lambda_1',\cdots,\lambda_{i_0}',\gamma_1,\cdots,\gamma_{i_0},\gamma_1',\cdots,\gamma_{i_0}'$, such that

$$\frac{C_1}{|x-a_i|^{\gamma_i}}\le u(x,t)\le\frac{C_2}{|x-a_i|^{\gamma_i'}}\quad\forall 0<|x-a_i|<\delta_2, 0<t<T, i=1,2,\dots,i_0 \tag{1.10}$$

holds. Moreover the following holds.

(i) If there exist constants $T_0'>T_0>0$ and $\mu_1>0$ such that

$$f\ge\mu_1\quad\text{on } \partial\Omega\times[T_0,T_0'), \tag{1.11}$$

then for any $T_1\in(T_0,T_0')$ there exists a constant $\mu_2\in(0,\mu_1)$ such that

$$u(x,t)\ge\mu_2\quad\forall x\in\widehat{\Omega}, T_1\le t<T_0'. \tag{1.12}$$

(ii) If there exists a constant $T_2 \geq 0$ *such that*

$$f(x,t) \text{ is monotone decreasing in } t \text{ on } \partial\Omega \times (T_2, \infty), \tag{1.13}$$

then u satisfies

$$u_t \leq \frac{u}{(1-m)(t-T_2)} \quad \text{in } \widehat{\Omega} \times (T_2, \infty). \tag{1.14}$$

Theorem 1.2. *Let* $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$, $0 \leq f \in L^\infty(\partial\Omega \times (0,\infty))$ *and* $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ *for some constant* $p > \frac{n(1-m)}{2}$ *be such that* (1.7) *holds for some constants* γ_i, γ_i', $i = 1, \ldots, i_0$, *satisfying* (1.8) *and* $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. *Let* $\widehat{\Omega}$ *be given by* (1.2). *Then there exists a unique maximal solution u of* (1.6) *such that for any constants* $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exist constants* $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, *depending only on* $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}', \gamma_1, \cdots, \gamma_{i_0}, \gamma_1', \cdots, \gamma_{i_0}'$, *such that* (1.10) *holds. Moreover the following holds.*

(i) If there exists constants $T_0' > T_0 > 0$ *and* $\mu_1 > 0$ *such that* (1.11) *holds, then for any* $T_1 \in (T_0, T_0')$ *there exists a constant* $\mu_2 \in (0, \mu_1)$ *such that* (1.12) *holds.*

(ii) If there exists a constant $T_2 \geq 0$ *such that* (1.13) *holds, then u satisfies* (1.14).

Theorem 1.3. *Let* $n \geq 3$ *and* $0 < m < \frac{n-2}{n}$. *Let* $0 \leq g \in C^3(\partial\Omega)$ *and* ϕ *be the solution of*

$$\begin{cases} \Delta\phi = 0 & \text{in } \Omega \\ \phi = g^m & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

Let $0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}\right)$ *for some constant* $p > \frac{n(1-m)}{2}$ *be such that* (1.7) *holds for some constants* γ_i, γ_i', $i = 1, \ldots, i_0$, *satisfying* (1.8) *and* $0 < \delta_1 < \min(1, \delta_0)$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. *Let* $f \in C^3(\partial\Omega \times (0,\infty))) \cap L^\infty(\partial\Omega \times (0,\infty)))$ *be such that*

$$f \to g \quad \text{uniformly in } C^3(\partial\Omega) \quad \text{as } t \to \infty. \tag{1.16}$$

Let $\widehat{\Omega}$ *be given by* (1.2). *Let u be the unique maximal solution of* (1.6) *given by Theorem 1.1. Then the following holds.*

(i) If $g > 0$ *on* $\partial\Omega$, *then*

$$u(x,t) \to \phi^{\frac{1}{m}} \quad \text{uniformly in } C^2(K) \quad \text{as } t \to \infty \tag{1.17}$$

holds for any compact subset K of $\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\}$.

(ii) If $g \not\equiv 0$ *on* $\partial\Omega$,

$$f \geq g \quad \text{on } \partial\Omega \times (0,\infty) \tag{1.18}$$

and

$$u_0 \geq \phi^{\frac{1}{m}} \quad \text{on } \widehat{\Omega} \tag{1.19}$$

holds, then (1.17) *holds for any compact subset K of* $\widehat{\Omega}$.

(iii) If $g \equiv 0$ on $\partial\Omega$, then

$$u(x,t) \to 0 \quad \text{uniformly in } K \quad \text{as } t \to \infty \tag{1.20}$$

for any compact subset K of $\widehat{\Omega}$.

Theorem 1.4. *Let $n \geq 3$ and $0 < m < \frac{n-2}{n}$. Let $0 \leq g \in C(\partial\Omega)$ and ϕ be the solution of* (1.15). *Let $0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}\right)$ for some constant $p > \frac{n(1-m)}{2}$ be such that* (1.7) *holds for some constants γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying* (1.8) *and $0 < \delta_1 < \min(1, \delta_0)$, $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. Let $f \in L^\infty(\partial\Omega \times (0,\infty)))$ be such that*

$$f \to g \quad \text{uniformly in } L^\infty(\partial\Omega) \quad \text{as } t \to \infty.$$

Let $\widehat{\Omega}$ be given by (1.2). *Let u be the unique maximal solution of* (1.6) *given by Theorem 1.2. Then the following holds.*

(i) If $g > 0$ on $\partial\Omega$, then (1.17) *holds for any compact subset K of $\widehat{\Omega}$.*

(ii) If $g \not\equiv 0$ on $\partial\Omega$ and both (1.18) *and* (1.19) *holds, then* (1.17) *holds for any compact subset K of $\widehat{\Omega}$.*

(iii) If $g \equiv 0$ on $\partial\Omega$, then (1.20) *holds for any compact subset K of $\widehat{\Omega}$.*

Theorem 1.5. *Let $n \geq 3$ and $0 < m < \frac{n-2}{n}$. Let $g_1, g_2 \in C(\partial\Omega)$, $g_2 > 0$, $g_1 > 0$, and ϕ_1, ϕ_2, be the solutions of* (1.15) *with $g = g_1, g_2$, respectively. Let*

$$0 < \mu_0 < \min\left(\min_{\partial\Omega} g_1, \min_{\partial\Omega} g_2\right)$$

be a constant. Let $0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}\right)$ for some constant $p > \frac{n(1-m)}{2}$ be such that (1.7) *holds for some constants γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying* (1.8) *and $0 < \delta_1 < \min(1, \delta_0)$, $\lambda_1, \cdots, \lambda_{i_0}$, $\lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$. Let $\widehat{\Omega}$ be given by* (1.2). *Then there exist a function $f \in L^\infty(\partial\Omega \times (0,\infty)))$ and an increasing sequence $\{t_i\}_{i=1}^\infty$, $t_i \to \infty$ as $i \to \infty$, such that if u is the maximal solution of* (1.6) *given by Theorem 1.2, then*

$$\begin{cases} u(x, t_{2i-1}) \to \phi_1^{\frac{1}{m}} & \text{in } C^2(K) \quad \text{as } i \to \infty \\ u(x, t_{2i}) \to \phi_2^{\frac{1}{m}} & \text{in } C^2(K) \quad \text{as } i \to \infty \end{cases}$$

for any compact subset K of $\widehat{\Omega}$.

Theorem 1.6. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$ and $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ be such that* (1.7) *holds for some constants γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying*

(1.9) *and* $\lambda_1, \cdots, \lambda_{i_0}, \lambda'_1, \cdots, \lambda'_{i_0} \in \mathbb{R}^+$. *Then there exists a unique minimal solution u of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = \infty & \text{on } \partial\Omega \times (0,\infty) \\ u(a_i,t) = \infty & \forall t > 0, i = 1,2\ldots,i_0 \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega} \end{cases} \tag{1.21}$$

such that for any constants $T > 0$ *and* $\delta_2 \in (0,\delta_1)$ *there exist constants* $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, *depending only on* $\lambda_1, \cdots, \lambda_{i_0}, \lambda'_1, \cdots, \lambda'_{i_0}, \gamma_1, \cdots, \gamma_{i_0}, \gamma'_1, \cdots, \gamma'_{i_0}$, *such that* (1.10) *holds. Moreover u satisfies* (1.14) *with* $T_2 = 0$ *and*

$$u(x,t) \to \infty \quad \text{uniformly on } \Omega_\delta \quad \text{as } t \to \infty \quad \forall 0 < \delta < \delta_0. \tag{1.22}$$

Remark 1.7. *The integrability condition* $0 \le u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ *for some constant* $p > \frac{n(1-m)}{2}$ *is necessary since this condition together with* $f \in L^\infty(\partial\Omega \times (0,\infty))$ *implies that the solution u of* (1.6) *locally satisfies a* $L^\infty - L^p$ *regularizing result in terms of the local* L^p *norm of the initial value* u_0 *and* L^∞ *norm of f (Lemma 3.2 and Lemma 3.3).*

Remark 1.8. *In the proof of Theorem 1.1 and Theorem 1.2 we will construct the solution of* (1.6) *as the limit of a monotone decreasing sequence of solutions of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u(a_i,t) = \infty & \forall t > 0, i = 1,2\ldots,i_0 \end{cases} \tag{1.23}$$

which have initial values strictly greater than u_0 *and lateral boundary values strictly greater than f. Since there is comparison results (Theorem 3.4 and Theorem 3.5) between any solution of* (1.6) *and this monotone decreasing sequence of solutions of* (1.23). *Hence this constructed solution of* (1.6) *must be maximal solution of* (1.6) *by comparison argument. On the other hand if we construct solution of* (1.6) *as the limit of a monotone increasing sequence of solutions of* (1.23) *which have initial values less than* u_0 *and lateral boundary values less than f. Since there is no comparison result between any solution of* (1.6) *and this monotone increasing sequence of solutions of* (1.23). *Hence it is not clear whether minimal solution of* (1.6) *exists.*

The plan of the paper is as follows. For the readers' convenience in section 2 we recall some results of [H1], [H2] and [HK2] that is cited in this paper. In section 3 we will prove Theorem 1.1 and Theorem 1.2. We will prove Theorem 1.3 and Theorem 1.4 in section 4. We will prove Theorem 1.5 and Theorem 1.6 in section 5. Unless stated otherwise we will assume that $n \ge 3$ and $0 < m < \frac{n-2}{n}$ for the rest of the paper.

We start with some definitions. For any $a \in \mathbb{R}^+$, let $a_+ = \max(a, 0)$.

Definition 1.9. *For any* $t_2 > t_1$, *we say that u is a solution (subsolution, supersolution respectively) of* (1.1) *in* $\Omega \times (t_1,t_2)$ *if* $u \in C^{2,1}(\Omega \times (t_1,t_2))$ *is positive in* $\Omega \times (t_1,t_2)$ *and satisfies*

$$u_t = \Delta u^m \quad \text{in } \Omega \times (t_1,t_2) \quad (\le, \ge, \text{ respectively}).$$

Definition 1.10. *For any* $0 \le f \in L^\infty(\partial\Omega \times (0,T))$ *and* $0 \le u_0 \in L^1_{loc}(\Omega)$, *we say that* u *is a solution (subsolution, supersolution respectively) of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \Omega \times (0,T) \\ u = f & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.24}$$

if u *is a solution (subsolution, supersolution respectively) of* (1.1) *in* $\Omega \times (0,T)$ *which satisfies*

$$\|u(\cdot,t) - u_0\|_{L^1(\Omega)} \to 0 \quad \text{as } t \to 0$$

and the boundary condition is satisfied in the sense that

$$\int_{t_1}^{t_2} \int_\Omega (u\eta_t + u^m \Delta\eta)\,dx\,dt = \int_{t_1}^{t_2} \int_{\partial\Omega} f^m \frac{\partial\eta}{\partial\nu}\,d\sigma\,dt + \int_\Omega u\eta\,dx \Big|_{t_1}^{t_2}$$

($\ge, \le$ *respectively) holds for any* $0 < t_1 < t_2 < T$ *and* $\eta \in C_c^2(\overline{\Omega} \times (0,T))$ *satisfying* $\eta = 0$ *on* $\partial\Omega \times (0,T)$.

Definition 1.11. *For any* $T > 0$, $0 \le f \in L^\infty(\partial\Omega \times (0,T))$ *and* $0 \le u_0 \in L^1_{loc}(\widehat{\Omega})$ *where* $\widehat{\Omega}$ *is given by* (1.2), *we say that* u *is a solution (subsolution, supersolution respectively) of*

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,T) \\ u(x,t) = f & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \widehat{\Omega}. \end{cases} \tag{1.25}$$

if u *is a solution (subsolution, supersolution respectively) of* (1.1) *in* $\widehat{\Omega} \times (0,T)$ *which satisfies*

$$\|u(\cdot,t) - u_0\|_{L^1(K)} \to 0 \quad \text{as } t \to 0 \tag{1.26}$$

for any compact set $K \subset \overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}$ *and*

$$\begin{aligned} &\int_{t_1}^{t_2} \int_{\widehat{\Omega}} (u\eta_t + u^m \Delta\eta)\; dxdt \\ &\qquad = \int_{t_1}^{t_2} \int_{\partial\Omega} f^m \frac{\partial\eta}{\partial\nu}\,d\sigma dt + \int_{\widehat{\Omega}} u(x,t_2)\eta(x,t_2)\,dx - \int_{\widehat{\Omega}} u(x,t_1)\eta(x,t_1)\,dx \end{aligned}$$

($\ge, \le$ *respectively) for any* $0 < t_1 < t_2 < T$ *and* $\eta \in C_c^2((\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}) \times (0,T))$ *satisfying* $\eta \equiv 0$ *on* $\partial\Omega \times (0,T)$.

Definition 1.12. *We say that* u *is a solution (subsolution, supersolution respectively) of* (1.6) *if* u *is a solution (subsolution, supersolution respectively) of* (1.25) *and*

$$u(x,t) \to \infty \quad \text{as } x \to a_i \quad \forall t > 0, i = 1, \dots, i_0. \tag{1.27}$$

Definition 1.13. *We say that u is a maximal solution of* (1.6) *if u is a solution of* (1.6) *and for any solution v of* (1.6), $v \le u$ *in* $\widehat{\Omega} \times (0, T)$.

Definition 1.14. *We say that u is a solution of* (1.21) *if u is a solution of* (1.1) *in* $\widehat{\Omega} \times (0, \infty)$ *which satisfies* (1.26) *for any compact set* $K \subset \widehat{\Omega}$, (1.27) *and*

$$\lim_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,\infty)}} u(y,s) = \infty \quad \forall (x,t) \in \partial\Omega \times (0, \infty).$$

Definition 1.15. *We say that u is a minimal solution of* (1.21) *if u is a solution of* (1.21) *and for any solution v of* (1.21), $v \ge u$ *in* $\widehat{\Omega} \times (0, T)$.

2 Preliminaries

In this section we recall some results of [H1], [H2] and [HK2] that are cited in this paper.

Theorem 2.1 (Theorem 1.1 of [H2]). *Let* $n \ge 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $f_1, f_2 \in C^3(\partial\Omega \times (0, \infty)) \cap L^\infty(\partial\Omega \times (0, \infty))$ *be such that* $f_2 \ge f_1 \ge \mu_0$ *on* $\partial\Omega \times (0, \infty)$ *and*

$$\mu_0 \le u_{0,1} \le u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}) \quad \text{for some constant } p > \frac{n(1-m)}{2} \tag{2.1}$$

be such that

$$\frac{\lambda_i}{|x - a_i|^{\gamma_i}} \le u_{0,1}(x) \le u_{0,2} \le \frac{\lambda_i'}{|x - a_i|^{\gamma_i'}} \qquad \forall 0 < |x - a_i| < \delta_1, i = 1, \cdots, i_0 \tag{2.2}$$

holds for some constants $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ *and*

$$\gamma_i' \ge \gamma_i > \frac{2}{1-m} \quad \forall i = 1, 2, \ldots, i_0.$$

Suppose u_1, u_2, *are the solutions of* (1.6) *with* $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, *respectively which satisfy*

$$u_j(x,t) \ge \mu_0 \quad \forall x \in \widehat{\Omega}, t > 0, j = 1, 2 \tag{2.3}$$

such that for any constants $T > 0$ *and* $\delta_2 \in (0, \delta_1)$ *there exist constants* $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, *such that*

$$\frac{C_1}{|x - a_i|^{\gamma_i}} \le u_j(x,t) \le \frac{C_2}{|x - a_i|^{\gamma_i'}} \tag{2.4}$$

holds for any $0 < |x - a_i| < \delta_2$, $0 < t < T$, $i = 1, 2, \ldots, i_0$, $j = 1, 2$. *Suppose* u_1, u_2, *also satisfy*

$$\|u_i(\cdot, t) - u_{0,i}\|_{L^1(\Omega_\delta)} \to 0 \quad \text{as } t \to 0 \quad \forall 0 < \delta < \delta_0, i = 1, 2. \tag{2.5}$$

Then

$$u_1(x,t) \le u_2(x,t) \quad \forall x \in \widehat{\Omega}, t > 0.$$

Theorem 2.2 (Theorem 1.2 of [H2]). *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $\mu_0 \leq f_1 \leq f_2 \in L^\infty(\partial\Omega \times (0, \infty))$ and* (2.1), (2.2), *hold for some constants $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ satisfying* (1.8). *Suppose u_1, u_2, are the solutions of* (1.6) *with $u_0 = u_{0,1}, u_{0,2}$, $f = f_1, f_2$, respectively which satisfy* (2.3) *and* (2.5) *such that for any constants $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that* (2.4) *holds. Then*

$$u_1(x,t) \leq u_2(x,t) \quad \forall x \in \widehat{\Omega}, t > 0.$$

For any $m \in \mathbb{R}$, we let $\phi_m(u) = u^m/m$ if $m \neq 0$ and $\phi_m(u) = \log u$ if $m = 0$.

Lemma 2.3 (Lemma 1.7 of [H1]). *Let $m_0 < 0 < \varepsilon_1 < 1$ and $m \in [m_0, 1 - \varepsilon_1]$. Suppose u is a solution of*

$$u_t = \Delta\phi_m(u)$$

in $\Omega \times (0, T)$ with initial value $0 \leq u_0 \in L^p_{loc}(\Omega)$ for some constant

$$p > \max\left(1, (1 - m_0)\max(1, n/2)\right).$$

Then for any $B_{R_1}(x_0) \subset \overline{B_{R_2}(x_0)} \subset \Omega$ there exists a constant $C > 0$ such that

$$\int_{B_{R_1}(x_0)} u(x,t)^p dx \leq C\left\{t^{p/(1-m_0)} + t^{p/\varepsilon_1} + \int_{B_{R_2}(x_0)} u_0^p dx\right\}$$

holds for any $0 \leq t < T$, $m \in [m_0, 1 - \varepsilon_1]$.

Theorem 2.4 (Theorem 1.1 of [HK2]). *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \delta_0$, $0 \leq f \in L^\infty(\partial\Omega \times [0, \infty))$ and $0 \leq u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ be such that*

$$u_0(x) \geq \frac{\lambda_i}{|x - a_i|^{\gamma_i}} \qquad \forall 0 < |x - a_i| < \delta_1, \ i = 1, \cdots, i_0$$

holds for some constants $\lambda_1, \cdots, \lambda_{i_0} \in \mathbb{R}^+$ and $\gamma_1, \cdots, \gamma_{i_0} \in (\frac{2}{1-m}, \infty)$. Then there exists a solution u of (1.6) *such that for any $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exists a constant $C_1 > 0$ such that*

$$u(x,t) \geq \frac{C_1}{|x - a_i|^{\gamma_i}} \quad \forall 0 < |x - a_i| < \delta_2, 0 < t < T.$$

Moreover if there exists a constant $T_2 \geq 0$ such that (1.13) *holds, then u satisfies* (1.14).

Theorem 2.5 (Theorem 1.5 of [HK2]). *Suppose that $n \geq 3$, $0 < m < \frac{n-2}{n}$ and $\mu_0 > 0$. Let $\mu_0 \leq u_0 \in L^p_{loc}\left(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}\right)$ for some constant $p > \frac{n(1-m)}{2}$ such that* (1.7) *holds for some constants $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ and γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying* (1.8). *Let $f \in L^\infty(\partial\Omega \times (0, \infty)) \cap C^3(\partial\Omega \times (T_1, \infty))$ for some constant $T_1 > 0$ satisfy*

$$f \geq \mu_0 \quad \text{on } \partial\Omega \times (0, \infty)$$

and (1.16) *for some function $g \in C^3(\partial\Omega)$, $g \geq \mu_0$ on $\partial\Omega$. Let u be the solution of* (1.6) *given by Theorem 2.4. Let ψ be the solution of* (1.15). *Then* (1.17) *holds for any compact subset K of $\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}$.*

Lemma 2.6 (Lemma 2.9 of [HK2]). *Let $n \ge 3$, $0 < m \le \frac{n-2}{n}$, $0 \le f \in L^\infty(\partial\Omega \times [0,\infty))$ and $0 \le u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$. Suppose u is a solution of* (1.25). *Then for any $0 < \delta_6 < \delta_5 < \delta_0$ and $0 < t_1 < T$ there exist constants $C > 0$ and $\theta > 0$ such that*

$$\|u\|_{L^\infty(\Omega_{\delta_5}\times[t_1,T))} \le C\left(k_f^p|\Omega| + \int_{\Omega_{\delta_6}} u_0^p\,dx\right)^{\theta/p} + k_f$$

where $k_f = \max(1, \|f\|_{L^\infty})$.

Lemma 2.7 (Lemma 3.2 of [HK2]). *Let $n \ge 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$, $0 \le f \in L^\infty(\partial\Omega \times [0,\infty))$ and $0 \le u_0 \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ such that* (1.7) *holds for some constants $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}' \in \mathbb{R}^+$ and γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying* (1.8). *Let u be the solution of* (1.6) *given by Theorem 2.4. Then for any $0 < \delta_2 < \delta_0$ and $t_0 > 0$ there exist constants $C_2 > 0$ and $C_3 > 0$ such that*

$$u(x,t) \le C_2 \quad \forall x \in \overline{\Omega_{\delta_2}} \times [t_0, \infty)$$

and

$$u(x,t) \le C_3|x - a_i|^{-\gamma_i'} \quad \forall 0 < |x - a_i| \le \delta_2, t \ge t_0, i = 1, \cdots, i_0$$

hold.

Remark 2.8 (Remark 3.7 of [HK2]). *If $f \in L^\infty(\partial\Omega \times (0,\infty))$, $g \in C(\partial\Omega)$ and*

$$f(x.t) \to g(x) \quad \text{uniformly in } L^\infty(\partial\Omega) \text{ as } t \to \infty,$$

then the solution u of (1.6) *given by Theorem 1.1 of [HK2] satisfy* (1.17) *for any compact set $K \subset \widehat{\Omega}$. Moreover*

$$u(x,t) \to \psi^{\frac{1}{m}} \quad \text{in } L^\infty_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}) \quad \text{as } t \to \infty.$$

3 Existence of maximal blow-up solutions

In this section we will use a modification of the argument of [HK2] and [H2] to prove the existence of maximal solution of (1.6). We first extend Theorem 1.1 and Theorem 1.2 of [H2]. We start with a technical lemma.

Lemma 3.1. *Let $n \ge 3$, $0 < m < 1$, $p > \frac{n(1-m)}{2}$, $0 \le f \in L^\infty(\partial\Omega \times (0,T))$ and $0 \le u_0 \in L^\infty(\Omega)$. Suppose $u \in L^\infty(\Omega \times (0,T))$ is a solution of* (1.24). *Then for any $0 < \delta' < \delta < \delta_0$ there exists a constant $C > 0$ depending only on p, m, δ and δ' such that*

$$\int_{\Omega_\delta} u(x,t)^p\,dx \le C\left(\int_{\Omega_{\delta'}} u_0^p\,dx + t^{\frac{p}{1-m}} + \|f\|^p_{L^\infty(\partial\Omega\times(0,T))}\right) \quad \forall 0 < t < T \tag{3.1}$$

where Ω_δ, $\Omega_{\delta'}$, is given by (1.3).

Proof. We will use a modification of the proof of Lemma 1.7 of [H1] to prove this lemma. Let $0 \le \phi_1 \in C_0^\infty(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$, $0 \le \phi_1 \le 1$, be such that $\phi_1(x) = 1$ for any $x \in \overline{\Omega_\delta}$ and $\phi_1(x) = 0$ for any $x \in \Omega \setminus \Omega_{\delta'}$. Let $\phi_2 = \phi_1^\alpha$ for some constant $\alpha > \frac{2p}{1-m}$ and $k > \|f\|_{L^\infty}$. Let $\widehat{\Omega}$ be given by (1.2). Then

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)\\
=&p\int_{\widehat{\Omega}}(u-k)_+^{p-1}u_t\phi_2\,dx\\
=&p\int_{\widehat{\Omega}}(u-k)_+^{p-1}\phi_2\Delta u^m\,dx\\
=&-p\int_{\widehat{\Omega}}\nabla u^m\cdot\nabla[(u-k)_+^{p-1}\phi_2]\,dx\\
=&-pm\left\{(p-1)\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p-2}|\nabla u|^2\phi_2\,dx+\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p-1}\nabla u\cdot\nabla\phi_2\,dx\right\}. \qquad (3.2)
\end{aligned}
$$

Since

$$
\begin{aligned}
\left|\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p-1}\nabla u\cdot\nabla\phi_2\,dx\right| \le&(p-1)\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p-2}|\nabla u|^2\phi_2\,dx\\
&+\frac{1}{4(p-1)}\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p}|\nabla\phi_2|^2\phi_2^{-1}\,dx,
\end{aligned}
$$

by (3.2) and Hölder's inequality with exponents $\frac{p}{1-m}$ and $\frac{p}{p+m-1}$,

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)\\
\le&\frac{pm}{4(p-1)}\int_{\widehat{\Omega}}u^{m-1}(u-k)_+^{p}|\nabla\phi_2|^2\phi_2^{-1}\,dx\\
\le&\frac{pm}{4(p-1)}\int_{\widehat{\Omega}}[(u-k)_+^p\phi_2]^{\frac{p+m-1}{p}}|\nabla\phi_2|^2\phi_2^{\frac{1-m}{p}-2}\,dx\\
\le&\frac{pm}{4(p-1)}\left(\int_{\widehat{\Omega}}\left(|\nabla\phi_2|^2\phi_2^{\frac{1-m}{p}-2}\right)^{\frac{p}{1-m}}\,dx\right)^{\frac{1-m}{p}}\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)^{1-\frac{1-m}{p}}\quad\forall 0<t<T. \qquad (3.3)
\end{aligned}
$$

Since

$$
|\nabla\phi_2|^2\phi_2^{\frac{1-m}{p}-2}=\alpha^2\phi_1^{\frac{(1-m)\alpha}{p}-2}|\nabla\phi_1|^2\le\alpha^2|\nabla\phi_1|^2,
$$

we have

$$
\int_{\widehat{\Omega}}\left(|\nabla\phi_2|^2\phi_2^{\frac{1-m}{p}-2}\right)^{\frac{p}{1-m}}\,dx\le\alpha^{\frac{2p}{1-m}}\int_{\widehat{\Omega}}|\nabla\phi_1|^{\frac{2p}{1-m}}\,dx<\infty.
$$

Then, by (3.3),

$$\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)\le C\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)^{1-\frac{1-m}{p}}$$

$$\Rightarrow\quad \left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)^{\frac{1-m}{p}-1}\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u-k)_+^p\phi_2\,dx\right)\le C. \tag{3.4}$$

where $\widehat{\Omega}$ is given by (1.2). Integrating (3.4) over (t_1,t), $0<t_1<t<T$,

$$\int_{\widehat{\Omega}}(u(x,t)-k)_+^p\phi_2\,dx\le\left\{Ct+\left(\int_{\widehat{\Omega}}(u(x,t_1)-k)_+^p\phi_2\,dx\right)^{\frac{1-m}{p}}\right\}^{\frac{p}{1-m}}.$$

Hence

$$\begin{aligned}\int_{\Omega_\delta}(u(x,t)-k)_+^p\,dx\le&\left\{Ct+\left(\int_{\Omega_{\delta'}}(u(x,t_1)-k)_+^p\,dx\right)^{\frac{1-m}{p}}\right\}^{\frac{p}{1-m}}\\ \le&C'\left(\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx+t^{\frac{p}{1-m}}+k^p\right)\end{aligned}$$

holds for any $0<t_1\le t<T$, $k>\|f\|_{L^\infty(\partial\Omega\times(0,T))}$. Thus

$$\int_{\Omega_\delta}u(x,t)^p\,dx\le C\left(\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx+t^{\frac{p}{1-m}}+k^p\right) \tag{3.5}$$

holds for any $0<t_1\le t<T$, $k>\|f\|_{L^\infty(\partial\Omega\times(0,T))}$. Letting $k\searrow\|f\|_{L^\infty(\partial\Omega\times(0,T))}$ in (3.5),

$$\int_{\Omega_\delta}u(x,t)^p\,dx\le C\left(\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx+t^{\frac{p}{1-m}}+\|f\|^p_{L^\infty(\partial\Omega\times(0,T))}\right)\quad\forall 0<t_1\le t<T. \tag{3.6}$$

Let $C_1=\max(\|u_0\|_{L^\infty(\Omega)},\|u\|_{L^\infty(\Omega\times(0,T))})$. By the mean value theorem for any $x\in\Omega_{\delta'}$ and $0<t_1<T$ there exists a constant ξ between $u(x,t_1)$ and $u_0(x)$ such that

$$|u(x,t_1)^p-u_0(x)^p|=p|\xi|^{p-1}|u(x,t_1)-u_0(x)|\le pC_1^{p-1}|u(x,t_1)-u_0(x)|.$$

Hence

$$\left|\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx-\int_{\Omega_{\delta'}}u_0^p\,dx\right|\le pC_1^{p-1}\int_{\Omega_{\delta'}}|u(x,t_1)-u_0(x)|\,dx \tag{3.7}$$

holds for any $0<t_1<T$. Since u is a solution of (1.24) with initial value u_0, letting $t_1\to 0$ in (3.7) by Definition 1.10 we have,

$$\lim_{t_1\to 0}\left|\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx-\int_{\Omega_{\delta'}}u_0^p\,dx\right|=0$$

$$\Rightarrow\quad\lim_{t_1\to 0}\int_{\Omega_{\delta'}}u(x,t_1)^p\,dx=\int_{\Omega_{\delta'}}u_0^p\,dx. \tag{3.8}$$

Since (3.6) holds for any $t_1 \in (0,t)$, letting $t_1 \to 0$ in (3.6), by (3.8) we get (3.1) and the lemma follows.

□

By Lemma 3.1 and an argument similar to the proof of Corollary 1.8 of [H1] and a compactness argument we have the following result.

Lemma 3.2. *Let $n \geq 3$, $0 < m \leq \frac{n-2}{n}$, $p > \frac{n(1-m)}{2}$, $0 \leq f \in L^\infty(\partial\Omega \times (0,T))$ and $0 \leq u_0 \in L^\infty(\Omega)$. Suppose $u \in L^\infty(\Omega \times (0,T))$ is a solution of* (1.24). *Then for any $0 < \delta' < \delta < \delta_0$ and $0 < t_1 < t_2 < T$ there exist constants $C > 0$ and $\theta > 0$ such that*

$$\|u\|_{L^\infty(\Omega_\delta \times [t_1,t_2))} \leq C\left(1 + \|f\|_{L^\infty}^p + \int_{\Omega_{\delta'}} u_0^p\,dx\right)^{\theta/p} + \|f\|_{L^\infty} \tag{3.9}$$

where Ω_δ, $\Omega_{\delta'}$, is given by (1.3).

By Lemma 1.7 of [H1] and a compactness argument we have the following result.

Lemma 3.3 (cf. Corollary 1.8 of [H1])**.** *Let $n \geq 3$, $0 < m \leq \frac{n-2}{n}$, $p > \frac{n(1-m)}{2}$, $0 \leq f \in L^\infty(\partial\Omega \times (0,T))$ and $0 \leq u_0 \in L^\infty(\Omega)$. Suppose $u \in L^\infty(\Omega \times (0,T))$ is a solution of* (1.24). *Then for any $0 < \delta' < \delta < \delta_0$ and $0 < t_1 < t_2 < T$ there exist constants $C > 0$ and $\theta > 0$ such that*

$$\|u\|_{L^\infty(D_\delta \times [t_1,t_2))} \leq C\left(1 + \int_{D_{\delta'}} u_0^p\,dx\right)^{\theta/p} \tag{3.10}$$

where D_δ, $D_{\delta'}$, is given by (1.3).

Theorem 3.4. *(cf. Theorem 1.1 of [H2]) Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1,\delta_0)$, $\mu_0 > 0$, $f_1, f_2 \in C^3(\partial\Omega \times (0,\infty)) \cap L^\infty(\partial\Omega \times (0,\infty))$ be such that*

$$f_2 \geq f_1 \geq 0 \quad \text{and } f_2 \geq \mu_0 \quad \text{on } \partial\Omega \times (0,\infty) \tag{3.11}$$

and

$$u_{0,1}, u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\}), u_{0,2} \geq u_{0,1} \geq 0, u_{0,2} \geq \mu_0 \ \text{ for some constant } p > \frac{n(1-m)}{2}. \tag{3.12}$$

Let $\widehat{\Omega}$ be given by (1.2). *Suppose $u_1, u_2 \in L^\infty_{loc}((\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\}) \times (0,\infty)) \cap C^{2,1}(\widehat{\Omega} \times (0,\infty))$ are subsolution and supersolution of* (1.6) *with $f = f_1, f_2$ and $u_0 = u_{0,1}, u_{0,2}$ respectively which satisfy*

$$u_2(x,t) \geq \mu_0 \quad \forall x \in \widehat{\Omega}, t > 0 \tag{3.13}$$

such that for any constants $T > 0$ and $\delta_2 \in (0,\delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that

$$u_1(x,t) \leq \frac{C_1}{|x-a_i|^{\gamma_i'}} \quad \text{and} \quad u_2(x,t) \geq \frac{C_2}{|x-a_i|^{\gamma_i}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T, i = 1,2,\ldots,i_0. \tag{3.14}$$

for some constants γ_i, γ_i', $i = 1,\ldots,i_0$, satisfying (1.9). *Then*

$$u_1(x,t) \leq u_2(x,t) \quad \forall x \in \widehat{\Omega}, t > 0. \tag{3.15}$$

Proof. Since the proof of the theorem is a modification of the proof of Theorem 1.1 of [H2], we will only sketch the proof here. Let

$$D_+ = \{(x,t) \in \widehat{\Omega} \times (0,\infty) : u_1(x,t) > u_2(x,t)\}$$

and $\alpha > \max(2+n, \gamma_1', \gamma_2', \dots, \gamma_{i_0}')$. Then by (3.13) for any $(x,t) \in D_+$,

$$u_1(x,t) > u_2(x,t) \geq \mu_0.$$

Hence by the mean value theorem,

$$(u_1^m - u_2^m)_+(x,t) \leq m\mu_0^{m-1}(u_1 - u_2)_+(x,t) \quad \forall x \in \widehat{\Omega}, t > 0. \tag{3.16}$$

As in [H2] we choose $\psi \in C^\infty(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ such that $\psi(x) = |x - a_i|^\alpha$ for any $x \in \cup_{i=1}^{i_0} B_{\delta_0}(a_i)$ and

$$\psi(x) \geq c_1 \quad \forall x \in \overline{\Omega} \setminus \cup_{i=1}^{i_0} B_{\delta_2}(a_i) \tag{3.17}$$

for some constant $c_1 > 0$. Let $T > 0$. Since

$$u_1, u_2 \in L^\infty_{loc}((\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}) \times (0,\infty)),$$

by (3.14) and the choice of α, we have for any $i = 1, \cdots, i_0$,

$$\begin{aligned}\int_{B_{\delta_2}(a_i)} |x - a_i|^\alpha (u_1 - u_2)_+(x,t)\,dx \leq & C_T \int_0^{\delta_2} \rho^{\alpha+n-\gamma_i'-1}\,d\rho \\ = & C_T' \delta_2^{\alpha+n-\gamma_i'} < \infty \quad \forall 0 < t < T\end{aligned} \tag{3.18}$$

for some constants $C_T > 0$, $C_T' > 0$. Since by the same argument as the proof of Proposition 2.2 of [H2], the result of Proposition 2.2 of [H2] remains valid for u_1, u_2. That is

$$\|u_i(\cdot,t) - u_{0,i}\|_{L^1(\Omega_\delta)} \to 0 \quad \text{as } t \to 0 \quad \forall 0 < \delta < \delta_0, i = 1, 2.$$

Hence there exists a constant $C_3(T) > 0$ such that

$$\begin{aligned} & \|u_i(\cdot,t) - u_{0,i}\|_{L^1(\overline{\Omega} \setminus \cup_{i=1}^{i_0} B_{\delta_2}(a_i))} \leq C_3(T) \quad \forall 0 < t < T, i = 1, 2 \\ \Rightarrow \quad & \|u_1(\cdot,t) - u_2(\cdot,t)\|_{L^1(\overline{\Omega} \setminus \cup_{i=1}^{i_0} B_{\delta_2}(a_i))} \leq 2C_3(T) \quad \forall 0 < t < T.\end{aligned} \tag{3.19}$$

By (3.18) and (3.19),

$$\int_{\widehat{\Omega}} \psi(x)(u_1 - u_2)_+(x,t)\,dx \leq C_T' \delta_2^{\alpha+n-\gamma_i'} + 2C_3(T) < \infty \quad \forall 0 < t < T. \tag{3.20}$$

By (3.14) and the mean value theorem for any $|x - a_i| \leq \delta_2$, $0 < t < T$, $i = 1, \cdots, i_0$,

$$\begin{aligned} & |x - a_i|^{\alpha-2}(u_1^m - u_2^m)_+(x,t) \\ \leq & m|x - a_i|^{\alpha-2} u_2(x,t)^{m-1}(u_1 - u_2)_+(x,t) \\ \leq & mC_2(T)^{m-1}|x - a_i|^{(1-m)\gamma_i - 2+\alpha}(u_1 - u_2)_+(x,t) \\ \leq & mC_2(T)^{m-1}\delta_0^{(1-m)\gamma_i-2}|x - a_i|^\alpha (u_1 - u_2)_+(x,t) \\ \leq & mC_2(T)^{m-1}\delta_0^{(1-m)\gamma_i-2}\psi(x)(u_1 - u_2)_+(x,t).\end{aligned} \tag{3.21}$$

As in [H2] we now choose a nonnegative monotone increasing function $\phi \in C^\infty(\mathbb{R})$ such that $\phi(s) = 0$ for any $s \le 1/2$ and $\phi(s) = 1$ for any $s \ge 1$. For any $0 < \delta < \delta_0$, let $\phi_\delta(x) = \phi(|x|/\delta)$ and

$$w_\delta(x) = \Pi_{i=1}^{i_0}\phi_\delta(x - a_i).$$

Then by (3.16), (3.17), (3.20) and (3.21) and an argument similar to the proof of Theorem 1.1 of [H2],

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\int_{\widehat{\Omega}}(u_1 - u_2)_+\psi w_\delta\,dx\right)\\
\le& C\int_{\Omega\setminus\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}(u_1^m - u_2^m)_+(x,t)\,dx\\
&\qquad + C\int_{\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}|x-a_i|^{\alpha-2}(u_1^m - u_2^m)_+(x,t)\,dx\\
\le& C\int_{\widehat{\Omega}}(u_1 - u_2)_+(x,t)\psi(x)\,dx\\
&\qquad + C\int_{\cup_{i=1}^{i_0}B_{\delta_2}(a_i)}|x-a_i|^{\alpha-2}(u_1^m - u_2^m)_+(x,t)\,dx\\
\le& C_T\int_{\widehat{\Omega}}(u_1 - u_2)_+(x,t)\psi(x)\,dx.
\end{aligned}\tag{3.22}$$

Integrating (3.22) over $(0,t)$ as letting $\delta \to 0$,

$$\int_{\widehat{\Omega}}(u_1 - u_2)_+(x,t)\psi(x)\,dx \le C_T\int_0^t\int_{\widehat{\Omega}}(u_1 - u_2)_+(x,t)\psi(x)\,dx\,dt \quad \forall 0 < t < T. \tag{3.23}$$

By (3.23) and the Gronwall inequality, we get (3.15) and the theorem follows. □

Theorem 3.5. *(cf. Theorem 1.2 of [H2]) Let $n \ge 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$, $\mu_0 > 0$, $f_1, f_2 \in L^\infty(\partial\Omega\times(0,\infty))$, $u_{0,1}, u_{0,2} \in L^p_{loc}(\overline{\Omega}\setminus\{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$, be such that* (3.11) *and* (3.12) *hold. Let $\widehat{\Omega}$ be given by* (1.2). *Suppose $u_1, u_2 \in L^\infty_{loc}((\overline{\Omega}\setminus\{a_1, \dots, a_{i_0}\})\times(0,\infty)) \cap C^{2,1}(\widehat{\Omega}\times(0,\infty))$ are subsolution and supersolution of* (1.6) *with $f = f_1, f_2$ and $u_0 = u_{0,1}, u_{0,2}$ respectively which satisfy* (3.13) *such that for any constants $T > 0$ and $\delta_2 \in (0, \delta_1)$ there exists a constant $C_2 = C_2(T) > 0$ such that*

$$u_j(x,t) \le \frac{C_2}{|x-a_i|^{\gamma_i'}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T, i = 1, 2, \dots, i_0, j = 1, 2 \tag{3.24}$$

holds for some constants

$$\frac{2}{1-m} < \gamma_i' < \frac{n-2}{m} \quad \forall i = 1, 2, \dots, i_0. \tag{3.25}$$

Then (3.15) *holds.*

Proof. Since the proof is similar to the proof of Theorem 1.2 of [H2], we will only sketch the argument here. Similar to the proof of Theorem 1.2 of [H2] we let

$$A(x,t)=\begin{cases}\dfrac{u_1(x,t)^m-u_2(x,t)^m}{u_1(x,t)-u_2(x,t)} & \forall x\in\widehat{\Omega}, t>0 \text{ satisfying } u_1(x,t)\neq u_2(x,t)\\ mu_2(x,t)^{m-1} & \forall x\in\widehat{\Omega}, t>0 \text{ satisfying } u_1(x,t)=u_2(x,t)\\ 0 & \forall x=a_i, i=1,\cdots,i_0, t>0.\end{cases}$$

For any $k\in\mathbb{Z}^+$, let

$$\alpha_k(x,t)=\begin{cases}\dfrac{|u_1(x,t)^m-u_2(x,t)^m|}{|u_1(x,t)-u_2(x,t)|+(1/k)} & \forall x\in\widehat{\Omega}, t>0\\ 0 & \forall x=a_i, i=1,\cdots,i_0, t>0\end{cases}$$

and $A_k(x,t)=\alpha_k(x,t)+k^{-1}$. We claim that the function $A(x,t)\in L^\infty(\widehat{\Omega}\times(0,\infty))$. We divide the proof of this claim into two cases.
Case 1: $u_2(x,t)\geq 2u_1(x,t)$.
By (3.13),

$$|A(x,t)|\leq\frac{u_2(x,t)^m}{\frac{1}{2}u_2(x,t)}=2u_2(x,t)^{m-1}\leq 2\mu_0^{m-1}.$$

Case 2: $u_2(x,t)<2u_1(x,t)$.
By (3.13) and the mean value theorem there exists a constant $\xi=\xi(x,t)$ lying between $u_1(x,t)$ and $u_2(x,t)$ such that

$$|A(x,t)|\leq m\xi^{m-1}\leq m(u_2(x,t)/2)^{m-1}\leq 2^{1-m}m\mu_0^{m-1}.$$

By case 1 and case 2, $A(x,t)\in L^\infty(\widehat{\Omega}\times(0,\infty))$. Since $|\alpha_k(x,t)|\leq|A(x,t)|$, we get $\alpha_k(x,t)\in L^\infty(\widehat{\Omega}\times(0,\infty))$ and hence one can apply the same argument as the proof of Theorem 1.2 of [H2] to conclude that the theorem holds.

□

Proof of Theorem 1.1: Since the proof is similar to the proof of Theorem 1.1 of [HK2], we will only sketch the argument here. For any $M>0$, $0<\varepsilon<1$, let

$$\begin{cases}u_{0,\varepsilon}(x)=(u_0(x)^m+\varepsilon^m)^{1/m}\\ u_{0,\varepsilon,M}(x)=(\min(u_0(x)^m,M^m)+\varepsilon^m)^{1/m}\end{cases}\tag{3.26}$$

and

$$f_\varepsilon(x,t)=(f(x,t)^m+\varepsilon^m)^{1/m}\qquad\forall(x,t)\in\partial\Omega\times(0,\infty).\tag{3.27}$$

Let $u_{\varepsilon,M}$ be the solution of

$$\begin{cases}u_t=\Delta u^m & \text{in }\Omega\times(0,T)\\ u=f_\varepsilon & \text{on }\partial\Omega\times(0,T)\\ u(x,0)=u_{0,\varepsilon,M}(x) & \text{in }\Omega.\end{cases}\tag{3.28}$$

Then

$$\begin{cases} u_{\varepsilon,M_2} \ge u_{\varepsilon,M_1} \ge \varepsilon & \text{in } \Omega\times(0,\infty) \quad \forall M_2 > M_1 > 0, \varepsilon > 0 \\ u_{\varepsilon_1,M} \ge u_{\varepsilon_2,M} & \text{in } \Omega\times(0,\infty) \quad \forall M > 0, \varepsilon_1 > \varepsilon_2 > 0. \end{cases} \tag{3.29}$$

By Lemma 3.2 for any $0 < \delta' < \delta < \delta_0$, $t_0' > t_0 > 0$, there exists a constant $C > 0$ such that

$$\left\|u_{\varepsilon,M}\right\|_{L^\infty(\Omega_\delta\times[t_0,t_0'])} \le C\left(1 + \|f\|_{L^\infty}^p + \int_{\Omega_{\delta'}} u_0^p\,dx\right)^{\theta/p} + \|f\|_{L^\infty} =: C_0 \tag{3.30}$$

holds for any $0 < \varepsilon \le 1$ and $M > 0$ where Ω_δ, $\Omega_{\delta'}$, is given by (1.3). As in [HK2], by (3.29) and (3.30), as $M \to \infty$, $u_{\varepsilon,M}$ will increase monotonically to some solution u_ε of

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega}\times(0,\infty) \\ u = f_\varepsilon & \text{on } \partial\Omega\times(0,\infty) \\ u(a_i,t) = \infty & \forall t > 0, i = 1,2\dots,i_0 \\ u(x,0) = u_{0,\varepsilon}(x) & \text{in } \widehat{\Omega}. \end{cases} \tag{3.31}$$

Letting $M \to \infty$ in (3.29) and (3.30),

$$\begin{cases} u_\varepsilon \ge \varepsilon & \text{in } \widehat{\Omega}\times(0,\infty) \\ u_{\varepsilon_1} \ge u_{\varepsilon_2} & \text{in } \widehat{\Omega}\times(0,\infty) \quad \forall \varepsilon_1 > \varepsilon_2 > 0 \\ u_\varepsilon \le C_0 & \text{in } \overline{\Omega_\delta}\times[t_0,t_0'] \quad \forall 0 < \varepsilon \le 1. \end{cases} \tag{3.32}$$

Moreover u_ε will decrease monotonically to a solution u of (1.6) as $\varepsilon \to 0$. By an argument similar to the proof of Theorem 1.1 of [HK2] for any $T > 0$, $\delta_2 \in (0,\delta_1)$, there exists constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, depending only on $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}', \gamma_1, \cdots, \gamma_{i_0}, \gamma_1', \cdots, \gamma_{i_0}'$, such that both u and u_ε satisfy (1.10) for any $0 < \varepsilon < 1$.

Suppose v is another solution of (1.6) which satisfies (1.10) for some constants $C_1 > 0$, $C_2 > 0$. Since by (3.26) and (3.27),

$$u_{0,\varepsilon} \ge \max(u_0,\varepsilon) \quad \text{and } f_\varepsilon \ge \max(f,\varepsilon),$$

by Theorem 3.4,

$$\begin{aligned} & v \le u_\varepsilon \quad \text{in } \widehat{\Omega}\times(0,\infty) \quad \forall 0 < \varepsilon < 1 \\ \Rightarrow \quad & v \le u \quad \text{in } \widehat{\Omega}\times(0,\infty) \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Hence u is the maximal solution of (1.6).

Proof of (i) of Theorem 1.1:

Suppose there exist constants $T_0' > T_0 > 0$ and $\mu_1 > 0$ such that (1.11) holds and $T_1 \in (T_0, T_0')$. Let $T_3 = (T_0+T_1)/2$, $T = (T_1 - T_0)/2$ and $C_3 = T^{\frac{2}{1-m}}/\mu_1^2$. Let q and $\lambda_1 > 0$ be the first positive eigenfunction and the first eigenvalue of $-\Delta$ on Ω. By the proof of Theorem 2.2 of [H1] there exists a constant $C_4 > 0$ such that the function

$$w(x,t) = \frac{[m(t-T_3)]^{1/(1-m)}}{(C_3 + C_4 q(x))^{1/2}} \tag{3.33}$$

is a subsolution of (1.1) in $\Omega\times(T_3,\infty)$. Since $w(x,T_3)=0$ in Ω and

$$w(x,t)=\left(\frac{m(t-T_3)}{T}\right)^{1/(1-m)}\mu_1\le f_\varepsilon(x,t)\quad\text{ on }\partial\Omega\times[T_3,T_1],$$

by Theorem 3.4,

$$\begin{aligned}
&u_\varepsilon(x,t)\ge w(x,t)\quad\forall x\in\widehat{\Omega}\times(T_3,T_1],0<\varepsilon<1\\
\Rightarrow\quad&u(x,t)\ge w(x,t)\quad\forall x\in\widehat{\Omega}\times(T_3,T_1]\quad\text{ as }\varepsilon\to0\\
\Rightarrow\quad&u(x,T_1)\ge\mu_3:=\left(\frac{m(T_1-T_0)}{2}\right)^{1/(1-m)}(C_3+C_4\|q\|_\infty)^{-1/2}\quad\forall x\in\widehat{\Omega}
\end{aligned}\tag{3.34}$$

Let $\mu_2=\min(\mu_1,\mu_3)$. Then by (1.11), (3.34) and Theorem 3.4,

$$\begin{aligned}
&u_\varepsilon(x,t)\ge\mu_2\quad\forall x\in\widehat{\Omega}\times[T_1,T_0'),0<\varepsilon<1\\
\Rightarrow\quad&u(x,t)\ge\mu_2\quad\forall x\in\widehat{\Omega}\times[T_1,T_0')\quad\text{ as }\varepsilon\to0
\end{aligned}$$

and (i) follows.

Proof of (ii) of Theorem 1.1:
Suppose there exists a constant $T_2\ge0$ such that (1.13) holds. Then f_ε is monotone decreasing in t on $\partial\Omega\times(T_2,\infty)$. Hence similar to Theorem 1.1 of [HK2] both $u_{\varepsilon,M}$ and u_ε satisfies (1.14). Putting $u=u_\varepsilon$ in (1.14) and letting $\varepsilon\to0$, we get that u satisfies (1.14) and (ii) follows. □

By Lemma 3.2, Lemma 3.3 and the construction of solution of (1.6) in Theorem 1.1 we recover Lemma 2.9 of [HK2] and have the following results.

Lemma 3.6 (cf. Lemma 2.9 of [HK2])**.** *Let $n\ge3$, $0<m\le\frac{n-2}{n}$, $0\le f\in L^\infty(\partial\Omega\times[0,\infty))$ and $0\le u_0\in L^p_{loc}(\overline{\Omega}\setminus\{a_1,\cdots,a_{i_0}\})$ for some constant $p>\frac{n(1-m)}{2}$. Suppose u is a solution of* (1.6)*. Then for any $0<\delta'<\delta<\delta_0$ and $0<t_1<t_2<T$ there exist constants $C>0$ and $\theta>0$ such that* (3.9) *holds.*

Lemma 3.7. *Let $n\ge3$, $0<m\le\frac{n-2}{n}$, $0\le f\in L^\infty(\partial\Omega\times[0,\infty))$ and $0\le u_0\in L^p_{loc}(\overline{\Omega}\setminus\{a_1,\cdots,a_{i_0}\})$ for some constant $p>\frac{n(1-m)}{2}$. Suppose u is a solution of* (1.6)*. Then for any $0<\delta'<\delta<\delta_0$ and $0<t_1<t_2<T$ there exist constants $C>0$ and $\theta>0$ such that* (3.10) *holds.*

Lemma 3.8. *Let $n\ge3$, $0<m\le\frac{n-2}{n}$, $0\le f\in L^\infty(\partial\Omega\times[0,\infty))$ and $0\le u_0\in L^p_{loc}(\overline{\Omega}\setminus\{a_1,\cdots,a_{i_0}\})$ for some constant $p>\frac{n(1-m)}{2}$. Suppose u is a solution of* (1.6)*. Then for any $0<\delta'<\delta<\delta_0$ there exists a constant $C>0$ depending only on p, m, δ and δ' such that* (3.1) *holds.*

Remark 3.9. *By an argument similar to the proof of Theorem 1.1 but with Theorem 3.5 replacing Theorem 3.4 in the proof we get Theorem 1.2.*

By Theorem 3.4, Theorem 3.5 and the construction of solution of (1.6) in Theorem 1.1 we have the following corollaries.

Corollary 3.10. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$ and $T_1 > 0$. Let $0 \leq u_{0,1} \leq u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ and $f_1, f_2 \in C^3(\partial\Omega \times (0, T_1)) \cap L^\infty(\partial\Omega \times (0, T_1))$ be such that*

$$f_2 \geq f_1 \geq 0 \quad \text{on } \partial\Omega \times (0, T_1)$$

holds. Let $\widehat{\Omega}$ be given by (1.2). *Suppose $u_1, u_2 \in L^\infty_{loc}((\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\}) \times (0, T_1)) \cap C^{2,1}(\widehat{\Omega} \times (0, T_1))$ are the maximal solutions of* (1.6) *in $\widehat{\Omega} \times (0, T_1)$ with $f = f_1, f_2$ and $u_0 = u_{0,1}, u_{0,2}$ respectively such that for any constants $0 < T < T_1$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that*

$$\frac{C_1}{|x - a_i|^{\gamma_i}} \leq u_j(x, t) \leq \frac{C_2}{|x - a_i|^{\gamma_i'}} \quad \forall 0 < |x - a_i| < \delta_2, 0 < t < T, i = 1, 2, \ldots, i_0, j = 1, 2 \tag{3.35}$$

holds for some constants γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying (1.9). *Then* (3.15) *holds for any $x \in \widehat{\Omega}$, $0 < t < T_1$.*

Corollary 3.11. *Let $n \geq 3$, $0 < m < \frac{n-2}{n}$, $0 < \delta_1 < \min(1, \delta_0)$ and $T_1 > 0$. Let $0 \leq u_{0,1} \leq u_{0,2} \in L^p_{loc}(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ for some constant $p > \frac{n(1-m)}{2}$ and $0 \leq f_1 \leq f_2 \in L^\infty(\partial\Omega \times (0, T_1))$. Let $\widehat{\Omega}$ be given by* (1.2). *Suppose $u_1, u_2 \in L^\infty_{loc}((\overline{\Omega} \setminus \{a_1, \ldots, a_{i_0}\}) \times (0, \infty)) \cap C^{2,1}(\widehat{\Omega} \times (0, \infty))$ are the maximal solutions of* (1.6) *in $\widehat{\Omega} \times (0, T_1)$ with $f = f_1, f_2$ and $u_0 = u_{0,1}, u_{0,2}$ respectively which satisfy* (3.13) *such that for any constants $0 < T < T_1$ and $\delta_2 \in (0, \delta_1)$ there exist constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, such that* (3.35) *holds for some constants γ_i, γ_i', $i = 1, \ldots, i_0$, satisfying* (1.8). *Then* (3.15) *holds for any $x \in \widehat{\Omega}$, $0 < t < T_1$.*

4 Asymptotic behaviour of blow-up solutions

In this section we will prove the asymptotic behaviour of the maximal finite blow-up points solutions.

***Proof of Theorem 1.3**:* For any $0 < \varepsilon < 1$, let $u_{0,\varepsilon}$, f_ε and u_ε as in the proof of Theorem 1.1. Then

$$u(x, t) \leq u_\varepsilon(x, t) \quad \forall x \in \widehat{\Omega}, t > 0. \tag{4.1}$$

By an argument similar to the proof of Theorem 1.1 of [HK2] for any $T > 0$, $\delta_2 \in (0, \delta_1)$, there exists constants $C_1 = C_1(T) > 0$, $C_2 = C_2(T) > 0$, depending only on $\lambda_1, \cdots, \lambda_{i_0}, \lambda_1', \cdots, \lambda_{i_0}', \gamma_1, \cdots, \gamma_{i_0}, \gamma_1', \cdots, \gamma_{i_0}'$, such that (1.10) holds with $u = u_\varepsilon$ for all $0 < \varepsilon < 1$. For any $0 < \delta < \delta_0$, let Ω_δ be given by (1.3). By (1.8) and Lemma 3.2 of [HK2] for any constants $0 < \delta < \delta_0$, $t_0 > 0$, there exists a constant $C_\delta > 0$ such that

$$u_\varepsilon(x, t) \leq C_\delta \quad \forall x \in \overline{\Omega_\delta} \times [t_0, \infty), 0 < \varepsilon < 1. \tag{4.2}$$

Let $\{t_i\}_{i=1}^\infty \subset \mathbb{R}^+$ be a sequence such that $t_i \to \infty$ as $i \to \infty$. Let $u_i(x, t) = u(x, t + t_i)$ and $u_{\varepsilon,i} = u_\varepsilon(x, t + t_i)$. Let ϕ_ε be the solution of (1.15) with g^m being replaced by $g^m + \varepsilon^m$. By Theorem 1.5 of [HK2] and (1.16), (3.26), (3.27),

$$u_\varepsilon \to \phi_\varepsilon^{\frac{1}{m}} \quad \text{uniformly in } C^2(K) \quad \text{as } t \to \infty \tag{4.3}$$

for any compact subset $K \subset \overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}$.

We now divide the proof into three cases.

Case (i): $g > 0$ on $\partial\Omega$.

Since $\min_{\partial\Omega} g > 0$, we can choose a constant $\mu_1 \in (0, \min_{\partial\Omega} g)$. Then by (1.16) there exists a constant $T_0 > 0$ such that (1.11) holds with $T_0' = \infty$. Let $T_1 > T_0$. Then by Theorem 1.1 there exists a constant $\mu_2 \in (0, \mu_1)$ such that (1.12) holds with $T_0' = \infty$. By Theorem 1.5 of [HK2], (1.16), (1.11) and (1.12), (1.17) holds for any compact subset K of $\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\}$ and (i) follows.

Case (ii): $g \not\equiv 0$ on $\partial\Omega$ and (1.18), (1.19) holds.

Since $f_\varepsilon \geq \max(g, \varepsilon)$ and the function $\phi^{1/m} \in C^1(\overline{\Omega}) \cap C^{2,1}(\Omega)$ satisfy (1.24) with $f = g$ and $u_0 = \phi$, by (1.18), (1.19) and Theorem 3.5,

$$\begin{aligned} & u_\varepsilon(x,t) \geq \phi(x)^{1/m} \quad \forall x \in \widehat{\Omega}, t > 0, 0 < \varepsilon < 1 \\ \Rightarrow \quad & u(x,t) \geq \phi(x)^{1/m} \quad \forall x \in \widehat{\Omega}, t > 0 \quad \text{as } \varepsilon \to 0. \end{aligned} \tag{4.4}$$

Since $\phi(x) > 0$ on Ω, by (4.1), (4.2) and (4.4) for any $N > 0$ the equation (1.1) for the sequence $\{u_i\}_{t_i > -N}$ is uniformly parabolic on any compact subset of $K \subset \widehat{\Omega} \times [-N, N]$. Hence by the parabolic Schauder estimates [LSU] the sequence $\{u_i\}_{t_i > -N}$ is uniformly continuous in $C^2(K)$ for any compact subset of $K \subset \widehat{\Omega} \times [-N, N]$. Thus by (4.1), (4.3), (4.4), the Ascoli Theorem and a diagonalization argument the sequence $\{u_i\}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in $C^2(K)$ for any compact subset of $K \subset \widehat{\Omega} \times (-\infty, \infty)$ to a solution v of (1.1) in $\widehat{\Omega} \times (-\infty, \infty)$ which satisfies

$$\begin{aligned} & \phi(x)^{1/m} \leq v(x,t) \leq \phi_\varepsilon(x)^{1/m} \quad \forall x \in \widehat{\Omega}, t > 0 \\ \Rightarrow \quad & v(x,t) = \phi(x)^{1/m} \quad \forall x \in \widehat{\Omega}, t > 0 \quad \text{as } \varepsilon \to 0 \\ \Rightarrow \quad & u(x,t_i) \to v(x,0) = \phi(x)^{1/m} \quad \text{uniformly on } C^2(K) \quad \text{as } i \to \infty \end{aligned}$$

for any compact subset $K \subset \widehat{\Omega}$. Since the sequence $\{t_i\}$ is arbitrary, we get (1.17) and (ii) follows.

Case (iii): $g = 0$ on $\partial\Omega$.

By (4.1), (4.2) and Theorem 1.1 of [S] for any $N > 0$ the sequence $\{u_i\}_{t_i > -N}$ is uniformly continuous in K for any compact subset of $K \subset \widehat{\Omega} \times [-N, N]$. Thus by (4.1), (4.2) , Theorem 1.1 of [S], the Ascoli Theorem and a diagonalization argument the sequence $\{u_i\}$ has a subsequence which we may assume without loss of generality to be the sequence itself that converges uniformly in K for any compact subset of $K \subset \widehat{\Omega} \times (-\infty, \infty)$ to a continuous function v which satisfies

$$\begin{aligned} & 0 \leq v(x,t) \leq \psi_\varepsilon(x)^{1/m} \quad \forall x \in \widehat{\Omega}, -\infty < t < \infty \\ \Rightarrow \quad & v(x,t) = 0 \quad \forall x \in \widehat{\Omega} \quad \text{as } \varepsilon \to 0. \end{aligned}$$

Hence

$$u(x,t_i) = u_i(x,0) \to 0 \quad \text{as } i \to \infty.$$

Since the sequence $\{t_i\}$ is arbitrary, we get (1.20) and (iii) follows.

□

By an argument similar to the proof of Theorem 1.3 but with Remark 3.7 of [HK2] replacing Theorem 1.5 of [HK2] in the argument Theorem 1.4 follows.

5 Existence of finite blow-up solutions that blow-up at the lateral boundary

In this section we will construct a solution of (1.6) with appropriate lateral boundary value such that the finite blow-up points solution will oscillate between two given harmonic functions as $t \to \infty$. We will also prove the existence of finite blow-up solutions that blow-up at the lateral boundary of the bounded cylindrical domain.

***Proof of Theorem 1.5**:* Let $f_1 = g_1$ and u_1 be the maximal solution of (1.6) given by Theorem 1.2 with $f = f_1$. For any $0 < \delta < \delta_0$, let D_δ be given by (1.3). Let $t_0 = 0$ and $\delta_k = \delta_1/k$ for any $k \in \mathbb{Z}^+$. Then by Theorem 1.4 there exists a constant $t_1 > 0$ such that

$$|u_1(x,t) - \phi_1(x)| < 1 \quad \forall x \in D_{\delta_1}, t \geq t_1. \tag{5.1}$$

Let $f_2(x,t) = g_1(x)$ for $0 < t \leq t_1$ and $f_2(x,t) = g_2(x)$ for $t > t_1$. Let u_2 be the maximal solution of (1.6) with $f = f_2$. Then by Theorem 1.4, there exists a constant $t_2 > t_1 + 1$ such that

$$|u_2(x,t) - \phi_2(x)| < \frac{1}{2} \quad \forall x \in D_{\delta_2}, t \geq t_2. \tag{5.2}$$

By repeating the above argument there exist sequences $\{t_i\}_{i=1}^\infty$, $t_i + 1 < t_{i+1}$ for all $i \in \mathbb{Z}^+$, $\{f_i\}_{i=1}^\infty \subset L^\infty(\partial\Omega)$, such that $\forall i \in \mathbb{Z}^+$,

$$f_{2i+1}(x,t) = \begin{cases} g_1(x) & \forall x \in \partial\Omega, t \in \cup_{k=0}^{i-1}(t_{2k}, t_{2k+1}] \cup (t_{2i}, \infty) \\ g_2(x) & \forall x \in \partial\Omega, t \in \cup_{k=1}^{i}(t_{2k-1}, t_{2k}] \end{cases} \tag{5.3}$$

and

$$f_{2i}(x,t) = \begin{cases} g_2(x) & \forall x \in \partial\Omega, t \in \cup_{k=1}^{i-1}(t_{2k-1}, t_{2k}] \cup (t_{2i-1}, \infty) \\ g_1(x) & \forall x \in \partial\Omega, t \in \cup_{k=1}^{i}(t_{2k-2}, t_{2k-1}] \end{cases} \tag{5.4}$$

and a sequence $\{u_i\}_{i=1}^\infty$ of maximal solutions of (1.6) with $f = f_i$ that satisfies

$$\begin{cases} |u_{2i+1}(x,t) - \phi_1(x)| < \dfrac{1}{2i+1} & \forall x \in D_{\delta_{2i+1}}, t \geq t_{2i+1}, i \in \mathbb{Z}^+ \\ |u_{2i}(x,t) - \phi_2(x)| < \dfrac{1}{2i} & \forall x \in D_{\delta_{2i}}, t \geq t_{2i}, i \in \mathbb{Z}^+. \end{cases} \tag{5.5}$$

Let u be the maximal solution of (1.6) with

$$f(x,t) = \begin{cases} g_1(x) & \forall x \in \partial\Omega, t \in \cup_{k=0}^{\infty}(t_{2k}, t_{2k+1}] \\ g_2(x) & \forall x \in \partial\Omega, t \in \cup_{k=1}^{\infty}(t_{2k-1}, t_{2k}]. \end{cases} \tag{5.6}$$

Then by (5.3), (5.4) and (5.6),

$$f(x,t) = f_i(x,t) \quad \forall x \in \partial\Omega, t \in (0,t_i), i \in \mathbb{Z}^+. \tag{5.7}$$

Hence by Corollary 3.11,

$$u(x,t) = u_i(x,t) \quad \forall x \in \partial\Omega, t \in (0,t_i], i \in \mathbb{Z}^+. \tag{5.8}$$

By (5.5) and (5.8),

$$\begin{cases} |u(x,t_{2i+1}) - \phi_1(x)| < \dfrac{1}{2i+1} & \forall x \in D_{\delta_{2i+1}}, i \in \mathbb{Z}^+ \\ |u(x,t_{2i}) - \phi_2(x)| < \dfrac{1}{2i} & \forall x \in D_{\delta_{2i}}, i \in \mathbb{Z}^+. \end{cases}$$

Since $D_{\delta_i} \subset D_{\delta_{i+1}}$ for all $i \in \mathbb{Z}^+$ and $\widehat{\Omega} = \cup_{i=1}^{\infty} D_{\delta_i}$, for any $0 < \varepsilon < 1$ and compact subset K of $\widehat{\Omega}$ there exists $k_0 \in \mathbb{Z}^+$, $k_0 > \varepsilon^{-1}$, such that

$$K \subset D_{\delta_{k_0}} \subset D_{\delta_i} \quad \forall i \geq k_0.$$

Hence

$$\begin{cases} |u(x,t_{2i+1}) - \phi_1(x)| < \varepsilon & \forall x \in K, i \geq k_0 \\ |u(x,t_{2i}) - \phi_2(x)| < \varepsilon & \forall x \in K, i \geq k_0 \end{cases}$$

and the theorem follows.

□

Proof of Theorem 1.6: For any $0 < \delta < \delta_0$, let D_δ be given by (1.3). For any $k \in \mathbb{Z}^+$, let u_k be the maximal solution of (1.6) with $f = k$ given by Theorem 1.1 which satisfies (1.14) with $T_2 = 0$. By Lemma 3.7 and Corollary 3.10, for any $0 < \delta < \min(1,\delta_0)$, $t_0' > t_0 > 0$, there exists a constant $C_\delta > 0$ such that

$$\begin{cases} u_1(x,t) \leq u_k(x,t) \leq u_{k+1}(x,t) & \forall x \in (\overline{\Omega} \setminus \{a_1,\dots,a_{i_0}\}) \times (0,\infty), k \in \mathbb{Z}^+ \\ u_k(x,t) \leq C_\delta & \forall x \in D_\delta, t_0 \leq t \leq t_0', k \in \mathbb{Z}^+. \end{cases} \tag{5.9}$$

By Theorem 1.1, for any $T > 0$ and $\delta_2 \in (0,\delta_1)$ there exist a constant $C_1 = C_1(T) > 0$ such that

$$u_1(x,t) \geq \frac{C_1}{|x-a_i|^{\gamma_i}} \quad \forall 0 < |x-a_i| < \delta_2, 0 < t < T, i = 1,2,\dots,i_0. \tag{5.10}$$

On the other hand by the proof of Lemma 2.3 of [HK2] there exists a constant $A_0 > 0$ such that

$$u_k(x,t) \leq \frac{A_0(1+t)^{\frac{1}{1-m}}}{|x-a_i|^{\gamma_i'}(\delta_1 - |x-a_i|)^{\frac{2}{1-m}}} \quad \forall 0 < |x-a_i| < \delta_1, i = 1,\dots,i_0, k \in \mathbb{Z}^+. \tag{5.11}$$

Let $\widehat{\Omega}$ be given by (1.2). By (5.9) the equation (1.1) for the sequence $\{u_k\}_{k=1}^{\infty}$ is uniformly parabolic on any compact subset K of $\widehat{\Omega} \times (0,\infty)$. Hence by the parabolic Schauder estimates [LSU] the sequence $\{u_k\}_{k=1}^\infty$ is uniformly continuous in $C^2(K)$ for any compact subset $K\subset\widehat{\Omega}\times(0,\infty)$. Thus by (5.9), (5.10), (5.11), the Ascoli Theorem and a diagonalization argument the sequence $\{u_k\}_{k=1}^\infty$ has a subsequence which we may assume without loss of generality to be the sequence itself that increases and converges uniformly in $C^2(K)$ for any compact subset $K\subset\widehat{\Omega}\times(0,\infty)$ to a solution u of (1.1) in $\widehat{\Omega}\times(0,\infty)$ which satisfies

$$u(x,t)\ge\frac{C_1}{|x-a_i|^{\gamma_i}}\quad\forall 0<|x-a_i|<\delta_2, 0<t<T, i=1,2,\dots,i_0, \tag{5.12}$$

$$u(x,t)\le\frac{A_0(1+t)^{\frac{1}{1-m}}}{|x-a_i|^{\gamma_i'}(\delta_1-|x-a_i|)^{\frac{2}{1-m}}}\quad\forall 0<|x-a_i|<\delta_1, i=1,\dots,i_0 \tag{5.13}$$

and

$$u_k\le u\quad\text{in }\widehat{\Omega}\times(0,\infty)\quad\forall k\in\mathbb{Z}^+ \tag{5.14}$$

and u also satisfies (1.14) with $T_2=0$.

By (5.12) and (5.13) u satisfies (1.10) for some constants $C_1>0$, $C_2>0$. Now by (i) of Theorem 1.1, Corollary 3.10 and (5.9), for any $T_0>0$ there exists a constant $0<\mu_{T_0}<1$ such that

$$u_k(x,t)\ge u_1(x,t)\ge\mu_{T_0}\quad\forall x\in(\overline{\Omega}\setminus\{a_1,\dots,a_{i_0}\})\times[T_0,\infty), k\in\mathbb{Z}^+. \tag{5.15}$$

By Lemma 3.6 and (5.15), for any $k\in\mathbb{Z}^+$ the equation (1.1) for u_k is uniformly parabolic on any compact subset $K\subset(\overline{\Omega}\setminus\{a_1,\dots,a_{i_0}\})\times(0,\infty)$. Hence by the parabolic Schauder estimates [LSU], $u_k\in C^{2,1}((\overline{\Omega}\setminus\{a_1,\dots,a_{i_0}\})\times(0,\infty))$ for any $k\in\mathbb{Z}^+$. Thus

$$\begin{aligned}&\lim_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,\infty)}}u(y,s)\ge\lim_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,\infty)}}u_k(y,s)=k\quad\forall(x,t)\in\partial\Omega\times(0,\infty)\\ \Rightarrow\quad&\lim_{\substack{(y,s)\to(x,t)\\(y,s)\in\Omega\times(0,\infty)}}u(y,s)=\infty\qquad\forall(x,t)\in\partial\Omega\times(0,\infty)\quad\text{as }k\to\infty.\end{aligned} \tag{5.16}$$

We will now show that u has initial value u_0. Since u_k has initial value u_0 for all $k\in\mathbb{Z}^+$, by Lemma 3.1 of [HP] and a compactness argument for any $0<\delta<\delta_0$ there exists a constant $C>0$ depending on δ such that

$$\begin{aligned}&\int_{D_\delta}(u_k(x,t)-u_1(x,t))\,dx\le Ct^{1/(1-m)}\quad\forall t>0, k\in\mathbb{Z}^+\\ \Rightarrow\quad&\int_{D_\delta}(u(x,t)-u_1(x,t))\,dx\le Ct^{1/(1-m)}\quad\forall t>0\quad\text{as }k\to\infty.\end{aligned} \tag{5.17}$$

Hence by (5.17),

$$\begin{aligned}\int_{D_\delta}|u(x,t)-u_0(x)|\,dx\le&\int_{D_\delta}(u(x,t)-u_1(x,t))\,dx+\int_{D_\delta}|u_1(x,t)-u_0(x)|\,dx\\ \le&Ct^{1/(1-m)}+\int_{D_\delta}|u_1(x,t)-u_0(x)|\,dx\quad\forall t>0\\ \Rightarrow\quad\lim_{t\to0}\int_{D_\delta}|u(x,t)-u_0(x)|\,dx=&0\quad\forall 0<\delta<\delta_0.\end{aligned} \tag{5.18}$$

Hence by (1.10), (5.16) and (5.18), u is a solution of (1.21).

We will now use a modification of the proof of Lemma 2.9 of [H1] and Theorem 1.1 of [H2] to show that u is the minimal solution of (1.21). Suppose v is another solution of (1.21). Let $0 < \psi \in C^\infty(\overline{\Omega} \setminus \{a_1, \cdots, a_{i_0}\})$ be as in the proof of Theorem 3.4 and let E_j be given by (1.4). Let $T > t_1 > 0$ and $k \in \mathbb{Z}^+$. By Lemma 3.6 $u_k \in L^\infty_{loc}((\overline{\Omega} \setminus \{a_1, \dots, a_{i_0}\} \times (0, \infty))$. Hence by a compactness argument there exists $j_0 \in \mathbb{Z}^+$, $j_0 > 1/\delta_0$, such that

$$\inf_{\substack{x\in\Omega\setminus E_j\\ t_1\le t\le T}} v(x,t) > \sup_{\substack{x\in\Omega\setminus E_j\\ t_1\le t\le T}} u_k(x,t) \quad \forall j \ge j_0 \tag{5.19}$$

where E_j is given by (1.4). Since $v > 0$ in $\widehat{\Omega} \times (0, \infty)$ and satisfies (1.10),

$$v(x,t) \ge \overline{\mu}_{j_0} \quad \forall x \in E_{j_0}, t_1 \le t \le T \tag{5.20}$$

for some constant $\overline{\mu}_{j_0} > 0$. Hence by (5.15), (5.19) and (5.20),

$$v(x,t) \ge \min(\overline{\mu}_{j_0}, \mu_{t_1}) > 0 \quad \forall x \in \widehat{\Omega}, t_1 \le t \le T. \tag{5.21}$$

By (5.21) and an argument similar to the proof of Theorem 1.1 of [H2] and the proof of Theorem 3.4 we get

$$\int_{E_j} (u_k - v)_+(x,t)\psi(x)\,dx \le \int_{E_j} (u_k - v)_+(x,t_1)\psi(x)\,dx + C\int_{t_1}^t \int_{E_j} (u_k - v)_+\psi(x)\,dx\,dt \tag{5.22}$$

where $C > 0$ is some constant for any $t_1 \le t < T$, $j \ge j_0$. By (5.22) and the Gronwall inequality,

$$\int_{E_j} (u_k - v)_+(x,t)\psi(x)\,dx \le \frac{e^{Ct}}{C} \int_{E_j} (u_k - v)_+(x,t_1)\psi(x)\,dx \quad \forall t_1 \le t < T, j \ge j_0. \tag{5.23}$$

Letting $j \to \infty$ in (5.22),

$$\int_{\widehat{\Omega}} (u_k - v)_+(x,t)\psi(x)\,dx \le \frac{e^{Ct}}{C} \int_{\widehat{\Omega}} (u_k - v)_+(x,t_1)\psi(x)\,dx \quad \forall t_1 \le t < T. \tag{5.24}$$

We now fix $0 < \delta_2 < \delta_1$. Then by (1.10) and (5.19), $\forall 0 < \delta < \delta_2$, $j > j_0$,

$$
\begin{aligned}
&\int_{\widehat{\Omega}} (u_k - v)_+(x, t_1)\psi(x)\, dx \\
\le &\int_{\cup_{i=1}^{i_0} B_\delta(a_i)} u_k(x, t_1)\psi(x)\, dx + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |u_k(x, t_1) - u_0(x)|\psi(x)\, dx \\
&\qquad + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |v(x, t_1) - u_0(x)|\psi(x)\, dx \quad \forall 0 < \delta < \delta_1 \\
\le &C_2 \sum_{i=1}^{i_0} \int_{\cup_{i=1}^{i_0} B_\delta(a_i)} |x - a_i|^{\alpha - \gamma_i'}\, dx + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |u_k(x, t_1) - u_0(x)|\psi(x)\, dx \\
&\qquad + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |v(x, t_1) - u_0(x)|\psi(x)\, dx \\
\le &C'\delta^{\alpha + n - \gamma_i'} + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |u_k(x, t_1) - u_0(x)|\psi(x)\, dx + \int_{E_j \setminus \cup_{i=1}^{i_0} B_\delta(a_i)} |v(x, t_1) - u_0(x)|\psi(x)\, dx
\end{aligned} \tag{5.25}
$$

for some constant $C' > 0$. Letting first $t_1 \to 0$ and then $\delta \to 0$, $j \to \infty$, in (5.25),

$$
\lim_{t_1 \to 0} \int_{\widehat{\Omega}} (u_k - v)_+(x, t_1)\psi(x)\, dx = 0. \tag{5.26}
$$

Letting $t_1 \to 0$ in (5.24), by (5.26),

$$
\begin{aligned}
&\int_{\widehat{\Omega}} (u_k - v)_+(x, t)\psi(x)\, dx = 0 \quad \forall 0 < t < T, k \in \mathbb{Z}^+ \\
\Rightarrow \quad & u_k(x, t) \le v(x, t) \quad \forall x \in \widehat{\Omega}, 0 < t < T, k \in \mathbb{Z}^+ \\
\Rightarrow \quad & u(x, t) \le v(x, t) \quad \forall x \in \widehat{\Omega}, 0 < t < T \qquad \text{as } k \to \infty.
\end{aligned}
$$

Since $T > 0$ is arbitrary,

$$
u(x, t) \le v(x, t) \quad \forall x \in \widehat{\Omega}, t > 0.
$$

Hence u is the minimal solution of (1.21).

We will now prove that (1.22) holds. We choose $\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_{i_0}$, such that

$$
\frac{2}{1 - m} < \widetilde{\gamma}_i < \min\left(\frac{n - 2}{m}, \gamma_i\right) \quad \forall i = 1, \dots, i_0.
$$

and let

$$
\widetilde{u}_0(x) = \begin{cases} u_0(x) & \forall x \in \Omega \setminus \cup_{i=1}^{i_0} B_{\delta_1}(a_i) \\ \lambda_i |x - a_i|^{-\widetilde{\gamma}_i} & \forall 0 < |x - a_i| < \delta_1, i = 1, \dots, i_0. \end{cases}
$$

Then

$$
\widetilde{u}_0(x) \le u_0(x) \quad \text{in } \widehat{\Omega}.
$$

For any $k \in \mathbb{Z}^+$, let v_k be the maximal solution of

$$\begin{cases} u_t = \Delta u^m & \text{in } \widehat{\Omega} \times (0,\infty) \\ u = k & \text{on } \partial\Omega \times (0,\infty) \\ u(a_i,t) = \infty & \forall t > 0, i = 1, 2 \dots, i_0 \\ u(x,0) = \widetilde{u_0}(x) & \text{in } \widehat{\Omega} \end{cases}$$

given by Theorem 1.1. Then by Corollary 3.10,

$$v_k \le u_k \quad \text{in } \widehat{\Omega} \times (0,\infty) \quad \forall k \in \mathbb{Z}^+. \tag{5.27}$$

Since by Theorem 1.3,

$$v_k(x,t) \to k \quad \text{uniformly on } \Omega_\delta \quad \text{as } t \to \infty \tag{5.28}$$

for any $0 < \delta < \delta_0$ and $k \in \mathbb{Z}^+$, by (5.14) and (5.27),

$$\liminf_{\substack{x \in \Omega_\delta \\ t \to \infty}} u(x,t) \ge k \quad \forall k \in \mathbb{Z}^+. \tag{5.29}$$

for any $0 < \delta < \delta_0$. Letting $k \to \infty$ in (5.29), we get (1.22) and the theorem follows.

□

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