GINZBURG-LANDAU EQUATIONS AND THEIR GENERALLIZATIONS

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Dedicated to 100th anniversary of Professor Jaap KOREVAAR

Introduction

The Ginzburg–Landau equations were proposed in the superconductivity theory to describe mathematically the intermediate state of superconductors in which the normal conductivity is mixed with the superconductivity (cf. [3]). It was understood later on that these equations play an important role also in various problems of mathematical physics. We mention here the extension of these equations to compact Riemann surfaces and Riemannian 4-manifolds. A separate interesting topic is the scattering theory of vortices reducing to the study of hyperbolic Ginzburg–Landau equations. In this review we tried to touch these interesting topics with many still unsolved problems.

Briefly on the content of the paper. We start from Section 1 in which we introduce the Ginzburg-Landau equations on the plane. The physical aspects of these equations are described in [3] (cf. also [9]). In Subsection 1.2 we describe the vortex solutions which are the local minima of the functional of potential energy. The main result here is the description of the d-vortex solutions due to Taubes (cf. [2]). The Subsection 1.3 is devoted to the generalization of the results of previous two subsections to compact Riemann surfaces. The generalization of Taubes theorem to this case was obtained by Bradlow in [1].

In Section 2 we switch on the time variable and consider the hyperbolic Ginzburg-Landau equations introduced in Subsection 2.1. We study the adiabatic or slow time limit in these equations. The hyperbolic Ginzburg-Landau equations in this limit convert into the adiabatic equations. Their solutions, called the adiabatic trajectories, are given by the geodesics on the moduli space of vortex solutions with respect to the metric generated by the kinetic energy functional. Solving the Euler equation for these geodesics, we can describe approximately solutions of the original Ginzburg-Landau equations with small kinetic energy.

In Section 3 we deal with the Seiberg-Witten equations which may be considered as an extension of Ginzburg-Landau equations to compact Riemannian 4-manifolds. A key idea is to use the Spin^c-structure existing on any Riemannian 4-manifold. Necessary notions from the spinor geometry are given in Subsection 3.1. In Subsection 3.2 we introduce the Seiberg-Witten equations on compact Riemannian 4-manifolds. In the next Subsection 3.3 we consider the model example of Seiberg-Witten equations on a compact Kähler surface. In this case the moduli space of solutions coincides with the space of holomorphic curves on the considered surface lying in a given topological class. In the last Subsection 3.4 we study the Seiberg-Witten equations on compact symplectic 4-manifolds. Again, as in the 3-dimensional case, we use the adiabatic limit construction with scale parameter $\lambda \to \infty$. In the limit the sequence of solutions of the Seiberg-Witten equations, depending on the scale parameter, converges (in the weak sense) to a pseudoholomorphic curve which may be considered as a complex analogue of the adiabatic trajectory in the 3-dimensional case. The parameter along this limiting curve plays the role of the "complex time". The Seiberg-Witten equations in this limit reduce to a family of vortex equations, defined in the normal planes to the limiting pseudoholomorphic curve. The limiting curve and a family of vortex solutions along this curve must satisfy the adiabatic equation analogous to the ∂ -equation which may be considered as a complex analogue of the Euler geodesic equation. Conversely, if we have a pseudoholomorphic curve and a family of vortex equations in normal planes, satisfying the adiabatic equation, then we can reconstruct from these data a solution of Seiberg-Witten equations which tends in the adiabatic limit to the original pseudoholomorphic curve and given family of vortex solutions.

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1. Ginzburg-Landau equations in dimension 2

1.1. **Ginzburg–Landau Lagrangian.** The Ginzburg–Landau equations on the plane $\mathbb{R}^2_{(x_1,x_2)}$ are the Euler–Lagrange equations for the potential energy functional of the form

(1)
$$U(A,\Phi) := \frac{1}{2} \int \mathcal{L}(A,\Phi) \, dx_1 dx_2$$

where $\mathcal{L}(A, \Phi)$ is the Ginzburg–Landau Lagrangian. This Lagrangian depends on two variables A and Φ . The first of them is the 1-form

$$A = A_1 dx_1 + A_2 dx_2$$

with smooth pure imaginary coefficients. The second one is a smooth complex-valued function $\Phi = \Phi_1 + i\Phi_2$.

The Lagrangian $\mathcal{L}(A,\Phi)$ has the form

(2)
$$\mathcal{L}(A,\Phi) = |F_A|^2 + |D_A\Phi|^2 + \frac{1}{4}(1-|\Phi|^2)^2.$$

Here

$$F_A = dA = \sum_{i,i=1}^{2} F_{ij} dx_i \wedge dx_j = 2F_{12} dx_1 \wedge dx_2$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ with $\partial_i := \partial/\partial x_i$,

$$d_A \Phi = d\Phi + A\Phi = \sum_{i=1}^{2} (\partial_i + A_i) \Phi \, dx_i.$$

The Euler–Lagrange equations for the potential energy functional (1) have the form

(3)
$$\begin{cases} \partial_i F_{ij} = 0, & j = 1, 2, \\ \nabla_A^2 \Phi = \frac{1}{2} \Phi(|\Phi|^2 - 1). \end{cases}$$

To satisfy the condition $U(A,\Phi)<\infty$ we shall require that $|\Phi|\to 1$ for $|x|\to\infty$. It follows from this asymptotic condition that our problem has a topological invariant given by the rotation number d of the map Φ sending the circles S^1_R of large radius R to topological circles $|\Phi|\approx 1$. This invariant takes on the integer values and is called the vortex number.

1.2. Vortex equations. Assume now that $d \ge 0$. It may be proved that

$$U(A, \Phi) \ge \pi d$$

and the equality here is attained only on solutions of the vortex equations which are written in complex coordinate $z = x_1 + ix_2$ in the form

(4)
$$\begin{cases} \bar{\partial}_A \Phi = 0, \\ iF_{12} = \frac{1}{2} (1 - |\Phi|^2) \end{cases}$$

where $\bar{\partial}_A \Phi = \bar{\partial} + A^{0,1}$ with $A^{0,1}$ being the (0,1)-component of the form A written in terms of complex coordinate like $A = A^{1,0} + A^{0,1}$.

Note that the vortex equations, as well as potential energy $U(A, \Phi)$, are invariant under gauge transforms given by

$$A \longmapsto A + id\chi, \quad \Phi \longmapsto e^{-i\chi}\Phi$$

where χ is a smooth real-valued function.

For d < 0 we have the similar inequality

$$U(A, \Phi) \ge -\pi d$$

where the equality is attained on solutions of anti-vortex equations

$$\begin{cases} \partial_A \Phi = 0, \\ iF_{12} = \frac{1}{2} \left(|\Phi|^2 - 1 \right). \end{cases}$$

Solutions of the vortex equations are described by the following theorem.

Theorem 1 (Taubes [2]). For any natural number d > 0 and arbitrary collection $\{z_1, z_2, \ldots z_k\}$ of different points in the complex plane \mathbb{C} , taken with multiplicities $d_1, d_2, \ldots d_k$ such that $\sum_{j=1}^k d_j = d$, there exists a unique (up to gauge transforms) d-vortex solution (A, Φ) with $U(A, \Phi) < \infty$, satisfying the condition: the divisor of zeros of the function Φ coincides with $\sum_{j=1}^k d_j z_j$.

An analogous theorem holds for solutions of anti-vortex equations for d < 0. Moreover, Taubes has proved that any critical point (A, Φ) of the potential energy (1) with $U(A, \Phi) < \infty$ and d > 0 is gauge equivalent to some d-vortex solution. It follows that any solution of the Euler–Lagrange equations with $U(A, \Phi) < \infty$ is either d-vortex, or |d|-anti-vortex.

The moduli space of d-vortices is by definition the quotient

$$\mathcal{M}_d = \frac{\{d\text{-vortices }(A, \Phi)\}}{\{\text{gauge transforms}\}}.$$

In the sequel we restrict to the case d > 0.

Theorem 1 implies that the moduli space of d-vortices coincides with the set of unordered collections of d points in the complex plane \mathbb{C} , i.e. with the dth $symmetric\ power$ of \mathbb{C} :

$$\mathcal{M}_d = \operatorname{Sym}^d \mathbb{C}.$$

Note that the symmetric power $\operatorname{Sym}^d\mathbb{C}$ may be identified with the space \mathbb{C}^d by assigning to the collection of d points in the complex plane \mathbb{C} the polynomial with the highest coefficient equal to 1, having its zeros at given points.

1.3. Vortex equations on compact Riemann surfaces. Let X be a compact Riemann surface provided with Riemannian metric g and Kähler form ω . We fix a complex Hermitian line bundle $L \to X$ with

Hermitian metric h and define the energy functional by analogy with the complex plane case

$$U(A,\Phi) = \frac{1}{2} \int_X \left\{ |F_A|^2 + |d_A \Phi|^2 + \frac{1}{4} (1 - |\Phi|^2)^2 \right\} \omega.$$

Here, A is a U(1)-connection on L, $F_A = dA$ is its curvature, d_A is the covariant exterior derivative, generated by A, Φ is a section of the bundle $L \to X$, its norm $|\Phi|$ being computed with respect to metric h. As in the complex plane case, this functional is invariant under gauge transforms, given by the maps $u \in \mathcal{G} = C^{\infty}(X, U(1))$.

The first Chern class $c_1(L)$ of the line bundle $L \to X$ is equal, according to Gauss-Bonnet formula, to

$$c_1(L) = \frac{i}{2\pi} \int_X F_A.$$

Let us assume that $c_1(L) > 0$. Then, as in Subsection 1.2, we have the lower estimate for the energy of the form

$$U(A, \Phi) \ge \pi c_1(L)$$
.

The equality here is attained on solutions of the equations

(5)
$$\begin{cases} \bar{\partial}_A \Phi = 0, \\ iF_A^{\omega} = \frac{1}{2} (1 - |\Phi|^2) \end{cases}$$

where $F_A^{\omega} = \omega \, \lrcorner \, F_A$ is the (1,1)-component of the curvature F_A , parallel to ω .

The obtained equations look the same as the vortex equations on the complex plane. However, in the case of a compact Riemann surface we have an evident obstruction to their solvability. Namely, by integrating the second equation over X, we get

$$\frac{i}{2\pi} \int_X F_A = \frac{1}{4\pi} \int_X \omega - \frac{1}{4\pi} \int_X |\Phi|^2 \omega,$$

which may be rewritten in the form

$$c_1(L) = \frac{1}{4\pi} \operatorname{Vol}_g(X) - \frac{1}{4\pi} \|\Phi\|_{L^2}^2.$$

So we arrive at the *necessary condition of the solvability* of the above equations:

$$c_1(L) \le \frac{1}{4\pi} \operatorname{Vol}_g(X).$$

This condition arises because of the non-invariance of the energy under the scale transform. The scale transform changes the metric g to the metric $g_t := t^2 g$. Simultaneously, the Kähler form and volume change to:

$$\omega_t = t^2 \omega, \quad \operatorname{Vol}_{g_t}(X) = t^2 \operatorname{Vol}_g(X).$$

The necessary solvability condition for the rescaled metric g_t looks like:

$$c_1(L) \le \frac{t^2}{4\pi} \operatorname{Vol}_g(X)$$

and is evidently satisfied for sufficiently large t. So we can always attain the necessary solvability condition by rescaling the original metric g.

It is, however, more convenient to fix the metric and introduce the scaling into the definition of the functional $U(A, \Phi)$. Namely, we replace the energy functional $U(A, \Phi)$ by its rescaled version

$$U_{\tau}(A,\Phi) = \frac{1}{2} \int_{X} \left\{ |F_{A}|^{2} + |d_{A}\Phi|^{2} + \frac{1}{2} (\tau - |\Phi|^{2})^{2} \right\}$$

where $\tau > 0$ is the scaling parameter.

Then we obtain the following lower estimate for the energy

$$U_{\tau}(A, \Phi) \ge \pi c_1(L),$$

where the equality is attained only on solutions of the equations

(6)
$$\begin{cases} \bar{\partial}_A \Phi = 0, \\ iF_A^{\omega} = \frac{1}{2} (\tau - |\Phi|^2). \end{cases}$$

These are the right vortex equations on a compact Riemann surface. For them the necessary solvability condition takes the form

$$c_1(L) \le \frac{\tau}{4\pi} \operatorname{Vol}_g(X).$$

For these equations we have the following analogue of the Taubes theorem.

Theorem 2 (Bradlow [1]). Let $d := c_1(L) > 0$ and D is an effective divisor on X of degree d, i.e. $D = \sum_{j=1}^k d_j z_j$ with $\sum_{j=1}^k d_j = d$. Then the condition

$$c_1(L) < \frac{\tau}{4\pi} \operatorname{Vol}(X)$$

is necessary and sufficient for the existence of a unique (up to gauge equivalence) d-vortex solution (A, Φ) such that the zero divisor of Φ coincides with D.

Moreover, the holomorphic line bundle L, provided with the complex structure determined by the operator $\bar{\partial}_A$, is isomorphic to the holomorphic line bundle [D], defined by the divisor D.

Note that the first vortex equation $\bar{\partial}_A \Phi = 0$ means, in other words, that Φ is a holomorphic section of the Hermitian line bundle $(L, \bar{\partial}_A)$.

According to Bradlow theorem, in the case when

$$c_1(L) < \frac{\tau}{4\pi} \operatorname{Vol}(X)$$

we have a bijective correspondence between the sets:

$$\{d\text{-vortex solutions }(A,\Phi)\}/\mathcal{G}$$

and

{ effective divisors
$$D$$
 of degree $d = c_1(L)$ }.

So the moduli space of d-vortex solutions coincides with the symmetric power $\operatorname{Sym}^d X$.

The inequality

$$\tau > \frac{4\pi c_1(L)}{\operatorname{Vol}(X)}$$

coincides with the stability condition for the pair (E, Φ) (cf. [1]).

2. Ginzburg-Landau equations in dimension 3

2.1. **Hyperbolic Ginzburg–Landau equations.** We add the time variable $x_0 = t$ to the variables (x_1, x_2) and denote by $\Phi = \Phi(t, x_1, x_2)$ a smooth complex-valued function on the space $\mathbb{R}^3 = \mathbb{R}^{1+2}$ with coordinates (t, x_1, x_2) . The form A from Section 1 is replaced by the form

$$\mathcal{A} = A_0 dt + A_1 dx_1 + A_2 dx_2$$

with coefficients $A_{\mu} = A_{\mu}(t, x_1, x_2)$, $\mu = 0, 1, 2$, being smooth functions with pure imaginary values on the space \mathbb{R}^{1+2} . Denote the time component of the form \mathcal{A} by $A^0 := A_0 dt$ and its space component by $A = A_1 dx_1 + A_2 dx_2$.

The potential energy of the system is given by the same formula, as in Subsection 1.1, i.e. $U(\mathcal{A}, \Phi) = U(A, \Phi)$.

We define the kinetic energy of the system by

$$T(\mathcal{A}, \Phi) = \frac{1}{2} \int \left\{ |F_{01}|^2 + |F_{02}|^2 + |d_{A^0}\Phi|^2 \right\} dx_1 dx_2$$

where F_{0j} , j = 1, 2, are given, as before, by the formula

$$F_{0j} = \partial_0 A_j - \partial_j A_0,$$

and $d_{A^0}\Phi = d\Phi + A_0 dt$.

Introduce the Ginzburg-Landau action functional

$$S(\mathcal{A}, \Phi) = \int_0^{T_0} \left(T(\mathcal{A}, \Phi) - U(\mathcal{A}, \Phi) \right) dt.$$

The Euler-Lagrange equations for this functional, called also the hyperbolic Ginzburg-Landau equations, have the form:

$$\begin{cases} \partial_1 F_{01} + \partial_2 F_{02} = i \operatorname{Im}(\bar{\Phi} \nabla_{A,0} \Phi) \\ \partial_0 F_{0j} + \sum_{k=1}^2 \varepsilon_{jk} \partial_k F_{12} = i \operatorname{Im}(\bar{\Phi} \nabla_{A,j} \Phi), \ j = 1, 2 \\ (\nabla_{A,0}^2 - \nabla_{A,1}^2 - \nabla_{A,2}^2) \Phi = \frac{1}{2} \Phi (1 - |\Phi|^2) \end{cases}$$

where $\nabla_{A,\mu} = \partial_{\mu} + A_{\mu}$, $\mu = 0, 1, 2$, $\varepsilon_{12} = -\varepsilon_{21} = 1$, $\varepsilon_{11}\varepsilon_{22} = 0$.

The first of these equations is of constraint type which means that it holds for any t if it is satisfied for the initial moment of time. The last equation, containing the covariant D'Alembertian in its left hand side, is a nonlinear wave equation.

These equations are invariant under the gauge transforms of the form

$$A \mapsto A + id\chi, \quad \Phi \mapsto e^{-i\chi}\Phi$$

where $\chi = \chi(t, x_1, x_2)$ is a smooth real-valued function on \mathbb{R}^{1+2} .

We can choose the gauge function χ so that $A_0 = 0$, such a choice is called the temporal gauge. (Note that after fixing the temporal gauge we still have the gauge freedom with respect to static gauge transforms, given by gauge functions χ which do not depend on time t.)

In the temporal gauge the kinetic energy is written in the form

$$T(A, \Phi) = \frac{1}{2} \{ \|\dot{\Phi}\|^2 + \|\dot{A}\|^2 \}$$

where "dot" denotes the time derivative $\partial/\partial t = \partial/\partial x_0$ and $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^2)}$ is the norm in the space $L^2(\mathbb{R}^2)$.

2.2. Adiabatic limit. Our goal is to describe the space of solutions of hyperbolic Ginzburg–Landau equations modulo dynamic gauge transforms. We shall call the solutions of these equations, for brevity, the dynamic solutions and the quotient of the space of dynamic solutions modulo gauge transforms is called the moduli space of dynamic solutions.

In contrast with the case of the moduli space of static solutions, which structure is completely described by Taubes theorems, we cannot expect to get anything similar in the dynamic case. However, we can hope to obtain an approximate description of at least some classes of dynamic solutions. We shall present here an heuristic approach, proposed by Manton (cf. [5]), to the approximate description of "slowly moving" dynamic solutions.

In the temporal gauge the dynamic solutions of hyperbolic Ginzburg– Landau equations are given by the smooth trajectories

$$\gamma: t \longmapsto [A(t), \Phi(t)]$$

in the static configuration space:

$$\mathcal{N}_d = \frac{\{\text{smooth data } (A, \Phi) \text{ with } U(A, \Phi) < \infty \text{ and vortex number } d\}}{\{\text{static gauge transforms}\}}$$

where $[A(t), \Phi(t)]$ denotes the gauge class of the pair $(A(t), \Phi(t))$ modulo static gauge transforms. It contains, in particular, the moduli space \mathcal{M}_d of d-vortex solutions.

The configuration space \mathcal{N}_d may be thought of as a horizontal canyon with a small ball with trajectory $\gamma(t)$, rolling inside it. The moduli space of d-vortex solutions \mathcal{M}_d , for which the potential energy is minimal, corresponds to the bottom of this canyon. The lower is the kinetic energy of the ball, the closer lies its trajectory to the bottom of the canyon. The ball can even hit this bottom but, having a non-zero kinetic energy, cannot stop there and is forced to assent the wall of the canyon.

Define the kinetic energy of the trajectory $\gamma(t) = [A(t), \Phi(t)]$ by

$$T(\gamma) := \frac{1}{2} \{ \|\dot{A}\|^2 + \|\dot{\Phi}\|^2 \}.$$

Consider the family of trajectories $\gamma_{\varepsilon}(t)$, depending on the parameter $\varepsilon > 0$, having the kinetic energy $T(\gamma_{\varepsilon})$ proportional to ε . For small ε the trajectories $\gamma_{\varepsilon}(t)$ are lying close to the moduli space \mathcal{M}_d and in the limit $\varepsilon \to 0$ they converge to a static solution, i.e. to a point on \mathcal{M}_d .

However, if we introduce the "slow time" $\tau := \varepsilon t$ on the trajectory γ_{ε} then in the limit $\varepsilon \to 0$ the "rescaled" trajectories $\gamma_{\varepsilon}(\tau)$ will converge not to a point but to some trajectory γ_0 , lying in \mathcal{M}_d .

The described construction is called the adiabatic limit and the equations, to which the original Ginzburg-Landau equations reduce in this limit, are called the adiabatic equations. Accordingly, their solutions are called the adiabatic trajectories.

The adiabatic trajectories admit the following intrinsic description in terms of the moduli space \mathcal{M}_d .

Theorem 3. The kinetic energy functional determines a Riemannian metric on the vortex space \mathcal{M}_d , called the kinetic or T-metric. Geodesics of this metric coincide precisely with the adiabatic trajectories.

Since any point of an adiabatic trajectory γ_0 is a static solution, the trajectory itself cannot be a dynamic solution. However, such trajectories describe approximately dynamic solutions with small kinetic energy.

Manton has formulated the following adiabatic principle: For any adiabatic trajectory γ_0 on the moduli space \mathcal{M}_d it should exist a sequence $\{\gamma_{\varepsilon}\}$ of dynamic trajectories (solutions of hyperbolic Ginzburg–Landau equations), converging for $\varepsilon \to 0$ to γ_0 in the adiabatic limit.

The rigorous formulation of this principle and its proof were given by Roman Palvelev [6] (cf. also [7]).

3. Seiberg-Witten equations

In this section we consider one more generalization of Ginzburg–Landau equations, this time to 4-dimensional Riemannian manifolds. These are the Seiberg–Witten equations. Let us start with some basic definitions from spinor geometry of 4-manifolds.

3.1. **Spinor geometry.** A key role in studying the 4-dimensional Riemannian manifolds is played by the Spin^c-structure which exists on any Riemannian 4-manifold. It can be considered as a replacement of the complex structure underlying the theory of 2-dimensional Riemann surfaces.

Leaving apart the general definition of Spin^c-structure (which may be found in the book [4]) we describe here its properties used in Seiberg–Witten theory.

Let (X,g) be a compact oriented Riemannian 4-manifold provided with Levi-Civita connection. Then we can define the Clifford multiplication ρ by differential forms on X, i.e. a representation of such forms by linear endomorphisms acting on smooth sections of the spinor bundle $W \to X$. It is a complex Hermitian vector bundle of rank 4 decomposed into the direct sum

$$W = W^+ \oplus W^-$$

of complex semispinor bundles of rank 2.

The spinor bundle W may be provided with spinor connection ∇ which is an extension of the Levi-Civita connection to a connection on W. The Dirac operator on smooth sections of W is given by the composition $\rho \circ \nabla$ of Clifford multiplication with spinor connection.

In the case when the manifold (X, g) is symplectic, i.e. provided with the symplectic form ω compatible with g, it also has an almost complex structure J compatible both with ω and g.

In this case we have a canonical construction of the spinor bundle W identified with

$$W_{\text{can}} = \Lambda^{0,*}(T^*X) = \bigoplus_{q=0}^{2} \Lambda^{0,q}(T^*X).$$

Accordingly,

$$W_{\text{can}}^+ = \Lambda^{0,0}(T^*X) \oplus \Lambda^{0,2}(T^*X), \quad W_{\text{can}}^- = \Lambda^{0,1}(T^*X).$$

In this case there is also a canonical spinor connection ∇_{can} on W_{can} and an explicit formula for the Clifford multiplication (cf. [4],[9]).

Moreover, for any Hermitian line bundle $E \to X$ with a Hermitian connection B on it we can construct the associated spinor bundle $W_E := W_{\operatorname{can}} \otimes E$ and the spinor connection ∇_A on W_E where $A = A_E$ is the tensor product of the canonical spinor connection $\nabla_{\operatorname{can}}$ on W_{can} and the given Hermitian connection B on E.

The Dirac operator

$$D_A = \rho \circ \nabla_A : \Gamma(X, W^+) \longrightarrow \Gamma(X, W^-)$$

coincides in this case with $\bar{\partial}_B + \bar{\partial}_B^*$ where $\bar{\partial}_B^*$ is the L^2 -adjoint of the operator $\bar{\partial}_B$.

3.2. Seiberg–Witten equations on Riemannian 4-manifolds. Let (X, g) be a compact oriented Riemannian 4-manifold provided with a Spin^c -structure and $E \to X$ is Hermitian line bundle provided with a Hermitian connection B.

Consider the following Seiberg-Witten action functional

$$S(A, \Phi) = \frac{1}{2} \int_{X} \left\{ |F_{A}|^{2} + |\nabla_{A}\Phi|^{2} + \left(s(g) + |\Phi|^{2}\right) \frac{|\Phi|^{2}}{4} \right\} \text{vol}$$

where s(g) is the scalar curvature of (X, g), F_A is the curvature of the connection ∇_A , Φ is a smooth section of W^+ and vol is the volume element on (X, g).

The local minima of this functional satisfy the Seiberg–Witten equations

(7)
$$\begin{cases} D_A \Phi = 0, \\ F_A^+ = \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2 \cdot \operatorname{Id} \end{cases}$$

where $\Phi \otimes \Phi^* - \frac{1}{2}|\Phi|^2 \cdot \text{Id}$ is the traceless Hermitian endomorphism of W^+ , associated with Φ and F_A^+ is the selfdual component of the curvature F_A (with respect to the Hodge *-operator). The section Φ , being a section of W^+ , is represented by two forms (φ_0, φ_2) where $\varphi_0 \in \Omega^0(X, E), \varphi_2 \in \Omega^{0.2}(X, E)$.

The Seiberg–Witten equations, as well as the Seiberg–Witten functional $S(A, \Phi)$, are invariant under the gauge transformations given by the formula

$$A \longmapsto A + u^{-1}u, \ \Phi \longmapsto u^{-1}\Phi$$

where $u = e^{i\chi}$ and χ is a smooth real-valued function so that $u \in \mathcal{G} = C^{\infty}(X, \mathrm{U}(1))$.

Along with Seiberg-Witten equations we shall consider their perturbed version given by

(8)
$$\begin{cases} D_A \Phi = 0, \\ F_A^+ + \eta = \Phi \otimes \Phi^* - \frac{1}{2} |\Phi|^2 \cdot \operatorname{Id} \end{cases}$$

where η is a self-dual 2-form on X. We shall call this perturbed version of Seiberg–Witten equations by the SW_{η} -equations. The introduced perturbation is necessary to guarantee the existence of a solution of Seiberg–Witten equations.

3.3. Seiberg–Witten equations on a K ahler surface. Suppose now that X is a Kähler surface, i.e. a smooth compact 2-dimensional complex manifold. Then the complexified bundle $\Lambda^2_+ \otimes \mathbb{C}$ of selfdual 2-forms on X is decomposed into the direct sum of subbundles

$$\Lambda^2_+ \otimes \mathbb{C} = \Lambda^{2,0} \oplus \mathbb{C}[\omega] \oplus \Lambda^{0,2}$$

Accordingly, the second Seiberg–Witten equation for the curvature decomposes into the sum of three equations — one for the component, parallel to ω , one for (0,2)-component and another one for (2,0)-component which is conjugate to (0,2)-component and by this reason is omitted below.

Then the SW_n -equations take the form

(9)
$$\begin{cases} \bar{\partial}_B \varphi_0 + \bar{\partial}_B^* \varphi_2 = 0, \\ F_B^{0,2} + \eta^{0,2} = \frac{\bar{\varphi}_0 \varphi_2}{2}, \\ F_{A_{\text{can}}}^{\omega} + F_B^{\omega} = \frac{i}{4} (|\varphi_0|^2 - |\varphi_2|^2) - \eta^{\omega}. \end{cases}$$

The first of these equations is the Dirac equation, the second one corresponds to the (0,2)-component of the curvature equation, and the third one corresponds to the component of the curvature equation, parallel to ω .

For the SW_{η} -equations on a Kähler surface an analogue of Bradlow theorem for vortex equations on a compact Riemann surface holds. Let

 $E \to X$ be a Hermitian line bundle over (X, ω, J) . Suppose that for some $\lambda > 0$ its first Chern class satisfies the inequality

(10)
$$0 \le c_1(E) \cdot [\omega] < \frac{c_1(K) \cdot [\omega]}{2} + \lambda \text{Vol}(X)$$

where K is the canonical bundle of X. This inequality plays the same role, as the stability condition $c_1(L) < \tau/4\pi \operatorname{Vol}_g(X)$ in Bradlow theorem.

Under this condition the moduli space of SW_{η} -solutions with the form $\eta = \pi i \lambda \omega$ and $Spin^c$ -structure W_E admits the following description: there exists a bijective correspondence between the gauge classes of SW_{η} -solutions and effective divisors of degree $c_1(E)$ on X.

3.4. Seiberg–Witten equations on a 4-dimensional symplectic manifold. Suppose now that X is a compact symplectic 4-manifold provided with symplectic 2-form ω and compatible almost complex structure J. Let $E \to X$ be a Hermitian line bundle with a Hermitian connection B and $W_E := W_{\text{can}} \otimes E$ is the associated spinor bundle.

We take the perturbation form η equal to $\eta = -F_{A_{\text{can}}}^+ + \pi i \lambda \omega$ with $\lambda > 0$.

The corresponding SW_n -equations have the form

(11)
$$\begin{cases} \bar{\partial}_B \varphi_0 + \bar{\partial}_B^* \varphi_2 = 0, \\ F_{A_{\text{can}}}^{0,2} + F_B^{0,2} + \eta^{0,2} = \frac{\bar{\varphi}_0 \varphi_2}{2}, \\ F_{A_{\text{can}}}^{\omega} + F_B^{\omega} + \eta^{\omega} = \frac{|\varphi_2|^2 - |\varphi_0|^2}{4} \end{cases}$$

where $(\varphi_0, \varphi_2) \in \Omega^0(X, E) \oplus \Omega^{0,2}(X, E)$.

We introduce now the normalized sections:

$$\alpha := \frac{\varphi_0}{\sqrt{\lambda}}, \quad \beta := \frac{\varphi_2}{\sqrt{\lambda}}.$$

Then the SW $_{\eta}$ -equations will rewrite as

(12)
$$\begin{cases} \bar{\partial}_B \alpha + \bar{\partial}_B^* \beta = 0, \\ \frac{2}{\lambda} F_B^{0,2} = \bar{\alpha} \beta, \\ \frac{4i}{\lambda} F_B^{\omega} = 4\pi + |\beta|^2 - |\alpha|^2. \end{cases}$$

According to Taubes [10], solutions ($\alpha \equiv \alpha_{\lambda}, \beta \equiv \beta_{\lambda}$) of the perturbed equations have the following behavior for $\lambda \to \infty$:

- (1) $|\alpha_{\lambda}| \to 1$ everywhere outside the set of zeros $\alpha_{\lambda}^{-1}(0)$;
- (2) $|\beta_{\lambda}| \to 0$ everywhere together with 1st order derivatives.

Denote by $C_{\lambda} := \alpha_{\lambda}^{-1}(0)$ the zero set of α_{λ} . The curves C_{λ} converge in the sense of currents to some pseudoholomorphic divisor, i.e. a chain $\sum d_k C_k$, consisting of connected pseudoholomorphic curves C_k taken with multiplicities d_k .

Simultaneously, the original Seiberg-Witten equations reduce to a family of Ginzburg-Landau vortex equations in the complex planes normal to the curves C_k . These families can be identified with sections of the d_k -vortex bundle over C_k (cf. [9]).

Conversely, in order to reconstruct the solution of Seiberg–Witten equations from the family of vortex solutions in normal planes it should satisfy a nonlinear equation of $\bar{\partial}$ -type.

Thus, we have for the Seiberg–Witten equations on symplectic 4-manifolds the following correspondence, established by the adiabatic limit:

$$\begin{cases} \text{solutions} & \text{of} \\ \text{Seiberg-Witten} \end{cases} \longmapsto \begin{cases} \text{families of vortex solutions} \\ \text{in normal planes of pseudo-} \\ \text{holomorphic divisors} \end{cases}.$$

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