

UNIFORMLY BOUNDED SPHERICAL HARMONICS AND QUANTUM ERGODICITY ON \mathbb{S}^2

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ABSTRACT. On the two-dimensional unit sphere, we construct uniformly bounded spherical harmonics of arbitrary degree, under a condition of point distribution on the sphere. It extends the results on odd-dimensional spheres by Bourgain [Bo1, Bo2], Shiffman [Sh], and Marzo-Ortega-Cerdà [MOC]. Moreover, we show that the spherical harmonics constructed in this paper are equidistributed in the phase space, i.e., they are quantum ergodic. It provides the first example of Laplacian eigenfunctions which are both uniformly bounded and quantum ergodic.

1. INTRODUCTION

Spherical harmonics on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ are homogeneous and harmonic polynomials in \mathbb{R}^{d+1} restricted to \mathbb{S}^d . We are concerned with the existence of uniformly bounded spherical harmonics (normalized in $L^2(\mathbb{S}^d)$) of arbitrary degree. Some results were previously known. In particular, \mathbb{S}^3 can be regarded as the boundary of the unit ball in the complex space \mathbb{C}^2 ; Bourgain constructed an orthonormal basis of uniformly bounded homogeneous and holomorphic polynomials, which are also spherical harmonics on \mathbb{S}^3 [Bo1]. He then extended his result to \mathbb{S}^5 , similarly regarded as the boundary of the unit ball in \mathbb{C}^3 [Bo2]. On more general compact Kähler manifolds, Shiffman [Sh] and Marzo-Ortega-Cerdà [MOC] proved the existence of uniformly bounded holomorphic sections. (However, the sections in [Sh, MOC] do not form a basis in the complex domain.) Applying to unit spheres in the complex space, we know that there are uniformly bounded spherical harmonics of arbitrary degree on any odd-dimensional sphere. The arguments in [Bo1, Sh, Bo2, MOC] do not apply to even-dimensional spheres.

In the first part of this paper, we develop a different approach to construct uniformly bounded spherical harmonics on \mathbb{S}^2 .

Theorem 1. *Assume Condition 4 of point distribution on \mathbb{S}^2 in Section 2. Then there exists an absolute constant $L > 0$ such that for any $N \in \mathbb{N}$, there is an L^2 -normalized spherical harmonic u_N of degree N on \mathbb{S}^2 which satisfies that $\|u_N\|_{L^\infty(\mathbb{S}^2)} \leq L$.*

Remark. Bourgain independently obtained the same result (unpublished) as in Theorem 1, see [Bo3]. We thank C. Demeter and P. Varjú for informing us.

We shall make some comments about the number of the uniformly bounded spherical harmonics in Theorem 1 and in [Bo1, Sh, Bo2, MOC]. Denote $\mathbb{S}\mathbb{H}_N^d$ the space of spherical harmonics of degree N on \mathbb{S}^d . It is well known that on \mathbb{S}^d ,

$$\dim \mathbb{S}\mathbb{H}_N^d = CN^{d-1} + O_d(N^{d-2}), \quad (1.1)$$

in which $C = C(d) > 0$, see Sogge [So, Section 3.4]. Since $d = 2$ in Theorem 1, the density of the uniformly bounded spherical harmonics in $\mathbb{S}\mathbb{H}_N^2$ is $O(N^{-1}) \rightarrow 0$ as $N \rightarrow \infty$.

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In comparison, the space of homogeneous and holomorphic polynomials of degree N on $\mathbb{S}^d \subset \mathbb{C}^{(d+1)/2}$ has dimension

$$\binom{N + \frac{d+1}{2} - 1}{N} = O_d \left(N^{\frac{d-1}{2}} \right) \quad \text{for odd } d. \quad (1.2)$$

Bourgain's construction provides an orthonormal basis of uniformly bounded holomorphic and homogeneous polynomials on the unit spheres in \mathbb{C}^2 and \mathbb{C}^3 [Bo1, Bo2]; in the general complex setting, Shiffman [Sh] and Marzo-Ortega-Cerdà [MOC] provide a family of uniformly bounded holomorphic sections which has (asymptotically) positive density in an orthonormal basis. However, due to the different dimensions of the bases in (1.1) and (1.2), these holomorphic functions can not form a basis for the spherical harmonics. In [Bo1, Sh, Bo2, MOC], the density of the uniformly bounded spherical harmonics in $\mathbb{S}\mathbb{H}_N^d$ for odd d is $O(N^{(d-1)/2}/N^{d-1}) = O(N^{-(d-1)/2}) \rightarrow 0$ as $N \rightarrow \infty$.

We see that the spherical harmonics in all of these examples (Theorem 1 and [Bo1, Sh, Bo2, MOC]) have density asymptotically zero. It is unclear whether these results can be improved.

Problem 2. *Determine whether there exist uniformly bounded spherical harmonics with (asymptotically) positive density in an orthonormal basis and whether there exists an orthonormal basis of uniformly bounded spherical harmonics.*

Spherical harmonics arise naturally in mathematics and physics as the eigenfunctions of the Laplacian $\Delta_{\mathbb{S}^d}$ on the sphere \mathbb{S}^d (with round metric). That is, if $u \in \mathbb{S}\mathbb{H}_N^d$, then

$$-\Delta_{\mathbb{S}^d} u = N(N + d - 1)u.$$

Hence, the L^∞ estimate of spherical harmonics is subject to Hörmander's (pointwise) Weyl law on compact manifolds [Ho]: For all $u \in \mathbb{S}\mathbb{H}_N^d$,

$$\|u\|_{L^\infty(\mathbb{S}^d)} \leq CN^{\frac{d-1}{2}} \|u\|_{L^2(\mathbb{S}^d)}, \quad (1.3)$$

in which $C = C(d) > 0$. See Sogge [So] and Zelditch [Ze4] for the board area of Laplacian eigenfunction studies on manifolds. In particular, a fundamental but largely unanswered problem is to find and characterize the eigenfunctions with minimal L^∞ norm growth, i.e., $\|u\|_{L^\infty} \leq C\|u\|_{L^2}$, see Toth-Zelditch [TZ] for an overview of this problem and some characterization results under certain condition of "quantum integrability". (The reverse inequality, $\|u\|_{L^2} \leq C\|u\|_{L^\infty}$, is automatically true on compact manifolds by Hölder's inequality.)

In the second part of this paper, we characterize the density distribution of the uniformly bounded spherical harmonics in Theorem 1.

The only manifolds other than the spheres currently known to support uniformly bounded eigenfunctions of arbitrarily large eigenvalues are the tori $\mathbb{T}^d = \mathbb{R}^d/\Lambda$, in which Λ is a lattice such as \mathbb{Z}^d . (One can also consider the eigenfunctions on a fundamental domain of \mathbb{R}^d/Λ with suitable boundary conditions.) In fact, there is a basis of uniformly bounded eigenfunctions $u_\lambda(x) = e^{i\langle \lambda, x \rangle}$ in $L^2(\mathbb{T}^d)$. Here, $x \in \mathbb{T}^d$ and $\lambda \in \Lambda^*$, the dual lattice of Λ . The apparent feature of their density distribution is that they are equidistributed on \mathbb{T}^d , i.e., for any open set $\Omega \subset \mathbb{T}^d$,

$$\lim_{\lambda \rightarrow \infty} \int_{\Omega} |u_\lambda|^2 dx = \text{Vol}(\Omega).$$

That is, $|u_\lambda|^2 dx$ as $\lambda \rightarrow \infty$ defines a measure that coincides with the Lebesgue measure on \mathbb{T}^d . The same conclusion also holds for the real-valued eigenfunctions $\cos\langle \lambda, x \rangle$ and $\sin\langle \lambda, x \rangle$. However, these eigenfunctions are *not* equidistributed in the phase space. In addition, there are other uniformly bounded eigenfunctions on the tori which fail equidistribution in the phase space, such as $\cos(2\pi Nx_1) \sin(2\pi y)$ as $N \rightarrow \infty$ on $\mathbb{R}^2/\mathbb{Z}^2$, see Jakobson [J] for more details.

To describe the distribution property of eigenfunctions in the phase space, we use the semiclassical measures. We recall the setup briefly here and refer to Section 4.1 for more background. Let \mathbb{M} be a compact manifold and $T^*\mathbb{M} = \{(x, \xi) : x \in \mathbb{M}, \xi \in T_x^*\mathbb{M}\}$ be its cotangent bundle. Fix $h_0 \in (0, 1)$ and denote $h \in (0, h_0)$ the semiclassical parameter. We say that a sequence of functions $\{u_{h_k}\}_{k=1}^\infty \subset L^2(\mathbb{M})$ induce a semiclassical measure μ in $T^*\mathbb{M}$ if for any $a \in C_0^\infty(T^*\mathbb{M})$,

$$\langle \text{Op}_{h_k}(a)u_{h_k}, u_{h_k} \rangle \rightarrow \int_{T^*\mathbb{M}} a d\mu \quad \text{as } h_k \rightarrow 0,$$

in which $\text{Op}_h(a)$ is the semiclassical pseudo-differential operator with symbol a . Taking $a = a(x) \in C^\infty(\mathbb{M})$, we have that $\text{Op}_h(a) = a$ so $\langle \text{Op}_h(a)u_h, u_h \rangle = \int_{\mathbb{M}} a|u_h|^2$. Therefore, the semiclassical measure is the lift of the measure defined by $|u_h|^2$ on the physical space \mathbb{M} to the phase space $T^*\mathbb{M}$, which then describes the distribution of $|u_h|^2$ in $T^*\mathbb{M}$.

Consider the eigenfunctions of the semiclassical Laplacian, $(h^2\Delta_{\mathbb{M}} + 1)u_h = 0$. It is well known that the semiclassical measure induced by any sequence of Laplacian eigenfunctions is supported on the cosphere bundle $S^*\mathbb{M} = \{(x, \xi) \in T^*\mathbb{M} : |\xi|_x = 1\}$ and is invariant under the geodesic flow on $S^*\mathbb{M}$. See Zworski [Zw, Section 5.2].

Example (Toral eigenfunctions). On the torus \mathbb{T}^d , consider the eigenfunctions

$$u_{h_k}(x) = e^{i\langle \lambda_k, x \rangle} = e^{i|\lambda_k| \langle \frac{\lambda_k}{|\lambda_k|}, x \rangle} = e^{ih_k^{-1} \langle \frac{\lambda_k}{|\lambda_k|}, x \rangle},$$

in which the semiclassical parameter $h_k = |\lambda_k|^{-1}$. If $\lambda_k/|\lambda_k| \rightarrow \xi_0$ as $|\lambda_k| \rightarrow \infty$ for some fixed $\xi_0 \in \mathbb{S}^{d-1}$, then

$$\langle \text{Op}_{h_k}(a)u_{h_k}, u_{h_k} \rangle \rightarrow \int_{\mathbb{T}^d} a(x, \xi_0) dx \quad \text{as } h_k \rightarrow 0,$$

that is, the corresponding semiclassical measure is $dx\delta_{\xi=\xi_0}$, which is supported on the invariant set $\mathbb{T}^d \times \{\xi_0\} \subset S^*\mathbb{T}^d$. This shows that while these toral eigenfunctions $e^{i\langle \lambda_k, x \rangle}$ are equidistributed on \mathbb{T}^d , they are highly localized in the frequency space so are not equidistributed in the phase space.

The canonical uniform measure on $S^*\mathbb{M}$ is the Liouville measure μ_L and we normalize it so that $\mu_L(S^*\mathbb{M}) = 1$. We say that a sequence of L^2 -normalized eigenfunctions $\{u_{h_k}\}_{k=1}^\infty$ is equidistributed in the phase space if the corresponding semiclassical measure is μ_L , i.e., for all $a \in C_0^\infty(T^*\mathbb{M})$,

$$\langle \text{Op}_{h_k}(a)u_{h_k}, u_{h_k} \rangle \rightarrow \int_{S^*\mathbb{M}} a d\mu_L \quad \text{as } h_k \rightarrow 0.$$

It is obvious that equidistribution in the phase space is a stronger condition than the one in the physical space.

The celebrated Quantum Ergodicity Theorem states that if the geodesic flow is ergodic on $S^*\mathbb{M}$, then any orthonormal basis of eigenfunctions contains a full density subsequence that is equidistributed in the phase space, see Šnirel'man [Sn], Zelditch [Ze1], and Colin de Verdière [CdV]. Because the Laplacian eigenfunctions are the stationary states in the quantum system of the geodesic flow, the eigenfunctions which are equidistributed in the phase space are also said to be *quantum ergodic*.

Let $u \in \mathbb{S}\mathbb{H}_N^2$. Then in the semiclassical setup,

$$(h_N^2\Delta_{\mathbb{S}^2} + 1)u = 0,$$

in which the semiclassical parameter

$$h_N = \frac{1}{\sqrt{N(N+1)}} \approx N^{-1} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

With the help of semiclassical measures, we now characterize the density distribution of the uniformly bounded spherical harmonics in Theorem 1: Our second main theorem asserts that they are quantum ergodic. This is in sharp contrast with the uniformly bounded toral eigenfunctions discussed above, which are highly localized in the phase space.

Theorem 3. *Assume Condition 4 of point distribution on \mathbb{S}^2 in Section 2. Then there is a sequence of uniformly bounded and quantum ergodic spherical harmonics $\{u_N\}_{N=1}^\infty$, $u_N \in \mathbb{S}\mathbb{H}_N^2$, that is, for any $a \in C_0^\infty(T^*\mathbb{S}^2)$,*

$$\lim_{N \rightarrow \infty} \langle \text{Op}_{h_N}(a)u_N, u_N \rangle = \int_{S^*\mathbb{S}^2} a d\mu_L.$$

To the author's knowledge, these spherical harmonics are the *first* example of Laplacian eigenfunctions which are both uniformly bounded and quantum ergodic. It is interesting to ask whether the uniformly bounded spherical harmonics in [Bo1, Sh, Bo2, MOC] are also quantum ergodic and whether the ones in Problem 2, if exist, can also be quantum ergodic.

Remark (Random spherical harmonics). We put Theorems 1 and 3 in the random setting to describe how typical the properties of uniform boundedness and quantum ergodicity are. First, there exists an absolute constant $c > 0$ such that almost surely the spherical harmonics u of degree N from a random orthonormal basis satisfy that $\|u\|_{L^\infty(\mathbb{S}^2)} \geq c\sqrt{\log N}$ for all $N \in \mathbb{N}$. Secondly, the whole sequence of the random basis is quantum ergodic almost surely. See Zelditch [Ze2], VanderKam [V], and Burq-Lebeau [BuLe] for these results in the precise probabilistic setup. Therefore, the uniformly bounded spherical harmonics in Theorems 1 are very atypical, yet they can still display quantum ergodicity as shown in Theorem 3.

Remark (Hecke eigenfunctions). On an arithmetic hyperbolic surface, there is an orthonormal basis of Laplacian eigenfunctions which satisfies certain symmetric conditions associated with the arithmetic structure, see Rudnick-Sarnak [RS]. The whole sequence of these ‘‘Hecke eigenfunctions’’ are known to be quantum ergodic. This phenomenon is called *arithmetic quantum unique ergodicity* and was proved by Lindenstrauss [L], Silberman-Venkatesh [SV], Holowinsky-Soundararajan [HS], and Brooks-Lindenstrauss [BrLi]. For the L^2 -normalized Hecke eigenfunctions, Iwaniec-Sarnak [IS, Theorem 0.1(b)] proved that there is an absolute constant $c > 0$ such that $\|u\|_{L^\infty} \geq c\sqrt{\log \log \lambda}$, in which λ is the eigenvalue of u . So the Hecke eigenfunctions are not uniformly bounded and this lower bound is consistent with the above random model.

Outline of the proofs. The proofs of Theorem 1 and 3 are both based on Gaussian beams, a special type of spherical harmonics that are highly localized around oriented great circles on the sphere. In particular, each oriented great circle $G_p \subset \mathbb{S}^2$ is uniquely determined by a pole $p \in \mathbb{S}^2$ and gives rise a (L^2 -normalized) Gaussian beam $Q_p \in \mathbb{S}\mathbb{H}_N^2$ that localizes around G_p . This means that $\langle Q_j, Q_k \rangle$ is smaller when their corresponding oriented great circles G_j and G_k are more separated, i.e., the two poles p_j and p_k have a larger distance on \mathbb{S}^2 . We construct $u_N \in \mathbb{S}\mathbb{H}_N^2$ as a linear combination of a family of Gaussian beams $\{Q_j\}_{j=1}^m \subset \mathbb{S}\mathbb{H}_N^2$ with properly chosen poles $\{p_j\}_{j=1}^m \subset \mathbb{S}^2$:

$$F_N = \sum_{j=1}^m Q_j \quad \text{and} \quad u_N = \frac{F_N}{\|F_N\|_{L^2(\mathbb{S}^2)}}.$$

- Theorem 1: To have uniformly bounded u_N , we need to control the L^2 and L^∞ norms of F_N . On one hand, we need sufficiently many Gaussian beams so that $\|F_N\|_{L^2(\mathbb{S}^2)}$ can be large in the denominator; on the other hand, we can not have too many of them so $|F_N|$ can potentially be too large in the numerator. Therefore, the choice of the number m of Gaussian beams is

a delicate balance to meet both requirements. In addition, the separation of the poles $\{p_j\}_{j=1}^m$ dictates the interactions among the Gaussian beams $\{Q_j\}_{j=1}^m$, which are crucial for the L^2 as well as the L^∞ estimates of F_N . In Section 3, we prove that

$$\|F_N\|_{L^2(\mathbb{S}^2)}^2 = m + O(N^{-\infty}) \quad \text{and} \quad \|F_N\|_{L^\infty(\mathbb{S}^2)} = O(\sqrt{m}), \quad (1.4)$$

for appropriate $m = O(N^{1/2})$ and under the condition that the poles are separated by distance at least $m^{-1/2}$ and never cluster around any great circle.

- Theorem 3: To have quantum ergodic u_N , we compute that

$$\begin{aligned} & \langle \text{Op}_{h_N}(a)u_N, u_N \rangle \\ &= \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j,k=1}^m \langle \text{Op}_{h_N}(a)Q_j, Q_k \rangle \\ &= \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j,k=1, j \neq k}^m \langle \text{Op}_{h_N}(a)Q_j, Q_k \rangle + \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j=1}^m \langle \text{Op}_{h_N}(a)Q_j, Q_j \rangle. \end{aligned}$$

Notice that $\text{Op}_{h_N}(a)$ is microlocal and does not change the localization of Q_j (which is around the oriented great circle G_j). Hence, the off-diagonal terms in the first summation are small, from the well-separation of the poles $\{p_j\}_{j=1}^m$. The diagonal terms $\langle \text{Op}_{h_N}(a)Q_j, Q_j \rangle$ induce a semiclassical measure as the normalized arclength measure $dl/(2\pi)$ on G_j . Therefore, the second summation above tends to

$$\frac{1}{m} \sum_{j=1}^m \frac{1}{2\pi} \int_{G_j} a \, dl,$$

in the view of the L^2 estimate in (1.4). Next, if the collection of poles $\{p_j\}_{j=1}^m$ tend to be equidistributed on \mathbb{S}^2 , then

$$\frac{1}{m} \sum_{j=1}^m \frac{1}{2\pi} \int_{G_j} a \, dl \rightarrow \frac{1}{\text{Area}(\mathbb{S}^2)} \int_{\mathbb{S}^2} \left(\frac{1}{2\pi} \int_{G_p} a \, dl \right) dp \quad \text{as } m \rightarrow \infty, \quad (1.5)$$

which in turn leads to $\int_{S^*\mathbb{S}^2} a \, d\mu_L$. This is because $S^*\mathbb{S}^2$ can be naturally identified by the space of oriented great circles $\{G_p\}_{p \in \mathbb{S}^2}$, each of which is indexed by $p \in \mathbb{S}^2$. Quantum ergodicity of $\{u_N\}_{N=1}^\infty$ thus follows.

From the outline of the proofs of Theorems 1 and 3, we see that the consequential condition of the Gaussian beams is how their poles are distributed on the sphere. In Section 2, we introduce such a condition about the collections of the poles. In Sections 3 and 4, this condition is used to construct the uniformly bounded spherical harmonics and to prove that they are quantum ergodicity, respectively.

Finally, we remark that the arguments in this paper are restricted to \mathbb{S}^2 . Extending to higher dimensions seems possible but requires new input.

2. POINT DISTRIBUTION ON THE SPHERE

The appropriate condition of distribution of poles for the Gaussian beams is summarized as follows.

Condition 4. *There exist absolute constants $c, C > 0$ such that for any positive integer m , there is a collection of points $\{p_j^m\}_{j=1}^m \subset \mathbb{S}^2$ such that the following holds.*

(i). [Well-separation] For any $j, k = 1, \dots, m$ and $j \neq k$,

$$\text{dist}_{\mathbb{S}^2}(p_j, p_k) \geq \frac{c}{\sqrt{m}},$$

in which $\text{dist}_{\mathbb{S}^2}$ is the geodesic distance on \mathbb{S}^2 .

(ii). [No clustering around great circles] For any great circle $G \subset \mathbb{S}^2$,

$$\# \left\{ j : \text{dist}_{\mathbb{S}^2}(G, p_j) \leq \frac{1}{m} \right\} \leq C,$$

(iii). [Equidistribution] For any $f \in C^\infty(\mathbb{S}^2)$,

$$\lim_{m \rightarrow \infty} \frac{1}{m} \sum_{j=1}^m f(p_j^m) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(p) dp.$$

There is a large literature about the point distribution on the sphere, see, e.g., Lubotzky-Phillips-Sarnak [LPS] and Saff-Kuijlaars [SK]. In particular, it is well known that points on the sphere which satisfy (i) well-separation and (iii) equidistribution in Condition 4 can be constructed explicitly. However, (ii) no clustering around great circles is a rather unconventional requirement in the current literature. Notice that the m^{-1} -neighborhood of any great circle G has area

$$\text{Area} \left(\left\{ p \in \mathbb{S}^2 : \text{dist}_{\mathbb{S}^2}(G, p) \leq \frac{1}{m} \right\} \right) \leq 2\pi \cdot \frac{2}{m} = \frac{4\pi}{m}.$$

Thus, on average of all great circles G , the number of points from $\{p_j\}_{j=1}^m$ which are contained in the m^{-1} -neighborhood of G is uniformly bounded. That is, Property (ii) is expected in a certain average sense. Nevertheless, the construction of points on the sphere which satisfy (i), (ii), and (iii) is unknown to the author. Therefore, Theorems 1 and 3 are both *conditional*. More precisely, Theorem 1 is a consequence of (i) and (ii), while the proof of Theorem 3 uses (i) and (iii).

Remark. We refer to Saff-Kuijlaars [SK] for explicit examples of collections of points on the sphere which satisfy (i) well-separation and (iii) equidistribution in Condition 4. Using these collections, one can construct explicit spherical harmonics that are quantum ergodic, by the same argument as the one for Theorem 3.

As mentioned in the Introduction, a random eigenbasis of spherical harmonics is quantum ergodic almost surely [Ze2, V, BuLe]. So the existence of quantum ergodic spherical harmonics was known, though only implicitly.

Property (ii) no clustering around great circles is necessary for our construction of the uniformly bounded and quantum ergodic spherical harmonics.

3. PROOF OF THEOREM 1

In this section, we construct uniformly bounded spherical harmonics of arbitrary degree. Denote the spherical coordinates $\mathbb{S}^2 \ni x = (x_1, x_2, x_3) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)$, in which $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$. Let $N \in \mathbb{N}$. Then $(x_1 + ix_2)^N \in \mathbb{S}\mathbb{H}_N^2$ when restricted to \mathbb{S}^2 . Write

$$Q_0(x) = C_N N^{\frac{1}{4}} (x_1 + ix_2)^N = C_N N^{\frac{1}{4}} (\sin \phi)^N e^{iN\theta}, \quad (3.1)$$

in which $C_N > 0$ is chosen so that $\|Q_0\|_{L^2(\mathbb{S}^2)} = 1$. Then there is an absolute constant $c_1 > 0$ (i.e., independent of N) such that $C_N \leq c_1$ for all $N \in \mathbb{N}$. See Zelditch [Ze4, Section 4.4.5].

It is obvious that the spherical harmonic Q_0 is concentrated around the equator $G_0 = \{\phi = \pi/2\}$, the great circle perpendicular to the north pole p_0 (i.e., $\phi = 0$), and decreases in ϕ

exponentially in the transverse direction. For this reason, it is usually referred as a Gaussian beam or a highest weight spherical harmonic. In addition, Q_0 propagates in the counter-clockwise direction by the right-hand rule and we say that Q_0 has pole as p_0 . (In Section 4, we describe the ‘‘microlocalization’’ of Gaussian beams in the phase space, which simultaneously characterizes the localization on \mathbb{S}^2 and in the frequency space.)

Since any rotation of Q_0 with respect to an element in $SO(3)$ remains a spherical harmonic, for any $p \in \mathbb{S}^2$, there is a Gaussian beam¹ $Q_p \in \mathbb{SH}_k$ with pole p . For example, the Gaussian beam

$$C_N N^{\frac{1}{4}}(x_1 - ix_2)^N = C_N N^{\frac{1}{4}}(\sin \phi)^N e^{-iN\theta}$$

has pole as the south pole $\phi = \pi$ and is orthogonal with Q_0 . (It has the same concentration around the equator G_0 as Q_0 but propagates in the opposite direction along G_0 .)

$$\text{Let } D > 0 \text{ be chosen later and } m \text{ be the nearest even integer to } \frac{N^{\frac{1}{2}}}{D^2}. \quad (3.2)$$

By Condition 4, there is a collection of points $\{p_j^m\}_{j=1}^m$ which satisfy all properties (i), (ii), (iii). However, we only need (i) well-separation and (ii) no clustering around great circles in this section for Theorem 1. For notational simplicity, we drop the superscript m . Assign Q_j as a Gaussian beam with pole p_j , $j = 1, \dots, m$. Write

$$F_N = \sum_{j=1}^m Q_j. \quad (3.3)$$

Set $u_N = F_N / \|F_N\|_{L^2(\mathbb{S}^2)}$. In the view of (1.3), Theorem 1 follows from the following L^2 and L^∞ estimates of F_N for sufficiently large N .

Proposition 5. *There exist absolute constants $D, N_0, c_2, c_3 > 0$ such that if $N \geq N_0$, then*

$$\|F_N\|_{L^2(\mathbb{S}^2)} \geq c_2 N^{\frac{1}{4}} \quad \text{and} \quad \|F_N\|_{L^\infty(\mathbb{S}^2)} \leq c_3 N^{\frac{1}{4}}.$$

Remark (Comparison with the arguments in [Bo1, Bo2]). Bourgain’s construction of uniformly bounded spherical harmonics on \mathbb{S}^3 and \mathbb{S}^5 uses the fact that the spheres are odd-dimensional. For example, use the coordinates (x_1, x_2, x_3, x_4) on $\mathbb{S}^3 \subset \mathbb{C}^2$. Then the space of the homogeneous and holomorphic polynomials is spanned by $\{z^j w^{N-j}, j = 0, \dots, N\}$. Here, $z = x_1 + ix_2$ and $w = x_3 + ix_4$. Let

$$F_N = \sum_{j=0}^N a_j z^j w^{N-j},$$

in which the coefficients are chosen explicitly that depend on the Rudin-Shapiro sequence [R]. Observe that the polynomials $z^j w^{N-j}$ are mutually orthogonal for different j ’s. Hence, the L^2 norm of F_N is straightforward to estimate. On the other hand, each of $z^j w^{N-j}$ is Gaussian (of different degrees) in both variables z and w . A carefully chosen set of coefficients depending on the Rudin-Shapiro sequence help to control the L^∞ norm of F_N . In this way, Bourgain was able to construct uniformly bounded homogeneous and holomorphic polynomials, which are of course also spherical harmonics.

On even-dimensional spheres such as \mathbb{S}^2 , these holomorphic polynomials are no longer available. In our construction, the Gaussian beams in (3.3) are *not* orthogonal. However, we show that they are almost orthogonal (so the L^2 norm of F_N can be controlled) when the poles are

¹In fact, one can find a family of Gaussian beams with pole p . These functions differ by a phase shift. See [Ha] for a detailed discussion.

well-separated. On the other hand, the L^∞ norm of F_N can be estimated by the properties of well-separation and no clustering around great circles of the poles.

Remark. The estimates in Proposition 5 highly depend on the well-separation and no clustering around great circles of the poles of the Gaussian beams. One can generate a rich variety of spherical harmonic behaviors by arranging the poles in different patterns. For example, one can put $\approx N^{1/2}$ poles on the equator with separation $\approx N^{-1/2}$ such that the resulting spherical harmonic in (3.3) has the maximal L^∞ growth rate of $N^{1/2}$ as the one in (1.3). See Guo-Han-Tacy [GHT, Section 4.2].

3.1. L^2 estimate. The L^2 estimate of F_N in (3.3) is a consequence of (i) well-separation of poles in Condition 4. That is, by (i), the distance between any pair p_j^m and p_k^m for $j \neq k$,

$$\beta_{jk}^m \geq \frac{c}{\sqrt{m}} \geq c_0 D N^{-\frac{1}{4}}, \quad (3.4)$$

in which $c_0 > 0$ is an absolute constant. Then the Gaussian beams $\{Q_j\}_{j=1}^m$ are ‘‘almost orthogonality’’. Indeed, from [Ha, Lemma 5],

$$|\langle Q_j, Q_k \rangle| \leq \left(\cos \frac{\beta_{jk}}{2} \right)^{2N}, \quad (3.5)$$

in which β_{jk} is the distance between the poles p_j and p_k on \mathbb{S}^2 . (In the special case when $\beta_{jk} = \pi$, the poles are antipodal and the two corresponding Gaussian beams are orthogonal.) Since Q_j 's are L^2 -normalized in (3.1),

$$\begin{aligned} \|F_N\|_{L^2(\mathbb{S}^2)}^2 &= \sum_{k,j=1}^m \langle Q_j, Q_k \rangle \\ &= \sum_{j=1}^m \|Q_j\|_{L^2(\mathbb{S}^2)}^2 + \sum_{k,j=1, j \neq k}^m \langle Q_j, Q_k \rangle \\ &= m + \sum_{k,j=1, j \neq k}^m \langle Q_j, Q_k \rangle. \end{aligned}$$

Since $\beta_{jk} \geq c_0 D N^{-1/4}$ by (3.4), we have by (3.5) that

$$\begin{aligned} \sum_{k,j=1, j \neq k}^m |\langle Q_j, Q_k \rangle| &\leq m^2 \left[\cos \left(c_0 D N^{-\frac{1}{4}} \right) \right]^{2N} \\ &\leq \frac{4N}{D^4} \exp \left[2N \log \left(1 - \frac{c_0^2 D^2 N^{-\frac{1}{2}}}{3} \right) \right] \\ &\leq \frac{4N}{D^4} \exp \left[-\frac{2c_0^2 D^2 N^{\frac{1}{2}}}{3} \right] \\ &= O_D(N^{-\infty}) \quad \text{as } N \rightarrow \infty \text{ for any fixed } D > 0. \end{aligned}$$

Here, we use the fact that

$$\cos(\beta) = 1 - \frac{\beta^2}{2} + O(\beta^4) \leq 1 - \frac{\beta^2}{3} \quad \text{for } 0 \leq \beta \leq \frac{1}{2}. \quad (3.6)$$

Therefore,

$$\|F_N\|_{L^2(\mathbb{S}^2)}^2 = m + \sum_{k,j=1, j \neq k}^m \langle Q_j, Q_k \rangle = m + O_D(N^{-\infty}) \quad \text{as } N \rightarrow \infty. \quad (3.7)$$

Recall that m is the nearest even integer to $N^{1/2}/D^2$ by (3.2). Hence, for any $D > 0$, there exist positive constants $N_0 = N_0(D)$ and $c_2 = c_2(D)$ such that if $N \geq N_0$, then

$$\|F_N\|_{L^2(\mathbb{S}^2)} = \sqrt{m + O_D(N^{-\infty})} \geq c_2 N^{\frac{1}{4}}.$$

3.2. L^∞ estimate. The L^∞ estimate of F_N in (3.3) is a consequence of (i) well-separation of poles and (ii) no clustering around great circles in Condition 4. To this end, we estimate $F_N(p_0)$ at the north pole p_0 without loss of generality. (One can always rotate F_N such that the maximum is achieved at p_0 .) By (3.1),

$$|F_N(p_0)| = \left| \sum_{j=1}^m Q_j(p_0) \right| \leq \sum_{j=1}^m |Q_j(p_0)| = C_N N^{\frac{1}{4}} \sum_{j=1}^m |\sin \phi_j|^N \leq c_1 N^{\frac{1}{4}} \sum_{j=1}^m (\cos \alpha_j)^N, \quad (3.8)$$

in which ϕ_j is the distance between the poles p_j and p_0 and $\alpha_j = |\pi/2 - \phi_j|$, i.e., the angle between p_j and the equator $\{\phi = \pi/2\}$. To prove the L^∞ estimate in Proposition 5, we need to show that the above summation is uniformly bounded if D is chosen to be sufficiently large. Recall that the distance between poles is at least $c_0 D N^{-1/4}$ by (3.4), this means that the poles are more separated as D increases while remaining in the order of $N^{-1/4}$.

We partition the poles $\{p_j\}_{j=1}^m$ into three groups:

- (I). $G_1 = \{\alpha_j \leq m^{-1}\}$. Since the poles are not clustered around the equator by (iii) in Condition 4, there are uniformly bounded number of poles in Group I. Since $(\cos \alpha_j)^N \leq 1$, the summation in Group I is uniformly bounded.
- (II). $G_2 = \{m^{-1} < \alpha_j \leq 1/3\}$. We further divide this group into poles from strips of width m^{-1} . Each strip also contains uniformly bounded number of poles from $\{p_j\}_{j=1}^m$. The estimate of the summation in this group follows from choosing D sufficiently large in $m \approx N^{1/2}/D^2$.
- (III). $G_3 = \{\alpha_j > 1/3\}$. Since $(\cos \alpha_j)^N$ decreases exponentially in N , the summation in Group III is well controlled.

Group I. By (iii) in Condition 4, we have that

$$\# \left\{ j : \alpha_j \leq \frac{1}{m} \right\} \leq C.$$

Hence,

$$\sum_{\alpha_j \in G_1} (\cos \alpha_j)^N \leq C.$$

Group II. We further divide G_2 into poles from strips S_l that are orthogonal to the z -axis and of width m^{-1} . Each strip S_l can be covered by W great circles (not necessarily from $\{G_k\}_{k=1}^m$), in which $W > 0$ is a universal constant. By (3.6), we then compute that

$$\begin{aligned} \sum_{\alpha_j \in G_2} (\cos \alpha_j)^N &\leq 2 \sum_{l=1}^{\lfloor m^{-1}/3 \rfloor + 1} \sum_{\alpha_j \in S_l} (\cos \alpha_j)^N \\ &\leq 2 \sum_{l=1}^{\infty} [\cos(lm^{-1})]^N \end{aligned}$$

$$\begin{aligned}
&\leq 2 \sum_{l=1}^{\infty} \exp \left[N \log \left(1 - \frac{l^2 m^{-2}}{3} \right) \right] \\
&\leq C \sum_{l=1}^{\infty} \exp \left[-\frac{cl^2 D^4}{3} \right] \\
&= o(1) \quad \text{as } D \rightarrow \infty.
\end{aligned}$$

Here, we used that fact that $m \approx N^{1/2}/D^2$.

Group III. The number of poles in Group III is trivially bounded by m . We have that

$$\sum_{\alpha_j \in G_3} (\cos \alpha_j)^N \leq m \left(\cos \frac{1}{3} \right)^N \leq \frac{2N^{\frac{1}{2}}}{D^2} \left(\cos \frac{1}{3} \right)^N = O_D(N^{-\infty})$$

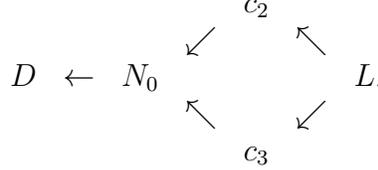
as $N \rightarrow \infty$ for any fixed $D > 0$.

Combining the three groups, there exists $D > 0$ such that

$$\sum_{j=1}^m (\cos \alpha_j)^N \leq c_3$$

for some $c_3 = c_3(D) > 0$ and $N \geq N_0 = N_0(D)$. So $\|F_N\|_{L^\infty(\mathbb{S}^2)} \leq c_3 N^{1/4}$ as stated in Proposition 5.

Remark. A diagram of dependence among the parameters might be helpful: Recall the constant L in Theorem 1. Then



4. PROOF OF THEOREM 3

In this section, we prove that the spherical harmonics $\{u_N\}_{N=1}^{\infty}$ constructed in Section 3 are quantum ergodic. Recall that in the semiclassical setting, $(h_N^2 \Delta_{\mathbb{S}^2} + 1)u_N = 0$ with $h_N = 1/\sqrt{N(N+1)} \approx N^{-1}$. For notational simplicity, we drop the subscript and set $h = N^{-1}$ without affecting the proof. We then need to show that

$$\langle \text{Op}_h(a)u_N, u_N \rangle = \int_{S^*\mathbb{S}^2} a d\mu_L + o_a(1) \quad \text{as } h \rightarrow 0,$$

in which μ_L is the Liouville measure on $S^*\mathbb{S}^2$ normalized such that $\mu_L(S^*\mathbb{S}^2) = 1$. By (3.3),

$$\begin{aligned}
&\langle \text{Op}_h(f)u_N, u_N \rangle \\
&= \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j,k=1, j \neq k}^m \langle \text{Op}_h(a)Q_j, Q_k \rangle + \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j=1}^m \langle \text{Op}_h(a)Q_j, Q_j \rangle. \quad (4.1)
\end{aligned}$$

Here, the poles $\{p_j\}_{j=1}^m$ of the Gaussian beams $\{Q_j\}_{j=1}^m \subset \text{SH}_N^2$ are chosen according to Condition 4. Indeed, we use (i) well-separation and (iii) equidistribution of these poles. That is, the poles are separated by distance at least $c_0 D N^{-1/4}$ and the collection $\{p_j\}_{j=1}^m$ tend to be equidistributed on \mathbb{S}^2 as $m \rightarrow \infty$. In Section 3, we prove that $\{u_N\}_{N=1}^{\infty}$ is uniformly bounded if we set the absolute constant $D > 0$ to be sufficiently large. In this section, D is fixed.

Proof of Theorem 3.

The estimates of the matrix elements $\langle \text{Op}_h(a)Q_j, Q_k \rangle$ in (4.1) are in order. If $\text{Op}_h(a)$ is replaced by the identity operator, then $\langle Q_j, Q_k \rangle$ is controlled by the explicit estimate (3.5) used in Section 3. For the general $\text{Op}_h(a)$ in this section, it is no longer available and the estimates are instead based on the concentration of the Gaussian beams Q_j in the phase space $T^*\mathbb{S}^2$, i.e., the microlocalization. For example, we identify the oriented great circle which is perpendicular to the north pole p_0 by

$$\left\{ (\phi, \theta, \xi_\phi, \xi_\theta) \in T^*\mathbb{S}^2 : \phi = \frac{\pi}{2}, \theta \in [0, 2\pi), \xi_\phi = 0, \xi_\theta = 1 \right\} \subset S^*\mathbb{S}^2,$$

and still denote it by G_0 , i.e., the equator with an orientation given by the covector $(\xi_\phi, \xi_\theta) = (0, 1)$. Then the Gaussian beams in (3.1)

$$Q_0(\phi, \theta) = C_N(\sin \phi)^N e^{iN\theta} = C_N(\sin \phi)^{\frac{1}{h}} e^{i\theta/h} \quad \text{as } h \rightarrow 0,$$

can be represented by a semiclassical Lagrangian distribution that is associated with G_0 . More precisely, Q_0 is microlocalized in the h^ρ -neighborhood of G_0 for any $\rho \in [0, 1/2)$, see Lemma 6 for the precise statement. Similar to Q_0 , the Gaussian beam G_j is microlocalized in the h^ρ -neighborhood of G_j , the oriented great circle with the pole p_j .

We now discuss the off-diagonal terms and the diagonal terms in (4.1), separately.

(I). Off-diagonal term estimate: Since pseudo-differential operators are microlocal, $\text{Op}_h(a)Q_j$ is also microlocalized in the h^ρ -neighborhood of G_j . From the construction in Section 3, the great circles G_j and G_k are separated by distance at least $c_0 DN^{-1/4} \approx h^{1/4} \gg h^\rho$ if we set $1/4 < \rho < 1/2$. This indicates that the microlocal regions of $\text{Op}_h(a)Q_j$ and Q_k are disjoint so the off-diagonal terms are small. In Proposition 7, we prove that for any $j \neq k$,

$$\langle \text{Op}_h(a)Q_j, Q_k \rangle = O_a(h^\infty) \quad \text{as } h \rightarrow 0. \quad (4.2)$$

(II). Diagonal term estimate: In Proposition 8, we prove that,

$$\langle \text{Op}_h(a)Q_j, Q_j \rangle = \frac{1}{2\pi} \int_{G_j} a \, dl + O_a(h) \quad \text{as } h \rightarrow 0, \quad (4.3)$$

in which dl is the arclength measure on G_j .

Putting (I) and (II) together, (4.1) becomes

$$\langle \text{Op}_h(a)u_N, u_N \rangle = \frac{1}{\|F_N\|_{L^2(\mathbb{S}^2)}^2} \sum_{j=1}^m \langle \text{Op}_h(a)Q_j, Q_j \rangle + O_a(h) = \frac{1}{m} \sum_{j=1}^m \frac{1}{2\pi} \int_{G_j} a \, dl + O_a(h),$$

in the view of (3.7) that $\|F_N\|_{L^2(\mathbb{S}^2)}^2 = m + O(h^\infty)$.

(III). Each oriented great circle G_p is uniquely determined by its pole $p \in \mathbb{S}^2$. According to (iii) in Condition 4, the poles $\{p_j\}_{j=1}^m$ tend to be equidistributed on \mathbb{S}^2 as $m \rightarrow \infty$:

$$\frac{1}{m} \sum_{j=1}^m f(p_j) = \frac{1}{4\pi} \int_{\mathbb{S}^2} f(p) \, dp + o_f(1) \quad \text{as } h \rightarrow 0.$$

Setting

$$f(p) = \frac{1}{2\pi} \int_{G_p} a \, dl,$$

we derive that

$$\frac{1}{m} \sum_{j=1}^m \frac{1}{2\pi} \int_{G_j} a \, dl = \frac{1}{8\pi^2} \int_{\mathbb{S}^2} \int_{G_p} a \, dl dp + o_a(1) \quad .$$

To finish the proof of Theorem 3, we observe that the cosphere bundle $S^*\mathbb{S}^2$ can be identified by the space of oriented great circles: For any $a \in C^\infty(S^*\mathbb{S}^2)$,

$$\frac{1}{8\pi^2} \int_{\mathbb{S}^2} \int_{G_p} a \, dl dp = \int_{S^*\mathbb{S}^2} a \, d\mu_L. \quad (4.4)$$

See Jakobson-Zelditch [JZ, Section 3]. The constant $1/(8\pi^2)$ on the left-hand-side of (4.4) is from the normalization that $\mu_L(S^*\mathbb{S}^2) = 1$ and can be easily checked by setting $a = 1$ on $S^*\mathbb{S}^2$. □

To complete the proof of Theorem 3, it is therefore left for us to prove the matrix element estimates (4.2) and (4.3). We review some necessary tools from semiclassical analysis in Subsection 4.1 and then prove these estimates in Subsection 4.2.

Remark. Step (III) above is largely inspired by Jakobson-Zelditch [JZ, Theorem 1.1], which states that any probability measure μ that is invariant under the geodesic flow on $S^*\mathbb{S}^d$ is a semiclassical measure for some sequence of spherical harmonics. To this end, they first reduce μ to the one on the space of oriented great circles. It then can be recovered by linear combinations of the measures on the oriented great circles, each of which is a semiclassical measure induced by the corresponding Gaussian beams. Finally, the linear combinations of the Gaussian beams with the same degree (which are still spherical harmonics) must induce a semiclassical measure as μ .

In our case on \mathbb{S}^2 when the oriented great circles tend to be equidistributed, we recover the Liouville measure on $S^*\mathbb{S}^2$. Of course, we also need the quantitative control on the separation (at the scale of $h^{1/4}$) of these great circles so the linear combination of the corresponding Gaussian beams can produce uniformly bounded spherical harmonics in Theorem 1. It in turn requires finer analysis (i.e., at proper scales depending on h) on the interactions between Gaussian beams than [JZ].

4.1. Semiclassical preliminaries. In this subsection, we collect some standard facts from semiclassical analysis and refer to Zworski [Zw] for a complete treatment on this topic. In particular, we only use symbols with compact support which are sufficient for our purpose.

Let $h_0 \in (0, 1)$ be fixed and $\rho \in [0, 1/2)$. We say that $a(x, \xi; h) \in C_0^\infty(T^*\mathbb{M} \times [0, h_0))$ is in the symbol class $S_\rho^{\text{comp}}(\mathbb{M})$ if for each multi-indices α and β ,

$$\sup_{x \in \mathbb{M}, \xi \in T_x \mathbb{M}} |\partial_x^\alpha \partial_\xi^\beta a| \leq Ch^{-\rho(|\alpha|+|\beta|)},$$

in which $C = C(\alpha, \beta) > 0$ is independent of h . The infimum $C(\alpha, \beta)$ for which the above estimates hold are called the seminorms of a . Clearly, if $a(x, \xi) \in C_0^\infty(T^*\mathbb{M})$ is independent of h , then $a \in S_0^{\text{comp}}(\mathbb{M})$.

We associate symbols in $S_\rho^{\text{comp}}(\mathbb{M})$ with semiclassical pseudo-differential operators as follows. First, for $a \in S_\rho^{\text{comp}}(\mathbb{R}^d)$, choose the (left-)quantization

$$\text{Op}_h(a)u(x) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\langle \xi, x-y \rangle/h} a(x, \xi) u(y) \, d\xi dy, \quad (4.5)$$

in which $u \in C_0^\infty(\mathbb{R}^d)$. Then, for $a \in S_\rho^{\text{comp}}(\mathbb{M})$, the semiclassical pseudo-differential operator $\text{Op}_h(a)$ can be defined via local charts on \mathbb{M} . The correspondence between the operators and the symbols depends on the quantization rule (4.5) and is not one-to-one. However, the following are true. See Dyatlov-Guillarmou [DG, Section 3] for a concise introduction of these results.

- [L^2 boundedness] Let $\rho \in [0, 1/2)$ and $a \in S_\rho^{\text{comp}}(\mathbb{M})$. Then

$$\|\text{Op}_h(a)\|_{L^2(\mathbb{M}) \rightarrow L^2(\mathbb{M})} \leq C, \quad (4.6)$$

in which $C = C(a) > 0$ depends on finite number of seminorms of a and is independent of h .

- [Product] Let $\rho_1, \rho_2 \in [0, 1/2)$ and $\rho_1 + \rho_2 < 1/2$. If $a_1 \in S_{\rho_1}^{\text{comp}}(\mathbb{M})$ and $a_2 \in S_{\rho_2}^{\text{comp}}(\mathbb{M})$, then there exists $a \in S_{\rho_1 + \rho_2}^{\text{comp}}(\mathbb{M})$ such that

$$\text{Op}_h(a_1)\text{Op}_h(a_2) = \text{Op}_h(a) + O_{L^2(\mathbb{M}) \rightarrow L^2(\mathbb{M})}(h^\infty), \quad (4.7)$$

in which $a - a_1 a_2 \in h^{1-\rho_1-\rho_2} S_{\rho_1+\rho_2}^{\text{comp}}(\mathbb{M})$ and $\text{supp } a \subset \text{supp } a_1 \cap \text{supp } a_2$.

- [Adjoint] Let $\rho \in [0, 1/2)$ and $a \in S_\rho^{\text{comp}}(\mathbb{M})$. Denote $\text{Op}_h(a)^*$ the adjoint of $\text{Op}_h(a)$ in $L^2(\mathbb{M})$. Then there exists $b \in S_\rho^{\text{comp}}(\mathbb{M})$ such that

$$\text{Op}_h(a)^* = \text{Op}_h(b) + O_{L^2(\mathbb{M}) \rightarrow L^2(\mathbb{M})}(h^\infty), \quad (4.8)$$

in which $b - \bar{a} \in h^{1-2\rho} S_\rho^{\text{comp}}(\mathbb{M})$ and $\text{supp } b \subset \text{supp } a$.

- [Microlocalization] Let $(h^2 \Delta_{\mathbb{M}} + 1)u_h = 0$ and $a \in C_0^\infty(T^*\mathbb{M})$ such that $a = 1$ in a neighborhood of $S^*\mathbb{M}$. Then

$$\|\text{Op}_h(a)u_h - u_h\|_{L^2(\mathbb{M})} = O(h^\infty). \quad (4.9)$$

The implied constants of the terms $O(h^\infty)$ in (4.7), (4.8), and (4.9) are all independent of h and only depend on finite number of seminorms of the symbols involved.

4.2. Estimates of the matrix elements. In this subsection, we provide the estimates of the matrix elements $\langle \text{Op}_h(a)Q_j, Q_k \rangle$ in (4.1). The main tool is the following description of microlocalization of Gaussian beams in the phase space $T^*\mathbb{M}$ at scales h^ρ with $0 \leq \rho < 1/2$.

Lemma 6. *Let $\rho \in [0, 1/2)$ and $b \in S_\rho^{\text{comp}}(\mathbb{S}^2)$.*

- (i). *If $\text{dist}(\text{supp } b, G_0) \geq h^\rho$, then*

$$\langle \text{Op}_h(b)Q_0, Q_0 \rangle = O(h^\infty).$$

- (ii). *If $b = 1$ in the h^ρ -neighborhood of G_0 , then*

$$\langle \text{Op}_h(b)Q_0, Q_0 \rangle = 1 + O(h^\infty).$$

The implied constants in $O(h^\infty)$ depend on finite number of seminorms of b and are independent of h .

Here, dist is the distance function in $T^*\mathbb{M}$ that is equipped with the Sasaki metric, see e.g., Ballmann [Ba, Section IV.1].

Proof. Use the coordinates $(\phi, \theta, \xi_\phi, \xi_\theta)$ for elements in $T^*\mathbb{M}$. Let $\chi_{\phi,h} \in C_0^\infty(\mathbb{S}^2 \times (0, h_0))$ be independent of θ such that $0 \leq \chi_{\phi,h} \leq 1$ and

$$\chi_{\phi,h} = \begin{cases} 1 & \text{if } \left| \phi - \frac{\pi}{2} \right| \leq \frac{1}{4}h^\rho, \\ 0 & \text{if } \left| \phi - \frac{\pi}{2} \right| \geq \frac{1}{3}h^\rho. \end{cases}$$

Then

$$\begin{aligned} & |\langle \text{Op}_h(b)Q_0, Q_0 \rangle| \\ & \leq |\langle \text{Op}_h(b)(\chi_{\phi,h}Q_0), \chi_{\phi,h}Q_0 \rangle| + |\langle \text{Op}_h(b)(\chi_{\phi,h}Q_0), (1 - \chi_{\phi,h})Q_0 \rangle| \end{aligned}$$

$$+ |\langle \text{Op}_h(b)((1 - \chi_{\phi,h})Q_0), \chi_{\phi,h}Q_0 \rangle| + |\langle \text{Op}_h(b)((1 - \chi_{\phi,h})Q_0), (1 - \chi_{\phi,h})Q_0 \rangle|. \quad (4.10)$$

The last three terms in (4.10) are straightforward to estimate: If $|\phi - \pi/2| \geq h^\rho/4$, then by (3.1) and (3.6) we have that

$$|Q_0| \leq C_N N^{\frac{1}{4}} |\sin \phi|^N \leq c_1 h^{-\frac{1}{4}} \cos \left(\frac{h^\rho}{4} \right)^{\frac{1}{h}} \leq c_1 h^{-\frac{1}{4}} e^{-\frac{h^{2\rho-1}}{48}} = O(h^\infty),$$

since $0 \leq \rho < 1/2$. Since $\text{supp}(1 - \chi_{\phi,h}) \subset \{|\phi - \pi/2| \geq h^\rho/4\}$,

$$\|(1 - \chi_{\phi,h})Q_0\|_{L^2(\mathbb{S}^2)} = O(h^\infty).$$

Together with the L^2 mapping norm estimate of $\text{Op}_h(b)$ in (4.6), we have that

$$|\langle \text{Op}_h(b)(\chi_{\phi,h}Q_0), (1 - \chi_{\phi,h})Q_0 \rangle| \leq C \|\chi_{\phi,h}Q_0\|_{L^2(\mathbb{S}^2)} \|(1 - \chi_{\phi,h})Q_0\|_{L^2(\mathbb{S}^2)} = O_b(h^\infty).$$

For the same reason, the last two terms in (4.10) are also $O_b(h^\infty)$.

It is left to estimate the first term in (4.10). Recall that $G_0 = \{(\phi, \theta, \xi_\phi, \xi_\theta) \in S^*\mathbb{S}^2 : \phi = \pi/2, \theta \in [0, 2\pi), \xi_\phi = 0, \xi_\theta = 1\}$. If $(\phi, \theta) \in \text{supp} \chi_{\phi,h}$ and $(\phi, \theta, \xi_\phi, \xi_\theta) \in \text{supp} b$, then $|\xi_\theta - 1| \geq h^\rho/3$ if $h < h_0$ for sufficiently small h_0 .

Let $\chi_\theta \in C_0^\infty(\mathbb{S}^2)$ depend only on the θ -variable so we can use a local chart on \mathbb{S}^2 that

$$\begin{aligned} & \text{Op}_h(b)(\chi_{\phi,h}Q_0\chi_\theta)(\phi, \theta) \\ &= \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i((\xi_\phi, \phi - \tilde{\phi}) + (\xi_\theta, \theta - \tilde{\theta}))/h} b(\phi, \theta, \xi_\phi, \xi_\theta) \chi_{\phi,h}(\tilde{\phi}) Q_0(\tilde{\phi}, \tilde{\theta}) \chi_\theta(\tilde{\theta}) d\xi_\phi d\xi_\theta d\tilde{\phi} d\tilde{\theta} \\ &= \frac{C_N h^{-\frac{1}{4}}}{(2\pi h)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{i((\xi_\phi, \phi - \tilde{\phi}) + (\xi_\theta, \theta - \tilde{\theta}))/h} b(\phi, \theta, \xi_\phi, \xi_\theta) \chi_{\phi,h}(\tilde{\phi}) \chi_\theta(\tilde{\theta}) (\sin \tilde{\phi})^{\frac{1}{h}} e^{i\tilde{\theta}/h} d\xi_\phi d\xi_\theta d\tilde{\phi} d\tilde{\theta}, \end{aligned}$$

in which the integral with respect to the $\tilde{\theta}$ -variable is

$$\int_{\mathbb{R}} e^{i(1 - \xi_\theta, \tilde{\theta})/h} \chi_\theta(\tilde{\theta}) d\tilde{\theta}.$$

Because $|1 - \xi_\theta| \geq h^\rho/3$ with $0 \leq \rho < 1/2$ and χ_θ is independent of h , we can perform the stationary phase argument to deduce that this integral is $O_b(h^\infty)$. See Zworski [Zw, Section 3.4]. Therefore, $\text{Op}_h(b)(\chi_{\phi,h}Q_0\chi_\theta)(\phi, \theta) = O_b(h^\infty)$ in any local chart so $\text{Op}_h(b)(\chi_{\phi,h}Q_0) = O_b(h^\infty)$. The first term in (4.10) becomes

$$|\langle \text{Op}_h(b)(\chi_{\phi,h}Q_0), \chi_{\phi,h}Q_0 \rangle| \leq \|\text{Op}_h(b)(\chi_{\phi,h}Q_0)\|_{L^2(\mathbb{S}^2)} \|\chi_{\phi,h}Q_0\|_{L^2(\mathbb{S}^2)} = O_b(h^\infty),$$

and (i) in this lemma follows.

To prove (ii), choose $\chi_0 \in C_0^\infty(T^*\mathbb{S}^2)$ such that $\chi_0 = 1$ in the $1/2$ -neighborhood of $S^*\mathbb{S}^2$. Since $b = 1$ in the h^ρ -neighborhood of G_0 , $b - \chi_0 = 0$ in this region. By (4.9),

$$\begin{aligned} \|\text{Op}_h(b)Q_0 - Q_0\|_{L^2(\mathbb{S}^2)} &\leq \|\text{Op}_h(b)Q_0 - \text{Op}_h(\chi_0)Q_0\|_{L^2(\mathbb{S}^2)} + \|\text{Op}_h(\chi_0)Q_0 - Q_0\|_{L^2(\mathbb{S}^2)} \\ &\leq \|\text{Op}_h(b - \chi_0)Q_0\|_{L^2(\mathbb{S}^2)} + O(h^\infty) \\ &\leq O_b(h^\infty), \end{aligned}$$

in which the last inequality follows (i) since $\text{dist}(\text{supp}(b - \chi_0), G_0) \geq h^\rho$. \square

Immediately, we have that

Proposition 7 (Off-diagonal term estimate). *Let $a \in C_0^\infty(T^*\mathbb{S}^2)$. Then for Q_j and Q_k in (4.2) with $j \neq k$,*

$$\langle \text{Op}_h(a)Q_j, Q_k \rangle = O_a(h^\infty).$$

Proof. We assume that $Q_k = Q_0$ without loss of generality. Since G_j and G_0 are separated in $T^*\mathbb{S}^2$ by distance at least $c_0DN^{-1/4} = c_0Dh^{1/4}$ by (3.4), their $2h^\rho$ -neighborhoods are disjoint if $1/4 < \rho < 1/2$ and $0 < h < h_0$ for some absolute constant $0 < h_0 \leq 1$. Therefore, there is a function $b \in S_\rho^{\text{comp}}(\mathbb{S}^2)$ such that b is supported in the $2h^\rho$ -neighborhood of G_0 , equals 1 in the h^ρ -neighborhood of G_0 , but vanishes in the h^ρ -neighborhood of G_j . By (ii) in Lemma 6, we have that

$$Q_0 = \text{Op}_h(b)Q_0 + O_{L^2(\mathbb{S}^2)}(h^\infty),$$

in which the implied constant depends on finite number of seminorms of b and is independent of h . Then

$$\begin{aligned} \langle \text{Op}_h(a)Q_j, Q_0 \rangle &= \langle \text{Op}_h(a)Q_j, \text{Op}_h(b)Q_0 \rangle + O(h^\infty) \\ &= \langle \text{Op}(\tilde{b})^* \text{Op}_h(a)Q_j, Q_0 \rangle + O(h^\infty) \\ &= \langle \text{Op}(g)Q_j, Q_0 \rangle + O_a(h^\infty). \end{aligned}$$

Here, $\tilde{b} \in S_\rho^{\text{comp}}(\mathbb{S}^2)$ and $\text{supp } \tilde{b} \subset \text{supp } b$ by (4.8), $g \in S_\rho^{\text{comp}}(\mathbb{S}^2)$ and $\text{supp } g \subset \text{supp } \tilde{b} \cap \text{supp } a \subset \text{supp } b$ by (4.7). It then follows that $\text{dist}(\text{supp } g, G_j) \geq \text{dist}(\text{supp } b, G_j) \geq h^\rho$. According to (i) in Lemma 6,

$$\langle \text{Op}_h(g)Q_j, Q_0 \rangle = O_a(h^\infty),$$

so $\langle \text{Op}_h(a)Q_j, Q_k \rangle = O_a(h^\infty)$ for all $j \neq k$. \square

Proposition 8 (Diagonal term estimate). *Let $a \in C_0^\infty(T^*\mathbb{S}^2)$. Then for Q_j in (4.3),*

$$\langle \text{Op}_h(a)Q_j, Q_j \rangle = \frac{1}{2\pi} \int_{G_j} a \, dl + O_a(h).$$

Proof. This is a special case of the semiclassical measure computed in Zelditch [Ze3, Proposition 12.1]:

$$\langle \text{Op}_h(a)Q_0, Q_0 \rangle = \frac{1}{2\pi} \int_{G_0} a \, dl + O_a(h).$$

See also Zworski [Zw, Example 2 in Section 5.1]. The constant $1/(2\pi)$ can be easily verified by choosing $a = 1$ in a neighborhood of G_0 . The case for Q_j in (4.3) follows by a rotation. \square

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