

ON LIST COLORING WITH SEPARATION OF THE COMPLETE GRAPH AND SET SYSTEM INTERSECTIONS

JEAN-CHRISTOPHE GODIN

Institut de Mathématiques de Toulon, Université de Toulon, France

RÉMI GRISOT

LEAT, Université Côte d'Azur

OLIVIER TOGNI

LIB, Université Bourgogne Franche-Comté, France

ABSTRACT. We consider the following list coloring with separation problem: Given a graph G and integers a, b , find the largest integer c such that for any list assignment L of G with $|L(v)| = a$ for any vertex v and $|L(u) \cap L(v)| \leq c$ for any edge uv of G , there exists an assignment φ of sets of integers to the vertices of G such that $\varphi(u) \subset L(u)$ and $|\varphi(v)| = b$ for any vertex u and $\varphi(u) \cap \varphi(v) = \emptyset$ for any edge uv . Such a value of c is called the separation number of (G, a, b) . Using a special partition of a set of lists for which we obtain an improved version of Poincaré's crible, we determine the separation number of the complete graph K_n for some values of a, b and n , and prove bounds for the remaining values.

1. INTRODUCTION

Let a, b, c be integers and let G be a graph. A a -list assignment L of G is a function which associates to each vertex a set of a integers. The list assignment L is c -separating if for any $uv \in E(G)$, $|L(u) \cap L(v)| \leq c$. The graph G is (a, b, c) -choosable if for any c -separating a -list assignment L , there exists an (L, b) -coloring of G , i.e. a coloring function φ on the vertices of G that assigns to each vertex v a subset

E-mail addresses: godinjeanchri@yahoo.fr, remi.grisot@etu.univ-cotedazur.fr, olivier.togni@u-bourgogne.fr.

2010 *Mathematics Subject Classification.* 05C15, 05C50.

Key words and phrases. Graph; list coloring; separation condition; set system intersections.

of b elements from $L(v)$ in such a way that $\varphi(u) \cap \varphi(v) = \emptyset$ for any $uv \in E(G)$.

The list coloring problem with restrictions on the list intersections has been introduced by Kratochvíl, Tuza and Voigt [13]. Notice that Kratochvíl et al. [13, 14] defined (a, b, c) -choosability a bit differently, requiring for a c -separating a -list assignment L that the lists of two adjacent vertices u and v satisfy $|L(u) \cap L(v)| \leq a - c$. Among the first results on the topic, a complexity dichotomy was presented [13] and general properties given [14]. These first papers were followed by a series of papers considering choosability with separation of planar graphs, mainly for the case $b = 1$ [3, 4, 5, 6, 7, 12, 16]. While the fact that planar graphs are $(4, 1, 2)$ -choosable was proved very recently [17], a still open question is whether all planar graphs are $(3, 1, 1)$ -choosable or not. Other recent papers concern balanced complete multipartite graphs and k -uniform hypergraphs (for the case $b = 1$) [10]; bipartite graphs (for the case $b = c = 1$) [9]; a study with an extended separation condition [15], and cycles and outerplanar graphs for arbitrary b [11].

In this paper, we concentrate on choosability with separation of complete graphs. As a (a, b, c) -choosable graph is also (a, b, c') -choosable for any $c' < c$, our aim is to determine, for given $a, b, a \geq b$, the largest c such that G is (a, b, c) -choosable. Following our previous work on cycles [11], we define the (*list*) *separation number* $\text{sep}(G, a, b)$ of G as

$$\text{sep}(G, a, b) = \max\{c, G \text{ is } (a, b, c)\text{-choosable}\}.$$

Notice that we have $0 \leq \text{sep}(G, a, b) \leq a$ for any graph G and $a \geq b$, hence this parameter is well defined.

In our setting, we know that any planar graph G satisfies $\text{sep}(G, 5, 1) = 5$ (Thomassen's Theorem), $\text{sep}(G, 4, 1) \geq 2$ [17]; but we do not know if $\text{sep}(G, 3, 1) \geq 1$ holds for all planar graphs G . For the complete graph, Kratochvíl et al. [14] proved that $\text{sep}(K_n, \lfloor \sqrt{n - 11/4} + 3/2 \rfloor, 1) \geq 1$. Moreover, the separation number of the cycle is determined and bounds are given for catuses and outerplanar graphs [11].

The following Hall-type condition that we call the *amplitude condition* is necessary for a graph G to be (L, b) -colorable:

$$\forall H \subset G, \sum_{k \in C} \alpha(H, L, k) \geq b|V(H)|,$$

where $C = \bigcup_{v \in V(H)} L(v)$ and $\alpha(H, L, k)$ is the independence number of the subgraph of H induced by the vertices containing k in their color list. Notice that H can be restricted to be a connected induced subgraph of G . As shown by Cropper et al. [8] (in the more general context of weighted list coloring), this condition is also sufficient when the graph is a complete graph or a path (or some other specific graphs).

For a list assignment L on a graph G of order n with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and for $S \subset [1, n]$, we write $\Sigma_S(L) = \sum_{k \in C} \alpha(H, L, k)$, where H is the subgraph of G induced by $\{v_i, i \in S\}$.

Remark that if G is a complete graph, then $\alpha(H, L, k) = 1$ for any k . Hence the amplitude condition for K_n becomes

$$(1) \quad \forall S \subset [1, n], \Sigma_S(L) = |\cup_{i \in S} L(x_i)| \geq b|S|$$

Example 1. For the complete graph K_4 with vertex set $\{v_1, v_2, v_3, v_4\}$, let L be the 3-separating 5-list assignment defined by:

$L(v_1) = \{1, 2, 3, 4, 5\}$, $L(v_2) = \{1, 2, 3, 6, 7\}$, $L(v_3) = \{3, 4, 6, 7, 8\}$, and $L(v_4) = \{4, 6, 8, 9, 10\}$.

We have $L(v_1) \cup L(v_2) \cup L(v_3) = \{1, 2, 3, 4, 5, 6, 7, 8\}$, hence $\Sigma_{\{1,2,3\}}(L) = |\{1, 2, 3, 4, 5, 6, 7, 8\}| = 8$. Therefore K_4 is not $(L, 3)$ -colorable since $\Sigma_{\{1,2,3\}}(L) = 8 < 3b = 9$. (Note that we also have $\Sigma_{[1,4]}(L) = 10 < 4b = 12$ in this case.)

For the separation number of the complete graph, the following properties are easy to prove:

Property 1. Let a, b, n be integers. Then

- for fixed b, n , the function $\text{sep}(K_n, a, b)$ is increasing with a ;
- for fixed a, b , the function $\text{sep}(K_n, a, b)$ is decreasing with n .

Moreover, we observed (and will prove it in the case $b \leq a < 2b$ and $(n-1)b \leq a < nb$) that for any a, b, n , $\text{sep}(K_n, a+1, b) \leq \text{sep}(K_n, a, b) + 2$.

In Section 2, we introduce proper intersections of set systems and show some of their properties and use them to partition the lists of colors of the vertices of K_n , allowing to simplify the computations. In Section 3, we propose a general coloring algorithm and two special types of list assignments with nice properties that will be used mainly for finding good counter-examples. Then in Section 4, we determine bounds and exact values for the separation number of the complete graph K_n , depending on a, b and n , and finish with a conjecture. The algebra that allowed us to find the counter examples of Section 4 is given in Appendix A.

2. ALGEBRAIC PRELIMINARIES ON SET SYSTEMS INTERSECTIONS

2.1. Proper intersections and Poincaré's crible improvement.

Let $n \geq 1$ be an integer and $[n] = [1, n]$. For sets of elements A_1, A_2, \dots, A_n , we define the following: For any $i \geq 1$ and $S_i = \{\alpha_1, \alpha_2, \dots, \alpha_i\} \subset [n]$, the *proper intersection* $I_p(S_i)$ of $A_{\alpha_1}, A_{\alpha_2}, \dots, A_{\alpha_i}$ is given by

$$I_p(S_i) = \left(\bigcap_{k=1}^i A_{\alpha_k} \right) \setminus \left(\bigcup_{\beta \in [n] \setminus S_i} A_{\beta} \right).$$

Notice that for any set $S \in [n]$, the whole n sets A_i are present in the formulae of the proper intersection of S . For example, for $n = 5$, $I_p(\{1\}) = A_1 \setminus (A_2 \cup A_3 \cup A_4 \cup A_5)$; $I_p(\{2, 3\}) = (A_2 \cap A_3) \setminus (A_1 \cup A_4 \cup A_5)$; $I_p(\{1, 2, 5\}) = (A_1 \cap A_2 \cap A_5) \setminus (A_3 \cup A_4)$; $I_p(\{2, 3, 4, 5\}) = (A_2 \cap A_3 \cap A_4 \cap A_5) \setminus A_1$ and $I_p(\{1, 2, 3, 4, 5\}) = A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5$.

Then the classical intersection can be described in terms of proper intersections:

Property 2. For any $S \subset [n]$,

$$\bigcap_{i \in S} A_i = \bigcup_{S' \subset [n]} I_p(S').$$

Proof. The proof is by double inclusion. First, by construction, For any $S' \supset S$, any x in $I_p(S')$ is also an element of $\bigcap_{i \in S} A_i$. Next, if $x \in \bigcap_{i \in S} A_i$, then either $x \in I_p(S)$ and we are done or $x \notin I_p(S)$. In this latter case, let \bar{S} be the set of indices j of $[n] \setminus S$ for which $x \in A_j$. Then, by construction, we have $x \in I_p(S')$, for $S' = S \cup \bar{S}$. \square

Thanks to this property, we obtain the fact that the proper intersections form a partition of $\bigcup_{i=1}^n A_i$.

Property 3. For any $S, S' \subset [n]$ such that $S \neq S'$, we have

$$I_p(S) \cap I_p(S') = \emptyset.$$

Proof. At least one of $S - S'$ or $S' - S$ is non empty. Let us say $S - S'$ is not empty and let $s \in S - S'$. Let $x \in I_p(S) \cap I_p(S')$. Then $x \in A_s$, but $x \in I_p(S')$ and $s \notin S'$. Hence $x \notin A_s$, a contradiction and therefore the intersection is empty. \square

We thus have the following *à-la-Poincaré* result:

Theorem 4.

$$\bigcup_{i=1}^n A_i = \bigcup_{S \subset [n], S \neq \emptyset} I_p(S).$$

Proof. By Property 2 for $|S| = 1$, we have $A_i = \bigcup_{S' \subset [n] \setminus \{i\}} I_p(\{i\} \cup S')$. Therefore $\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n \bigcup_{S' \subset [1, n] \setminus \{i\}} I_p(\{i\} \cup S') = \bigcup_{S \subset [n], S \neq \emptyset} I_p(S)$. \square

By Property 3 and Theorem 4, we obtain the following crible which can be seen as an improvement of Poincaré's crible as we only do additions (compared to Poincaré's crible where additions and subtractions alternate):

Corollary 5.

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{S \subset [n], S \neq \emptyset} |I_p(S)|.$$

2.2. List assignments of K_n and proper intersections. We will use proper intersections on list assignments of K_n . Let $V(K_n) = [n]$. For a list assignment L of K_n , we let $A_i = L(i)$, for $1 \leq i \leq n$.

In a basic way, L is a set of n lists of size a . Using proper intersection tools from the previous subsection, the list assignment L can now be represented by the vector of proper intersections cardinals $\mathbf{V}(L)$ of dimension $2^n - 1$ (ordered by set inclusion and alphanumeric order).

Then we have

$$(2) \quad \forall i \in [n] \quad \sum_{S \subset [n], i \in S} |I_p(S)| = |L(i)|.$$

And the cardinality of the intersection between two lists is given by

$$(3) \quad \forall i, j \in [n], i \neq j, \quad \sum_{S \subset [n], i, j \in S} |I_p(S)| = |L(i) \cap L(j)|.$$

Moreover, thanks to Corollary 5, the total amplitude of L is given by

$$(4) \quad \Sigma_{[n]}(L) = \sum_{S \subset [n]} |I_p(S)|.$$

Example 2. For the complete graph K_4 and the list assignment L defined in Example 1, we have : $I_p(\{1\}) = \{5\}$, $I_p(\{4\}) = \{9, 10\}$, $I_p(\{1, 2\}) = \{1, 2\}$, $I_p(\{2, 3\}) = \{7\}$, $I_p(\{3, 4\}) = \{8\}$, $I_p(\{1, 2, 3\}) = \{3\}$, $I_p(\{1, 3, 4\}) = \{4\}$, and $I_p(\{2, 3, 4\}) = \{6\}$. Hence the vector $\mathbf{V}(L) = (1, 0, 0, 2, 2, 0, 0, 1, 0, 1, 1, 0, 1, 1, 0)$.

Consequently, from these equations, we obtain an ILP-formulation of the problem of finding smallest counter examples (c -separating a -list assignments L of K_n for which no (L, b) -coloring exists): For $S \subset [n]$, we consider the variable $x_S = |I_p(S)|$. Then the goal is to minimize c subject to the constraints:

$$(5) \quad \forall i \in [n], \quad \sum_{S \subset [n], i \in S} x_S = a,$$

$$(6) \quad \forall i, j \in [n], i \neq j, \quad \sum_{S \subset [n], i, j \in S} x_S \leq c,$$

and

$$\sum_{S \subset [n]} x_S < nb.$$

This formulation allows to use ILP-solvers to help us finding counter-examples.

Remark 6. Assuming that the colors are taken from the set $\{1, \dots, na\}$ (this is always possible, up to a color renumbering), we can count the number of different list assignments associated to the same vector of proper intersections $\mathbf{V}(L)$: Considering any ordering $S_1, S_2, \dots, S_{2^n-1}$

of the non-empty subsets of $[n]$, this number of list assignments is given by

$$\prod_{i=1}^{2^n-1} \binom{na - \sum_{j<i} |I_p(S_j)|}{|I_p(S_i)|} = \frac{(na)!}{(na - \sum_i |I_p(S_i)|)! (\prod_i |I_p(S_i)|)!}.$$

For instance, if all proper intersections are equal to zero except $I_p([n]) = a$, then we obtain $\binom{na}{a}$ lists.

2.3. Counting list assignments up to proper intersection equivalence. In order to compute the gain of working with proper intersections instead of list assignments, we are now going to count the total number of a -list assignments up to proper intersection equivalence. Let $\mathcal{L}(n, a)$ be the set of all a -list assignments on n vertices up to proper intersection. The cardinal of $\mathcal{L}(n, a)$ can be compared with the total number of a -list assignments on na colors, which is $\binom{na}{a}^n$.

For $n = 2$, it is easy to see that there are only $a+1$ different lists up to proper intersections, hence $|\mathcal{L}(2, a)| = a+1$. We define $\mathcal{L}(n, a, [n] = 0)$ as the subset of $\mathcal{L}(n, a)$ for which $I_p([n]) = \emptyset$ and $\mathcal{L}(n, a, [n] > 0)$ as the subset of $\mathcal{L}(n, a)$ for which $|I_p([n])| > 0$.

Lemma 7. *For integers $a \geq 1, n \geq 1$, we have :*

$$|\mathcal{L}(n, a)| = |\mathcal{L}(n, a-1)| + |\mathcal{L}(n, a, [n] = 0)|.$$

Proof. It is easily seen that $\mathcal{L}(n, a) = \mathcal{L}(n, a, [n] > 0) \cup \mathcal{L}(n, a, [n] = 0)$. Now, note that there is a trivial isomorphism between lists of $\mathcal{L}(n, a, [n] > 0)$ and that of $\mathcal{L}(n, a-1)$. To see this, represent each list by the vector of the cardinals of their proper intersections, ordered by the size. Then since the last element of each vector is positive, its value can be decreased by one, obtaining a vector corresponding with a list assignment of $\mathcal{L}(n, a-1)$. \square

Remark that any element of $\mathcal{L}(n, a)$ must satisfy Equation 5 for each vertex and reciprocally any solution of this set of n equations gives a list assignment $\mathcal{L}(n, a)$. Hence the set of all solutions is a subspace of dimension $2^n - 1 - n$ of a vectorial space of this equation set.

Lemma 8. *For all $S = \{a_1, a_2, \dots, a_i\} \subset [n], |S| \geq 2$, the vectors $\mathbf{v}_S = \mathbf{e}_S - \sum_{j=1}^i \mathbf{e}_{a_j}$ form a canonical base of $\mathcal{L}(n, a)$.*

Proof. As we have $2^n - 1$ proper intersections, then the space is of dimension $2^n - 1$. But we have n linear equations (Equation 5) and since they are linearly independent, thus $\mathcal{L}(n, a)$ is of dimension $2^n - 1 - n$. It then suffices to prove that the family of \mathbf{v}_S is free. For any $S \subset [n], |S| \geq 2$, let α_S , such that $\sum \alpha_S \mathbf{v}_S = \mathbf{0}$. We have $\sum \alpha_S \mathbf{v}_S = \sum_{i=1}^n \beta_i \mathbf{e}_i + \sum_{|S| \geq 2} \alpha_S \mathbf{e}_S = \mathbf{0}$. As the \mathbf{e}_S are vectors of the canonical base, then $\alpha_S = 0$ for every S . Hence the family is free. \square

There exists a trivial solution to the set of equations, it is the list assignment with vector $\mathbf{T} = (a, a, \dots, a, 0, \dots, 0)$ with n values equal to a . Then any list assignment solution can be written as the trivial solution plus a linear combination of these base vectors \mathbf{e}_S . Hence a list assignment solution can be identified by a linear sum of base vectors. We therefore only have to work on linear sums of base vectors. But, as the entries of a solution have to be non negative, we only have a finite number of solutions. We define the set of list assignments $\mathcal{L}(n, = a)$ as the set of list assignments such that there exists some $i \in [n]$ for which the sum of coefficients α_S over all S containing i in the linear combination is equal to a in the vectorial representation (i.e., such that $I_p(\{i\}) = \emptyset$) and for which $I_p([n]) = \emptyset$.

Lemma 9.

$$|\mathcal{L}(n, a, [n] = 0)| = |\mathcal{L}(n, a - 1, [n] = 0)| + |\mathcal{L}(n, = a)|.$$

Proof. The set $\mathcal{L}(n, a, [n] = 0)$ trivially decomposes into the set \mathcal{A} of list assignments for which all the first n entries of the vector are strictly positive and the set of which there is at least a zero in the first n entries of the vector, i.e., $\mathcal{L}(n, = a)$. But there is a natural bijection between \mathcal{A} and $\mathcal{L}(n, a - 1, [n] = 0)$, hence the recursion formulae. \square

Lemma 10.

$$|\mathcal{L}(3, = a)| = \begin{cases} 3/4a^2 + 3/2a + 1 & a \text{ even} \\ 3/4a^2 + 3/2a + 3/4 & a \text{ odd.} \end{cases}$$

Proof. For $n = 3$, we have only 3 vectors: $S = (1, 2), (1, 3)$, and $(2, 3)$. Hence any solution can be represented by a triplet of vectors $(\alpha_1, \alpha_2, \alpha_3)$. Starting from $(a, 0, 0)$, we count the triplets by equivalence classes such that the first coefficient is maximum, the second one is greater than or equal to the third one and then we count the number of elements in the class of equivalence. We then have three cases: if the three numbers are different then there are $3! = 6$ different combinations; if exactly two numbers are different then we have 3 combinations and only one if the three numbers are all identical. Hence each solution is of the form $(a - x, x, y)$ with $0 \leq x \leq a/2$ and $y \leq x$. Therefore, we obtain the following formulae:

$$|\mathcal{L}(3, = a)| = \begin{cases} 1 + 3a/2 + \sum_{i=a/2+1}^a (3 + 6(a - i)), & a \text{ even} \\ \sum_{i=(a+1)/2}^a (3 + 6(a - i)), & a \text{ odd.} \end{cases}$$

Hence, using simple calculations, we obtain the formulae of the lemma. \square

Proposition 11. *For $n = 3$, we have*

$$|\mathcal{L}(n, a)| = \frac{1}{16}(a^4 + 8a^3 + 24a^2 + 32a + 16 - \epsilon(a)),$$

where $\epsilon(a) = 1$ if a is odd and $\epsilon(a) = 0$ otherwise.

Proof. Combining the two recursion formulas of Lemmas 7 and 9, we obtain the formulae for the residue, allowing to find the general polynomial. \square

Conjecture 12. $|\mathcal{L}(n, a)|$ is a polynomial in a of degree $n + 1$, with coefficients being functions of Bernoulli's numbers.

Proposition 11 (and Conjecture 12, if true) says that using proper intersections allows to go from an exponential number of list assignments to treat to a polynomial number.

3. (a, b, c) -CHOOSABILITY OF K_n

3.1. General coloring algorithm. We provide an algorithm that, given a list assignment L of K_n , produces a multi-coloring. We conjecture that this algorithm is optimal (i.e., produces an (L, b) -coloring if K_n is (L, b) -colorable).

Algorithm ColorSym: Taking an arbitrary list-assignment L as input, we consider at each step the vector $\mathbf{w} = (w_1, \dots, w_n)$, with $w_i =$ being the number of colors given to v_i so far.

Step 1. Color every proper intersections of sets of size 1 : v_i gets $|I_p(\{i\})|$ colors. Thus $w_i = |I_p(\{i\})|$.

Step $i \geq 2$. If $w_i \geq b$ for each i or $i = n + 1$, then Stop. Otherwise, for each set S with $|S| = i$, consider $|I_p(S)|$ and let $j \in S$ be the index for which $w(j)$ is minimum (if more than one j with minimum w , then take the smallest index). Remove a color from $I_p(S)$ and assign it to v_j . Goto Step i until $I_p(S)$ is empty.

3.2. Symmetrical list assignments on K_n . Let L be a a -list assignment on K_n . We say that L is *symmetrical* if for any i , $1 \leq i \leq n$, and any $S, S' \subset [n]$ such that $|S| = |S'|$, we have $|I_p(S)| = |I_p(S')|$. For such a list assignment, we let $x_i = |I_p(S)|$ for $S \subset [n], |S| = i$ and $\mathbf{x}(L) = (x_1, x_2, \dots, x_n)$.

Considering symmetrical list assignments allows to reduce the number of variables in the linear program from $2^n - 1$ to n and to simplify Equation 5 into

$$(7) \quad \sum_{i=1}^n \binom{n-1}{i-1} x_i = a,$$

and Equation 6 into

$$(8) \quad \sum_{i=2}^n \binom{n-2}{i-2} x_i \leq c,$$

and for the total amplitude:

$$(9) \quad \sum_{i=1}^n \binom{n}{i} x_i \geq nb.$$

As we will see, symmetrical list assignments have the nice property that Equation 9 is sufficient to guarantee an (L, b) -coloring of K_n . We will first prove how to find a balanced coloring of a set of subsets of $[n]$. We define $\mathcal{P}(i, n)$ as the set of all subsets of cardinality i of the set $\{1, \dots, n\}$. Observe that $|\mathcal{P}(i, n)| = \binom{n}{i}$. Let also $\mathcal{P}(n) = \cup_{i=1}^n \mathcal{P}(i, n)$. For a weight vector $\mathbf{w} = (w_1, \dots, w_n)$, with $w_i \geq 0$ for all i , $1 \leq i \leq n$, let $\mathcal{P}(n)^{\mathbf{w}}$ be the multiset obtained from the empty set by adding to it w_i times the set $\mathcal{P}(i, n)$ for any i , $1 \leq i \leq n$.

A partition (or coloring) P_1, \dots, P_n of a subset S of $\mathcal{P}(n)^{\mathbf{w}}$ is *balanced* if for any i, j , $1 \leq i \neq j \leq n$, $||P_i| - |P_j|| \leq 1$ and for any $X \in P_j$, $j \in X$.

Lemma 13. *For any integer $n \geq 3$ and any weight vector \mathbf{w} of size n , there exists a balanced partition of $\mathcal{P}(n)^{\mathbf{w}}$.*

Proof. Observe that $\mathcal{P}(n)^{\mathbf{w}}$ is the union of sets $\mathcal{P}(i, n)$. Let S_n be the group of permutations of n elements. Remark that S_n acts on $\mathcal{P}(i, n)$. Let g be the n -elements cycle. Hence $\mathcal{P}(i, n)$ can be viewed as the set of its orbits by g and $\mathcal{P}(n)^{\mathbf{w}}$ as a multiset of orbits.

The general idea to prove the lemma is to color $\mathcal{P}(n)^{\mathbf{w}}$ by coloring each orbit of each set $\mathcal{P}(i, n)$ one after the other in any order while ensuring that at each step, the partial coloring is balanced. The key idea for showing this is possible is the following observation:

Claim 14. *For any integers i and j , $1 \leq i, j \leq n$, any orbit O of $\mathcal{P}(i, n)$ can be colored with colors $j, j+1, \dots, j+|O|$ (with $n+1=1$).*

proof. Observe first that all elements of $[n]$ are present at least once in O . Hence, there exists $o_j \in O$ such that o_j contains j . Color o_j by color j and color each remaining element $g^k(o_j)$ of O with color $j+k$ for $k \in \{1, \dots, n-1\}$ ($j+k-n$ if $j+k > n$).

The algorithm for obtaining a balanced coloring φ consists in coloring the orbits of $\mathcal{P}(n)^{\mathbf{w}}$ in any order and, when coloring the elements of a new orbit, to start with the color j such that $|\varphi^{-1}(j)| < |\varphi^{-1}(j-1)|$ if such a j exists and $j=1$ otherwise and use Claim 14. After each such step, we can observe that the (partial) coloring φ remains balanced. Therefore, after having colored the last element of the last orbit, φ is a balanced coloring of $\mathcal{P}(n)^{\mathbf{w}}$. □

Corollary 15. *Let L be a symmetrical list assignment of K_n . If K_n is (L, b) -colorable, then Algorithm ColorSym produces an (L, b) -coloring.*

Proof. To any symmetrical list assignment L , we can make correspond a set $\mathcal{S} \subset \mathcal{P}(n)^{\times}$ (each set $S \subset [n]$, $|S| = i$, is present x_i times in

\mathcal{S}). Hence by Lemma 13, there exists a balanced coloring of \mathcal{S} , and thus a coloring φ from L which is balanced, i.e., for any $i, j \in [n]$, $||\varphi(v_i)| - |\varphi(v_j)|| \leq 1$. Therefore the amplitude condition has only to be verified globally, i.e., it suffices to show that Equation 9 is satisfied in order K_n to be (L, b) -colorable. As Equation 9 is a necessary condition, if an (L, b) -coloring of K_n exists then Algorithm ColorSym ends up with such a coloring. \square

3.3. Quasi-symmetrical list assignments. Let L be a a -list assignment on K_n . We say that L is *quasi-symmetrical* if for any i , $1 \leq i \leq n$, and any $S, S' \subset [n]$ such that $|S| = |S'|$, we have $-1 \leq |I_p(S)| - |I_p(S')| \leq 1$.

The *vector of cardinality-support* $\mathbf{cs}(S)$ of a set $S = \{S_1, S_2, \dots, S_k\}$ of subsets of $[n]$ is defined by $\mathbf{cs}(S) = (\beta_1, \dots, \beta_n)$, with $\beta_i = |\{j \in \{1, \dots, k\} : i \in S_j\}|$.

Lemma 16. *For any n, k with $n \geq 4$, $k \geq 2$ and $n \geq (2k - 1 + \sqrt{8k + 1})/2$, there exist a set of $k + 1$ k -subsets $\{S'_1, S'_2, \dots, S'_{k+1}\}$ of $[n]$ and a set of k $(k + 1)$ -subsets $\{S_1, S_2, \dots, S_k\}$ of $[n]$ having the same cardinality-support vector.*

Proof. Let n, k be as in the statement of the Lemma. We consider two cases : $2 \leq k \leq n/2$ and $n/2 < k \leq n - 2$.

Case 1 : $2 \leq k \leq n/2$. For $1 \leq i \leq k$, set $S'_i = \{1, \dots, k, k + i\}$

For $1 \leq j \leq k$, set $S_j = S'_j \setminus \{j\}$ and $S_{k+1} = \{1, \dots, k\}$.

By construction, they have the same cardinality support vector that starts with k times the entry k and k times the entry 1, i.e., $\mathbf{cs}(S) = (k, k, \dots, k, 1, \dots, 1, 0, \dots, 0)$.

Case 2 : $k \geq n/2$. For $1 \leq i \leq k$, set $S'_i = \{1, \dots, k - 1, x_i, y_i\}$, with $\{x_i, y_i\} \subset [k, n]$ and $x_i \neq y_i$. We choose for each S'_i a different couple. This is always possible since there are $\binom{n - k + 1}{2}$ such couples and

we have $\binom{n - k + 1}{2} \geq k$ since $n \geq (2k - 1 + \sqrt{8k + 1})/2$.

For $1 \leq j \leq k - 1$, $S_j = S'_j \setminus \{j\}$ and $S_k = S'_k \setminus \{x_k\}$ and $S_{k+1} = \{1, \dots, k - 1, x_k\}$. Again, by construction, both families of sets have the same cardinality support vector. \square

Proposition 17. *Let a, b, n, x be integers such that $x \leq n$ and $a = xb$ is a multiple of $\binom{n-1}{x-1}$. Then K_n is not $(a, b, a \frac{x-1}{n-1} + 1)$ -colorable.*

Proof. Let $a = p \binom{n-1}{x-1}$ for some $p \geq 1$. We first construct a symmetrical list assignment L by setting $|I_p(S_x)| = p$ for every set $S_x \subset [n]$ with $|S_x| = x$ and $|I_p(S)| = 0$ for $|S| \neq x$.

We check that for every $j \in [n]$, $|L(j)| = a$ and that $\sum(L) = nb$. In fact, there are $\binom{n}{x}$ x -subsets of $[n]$ and $\binom{n-1}{x-1}$ of them containing

some $j \in [n]$. Hence $|L(j)| = p \binom{n-1}{x-1} = a$ and $\Sigma(L) = p \binom{n}{x} = p \binom{n-1}{x-1} \frac{n}{x} = \frac{an}{x} = bn$. Moreover, observe that for any $i, j \in [n]$, we have $|L(i) \cap L(j)| = p \binom{n-2}{x-2} = p \binom{n-1}{x-1} \frac{x-1}{n-1} = a \frac{x-1}{n-1}$.

Now, by Lemma 16, there exists a set of $x+1$ x -subsets of $[n]$ and a set of x $(x+1)$ -subsets of $[n]$ with the same cardinality support vector \mathbf{cs} . Let \mathcal{S} and \mathcal{S}' , respectively, be such sets as defined in the proof of Lemma 16. Hence, we can decrease $|I_p(S)|$ by one for every $S \in \mathcal{S}$ (hence $|I_p(S)| = p-1$ for such set) and increase $|I_p(S')|$ by one for every $S' \in \mathcal{S}'$ (hence $|I_p(S')| = 1$ for such set). Let L' be the quasi-symmetrical resulting list assignment. Since the cardinality support vector is the same for \mathcal{S} and \mathcal{S}' , L' is a a -list assignment. We now prove that the amplitude of L' is $\Sigma(L') = nb-1$ and that L' is c' -separating, with $c' = a \frac{x-1}{n-1} + 1$. For the amplitude, we have $\Sigma(L') = p \binom{n}{x} - (x+1) + x = p \binom{n}{x} - 1 = \Sigma(L) - 1 = nb-1$. For the separation condition, if $i \in [n]$ such that $\mathbf{cs}(i) = 0$, then we have, for any $j \in [n]$, $|L'(i) \cap L'(j)| = |L(i) \cap L(j)| \leq c$. Otherwise, if $i, j \in [n]$ are such that $\mathbf{cs}(i) > 0$ and $\mathbf{cs}(j) > 0$, then we are going to prove that $|L'(i) \cap L'(j)| \leq |L(i) \cap L(j)| + 1 \leq c+1$. First, consider the case $2 \leq x \leq n/2$. If $1 \leq i, j, \leq x$, then, by the construction of the proof of Lemma 16, there are $x-2+1 = x-1$ sets in \mathcal{S} and x sets in \mathcal{S}' containing i, j , hence $|L'(i) \cap L'(j)| = |L(i) \cap L(j)| - (x-1) + x = |L(i) \cap L(j)| + 1 = c'$. Otherwise ($i > x$ or $j > x$), then $|L'(i) \cap L'(j)| = |L(i) \cap L(j)|$. Second, consider the case $x > n/2$. If $1 \leq i, j, \leq x-1$, then, by the construction of the proof of Lemma 16, there are $x-2+1 = x-1$ sets in \mathcal{S} and x sets in \mathcal{S}' containing i, j , hence $|L'(i) \cap L'(j)| = |L(i) \cap L(j)| - (x-1) + x = |L(i) \cap L(j)| + 1 = c'$. Otherwise, if $1 \leq i \leq x-1$ and $j \geq x$, then there is one more set in \mathcal{S}' than in \mathcal{S} containing both i and j , hence $|L'(i) \cap L'(j)| = |L(i) \cap L(j)| + 1 = c'$. Otherwise ($i \geq x$ and $j \geq x$), we have $|L'(i) \cap L'(j)| = |L(i) \cap L(j)|$.

In conclusion, we have built a c' -separating a -list assignment L' for which the amplitude condition is not fulfilled and thus $\text{sep}(K_n, a, b) < c'$. \square

Proposition 18. *For integers a, b, n such that $a \geq 2b$ and a is a multiple of $\lfloor \frac{a^2}{2b(n-1)} \rfloor$, K_n is $(a, b, \lfloor \frac{a^2}{2b(n-1)} \rfloor)$ -choosable.*

Proof. Let $c = \lfloor \frac{a^2}{2b(n-1)} \rfloor$ and L be a c -separating a -list assignment with $a = \lambda c$ for some $\lambda \geq 1$. Let S be an i -subset of $[n]$. We are going to show that the amplitude condition is satisfied for S . We consider two cases.

If $i \leq \lambda$, then since L is c -separating, the amplitude satisfies

$$\Sigma_S(L) \geq a + (a - c) + \dots + (a - (i - 1)c) = ia - \frac{1}{2}i(i - 1)c.$$

Since $ic \leq \lambda c \leq a$ and $a \geq 2b$, we obtain $\Sigma_S(L) \geq ia - \frac{1}{2}(i - 1)a = \frac{1}{2}(i + 1)a \geq ib$.

If $i > \lambda$, then the amplitude satisfies

$$\Sigma_S(L) \geq a + (a - c) + \dots + (a - \lambda c) = (\lambda + 1)a - \frac{1}{2}\lambda(\lambda + 1)c = (\lambda + 1)a - \frac{1}{2}\lambda a = \frac{1}{2}(\lambda + 1)a.$$

Since $c = \lfloor \frac{a^2}{2b(n-1)} \rfloor \leq \frac{a^2}{2b(n-1)}$, then $\lambda = \frac{a}{c} \geq \frac{2b(n-1)}{a}$.

The above inequality with $a \geq 2b$ induces

$$\Sigma_S(L) \geq \frac{1}{2}(\lambda + 1)a \geq \frac{2b(n-1) + a}{2a}a = bn + \frac{a - 2b}{2} \geq bn.$$

Therefore the amplitude condition is satisfied and thus K_n is (L, b) -colorable. \square

4. SEPARATION NUMBER OF K_n

For K_2 , it is easily seen that $\text{sep}(K_2, a, b) = a - b$ if $a \leq 2b$ and $\text{sep}(K_2, a, b) = a$ if $a \geq 2b$.

For $K_3 = C_3$, the separation number follows from results about the cycle from [11]:

Theorem 19 ([11]). *For any $p \geq 1$ and any a, b such that $a \geq b \geq 1$,*

$$\text{sep}(C_{2p+1}, a, b) = \begin{cases} a - b, & b \leq a < 2b \\ b + (p + 1)(a - 2b), & 2b \leq a \leq 2b + \frac{b}{p} \\ a, & a \geq 2b + \frac{b}{p}. \end{cases}$$

Hence for $K_3 = C_3$ we have:

Corollary 20.

$$\text{sep}(K_3, a, b) = \begin{cases} a - b, & b \leq a < 2b \\ 2a - 3b, & 2b \leq a < 3b \\ a, & a \geq 3b. \end{cases}$$

For arbitrary values of n , we are (only) able to prove some bounds and two exact results for the remaining cases.

First, combining Propositions 17 and 18 allows to obtain the following bounds for the separation number when $2b \leq a \leq nb$ (note the (roughly) factor two between the lower and upper bound):

Proposition 21. *For integers a, b, n, x such that $a = xb$ is a multiple of both $\binom{n-1}{x-1}$ and $\lfloor \frac{a^2}{2b(n-1)} \rfloor$, we have*

$$\left\lfloor a \frac{x}{2(n-1)} \right\rfloor \leq \text{sep}(K_n, a, b) \leq a \frac{x-1}{n-1}.$$

Proof. Proposition 17 gives $\text{sep}(K_n, a, b) \leq a \frac{x-1}{n-1}$ and Proposition 18 with $x = a/b$ gives the lower bound. \square

Proposition 22. *For any $n \geq 3$ and a, b such that $b \leq a \leq 2b$, we have $\text{sep}(K_n, a, b) = \lfloor \frac{2(a-b)}{n-1} \rfloor$.*

Proof. Let a, b such that $b \leq a \leq 2b$ and let $c = \lfloor \frac{2(a-b)}{n-1} \rfloor$. Consider a c -separating a -list assignment L of K_n , for $n \geq 3$.

From the hypothesis, we have $(n-1)c \leq 2(a-b) < a$.

By the separation condition, the amplitude of L on vertices from $S \subset [n]$, $|S| = i$, satisfies

$$\Sigma_S(L) \geq a + (a-c) + \dots + (a - (i-1)c) = ia - \frac{1}{2}i(i-1)c.$$

Therefore, as $c \leq 2(a-b)/(n-1)$, we obtain

$$\Sigma_S(L) \geq ia - \frac{1}{2}i(i-1) \frac{2(a-b)}{n-1} = ia \frac{n-i}{n-1} + ib \frac{i-1}{n-1} \geq ib.$$

Hence the amplitude condition is satisfied for any i , $1 \leq i \leq n$ and thus K_n is (L, b) -colorable.

For proving the upper bound, we construct counter-examples to show that, for $c' = c + 1$, there exists c' -separating a -list assignment L' that do not satisfy the amplitude condition, hence for which no (L', b) -coloring exists.

First, if $a \geq (n-1)c'$, then L' is the quasi-symmetrical list assignment constructed by setting for any $i \in [n]$, $|I_p(\{i\})| = a - (n-1)c'$ and for any $i, j \in [n], i \neq j$, $|I_p(\{i, j\})| = c'$, the other proper intersections being empty.

Then, by Equation 1 and Corollary 5, we have

$$\Sigma_{[n]}(L') = \sum_{i \in [n]} |I_p(\{i\})| + \sum_{i, j \in [n], i < j} |I_p(\{i, j\})| = na - n(n-1)c' + n \frac{n-1}{2} c' = na - n \frac{n-1}{2} c'.$$

As $c' = c + 1 > 2(a-b)/(n-1)$, we obtain

$$\Sigma_{[n]}(L') < na - n \frac{n-1}{2} \frac{2(a-b)}{n-1} = nb.$$

Hence the amplitude condition is not satisfied and K_n is not (L', b) -colorable.

Now, if $(n-1)c \leq a < (n-1)c'$, then let α such that $a = (n-1)c' - \alpha$, with $1 \leq \alpha \leq n-1$. Then L' is the quasi-symmetrical list assignment constructed by setting for any $i \in [n]$, $|I_p(\{i\})| = a - (n-1)c' + \alpha$ and for any $i, j \in [n], i \neq j$, $|I_p(\{i, j\})| = c'$ if $i < j \leq i + \alpha$ and $|I_p(\{i, j\})| = c' - 1$ otherwise, the other proper intersections being empty.

It is easy to observe that each vertex has a list of $a - (n-1)c' + \alpha + \alpha(c' - 1) + (n-1 - \alpha)c' = a$ colors and that the list assignment is

c' -separating. But, the amplitude of the full list is

$$\begin{aligned}\Sigma_{[n]}(L') &= \sum_{i \in [n]} |I_p(\{i\})| + \sum_{i, j \in [n], i < j} |I_p(\{i, j\})| = na - n(n-1)c' + n\alpha + \frac{n}{2}(c'-1)\alpha + \frac{n}{2}(n-1-\alpha)c' \\ &= na - \frac{n}{2}(n-1)c' - \frac{n}{2}\alpha.\end{aligned}$$

As $c' = c + 1 > 2(a-b)/(n-1)$, we obtain

$$\Sigma_{[n]}(L') < na - \frac{n}{2}(n-1)\frac{2(a-b)}{n-1} - \frac{n}{2}\alpha = nb - \frac{n}{2}\alpha.$$

Hence the amplitude condition is not satisfied and K_n is not (L', b) -colorable. \square

Proposition 23. *For any $n \geq 3$ and , a, b such that $(n-1)b \leq a \leq nb$, we have $\text{sep}(K_n, a, b) = 2a - nb$.*

Proof. Let a, b such that $(n-1)b \leq a \leq nb$ and $c = 2a - nb$. Consider a c -separating a -list assignment L of K_n , for $n \geq 3$.

By the separation condition, the amplitude of L on vertices from $S \subset [n]$, $|S| = 2$, satisfies

$$\Sigma_S(L) \geq a + (a - c) = 2a - c = 2a - 2a + nb = nb.$$

Hence, for any $S \subset [n]$, $|S| \geq 1$, $\Sigma_S(L) \geq |S|b$ and the amplitude condition is satisfied.

Now, we construct a counter example for $c' = c + 1 = 2a - nb + 1$ of a c' -separating a -list assignment L' for which no (L', b) -coloring exists. Let L' be constructed by setting for any $i \in [n]$, $|I_p([n] \setminus \{i\})| = a - c'$ and $|I_p(\{i\})| = (n-1)c' - (n-2)a$, the other proper intersections being empty. Note that $(n-1)c' - (n-2)a = (n-1)(2a - nb + 1) - (n-2)a = a(2n - 2 - n + 2) - (n-1)(nb - 1) = n(a - (n-1)b) + n - 1 \geq 0$ since $a \geq (n-1)b$ by the hypothesis. Then, by Equation 1 and Corollary 5, we have

$$\Sigma_{[n]}(L') = n(a - c') + (n-1)c' - (n-2)a = 2a - c' = 2a - 2a + nb - 1 = nb - 1 < nb.$$

Thus the amplitude condition is not satisfied and K_n is not (L', b) -colorable. \square

Remark that Corollary 20 can also be deduced from Propositions 22 and 23.

Putting all the partial results together and the computations made, we propose the following conjecture:

Conjecture 24. *for any $n \geq 4, a, b, p$ with $2 \leq p \leq n - 2$ and $pb \leq a < (p + 1)b$, we have*

$$\text{sep}(K_n, a, b) = \left\lceil \frac{2pa - p(p+1)b}{n-1} \right\rceil + \epsilon,$$

with $\epsilon \in \{-1, 0\}$.

Let us explain how we arrive to this conjecture: First, as the degree of every vertex in K_n is $n - 1$, then, for a balanced distribution of colors among vertices, they should be grouped by packets of $n - 1$. Hence the separation number must be (the ceiling of) a certain function of a, b, p divided by $n - 1$. By Property 1, this function must also be close to an affine function. Second, supported by Proposition 17, we conjecture that when $a = pb$, then $\text{sep}(K_n, a, b) = a(p - 1)/(n - 1)$. All this together lead us to propose the above conjecture, with ϵ being the correcting term depending on a, b, n .

In particular, for $n = 4$, the following refinement is conjectured, where only the case $2b \leq a < 3b$ remains to be verified:

Conjecture 25.

$$\text{sep}(K_4, a, b) = \begin{cases} \lfloor \frac{2(a-b)}{3} \rfloor, & b \leq a < 2b \\ \lceil \frac{4a-6b-1}{3} \rceil, & 2b \leq a < 3b \\ 2a - 4b, & 3b \leq a < 4b \\ a, & a \geq 4b. \end{cases}$$

REFERENCES

- [1] Y. Aubry, J.-C. Godin and O. Togni, *Every triangle-free induced subgraph of the triangular lattice is $(5m, 2m)$ -choosable*, Discrete Applied Math. 166, (2014), 51-58. <https://doi.org/10.1016/j.dam.2013.09.028>
- [2] Y. Aubry, J.-C. Godin and O. Togni, *Free choosability of outerplanar graphs*, Graphs and Combinatorics, 32(3), 851-859. <https://doi.org/10.1007/s00373-015-1625-3>
- [3] Z. Berikkyzy, C. Cox, M. Dairyko, K. Hogenson, M.M Kumbhat, B. Lidický, K. Messerschmidt, K. Moss, K. Nowak, K. F. Palmowski, D. Stolee, *$(4, 2)$ -Choosability of Planar Graphs with Forbidden Structures*, Graphs and Combinatorics (2017) 33: 751. <https://doi.org/10.1007/s00373-017-1812-5>
- [4] M. Chen, K.-W. Lih, W. Wang *On choosability with separation of planar graphs without adjacent short cycles*, Bull. Malays. Math. Sci. Soc., 41 (2018), 1507-1518. <https://doi.org/10.1007/s40840-016-0409-0>
- [5] M. Chen, Y. Fan, A. Raspaud, W. C. Shiu, W. Wang, *Choosability with separation of planar graphs without prescribed cycles*, Applied Mathematics and Computation 367, (2020). <https://doi.org/10.1016/j.amc.2019.124756>
- [6] I. Choi, B. Lidický, D. Stolee *On choosability with separation of planar graphs with forbidden cycles*, J. Graph Theory, 81 (2016), 283-306. <https://doi.org/10.1002/jgt.21875>
- [7] M. Chen, Y. Fan, Y. Wang, W. Wang *A sufficient condition for planar graphs to be $(3,1)$ -choosable*, J. Comb. Optim., 34 (2017), 987-1011. <https://doi.org/10.1007/s10878-017-0124-2>
- [8] M. M. Cropper, J. L. Goldwasser, A. J. W. Hilton, D. G. Hoffman, P. D. Johnson, *Extending the disjoint-representatives theorems of Hall, Halmos, and Vaughan to list-multicolorings of graphs*. J. Graph Theory 33 (2000), no. 4, 199-219.
- [9] L. Esperet, R.J. Kang, S. Thomassé *Separation Choosability and Dense Bipartite Induced Subgraphs*, Combinatorics, Probability and Computing (2019), 28, 720-732. <https://doi.org/10.1017/S0963548319000026>

- [10] Z. Füredi, A. Kostochka, M. Kumbhat *Choosability with separation of complete multipartite graphs and hypergraphs*, J. Graph Theory, 76 (2014), 129-137. <https://doi.org/10.1002/jgt.21754>
- [11] J.-C. Godin, O. Togni *Choosability with Separation of cycles and outerplanar graphs*, Discuss. Math. Graph Theory, (2021) in press. <https://doi.org/10.7151/dmgt.2398>
- [12] H.A. Kierstead, B. Lidický, *On choosability with separation of planar graphs with lists of different sizes*, Discrete Mathematics 338:10 (2015), 1779-1783. <https://doi.org/10.1016/j.disc.2015.01.008>
- [13] J. Kratochvíl, Z. Tuza, M. Voigt, *Complexity of choosing subsets from color sets*, Discrete Math., 191:1-3 (1998), 139-148. [https://doi.org/10.1016/S0012-365X\(98\)00101-0](https://doi.org/10.1016/S0012-365X(98)00101-0)
- [14] J. Kratochvíl, Z. Tuza, M. Voigt *Brooks-type theorems for choosability with separation*, J. Graph Theory, 27 (1998), 43-49. [https://doi.org/10.1002/\(SICI\)1097-0118\(199801\)27:1<43::AID-JGT7>3.0.CO;2-G](https://doi.org/10.1002/(SICI)1097-0118(199801)27:1<43::AID-JGT7>3.0.CO;2-G)
- [15] M. Kumbhat, K. Moss, D. Stolee, *Choosability with union separation*, Discrete Math. 341: 3 (2018), 600-605. <https://doi.org/10.1016/j.disc.2017.10.022>
- [16] R. Škrekovski *A note on choosability with separation for planar graphs*, Ars Comb., 58 (2001), 169-174.
- [17] X. Zhu *List 4-colouring of planar graphs*, arXiv, 2022, <https://arxiv.org/abs/2203.16314>.

APPENDIX A. VECTORIAL SOLUTION OF THE ILP PROBLEM FOR SYMMETRICAL LIST ASSIGNMENTS

In this section, we show how the counter examples can be found using an algebraic formulation.

Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a canonical base of the vectorial space of dimension n .

We define the vector

$$\mathbf{a} = \sum_{i=1}^n \binom{n-1}{i-1} \mathbf{e}_i.$$

We can remark that \mathbf{a} is symmetrical. For a vector $\mathbf{x} \in \mathbb{R}^n$, we define the linear application $\varphi_a(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$.

We also define the antisymmetrical vectors $\forall i \in [1, \lfloor \frac{n}{2} \rfloor]$,

$\mathbf{as}(i) = \mathbf{e}_i - \mathbf{e}_{n+1-i}$ and, $\forall i \in [3, \lceil \frac{n}{2} \rceil]$, the Pascal's triangle vectors

$\mathbf{tp}(i) = \mathbf{e}_i + \mathbf{e}_{i-1} - \binom{n}{i-1} \mathbf{e}_1$.

Lastly, we define the binomial vector $\mathbf{bn} = \sum_{i=1}^n \mathbf{e}_i - 2^{n-1} \mathbf{e}_1$.

Lemma 26. *The kernel $\ker(\varphi_a)$ has $\{\{\mathbf{as}(i)\}, \{\mathbf{tp}(i)\}, \mathbf{bn}\}$ as basis and is thus of dimension $n-1$.*

Proof. By construction, it is easy to observe that $F_a = \{\{\mathbf{as}(i)\}, \{\mathbf{tp}_i\}, \mathbf{bn}\} \subset \ker(\varphi_a)$. As the number of elements of F_a is equal to the dimension of $\ker(\varphi_a)$, it suffices to show that F_a is a family of free vectors. We consider two cases depending on the parity of n .

- Case n even. Let $\alpha_i, 1 \leq i \leq \frac{n}{2}$ and $\beta_j, 3 \leq j \leq \frac{n}{2}$ and $\gamma \in \mathbb{R}$ such that

$$\sum_{i=1}^{n/2} \alpha_i \mathbf{as}(i) + \sum_{j=3}^{n/2} \beta_j \mathbf{tp}(j) + \gamma \mathbf{bn} = \mathbf{0}.$$

For the k -th coordinate of the left side of the above equality, with $k \in [\frac{n}{2} + 1, n]$, we have $-\alpha_{n+1-k} + \gamma = 0$, thus $\gamma = \alpha_1 = \alpha_2 = \dots = \alpha_{n/2}$.

For $k = 2$, $\alpha_2 + \beta_3 + \gamma = 0 \Rightarrow \beta_3 = -2\gamma$.

For $k \in [3, \frac{n}{2} - 1]$, $\alpha_k + \beta_k + \beta_{k+1} + \gamma = 0 \Rightarrow \beta_4 = \dots = \beta_{n/2} = 0$.

For $k = n/2$, $\alpha_k + \beta_k + \gamma = 0 \Rightarrow \alpha_{n/2} = -\gamma$. But we have seen that $\alpha_{n/2} = \gamma$, hence $\gamma = 0$.

Consequently, we obtain that all coefficients are equal to zero.

- Case n odd. Let α_i , $1 \leq i \leq \frac{n-1}{2}$ and β_j , $3 \leq j \leq \frac{n+1}{2}$ and $\gamma \in \mathbb{R}$ such that

$$\sum_{i=2}^{(n-1)/2} \alpha_i \mathbf{as}(i) + \sum_{j=3}^{(n+1)/2} \beta_j \mathbf{tp}(j) + \gamma \mathbf{bn} = \mathbf{0}.$$

For the k -th coordinate of the left side of the above equality, $k \in [\frac{n+3}{2}, n-1]$, we have $-\alpha_{n+1-k} + \gamma = 0$. Thus $\gamma = \alpha_1 = \dots = \alpha_{(n-1)/2}$.

For $k = (n+1)/2$, $\beta_k + \gamma = 0 \Rightarrow \beta_{(n+1)/2} = -\gamma$.

For $k \in [3, \frac{n-1}{2} - 1]$, $\alpha_k + \beta_k + \beta_{k+1} + \gamma = 0$. Hence $\beta_k = -\alpha_k = -\gamma$.

Finally, for $k = 2$, $\alpha_2 + \beta_3 + \gamma = 0 \Rightarrow \gamma = 0$.

□

Similarly, for the separation condition, we define the vector

$$\mathbf{c} = \sum_{i=2}^n \binom{n-2}{i-2} \mathbf{e}_i$$

and the linear application $\varphi_c(\mathbf{x}) = \mathbf{c} \times \mathbf{x}$.

We also define the vectors

- $\forall i \in [2, \lfloor \frac{n}{2} \rfloor]$, $\mathbf{asc}(i) = \Delta(i, n) \mathbf{as}(1) + \mathbf{as}(i)$,
with $\Delta(i, n) = \binom{n-2}{i-2} - \binom{n-2}{n-1-i} = \binom{n-2}{i-2} - \binom{n-2}{i-1}$.
- $\forall i \in [3, \lceil \frac{n}{2} \rceil]$, $\mathbf{tpc}(i) = \binom{n-1}{i-2} \mathbf{as}(1) + \mathbf{tp}(i)$.
- $\mathbf{bnc} = 2^{n-2} \mathbf{as}(1) + \mathbf{bn}$.

Lemma 27. *The intersection of kernels $\ker(\varphi_a) \cap \ker(\varphi_c)$ has $\{\{\mathbf{asc}(i)\}, \{\mathbf{tpc}(i)\}, \mathbf{bnc}\}$ as a basis and is thus of dimension $n-2$.*

Proof. It is easy to see that $F_{ac} = \{\{\mathbf{asc}(i)\}, \{\mathbf{tpc}(i)\}, \mathbf{bnc}\} \subset \ker(\varphi_a) \cap \ker(\varphi_c)$ and has $n-2$ vectors. Since the dimension of $\ker(\varphi_a) \cap \ker(\varphi_c)$ is equal to $n-2$ too, it suffices to show that F_{ac} is a free family. We consider two cases depending on the parity of n .

- Case n even. Let α_i , $2 \leq i \leq \frac{n}{2}$ and β_j , $3 \leq j \leq \frac{n}{2}$ and $\gamma \in \mathbb{R}$ such that

$$\sum_{i=2}^{n/2} \alpha_i \mathbf{asc}(i) + \sum_{j=3}^{n/2} \beta_j \mathbf{tpc}(j) + \gamma \mathbf{bnc} = \mathbf{0}.$$

For $k \in [\frac{n}{2} + 1, n - 1]$, $-\alpha_{n+1-k} + \gamma = 0 \Rightarrow \gamma = \alpha_2 = \alpha_3 = \dots = \alpha_{n/2-1} = \alpha_{n/2}$.

For $k = n/2$, $\alpha_k + \beta_k + \gamma = 0 \Rightarrow \beta_{n/2} = -2\gamma$.

For $k \in [3, \frac{n}{2} - 1]$, $\alpha_k + \beta_k + \beta_{k+1} + \gamma = 0 \Rightarrow \beta_3 = \dots = \beta_{n/2-1} = 0$.

Finally, for $k = 2$, $\alpha_2 + \beta_3 + \gamma = 0 \Rightarrow 2\gamma = 0 \Rightarrow \gamma = 0$.

- Case n odd. Let α_i , $2 \leq i \leq \frac{n+1}{2}$ and β_j , $3 \leq j \leq \frac{n+1}{2}$ and $\gamma \in \mathbb{R}$ such that

$$\sum_{i=2}^{(n+1)/2} \alpha_i \mathbf{asc}(i) + \sum_{j=3}^{(n+1)/2} \beta_j \mathbf{tpc}(j) + \gamma \mathbf{bnc} = \mathbf{0}.$$

For $k \in [\frac{n+3}{2}, n - 1]$, $-\alpha_{n+1-k} + \gamma = 0 \Rightarrow \gamma = \alpha_2 = \alpha_3 = \dots = \alpha_{(n-1)/2}$.

For $k = (n+1)/2$, $-\alpha_{(n-1)/2} + \beta_k + \gamma = 0 \Rightarrow \beta_{(n+1)/2} = 0$.

For $k \in [3, \frac{n-1}{2} - 1]$, $\alpha_k + \beta_k + \beta_{k+1} + \gamma = 0 \Rightarrow \beta_{(n-1)/2} = 2\gamma, \beta_{(n-3)/2} = 0, \beta_{(n-5)/2} = 2\gamma, \dots$

Finally, for $k = 2$, $\alpha_2 + \beta_3 + \gamma = 0 \Rightarrow \beta_3 = -2\gamma$. If $(n+1)/2 \equiv 3 \pmod{4}$ then $\gamma = 0$. Otherwise $((n+1)/2 \not\equiv 3 \pmod{4})$, then $2\gamma = -2\gamma$ and thus $\gamma = 0$.

In both cases, we have shown that all coefficients are equal to zero, proving that the set of vectors forms a free family. \square

We define $\psi = \sum_{i=1}^n \binom{n}{i} \mathbf{e}_i$ which is the measure of the amplitude and $E(a, c) = \{\mathbf{x} \text{ such that } \varphi_a(\mathbf{x}) = a \text{ and } \varphi_c(\mathbf{x}) = c\}$.

Now, using the maximum principle, we define the two optimal solution vectors \mathbf{x}_i , $i = 1, 2$.

- If $a \geq (n-1)c$, we define $\mathbf{x}_1 = (a - (n-1)c)\mathbf{e}_1 + c\mathbf{e}_2$.

We can observe that $\mathbf{x}_1 \in E(a, c)$. In fact $E(a, c) = \mathbf{x}_1 + \ker(\varphi_a) \cap \ker(\varphi_c)$. This solution corresponds to the counterexample used in proof of Proposition 22.

- If $a \leq \frac{n-1}{n-2}c$, $\mathbf{x}_2 = (a - c)\mathbf{e}_{n-1} + ((n-1)c - (n-2)a)\mathbf{e}_n$.

We can observe that $\mathbf{x}_2 \in E(a, c)$. In fact $E(a, c) = \mathbf{x}_2 + \ker(\varphi_a) \cap \ker(\varphi_c)$. This solution corresponds to the counterexample used in proof of Proposition 23.