Wasserstein-p Bounds in the Central Limit Theorem under Weak Dependence

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Abstract

The central limit theorem is one of the most fundamental results in probability and has been successfully extended to locally dependent data and strongly mixing random fields. In this paper, we establish the rate of convergence in the central limit theorem in terms of transport distances. In specific, for arbitrary $p \geq 1$ we obtain an upper bound on the Wasserstein-p distance between the law of the scaled sum and the limiting normal distribution for (i) locally dependent random variables and (ii) strongly mixing stationary random fields. Our proofs extend the Stein's dependency neighborhood method for the Wasserstein distance of a general order $p \geq 1$ and provide new tools to study the deviation behaviors for dependent random variables. Moreover, as an application we demonstrate how our results can be used to obtain tail bounds that are asymptotically tight and decrease polynomially fast for the empirical average under weak dependence.

Keywords: Normal approximation, Wasserstein distance, Stein's method, dependency graph, strong mixing coefficients.

1 Introduction

The central limit theorem is one of the most fundamental theorems in probability theory. Initially formulated for independent and identically distributed random variables, it has since then been generalized to triangular arrays [Feller 1945], martingales [Lévy 1935], U-statistics [Hoeffding 1948], locally dependent random variables [Hoeffding & Robbins 1948; Heinrich 1982; Petrovskaya & Leontovich 1983], and mixing random fields [Rosenblatt 1956; Bolthausen 1982]. It states as follows: Let (I_n) be an increasing sequence of subsets $I_1 \subseteq I_2 \subseteq \cdots \subseteq I$, whose sizes increase to infinity $|I_n| \to \infty$. If $(X_i)_{i\in I}$ are (dependent) centered random variables, then under certain conditions on the moments of (X_i) and on its dependence structure the scaled sum is asymptotically normal, i.e.,

$$W_n := \sigma_n^{-1} \sum_{i\in I_n} X_i \xrightarrow{d} \mathcal{N}(0,1),$$

where we write $\sigma_n^2 = \mathrm{Var}\big(\sum_{i\in I_n} X_i\big)$. While the central limit theorem is an asymptotic result, there is a long history of quantifying how far W_n is from being normally distributed. One of the most important metrics to do so is the Wasserstein-p distance, which originated in optimal transport theory [Villani 2009]. For two probability measures ν and μ over the real line $\mathbb{R}$, we denote by $\Gamma(\nu,\mu)$ the set of all couplings of ν and μ, and the Wasserstein-p distance between ν and μ is defined as

$$\mathcal{W}_p(\nu,\mu) := \inf_{\gamma\in\Gamma(\nu,\mu)} \left[\mathbb{E}_{(X,Y)\sim\gamma}[|X-Y|^p]\right]^{1/p}.$$

In the case of independent observations, the first rates for $p \geq 1$ were obtained by Bártfai [1970]. They are, however, sub-optimal in terms of the sample size $|I_n|$, decreasing at a slower rate of $\mathcal{O}\left(|I_n|^{-\frac{1}{2}+\frac{1}{p}}\right)$. Under some additional necessary moment conditions, Rio [2009] obtained that, for $1 \leq p \leq 2$, the Wasserstein distance converges at the optimal rate of $\mathcal{O}\left(1/\sqrt{|I_n|}\right)$. They conjectured that such a rate would be extendable to arbitrary $p \geq 1$, which was recently proven to be true by Bobkov [2018]; Bonis [2020] using a series of methods including the Edgeworth expansion and the exchangeable pair method. They showed that if $\max_i \|X_i\|_{p+2} < \infty$ and if $\mathrm{Var}(X_1) = \mathrm{Var}(X_i) = 1$, then there is a constant $K_p < \infty$ such that

$$\mathcal{W}_p\big(\mathcal{L}(W_n), \mathcal{N}(0,1)\big) \leq \frac{K_p \|X_1\|_{p+2}^{1+2/p}}{\sqrt{|I_n|}},$$

where $\mathcal{L}(\,\cdot\,)$ designates the distribution of the given random variable.

While the central limit theorem is known to hold both for locally dependent and for mixing random fields, no analogous bound is known for general Wasserstein-p distances ($p \geq 1$) for dependent data. This is the gap that we fill in this paper. We do so in two different settings (i) for the locally dependent random variables (including m-dependent random fields and U-statistics), and (ii) for strongly mixing stationary random fields. We obtain that for a d-dimensional stationary random fields, if the mixing coefficients decrease as $\alpha_\ell = \mathcal{O}(\ell^{-\beta})$ with $\beta > d(p+1)$ then we have

$$\mathcal{W}_p\big(\mathcal{L}(W_n), \mathcal{N}(0,1)\big) = \mathcal{O}\left(\frac{1}{\sqrt{|I_n|}}\right) \qquad \text{(Corollary 5.4)},$$

where I_n is the index set of the random field. The constants here depend on p, the dependence structure, and the moments of the random variables (X_i). If the mixing coefficients decrease as $\alpha_\ell = \ell^{-\beta}$ with $\beta \in \left(\frac{d(p+1)}{2},\ d(p+1)\right]$ then we have

$$\mathcal{W}_p\big(\mathcal{L}(W_n), \mathcal{N}(0,1)\big) = \mathcal{O}\left(\frac{1}{|I_n|^{-\gamma}}\right) \qquad \text{(Corollary 5.7)},$$

where $\gamma < 1/2$ is an explicit constant that depends on β. For locally dependent random variables, if the sizes of dependency neighborhoods are bounded and certain moment conditions are satisfied, then we obtain a similar rate of $\mathcal{O}\left(1/\sqrt{|I_n|}\right)$ (see Theorem 3.3). Note that for m-dependent random fields we do not make any assumption of stationarity for the distribution of (X_i), but only impose non-degeneracy conditions on the variances (σ_n) and moment conditions on the random variables. Further remark that we generalize our results to triangular arrays where the random variables $\left(X_i^{(n)}\right)$ are allowed to change with n.

Another contribution of our paper is that we propose a new way to adapt the Stein's method to obtain general Wasserstein-p bounds. This is the first technique that successfully handles a large class of dependent random variables. Indeed, the results in Bobkov [2018] were obtained by exploiting Edgeworth expansions. This is sadly not amenable to handling dependence as in this case the characteristic function of W_n does not decompose into a product of characteristic functions. Ledoux et al. [2015], on the other hand, showed that the Stein kernel can also be used to obtain Wasserstein-p bounds. However, the Stein kernel is not always guaranteed to exist, and furthermore, even when it does there is no known way to obtain a Stein kernel for W_n if the random variables are dependent. Fang [2019] proposed a different variant of the Stein's method to obtain a Wasserstein-2 bound for locally dependent random variables by relating to Zolotarev's metrics. Contrary to Fang [2019] we obtain bounds that are valid for any real value $p \geq 1$ and generalize to mixing random fields. Another approach proposed by Bonis [2020] successfully adapted the Stein exchangeable pair idea to obtain Wasserstein-p bounds, and was later used by Fang & Koike [2022] to deal with local dependence. However, Fang & Koike [2022] only obtained sub-optimal rates of convergence, and this type of argument is once again not amenable to handling mixing random fields. In this paper, we not only obtain optimal rate for local dependence, but also show how it can be used to handle the strong mixing case.

1.1 Application to tail bounds

In this subsection, we show a specific application of our results to elaborate the importance of considering Wasserstein distance of order $p > 1$. To be specific, we explain how $\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1))$ $(p > 1)$ can be used to obtain tail bounds for W_n. Indeed, letting $t \geq 0$, an important goal for statistical inference is to obtain tight upper bounds for $\mathbb{P}(W_n \geq t)$. Notably, among many other applications, such inequalities lie at the heart of decision making in reinforcement learning [Mnih et al. 2008; Audibert et al. 2009] and generalization guarantees for high dimensional statistics [Wainwright 2019; Bartlett & Mendelson 2002]. When the observations (X_i) are i.i.d and bounded, general well-known concentration inequalities such as Azuma's or Bernstein's inequalities allow one to upper-bound $\mathbb{P}(W_n \geq t)$ with a sub-Gaussian decay in t. Notably if $\|X_1\|_\infty \leq R$ and $\text{Var}(X_1) = 1$, then Azuma's inequality states that $\mathbb{P}(W_n \geq t) \leq e^{-\frac{nt^2}{2R^2}}$. However, no such concentration inequality is known to hold for strongly mixing sequences. Moreover, even when sub-Gaussian bounds hold, those are known to be significantly looser than the asymptotically valid tail bound $\Phi^c(t)$ [Austern & Mackey 2022], where $\Phi^c(\cdot) = 1 - \Phi(\cdot)$ and $\Phi(\cdot)$ denotes the cumulative distribution function (CDF) of the standard normal. In this subsection we see that our results can be used to obtain a tail bound that is asymptotically tight, decreases polynomially fast in t, and is valid for mixing sequences.

To this goal, choose $p \geq 1$ and $\rho \in (0,1)$. Then remark that for all $\epsilon > 0$ there is $G \sim \mathcal{N}(0,1)$ such that $\|G - W_n\|_p \leq \mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) + \epsilon$. Therefore, by the union bound we have

$$\begin{aligned}
\mathbb{P}\big(W_n \geq t\big) &= \mathbb{P}\big(W_n - G + G \geq t\big) \\
&\leq \mathbb{P}\big(W_n - G \geq (1-\rho)t\big) + \mathbb{P}\big(G \geq \rho t\big) \\
&\overset{(a)}{\leq} \Phi^c(\rho t) + \frac{\|W_n - G\|_p^p}{(\rho t)^p}
\end{aligned}$$

$$\leq \Phi^c(\rho t) + \frac{\left(\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) + \epsilon\right)^p}{(\rho t)^p}$$

where to obtain (a) we have used Markov's inequality. Now as this holds for any arbitrary choice of $\epsilon > 0$ we conclude that

$$\mathbb{P}\big(W_n \geq t\big) \leq \Phi^c(\rho t) + \frac{\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1))^p}{(\rho t)^p}.$$

Now imagine that there is a constant K_p such that $\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq \frac{K_p}{\sqrt{|I_n|}}$, then we obtain that

$$\mathbb{P}\big(W_n \geq t\big) \leq \inf_{\rho} \left\{ \Phi^c(\rho t) + \frac{K_p^p}{\left(\rho t \sqrt{|I_n|}\right)^p} \right\}. \tag{1.1}$$

Note that this upper bound is asymptotically tight. Indeed as n grows to infinity, the upper bound given by (1.1) converges to $\mathbb{P}\big(W_n \geq t\big) \xrightarrow{n\to\infty} \Phi^c(t)$. Moreover, the remainder term $\frac{K_p^p}{(\rho t \sqrt{|I_n|})^p}$ decreases polynomially fast in t and $\sqrt{|I_n|}$. This leads to interesting new concentration inequalities since there has not been any such non-uniform Berry-Esseen bounds for mixing fields in the literature (see Nagaev [1965]; Bikelis [1966] for examples in the i.i.d setting and Chen & Shao [2004] for weaker results in the local dependence setting).

1.2 Related literature

Agnew [1957]; Esseen [1958] established that the convergence rate in the central limit theorem is $\mathcal{O}\big(1/\sqrt{|I_n|}\big)$ in terms of the Wasserstein-1 distance. Since then many papers have tightened this result and generalized it to dependent observations. Notably, the Stein's method offers a series of powerful techniques for obtaining Wasserstein-1 bounds in the dependence setting. See Ross [2011] for a survey of those methods. Baldi & Rinott [1989]; Barbour et al. [1989] obtained Wasserstein-1 bounds under local dependence conditions and Sunklodas [2007] showed a similar bound for strongly mixing sequences, which is $\mathcal{O}\big(1/\sqrt{|I_n|}\big)$ when the mixing coefficients decay fast enough.

For $p \geq 1$, the first paper to propose a rate for the central limit theorem was Bártfai [1970]. Under the hypothesis that the random variables have finite exponential moments, they obtained a rate of $\mathcal{O}\left(|I_n|^{-\frac{1}{2}+\frac{1}{p}}\right)$ for the Wasserstein-p distance. Sakhanenko [1985] obtained a similar rate but only required the existence of p-th moments. Rio [1998, 2009] showed that in order to obtain a convergence rate of $\mathcal{O}\big(1/\sqrt{|I_n|}\big)$, it is necessary to require finite (p+2)-th moments of the random variables. They also obtained the expected rate for $p \leq 2$ and conjectured that a similar rate should be valid for any arbitrary $p > 2$. This conjecture was demonstrated to be true by Bobkov [2018]; Bonis [2020]. Those two papers took different approaches. Bobkov [2018] used an Edgeworth expansion argument. Bonis [2020], on the other hand, used the Ornstein-Uhlenbeck interpolation combined with a Stein exchangeable pair argument. Previous to that, Ledoux et al. [2015] had already obtained the optimal rate for the Wasserstein-p distance using the Ornstein-Uhlenbeck interpolation but required significantly stronger assumptions on the distribution of the random variables by requiring the existence of a Stein kernel. Moreover, for the special case $p = 2$, the celebrated HWI inequality [Otto & Villani 2000] and Talagrand quadratic transport inequality [Talagrand 1996] can help obtain Wasserstein-2 bounds by relating it to the Kullback-Leibler divergence.

Contrary to the independent case, much less is known for the general Wasserstein-p distance for dependent data. Barbour et al. [1989]; Fang [2019] adapted the Stein's method to obtain Wasserstein-2 bounds with the optimal rate for locally dependent variables. Fang & Koike [2022] modified the approach of Bonis [2020] and obtained a rate $\mathcal{O}\left(\log|I_n|/\sqrt{|I_n|}\right)$ for the Wasserstein-p distance under local dependence. Our results propose significant extensions to both of those results by generalizing them to arbitrary $p \geq 1$ and mixing random fields. When the mixing coefficients decrease fast enough we obtain that the Wasserstein distance decreases at the optimal rate $\mathcal{O}\left(1/\sqrt{|I_n|}\right)$. Otherwise, the rate of convergence is slower and depends on the mixing coefficients.

Our proofs rely on the Stein's method and a result of Rio [2009] that allows to upper the Wasserstein-p distance by an integral probability metric [Zolotarev 1984]. As those metrics are defined as the supremum of differences of expectation of functions, the Stein's method lends itself nicely to this problem. The Stein's method was first introduced in Stein [1972] as a new method to obtain a Berry-Esseen bound and prove the central limit theorem for weakly dependent data. It has since then become one of the most popular and powerful methods to prove asymptotic normality for dependent data, and different adaptations of it have been proposed, notably the dependency neighborhoods, the exchangeable pairs, the zero-bias coupling and the size-bias coupling [Ross 2011]. In addition to being used to prove the central limit theorem, it has since then been adapted to obtain limit theorems for the Poisson distribution [Chen 1975] or for the exponential distribution [Chatterjee et al. 2011; Peköz & Röllin 2011]. It has also been used as a tool for comparing different univariate distributions [Ley et al. 2017]. Our use of the Stein's method is closely related to the dependency neighborhood method described in Ross [2011].

We further remark that the theory we have developed has an interesting by-product. We prove upper bounds on the absolute values of the cumulants of W_n (see Corollaries B.5 and F.11). Previously, Janson [1988] showed a similar bound under the dependency graph conditions and Heinrich [1990]; Götze et al. [1995] tightened the results for m-dependent random fields. Götze & Hipp [1983]; Lahiri [1993, 1996] obtained similar cumulant bounds for the strongly mixing sequences in an effort to obtain Edgeworth expansions. Note that we also provide the bounds on the cumulants of W_n for strongly mixing random fields in Appendix F, which is more general than the results mentioned above. Furthermore, Döring & Eichelsbacher [2013]; Döring et al. [2022] showed that cumulant bounds can be useful in problems including the analysis of moderate deviations.

1.3 Paper outline

In Section 2 we clarify some notations that we use throughout the paper. Then we present our results under the hypothesis that the random variables under two different local dependence conditions in Section 3, and provide some applications in Section 4. In Section 5 we show upper bounds for the Wasserstein-p distance for mixing random fields. In Section 6, we make an overview of our proof techniques. Finally all proofs are presented in the appendices.

2 Notations

Notations concerning integers and sets In this paper, we will write $\lceil x \rceil$ to denote the smallest integer that is bigger or equal to x and $\lfloor x \rfloor$ denotes the largest integer smaller or equal to x. We use $\mathbb{N}$ to denote the set of non-negative integers and let $\mathbb{N}_+$ be the set of positive integers. For any $n \in \mathbb{N}_+$, denote $[n] := \{\ell \in \mathbb{N}_+ : 1 \leq \ell \leq n\}$. Moreover, for a finite set B we denote by $|B|$ its cardinality.

Notations for sequences Given a sequence (x_i) we will shorthand $x_{1:\ell} = (x_1, \cdots, x_\ell)$ and similarly for any subset $B \subseteq \mathbb{N}_+$ we denote $x_B := (x_i)_{i \in B}$.

Notations for functions For any real valued functions $f(\cdot), g(\cdot) : \mathbb{N}_+ \to \mathbb{R}$, we write $f(n) \lesssim g(n)$ or $f(n) = \mathcal{O}(g(n))$ if there exists some constant C (with dependencies that are fixed in the contexts) and an integer $N > 0$ such that the inequality $f(n) \leq Cg(n)$ holds for all $n \geq N$. We further write $f(n) \asymp g(n)$ as shorthand for $f(n) \lesssim g(n)$ and $g(n) \lesssim f(n)$.

Notations for probability distributions For a random variable X we write by $\mathcal{L}(X)$ the distribution of X.

3 $\mathcal{W}_p$ bounds under local dependence

Let $p \geq 1$ be a positive real number and write $\omega := p + 1 - \lceil p \rceil \in [0, 1]$. We choose I to be an infinite index set I and $(I_n)_{n=1}^{\infty}$ to be a sequence of finite subsets of $I_1 \subseteq I_2 \subseteq \cdots \subsetneq I$ that satisfy $|I_n| \xrightarrow{n \to \infty} \infty$. Let $\big(X_i^{(n)}\big)_{i \in I_n}$ be a triangular array of random variables, each row indexed by $i \in I_n$ ($n = 1, 2, \cdots$), and let W_n be the following empirical average empirical average

$$W_n := \sigma_n^{-1} \sum_{i \in I_n} X_i^{(n)}, \quad \text{with } \sigma_n^2 := \mathrm{Var}\Big(\sum_{i \in I_n} X_i^{(n)}\Big).$$

Under the hypothesis that the random variables $\big(X_i^{(n)}\big)$ are locally dependent we will, in this section, bound the Wasserstein-p distance between W_n and its normal limit. The bound we obtain depends on the size of the index set I_n, the moments of the random variables and the structure of local dependence in question.

To capture the local dependence structure, we first provide the definition of dependency neighborhoods following Ross [2011]. Given random variables $(Y_i)_{i \in I}$ we call $N(J)$ a **dependency neighborhood** for the index subset $J \subseteq I$ if $\{Y_j : j \notin N(J)\}$ is independent of $\{Y_j : j \in J\}$. In our setting of the triangular array $\big(X_i^{(n)}\big)$, we define $(N_n(i_{1:q}))_q$ by choosing the subsets of I_n that satisfy the following conditions:

[LD-1]: For each $i_1 \in I_n$, there exists a subset $N_n(i_1) \subseteq I_n$ such that $\big\{X_j^{(n)} : j \notin N_n(i_1)\big\}$ is independent of $X_{i_1}^{(n)}$.

[LD-*q*] ($q \geq 2$): For each $i_1 \in I_n$, $i_2 \in N_n(i_1)$, $\cdots$, $i_q \in N_n(i_{1:(q-1)})$, there exists a subset $N_n(i_{1:q})$ such that $\big\{X_j^{(n)} : j \notin N_n(i_{1:q})\big\}$ is independent of $\big(X_{i_1}^{(n)}, \cdots, X_{i_q}^{(n)}\big)$.

We remark that the sequence of subsets $(N_n(i_{1:q}))_q$ is increasing, i.e., $N_n(i_{1:(q-1)}) \subseteq N_n(i_{1:q})$ in q; and that the neighborhoods $N_n(i_{1:q})$ are allowed to be different for different values of n–which reflects the triangular array structure of our problem. The condition of dependency neighborhoods here generalizes the one in Ross [2011] and was also adopted in Fang [2019], inspired by Barbour et al. [1989]; Chen & Shao [2004]. Barbour et al. [1989] obtained a Wasserstein-1 bound under "decomposable" conditions similar to [LD-1] and [LD-2], and Chen & Shao [2004] showed a Berry–Esseen type result under slightly stronger assumptions for local dependence, while finally Fang [2019] obtained a Wasserstein-2 bound.

In order to define the remainder terms that will appear in our bounds, we introduce the following notions. Given $t \in \mathbb{N}_+$ such that $k \geq 2$, we say that the tuple $(\eta_1, \eta_2, \cdots, \eta_\ell)$ where $\ell \in \mathbb{N}_+$ is an **integer composition** of t if and only if $\eta_{1:\ell}$ are positive integers such that $\eta_1 + \eta_2 + \cdots + \eta_\ell = t$. We denote by $C(t)$ the set of those integer composition

$$C(t) := \left\{\ell, \eta_{1:\ell} \in \mathbb{N}_+ : \textstyle\sum_{j=1}^{\ell} \eta_j = t\right\}.$$

Moreover, for any random variables $(Y_i)_{i=1}^t$, we define the order-t **compositional expectation** with respect to $\eta_{1:\ell}$ as

$$[\eta_1, \cdots, \eta_\ell] \triangleright (Y_1, \cdots, Y_t) := \mathbb{E}\big[Y_1 \cdots Y_{\eta_1}\big]\, \mathbb{E}\big[Y_{\eta_1+1} \cdots Y_{\eta_1+\eta_2}\big] \cdots \mathbb{E}\big[Y_{\eta_1+\cdots+\eta_{\ell-1}+1} \cdots Y_t\big]. \quad (3.1)$$

Note that if $\eta_\ell = 1$, the last expectation reduces to $\mathbb{E}[Y_t]$.

Next for any positive integer k and real value $\omega \in (0,1]$, we define

$$R_{k,\omega,n} := \sum_{(\ell,\eta_{1:\ell}) \in C^*(k+2)} \sum_{i_1 \in I_n} \sum_{i_2 \in N_n(i_1)} \cdots \sum_{i_{k+1} \in N_n(i_{1:k})} [\eta_1, \cdots, \eta_\ell] \triangleright \left(\left|X_{i_1}^{(n)}\right|, \cdots, \left|X_{i_{k+1}}^{(n)}\right|, \Big(\sum_{i_{k+2} \in N_n(i_{1:(k+1)})} \left|X_{i_{k+2}}^{(n)}\right| \Big)^{\omega} \right), \quad (3.2)$$

where $C^*(k+2)$ is given by

$$C^*(t) := \left\{(\ell, \eta_{1:\ell}) \in C(t) :\ \eta_j \geq 2 \ \text{ for } 1 \leq j \leq \ell - 1, \right\} \subseteq C(t).$$

The terms $(R_{k,\omega,n})$ are remainder terms that appear in our bound of the Wasserstein-p distance between W_n and its normal limit.

Theorem 3.1. *Let $\big(X_i^{(n)}\big)_{i \in I_n}$ be a triangular array of mean zero random variables and suppose that they satisfy* [LD-1] *to* [LD-($\lceil p \rceil$+1)]. *Let $\sigma_n^2 := \mathrm{Var}\Big(\sum_{i \in I_n} X_i^{(n)}\Big)$ and define $W_n := \sigma_n^{-1} \sum_{i \in I_n} X_i^{(n)}$. Further suppose for any $j \in \mathbb{N}_+$ such that $j \leq \lceil p \rceil - 1$, it holds that $R_{j,1,n} \xrightarrow{n\to\infty} 0$ as $n \to \infty$. Then there exists an integer $N \in \mathbb{N}_+$ such that for all $n \geq N$, we have the following Wasserstein bounds:*

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq C_p \left(\sum_{j=1}^{\lceil p \rceil - 1} R_{j,1,n}^{1/j} + \sum_{j=1}^{\lceil p \rceil} R_{j,\omega,n}^{1/(j+\omega-1)} \right), \quad (3.3)$$

where $\omega = p + 1 - \lceil p \rceil$ and C_p is a constant that only depends on p.

Remark. We note that the condition that the remainder terms $R_{j,1,n}$ shrink to 0 for all $j \leq \lceil p \rceil - 1$ impose an implicit constraint on the size of the sets $N_n(i_{1:q})$. In particular, for $p = 1, 2$ we have

$$\mathcal{W}_1(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq C_1 R_{1,1,n}, \tag{3.4}$$

$$\mathcal{W}_2(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq C_2\big(R_{1,1,n} + R_{2,1,n}^{1/2}\big). \tag{3.5}$$

where the remainders are given by

$$\begin{aligned} R_{1,1,n} =& \sigma_n^{-3} \sum_{i \in I_n} \sum_{j \in N_n(i)} \sum_{k \in N_n(i,j)} \Big(\mathbb{E}\big[\big|X_i^{(n)} X_j^{(n)} X_k^{(n)}\big|\big] + \mathbb{E}\big[\big|X_i^{(n)} X_j^{(n)}\big|\big]\,\mathbb{E}\big[\big|X_k^{(n)}\big|\big]\Big), \\ R_{2,1,n} =& \sigma_n^{-4} \sum_{i \in I_n} \sum_{j \in N_n(i)} \sum_{k \in N_n(i,j)} \sum_{\ell \in N_n(i,j,k)} \Big(\mathbb{E}\big[\big|X_i^{(n)} X_j^{(n)} X_k^{(n)} X_\ell^{(n)}\big|\big] \\ &+ \mathbb{E}\big[\big|X_i^{(n)} X_j^{(n)} X_k^{(n)}\big|\big]\,\mathbb{E}\big[\big|X_\ell^{(n)}\big|\big] + \mathbb{E}\big[\big|X_i^{(n)} X_j^{(n)}\big|\big]\,\mathbb{E}\big[\big|X_k^{(n)} X_\ell^{(n)}\big|\big]\Big). \end{aligned}$$

Note that (3.4) was proven by Barbour et al. [1989] and (3.5) is a corollary of Theorem 2.1, Fang [2019]. The bound (3.3) with an integer p was also proposed as a conjecture in Fang [2019]. As p grows, the right-hand-side of (3.3) becomes more and more complicated, which suggests the necessity of new assumptions in order to be able to obtain a simplified result.

We further remark that the choice of $N_n(i_{1:q})$ might not be unique (even if we require that it has the smallest cardinality among all possible index sets that fulfill the assumption [LD-q]). Therefore, to be able to obtain more interpretable upper-bounds for the remainder terms $(R_{j,\omega,n})$, we impose a slightly stronger assumption on the dependence structure:

[LD*]: We suppose that there exists a graph $G_n = (V_n, E_n)$, with $V_n := I_n$ being the vertex set and E_n being the edge set, such that for any two disjoint subsets $J_1, J_2 \subseteq I_n$ if there is no edge between J_1 and J_2, then $\{X_j^{(n)} : j \in J_1\}$ is independent of $\{X_j^{(n)} : j \in J_2\}$.

Introduced by Petrovskaya & Leontovich [1983] the graph G_n defined above is known as the **dependency graph** and was later adopted in Janson [1988]; Baldi & Rinott [1989]; Ross [2011]. Please refer to Féray et al. [2016] for a detailed discussion.

If [LD*] is satisfied, for any subset $J \subseteq I_n$, we define $N_n(J)$ to be the set of vertices in the neighborhood of $J \subseteq I_n$ in the graph G. To be precise, this is

$$N_n(J) := J \cup \{i \in I_n : e(i,j) \in E_n \text{ for some } j \in J\},$$

where $e(i,j)$ denotes an edge between the vertices i and j. To simplify the notations, we further denote $N_n(J)$ by $N_n(i_{1:q})$ if $J = \{i_1, \cdots, i_q\}$ for any $1 \leq q \leq \lceil p \rceil + 1$. Then $(N_n(i_{1:q}))$ not only satisfies [LD-1] to [LD-($\lceil p \rceil$+1)], but has the following properties as well:

(a) $N_n(i_{1:q}) = N_n\big(i_{\pi(1)}, \cdots, i_{\pi(q)}\big)$ for any permutation π on $\{1, \cdots, q\}$;

(b) $i_q \in N_n(i_{1:(q-1)}) \Leftrightarrow i_1 \in N_n(i_{2:q})$.

We point out that by definition of the dependency graph even if $\left\{X_j^{(n)} : j \in J_1\right\}$ is independent of $\left\{X_j^{(n)} : j \in J_2\right\}$, there can still be edges between the vertex sets J_1 and J_2. In fact, there might not exist G_n such that there is no edge between J_1 and J_2 as long as $\left\{X_j^{(n)} : j \in J_1\right\}$ is independent of $\left\{X_j^{(n)} : j \in J_2\right\}$ since pairwise independence does not imply joint dependence.

The condition [LD*] provides us with a tractable bound on $R_{k,\omega,n}$, which is applicable in most of the commonly encountered settings, including m-dependent random fields and U-statistics.

Proposition 3.2. *Given $M \in \mathbb{N}_+$ and real number $\omega \in (0,1]$, suppose that $\left(X_i^{(n)}\right)_{i\in I_n}$ satisfies* [LD*] *and that the cardinality of $N_n(i_{1:(k+1)})$ is upper-bounded by $M < \infty$ for any $i_1, \cdots, i_{k+1} \in I_n$. Then there exists a constant $C_{k+\omega}$ only depending on $k+\omega$ such that*

$$R_{k,\omega,n} \leq C_{k+\omega} M^{k+\omega} \sum_{i\in I_n} \sigma_n^{-(k+1+\omega)} \mathbb{E}\left[\left|X_i^{(n)}\right|^{k+1+\omega}\right].$$

We remark that the upper bound on $(R_{k,\omega,n})$ depends on the moments of the random variables $\left(X_i^{(n)}\right)$ and the maximum size of the dependency neighborhoods. The results of Proposition 3.2 can be used to propose a more interpretable upper bound for the Wasserstein-p distance.

Theorem 3.3. *Suppose that $\left(X_i^{(n)}\right)$ is a triangular array of mean zero random variables satisfying* [LD*], *and that the cardinality of index set $N_n\left(i_{1:(\lceil p\rceil+1)}\right)$ is upper-bounded by $M_n < \infty$ for any $i_1, \cdots, i_{\lceil p\rceil+1} \in I_n$. Furthermore, assume that*

$$M_n^{1+\omega} \sigma_n^{-(\omega+2)} \sum_{i\in I_n} \mathbb{E}\left[\left|X_i^{(n)}\right|^{\omega+2}\right] \to 0, \quad M_n^{p+1} \sigma_n^{-(p+2)} \sum_{i\in I_n} \mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right] \to 0.$$

Then there is N such that for all $n \geq N$ we have

$$\begin{aligned} &\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \\ \leq & C_p \left(M_n^{1+\omega} \sigma_n^{-(\omega+2)} \sum_{i\in I_n} \mathbb{E}\left[\left|X_i^{(n)}\right|^{\omega+2}\right]\right)^{1/\omega} + C_p \left(M_n^{p+1} \sigma_n^{-(p+2)} \sum_{i\in I_n} \mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right]\right)^{1/p}, \end{aligned} \tag{3.6}$$

for some constant C_p that only depends on p.

We notably remark that if the moments are nicely behaved in the sense that

$$B_1 := \sup_{i,j\in I_n, n\in\mathbb{N}_+} \frac{\|X_i^{(n)}\|_{p+2}}{\|X_j^{(n)}\|_2} < \infty,$$

and that the size of the dependency neighborhood are universally bounded, i.e.,

$$B_2 := \sup_n \sup_{i_{1:(\lceil p\rceil+1)} \in I_n} \left|N_n(i_{1:(\lceil p\rceil+1)})\right| < \infty,$$

then there is a constant K_p that only depends on B_1, B_2 and $p \geq 1$ such that for n large enough we have

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq \frac{K_p}{\sqrt{|I_n|}}.$$

The rate of convergence matches the known rate for independent random variables (see Bobkov [2018]).

4 Applications

4.1 m-dependent random fields

Let $d \in \mathbb{N}_+$ be a positive integer, in this subsection we will study d-dimensional random fields.

Definition 4.1 (m-Dependent Random Field)**.** *A random field $(X_i)_{i\in T}$ on $T \subseteq \mathbb{Z}^d$ is m-dependent if and only if for any subsets $U_1, U_2 \subseteq \mathbb{Z}^d$, the random variables $(X_{i_1})_{i_1\in U_1\cap T}$ and $(X_{i_2})_{i_2\in U_2\cap T}$ are independent whenever $\|i_1 - i_2\| > m$ for all $i_1 \in U_1$ and $i_2 \in U_2$.*

Here $\|\cdot\|$ denotes the maximum norm on $\mathbb{Z}^d$, that is $\|z\| = \max_{1\le j\le d}|z_j|$ for $z = (z_1, \cdots, z_d)$.

Now we consider an increasing sequence $T_1 \subseteq T_2 \subseteq \cdots$ of finite subsets of $\mathbb{Z}^d$ that satisfy $|T_n| \xrightarrow{n\to\infty} \infty$. We have the following result as a corollary of Theorem 3.3.

Corollary 4.2. *Let $p \in \mathbb{N}_+$ and $m \in \mathbb{N}_+$ be positives integer. Suppose that $\left(X_i^{(n)}\right)$ is a triangular array where each row is an m-dependent random field indexed by finite subsets $T_n \subseteq \mathbb{Z}^d$ such that $|T_n| \xrightarrow{n\to\infty} \infty$. Let $\sigma_n^2 := \mathrm{Var}\left(\sum_{i\in T_n} X_i^{(n)}\right)$ and define $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i^{(n)}$. Further suppose that $\mathbb{E}\left[X_i^{(n)}\right] = 0$ for any $i \in T_n$ and that the following conditions hold:*

- *Moment condition: $\sigma_n^{-(p+2)}\sum_{i\in T_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right] \to 0 \quad$ as $n \to \infty$;*
- *Non-degeneracy condition: $\limsup_n \sigma_n^{-2}\sum_{i\in T_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^2\right] \le M < \infty$ for some $M \ge 1$.*

Then for n large enough, we have

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \le C_{p,d}\, m^{\frac{(1+\omega)d}{\omega}} M^{\frac{p-\omega}{p\omega}} \sigma_n^{-\frac{p+2}{p}} \left(\sum_{i\in T_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right]\right)^{1/p}, \tag{4.1}$$

where $C_{p,d}$ only depends on p and d.

In particular, for a triangular array of m-dependent stationary *random fields, suppose that we have $\sup_n \mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right] < \infty$, and that the non-degeneracy condition $\liminf_n \sigma_n^2/|T_n| > 0$ holds. Then we have*

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2}).$$

4.2 U-statistics

Definition 4.3 (U-Statistic)**.** *Let $(X_i)_{i=1}^n$ be a sequence of i.i.d. random variables. Fix $m \in \mathbb{N}_+$ such that $m \ge 2$. Let $h : \mathbb{R}^m \to \mathbb{R}$ be a fixed Borel-measurable function. The Hoeffding U-statistic is defined as*

$$\sum_{1\le i_1\le\cdots\le i_m\le n} h\left(X_{i_1}, \cdots, X_{i_m}\right).$$

Corollary 4.4. *Given $p \geq 1$, suppose that the U-statistic of an i.i.d. sequence $(X_i)_{i=1}^n$ induced by a symmetric function $h : \mathbb{R}^m \to \mathbb{R}$ satisfies the following conditions*

- *Mean zero:* $\mathbb{E}\big[h(X_1, \cdots, X_m)\big] = 0$;
- *Moment condition:* $\mathbb{E}\Big[\big|h(X_1, \cdots, X_m)\big|^{p+2}\Big] < \infty$;
- *Non-degeneracy condition:* $\mathbb{E}[g(X_1)^2] > 0$, *where* $g(x) := \mathbb{E}\big[h(X_1, \cdots, X_m) \,\big|\, X_1 = x\big]$.

If we let

$$W_n := \frac{1}{\sigma_n} \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} h\big(X_{i_1}, \cdots, X_{i_m}\big), \quad \text{where } \sigma_n^2 := \operatorname{Var}\Big(\sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} h\big(X_{i_1}, \cdots, X_{i_m}\big)\Big),$$

the following Wasserstein bound holds:

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(n^{-1/2}).$$

5 $\mathcal{W}_p$ bounds for strongly mixing random fields

Let $p \geq 1$ be an arbitrary real number and write $\omega := p + 1 - \lceil p \rceil \in (0, 1]$. In this section we will characterize the rate of convergence of the Wasserstein-p distance beyond the case of local dependence. To do so we need to quantify the amount of dependence between random variables. Introduced by Rosenblatt [1956], the strong mixing coefficient is one of the most widely-adopted measurement for this purpose. See Bradley [2005] for a discussion of the different notions of mixing coefficients.

Definition 5.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Given two sub-σ-algebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{F}$, the strong mixing coefficient or α-mixing coefficient between $\mathcal{A}$ and $\mathcal{B}$ is defined by*

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} \Big|\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)\Big|. \tag{5.1}$$

Let $d \in \mathbb{N}_+$ be an integer, in this subsection we will study d-dimensional random fields. Let (T_n) be an increasing sequence $T_1 \subseteq T_2 \subseteq \cdots$ of finite subsets of $\mathbb{Z}^d$ that satisfy $|T_n| \xrightarrow{n \to \infty} \infty$. Let $\big(X_i^{(n)}\big)_{i \in T_n}$ be a triangular array where each row is a random field indexed by T_n.

We define the strong mixing coefficients associated to a random field as follows:

Definition 5.2. *Given a finite index set $T \subseteq \mathbb{Z}^d$, suppose $(X_i)_{i \in T}$ is a random field on T. For any $U \subseteq T$, denote by $\mathcal{F}_U := \sigma(X_i : i \in U)$. For positive integers ℓ, k_1, k_2, define the strong mixing coefficients of $(X_i)_{i \in T}$ by*

$$\alpha_{k_1, k_2; \ell} := 0 \vee \sup\big\{\alpha(\mathcal{F}_{U_1}, \mathcal{F}_{U_2}) : U_1, U_2 \subseteq T, |U_1| \leq k_1, |U_2| \leq k_2, d(U_1, U_2) \geq \ell\big\}, \tag{5.2}$$

where $d(U_1, U_2) := \min\{\|i_1 - i_2\| : i_1 \in U_1, i_2 \in U_2\}$. Here $\|\cdot\|$ denotes the maximum norm on $\mathbb{Z}^d$.

Let $(\alpha_{k_1,k_2;\ell,n})$ be the strong mixing coefficients associated with $\left(X_i^{(n)}\right)_{i\in T_n}$. In this section, we will always consider the strong mixing coefficients with $k_1 = \lceil p\rceil + 1$ and $k_2 = |T_n|$. For convenience, when there is no ambiguity we will denote $\alpha_{\ell,n} := \alpha_{\lceil p\rceil+1,|T_n|;\ell,n}$.

Note that the strongly mixing random field is a natural extension of the m-dependent random field discussed in Section 4 as m-dependence corresponds to the case where $\alpha_{\ell,n} = 0$ for all $\ell \geq m+1$. In the rest of this section, we will extend the Wasserstein-p normal approximation results to random fields whose strong mixing coefficients converge to 0 fast enough (uniformly on n).

Theorem 5.3. *Let $\left(X_i^{(n)}\right)_{i\in T_n}$ be a triangular array of real-valued stationary random fields with strong mixing coefficients $(\alpha_{\ell,n})_{\ell\geq 1}$. Let $\sigma_n^2 := \operatorname{Var}\left(\sum_{i\in T_n} X_i^{(n)}\right)$ and define $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i^{(n)}$. Suppose that $\mathbb{E}\left[X_i^{(n)}\right] = 0$ for any $i \in T_n$, and that the following conditions hold:*

- *Moment condition: There exists $r > p+2$ such that $\sup_n \mathbb{E}\left[\left|X_i^{(n)}\right|^r\right] < \infty$;*
- *Non-degeneracy condition: $\liminf_{n\to\infty} \sigma_n^2/|T_n| > 0$;*
- *Mixing condition I: $\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-p-2)/r} < \infty$;*
- *Mixing condition II:*

$$|T_n|^{-p/2}\sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega}\alpha_{\ell,n}^{(r-p-2)/r} \to 0, \quad \text{as } n\to\infty.$$

Then the Wasserstein-p distance $\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))$ converges to 0 and we have

$$\mathcal{W}_p(\mathcal{L}(W),\mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2}) + \mathcal{O}\left(|T_n|^{-1/2}\Big(\sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega}{\alpha_{\ell,n}}^{(r-p-2)/r}\Big)^{1/p}\right). \tag{5.3}$$

In particular, $\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2})$ if the condition $\sup_n \sum_{\ell=1}^{\infty} \ell^{d(p+1)-\omega}{\alpha_{\ell,n}}^{(r-p-2)/r} < \infty$ holds.

If we assume that the mixing coefficients decrease polynomialy fast with ℓ then the statement of Theorem 5.3 can be simplified in the following way.

Corollary 5.4. *Assume that the conditions of Theorem 5.3 hold. Furthermore, suppose that $\alpha_{\ell,n} \leq C\ell^{-v}$ for some constants $v > 0$ and $C > 0$ that do not depend on n. Let $u := (r-p-2)v/r-(1-\omega)$. Then the converging rate of the Wasserstein-p distance $\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))$ ($p \geq 1$) is given by*

$$\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-\beta})$$

where

$$\beta = \begin{cases} \frac{1}{2} & \text{if } u > d(p+1) \\ \frac{1}{2}-\epsilon & \text{if } u = d(p+1) \\ \frac{1}{2}-\left(\frac{p+1}{p}-\frac{u}{dp}\right) & \text{if } d(p/2+1) < u < d(p+1) \end{cases},$$

for any $\epsilon > 0$.

As we can see, Corollary 5.4 implies that the Wasserstein-p distance converges if $u > d(p/2+1)$. In particular, if $p = 1$, we need $u > 3d/2$. We remark that this condition is sufficient but not necessary. When p is integer valued the conditions on the mixing coefficients can be weakened in the following way:

Theorem 5.5. *Let $p \in \mathbb{N}_+$ and $\left(X_i^{(n)}\right)_{i\in T_n}$ be a triangular array of real-valued stationary random fields with strong mixing coefficients $(\alpha_{\ell,n})_{\ell\geq 1}$. Let $\sigma_n^2 := \mathrm{Var}\left(\sum_{i\in T_n} X_i^{(n)}\right)$ and define $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i^{(n)}$. Suppose $\mathbb{E}\left[X_i^{(n)}\right] = 0$ for any $i \in T_n$, and they satisfy*

- *Moment condition: There exists $r \geq p+2$ such that $\sup_n \mathbb{E}[|X_i^{(n)}|^r] < \infty$;*
- *Non-degeneracy condition: $\liminf_{n\to\infty} \sigma_n^2/|T_n| > 0$;*
- *Mixing condition I: $\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-p-1)/r} < \infty$;*
- *Mixing condition II: For some $m \in \mathbb{N}_+$ and $\delta \in [0,1]$ (that can depend on n) we have that each of the following terms converges to 0 as $n \to \infty$:*

$$|T_n|^{-1/2}m^{2d}, \qquad |T_n|^{-1/2+(1-\delta)/(2p)}m^d\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}{\alpha_{\ell,n}}^{(r-p-1-\delta)/r}\Big)^{1/p},$$

$$|T_n|^{-1/2+1/(2p)}\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{dp-1}\alpha_{\ell,n}^{(r-p-1)/r}\Big)^{1/p}.$$

Then the Wasserstein-p distance $\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))$ converges to 0 and we have

$$\begin{aligned}&\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))\\ =&\mathcal{O}\big(|T_n|^{-1/2}m^{2d}\big) + \mathcal{O}\bigg(|T_n|^{-1/2+(1-\delta)/(2p)}m^d\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}{\alpha_{\ell,n}}^{(r-p-1-\delta)/r}\Big)^{1/p}\bigg)\\ &+\mathcal{O}\bigg(|T_n|^{-1/2+1/(2p)}\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{dp-1}{\alpha_{\ell,n}}^{(r-p-1)/r}\Big)^{1/p}\bigg).\end{aligned} \tag{5.4}$$

Note that in general the bound (5.4) is not comparable to (5.3). However, supposing that the strong mixing coefficients converge with a polynomial rate, Theorem 5.3 leads to better convergence rate for the Wasserstein-p distance if the mixing coefficients converge to 0 sufficiently fast while Theorem 5.5 tends to give faster convergence when the mixing coefficients converge slower. Theorem 5.5 also requires weaker conditions for the Wasserstein-p distance to converge. In specific, we show the following two results.

Corollary 5.6. *Let $\left(X_i^{(n)}\right)_{i\in T_n}$ be a triangular array of real-valued stationary random fields with strong mixing coefficients $(\alpha_{\ell,n})_{\ell\geq 1}$. Let $\sigma_n^2 := \mathrm{Var}\left(\sum_{i\in T_n} X_i^{(n)}\right)$ and define $W_n := \sigma^{-1}\sum_{i\in T_n} X_i^{(n)}$. Suppose that $\mathbb{E}\left[X_i^{(n)}\right] = 0$ for any $i \in T_n$, and that the following conditions hold*

- *Moment condition: There exists* $r \geq 3$ *such that* $\sup_n \mathbb{E}\Big[\big|X_i^{(n)}\big|^r\Big] < \infty$*;*
- *Non-degeneracy condition:* $\liminf_{n\to\infty} \sigma_n^2/|T_n| > 0$*;*
- *Mixing condition I:* $\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-2)/r} < \infty$*;*
- *Mixing condition II (uniformly Cauchy):* $\sup_n \sum_{\ell=m}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-2-\epsilon)/r} \xrightarrow{m\to\infty} 0$ *for some* $\epsilon > 0$.

Then the Wasserstein-1 *distance* $\mathcal{W}_1(\mathcal{L}(W_n), \mathcal{N}(0,1))$ *converges to* 0 *as* $n \to \infty$.

Corollary 5.7. *Let* $p \in \mathbb{N}_+$ *and* $\left(X_i^{(n)}\right)_{i\in T_n}$ *be a triangular array of real-valued stationary random fields with strong mixing coefficients* $(\alpha_{\ell,n})_{\ell\geq 1}$*. Let* $\sigma_n^2 := \mathrm{Var}\left(\sum_{i\in T_n} X_i^{(n)}\right)$ *and define* $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i^{(n)}$*. Suppose* $\mathbb{E}\big[X_i^{(n)}\big] = 0$ *for any* $i \in T_n$*, and they satisfy*

- *Moment condition: There exists* $r > p+2$ *such that* $\sup_n \mathbb{E}\Big[\big|X_i^{(n)}\big|^r\Big] < \infty$*;*
- *Non-degeneracy condition:* $\liminf_{n\to\infty} \sigma_n^2/|T_n| > 0$*;*
- *Mixing condition I:* $\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-p-2)/r} < \infty$*;*
- *Mixing condition II:* $\alpha_{\ell,n}^{(r-p-2)/r} \leq C\ell^{-u}$ *holds for some constants* $u > d(p+1)/2$ *and* $C > 0$.

Then the convergence rate of the Wasserstein-p distance $\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1))$ *(*$p \geq 1$*) is given by*

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-\beta}),$$

where

$$\beta = \begin{cases} \frac{1}{2} & \text{if } u > d(p+1) \\ \frac{1}{2} - \epsilon & \text{if } u = d(p+1) \\ \frac{1}{2} - \min\left\{\frac{p+1}{p} - \frac{u}{dp}, \frac{d}{u+dp}\right\} & \text{if } dp < u < d(p+1) \\ \frac{1}{2} - \left(\frac{1}{2p} + \epsilon\right) & \text{if } u = dp \\ \frac{1}{2} - \left(\frac{2p+1}{2p} - \frac{u}{dp}\right) & \text{if } d(p+1)/2 < u < dp \end{cases},$$

for any $\epsilon > 0$.

In particular, for $p = 1$ *and* $\alpha_{\ell,n}^{(r-3)/r} = \mathcal{O}(\ell^{-u})$*,* β *is given by*

$$\beta = \begin{cases} \frac{1}{2} & \text{if } u > 2d \\ \frac{1}{2} - \epsilon & \text{if } u = 2d \\ \frac{1}{2} - \min\left\{2 - \frac{u}{d}, \frac{d}{u+d}\right\} & \text{if } d < u < 2d \end{cases},$$

for any $\epsilon > 0$.

We make a final remark that there are also measurements of dependence besides strong mixing coefficients. Volkonskii & Rozanov [1959] considered the assumption called absolute regularity (also known as β-mixing), Kolmogorov & Rozanov [1960] introduced the ρ-mixing coefficients, and Ibragimov [1959]; Cogburn [1960] studied the ϕ-mixing conditions. Only strong mixing conditions are discussed in this paper because they are the most commonly studied ones and can be upper-bounded by the other coefficients mentioned above. Please refer to Bradley [2005] for more details on the comparison between these different conditions. We further suggest that our proof method is potentially applicable to other dependence measurements as well.

6 Overview of the proof techniques

The key idea of our proofs is to approximate the sum of weakly dependent random variables $\left(X_i^{(n)}\right)_{i\in I_n}$ by the empirical average of q_n i.i.d. random variables $\xi_1^{(n)},\cdots,\xi_{q_n}^{(n)}$. Specifically, we show (Lemma A.8) that as long as the third and higher-order cumulants of W_n decay then there exist integers (q_n) and i.i.d. random variables such that the first k ($k\in\mathbb{N}_+$) cumulants of

$$V_n := \frac{1}{\sqrt{q_n}}\sum_{i=1}^{q_n}\xi_i^{(n)}$$

matches those of W_n for n large enough. The decay of the cumulants can be proven to hold by exploiting the weak dependence assumptions (see Corollaries B.5 and F.11).

We then relate the cumulants to the Wasserstein-p bound thanks to the fact that the Wasserstein-p distance can be upper-bounded by integral probability metrics (Lemma A.3) and the well-known Stein equation. Indeed we establish local expansions of the latter to orders that will depend on our choice of $p\ge 1$. Notably for i.i.d. random variables $\left(\xi_i^{(n)}\right)_{i=1}^{q_n}$, Barbour [1986] showed that the following approximation holds (restated in Lemma A.5)

$$\begin{aligned}&\mathbb{E}[h(V_n)]-\mathcal{N}h=\mathbb{E}[f'(V_n)-V_nf(V_n)]\\&=\sum_{(r,s_{1:r})\in\Gamma(\lceil p\rceil-1)}(-1)^r\prod_{j=1}^r\frac{\kappa_{s_j+2}(V_n)}{(s_j+1)!}\mathcal{N}\left[\prod_{j=1}^r(\partial^{s_j+1}\Theta)\,h\right]+\text{ Remainders,}\end{aligned}\tag{6.1}$$

where f is the solution of the Stein equation (A.2) and $\kappa_j(\cdot)$ denotes the j-th cumulant of a random variable. (All the other notations in (6.1) will be made clear in Appendix A.) We show that we can obtain similar expansions for $\mathbb{E}[f'(W_n)-W_nf(W_n)]$ (see Lemma A.7):

$$\begin{aligned}&\mathbb{E}[h(W_n)]-\mathcal{N}h=\mathbb{E}[f'(W_n)-W_nf(W_n)]\\&=\sum_{(r,s_{1:r})\in\Gamma(\lceil p\rceil-1)}(-1)^r\prod_{j=1}^r\frac{\kappa_{s_j+2}(W_n)}{(s_j+1)!}\mathcal{N}\left[\prod_{j=1}^r(\partial^{s_j+1}\Theta)\,h\right]+\text{ Remainders,}\end{aligned}\tag{6.2}$$

As mentioned in the previous paragraph, q_n and $\xi_i^{(n)}$ can be chosen to be such that $\kappa_j(V_n)=\kappa_j(W_n)$ for $j=1,\cdots,\lceil p\rceil+1$. Thus, by taking the difference of (6.1) and (6.2), we get an upper bound on $\left|\mathbb{E}[h(W_n)]-\mathbb{E}[h(V_n)]\right|$ for a large class of function h. As shown in Lemma A.3, this allows us to obtain an upper bound of the Wasserstein-p distance between $\mathcal{L}(W_n)$ and $\mathcal{L}(V_n)$ for a general $p \geq 1$. The desired result is therefore implied by the triangle inequality of the Wasserstein-p distance

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \leq \mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) + \mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)),$$

and the already known Wasserstein-p bounds for i.i.d. random variables (Lemma A.6).

To be able to show that (6.2) holds, we develop two new techniques to obtain such expansions. These techniques will be carefully elaborated and discussed in Appendices B and F to H.

References

Agnew, R. P. Estimates for global central limit theorems. *The Annals of Mathematical Statistics*, 28(1):26–42, 1957.

Akhiezer, N. I. *The classical moment problem and some related questions in analysis*. SIAM, 2020.

Audibert, J.-Y., Munos, R., and Szepesvári, C. Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. *Theoretical Computer Science*, 410(19):1876–1902, 2009.

Austern, M. and Mackey, L. Efficient concentration with gaussian approximation. *arXiv preprint arXiv:2208.09922*, 2022.

Baldi, P. and Rinott, Y. On normal approximations of distributions in terms of dependency graphs. *The Annals of Probability*, 17(4):1646–1650, 1989.

Barbour, A. D. Asymptotic expansions based on smooth functions in the central limit theorem. *Probability Theory and Related Fields*, 72(2):289–303, 1986.

Barbour, A. D., Karoński, M., and Ruciński, A. A central limit theorem for decomposable random variables with applications to random graphs. *Journal of Combinatorial Theory, Series B*, 47 (2):125–145, 1989.

Bártfai, P. über die entfernung der irrfahrtswege. *Studia Sci. Math. Hungar*, 5:41–49, 1970.

Bartlett, P. L. and Mendelson, S. Rademacher and gaussian complexities: Risk bounds and structural results. *Journal of Machine Learning Research*, 3(Nov):463–482, 2002.

Bender, E. A. and Williamson, S. G. *Lists, decisions and graphs*. S. Gill Williamson, 2010.

Bentkus, V. and Sunklodas, J. K. On normal approximations to strongly mixing random fields. *Publicationes Mathematicae Debrecen*, 70(3-4):253–270, 2007.

Berg, C. Indeterminate moment problems and the theory of entire functions. *Journal of Computational and Applied Mathematics*, 65(1-3):27–55, 1995.

Bikelis, A. On estimates of the remainder term in the central limit theorem. *Lithuanian Mathematical Journal*, 6(3):323–346, Sep. 1966. doi: 10.15388/LMJ.1966.19732.

Bobkov, S. G. Berry–esseen bounds and edgeworth expansions in the central limit theorem for transport distances. *Probability Theory and Related Fields*, 170(1):229–262, 2018.

Bolthausen, E. On the central limit theorem for stationary mixing random fields. *The Annals of Probability*, pp. 1047–1050, 1982.

Bonis, T. Stein's method for normal approximation in wasserstein distances with application to the multivariate central limit theorem. *Probability Theory and Related Fields*, 178(3):827–860, 2020.

Bradley, R. C. Basic properties of strong mixing conditions. a survey and some open questions. *Probability surveys*, 2:107–144, 2005.

Chatterjee, S., Fulman, J., and Röllin, A. Exponential approximation by stein's method and spectral graph theory. *ALEA Lat. Am. J. Probab. Math. Stat*, 8(1):197–223, 2011.

Chen, L. H. Poisson approximation for dependent trials. *The Annals of Probability*, 3(3):534–545, 1975.

Chen, L. H. and Shao, Q.-M. Normal approximation under local dependence. *The Annals of Probability*, 32(3):1985–2028, 2004.

Chen, L. H. and Shao, Q.-M. Normal approximation for nonlinear statistics using a concentration inequality approach. *Bernoulli*, 13(2):581–599, 2007.

Cogburn, R. *Asymptotic properties of stationary sequences*, volume 3. University of California Press, 1960.

Döring, H. and Eichelsbacher, P. Moderate deviations via cumulants. *Journal of Theoretical Probability*, 26(2):360–385, 2013.

Döring, H., Jansen, S., and Schubert, K. The method of cumulants for the normal approximation. *Probability Surveys*, 19:185–270, 2022.

Doukhan, P. *Mixing*. Springer, 1994.

Esseen, C.-G. On mean central limit theorems. *Trans. Roy. Inst. Tech., Stockholm*, 2(120):1–30, 1958.

Fang, X. Wasserstein-2 bounds in normal approximation under local dependence. *Electronic Journal of Probability*, 24:1–14, 2019.

Fang, X. and Koike, Y. From p-wasserstein bounds to moderate deviations. *arXiv preprint arXiv:2205.13307*, 2022.

Feller, W. The fundamental limit theorems in probability. *Bulletin of the American Mathematical Society*, 51(11):800–832, 1945.

Féray, V., Méliot, P.-L., and Nikeghbali, A. *Mod-φ convergence: Normality zones and precise deviations*. Springer, 2016.

Gilbert, G. T. Positive definite matrices and sylvester's criterion. *The American Mathematical Monthly*, 98(1):44–46, 1991.

Götze, F. and Hipp, C. Asymptotic expansions for sums of weakly dependent random vectors. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 64(2):211–239, 1983.

Götze, F., Heinrich, L., and Hipp, C. m-dependent random fields with analytic cumulant generating function. *Scandinavian journal of statistics*, pp. 183–195, 1995.

Heinrich, L. A method for the derivation of limit theorems for sums of m-dependent random variables. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 60(4):501–515, 1982.

Heinrich, L. Some bounds of cumulants of m-dependent random fields. *Mathematische Nachrichten*, 149(1):303–317, 1990.

Heubach, S. and Mansour, T. *Combinatorics of compositions and words*. Chapman and Hall/CRC, 2009.

Hoeffding, W. A Class of Statistics with Asymptotically Normal Distribution. *The Annals of Mathematical Statistics*, 19(3):293 – 325, 1948. doi: 10.1214/aoms/1177730196.

Hoeffding, W. and Robbins, H. The central limit theorem for dependent random variables. *Duke Mathematical Journal*, 15(3):773–780, 1948.

Ibragimov, I. Some limit theorems for stochastic processes stationary in the strict sense. *Dokl. Akad. Nauk SSSR*, 125(4):711–714, 1959.

Janson, S. Normal convergence by higher semiinvariants with applications to sums of dependent random variables and random graphs. *The Annals of Probability*, pp. 305–312, 1988.

Kolmogorov, A. N. and Rozanov, Y. A. On strong mixing conditions for stationary gaussian processes. *Theory of Probability & Its Applications*, 5(2):204–208, 1960.

Lahiri, S. N. Refinements in asymptotic expansions for sums of weakly dependent random vectors. *The Annals of Probability*, pp. 791–799, 1993.

Lahiri, S. N. Asymptotic expansions for sums of random vectors under polynomial mixing rates. *Sankhyā: The Indian Journal of Statistics, Series A*, pp. 206–224, 1996.

Lang, S. *Introduction to linear algebra*. Springer Science & Business Media, 2012.

Ledoux, M., Nourdin, I., and Peccati, G. Stein's method, logarithmic sobolev and transport inequalities. *Geometric and Functional Analysis*, 25(1):256–306, 2015.

Lévy, P. Propriétés asymptotiques des sommes de variables indépendantes ou enchainées. *Journal des mathématiques pures et appliquées. Series*, 9(14):4, 1935.

Ley, C., Reinert, G., and Swan, Y. Stein's method for comparison of univariate distributions. *Probability Surveys*, 14:1–52, 2017.

Lukacs, E. *Characteristic Functions*. A Charles Griffin Book. Griffin, 1970. ISBN 9780195205787.

Mnih, V., Szepesvári, C., and Audibert, J.-Y. Empirical bernstein stopping. In *Proceedings of the 25th International Conference on Machine Learning*, pp. 672–679, 2008.

Nagaev, S. V. Some limit theorems for large deviations. *Theory of Probability & Its Applications*, 10(2):214–235, 1965.

Otto, F. and Villani, C. Generalization of an inequality by talagrand and links with the logarithmic sobolev inequality. *Journal of Functional Analysis*, 173(2):361–400, 2000.

Peköz, E. A. and Röllin, A. New rates for exponential approximation and the theorems of rényi and yaglom. *The Annals of Probability*, 39(2):587–608, 2011.

Petrovskaya, M. and Leontovich, A. M. The central limit theorem for a sequence of random variables with a slowly growing number of dependences. *Theory of Probability & Its Applications*, 27(4):815–825, 1983.

Rio, E. Distances minimales et distances idéales. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 326(9):1127–1130, 1998.

Rio, E. Upper bounds for minimal distances in the central limit theorem. *Annales de l'IHP Probabilités et statistiques*, 45(3):802–817, 2009.

Roman, S. *An introduction to Catalan numbers*. Springer, 2015.

Rosenblatt, M. A central limit theorem and a strong mixing condition. *Proceedings of the national Academy of Sciences*, 42(1):43–47, 1956.

Ross, N. Fundamentals of stein's method. *Probability Surveys*, 8:210–293, 2011.

Sakhanenko, A. I. Estimates in an invariance principle. *Matematicheskie Trudy*, 5:27–44, 1985.

Smith, P. J. A recursive formulation of the old problem of obtaining moments from cumulants and vice versa. *The American Statistician*, 49(2):217–218, 1995.

Stanley, R. P. *Enumerative Combinatorics*, volume 1 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2 edition, 2011. doi: 10.1017/CBO9781139058520.006.

Stein, C. A bound for the error in the normal approximation to the distribution of a sum of dependent random variables. In *Proceedings of the sixth Berkeley symposium on mathematical statistics and probability, volume 2: Probability theory*, volume 6, pp. 583–603. University of California Press, 1972.

Sunklodas, J. On normal approximation for strongly mixing random variables. *Acta Applicandae Mathematicae*, 97(1):251–260, 2007.

Talagrand, M. Transportation cost for gaussian and other product measures. *Geometric & Functional Analysis GAFA*, 6(3):587–600, 1996.

Tamarkin, J. D., Shohat, J. A., and Tamarkin, J. *The Problem of Moments*. American Mathematical Society, 1943.

Villani, C. *Optimal transport: old and new*, volume 338. Springer, 2009.

Volkonskii, V. and Rozanov, Y. A. Some limit theorems for random functions. i. *Theory of Probability & Its Applications*, 4(2):178–197, 1959.

Wainwright, M. J. *High-dimensional statistics: A non-asymptotic viewpoint*, volume 48. Cambridge University Press, 2019.

Zolotarev, V. M. Probability metrics. *Theory of Probability & Its Applications*, 28(2):278–302, 1984.

A Proof of Theorem 3.1

In this section, we provide the proofs of Theorems 3.1 and 3.3 using Stein's method. We first introduce some background definitions and lemmas before showing the proofs of the main theorems.

A.1 Preliminary definitions and notations

Definition A.1 (Hölder Space)**.** *For any $k \in \mathbb{N}$ and real number $\omega \in (0,1]$, the Hölder space $\mathcal{C}^{k,\omega}(\mathbb{R})$ is defined as the class of k-times continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that the k-times derivative of f is ω-Hölder continuous, i.e.,*

$$|f|_{k,\omega} := \sup_{x \neq y \in \mathbb{R}} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{\omega}} < \infty,$$

where ∂ denotes the differential operator. Here ω is called the Hölder exponent and $|f|_{k,\omega}$ is called the Hölder coefficient.

Using the notions of Hölder spaces, we define the Zolotarev's ideal metrics, which are related to the Wasserstein-p distances via Lemma A.3.

Definition A.2 (Zolotarev Distance)**.** *Suppose μ and ν are two probability distributions on $\mathbb{R}$. For any $p > 0$ and $\omega := p + 1 - \lceil p \rceil \in (0,1]$, the Zolotarev-p distance between μ and ν is defined by*

$$\mathcal{Z}_p(\mu, \nu) := \sup_{f \in \Lambda_p} \left(\int_{\mathbb{R}} f(x)\,\mathrm{d}\mu(x) - \int_{\mathbb{R}} f(x)\,\mathrm{d}\nu(x) \right),$$

where $\Lambda_p := \{f \in \mathcal{C}^{\lceil p \rceil - 1,\omega}(\mathbb{R}) : |f|_{\lceil p \rceil - 1,\omega} \leq 1\}$

To bound $\mathcal{Z}_p(\cdot,\cdot)$ we rely on the Stein's method which was introduced by Stein [1972] in order to prove the central limit theorem for dependent data. It has been widely adapted to all kinds of normal approximation problems. See Ross [2011] for a detailed exposition.

Stein equation and its solutions

Let $Z \sim \mathcal{N}(0,1)$ be a standard normal random variable. For any measurable function $h : \mathbb{R} \to \mathbb{R}$, if $h(Z) \in \mathcal{L}^1(\mathbb{R})$, we write $\mathcal{N}h := \mathbb{E}[h(Z)]$. Thus, $h(Z) \in \mathcal{L}^1(\mathbb{R})$ if and only if $\mathcal{N}|h| < \infty$. Moreover, we define $f_h(\cdot)$ by

$$f_h(W_n) := \int_{-\infty}^{w} e^{(w^2-t^2)/2}(h(t) - \mathcal{N}h)\,\mathrm{d}t = -\int_{w}^{\infty} e^{(w^2-t^2)/2}(h(t) - \mathcal{N}h)\,\mathrm{d}t. \tag{A.1}$$

We remark that $f_h(\cdot)$ is a *solution* of the Stein equation meaning that it satisfies

$$f'(w) - wf(w) = h(w) - \mathcal{N}h, \qquad \forall w \in \mathbb{R}. \tag{A.2}$$

For the convenience of further discussion, we denote by Θ the operator that maps h to f_h for any h such that $\mathcal{N}|h| < \infty$, i.e.,

$$\Theta h = f_h.$$

Note that $\Theta h(\cdot)$ is a function. If $h \in \Lambda_p$, then we will see in Lemma A.4 that Θh can be bounded.

A.2 Preliminary lemmas

An important fact proven by Rio [2009] is that the Wasserstein-p distance can be controlled in terms of the Zolotarev distance.

Lemma A.3 (Theorem 3.1 of Rio [2009])**.** *For any $p \geq 1$, there exists a positive constant C_p, such that for any pair of distributions μ, ν on $\mathbb{R}$ with finite absolute moments of order p such that*

$$\mathcal{W}_p(\mu,\nu) \leq C_p\big(\mathcal{Z}_p(\mu,\nu)\big)^{1/p}.$$

In particular, $\mathcal{W}_1(\mu,\nu) = \mathcal{Z}_1(\mu,\nu)$ by Kantorovich–Rubinstein duality.

To bound $\mathcal{Z}_p(\cdot,\cdot)$ we will use the Stein's method and exploit the fact Θh can be controlled for any function $h \in \Lambda_p$.

Lemma A.4 (Part of Lemma 6 of Barbour [1986])**.** *For any $p > 0$, let $h \in \Lambda_p$ be as defined in Definition A.2. Then $\Theta h = f_h$ in* (A.1) *is a solution to* (A.2)*. Moreover, $\Theta h \in \mathcal{C}^{\lceil p\rceil-1,\omega}(\mathbb{R}) \cap \mathcal{C}^{\lceil p\rceil,\omega}(\mathbb{R})$ and the Hölder coefficients $|\Theta h|_{\lceil p\rceil-1,\omega}$ and $|\Theta h|_{\lceil p\rceil,\omega}$ are bounded by some constant only depending on p.*

Next we present two lemmas on the normal approximation for independent random variables. Lemma A.5 provides an expansion for the difference between $\mathbb{E}[h(S_n)]$, where S_n is an empirical average, and $\mathcal{N}h$. Lemma A.6 gives an upper bound on the Wasserstein distance between the distribution of this empirical average, S_n, and the standard normal distribution.

Lemma A.5 (Theorem 1 of Barbour [1986])**.** *For any $p > 0$, let $h \in \Lambda_p$ and $S_n := \sum_{i=1}^n X_i$ where $\{X_1, \cdots, X_n\}$ are independent, with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[S_n^2] = 1$. Then it follows that*

$$\mathbb{E}[h(S_n)] - \mathcal{N}h = \sum_{(r,s_{1:r}) \in \Gamma(\lceil p \rceil - 1)} (-1)^r \prod_{j=1}^r \frac{\kappa_{s_j+2}(S_n)}{(s_j+1)!} \mathcal{N}\left[\prod_{j=1}^r (\partial^{s_j+1}\Theta)\, h\right] + \mathcal{O}\Big(\sum_{i=1}^n \mathbb{E}[|X_i|^{p+2}]\Big), \quad \text{(A.3)}$$

where the first sum is over $\Gamma(\lceil p \rceil - 1) := \left\{r, s_{1:r} \in \mathbb{N}_+ : \sum_{j=1}^r s_j \le \lceil p \rceil - 1\right\}$.

Note that there is a slight abuse of notation in (A.3). The last $\prod$ indicates the composition of the operators in the parentheses rather than the product.

Lemma A.6 (Theorem 1.1 of Bobkov [2018])**.** *For any $p \ge 1$, let $S_n := \sum_{i=1}^n X_i$ where $\{X_1, \cdots, X_n\}$ are independent, with $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[S_n^2] = 1$. Then it follows that*

$$\mathcal{W}_p(\mathcal{L}(S_n), \mathcal{N}(0,1)) \le C_p \Big(\sum_{i=1}^n \mathbb{E}[|X_i|^{p+2}]\Big)^{1/p}, \quad \text{(A.4)}$$

where C_p continuously depends on p.

A.3 Main lemmas

We introduce two new lemmas crucial in the proof of Theorem 3.1. They will be proven in Appendix B and Appendix C.

Lemma A.7 (Local Expansion)**.** *Suppose that $\big(X_i^{(n)}\big)_{i \in I_n}$ is a triangular array of random variables with dependency neighborhoods satisfying the local dependence conditions* [LD-1] *to* [LD-($\lceil p \rceil$+1)]*. Let $W_n := \sum_{i \in I_n} X_i^{(n)}$ with $\mathbb{E}\big[X_i^{(n)}\big] = 0$, $\mathbb{E}[W_n^2] = 1$. Then for any $p > 0$ and $h \in \Lambda_p$, we have*

$$\begin{aligned}\mathbb{E}[h(W_n)] - \mathcal{N}h = &\sum_{(r,s_{1:r}) \in \Gamma(\lceil p \rceil - 1)} (-1)^r \prod_{j=1}^r \frac{\kappa_{s_j+2}(W_n)}{(s_j+1)!} \mathcal{N}\left[\prod_{j=1}^r (\partial^{s_j+1}\Theta)\, h\right] \\ &+ \mathcal{O}\Big(\sum_{j=1}^{\lceil p \rceil - 1} R_{j,1,n}^{p/j} + \sum_{j=1}^{\lceil p \rceil} R_{j,\omega,n}^{p/(j+\omega-1)}\Big),\end{aligned} \quad \text{(A.5)}$$

where the first sum is over $\Gamma(\lceil p \rceil - 1) := \left\{r, s_{1:r} \in \mathbb{N}_+ : \sum_{j=1}^r s_j \le \lceil p \rceil - 1\right\}$.

We can see that Lemmas A.5 and A.7 look quite similar to one another with the only differences being the dependence structures of $\big(X_i^{(n)}\big)$ and the remainder terms in the expansions. This similarity inspires the proof of Theorem 3.1. To illustrate this, imagine that there would exist some i.i.d. random variables $\big(\xi_i^{(n)}\big)_{i=1}^{q_n}$ and a large sample size q_n such that the first $\lceil p \rceil$+1 cumulants of $V_n := q_n^{-1/2} \sum_{i=1}^{q_n} \xi_i^{(n)}$ match with those of W_n, then the expansion (A.5) and in (A.3) would be almost identical, and the difference between those would be controlled by the remainder terms $(R_{j,1,n})$ and $(R_{j,\omega,n})$. If those remainder terms are small then we could exploit the asymptotic normality of V_n to obtain the asymptotic normality of W_n. We show that such a sequence exists when $|I_n|$ is large.

Lemma A.8 (Cumulant Matching). *Let $p \geq 1$ and $k := \lceil p \rceil$. If $p > 1$, let $\left(u_j^{(n)}\right)_{j=1}^{k-1}$ be a sequence of real numbers. Suppose that for any $j = 1, \cdots, k-1$, we have $u_j^{(n)} \to 0$ as $n \to \infty$. Then there exist constants C_p, C_p' only depending on p and a positive value $N > 0$ (that might depend on $\left(u_j^{(n)}\right)$) such that for any $n > N$, there exists $q_n \in \mathbb{N}_+$ and a random variable $\xi^{(n)}$ such that*

(a) $\mathbb{E}[\xi^{(n)}] = 0, \quad \mathbb{E}[(\xi^{(n)})^2] = 1$;

(b) $\kappa_{j+2}(\xi^{(n)}) = q_n^{j/2} u_j^{(n)}$ *for* $j = 1, \cdots, k-1$;

(c) Either $\max_{1 \leq j \leq k-1} \left|\kappa_{j+2}(\xi^{(n)})\right| = 0$ *or* $\max_{1 \leq j \leq k-1} \left|\kappa_{j+2}(\xi^{(n)})\right| \geq C_p > 0$;

(d) $\mathbb{E}[|\xi^{(n)}|^{p+2}] \leq C_p'$.

Furthermore, q_n can be chosen to be such that $q_n \to \infty$ as $|I| \to \infty$.

We note that the condition that $u_j^{(n)} \to 0$ as $n \to \infty$ is crucial. Lemma A.8 is an asymptotic statement in the sense that for a given $n \leq N$, q_n and $\xi^{(n)}$ might not exist.

Intuitively, Lemma A.8a and Lemma A.8b determines the cumulants of $\xi^{(n)}$ and relates them to the cumulants of W_n. Lemma A.8c requires that the maximum $\max_{1 \leq j \leq k} \left|\kappa_{j+2}(\xi^{(n)})\right|$ is either 0 or bounded away from 0 as n grows. And Lemma A.8d indicates that the (p+2)-th absolute moment is upper-bounded.

A.4 Proof of Theorem 3.1

The proof of Theorem 3.1 works in three stages:

1. Using Lemma A.8 we find a sequence of i.i.d. random variables $\left(\xi_\ell^{(n)}\right)_\ell$ and a sample size q_n such that the first k+1 cumulants of W_n match the first k+1 cumulants of $V_n := q_n^{-1/2} \sum_{i=1}^{q_n} \xi_i^{(n)}$;
2. Using Lemma A.3 we remark that we can bound the Wasserstein distance between the distributions of W_n and an empirical average, V_n, of i.i.d. observations in terms of $\left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right|$ for a large class of functions h. We do so by exploiting Lemmas A.5 and A.7;
3. We remark that Lemma A.6 provides us with the bound on the Wasserstein distance between the distribution of V_n and the standard normal.

Then Theorem 3.1 follows from the triangle inequality of the Wasserstein metric:

$$\mathcal{W}_p(W_n, \mathcal{N}(0,1)) \leq \mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) + \mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)).$$

Proof of Theorem 3.1. Without loss of generality, we assume $\sigma_n = 1$ and denote $W_n := \sum_{i \in I_n} X_i^{(n)}$.

Firstly, we remark that according to Corollary B.5, for all $1 \leq j \leq k-1$ we have $\left|\kappa_{j+2}(W_n)\right| \lesssim R_{j,1,n}$. Moreover, by assumption we have $R_{j,1,n} \to 0$ as $n \to \infty$. Therefore, $\left|\kappa_{j+2}(W_n)\right| \to 0$ as $n \to \infty$ and the assumptions of Lemma A.8 hold, which implies that there exist constants C_p and C'_p such that for any n large enough there are positive integers (q_n) and random variables $(\xi^{(n)})$ such that

(a) $\mathbb{E}[\xi^{(n)}] = 0, \quad \mathbb{E}[(\xi^{(n)})^2] = 1$;

(b) $\kappa_{j+2}(\xi^{(n)}) = q_n^{j/2}\kappa_{j+2}(W_n)$ for $j = 1, \cdots, k-1$;

(c) Either $\max_{1\leq j\leq k-1}\left|\kappa_{j+2}(\xi^{(n)})\right| = 0$ or $\max_{1\leq j\leq k-1}\left|\kappa_{j+2}(\xi^{(n)})\right| \geq C_p > 0$;

(d) $\mathbb{E}[|\xi^{(n)}|^{p+2}] \leq C'_p$.

Furthermore, we know that (q_n) satisfies that $q_n \to \infty$ as $n \to \infty$.

As presented in the proof sketch we will use this to bound the distance between the distribution of W_n to the one of an empirical average of at least q_n i.i.d. random variables. Note that when $\max_{1\leq j\leq k-1}|\kappa_{j+2}(\xi^{(n)})| > 0$ then we can obtain (by combining items (b) and (c)) a lower bound on q_n which will be crucial in our arguments as it will allow us to control the distance between this empirical average and its normal limit. When $\kappa_3(W_n) = \cdots = \kappa_{k+1}(W_n) = 0$, such a lower bound on q_n cannot be obtained in a similar way. Thus, we introduce an alternative sequence $(\widetilde{q}_n)$ by setting $\widetilde{q}_n := |I_n|^{2(p+1)/p} \vee q_n$ if $\kappa_3(W_n) = \cdots = \kappa_{k+1}(W_n) = 0$, and $\widetilde{q}_n := q_n$ otherwise. We remark that the sequence $(\widetilde{q}_n)$ still respects $\widetilde{q}_n \to \infty$ as $n \to \infty$.

Let $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ be i.i.d. copies of $\xi^{(n)}$. Define $V_n := \widetilde{q}_n^{-1/2}\sum_{i=1}^{\widetilde{q}_n}\xi_i^{(n)}$.

By construction, for any $j \in \mathbb{N}_+$ such that $j \leq k-1 = \lceil p \rceil - 1$ we have

$$\kappa_{j+2}(V_n) \overset{(*)}{=} \widetilde{q}_n^{-(j+2)/2}\sum_{i=1}^{\widetilde{q}_n}\kappa_{j+2}(\xi_i^{(n)}) = \widetilde{q}_n^{-j/2}\kappa_{j+2}(\xi^{(n)}) = \kappa_{j+2}(W_n).$$

Here in $(*)$ we have used the fact that cumulants are cumulative for independent random variables, which is directly implied by their definition. For more details on this, please refer to Lukacs [1970].

Thus, by Lemma A.5 and Lemma A.7, for any $h \in \Lambda_p$ we have

$$\left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right| \lesssim \sum_{j=1}^{k-1} R_{j,1,n}^{p/j} + \sum_{j=1}^{k} R_{j,\omega,n}^{p/(j+\omega-1)} + \widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}\Big[\left|\xi_i^{(n)}\right|^{p+2}\Big]. \tag{A.6}$$

To be able to have this upper bound not depend on $\xi_i^{(n)}$ we will upper-bound

$$\widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}[|\xi_i^{(n)}|^{p+2}]$$

in terms of the remainders $(R_{j,1,n})$ and $(R_{j,\omega,n})$. To do so we use the lower bounds on $(\widetilde{q}_n)$ implied by the specific form we chose.

If $\max_{1\le j\le k-1}\left|\kappa_{j+2}(W_n)\right| > 0$, item (c) implies that

$$C_p \le \max_{1\le j\le k-1}\left|\kappa_{j+2}(\xi^{(n)})\right| \overset{(*)}{=} \max_{1\le j\le k-1}\left\{\widetilde{q}_n^{j/2}\left|\kappa_{j+2}(W_n)\right|\right\} \overset{(**)}{\lesssim} \max_{1\le j\le k-1}\left\{\widetilde{q}_n^{j/2}R_{j,1,n}\right\}.$$

where to get $(*)$ we used item (b) and to get $(**)$ we used Corollary B.5. Thus, the following holds

$$\widetilde{q}_n^{-p/2} = (\widetilde{q}_n^{-j_0/2})^{p/j_0} \lesssim R_{j_0,1,n}^{p/j_0} \le \sum_{j=1}^{k-1} R_{j,1,n}^{p/j},$$

where j_0 is the integer satisfying that $\left|\kappa_{j_0+2}(\xi^{(n)})\right| = \max_{1\le j\le k-1}\left|\kappa_{j+2}(\xi^{(n)})\right|$.

On the other hand, if $\kappa_{j+2}(W_n) = 0$ for all $1 \le j \le k-1$, then by definitions we have $\widetilde{q}_n \ge |I_n|^{2(p+1)/p}$, and therefore, $\widetilde{q}_n^{-p/2} \le |I_n|^{-(p+1)}$. Moreover, by Hölder's inequality we know that the following holds

$$\sum_{i\in I_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^2\right] \le |I_n|^{p/(p+2)}\Big(\sum_{i\in I_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right]\Big)^{2/(p+2)}. \tag{A.7}$$

and

$$\Big(\sum_{i\in I_n}X_i^{(n)}\Big)^2 \le |I_n|\sum_{i\in I_n}\left|X_i^{(n)}\right|^2. \tag{A.8}$$

Since $\mathbb{E}\left[\left(\sum_{i\in I_n}X_i^{(n)}\right)^2\right] = \sigma_n^2 = 1$, we have

$$\begin{aligned}
\widetilde{q}_n^{-p/2} \le& |I_n|^{-(p+1)}\Big(\mathbb{E}\Big[\Big(\sum_{i\in I}X_i^{(n)}\Big)^2\Big]\Big)^{(p+2)/2}\\
\overset{(*)}{\le}& |I_n|^{-p/2}\Big(\sum_{i\in I}\mathbb{E}\left[\left|X_i^{(n)}\right|^2\right]\Big)^{(p+2)/2}\\
\overset{(**)}{\le}& \sum_{i\in I_n}\mathbb{E}\left[\left|X_i^{(n)}\right|^{p+2}\right] \le R_{k,\omega,n},
\end{aligned}$$

where to obtain $(*)$ we used (A.8) and to obtain $(**)$ we used (A.7).

Thus, using item (d) and the fact that $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ are i.i.d., we obtain

$$\widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}\left[\left|\xi_i^{(n)}\right|^{p+2}\right] \le C_p'\widetilde{q}_n^{-p/2} \lesssim \sum_{j=1}^{k-1}R_{j,1,n}^{p/j} + \sum_{j=1}^{k}R_{j,\omega,n}^{p/(j+\omega-1)}. \tag{A.9}$$

Therefore, by combining this with (A.6) we obtain that there is a constant $K > 0$ that does not depend on h such that

$$\left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right| \le K\Big(\sum_{j=1}^{k}R_{j,1,n}^{p/j} + \sum_{j=1}^{k+1}R_{j,\omega,n}^{p/(j+\omega-1)}\Big).$$

By taking supremum over $h \in \Lambda_p$ and by Lemma A.3 we obtain that

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) \lesssim \sup_{h\in\Lambda_p} \left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right|^{1/p} \lesssim \sum_{j=1}^{k-1} R_{j,1,n}^{1/j} + \sum_{j=1}^{k} R_{j,\omega,n}^{1/(j+\omega-1)}.$$

Moreover, by combining Lemma A.6 and (A.9) we have

$$\mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)) \lesssim \left(\widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\left[\left|\xi_i^{(n)}\right|^{p+2}\right]\right)^{1/p} \lesssim \sum_{j=1}^{k-1} R_{j,1,n}^{1/j} + \sum_{j=1}^{k} R_{j,\omega,n}^{1/(j+\omega-1)}.$$

Therefore, as the Wasserstein distance $\mathcal{W}_p$ satisfies the triangle inequality we conclude that

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \le \mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) + \mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)) \lesssim \sum_{j=1}^{k-1} R_{j,1,n}^{1/j} + \sum_{j=1}^{k} R_{j,\omega,n}^{1/(j+\omega-1)}.$$

■

B Proof of Lemma A.7

For ease of notation, when there is no ambiguity we will drop the dependence on n in our notation and write W, $N(\cdot)$, σ, X_i, I and $R_{j,\omega}$ for respectively W_n, $N_n(\cdot)$, σ_n, $X_i^{(n)}$, I_n and $R_{j,\omega,n}$.

B.1 Example & Roadmap

Given the form of expression in Lemma A.7, it is natural to consider performing induction on $\lceil p \rceil$. In fact, Barbour [1986] used a similar induction idea to prove Lemma A.5, the analogous result to Lemma A.7 for independent variables. As Fang [2019] suggested, the key of each inductive step is the following expansion of $\mathbb{E}[Wf(W)]$.

Proposition B.1 (Expansion of $\mathbb{E}[Wf(W)]$)**.** *Denote by $\kappa_j(W)$ the j-th cumulant of W. Given $k \in \mathbb{N}_+$ and real number $\omega \in (0,1]$, for any $f \in \mathcal{C}^{k,\omega}(\mathbb{R})$, we have*

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!} \mathbb{E}[\partial^j f(W)] + \mathcal{O}(|f|_{k,\omega} R_{k,\omega}). \tag{B.1}$$

The case $k = \omega = 1$ is a well-known result in the literature of Stein's method (for example see Barbour et al. [1989]; Ross [2011]). The case $k = 2, \omega = 1$ was first proven by Fang [2019], and they also conjectured that it was true for any positive integer k with $\omega = 1$. Inspired by Fang [2019]'s method, we confirm that this conjecture is correct by proving Proposition B.1.

To help better understand the intuition behind our proof for the general settings, let's first consider the simplest case with $k = \omega = 1$. Given a positive integer m, suppose that $(X_i)_{i=1}^n$ is an m-dependent random sequence (the special case of $d = 1$ in Definition 4.1). We let $W := \sum_{i=1}^n X_i$

and require that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[W^2] = 1$. For simplicity, we further assume $f \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^{1,1}(\mathbb{R})$ meaning that f'' is a continuous and bounded function.

For any indexes $i, j \in [n]$ (by convention $[n] := \{1, 2, \cdots, n\}$), we write

$$N(i) = \{\ell \in [n] : |\ell - i| \le m\}, \quad N(i,j) := \{\ell \in [n] : |\ell - i| \le m \text{ or } |\ell - j| \le m\}.$$

Denote $W_{i,m} := \sum_{j \notin N(i)} X_j$ and $W_{i,j,m} := \sum_{\ell \notin N(i,j)} X_\ell$. The idea is that for each i, we split W into two parts, $W_{i,m}$ and $W - W_{i,m}$. The former is independent of X_i while the latter is the sum of X_j's in the neighborhood of X_i and will converge to 0 when n grows to ∞. Thus, we perform the Taylor expansion for $f(W)$ around $W_{i,m}$.

We have

$$\begin{aligned}
\mathbb{E}\big[Wf(W) - f'(W)\big] &= \sum_{i=1}^n \mathbb{E}\Big[X_i\big(f(W) - f(W_{i,m}) - f'(W_{i,m})(W - W_{i,m})\big)\Big] \\
&\quad + \sum_{i=1}^n \mathbb{E}[X_i f(W_{i,m})] + \sum_{i=1}^n \mathbb{E}\big[X_i(W - W_{i,m}) f'(W_{i,m})\big] - \mathbb{E}[f'(W)] \\
&= \sum_{i=1}^n \mathbb{E}\Big[X_i\big(f(W) - f(W_{i,m}) - f'(W_{i,m})(W - W_{i,m})\big)\Big] \\
&\quad + \sum_{i=1}^n \mathbb{E}[X_i]\,\mathbb{E}[f(W_{i,m})] + \sum_{i=1}^n \sum_{j \in N(i)} \mathbb{E}\big[X_i X_j f'(W_{i,m})\big] - \mathbb{E}[f'(W)] \\
&= \sum_{i=1}^n \mathbb{E}\Big[X_i\big(f(W) - f(W_{i,m}) - f'(W_{i,m})(W - W_{i,m})\big)\Big] \\
&\quad + \Big(\sum_{i=1}^n \sum_{j \in N(i)} \mathbb{E}\big[X_i X_j f'(W_{i,m})\big] - \mathbb{E}[f'(W)]\Big) =: E_1 + E_2. \qquad \text{(B.2)}
\end{aligned}$$

By assumption, $\|f''\|$ is bounded and we have

$$\begin{aligned}
|E_1| &= \left|\sum_{i=1}^n \mathbb{E}\Big[X_i\big(f(W) - f(W_{i,m}) - f'(W_{i,m})(W - W_{i,m})\big)\Big]\right| \\
&\le \frac{\|f''\|}{2} \sum_{i=1}^n \mathbb{E}\Big[\big|X_i (W - W_{i,m})^2\big|\Big] = \frac{\|f''\|}{2} \sum_{i=1}^n \mathbb{E}\Big[|X_i|\Big(\sum_{j \in N(i)} X_j\Big)^2\Big] \\
&= \frac{\|f''\|}{2} \sum_{i=1}^n \sum_{j \in N(i)} \sum_{\ell \in N(i)} \mathbb{E}[|X_i X_j X_\ell|] \le \frac{\|f''\|}{2} \sum_{i=1}^n \sum_{j \in N(i)} \sum_{\ell \in N(i,j)} \mathbb{E}[|X_i X_j X_\ell|]. \qquad \text{(B.3)}
\end{aligned}$$

Now we bound E_2.

$$E_2 = \sum_{i=1}^n \sum_{j \in N(i)} \mathbb{E}\big[X_i X_j f'(W_{i,m})\big] - \mathbb{E}[f'(W)]$$

$$
\begin{aligned}
&=\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}\big[X_iX_j\big(f'(W_{i,m})-f'(W_{i,j,m})\big)\big]+\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}\big[X_iX_jf'(W_{i,j,m})\big]-\mathbb{E}[f'(W)]\\
&\overset{(*)}{=}\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}\big[X_iX_j\big(f'(W_{i,m})-f'(W_{i,j,m})\big)\big]+\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}[X_iX_j]\,\mathbb{E}[f'(W_{i,j,m})]-\mathbb{E}[f'(W)]\\
&\overset{(**)}{=}\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}\big[X_iX_j\big(f'(W_{i,m})-f'(W_{i,j,m})\big)\big]+\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}[X_iX_j]\,\mathbb{E}\big[f'(W_{i,j,m})-f'(W)\big] \quad \text{(B.4)}\\
&=(t_1)+(t_2), \quad \text{(B.5)}
\end{aligned}
$$

where to obtain $(*)$ we have used the fact that $W_{i,j,m}$ is independent of (X_i,X_j) in the second equation and to obtain $(**)$ we have assumed hat $\mathbb{E}(W^2)=1$.

The first term in (B.4), namely (t_1), can be upper-bounded by the mean value theorem as

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}\big[X_iX_j\big(f'(W_{i,m})-f'(W_{i,j,m})\big)\big]\right|\\
\le&\sum_{i=1}^{n}\sum_{j\in N(i)}\|f''\|\,\mathbb{E}\big[\big|X_iX_j(W_{i,m}-W_{i,j,m})\big|\big]\\
\le&\|f''\|\sum_{i=1}^{n}\sum_{j\in N(i)}\sum_{\ell\in N(i,j)}\mathbb{E}[|X_iX_jX_\ell|].
\end{aligned}
$$

By another application of the mean-value theorem, we remark that the second term in (B.4), namely (t_2), is controlled by

$$
\begin{aligned}
&\left|\sum_{i=1}^{n}\sum_{j\in N(i)}\mathbb{E}[X_iX_j]\,\mathbb{E}\big[f'(W_{i,j,m})-f'(W)\big]\right|\\
\le&\sum_{i=1}^{n}\sum_{j\in N(i)}\|f''\|\,\mathbb{E}[|X_iX_j|]\,\mathbb{E}\big[\big|W_{i,j,m}-W\big|\big]\\
\le&\|f''\|\sum_{i=1}^{n}\sum_{j\in N(i)}\sum_{\ell\in N(i,j)}\mathbb{E}[|X_iX_j|]\,\mathbb{E}[|X_\ell|].
\end{aligned}
$$

Thus,

$$
\big|\mathbb{E}\big[Wf(W)-f'(W)\big]\big|\le\|f''\|\sum_{i=1}^{n}\sum_{j\in N(i)}\sum_{\ell\in N(i,j)}\left(\frac{3}{2}\mathbb{E}[|X_iX_jX_\ell|]+\mathbb{E}[|X_iX_j|]\,\mathbb{E}[|X_\ell|]\right)\le\frac{3\|f''\|}{2}R_{1,1}.
$$

This gives us a bound that matches with (B.1).

For $k\ge 2$, we would like to carry out the expansion in the same spirit. However, it would be too tedious to write out every sum in the process. Thus, in Appendix B.2, we introduce the terms called $\mathcal{S}$-sums, $\mathcal{T}$-sums, and $\mathcal{R}$-sums, which serve as useful tools in tracking different quantities when we approximate $\mathbb{E}[f'(W) - Wf(W)]$ with respect to locally dependent random variables. Instead of performing the expansion to get (B.1) for $\mathbb{E}[Wf(W)]$, we first do it for any $\mathcal{T}$-sum and use induction to prove a more general result for the existence of such expansions (see Theorem B.3). In the general situation of $\mathcal{T}$-sums, the cumulants are replaced by other constants that only depend on the specific $\mathcal{T}$-sum in consideration and the joint distribution of $(X_i)_{i\in I}$. Finally, we prove that in particular, those constants for $\mathbb{E}[Wf(W)]$ are precisely the cumulants of W. This will be done by direct calculation when f is a polynomial and then extended to more general f's by applying Lemma B.4.

B.2 Notations and definitions

As in Section 3, given an integer $k \geq 1$, suppose $(X_i)_{i\in I}$ is a class of mean zero random variables indexed by I that satisfy the local dependence assumptions [LD-1] to [LD-k]. Without loss of generality, we always assume that $\sigma^2 := \mathrm{Var}\left(\sum_{i\in I} X_i\right) = 1$. We denote $W := \sigma^{-1}\sum_{i\in I} X_i = \sum_{i\in I} X_i$.

$\mathcal{S}$-sums

Fix $k \in \mathbb{N}_+$ and $t_1, \cdots, t_k \in \mathbb{Z}$ be integers such that $|t_j| \leq j-1$ for any $j \in [k]$. We set $t_1 = 0$. Let $z = \left|\{j : t_j > 0\}\right|$ be the number of indexes j for which t_j is strictly positive. If $z \geq 1$, we write $\{j : t_j > 0\} = \{q_1, \cdots, q_z\}$. Without loss of generality, we suppose that the sequence $2 \leq q_1 < \cdots < q_z \leq k$ is increasing. We further let $q_0 := 1$ and $q_{z+1} := k+1$. We define an order-k **$\mathcal{S}$-sum** with respect to the sequence $t_{1:k}$ as

$$\begin{aligned}\mathcal{S}[t_1, \cdots, t_k] &:= \sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_k\in N_k}[q_1-q_0, \cdots, q_{z+1}-q_z] \triangleright \left(X_{i_1}, \cdots, X_{i_k}\right) \\ &= \sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_k\in N_k}\mathbb{E}\left[X_{i_{q_0}}\cdots X_{i_{q_1-1}}\right]\mathbb{E}\left[X_{i_{q_1}}\cdots X_{i_{q_2-1}}\right]\cdots\mathbb{E}\left[X_{i_{q_z}}\cdots X_{i_{q_{z+1}-1}}\right], \quad \text{(B.6)}\end{aligned}$$

where $N_1 := I$, and for $j \in \mathbb{N}_+$ such that $j \geq 2$, we let

$$N_j := \begin{cases} N(i_{1:|t_j|}) = N(i_1, \cdots, i_{|t_j|}) & \text{if } t_j \neq 0 \\ \emptyset & \text{if } t_j = 0 \end{cases}.$$

Note that N_j depends on t_j and the sequence $i_{1:(j-1)}$. For ease of notation, we do not explicitly write out the dependencies on $i_{1:(j-1)}$ when there is no ambiguity. Further note that if any t_j, that is not t_1, is null then $N_j = \emptyset$ therefore, the $\mathcal{S}$-sum $\mathcal{S}[t_1, \cdots, t_k] = 0$.

By definition all $\mathcal{S}$-sums are deterministic quantities, the value of which only depends on $t_{1:k}$, and the joint distribution of $(X_i)_{i\in I}$. We also remark that the signs of t_j's determine how a $\mathcal{S}$-sum factorizes into different expectations. Notably if $z = 0$ (meaning that all the t_j are negative) then the $\mathcal{T}$-sum is

$$\mathcal{T}_{f,s}[t_1, \cdots, t_k] = \sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_k\in N_k}\mathbb{E}\left[X_{i_1}\cdots X_{i_k}\partial^{k-1}f\left(W_{i.}[k-s]\right)\right].$$

Since by assumption, X_i's are centered random variables, the $\mathcal{S}$-sum vanishes if $q_{j+1} = q_j + 1$ for some $0 \le j \le z$:

$$\mathcal{S}[t_1, \cdots, t_k] = \sum_{i_1 \in N_1} \sum_{i_2 \in N_2} \cdots \sum_{i_k \in N_k} \mathbb{E}\big[X_{i_{q_0}} \cdots X_{i_{q_1 - 1}}\big] \cdot \\ \mathbb{E}\big[X_{i_{q_1}} \cdots X_{i_{q_2-1}}\big] \cdots \mathbb{E}[X_{i_{q_j}}] \cdots \mathbb{E}\big[X_{i_{q_z}} \cdots X_{i_{q_{z+1}-1}}\big] = 0. \tag{B.7}$$

Furthermore, the absolute value of t_j's influences the range of running indexes. The bigger $|t_j|$ is the larger the set N_j is. The largest possible index set for i_{j+1} is $N(i_{1:(j-1)})$, which corresponds to the case $|t_j| = j-1$. On the other hand, if $t_j = 0$, the sum is over an empty set and vanishes. In particular, if we require that the $\mathcal{S}$-sum is not always zero, then t_2 is always taken to be -1 and $i_2 \in N(i_1)$.

$\mathcal{T}$-sums

For any function $f \in \mathcal{C}^{k-1}(\mathbb{R})$ and integer $s \in \mathbb{N}$ such that $s \le k$, the order-k **$\mathcal{T}$-sum**, with respect to the sequence $t_{1:k}$, is defined as

$$\begin{aligned} &\mathcal{T}_{f,s}[t_1, \cdots, t_k] := \\ &\sum_{i_1 \in N_1} \sum_{i_2 \in N_2} \cdots \sum_{i_k \in N_k} [q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \big(X_{i_1}, \cdots, X_{i_{k-1}}, X_{i_k} \partial^{k-1} f\big(W_{i.}[k-s]\big)\big) \\ &= \begin{cases} \sum_{i_1 \in N_1} \sum_{i_2 \in N_2} \cdots \sum_{i_k \in N_k} \mathbb{E}\Big[X_{i_1} \cdots X_{i_k} \partial^{k-1} f\big(W_{i.}[k-s]\big)\Big] & \text{if } z = 0 \\ \sum_{i_1 \in N_1} \sum_{i_2 \in N_2} \cdots \sum_{i_k \in N_k} \mathbb{E}\big[X_{i_{q_0}} \cdots X_{i_{q_1-1}}\big] \cdots \mathbb{E}\big[X_{i_{q_{(z-1)}}} \cdots X_{i_{q_z - 1}}\big] \cdot \\ \qquad \mathbb{E}\Big[X_{i_{q_z}} \cdots X_{i_k} \partial^{k-1} f\big(W_{i.}[k-s]\big)\Big] & \text{if } z \ge 1, \end{cases} \end{aligned} \tag{B.8}$$

where $N_{1:k}, z, q_{0:(z+1)}$ are defined as in the definition of $\mathcal{S}$-sums and $W_{i.}[j]$ is defined as

$$W_{i.}[j] := \begin{cases} W & \text{if } j = 0 \\ \sum_{i \in I \setminus N(i_{1:j})} X_i & \text{if } 1 \le j \le k \end{cases}.$$

Note that the bigger s is, the larger the set $I \setminus N(i_{1:(k-s)})$ is, which means that $W_{i.}[k-s]$ is the sum of more X_i's. Again we remark that the values of $\mathcal{T}$-sums can depend on the values of s and the sequences $t_{1:k}$. In particular, if $s = 0$, then we have $W_{i.}[k-s] = W_{i.}[k] = \sum_{i \in I \setminus N(i_{1:k})} X_i$, which implies that $W_{i.}[k-s]$ is independent of $X_{i_1}, \cdots, X_{i_k}$ by the assumption **[LD-k]**. Thus, we have

$$\mathbb{E}\big[X_{i_{q_z}} \cdots X_{i_k} \partial^{k-1} f\big(W_{i.}[k-s]\big)\big] = \mathbb{E}[X_{i_{q_z}} \cdots X_{i_k}]\, \mathbb{E}\big[\partial^{k-1} f\big(W_{i.}[k-s]\big)\big].$$

By definitions (B.6) and (B.8) we get

$$\mathcal{T}_{f,0}[t_1, \cdots, t_k] = \mathcal{S}[t_1, \cdots, t_k]\, \mathbb{E}[\partial^{k-1} f(W_{i.}[k])]. \tag{B.9}$$

This equation will be useful in our discussion later. In general if $z > 0$ then

$$\mathcal{T}_{f,s}[t_1, \cdots, t_k] = \mathcal{S}[t_1, \cdots, t_{q_z - 1}] \sum_{i_{q_z} \in N_{q_z}} \sum_{i_{q_z+1} \in N_{q_z+1}} \cdots \sum_{i_k \in N_k} \mathbb{E}\big[X_{i_{q_z}} \cdots X_{i_k} \partial^{k-1} f\big(W_{i.}[k-s]\big)\big]. \tag{B.10}$$

$\mathcal{R}$-sums

For $k \geq 2$ and given a real number $\omega \in (0,1]$, we further define an order-k $\mathcal{R}$**-sum** with respect to the sequence $t_{1:k}$ as

$$
\begin{aligned}
&\mathcal{R}_\omega[t_1,\cdots,t_k] := \\
&\quad \sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k-1}\in N_{k-1}}[q_1-q_0,\cdots,q_{z+1}-q_z]\triangleright\Big(|X_{i_1}|,\cdots,|X_{i_{k-1}}|,\big(\sum_{i_k\in N_k}|X_{i_k}|\big)^\omega\Big) \\
&= \begin{cases}
\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_k\in N_k}\mathbb{E}\Big[X_{i_1}\cdots X_{i_{k-1}}\big(\sum_{i_k\in N_k}|X_{i_k}|\big)^\omega\Big] & \text{if } z=0 \\
\displaystyle\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_k\in N_k}\mathbb{E}\big[X_{i_{q_0}}\cdots X_{i_{q_1-1}}\big]\cdots\,\mathbb{E}\big[X_{i_{q_{(z-1)}}}\cdots X_{i_{q_z-1}}\big]\cdot & \\
\qquad\mathbb{E}\Big[X_{i_{q_z}}\cdots X_{i_{k-1}}\big(\sum_{i_k\in N_k}|X_{i_k}|\big)^\omega\Big] & \text{if } z\geq 1
\end{cases}
\end{aligned}
\tag{B.11}
$$

We again remark that if $z \geq 1$ then

$$
\mathcal{R}_\omega[t_1,\cdots,t_k] = \mathcal{R}_1[t_1,\cdots,t_{q_z-1}]\sum_{i_{q_z}\in N_{q_z}}\sum_{i_{q_z+1}\in N_{q_z+1}}\cdots\sum_{i_k\in N_k}\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_{k-1}}\big(\sum_{i_k\in N_k}|X_{i_k}|\big)^\omega\big].
$$

We call ω the exponent of the $\mathcal{R}$-sum. If $\omega = 1$, the only difference between a $\mathcal{R}$-sum and a $\mathcal{S}$-sum is that the X_{i_j}'s in (B.6) are replaced by $|X_{i_j}|$'s in (B.11). Thus, a $\mathcal{S}$-sum is always upper-bounded by the corresponding compositional 1-sum, i.e.,

$$
\Big|\mathcal{S}[t_1,\cdots,t_k]\Big| \leq \mathcal{R}_1[t_1,\cdots,t_k]. \tag{B.12}
$$

Another important observation is that we can compare the values of $\mathcal{R}$-sums with respect to two different sequences $t_1,\cdots,t_k$ and $t_1',\cdots,t_k'$ in certain situations. In specific, if for any $j \in [k]$ we have that if t_j and t_j' are of the same sign and $|t_j| \leq |t_j'|$, then

$$
\mathcal{R}_\omega[t_1,\cdots,t_k] \leq \mathcal{R}_\omega[t_1',\cdots,t_k']. \tag{B.13}
$$

In fact, the sequences (t_j) and (t_j') having the same sign indicates that $\{j : t_j > 0\} = \{j : t_j' > 0\}$. Thus, we can write

$$
\mathcal{R}_\omega[t_1',\cdots,t_k'] = \sum_{i_1\in N_1'}\sum_{i_2\in N_2'}\cdots\sum_{i_{k-1}\in N_{k-1}'}[q_1-q_0,\cdots,q_{z+1}-q_z]\triangleright\Big(|X_{i_1}|,\cdots,|X_{i_{k-1}}|,\big(\sum_{i_k\in N_k'}|X_{i_k}|\big)^\omega\Big), \tag{B.14}
$$

where we note that $N_1' = I = N_1$ and for $j = 2,\cdots,k$ we have

$$
N_j' = N(i_1,\cdots,i_{|t_j'|}) \supseteq N(i_1,\cdots,i_{|t_j|}) = N_j.
$$

By comparing (B.11) with (B.14), we obtain (B.13).

Re-expression of the remainder terms $R_{k,\omega}$

Using the notion of $\mathcal{R}$-sums, we rewrite the $R_{k,\omega}$ in Section 3 as

$$R_{k,\omega} := \sum_{(\ell,\eta_{1:\ell})\in C^*(k+2)} \sum_{i_1\in N_1'} \sum_{i_2\in N_2'} \cdots \sum_{i_{k+1}\in N_{k+1}'} [\eta_1,\cdots,\eta_\ell] \triangleright \Big(|X_{i_1}|,\cdots,|X_{i_{k+1}}|, \Big(\sum_{i_{k+2}\in N_{k+2}'} |X_{i_{k+2}}|\Big)^\omega \Big) \tag{B.15}$$

$$= \sum_{t_{1:(k+2)}\in\mathcal{M}_{1,k+2}} \mathcal{R}_\omega[t_1,t_2,\cdots,t_{k+2}]. \tag{B.16}$$

where $N_1' := I$ and $N_j' := N(i_{1:(j-1)})$ for $j \ge 2$. $C^*(k+2)$ and $\mathcal{M}_{1,k+2}$ are given by

$$C^*(k+2) = \Big\{\ell,\eta_{1:\ell} \in \mathbb{N}_+ : \eta_j \ge 2 \ \forall j \in [\ell-1],\ \sum_{j=1}^{\ell} \eta_j = k+2\Big\},$$

and

$$\mathcal{M}_{1,k+2} := \Big\{ t_{1:(k+2)} :\ t_{j+1} = \pm j \ \ \&\ \ t_j \wedge t_{j+1} < 0 \ \ \forall 1 \le j \le k+1 \Big\}.$$

Note that $t_j \wedge t_{j+1} < 0$ for any $j \in [k+1]$ means that there is at least one -1 in any two consecutive signs, which corresponds to the requirement that $\eta_j \ge 2$ for $j \in [\ell-1]$ in (B.15).

B.3 Proofs of Proposition B.1 and Lemma A.7

In this section, we carry out the local expansion technique and prove Proposition B.1 and Lemma A.7.

Firstly, we establish the following lemma, which will be crucial in the inductive step of proving the main theorem.

Lemma B.2. *Fix $k \in \mathbb{N}_+$. For any $s \in [k] \cup \{0\}$ and $f \in \mathcal{C}^{k,\omega}(\mathbb{R})$, we have*

$$\begin{aligned}&\Big|\mathcal{T}_{f,s}[t_1,\cdots,t_{k+1}] - \mathcal{S}[t_1,\cdots,t_{k+1}]\,\mathbb{E}[\partial^k f(W)]\Big| \\ \le& |f|_{k,\omega}\Big(-(\mathbb{I}(t_{k+1}<0)\cdot\mathcal{R}_\omega[t_1,\cdots,t_{k+1},k+1] + \mathbb{I}(s\ge 1)\mathcal{R}_\omega[t_1,\cdots,t_{k+1},-(k+1)]\Big).\end{aligned} \tag{B.17}$$

Given any $\ell \in [k]$ and $s \in [\ell] \cup \{0\}$, we further have

$$\begin{aligned}&\Bigg|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell] - \mathcal{S}[t_1,\cdots,t_\ell]\,\mathbb{E}[\partial^{\ell-1} f(W)] \\ &- \mathbb{I}(s\ge 1)\cdot \sum_{j=1}^{k-\ell+1}\sum_{h=0}^{j} (-1)^h \frac{1}{h!(j-h)!} \mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h \text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h) \text{ times}}] \\ &- (\mathbb{I}(t_\ell<0) \sum_{j=1}^{k-\ell+1} \frac{1}{j!} \mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(j-1) \text{ times}}]\Bigg| \\ \le& \frac{|f|_{k,\omega}}{(k-\ell+1)!}\Big(-\mathbb{I}(t_\ell<0)\cdot\mathcal{R}_\omega[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+1) \text{ times}}] + \mathbb{I}(s\ge1)\cdot\mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2) \text{ times}}]\Big).\end{aligned} \tag{B.18}$$

Proof. Firstly, we remark that the definition of Hölder continuity implies that

$$\left|\partial^k f(y)-\partial^k f(x)\right| \le |f|_{k,\omega}|y-x|^{\omega}, \tag{B.19}$$

where ω is the Hölder exponent of f and $|f|_{k,\omega}$ is the Hölder constant (see Definition A.1). Let $z=\left|\{j\in[k+1]:t_j>0\}\right|$ be the number of positive indexes (t_j). If $z\ge 1$, we write $\{j\in[k+1]:t_j>0\}=\{q_1,\cdots,q_z\}$. Without loss of generality, we suppose that the sequence $2\le q_1<\cdots<q_z\le k+1$ is increasing. We further let $q_0:=1$ and $q_{z+1}:=k+2$. Applying (B.19) we have

$$\begin{aligned}
&\left|\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_{k+1}}\partial^k f\big(W_{i.}[k+1-s]\big)\big]-\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_{k+1}}\partial^k f\big(W_{i.}[k+1]\big)\big]\right| \qquad \text{(B.20)}\\
\le&|f|_{k,\omega}\mathbb{E}\Big[\left|X_{i_{q_z}}\cdots X_{i_{k+1}}\right|\cdot\left|W_{i.}[k+1-s]-W_{i.}[k+1]\right|^{\omega}\Big]\\
\le&|f|_{k,\omega}\mathbb{E}\Bigg[\left|X_{i_{q_z}}\cdots X_{i_{k+1}}\right|\cdot\Bigg|\sum_{i\in N(i_{1:(k+1)})\backslash N(i_{1:(k+1-s)})}X_i\Bigg|^{\omega}\Bigg]\\
\le&|f|_{k,\omega}\mathbb{E}\Bigg[\left|X_{i_{q_z}}\cdots X_{i_{k+1}}\right|\cdot\Bigg|\sum_{i\in N(i_{1:(k+1)})}X_i\Bigg|^{\omega}\Bigg],
\end{aligned}$$

where in the last inequality we have used the fact that $N(i_{1:(k+1)})\backslash N(i_{1:(k+1-s)})\subseteq N(i_{1:(k+1)})$. If $z=0$, this directly implies that

$$\left|\mathcal{T}_{f,s}[t_1,\cdots,t_{k+1}]-\mathcal{T}_{f,0}[t_1,\cdots,t_{k+1}]\right|\le\mathbb{I}(s\ge 1)\cdot|f|_{k,\omega}\mathcal{R}_{\omega}[t_1,\cdots,t_{k+1},-(k+1)]. \tag{B.21}$$

If $z\ge 1$, by definition (B.8) we have for $s\ge 1$

$$\begin{aligned}
&\left|\mathcal{T}_{f,s}[t_1,\cdots,t_{k+1}]-\mathcal{T}_{f,0}[t_1,\cdots,t_{k+1}]\right|\\
=&\Bigg|\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k+1}\in N_{k+1}}\mathbb{E}\big[X_{i_{q_0}}\cdots X_{i_{q_1-1}}\big]\cdots\mathbb{E}\big[X_{i_{q_{z-1}}}\cdots X_{i_{q_z-1}}\big]\cdot\\
&\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_{k+1}}\big(\partial^k f(W_{i.}[k+1-s])-\partial^k f(W_{i.}[k+1])\big)\big]\Bigg|\\
\le&\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k+1}\in N_{k+1}}\mathbb{E}\big[\big|X_{i_{q_0}}\cdots X_{i_{q_1-1}}\big|\big]\cdots\mathbb{E}\big[\big|X_{i_{q_{z-1}}}\cdots X_{i_{q_z-1}}\big|\big]\cdot\\
&\left|\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_{k+1}}\partial^k f(W_{i.}[k+1-s])-\partial^k f(W_{i.}[k+1])\big]\right|\\
\overset{\text{(B.20)}}{\le}&|f|_{k,\omega}\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k+1}\in N_{k+1}}\mathbb{E}\big[\big|X_{i_{q_0}}\cdots X_{i_{q_1-1}}\big|\big]\cdots\mathbb{E}\big[\big|X_{i_{q_{z-1}}}\cdots X_{i_{q_z-1}}\big|\big]\cdot\\
&\mathbb{E}\Bigg[\left|X_{i_{q_z}}\cdots X_{i_{k+1}}\right|\cdot\Bigg|\sum_{i\in N(i_{1:(k+1)})}X_i\Bigg|^{\omega}\Bigg]\\
=&|f|_{k,\omega}\mathcal{R}_{\omega}[t_1,\cdots,t_{k+1},-(k+1)].
\end{aligned}$$

Here the last equality is due to the definition (B.11). Thus, (B.21) is proven for both $z=0$ and $z\ge 1$. Next we show that

$$\left|\mathcal{S}[t_1,\cdots,t_{k+1}]\big(\mathbb{E}[\partial^k f(W)]-\mathbb{E}[\partial^k f(W_{i.}[k+1])]\big)\right|\le-\mathbb{I}(t_{k+1}<0)|f|_{k,\omega}\mathcal{R}_{\omega}[t_1,\cdots,t_{k+1},k+1]. \tag{B.22}$$

In this goal, we first note that if $t_{k+1} \geq 0$, by definition (B.6) we know that $q_z = k+1$ and therefore, according to (B.7) we know that

$$\mathcal{S}[t_1, \cdots, t_{k+1}] = 0,$$

and so (B.22) holds. Otherwise, we note that we have

$$\begin{aligned}
&\Big|\mathbb{E}\big[\partial^k f\big(W\big)\big] - \mathbb{E}\big[\partial^k f\big(W_{i.}[k+1]\big)\big]\Big| \leq |f|_{k,\omega}\mathbb{E}\Big[\Big|W_{i.}[k+1-s] - W_{i.}[k+1]\Big|^{\omega}\Big] \\
\leq &|f|_{k,\omega}\mathbb{E}\Big[\Big|\sum_{i\in N(i_{1:(k+1)})\backslash N(i_{1:(k+1-s)})} X_i\Big|^{\omega}\Big] \leq |f|_{k,\omega}\mathbb{E}\Big[\Big|\sum_{i\in N(i_{1:(k+1)})} X_i\Big|^{\omega}\Big].
\end{aligned} \tag{B.23}$$

This implies that

$$\begin{aligned}
&\Big|\mathcal{S}[t_1, \cdots, t_{k+1}]\big(\mathbb{E}[\partial^k f(W)] - \mathbb{E}[\partial^k f(W_{i.}[k+1])]\big)\Big| \\
\leq &\Big|\mathcal{S}[t_1, \cdots, t_{k+1}]\Big| \cdot \Big|\mathbb{E}\big[\partial^k f\big(W\big)\big] - \mathbb{E}\big[\partial^k f\big(W_{i.}[k+1]\big)\big]\Big| \\
\overset{(*)}{\leq} &|f|_{k,\omega}\mathcal{R}_1[t_1, \cdots, t_{k+1}]\, \mathbb{E}\Big[\Big|\sum_{i\in N(i_{1:(k+1)})} X_i\Big|^{\omega}\Big] \\
= &|f|_{k,\omega}\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k+1}\in N_{k+1}} [q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \big(|X_{i_1}|, \cdots, |X_{i_{k+1}}|\big)\, \mathbb{E}\Big[\Big|\sum_{i\in N(i_{1:(k+1)})} X_i\Big|^{\omega}\Big] \\
= &|f|_{k,\omega}\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k+1}\in N_{k+1}} [q_1 - q_0, \cdots, q_{z+1} - q_z, 1] \triangleright \Big(|X_{i_1}|, \cdots, |X_{i_{k+1}}|, \big(\sum_{i_{k+2}\in N(i_{1:(k+1)})} |X_{i_{k+2}}|\big)^{\omega}\Big) \\
= &|f|_{k,\omega}\mathcal{R}_{\omega}[t_1, \cdots, t_{k+1}, k+1].
\end{aligned}$$

where $(*)$ is due to (B.12) and (B.23). Taking the difference of (B.21) and (B.22), we obtain (B.17) by applying the equation (B.9).

For $\ell \leq k$, we apply the Taylor expansion with remainders taking the integral form and obtain that

$$\begin{aligned}
&\partial^{\ell-1} f(y) - \partial^{\ell-1} f(x) = \sum_{j=1}^{m-\ell} \frac{1}{j!}(y-x)^j \partial^{\ell-1+j} f(x) \\
&\quad + \frac{1}{(k-\ell+1)!}(y-x)^{k-\ell+1}\int_0^1 (k-\ell+1)v^{k-\ell}\partial^k f(vx + (1-v)y)\,\mathrm{d}v \\
\overset{(*)}{=} &\sum_{j=1}^{k-\ell+1} \frac{1}{j!}(y-x)^j \partial^{\ell-1+j} f(x) \\
&\quad + \frac{1}{(k-\ell+1)!}(y-x)^{k-\ell+1}\int_0^1 (k-\ell+1)v^{k-\ell}\big(\partial^k f(vx + (1-v)y) - \partial^k f(x)\big)\,\mathrm{d}v,
\end{aligned} \tag{B.24}$$

where to obtain $(*)$ we added and subtracted $\frac{(y-x)^{k-\ell+1}}{(k-\ell+1)!}\partial^k f(x)$. Moreover, using the fact that $\partial^k f(\cdot)$ is assumed to be Hölder continuous we obtain that

$$\Big|\partial^k f(vx + (1-v)y) - \partial^k f(x)\Big| \leq |f|_{k,\omega}(1-v)^{\omega}\ |y-x|^{\omega} \leq |f|_{k,\omega}|y-x|^{\omega}. \tag{B.25}$$

Therefore, as $\int_0^1 (k-\ell+1)v^{k-\ell}\,\mathrm{d}v = 1$, by combining (B.25) with (B.24) we get that

$$\left|\partial^{\ell-1}f(y)-\partial^{\ell-1}f(x)-\sum_{j=1}^{k-\ell+1}\frac{1}{j!}(y-x)^j\partial^{\ell-1+j}f(x)\right| \le \frac{|f|_{k,\omega}}{(k-\ell+1)!}|y-x|^{k-\ell+1+\omega}. \tag{B.26}$$

We prove that the following inequality holds:

$$\begin{aligned}&\Bigg|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\mathcal{T}_{f,0}[t_1,\cdots,t_\ell]\\ &\quad-\mathbb{I}(s\ge 1)\cdot\sum_{j=1}^{k-\ell+1}\sum_{h=0}^{j}(-1)^h\frac{1}{h!(j-h)!}\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h\text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h)\text{ times}}]\Bigg|\\ &\le\frac{\mathbb{I}(s\ge 1)\cdot|f|_{k,\omega}}{(k-\ell+1)!}\mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2)\text{ times}}],\end{aligned} \tag{B.27}$$

First, let's establish (B.27). Let $z=\big|\{j\in[\ell]:t_j>0\}\big|$. If $z\ge 1$, we write $\{j\in[\ell]:t_j>0\}=\{q_1,\cdots,q_z\}$. Without loss of generality, we suppose that the sequence $2\le q_1<\cdots<q_z\le\ell$ is increasing. We further let $q_0:=1$ and $q_{z+1}:=\ell+1$. Applying (B.26) we have

$$\begin{aligned}&\Big|\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W_{i.}[\ell-s]\big)\big]-\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W_{i.}[\ell]\big)\big]\\ &\quad-\sum_{j=1}^{k-\ell+1}\frac{1}{j!}\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}(W_{i.}[\ell-s]-W_{i.}[\ell])^j\partial^{\ell-1+j}f(W_{i.}[\ell])\big]\Big|\\ &\le\frac{|f|_{k,\omega}}{(k-\ell+1)!}\mathbb{E}\Big[\big|X_{i_{q_z}}\cdots X_{i_\ell}\big|\cdot\big|W_{i.}[\ell-s]-W_{i.}[\ell]\big|^{k-\ell+1+\omega}\Big].\end{aligned} \tag{B.28}$$

For convenience let

$$\begin{aligned}E_1&:=\sum_{i_{q_z}\in N_{q_z}}\cdots\sum_{i_\ell\in N_\ell}\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W_{i.}[\ell-s]\big)\big]-\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W_{i.}[\ell]\big)\big],\\ E_{2,j}&:=\sum_{i_{q_z}\in N_{q_z}}\cdots\sum_{i_\ell\in N_\ell}\mathbb{E}\big[X_{i_{q_z}}\cdots X_{i_\ell}(W_{i.}[\ell-s]-W_{i.}[\ell])^j\partial^{\ell-1+j}f(W_{i.}[\ell])\big],\\ E_3&:=\sum_{i_{q_z}\in N_{q_z}}\cdots\sum_{i_\ell\in N_\ell}\mathbb{E}\Big[\big|X_{i_{q_z}}\cdots X_{i_\ell}\big|\cdot\big|W_{i.}[\ell-s]-W_{i.}[\ell]\big|^{k-\ell+1+\omega}\Big].\end{aligned}$$

Then (B.28) reduces to

$$\Big|E_1-\textstyle\sum_{j=1}^{k-\ell+1}E_{2,j}/j!\Big|\le|f|_{k,\omega}E_3/(k-\ell+1)!. \tag{B.29}$$

Then we obersve that by definition of $W_{i.}[\cdot]$ we have

$$\begin{aligned}
&\mathbb{E}\Big[X_{i_{q_z}}\cdots X_{i_\ell}(W_{i.}[\ell-s]-W_{i.}[\ell])^j\partial^{\ell-1+j}f(W_{i.}[\ell])\Big]\\
=&\mathbb{E}\Bigg[X_{i_{q_z}}\cdots X_{i_\ell}\Big(\sum_{i\in N(i_{1:\ell})}X_i-\sum_{i\in N(i_{1:\ell-s})}X_i\Big)^j\partial^{\ell-1+j}f(W_{i.}[\ell])\Bigg]\\
=&\sum_{h=0}^{j}(-1)^h\binom{j}{h}\,\mathbb{E}\Bigg[X_{i_{q_z}}\cdots X_{i_\ell}\Big(\sum_{i\in N(i_{1:\ell-s})}X_i\Big)^h\Big(\sum_{i\in N(i_{1:\ell})}X_i\Big)^{j-h}\partial^{\ell-1+j}f(W_{i.}[\ell])\Bigg],
\end{aligned}\tag{B.30}$$

and that

$$\begin{aligned}
&\mathbb{E}\Big[\big|X_{i_{q_z}}\cdots X_{i_\ell}\big|\cdot\big|W_{i.}[\ell-s]-W_{i.}[\ell]\big|^{k-\ell+1+\omega}\Big]\\
\le&\mathbb{E}\Bigg[\big|X_{i_{q_z}}\cdots X_{i_{k+1}}\big|\cdot\Big|\sum_{i\in N(i_{1:(k+1)})\setminus N(i_{1:(k+1-s)})}X_i\Big|^{k-\ell+1+\omega}\Bigg]\\
\le&\mathbb{E}\Bigg[\big|X_{i_{q_z}}\cdots X_{i_{k+1}}\big|\cdot\Big|\sum_{i\in N(i_{1:(k+1)})}X_i\Big|^{k-\ell+1+\omega}\Bigg]\\
\le&\mathbb{E}\Bigg[\big|X_{i_{q_z}}\cdots X_{i_{k+1}}\big|\cdot\Big(\sum_{i\in N(i_{1:(k+1)})}|X_i|\Big)^{k-\ell+1}\cdot\Big|\sum_{i\in N(i_{1:(k+1)})}X_i\Big|^{\omega}\Bigg].
\end{aligned}\tag{B.31}$$

If $z=0$, we take the sum of (B.30) or (B.31) over $i_{q_z}\in N_{q_z},\cdots,i_\ell\in N_\ell$. By definition (B.8) and (B.11) we have

$$\begin{aligned}
&E_1=\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\mathcal{T}_{f,0}[t_1,\cdots,t_\ell],\\
&E_{2,j}=\sum_{h=0}^{j}(-1)^h\binom{j}{h}\,\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h\text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h)\text{ times}}],\\
&E_3\le\mathcal{R}_\omega[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+1)\text{ times}}].
\end{aligned}\tag{B.32}$$

Combining (B.32) and (B.29), we have for $s\ge1$

$$\begin{aligned}
&\Bigg|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\mathcal{T}_{f,0}[t_1,\cdots,t_\ell]\\
&\quad-\sum_{j=1}^{k-\ell+1}\sum_{h=0}^{j}(-1)^h\frac{1}{h!(j-h)!}\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h\text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h)\text{ times}}]\Bigg|\\
\overset{\text{(B.32)}}{=}&\Big|E_1-\sum_{j=1}^{k-\ell+1}E_{2,j}/j!\Big|\overset{\text{(B.29)}}{\le}|f|_{k,\omega}E_3/(k-\ell+1)!\\
\overset{\text{(B.32)}}{\le}&\frac{|f|_{k,\omega}}{(k-\ell+1)!}\mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2)\text{ times}}].
\end{aligned}$$

Thus, (B.27) holds for $z=0$.

If $z \geq 1$, similar to (B.31) we have

$$\begin{aligned}
&\mathcal{S}[t_1,\cdots,t_{q_z-1}]\cdot E_1 = \mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\mathcal{T}_{f,0}[t_1,\cdots,t_\ell],\\
&\mathcal{S}[t_1,\cdots,t_{q_z-1}]\cdot E_{2,j} = \sum_{h=0}^{j}(-1)^h\binom{j}{h}\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h\text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h)\text{ times}}],\\
&\mathcal{R}_1[t_1,\cdots,t_{q_z-1}]\cdot E_3 \leq \mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2)\text{ times}}].
\end{aligned} \tag{B.33}$$

Combining (B.33) and (B.29) we get for $s \geq 1$

$$\begin{aligned}
&\Bigg|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\mathcal{T}_{f,0}[t_1,\cdots,t_\ell]\\
&\quad-\sum_{j=1}^{k-\ell+1}\sum_{h=0}^{j}(-1)^h\frac{1}{h!(j-h)!}\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\underbrace{s-\ell,\cdots,s-\ell}_{h\text{ times}},\underbrace{-\ell,\cdots,-\ell}_{(j-h)\text{ times}}]\Bigg|\\
&\overset{(B.33)}{=}\big|\mathcal{S}[t_1,\cdots,t_{q_z-1}]\big|\cdot\Big|E_1-\sum_{j=1}^{k-\ell+1}E_{2,j}/j!\Big| \overset{(B.12)}{\leq} \mathcal{R}_1[t_1,\cdots,t_{q_z-1}]\cdot\Big|E_1-\sum_{j=1}^{k-\ell+1}E_{2,j}/j!\Big|\\
&\overset{(B.29)}{\leq}\mathcal{R}_1[t_1,\cdots,t_{q_z-1}]\cdot|f|_{k,\omega}E_3/(k-\ell+1)! \overset{(B.33)}{\leq}\frac{|f|_{k,\omega}}{(k-\ell+1)!}\mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2)\text{ times}}].
\end{aligned}$$

Thus, we have shown (B.27) for both $z=0$ and $z\geq 1$.

Next we prove that the following inequality holds:

$$\begin{aligned}
&\Bigg|\mathcal{S}[t_1,\cdots,t_\ell]\big(\mathbb{E}[\partial^{\ell-1}f(W)]-\mathbb{E}[\partial^{\ell-1}f(W_{i.}[\ell])]\big)\\
&\quad-\mathbb{I}(t_\ell<0)\cdot\sum_{j=1}^{k-\ell+1}\frac{1}{j!}\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(j-1)\text{ times}}]\Bigg|\\
&\leq\frac{\mathbb{I}(t_\ell<0)\cdot|f|_{k,\omega}}{(k-\ell+1)!}\mathcal{R}_\omega[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+1)\text{ times}}].
\end{aligned} \tag{B.34}$$

For (B.34), we apply (B.26) again and get that

$$\begin{aligned}
&\Bigg|\mathbb{E}\big[\partial^k f(W)\big]-\mathbb{E}\big[\partial^k f(W_{i.}[\ell])\big]\\
&-\sum_{j=1}^{k-\ell+1}\frac{1}{j!}\mathbb{E}\big[(W_{i.}-W_{i.}[\ell])^j\partial^{\ell-1+j}f(W_{i.}[\ell])\big]\Bigg| \leq \frac{|f|_{k,\omega}}{(k-\ell+1)!}\mathbb{E}\Big[\big|W-W_{i.}[\ell]\big|^{k-\ell+1+\omega}\Big].
\end{aligned} \tag{B.35}$$

For convenience let

$$E_4 := \mathbb{E}[\partial^{\ell-1}f(W)]-\mathbb{E}[\partial^{\ell-1}f(W_{i.}[\ell])],$$

$$E_{5,j} := \mathbb{E}\big[(W_{i.} - W_{i.}[\ell])^j \partial^{\ell-1+j} f(W_{i.}[\ell])\big],$$
$$E_6 := \mathbb{E}\big[\big|W - W_{i.}[\ell]\big|^{k-\ell+1+\omega}\big].$$

Then (B.35) reduces to

$$\Big|E_4 - \textstyle\sum_{j=1}^{k-\ell+1} E_{5,j}/j!\Big| \leq |f|_{k,\omega} E_6/(k-\ell+1)!. \tag{B.36}$$

We first note that if $t_\ell \geq 0$ then $\mathcal{S}[t_1, \cdots, t_\ell] = 0$ therefore, (B.34) holds. Moreover, similar to (B.33), we have for $t_\ell < 0$

$$\begin{aligned}
&\mathcal{S}[t_1, \cdots, t_\ell] \cdot E_4 = \mathcal{S}[t_1, \cdots, t_\ell]\big(\mathbb{E}[\partial^{\ell-1} f(W)] - \mathbb{E}[\partial^{\ell-1} f(W_{i.}[\ell])]\big),\\
&\mathcal{S}[t_1, \cdots, t_\ell] \cdot E_{5,j} = \mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(j-1) \text{ times}}],\\
&\mathcal{R}_1[t_1, \cdots, t_\ell] \cdot E_6 \leq \mathcal{R}_\omega[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(k-\ell+1) \text{ times}}].
\end{aligned} \tag{B.37}$$

Combining (B.37) and (B.36), we have

$$\begin{aligned}
&\Bigg|\mathcal{S}[t_1, \cdots, t_\ell]\big(\mathbb{E}[\partial^{\ell-1} f(W)] - \mathbb{E}[\partial^{\ell-1} f(W_{i.}[\ell])]\big) - \sum_{j=1}^{k-\ell+1} \frac{1}{j!} \mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(j-1) \text{ times}}]\Bigg|\\
&\overset{\text{(B.37)}}{=} \big|\mathcal{S}[t_1, \cdots, t_\ell]\big| \cdot \Big|E_4 - \sum_{j=1}^{k-\ell+1} E_{5,j}/j!\Big| \overset{\text{(B.12)}}{\leq} \mathcal{R}_1[t_1, \cdots, t_\ell] \cdot \Big|E_4 - \sum_{j=1}^{k-\ell+1} E_{5,j}/j!\Big|\\
&\overset{\text{(B.36)}}{\leq} \mathcal{R}_1[t_1, \cdots, t_\ell] \cdot |f|_{k,\omega} E_6/(k-\ell+1)! \overset{\text{(B.37)}}{\leq} \frac{|f|_{k,\omega}}{(k-\ell+1)!} \mathcal{R}_\omega[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(k-\ell+1) \text{ times}}].
\end{aligned}$$

Therefore, we have established both (B.27) and (B.34). Taking their difference and applying (B.9), we obtain (B.18).

■

Equipped with the tools in Lemma B.2, we approximate any $\mathcal{T}$-sum $\mathcal{T}_{f,s}[t_1, \cdots, t_\ell]$ by order-j $\mathcal{S}$-sums ($j = \ell, \cdots, k+1$) with remainder terms being order-$(k+2)$ $\mathcal{R}$-sums.

Theorem B.3. *Fix $k \in \mathbb{N}_+$. For any $\ell \in [k+1], s \in [\ell] \cup \{0\}$, and $t_1, \cdots, t_\ell \in \mathbb{Z}$ such that $|t_j| \leq j-1$ for any $j \in [\ell]$, there exist $Q_\ell, \cdots, Q_{k+1}$ (which depend on s and $t_{1:\ell}$ and the joint distribution of $(X_i)_{i\in I}$) and a constant $C_{k,\ell}$ ($C_{k,\ell} \leq 4^{k-\ell+1}$) such that for any $f \in \mathcal{C}^{k,\omega}(\mathbb{R})$, we have*

$$\Bigg|\mathcal{T}_{f,s}[t_1, \cdots, t_\ell] - \sum_{j=\ell}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)]\Bigg| \leq C_{k,\ell} |f|_{k,\omega} R_{k,\omega}. \tag{B.38}$$

Note that by (B.16) *$R_{k,\omega}$ is given as*

$$R_{k,\omega} = \sum_{t_{1:(k+2)} \in \mathcal{M}_{1,k+2}} \mathcal{R}_\omega[t_1, t_2, \cdots, t_{k+2}].$$

where

$$\mathcal{M}_{1,k+2} := \Big\{t_{1:(k+2)} : \ t_{j+1} = \pm j \ \ \& \ \ t_j \wedge t_{j+1} < 0 \ \ \forall 1 \leq j \leq k+1\Big\}.$$

Proof. If there exists an integer $2 \le j \le \ell$ such that $t_j = 0$ or there exists $j \in [\ell-1]$ such that t_j and t_{j+1} are both positive, then $\mathcal{T}_{f,s}[t_1, \cdots, t_\ell] = 0$ by definition and the theorem already holds by setting $Q_j = \cdots = Q_{k+1} = 0$.

Otherwise, we claim:

Claim. *Let $\mathcal{T}_{f,s}[t_1, \cdots, t_\ell]$ be a $\mathcal{T}$-sum. For any $j = \ell+1, \cdots, k+1$, let*

$$\mathcal{E}_{\ell+1,j} := \left\{ t_{(\ell+1):j} : \ |t_{h+1}| \le h \ \& \ t_h \wedge t_{h+1} \ \ \forall \ell \le h \le j-1 \right\}.$$

For all $j = \ell+1, \cdots, k+1$, $\nu \in [j] \cup \{0\}$, and $(t_{\ell+1}, \cdots, t_j) \in \mathcal{E}_{\ell,j}$, there are coefficients $a_{j,\nu,t_{(\ell+1):j}}$ (additionally depending on s) such that if we write

$$Q_j = \sum_{t_{(l+1):j} \in \mathcal{E}_{\ell,j}} \sum_{\nu=0}^{j} a_{j,\nu,t_{(\ell+1):j}} \mathcal{T}_{f,\nu}[t_1, \cdots, t_\ell, t_{\ell+1}, \cdots, t_j], \tag{B.39}$$

then the following holds

$$\left| \mathcal{T}_{f,s}[t_1, \cdots, t_\ell] - \sum_{j=\ell}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)] \right| \le 4^{k-\ell+1} |f|_{k,\omega} \sum_{t_{(\ell+1):(k+2)} \in \mathcal{M}_{\ell,k+1}} \mathcal{R}_\omega\big[t_1, \cdots, t_\ell, \cdots, t_{k+2}\big], \tag{B.40}$$

where

$$\mathcal{M}_{\ell+1,k+2} := \left\{ t_{(\ell+1):(k+2)} : \ t_{j+1} = \pm j \ \& \ t_j \wedge t_{j+1} < 0 \quad \forall \ell \le j \le k+1 \right\}.$$

We establish this claim by performing induction on ℓ with ℓ taking the value $k+1, k, \cdots, 1$ in turn.

For $\ell = k+1$, by (B.17) we have

$$\begin{aligned} &\Big| \mathcal{T}_{f,s}[t_1, \cdots, t_{k+1}] - \mathcal{S}[t_1, \cdots, t_{k+1}] \, \mathbb{E}[\partial^k f(W)] \Big| \\ \le & |f|_{k,\omega} \Big(\mathbb{I}(t_{k+1} < 0) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, k+1] + \mathbb{I}(s \ge 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)] \Big). \end{aligned}$$

If there exists $j \in [k]$ such that t_j and t_{j+1} are both positive, then $\mathcal{T}_{f,s}[t_1, \cdots, t_{k+1}] = 0$ and the claim holds with all $a_{j,\nu,t_{\ell:(k+1)}} = 0$. Otherwise, for all $j \le k$ either t_j is negative or t_{j+1} is negative for $j \in [k]$. If $t_{k+1} < 0$, then we have

$$\begin{aligned} & \mathbb{I}(t_{k+1} < 0) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, k+1] + \mathbb{I}(s \ge 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)] \\ = & \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, k+1] + \mathbb{I}(s \ge 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)] \\ \overset{(*)}{\le} & \mathcal{R}_\omega[0, \operatorname{sgn}(t_2), 2\operatorname{sgn}(t_3), \cdots, k \cdot \operatorname{sgn}(t_{k+1}), k+1] \\ & \quad + \mathcal{R}_\omega[0, \operatorname{sgn}(t_2), 2\operatorname{sgn}(t_3), \cdots, k \cdot \operatorname{sgn}(t_{k+1}), -(k+1)] \\ \le & \sum_{\substack{t_{k+2} = \pm(k+1): \\ t_{k+1} \wedge t_{k+2} < 0}} \mathcal{R}_\omega\big[t_1, \cdots, t_{k+1}, t_{k+2}\big]. \end{aligned}$$

where $(*)$ is a consequence of (B.13) and $\operatorname{sgn}(x) = 0, 1,$ or -1 denotes the sign of a real number x.

Further note that if $t_{k+1} > 0$, then $\mathbb{I}(t_{k+1} < 0) = 0$ and we get

$$\begin{aligned}
&\mathbb{I}(t_{k+1} < 0) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, k+1] + \mathbb{I}(s \geq 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)] \\
=&\mathbb{I}(s \geq 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)] \\
\overset{(*)}{\leq}&\mathcal{R}_\omega[0, \operatorname{sgn}(t_2), 2\operatorname{sgn}(t_3), \cdots, k \cdot \operatorname{sgn}(t_{k+1}), -(k+1)] \\
\leq& \sum_{\substack{t_{k+2} = \pm(k+1): \\ t_{k+1} \wedge t_{k+2} < 0}} \mathcal{R}_\omega\big[t_1, \cdots, t_{k+1}, t_{k+2}\big],
\end{aligned}$$

where $(*)$ is a consequence of (B.13). Thus, we have shown that

$$\begin{aligned}
&\Big|\mathcal{T}_{f,s}[t_1, \cdots, t_{k+1}] - \mathcal{S}[t_1, \cdots, t_{k+1}]\, \mathbb{E}[\partial^k f(W)]\Big| \\
\leq&|f|_{k,\omega}\big(\mathbb{I}(t_{k+1} < 0) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, k+1] + \mathbb{I}(s \geq 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_{k+1}, -(k+1)]\big) \\
\leq&|f|_{k,\omega} \sum_{\substack{t_{k+2} = \pm(k+1): \\ t_{k+1} \wedge t_{k+2} < 0}} \mathcal{R}_\omega\big[t_1, \cdots, t_{k+1}, t_{k+2}\big].
\end{aligned}$$

Now suppose the claim holds for $\ell + 1$ and consider the case of ℓ. By (B.18) we have

$$\begin{aligned}
&\Bigg|\mathcal{T}_{f,s}[t_1, \cdots, t_\ell] - \mathcal{S}[t_1, \cdots, t_\ell]\, \mathbb{E}[\partial^{\ell-1} f(W)] \\
&- \mathbb{I}(s \geq 1) \cdot \sum_{j=1}^{k-\ell+1} \sum_{h=0}^{j} (-1)^h \frac{1}{h!(j-h)!} \mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \underbrace{s-\ell, \cdots, s-\ell}_{h \text{ times}}, \underbrace{-\ell, \cdots, -\ell}_{(j-h) \text{ times}}] \\
&+ \mathbb{I}(t_\ell < 0) \sum_{j=1}^{k-\ell+1} \frac{1}{j!} \mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(j-1) \text{ times}}]\Bigg| \\
\leq&\frac{|f|_{k,\omega}}{(k-\ell+1)!}\Big(\mathbb{I}(t_\ell < 0) \cdot \mathcal{R}_\omega[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(k-\ell+1) \text{ times}}] + \mathbb{I}(s \geq 1) \cdot \mathcal{R}_\omega[t_1, \cdots, t_\ell, \underbrace{-\ell, \cdots, -\ell}_{(k-\ell+2) \text{ times}}]\Big).
\end{aligned}$$

Note that $\mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \underbrace{s-\ell, \cdots, s-\ell}_{h \text{ times}}, \underbrace{-\ell, \cdots, -\ell}_{(j-h) \text{ times}}]$ and $\mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \ell, \underbrace{-\ell, \cdots, -\ell}_{(j-1) \text{ times}}]$ are $\mathcal{T}$-sums of order at least $\ell + j$ $(j \geq 1)$. Therefore, we can apply inductive hypothesis on them. In specific, the remainder term ($\mathcal{R}$-sums) in the expansion of

$$\mathcal{T}_{f,j}[t_1, \cdots, t_\ell, \underbrace{s-\ell, \cdots, s-\ell}_{h \text{ times}}, \underbrace{-\ell, \cdots, -\ell}_{(j-h) \text{ times}}]$$

is given by this

$$\begin{aligned}
&4^{k-\ell-j+1}|f|_{k,\omega} \sum_{t_{(\ell+j+1):(k+2)} \in \mathcal{M}_{\ell+j+1,k+2}} \mathcal{R}_\omega\big[t_1, \cdots, t_\ell, \underbrace{s-\ell, \cdots, s-\ell}_{h \text{ times}}, \underbrace{-\ell, \cdots, -\ell}_{(j-h) \text{ times}}, t_{\ell+j+1}, \cdots, t_{k+2}\big] \\
\overset{\text{(B.13)}}{\leq}&4^{k-\ell-j+1}|f|_{k,\omega} \sum_{t_{(\ell+j+1):(k+2)} \in \mathcal{M}_{\ell+j+1,k+2}} \mathcal{R}_\omega\big[t_1, \cdots, t_\ell, -\ell, -(\ell+1), \cdots, -(\ell+j-1), t_{\ell+j+1}, \cdots, t_{k+2}\big]
\end{aligned}$$

$$\leq 4^{k-\ell-j+1}|f|_{k,\omega} \sum_{t_{(\ell+2):(k+2)}\in\mathcal{M}_{\ell+2,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,-\ell,t_{\ell+2},\cdots,t_{k+2}\big] =: 4^{k-\ell-j+1}|f|_{k,\omega}\cdot U_1.$$

Similarly the remainder term in the expansion of $\mathcal{T}_{f,j}[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(j-1)\text{ times}}]$ is given by

$$\begin{aligned}
&4^{k-\ell-j+1}|f|_{k,\omega} \sum_{t_{(\ell+j+1):(k+2)}\in\mathcal{M}_{\ell+j+1,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(j-1)\text{ times}},t_{\ell+j+1},\cdots,t_{k+2}\big]\\
\leq &4^{k-\ell-j+1}|f|_{k,\omega} \sum_{t_{(\ell+2):(k+2)}\in\mathcal{M}_{\ell+2,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,\ell,t_{\ell+2},\cdots,t_{k+2}\big] =: 4^{k-\ell-j+1}|f|_{k,\omega}\cdot U_2.
\end{aligned}$$

Note that $U_1+\mathbb{I}(t_\ell<0)\cdot U_2$ is controlled by

$$\begin{aligned}
&U_1+\mathbb{I}(t_\ell<0)\cdot U_2\\
=&\sum_{t_{(\ell+2):(k+2)}\in\mathcal{M}_{\ell+2,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,-\ell,t_{\ell+2},\cdots,t_{k+2}\big]\\
&+\mathbb{I}(t_\ell<0)\cdot \sum_{t_{(\ell+2):(k+2)}\in\mathcal{M}_{\ell+2,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,\ell,t_{\ell+2},\cdots,t_{k+2}\big]\\
\leq&\sum_{t_{(\ell+1):(k+2)}\in\mathcal{M}_{\ell+1,k+2}} \mathcal{R}_\omega\big[t_1,\cdots,t_\ell,t_{\ell+1},\cdots,t_{k+2}\big].
\end{aligned}\tag{B.41}$$

As we mentioned above, by inductive hypothesis we have that there exist coefficients Q_j satisfying (B.39) such that

$$\begin{aligned}
&\left|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\sum_{j=\ell}^{k+1}Q_j\,\mathbb{E}[\partial^{j-1}f(W)]\right|\\
\leq&\sum_{j=1}^{k-\ell+1}\sum_{h=0}^{j}\frac{1}{h!(j-h)!}4^{k-\ell-j+1}|f|_{k,\omega}\cdot U_1+\mathbb{I}(t_\ell<0)\sum_{j=1}^{k-\ell+1}\frac{1}{j!}4^{k-\ell-j+1}|f|_{k,\omega}\cdot U_2\\
&+\frac{|f|_{k,\omega}}{(k-\ell+1)!}\Big(\mathbb{I}(t_\ell<0)\cdot\mathcal{R}_\omega[t_1,\cdots,t_\ell,\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+1)\text{ times}}]+\mathcal{R}_\omega[t_1,\cdots,t_\ell,\underbrace{-\ell,\cdots,-\ell}_{(k-\ell+2)\text{ times}}]\Big).
\end{aligned}$$

Noting that $\sum_{h=0}^{j}1/(h!(j-h)!)=2^j/j!$, we have

$$\begin{aligned}
&\left|\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]-\sum_{j=\ell}^{k+1}Q_j\,\mathbb{E}[\partial^{j-1}f(W)]\right|\\
\leq&\sum_{j=1}^{k-\ell+1}\frac{2^j\cdot4^{k-\ell-j+1}}{j!}|f|_{k,\omega}\cdot\big(U_1+\mathbb{I}(t_\ell<0)\cdot U_2\big)\\
&+\frac{|f|_{k,\omega}}{(k-\ell+1)!}\big(\mathbb{I}(t_\ell<0)\cdot U_2+U_1\big)\\
\leq&\ \big(1+\textstyle\sum_{j=1}^{k-\ell+1}2^{2k-2\ell-j+2}\big)|f|_{k,\omega}\big(U_1+\mathbb{I}(t_\ell<0)\cdot U_2\big)
\end{aligned}$$

$$
\begin{aligned}
&\le 4^{k-\ell+1}|f|_{k,\omega}\big(U_1+\mathbb{I}(t_\ell<0)\cdot U_2\big)\\
&\overset{(B.41)}{\le} 4^{k-\ell+1}|f|_{k,\omega}\sum_{t_{(\ell+1):(k+2)}\in\mathcal{M}_{\ell+1,k+2}}\mathcal{R}_\omega\big[t_1,\cdots,t_\ell,t_{\ell+1},\cdots,t_{k+2}\big].
\end{aligned}
$$

Thus, we have shown (B.40).

Finally we note that for all $t_{1:\ell}\in\mathcal{M}_{1,\ell}$ and then by (B.13) we have

$$
\begin{aligned}
&\sum_{t_{(\ell+1):(k+1)}\in\mathcal{M}_{\ell+1,k+2}}\mathcal{R}_\omega\big[t_1,\cdots,t_\ell,\cdots,t_{k+2}\big]\\
\le&\sum_{t_{(\ell+1):(k+2)}\in\mathcal{M}_{\ell+1,k+2}}\mathcal{R}_\omega\big[0,\operatorname{sgn}(t_2),2\operatorname{sgn}(t_3),\cdots,(\ell-1)\operatorname{sgn}(t_\ell),t_{\ell+1}\cdots,t_{k+2}\big]\\
\le&\sum_{t_{1:(k+2)}\in\mathcal{M}_{1,k+2}}\mathcal{R}_\omega[t_1,t_2,\cdots,t_{k+2}]=R_{k,\omega}.
\end{aligned}
$$

■

We remark that if f is a polynomial of degree at most k, then the Hölder constant $|f|_{k,\omega}=0$ and hence the remainder $C_{k,\ell}|f|_{k,\omega}R_{k,\omega}$ vanishes.

For any $\mathcal{T}$-sum, we have established the existence of expansions in Theorem B.3. Next we show the uniqueness of such expansions.

Lemma B.4 (Uniqueness)**.** *Under the same settings as Theorem B.3, suppose that there exist two sets of coefficients $Q_\ell,\cdots,Q_{k+1}$ and $Q'_\ell,\cdots,Q'_{k+1}$ only depending on s and $t_{1:\ell}$, and the joint distribution of $(X_i)_{i\in I}$ such that for any polynomial f of degree at most ℓ, we have*

$$
\begin{aligned}
\mathcal{T}_{f,s}[t_1,\cdots,t_\ell]=&Q_\ell\mathbb{E}[\partial^{\ell-1}f(W)]+\cdots+Q_{k+1}\mathbb{E}[\partial^k f(W)]\\
=&Q'_\ell\mathbb{E}[\partial^{\ell-1}f(W)]+\cdots+Q'_{k+1}\mathbb{E}[\partial^k f(W)],
\end{aligned}
$$

Then $Q_j=Q'_j$ for any $j=\ell,\cdots,k+1$.

Proof. We prove this lemma by contradiction.

Let j be the smallest number such that $Q_j\neq Q'_j$. Since the coefficients $Q_\ell,\cdots,Q_{k+1}$ do not depend on f, we can choose $f(x)=cx^{j-1}$ such that $\partial^{j-1}f(x)=c(j-1)!\neq 0$. But $Q_{j+1}\mathbb{E}[\partial^j f(W)]=\cdots=Q_{k+1}\mathbb{E}[\partial^k f(W)]=0$, which implies $cQ_j=cQ'_j$. This is a contradiction. Therefore, $Q_j=Q'_j$ for any $j=\ell,\cdots,k+1$. ■

Proof of Proposition B.1. Applying Theorem B.3 with $\ell=1$, and $s=t_1=t_2=0$, we have for any $f\in\mathcal{C}^{k,\omega}(\mathbb{R})$,

$$
\mathbb{E}[Wf(W)]=\sum_{i_1\in I}\mathbb{E}[X_{i_1}f(W)]=\mathcal{T}_{f,0}[0]=\sum_{j=1}^{k+1}Q_j\mathbb{E}[\partial^{j-1}f(W)]+\mathcal{O}(|f|_{k,\omega}R_{k,\omega}),
$$

for some $Q_1,\cdots,Q_{k+1}$ that only depend on the distribution of $(X_i)_{i\in I}$ and where $R_{k,\omega}$ is defined in (B.16). Suppose that f is a polynomial of degree at most k, then we observe that $f\in\mathcal{C}^{k,\omega}(\mathbb{R})$ and

$|f|_{k,\omega} = 0$. Thus, this implies that

$$\mathcal{T}_{f,0}[0] = \mathbb{E}[Wf(W)] = \sum_{j=1}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)]. \tag{B.42}$$

On the other hand, for any random variable, the moments $(\mu_j)_{j\geq 0}$ and cumulants $(\kappa_j)_{j\geq 0}$, provided that they exist, are connected through the following relations [Smith 1995]:

$$\mu_n = \sum_{j=1}^{n} \binom{n-1}{j-1} \kappa_j \mu_{n-j}. \tag{B.43}$$

Using this we will obtain a similar expansion to (B.42) by using the cumulants (κ_j). In this goal, we first remark that if $f(x) = x^j$ where $j \leq k$, then by using (B.43) we obtain that

$$\begin{aligned}\mathbb{E}[Wf(W)] = \mu_{j+1}(W) = \sum_{h=1}^{j+1} \binom{j}{h-1} \kappa_h(W) \mu_{j+1-h}(W) \\ = \sum_{h=0}^{j} \binom{j}{h} \kappa_{h+1}(W) \mu_{j-h}(W) = \sum_{h=0}^{k} \frac{\kappa_{h+1}(W)}{h!} \mathbb{E}[\partial^h f(W)].\end{aligned}$$

Moreover, we remark that this can be generalized to arbitrary polynomials f of degree k. Indeed, any polynomial f of degree k can be written as $f(x) = \sum_{j=0}^{k} a_j x^j$ for certain coefficients (a_j). By the linearity of expectations, we know that

$$\mathbb{E}[Wf(W)] = \sum_{j=0}^{k} \frac{\kappa_{j+1}(W)}{j!} \mathbb{E}[\partial^j f(W)].$$

Compare this to (B.42) and apply Lemma B.4. We conclude that $Q_j = \kappa_j(W)/(j-1)!$ for any $j \in [k+1]$. In particular, $Q_1 = 0 = \kappa_1(W)$. ∎

Next we upper-bound the cumulants of W using $R_{k,1}$.

Corollary B.5 (Bounds for Cumulants). *For any $k \in \mathbb{N}_+$, there exists a constant C_k that only depends on k such that $\left|\kappa_{k+2}(W)\right| \leq C_k R_{k,1}$.*

Proof. Let $f(x) = x^{k+1}/(k+1)!$. We remark that $f \in \Lambda_{k+1}$ where $\Lambda_{k+1} := \{f \in \mathcal{C}^{k,1}(\mathbb{R}) : |f|_{k,1} \leq 1\}$. Moreover, by using Proposition B.1 we have

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!} \mathbb{E}[\partial^j f(W)] + \mathcal{O}(R_{k,1}).$$

Here the constant dropped from the big $\mathcal{O}$ analysis is controlled by 4^k. On the other hand, by (B.43) we have

$$\mathbb{E}[Wf(W)] = \frac{1}{(k+1)!} \mu_{k+2}(W)$$

$$=\sum_{j=1}^{k+1}\binom{k+1}{j}\kappa_{j+1}(W)\mu_{k+1-j}(W)$$
$$=\sum_{j=1}^{k}\frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)]+\frac{\kappa_{k+2}(W)}{(k+1)!}.$$

Thus, there exists C_k such that $\left|\kappa_{k+2}(W)\right| \le C_k R_{k,1}$. ■

Finally, we are able to prove Lemma A.7 based on Proposition B.1 and Corollary B.5.

Proof of Lemma A.7. We perform induction on $k := \lceil p \rceil$. We start with $k=1$. In this goal, we first remark that by Lemma A.4, we have $f=\Theta h \in \mathcal{C}^{1,\omega}(\mathbb{R})$ and that $|f|_{1,\omega}$ is bounded by a constant. Moreover, as $f=\Theta h$ is the solution to the Stein equation (A.2). By Proposition B.1 we obtain that

$$\mathbb{E}[h(W)]-\mathcal{N}h=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]=\mathcal{O}(R_{1,\omega}).$$

Therefore, the desired result is established for 1. Suppose that the proposition holds for $1,\cdots,k-1$, we want to prove that it will also hold for k. Let $f=\Theta h$, then by Lemma A.4 we know that $f\in\mathcal{C}^{k,\omega}(\mathbb{R})$ and that $|f|_{k,\omega}$ is bounded by some constant that only depends on k,ω. Thus, by Proposition B.1, we have

$$\mathbb{E}[Wf(W)]=\sum_{j=1}^{k}\frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)]+\mathcal{O}(R_{k,\omega}).$$

Hence we have the following expansion of the Stein equation

$$\begin{aligned}\mathbb{E}[h(W)]-\mathcal{N}h=&\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]=-\sum_{j=2}^{k}\frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)]+\mathcal{O}(R_{k,\omega})\\ =&-\sum_{j=1}^{k-1}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)]+\mathcal{O}(R_{k,\omega}).\end{aligned} \tag{B.44}$$

Noting that $\partial^{j+1}\Theta h\in\mathcal{C}^{k-j-1,\omega}(\mathbb{R})$ and $|\partial^{j+1}\Theta h|_{k-j-1,\omega}$ is bounded by a constant only depending on k,ω, then by inductive hypothesis we obtain that

$$\begin{aligned}\mathbb{E}[\partial^{j+1}\Theta h(W)]-\mathcal{N}[\partial^{j+1}\Theta h]=&\sum_{(r,s_{1:r})\in\Gamma(k-j-1)}(-1)^r\prod_{\ell=1}^{r}\frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^{r}(\partial^{s_\ell+1}\Theta)\circ\partial^{j+1}\Theta\, h\right]\\ &+\mathcal{O}\left(\sum_{\ell=1}^{k-j-1}R_{\ell,1}^{(k-j-1+\omega)/\ell}+\sum_{\ell=1}^{k-j}R_{\ell,\omega}^{(k-j-1+\omega)/(\ell+\omega-1)}\right),\end{aligned} \tag{B.45}$$

where we denoted $\Gamma(k-j-1):=\left\{r,s_{1:r}\in\mathbb{N}_+:\sum_{\ell=1}^{r}s_\ell\le k-j-1\right\}$.

By Corollary B.5 and Young's inequality, we have

$$\begin{aligned}|\kappa_{j+2}(W)R_{\ell,\omega}^{\frac{k-j+\omega-1}{\ell+\omega-1}}|\lesssim R_{j,1}R_{\ell,\omega}^{\frac{k-j+\omega-1}{\ell+\omega-1}}&\le\frac{j}{k+\omega-1}R_{j,1}^{\frac{k+\omega-1}{j}}+\frac{k-j+\omega-1}{k+\omega-1}R_{\ell,\omega}^{\frac{k+\omega-1}{\ell+\omega-1}},\\ |\kappa_{j+2}(W)R_{\ell,1}^{\frac{k-j+\omega-1}{\ell}}|\lesssim R_{j,1}R_{\ell,1}^{\frac{k-j+\omega-1}{\ell}}&\le\frac{j}{k+\omega-1}R_{j,1}^{\frac{k+\omega-1}{j}}+\frac{k-j+\omega-1}{k+\omega-1}R_{\ell,1}^{\frac{k+\omega-1}{\ell}}.\end{aligned} \tag{B.46}$$

Thus, we derive that

$$
\begin{aligned}
&\mathbb{E}[h(W)] - \mathcal{N}h \\
\overset{\text{(B.44)}}{=} & -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \mathbb{E}[\partial^{j+1}\Theta h(W)] + \mathcal{O}(R_{k,\omega}) \\
\overset{\text{(B.45)}}{=} & -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \mathcal{N}[\partial^{j+1}\Theta h] \\
& + \sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \sum_{(r,s_{1:r})\in\Gamma(k-j-1)} (-1)^r \prod_{\ell=1}^{r} \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!} \mathcal{N}\left[\prod_{\ell=1}^{r} (\partial^{s_\ell+1}\Theta) \circ \partial^{j+1}\Theta\, h\right] \\
& + \mathcal{O}\left(R_{k,\omega} + \sum_{j=1}^{k-1} |\kappa_{j+2}(W)| \sum_{\ell=1}^{k-j-1} R_{\ell,1}^{(k+\omega-j-1)/\ell} + \sum_{j=1}^{k-1} |\kappa_{j+2}(W)| \sum_{\ell=1}^{k-j} R_{\ell,\omega}^{(k+\omega-j-1)/(\ell+\omega-1)} \right) \\
\overset{\text{(B.46)}}{=} & -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \mathcal{N}[\partial^{j+1}\Theta h] \\
& + \sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \sum_{(r,s_{1:r})\in\Gamma(k-j-1)} (-1)^r \prod_{\ell=1}^{r} \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!} \mathcal{N}\left[\prod_{\ell=1}^{r} (\partial^{s_\ell+1}\Theta) \circ \partial^{j+1}\Theta\, h\right] \\
& + \mathcal{O}\left(R_{k,\omega} + \sum_{j=1}^{k-1} R_{j,1}^{(k+\omega-1)/j} + \sum_{j=1}^{k-1} \sum_{\ell=1}^{k-j-1} R_{\ell,1}^{(k+\omega-1)/\ell} + \sum_{j=1}^{k-1} \sum_{\ell=1}^{k-j} R_{\ell,\omega}^{(k+\omega-1)/(\ell+\omega-1)} \right) \\
= & \sum_{(r,s_{1:r})\in\Gamma(k-1)} (-1)^r \prod_{\ell=1}^{r} \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!} \mathcal{N}\left[\prod_{\ell=1}^{r} (\partial^{s_\ell+1}\Theta)\, h\right] \\
& + \mathcal{O}\left(\sum_{\ell=1}^{k-1} R_{\ell,1}^{(k+\omega-1)/\ell} + \sum_{\ell=1}^{k} R_{\ell,\omega}^{(k+\omega-1)/(\ell+\omega-1)} \right).
\end{aligned}
$$

Therefore, the desired property was established by induction. ■

C Proof of Lemma A.8

In Lemma A.8, we would like to find a random variable with a given sequence of real numbers as its cumulants. Constructing a random variable from its cumulants can be difficult in practice. However, there is a rich literature on establishing the existence of a random variable given the moment sequence. And it is well-known that the moments can be recovered from the cumulants, and vice versa. The explicit expression between moments μ_n and cumulants κ_n is achieved by using the Bell polynomials, i.e.,

$$\mu_n = B_n(\kappa_1, \cdots, \kappa_n) = \sum_{j=1}^{n} B_{n,j}(\kappa_1, \cdots, \kappa_{n-j+1}), \tag{C.1}$$

$$\kappa_n = \sum_{j=1}^{n} (-1)^{j-1}(j-1)! B_{n,j}(\mu_1, \cdots, \mu_{n-j+1}), \tag{C.2}$$

where B_n and $B_{n,j}$ are the exponential Bell polynomial defined by

$$\begin{aligned} B_n(x_1, \cdots, x_n) &:= \sum_{j=1}^{n} B_{n,j}(x_1, x_2, \cdots, x_{n-j+1}), \\ B_{n,j}(x_1, x_2, \cdots, x_{n-j+1}) &:= \sum \frac{n!}{i_1! i_2! \cdots i_{n-j+1}!} \Big(\frac{x_1}{1!}\Big)^{i_1} \Big(\frac{x_2}{2!}\Big)^{i_2} \cdots \Big(\frac{x_{n-j+1}}{(n-j+1)!}\Big)^{i_{n-j+1}}. \end{aligned} \tag{C.3}$$

The sum here is taken over all sequences $i_1, \cdots, i_{n-j+1}$ of non-negative integers such that the following two conditions are satisfied:

$$\begin{aligned} i_1 + i_2 + \cdots + i_{n-j+1} &= j, \\ i_1 + 2i_2 + \cdots + (n-j+1) i_{n-j+1} &= n. \end{aligned}$$

In mathematics, the classical *moment problem* is formulated as follows: Given a sequence $(\mu_i)_{i \geq 0}$, does there exist a random variable defined on a given interval such that $\mu_j = \mathbb{E}[X^j]$ for any non-negative integer j? There are three essentially different types of (closed) intervals. Either two end-points are finite, one end-point is finite, or no end-points are finite, which corresponds to the *Hamburger*, *Hausdorff*, and *Stieltjes* moment problem respectively. See Akhiezer [2020]; Berg [1995] or Tamarkin et al. [1943] for a detailed discussion. For our purpose, there is no restriction on the support of random variables. Thus, the following lemma for the Hamburger moment problem is all we need.

Lemma C.1. *The Hamburger moment problem is solvable, i.e., $(\mu_j)_{j \geq 0}$ is a sequence of moments if and only if $\mu_0 = 1$ and the corresponding Hankel kernel*

$$H = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdots \\ \mu_1 & \mu_2 & \mu_3 & \cdots \\ \mu_2 & \mu_3 & \mu_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{C.4}$$

is positive definite, i.e.,

$$\sum_{j,k \geq 0} \mu_{j+k} c_j c_k \geq 0$$

for every real sequence $(c_j)_{j \geq 0}$ with finite support, i.e., $c_j = 0$ except for finitely many j's.

If we define the $(j{+}1)$-th upper-left determinant of a Hankel matrix by

$$H_j(x_0, x_1, \cdots, x_{2j}) := \begin{vmatrix} x_0 & x_1 & \cdots & x_j \\ x_1 & x_2 & \cdots & x_{j+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_j & x_{j+1} & \cdots & x_{2j} \end{vmatrix}, \tag{C.5}$$

by Sylvester's criterion in linear algebra [Gilbert 1991], the positive-definite condition above is equivalent to $H_j(\mu_0, \cdots, \mu_{2j}) > 0$ for any $j \in \mathbb{N}_+$.

In order to prove Lemma A.8, we construct a Hankel matrix from given values of cumulants and ensure that the upper-left determinants of (C.4) are all positive. Then by Lemma C.1, there exists a random variable that has matched moments with the ones in (C.4) and hence it also has the required cumulants by (C.2).

For convenience, we write

$$L_j(x_1, \cdots, x_{2j}) := H_j(1, B_1(x_1), B_2(x_1, x_2), \cdots, B_{2j}(x_1, \cdots, x_{2j})).$$

Taking $x_1 = 0$, from the definitions (C.3) and (C.5), there is an expansion

$$L_j(0, x_2, \cdots, x_{2j}) = H_j(1, 0, B_2(0, x_2), \cdots, B_{2j}(0, x_2, \cdots, x_{2j})) = \sum a^{(j)}_{t_2, \cdots, t_{2j}} x_2^{t_2} \cdots x_{2j}^{t_{2j}}, \quad \text{(C.6)}$$

where the sum is taken over

$$t_2 + t_3 + \cdots + t_{2j} \geq j,$$
$$2t_2 + 3t_3 + \cdots + (2j)t_{2j} = j(j+1).$$

We further define in the following way a sequence of univariate polynomials which will be essential in our construction in Lemma A.8, by setting

$$P_j(x) := L_j(0, 1, x, x^2, x^3, \cdots, x^{2j-2}).$$

Firstly, we present a lemma on the properties of $P_j(x)$.

Lemma C.2. *$P_j(x)$ is a polynomial of degree at most $j(j-1)$ with only even-degree terms and if we write*

$$P_j(x) = \sum_{\ell=0}^{j(j-1)/2} b^{(j)}_{2\ell} x^{2\ell},$$

we have $b^{(j)}_0 = a^{(j)}_{j(j+1)/2, 0, \cdots, 0} \geq 2$ for any $j \geq 2$, $j \in \mathbb{N}_+$.

Proof. Note that by applying (C.6) we obtain that

$$P_j(x) = L_j(0, 1, x, \cdots, x^{2j-2}) = \sum a^{(j)}_{t_2, \cdots, t_{2j}} x^{t_3 + 2t_4 + \cdots + (2j-2)t_{2j}}, \quad \text{(C.7)}$$

where the sum is taken over

$$t_2 + t_3 + \cdots + t_{2j} \geq j,$$
$$2t_2 + 3t_3 + \cdots + (2j)t_{2j} = j(j+1).$$

The degree of each term in (C.7) is

$$t_3 + 2t_4 + \cdots + (2j-2)t_{2j}$$

$$=(2t_2+3t_3+\cdots+(2j)t_{2j})-2(t_2+t_3+\cdots+t_{2j})$$
$$=j(j+1)-2(t_2+t_3+\cdots+t_{2j}).$$

This is even and no greater than $j(j-1)$ since $t_2+t_3+\cdots+t_{2j}\geq j$.

Then we show the constant term $b_0^{(j)}\geq 2$. Consider a standard normal random variable $\xi\sim\mathcal{N}(0,1)$. Then $\kappa_j(\xi)=0$ for all $j\geq 1, j\neq 2$, and $\kappa_2(\xi)=1$, which is straightforward by checking that the moment generating function of ξ is $\exp(t^2/2)$. By Lemma C.1, we have

$$\begin{aligned}b_0^{(j)}&=P_j(0)=L_j(0,1,0,\cdots,0)\\&=L_j(\kappa_1(\xi),\kappa_2(\xi),\cdots,\kappa_{2j}(\xi))\\&=H_j(\mu_0(\xi),\mu_1(\xi),\cdots,\mu_{2j}(\xi))>0.\end{aligned}$$

Since $\mu_{2\ell}(\xi)=(2\ell-1)!!$ and $\mu_{2\ell-1}(\xi)=0$ are integers for $\ell\in\mathbb{N}_+$, $b_0^{(j)}$ is also an integer. Checking Leibniz formula of the determinant for the Hankel matrix H_j [Lang 2012], we observe that there is an even number of terms and that each term is odd. In specific, the determinant for the Hankel matrix is given by

$$b_0^{(j)}=H_j(\mu_0(\xi),\mu_1(\xi),\cdots,\mu_{2j}(\xi))=\sum_{\tau\in S_j}\operatorname{sgn}(\tau)\prod_{i=1}^{j}\mu_{\tau(i)+i-2}(\xi),$$

where by abuse of notation sgn is the sign function of permutations in the j-th permutation group S_j, which returns $+1$ and -1 for even and odd permutations, respectively. Since $\mu_{2\ell}(\xi)=(2\ell-1)!!$ and $\mu_{2\ell-1}(\xi)=0$ for all $\ell\in\mathbb{N}_+$, we have

$$\operatorname{sgn}(\tau)\prod_{i=1}^{j}\mu_{\tau(i)+i-2}(\xi)\begin{cases}\text{is odd} & \text{if } \tau(i)+i \text{ is even } \forall i=1,\cdots,j\\ =0 & \text{otherwise}\end{cases}.$$

Noting that the number of permutations τ that satisfies $\tau(i)+i$ is even for all $i=1,\cdots,j$ is $(j!)^2$, which is even when $j\geq 2$, we conclude that $b_0^{(j)}$ is even, and thus, it should be at least 2. ■

As we have explained at the beginning of this section, we would like to construct a 'moment' sequence such that the corresponding Hankel kernel is positive definite. The following lemma offers one single step in the construction.

Lemma C.3. *Suppose there is some constant C such that $|\mu_\ell|\leq C$ for $\ell=1,\cdots,2j+1$ and $H_j(\mu_0,\cdots,\mu_{2j})\geq 1$. Then there exists C' only depending on j and C such that*

$$H_{j+1}(\mu_0,\cdots,\mu_{2j},\mu_{2j+1},C')\geq 1.$$

Proof. Let $C'=(j+1)(j+1)!C^{j+2}+1$. Then by the Laplace expansion [Lang 2012] of the determinant, we have

$$H_{j+1}(\mu_0,\cdots,\mu_{2j},\mu_{2j+1},C')=C'H_j(\mu_0,\cdots,\mu_{2j})+\sum_{\ell=0}^{j}(-1)^{j+1+\ell}\mu_{j+1+\ell}A_{j+2,\ell+1}$$

$$\geq C' - (j+1)C \cdot (j+1)!C^{j+1} \geq 1,$$

where $A_{j+2,\ell+1}$ is the determinant of the $(j$+1$)\times$ $(j$+1$)$ submatrix obtained by deleting the $(j$+2$)$-th row and $(\ell$+1$)$-th column of

$$A = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_{j+1} \\ \mu_1 & \mu_2 & \cdots & \mu_{j+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{j+1} & \mu_{j+2} & \cdots & C' \end{pmatrix}.$$

■

Now we prove Lemma A.8.

Proof of Lemma A.8. The key of the proof will be to use Lemma C.1. To do so we need to postulate an infinite sequence that will be our candidates for of potential moments and check that the conditions of Lemma C.1 hold. We remark that as we already know what we want the first k+1 cumulants to be, we already know what the candidates are for the first k+1 moments; and we only to find adequate proposal for the $(k$+2$)$-th moment onward. We will do so by iteratively using Lemma C.3.

In this goal, we remark that since by Lemma C.2 we know that $b_0^{(j)} \geq 2$. Therefore, we can choose a small enough constant $0 < C_p < 1$ only depending on $k = \lceil p \rceil$ such that

$$b_0^{(j)} - \sum_{\ell=1}^{j(j-1)/2} \sum_{\substack{2t_2+2t_3+\cdots+2t_{2j}=j(j+1)-2\ell \\ 2t_2+3t_3+\cdots+2jt_{2j}=j(j+1)}} |a^{(j)}_{t_2,\cdots,t_{2j}}| C_p^{2\ell} \geq 1, \tag{C.8}$$

for any integer $j = 1, \cdots, (k+1)/2$. Given an index set I_n, if $u_j^{(n)} = 0$ for all $j = 1, \cdots, k-1$, let $\xi^{(n)} \sim \mathcal{N}(0,1)$ and $q_n \gg |I_n|$. Then q_n and $\xi^{(n)}$ satisfy all the requirements since $\kappa_j(\xi^{(n)}) = 0$ for all $j \in \mathbb{N}_+, j \neq 2$ and $\kappa_2(\xi^{(n)}) = 1$, which is straightforward by checking that the momemt generating function of $\xi^{(n)}$ is $\exp(t^2/2)$.

Otherwise, let

$$q_n := \Big\lfloor \min_{1\leq j\leq k-1, u_j^{(n)}\neq 0} \big\{ C_p^2 |u_j^{(n)}|^{-2/j} \big\} \Big\rfloor, \tag{C.9}$$

where $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Since by assumption, for any $j = 1, \cdots, k-1$, $u_j^{(n)} \to 0$ as $n \to \infty$, then we know that there exists $N > 0$ such that (i) $q_n \geq 1$ for any $n > N$ and (ii) $q_n \to \infty$ as $n \to \infty$. We note that by definition $\min_{1\leq j\leq k-1, u_j^{(n)}\neq 0}\big\{C_p^2|u_j^{(n)}|^{-2/j}\big\} < q_n + 1$, which implies

$$\max_{1\leq j\leq k-1} \big\{ q_n^{j/2} |u_j^{(n)}| \big\} > C_p^j \big(q_n/(q_n+1)\big)^{j/2} > C_p^p / 2^{p/2}. \tag{C.10}$$

On the other hand, (C.9) also implies that $C_p^2 |u_j^{(n)}|^{-2/j} \geq q_n$. Thus, $q_n^{j/2}|u_j^{(n)}| \leq C_p^j$. Now let $\widetilde{\kappa}_{j+2} := q_n^{j/2} u_j^{(n)}$. We remark that $\widetilde{\kappa}_{j+2} \leq C_p^j$ and $\widetilde{\kappa}_{j+2} \geq C_p^p/2^{p/2}$. We write $\widetilde{\mu}_{j+2} := B_{j+2}(0, \widetilde{\kappa}_2, \cdots, \widetilde{\kappa}_{j+2})$ for $j = 1, \cdots, k-1$. Those will be our candidates for the first k+1 moments. Moreover, if k is odd, we also propose a candidate for $(k$+2$)$-th moment by setting $\widetilde{\mu}_{k+2} := 0$.

For $j = 1, \cdots, \lceil k/2 \rceil$ by (C.6) we have

$$\begin{aligned}
&H_j(1, 0, \widetilde{\mu}_2, \widetilde{\mu}_3, \cdots, \widetilde{\mu}_{2j}) = L_j(0, \widetilde{\kappa}_2, \widetilde{\kappa}_3, \cdots, \widetilde{\kappa}_{2j}) \\
&= \sum_{2t_2+3t_3+\cdots+2jt_{2j}=j(j+1)} a^{(j)}_{t_2,\cdots,t_{2j}} \widetilde{\kappa}_2^{t_2} \cdots \widetilde{\kappa}_{2j}^{t_{2j}} = \sum_{\ell=0}^{j(j-1)/2} \sum_{\substack{2t_2+2t_3+\cdots+2t_{2j}=j(j+1)-2\ell \\ 2t_2+3t_3+\cdots+2jt_{2j}=j(j+1)}} a^{(j)}_{t_2,\cdots,t_{2j}} \widetilde{\kappa}_2^{t_2} \cdots \widetilde{\kappa}_{2j}^{t_{2j}} \\
&\overset{(a)}{\geq} b_0^{(j)} - \sum_{\ell=1}^{j(j-1)/2} \sum_{\substack{2t_2+2t_3+\cdots+2t_{2j}=j(j+1)-2\ell \\ 2t_2+3t_3+\cdots+2jt_{2j}=j(j+1)}} \left| a^{(j)}_{t_2,\cdots,t_{2j}} \widetilde{\kappa}_2^{t_2} \cdots \widetilde{\kappa}_{2j}^{t_{2j}} \right| \\
&\overset{(b)}{\geq} b_0^{(j)} - \sum_{\ell=1}^{j(j-1)/2} \sum_{\substack{2t_2+2t_3+\cdots+2t_{2j}=j(j+1)-2\ell \\ 2t_2+3t_3+\cdots+2jt_{2j}=j(j+1)}} |a^{(j)}_{t_2,\cdots,t_{2j}}| C_p^{2\ell} \overset{(c)}{\geq} 1.
\end{aligned}$$

where to get (a) we used the definition of $b_0^{(j)}$, and where to obtain (b) we used the fact that $|\widetilde{\kappa}_{j+2}| \leq C_p^j$, and where to get (c) we used (C.8). Moreover, as $|\widetilde{\kappa}_{j+2}| \leq C_p^j$, then we know that there exists some constant C_p' such that $|\widetilde{\mu}_{j+2}| = |B_{j+2}(0, \widetilde{\kappa}_2, \cdots, \widetilde{\kappa}_{j+2})| \leq C_p'$ for any integer $j = 1, \cdots, 2\lceil k/2 \rceil - 1$. Therefore, by Lemma C.3, there exists C_p'' depending on $k = \lceil p \rceil$ and C_p' such that

$$H_{\lceil k/2 \rceil+1}(1, 0, \widetilde{\mu}_2, \cdots, \widetilde{\mu}_{2\lceil k/2 \rceil+1}, C_p'') \geq 1.$$

Let $\widetilde{\mu}_{2\lceil k/2 \rceil+2} := C_p''$. Applying Lemma C.3 repeatedly, we get a sequence $(\widetilde{\mu}_j)_{j\geq 1}$ such that $\widetilde{\mu}_0 = 1$ and $H_j(\widetilde{\mu}_0, \widetilde{\mu}_1, \cdots, \widetilde{\mu}_{2j}) \geq 1 > 0$ for any $j \in \mathbb{N}_+$. The sequence $(\tilde{\mu}_j)$ is then our candidate for the moments and we remark that they satisfy the conditions of Lemma C.1. Therefore, by Lemma C.1, we conclude that there exists $\xi^{(n)}$ such that $\mu_j(\xi^{(n)}) = \widetilde{\mu}_j$ for any $j \in \mathbb{N}_+$. As the first k+1 moments uniquely define the first k+1 cumulants of a random variable we have $\kappa_{j+2}(\xi^{(n)}) = \widetilde{\kappa}_{j+2} = q_n^{j/2} u_j^{(n)}$ for all $j = 1, \cdots, k-1$. Thus, the q_n and $\xi^{(n)}$ that we have constructed meet the requirements of Lemmas A.8a and A.8b. Moreover, (C.10) implies that Lemma A.8c is also satisfied. Lastly, to show Lemma A.8d we note that

$$\mathbb{E}[|\xi^{(n)}|^{p+2}] = \|\xi^{(n)}\|_{p+2}^{p+2} \overset{(*)}{\leq} \|\xi^{(n)}\|_{2\lceil k/2 \rceil+2}^{p+2} = \left(\mu_{2\lceil k/2 \rceil+2}(\xi^{(n)})\right)^{(p+2)/(2\lceil k/2 \rceil+2)} \leq (C_p'')^{(p+2)/(2\lceil k/2 \rceil+2)}.$$

Here $(*)$ is due to the fact that $k = \lceil p \rceil \geq p$. ∎

D Proofs of other results in Sections 3 and 4

In this section, we provide the proofs of Proposition 3.2, Theorem 3.3, and Corollaries 4.2 and 4.4.

D.1 Proof of Proposition 3.2

For ease of notation, in this subsection we will drop the dependence on n in our notation and write W, $N(\cdot)$, σ, X_i, I and $R_{j,\omega}$ for respectively W_n, $N_n(\cdot)$, σ_n, $X_i^{(n)}$, I_n and $R_{j,\omega,n}$.

Before we prove the bounds for $R_{k,\omega}$, we note that $R_{k,\omega}$ can be defined without assuming local dependence [LD*]. Thus, we first aim to generalize this concept, which makes the result derived in Proposition D.1 also applicable in general dependent situations. Let $(X_i)_{i\in I}$ be a class of mean zero random variables indexed by I. For any graph G (not necessarily the dependency graph) with the vertex set I and a subset $J \subseteq I$, we define $N(J)$ to be vertex set of the neighborhood of J. As in Appendix B, we assume $\mathrm{Var}\left(\sum_{i\in I} X_i\right) = 1$, without loss of generality. Let $W = \sum_{i\in I} X_i$.

We extend the notation of $\mathcal{R}$-sums defined in (B.11) to this general setting. Given an integer $k \in \mathbb{N}_+$ such that $k \geq 2$, for any $t_{1:k} \in \mathbb{Z}$ such that $|t_j| \leq j-1$ for any $j \in [k]$, let $z = \left|\{j : t_j > 0\}\right|$. If $z \geq 1$, we write $\{j : t_j > 0\} = \{q_1, \cdots, q_z\}$, where the sequence $2 \leq q_1 < \cdots < q_z \leq k$ is taken to be increasing. We further let $q_0 := 1$ and $q_{z+1} := k+1$. Then we could still define the $\mathcal{R}$-sums by

$$\begin{aligned}&\mathcal{R}_\omega[t_1, t_2, \cdots, t_k] :=\\ &\qquad\sum_{i_1\in N_1}\sum_{i_2\in N_2}\cdots\sum_{i_{k-1}\in N_{k-1}}[q_1-q_0, \cdots, q_{z+1}-q_z] \triangleright \Big(|X_{i_1}|, \cdots, |X_{i_{k-1}}|, \big(\sum_{i_k\in N_k}|X_{i_k}|\big)^\omega\Big),\end{aligned}$$

where $N_1 := I$, and for $2 \leq j \leq k$

$$N_j := \begin{cases} N(i_{1:|t_j|}) = N(i_1, \cdots, i_{|t_j|}) & \text{if } t_j \neq 0 \\ \emptyset & \text{if } t_j = 0 \end{cases}.$$

Now the remainder term $R_{k,\omega}$ is defined as

$$\begin{aligned}R_{k,\omega} :=& \sum_{(\ell,\eta_{1:\ell})\in C^*(k+2)}\sum_{i_1\in N_1'}\sum_{i_2\in N_2'}\cdots\sum_{i_{k+1}\in N_{k+1}'}[\eta_1, \cdots, \eta_\ell] \triangleright \Bigg(|X_{i_1}|, \cdots, |X_{i_{k+1}}|, \Big(\sum_{i_{k+2}\in N_{k+2}'}|X_{i_{k+2}}|\Big)^\omega\Bigg)\\ =& \sum_{t_{1:(k+2)}\in\mathcal{M}_{1,k+2}}\mathcal{R}_\omega[t_1, t_2, \cdots, t_{k+2}].\end{aligned} \tag{D.1}$$

where $N_1' := I$ and $N_j' := N(i_{1:(j-1)})$ for $j \geq 2$. $C^*(k+2)$ and $\mathcal{M}_{1,k+2}$ are given by

$$C^*(k+2) = \Big\{\ell, \eta_{1:\ell} \in \mathbb{N}_+ : \eta_j \geq 2\ \forall j \in [\ell-1],\ \sum_{j=1}^{\ell}\eta_j = k+2\Big\},$$

and

$$\mathcal{M}_{1,k+2} := \Big\{t_{1:(k+2)} :\ t_{j+1} = \pm j\ \ \&\ \ t_j \wedge t_{j+1} < 0\ \ \forall 1 \leq j \leq k+1\Big\}.$$

Note that the expressions of $\mathcal{R}$-sums and $R_{k,\omega}$ have the same forms as those in Appendix B.2, but here we do not impose the assumption of the local dependence of $(X_i)_{i\in I}$ anymore as $N(i_{1:q})$'s are defined directly from the graph structure we constructed on I. The main goal of this section is to prove the following proposition.

Proposition D.1. *Fix $k \in \mathbb{N}_+$ such that $k \geq 2$ and real number $\omega \in (0,1]$. Let $N(J)$ be defined as above and suppose the cardinality of $N(J)$ is upper-bounded by M for any $|J| \leq k$. Then there exists a constant $C_{k+\omega}$ only depending on $k+\omega$ such that*

$$\mathcal{R}_\omega[t_1, t_2, \cdots, t_k] \leq C_{k+\omega} M^{k-2+\omega} \sum_{i\in I} \mathbb{E}[|X_i|^{k-1+\omega}].$$

Before proving Proposition D.1, we need the following two lemmas. Lemma D.2 helps us change the order of summation in $\mathcal{R}_\omega[t_1, \cdots, t_k]$ and Lemma D.3 is a generalized version of Young's inequality, which allows us to bound the expectations of products by sums of moments.

Lemma D.2. *Fix $k \in \mathbb{N}_+$ such that $k \geq 2$. For any $J \subseteq I$, let $N(J)$ be defined as above. Suppose $(i_1, \cdots, i_k)$ is a tuple such that $i_1 \in I$, $i_2 \in N(i_1)$, $\cdots$, $i_k \in N(i_{1:(k-1)})$. Then for any $1 \leq h \leq k$, there exists a permutation π on $[k]$ such that $\pi(1) = h$, $i_{\pi(1)} \in I$, $i_{\pi(2)} \in N\big(i_{\pi(1)}\big)$, $\cdots$, $i_{\pi(k)} \in N\big(i_{\pi(1)}, \cdots, i_{\pi(k-1)}\big)$.*

Proof. We perform induction on k.

Firstly, suppose that $k = 2$, then we remark that $i_2 \in N(i_1) \Leftrightarrow i_1 \in N(i_2)$. For $h = 1$, we can choose π to be the identity and the desired identity holds. For $h = 2$, we let $\pi(1) := 2$ and $\pi(2) := 1$ and remark than once again the desired result holds.

Suppose that the proposition is true for $2, \cdots, k-1$. We want to prove that it holds for k. If $h < k$, consider the tuple $(i_1, \cdots, i_h)$. By inductive hypothesis, there is a permutation $\widetilde{\pi}$ on $\{1, 2, \cdots, h\}$ such that $\widetilde{\pi}(1) = h$, $i_{\widetilde{\pi}(2)} \in N\big(i_{\widetilde{\pi}(1)}\big)$, $\cdots$, $i_{\widetilde{\pi}(h)} \in N\big(i_{\widetilde{\pi}(1)}, \cdots, i_{\widetilde{\pi}(q-1)}\big)$. Define

$$\pi(j) := \begin{cases} \widetilde{\pi}(j) & \text{if } 1 \leq j \leq h \\ j & \text{if } h+1 \leq j \leq k \end{cases}.$$

Then π satisfies the requirements in the lemma.

Now suppose $h = k$. $i_k \in N(i_{1:(k-1)})$ indicates that i_k is a neighbor of $\{i_1, \cdots, i_{k-1}\}$. Then there exists $1 \leq \ell \leq k-1$ such that there is an edge between i_k and i_ℓ in the graph $G = (I, E)$. Thus, $i_h \in N(i_\ell)$.

By inductive hypothesis, there is a permutation $\widetilde{\pi}$ on $[\ell]$ such that $\widetilde{\pi}(1) = \ell$, $i_{\widetilde{\pi}(2)} \in N\big(i_{\widetilde{\pi}(1)}\big)$, $\cdots$, $i_{\widetilde{\pi}(\ell)} \in N\big(i_{\widetilde{\pi}(1)}, \cdots, i_{\widetilde{\pi}(\ell-1)}\big)$.

Define

$$\pi(j) := \begin{cases} k & \text{if } j = 1 \\ \widetilde{\pi}(j-1) & \text{if } 2 \leq j \leq \ell+1 \\ j-1 & \text{if } \ell+2 \leq j \leq k \end{cases}.$$

Then $\pi(1) = h = k$, and moreover, we have $i_{\pi(2)} = i_\ell \in N(i_k) = N\big(i_{\pi(1)}\big)$. Moreover, we note that for all $j = 3, \cdots, \ell$ we have $i_{\pi(j+1)} = i_{\widetilde{\pi}(j)} \in N\big(i_{\widetilde{\pi}(1)}, \cdots, i_{\widetilde{\pi}(j-1)}\big) = N\big(i_{\pi(1)}, \cdots, i_{\pi(j)}\big)$. Finally for all $j \geq \ell+1$ we have $i_{\pi(j+1)} = i_j \in N(i_{1:(j-1)}) \subseteq N\big(i_1, \cdots, i_{j-1}, i_k\big) = N\big(i_{\pi(1)}, \cdots, i_{\pi(j)}\big)$. Thus, the lemma holds for k as well. By induction, the proof is complete. ∎

Also, we need a generalization of Young's inequality.

Lemma D.3. *Given $t \in \mathbb{N}_+$, let $Y_1, \cdots, Y_t$ be a sequence of random variables, and real numbers $p_1, \cdots, p_t > 1$ satisfy that $1/p_1 + \cdots + 1/p_t = 1$. Then for any $(\ell, \eta_{1:\ell}) \in C(t) := \{\ell, \eta_{1:\ell} \in \mathbb{N}_+ : \sum_{j=1}^{\ell} \eta_j = t\}$, we have that*

$$[\eta_1, \cdots, \eta_\ell] \triangleright (|Y_1|, \cdots, |Y_t|) \leq \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|^{p_t}]. \tag{D.2}$$

Proof. First, we prove

$$\mathbb{E}[|Y_1 \cdots Y_t|] \le \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|^{p_t}], \tag{D.3}$$

$$\mathbb{E}[|Y_1|] \cdots \mathbb{E}[|Y_t|] \le \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|^{p_t}]. \tag{D.4}$$

In this goal, note that Young's inequality is stated as follows: For any $a_1, \cdots, a_t \ge 0$, and $p_1, \cdots, p_t > 1$ such that $1/p_1 + \cdots + 1/p_t = 1$, we have

$$a_1 \cdots a_t \le \frac{1}{p_1} a_1^{p_1} + \cdots + \frac{1}{p_t} a_t^{p_t}.$$

Thus, by Young's inequality we know that

$$|Y_1 \cdots Y_t| \le \frac{1}{p_1}|Y_1|^{p_1} + \cdots + \frac{1}{p_t}|Y_t|^{p_t}.$$

Taking the expectation, we have

$$\mathbb{E}[|Y_1 \cdots Y_t|] \le \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|^{p_t}].$$

Again by Young's inequality, we obtain that

$$\mathbb{E}[|Y_1|] \cdots \mathbb{E}[|Y_t|] \le \frac{1}{p_1}\mathbb{E}[|Y_1|]^{p_1} + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|]^{p_t}.$$

By Jensen's inequality, $\mathbb{E}[|Y_i|]^{p_i} \le \mathbb{E}[|Y_i|^{p_i}]$ for $i \in [t]$. This implies that

$$\mathbb{E}[|Y_1|] \cdots \mathbb{E}[|Y_t|] \le \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_t}\mathbb{E}[|Y_t|^{p_t}].$$

Finally, we prove (D.2). Let $1/q_j := \sum_{i=\eta_{j-1}+1}^{\eta_j} 1/p_i$ for $1 \le j \le k$.

$$\begin{aligned}
&[\eta_1, \cdots, \eta_\ell] \triangleright (|Y_1|, \cdots, |Y_k|) \\
=&\mathbb{E}\Big[\big|Y_1 \cdots Y_{\eta_1}\big|\Big]\, \mathbb{E}\Big[\big|Y_{\eta_1+1} \cdots Y_{\eta_2}\big|\Big] \,\cdots\, \mathbb{E}\Big[\big|Y_{\eta_1+\cdots+\eta_{\ell-1}+1} \cdots Y_k\big|\Big] \\
\overset{(D.4)}{\le}& \frac{1}{q_1}\mathbb{E}\Big[\big|Y_1 \cdots Y_{\eta_1}\big|^{q_1}\Big] + \cdots + \frac{1}{q_k}\mathbb{E}\Big[\big|Y_{\eta_1+\cdots+\eta_{\ell-1}+1} \cdots Y_k\big|^{q_k}\Big] \\
\overset{(D.3)}{\le}& \frac{1}{p_1}\mathbb{E}[|Y_1|^{p_1}] + \cdots + \frac{1}{p_{\eta_1}}\mathbb{E}[|Y_{\eta_1}|^{p_{\eta_1}}] + \cdots \\
&\quad + \frac{1}{p_{\eta_1+\cdots+\eta_{\ell-1}+1}}\mathbb{E}[|Y_{k+1-u_\ell}|^{p_{\eta_1+\cdots+\eta_{\ell-1}+1}}] + \cdots + \frac{1}{p_k}\mathbb{E}[|Y_k|^{p_k}].
\end{aligned}$$

■

Now we are ready to prove Proposition D.1.

Proof of Proposition D.1. By (B.13), we only need to prove that the following inequality holds for any $k \in \mathbb{N}_+$:

$$\mathcal{R}_\omega[0, \pm 1, \cdots, \pm k] \lesssim M^{k-1+\omega} \sum_{i \in I} \mathbb{E}[|X_i|^{k+\omega}].$$

Once again we write $z := \left|\{j : t_j > 0\}\right|$. If $z \geq 1$, we write $\{j : t_j > 0\} = \{q_1, \cdots, q_z\}$, where $2 \leq q_1 < \cdots < q_z \leq k$ is increasing. Further let $q_0 := 1$ and $q_{z+1} := k+1$.

Noticing that

$$\underbrace{\frac{1}{k+\omega} + \cdots + \frac{1}{k+\omega}}_{k \text{ times}} + \frac{\omega}{k+\omega} = 1,$$

we apply Lemma D.3 and obtain that

$$\begin{aligned} &[q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \left(|X_{i_1}|, \cdots, |X_{i_k}|, \left(\frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{\omega}\right) \\ \lesssim &\mathbb{E}[|X_{i_1}|^{k+\omega}] + \cdots + \mathbb{E}[|X_{i_k}|^{k+\omega}] + \mathbb{E}\left[\left(\frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{k+\omega}\right]. \end{aligned} \tag{D.5}$$

Now by Jensen's inequality and the fact that $\left|N(i_{1:k})\right| \leq M$, we get that

$$\mathbb{E}\left[\left(\frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{k+\omega}\right] \leq \frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \mathbb{E}[|X_{i_{k+1}}|^{k+\omega}].$$

Moreover, we remark that

$$\begin{aligned} &M^{\omega}[q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \left(|X_{i_1}|, \cdots, |X_{i_k}|, \left(\frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{\omega}\right) \\ = &[q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \left(|X_{i_1}|, \cdots, |X_{i_k}|, \left(\sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{\omega}\right) \end{aligned} \tag{D.6}$$

Thus, this implies that

$$\begin{aligned} &\mathcal{R}_\omega[0, \pm 1, \cdots, \pm k] \qquad \text{(D.7)} \\ = &\sum_{i_1 \in I} \cdots \sum_{i_k \in N(i_{1:(k-1)})} [q_1 - q_0, \cdots, q_{z+1} - q_z] \triangleright \left(|X_{i_1}|, \cdots, |X_{i_k}|, \left(\sum_{i_{k+1} \in N(i_{1:k})} \left|X_{i_{k+1}}\right|\right)^{\omega}\right) \\ \lesssim &M^{\omega} \sum_{i_1 \in I} \cdots \sum_{i_k \in N(i_{1:(k-1)})} \left(\mathbb{E}[|X_{i_1}|^{k+\omega}] + \cdots + \mathbb{E}[|X_{i_k}|^{k+\omega}] + \frac{1}{M} \sum_{i_{k+1} \in N(i_{1:k})} \mathbb{E}[|X_{i_{k+1}}|^{k+\omega}]\right). \qquad \text{(D.8)} \end{aligned}$$

Since the cardinality of $N(i_1), \cdots, N(i_{1:k})$ are bounded by M, for $j = 1$ we have

$$\sum_{i_1 \in I} \sum_{i_2 \in N(i_1)} \cdots \sum_{i_k \in N(i_{1:(k-1)})} \mathbb{E}[|X_{i_j}|^{k+\omega}] \leq M^{k-1} \sum_{i \in I} \mathbb{E}[|X_i|^{k+\omega}]. \tag{D.9}$$

Now we bound
$$\sum_{i_1\in I}\sum_{i_2\in N(i_1)}\cdots\sum_{i_k\in N(i_{1:(k-1)})}\mathbb{E}[|X_{i_j}|^{k+\omega}],$$
where $j=2,\cdots,k$.

By Lemma D.2, for any tuple $(i_1,\cdots,i_k)$ in the summation, there exists a permutation π such that $\pi(1)=j$, $i_{\pi(2)}\in N\big(i_{\pi(1)}\big)$, $\cdots$, $i_{\pi(k)}\in N\big(i_{\pi(1)},\cdots,i_{\pi(k-1)}\big)$. Let ϕ_j be a map that sends $(i_1,\cdots,i_k)$ to $\big(i_{\pi(1)},\cdots,i_{\pi(k)}\big)$. Then no more than $(k-1)!$ tuples are mapped to the same destination since $(i_1,\cdots,i_k)$ is a permutation of $\big(i_{\pi(1)},\cdots,i_{\pi(k)}\big)$ and i_j is fixed to be $i_{\pi(1)}$. Thus, we obtain that

$$\begin{aligned}
&\sum_{i_1\in I}\sum_{i_2\in N(i_1)}\cdots\sum_{i_k\in N(i_{1:(k-1)})}\mathbb{E}[|X_{i_j}|^{k+\omega}]\\
\le&(k-1)!\sum_{\pi:\pi(1)=j}\sum_{i_{\pi(1)}\in I}\sum_{i_{\pi(2)}\in N\left(i_{\pi(1)}\right)}\cdots\sum_{i_{\pi(k)}\in N\left(i_{\pi(1)},\cdots,i_{\pi(k-1)}\right)}\mathbb{E}[|X_{i_{\pi(1)}}|^{k+\omega}]\\
\le&(k-1)!\sum_{\pi:\pi(1)=j}\sum_{i_1\in I}\sum_{i_2\in N(i_1)}\cdots\sum_{i_k\in N(i_{1:(k-1)})}\mathbb{E}[|X_{i_j}|^{k+\omega}]\\
\le&((k-1)!)^2M^{k-1}\sum_{i\in I}\mathbb{E}[|X_i|^{k+\omega}]\lesssim M^{k-1}\sum_{i\in I}\mathbb{E}[|X_i|^{k+\omega}].
\end{aligned}\tag{D.10}$$

Similarly,
$$\sum_{i_1\in I}\sum_{i_2\in N(i_1)}\cdots\sum_{i_{k+1}\in N(i_{1:k})}\mathbb{E}[|X_{i_{k+1}}|^{k+\omega}]\lesssim M^k\sum_{i\in I}\mathbb{E}[|X_i|^{k+\omega}].\tag{D.11}$$
Substituting (D.9), (D.10), and (D.11) into (D.7), we conclude

$$\begin{aligned}
\mathcal{R}_\omega[t_1,t_2,\cdots,t_k]\le&\mathcal{R}_\omega\big[0,\operatorname{sgn}(t_2),2\cdot\operatorname{sgn}(t_3),\cdots,(k-1)\operatorname{sgn}(t_{k-1})\big]\\
\lesssim&M^{k-2+\omega}\sum_{i\in I}\mathbb{E}[|X_i|^{k-1+\omega}].
\end{aligned}$$

■

Proof of Proposition 3.2. By Proposition D.1, we have

$$R_{k,\omega}\overset{(D.1)}{=}\sum_{t_{1:(k+2)}\in\mathcal{M}_{1,k+2}}\mathcal{R}_\omega[t_1,t_2,\cdots,t_{k+2}]\lesssim\sum_{t_{1:(k+2)}\in\mathcal{M}_{1,k+2}}M^{k+\omega}\sum_{i\in I}\mathbb{E}[|X_i|^{k+1+\omega}].$$

Noting that $|\mathcal{M}_{1,k+2}|<2^{k+1}$ [Heubach & Mansour 2009], we conclude that

$$R_{k,\omega}\lesssim M^{k+\omega}\sum_{i\in I}\mathbb{E}[|X_i|^{k+1+\omega}].$$

■

D.2 Proof of Theorem 3.3

The proof of Theorem 3.3 relies on Theorem 3.1 and Proposition 3.2.

Proof of Theorem 3.3. Let $k := \lceil p \rceil$. Then $p = k+\omega-1$. Without loss of generality, we assume $\sigma_n = 1$. By Proposition 3.2,

$$R_{j,\omega,n} \lesssim M_n^{j+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{j+1+\omega}\big].$$

If we let $q_1 = (k-1)/(k-j)$ and $q_2 = (k-1)/(j-1)$, then $1/q_1 + 1/q_2 = 1$ and $(2+\omega)/q_1 + (k+1+\omega)/q_2 = j+1+\omega$. Thus,

$$\big|X_i^{(n)}\big|^{j+1+\omega} = \big|X_i^{(n)}\big|^{(2+\omega)/q_1} \cdot \big|X_i^{(n)}\big|^{(k+1+\omega)/q_2}.$$

By Hölder's inequality,

$$\begin{aligned}
&M_n^{j+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{j+1+\omega}\big] \\
\leq &\Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{1/q_1} \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{1/q_2} \\
= &\Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{(k-j)/(k-1)} \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{(j-1)/(k-1)}.
\end{aligned}$$

Since

$$\frac{\omega(k-j)}{(k-1)(j+\omega-1)} + \frac{(j-1)(k+\omega-1)}{(k-1)(j+\omega-1)} = 1,$$

by Young's inequality (See Lemma D.3 for details), we get

$$\begin{aligned}
&\Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{\frac{k-j}{(k-1)(j+\omega-1)}} \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{\frac{j-1}{(k-1)(j+\omega-1)}} \\
\leq &\frac{\omega(k-j)}{(k-1)(j+\omega-1)} \Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{1/\omega} \\
&+ \frac{(j-1)(k+\omega-1)}{(k-1)(j+\omega-1)} \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{1/(k+\omega-1)}.
\end{aligned}$$

Thus, we have

$$R_{j,\omega,n}^{1/(j+\omega-1)} \lesssim \Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{1/\omega} + \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{1/(k+\omega-1)}.$$

Similarly, we derive that

$$\begin{aligned}
R_{j,1,n}^{1/j} \lesssim &\Big(M_n^{j+1} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{j+2}\big]\Big)^{1/j} \leq \Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{\frac{k+\omega-j-1}{kj}} \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{\frac{j-\omega}{(k-1)j}} \\
\lesssim &\Big(M_n^{1+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{2+\omega}\big]\Big)^{1/\omega} + \Big(M_n^{k+\omega} \sum_{i\in I_n} \mathbb{E}\big[\big|X_i^{(n)}\big|^{k+1+\omega}\big]\Big)^{1/(k+\omega-1)}.
\end{aligned}$$

Since by assumption $M_n^{1+\omega}\sum_{i\in I_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{\omega+2}\big]\to 0$ and $M_n^{k+\omega}\sum_{i\in I_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\to 0$ as $n\to\infty$, we have that $R_{j,1,n}\to 0$ as $n\to\infty$. Therefore, by Theorem 3.1 and noting the fact that $p=k+\omega-1$, we conclude

$$\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))\le C_p\Bigg(\Big(M_n^{1+\omega}\sum_{i\in I_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{\omega+2}\big]\Big)^{1/\omega}+\Big(M_n^{p+1}\sum_{i\in I_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\Big)^{1/p}\Bigg),$$

where C_p only depends on p. ■

D.3 Proofs of Corollaries 4.2 and 4.4

Proof of Corollary 4.2. Define the graph (T_n,E_n) to be such that there is an edge between $i_1,i_2\in T_n$ if and only if $\|i_1-i_2\|\le m$. From the definition of the m-dependent random field, $\big(X_i^{(n)}\big)_{i\in T_n}$ satisfies [LD*]. We will therefore apply Theorem 3.3 to obtain the desired result. We remark that $j\in N_n\big(i_{1:(\lceil p\rceil+1)}\big)$ only if there is $\ell\in[\lceil p\rceil+1]$ such that $\|i_\ell-j\|\le m$, which directly implies that $\big|N_n\big(i_{1:(\lceil p\rceil+1)}\big)\big|\le(2m+1)^d(\lceil p\rceil+1)$.

Moreover, by Hölder's inequality, we have

$$\begin{aligned}\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{\omega+2}\big]\le&\Big(\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{2}\big]\Big)^{(p-\omega)/p}\Big(\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\Big)^{\omega/p}\\ \overset{(a)}{\le}&M^{(p-\omega)/p}\sigma^{2(p-\omega)/p}\Big(\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\Big)^{\omega/p}.\end{aligned}$$

Here (a) is due to the non-degeneracy condition. And this directly implies that

$$\begin{aligned}&m^{(1+\omega)d/\omega}\Big(\sigma_n^{-(\omega+2)}\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{\omega+2}\big]\Big)^{1/\omega}\\ \le&m^{\frac{(1+\omega)d}{\omega}}M^{\frac{p-\omega}{p\omega}}\Big(\sigma_n^{-(p+2)}\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\Big)^{1/p}\to 0\quad\text{ as } n\to\infty.\end{aligned}$$

Therefore, by Theorem 3.3, there exists $C_{p,d}>0$ such that for n large enough we have

$$\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1))\le C_{p,d}m^{\frac{(1+\omega)d}{\omega}}M^{\frac{p-\omega}{p\omega}}\sigma_n^{-\frac{p+2}{p}}\Big(\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\Big)^{1/p}.$$

Moreover, if $\big(X_i^{(n)}\big)$ is in addition assumed to be stationary, then by assumption there is a constant K such that $\liminf_{n\to\infty}\sigma_n^2/|T_n|\ge K$. Therefore, we get that

$$\sigma_n^{-(p+2)}\sum_{i\in T_n}\mathbb{E}\big[\big|X_i^{(n)}\big|^{p+2}\big]\asymp|T_n|^{-(p+2)/2}\cdot|T_n|=|T_n|^{-p/2}\to 0,$$

and

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \le C_{p,d} m^{\frac{(1+\omega)d}{\omega}} M^{\frac{p-\omega}{p\omega}} \sigma_n^{-\frac{p+2}{p}} \Big(\sum_{i \in T_n} \mathbb{E}\Big[\big|X_i^{(n)}\big|^{p+2}\Big]\Big)^{1/p} = \mathcal{O}(|T_n|^{-1/2}).$$

■

Proof of Corollary 4.4. Consider the index set $I_n = \{\boldsymbol{i} = (i_1, \cdots, i_m) : 1 \le i_1 \le \cdots \le i_m \le n\} \subseteq \mathbb{Z}^m$. For each $\boldsymbol{i} \in I_n$, let $\xi_{\boldsymbol{i}} := h(X_{i_1}, \cdots, X_{i_m})$. Then $W_n = \sigma_n^{-1} \sum_{\boldsymbol{i} \in I} \xi_{\boldsymbol{i}}$. Let (I_n, E_n) be the graph such that there is an edge between $\boldsymbol{i}, \boldsymbol{j} \in I_n$ if and only if $\{i_1, \cdots, i_m\} \cap \{j_1, \cdots, j_m\} \neq \emptyset$.

Then we remark that the conditions [LD*] holds. Moreover, note that $\boldsymbol{j}$ is in $N_n(\boldsymbol{i}_{1:(\lceil p \rceil+1)})$ only if there is $\ell \in [\lceil p \rceil + 1]$ and $k_1, k_2 \in [m]$ such that $j_{k_1} = (\boldsymbol{i}_\ell)_{k_2}$, where $(\boldsymbol{i}_\ell)_{k_2}$ denotes the k_2-th component of the vector $\boldsymbol{i}_\ell$. This directly implies that the cardinality of the dependency neighborhoods are bounded by $n^m - \big(n - m(\lceil p \rceil + 1)\big)^m \asymp n^{m-1}$. Moreover, the non-degeneracy condition of the U-statistic implies that $\sigma_n^2 \asymp n^{2m-1}$ [Chen & Shao 2007]. Applying Theorem 3.3, we get that

$$\begin{aligned}
&\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \\
\lesssim&\Big(n^m (n^{m-1})^{1+\omega} \frac{1}{\sigma_n^{\omega+2}} \mathbb{E}\Big[\big|h(X_1, \cdots, X_m)\big|^{\omega+2}\Big]\Big)^{1/\omega} \\
&+ \Big(n^m (n^{m-1})^{p+1} \frac{1}{\sigma_n^{p+2}} \mathbb{E}\Big[\big|h(X_1, \cdots, X_m)\big|^{p+2}\Big]\Big)^{1/p} \\
\lesssim& n^{-1/2}\Big(\mathbb{E}\Big[\big|h(X_1, \cdots, X_m)\big|^{\omega+2}\Big]\Big)^{1/\omega} + n^{-1/2}\Big(\mathbb{E}\Big[\big|h(X_1, \cdots, X_m)\big|^{p+2}\Big]\Big)^{1/p} \\
\le& n^{-1/2} \big\|h(X_1, \cdots, X_m)\big\|_{p+2}^{(\omega+2)/\omega} + n^{-1/2} \big\|h(X_1, \cdots, X_m)\big\|_{p+2}^{(p+2)/p}.
\end{aligned}$$

By the moment condition, $\big\|h(X_1, \cdots, X_m)\big\|_{p+2} < \infty$. Thus, we conclude $\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(n^{-1/2})$. ■

E Proofs of results in Section 5

In this section, we establish Theorems 5.3 and 5.5 and Corollaries 5.4, 5.6 and 5.7. Similar to the role of Lemma A.7 plays in the proof of Theorem 3.1, we need lemmas on the high-order expansion of the Stein equation with respect to the strongly mixing fields. Thus, we introduce Lemma E.1 for the proof of Theorem 5.3 and Lemma E.2 for the proof of Theorem 5.5. Furthermore, Corollaries 5.4 and 5.7 are directly applications of Theorems 5.3 and 5.5 for random fields with strong mixing coefficients converging at a polynomial rate.

Lemma E.1. *Let $p \ge 1$ be a real number and $\omega := p + 1 - \lceil p \rceil \in (0,1]$. Set $(T_n) \subseteq \mathbb{Z}^d$ be an increasing sequence of finite index sets such that $|T_n| \to \infty$ and let $\big(X_i^{(n)}\big)_{i \in T_n}$ be a real-valued stationary random field with strong mixing coefficients $(\alpha_{\ell,n})_{\ell \ge 1}$. Suppose that $\mathbb{E}\big[X_i^{(n)}\big] = 0$, $\sup_n \mathbb{E}\Big[\big|X_i^{(n)}\big|^r\Big] < \infty$ for some $r > p + 2$, and that the non-degeneracy condition holds that*

$\liminf_{n\to\infty} \sigma_n^2/|T_n| > 0$, *where* $\sigma_n^2 := \mathrm{Var}\Big(\sum_{i\in T_n} X_i^{(n)}\Big)$. *Denote* $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i^{(n)}$. *Furthermore, suppose that the mixing coefficients satisfy that* $\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-p-2)/r} \le M < \infty$. *Define*

$$M_{1,n} := |T_n|^{-p/2} + |T_n|^{-p/2} \sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega}\alpha_{\ell,n}^{(r-p-2)/r}. \tag{E.1}$$

If $M_{1,n} \to 0$ *as* $n\to\infty$, *then for any* $j \in [p-1]$, *we have* $\kappa_{j+2}(W_n) = \mathcal{O}\big(M_{1,n}^{j/p}\big)$, *and for any* $h \in \Lambda_p$, *we have the expansion*

$$\mathbb{E}[h(W_n)] - \mathcal{N}h = \sum_{(r,s_{1:r})\in\Gamma(\lceil p\rceil-1)} (-1)^r \prod_{j=1}^{r} \frac{\kappa_{s_j+2}(W_n)}{(s_j+1)!}\mathcal{N}\Big[\prod_{j=1}^{r}(\partial^{s_j+1}\Theta)\, h\Big] + \mathcal{O}(M_{1,n}), \tag{E.2}$$

where $\Gamma(\lceil p\rceil-1) = \big\{r, s_{1:r} \in \mathbb{N}_+ : \sum_{j=1}^{r} s_j \le \lceil p\rceil - 1\big\}$.

Now we provide the proof of Theorem 5.3 by utilizing Lemma E.1. Lemma E.1 is also a novel result, whose proof is shown in Appendix F.6. The proof of Lemma E.1 requires new tools and theories that will be established throughout Appendix F.

Proof of Theorem 5.3. By Lemma E.1, we know that $\big|\kappa_{j+2}(W_n)\big| \lesssim M_{1,n}^{j/p} \to 0$ as $n\to\infty$, where $M_{1,n}$ is defined in (E.1). Apply Lemma A.8 with $u_j^{(n)} = \kappa_{j+2}(W_n)$ where $j \in [k-1]$. For n large enough, there exist constants C_p and C_p' (that do not depend on n) and positive integers (q_n) and random variables $(\xi^{(n)})$ such that

(a) $\mathbb{E}[\xi^{(n)}] = 0, \quad \mathbb{E}[(\xi^{(n)})^2] = 1$;

(b) $\kappa_{j+2}(\xi^{(n)}) = q_n^{j/2}\kappa_{j+2}(W_n)$ for $j \in [k-1]$;

(c) Either $\max_{1\le j\le k-1}\big|\kappa_{j+2}(\xi^{(n)})\big| = 0$ or $\max_{1\le j\le k-1}\big|\kappa_{j+2}(\xi^{(n)})\big| \ge C_p > 0$;

(d) $\mathbb{E}[|\xi^{(n)}|^{p+2}] \le C_p'$.

Furthermore, we know that (q_n) satisfy $q_n \to \infty$ diverges to infinity as $n\to\infty$.

Similar to the proof of Theorem 3.1, we will use this to bound the distance between the distribution of W_n to the one of an empirical average of at least q_n i.i.d. random variables. Again we introduce an alternative sequence $(\widetilde{q}_n)$ that can be lower-bounded for all cases. In specific, we let $\widetilde{q}_n := |T|^{2(p+1)/p} \vee q_n$ if $\kappa_3(W_n) = \cdots = \kappa_{k+1}(W_n) = 0$, and $\widetilde{q}_n := q_n$ otherwise. Then we still have $\widetilde{q}_n \to \infty$ as $n \to \infty$.

Let $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ be i.i.d. copies of $\xi^{(n)}$. Define $V_n := \widetilde{q}_n^{-1/2}\sum_{i=1}^{\widetilde{q}_n}\xi_i^{(n)}$.

By construction for any $j \in [k-1]$ we have

$$\kappa_{j+2}(V_n) = \widetilde{q}_n^{-(j+2)/2}\sum_{i=1}^{\widetilde{q}_n}\kappa_{j+2}(\xi_i^{(n)}) = \widetilde{q}_n^{-j/2}\kappa_{j+2}(\xi^{(n)}) = \kappa_{j+2}(W_n).$$

Thus, by Lemma A.5 and Lemma A.7, for any $h \in \Lambda_p$ we have

$$\Big|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\Big| \lesssim M_{1,n} + \widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\Big[\Big|\xi_i^{(n)}\Big|^{p+2}\Big]. \tag{E.3}$$

To be able to have this upper bound not depend on $\xi^{(n)}$ we will upper-bound

$$\widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\Big[\Big|\xi_i^{(n)}\Big|^{p+2}\Big]$$

in terms of $M_{1,n}$. To do so we use the lower bounds on $(\widetilde{q}_n)$ implied by their choice.

If $\max_{1\le j\le k-1}\Big|\kappa_{j+2}(W_n)\Big| > 0$, item (c) implies that there exists

$$C_p \le \max_{1\le j\le k-1}\Big|\kappa_{j+2}(\xi_1^{(n)})\Big| \overset{(*)}{=} \max_{1\le j\le k-1}\Big\{\widetilde{q}_n^{j/2}\Big|\kappa_{j+2}(W_n)\Big|\Big\} \overset{(**)}{\lesssim} \max_{1\le j\le k-1}\Big\{\widetilde{q}_n^{j/2} M_{1,n}^{j/p}\Big\},$$

where to get $(*)$ we use item (b) and to get $(**)$ we use Lemma E.1. Thus, the following holds

$$\widetilde{q}_n^{-p/2} = (\widetilde{q}_n^{-j_0/2})^{p/j_0} \lesssim M_{1,n},$$

where j_0 is the integer satisfying that $\Big|\kappa_{j_0+2}(\xi^{(n)})\Big| = \max_{1\le j\le k-1}\Big|\kappa_{j+2}(\xi^{(n)})\Big|$. Note that $M_{1,n}$ does not depend on the value of j_0 anymore.

On the other hand, if $\kappa_{j+2}(W_n) = 0$ for any $j \in [k-1]$, then by definitions we have $\widetilde{q}_n \ge |T_n|^{2(p+1)/p}$. Moreover, by Hölder's inequality we obtain that

$$\sum_{i\in T_n} \mathbb{E}\Big[\Big|X_i^{(n)}\Big|^2\Big] \le |T_n|^{p/(p+2)} \Big(\sum_{i\in T_n} \mathbb{E}\Big[\Big|X_i^{(n)}\Big|^{p+2}\Big]\Big)^{2/(p+2)}, \tag{E.4}$$

and that

$$\Big(\sum_{i\in T_n} X_i^{(n)}\Big)^2 \le |T_n| \sum_{i\in T_n} \Big|X_i^{(n)}\Big|^2. \tag{E.5}$$

Since $\sigma_n^2 = \mathbb{E}\Big[\Big(\sum_{i\in T_n} X_i^{(n)}\Big)^2\Big]$, we have

$$\begin{aligned}
\widetilde{q}_n^{-p/2} \le& |T_n|^{-(p+1)} \sigma_n^{-(p+2)} \Big(\mathbb{E}\Big[\Big(\sum_{i\in T_n} X_i^{(n)}\Big)^2\Big]\Big)^{(p+2)/2} \\
\overset{(*)}{\le}& \sigma_n^{-(p+2)} |T_n|^{-p/2} \Big(\sum_{i\in T_n} \mathbb{E}\Big[\Big|X_i^{(n)}\Big|^2\Big]\Big)^{(p+2)/2} \\
\overset{(**)}{\le}& \sigma_n^{-(p+2)} \sum_{i\in T_n} \mathbb{E}\Big[\Big|X_i^{(n)}\Big|^{p+2}\Big] \lesssim |T_n|^{-p/2} \le M_{1,n},
\end{aligned}$$

where to obtain $(*)$ we use (E.5) and to obtain $(**)$ we use (E.4).

Thus, using item (d) and the fact that $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ are i.i.d., we obtain

$$\widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\Big[\Big|\xi_i^{(n)}\Big|^{p+2}\Big] \le C_p' \widetilde{q}_n^{-p/2} \lesssim M_{1,n}. \tag{E.6}$$

Therefore, by combining this with (E.3) we have that there is a constant $K > 0$ that does not depend on h such that

$$\Big|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\Big| \le K M_{1,n}.$$

By taking supremum over $h \in \Lambda_p$ and by Lemma A.3, we obtain that

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) \lesssim \sup_{h\in\Lambda_p}\Big|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\Big|^{1/p} \lesssim \Big(M_{1,n} + \widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\Big[\Big|\xi_i^{(n)}\Big|^{p+2}\Big]\Big)^{1/p} \lesssim M_{1,n}^{1/p}.$$

Moreover, by combining Lemma A.6 and (E.6) we have

$$\mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)) \lesssim \Big(\widetilde{q}_n^{-(p+2)/2} \sum_{i=1}^{\widetilde{q}_n} \mathbb{E}\Big[\Big|\xi_i^{(n)}\Big|^{p+2}\Big]\Big)^{1/p} \lesssim M_{1,n}^{1/p}.$$

Therefore, as the Wasserstein distance $\mathcal{W}_p$ satisfies the triangle inequality we conclude that

$$\begin{aligned}\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) \le& \mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) + \mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)) \\ \lesssim& M_{1,n}^{1/p} \lesssim |T_n|^{-1/2} + |T_n|^{-1/2}\Big(\sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega} \alpha_{\ell,n}^{(r-p-2)/r}\Big)^{1/p}.\end{aligned}$$

■

Now we consider random fields with strong mixing coefficients converging to zero at a polynomial rate as a special case of Theorem 5.3.

Proof of Corollary 5.4. By assumption we know $\alpha_{\ell,n}^{(r-p-2)/r} \le C\ell^{-(u-\omega+1)}$. Thus, we have

$$\sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega} \alpha_{\ell,n}^{(r-p-2)/r} \lesssim \sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-\omega} \ell^{u-\omega+1} = \sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-u-1}.$$

If $u > d(p+1)$, then we have $d(p+1) - u - 1 < -1$. Thus, the sum is finite and does not depend on n, which implies that $\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2})$.

If $u = d(p+1)$, then similarly we have

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2}) + \mathcal{O}\Big(|T_n|^{-1/2} \sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{-1}\Big) = \mathcal{O}\big(|T_n|^{-1/2} \log|T_n|\big).$$

Lastly if $d(p/2+1) < u < d(p+1)$, we derive that

$$\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2}) + \mathcal{O}\Big(|T_n|^{-1/2}\Big(\sum_{\ell=1}^{\lfloor |T_n|^{1/d}\rfloor} \ell^{d(p+1)-u-1}\Big)^{1/p}\Big)$$

$$=\mathcal{O}\Big(|T_n|^{-1/2}\lfloor |T_n|^{1/d}\rfloor^{\frac{d(p+1)-u}{p}}\Big) = \mathcal{O}\Big(|T_n|^{-1/2-u/(dp)+(p+1)/p}\Big),$$

which concludes the proof. ■

In order to prove Theorem 5.5, we need the following lemma similar to Lemma E.1.

Lemma E.2. *Let $p \in \mathbb{N}_+$. Set $(T_n) \subseteq \mathbb{Z}^d$ be an increasing sequence of finite index sets such that $|T_n| \to \infty$ and let $\big(X_i^{(n)}\big)_{i\in T_n}$ be a real-valued stationary random field with strong mixing coefficients $(\alpha_{\ell,n})_{\ell\geq 1}$. Suppose that $\mathbb{E}\big[X_i^{(n)}\big] = 0$, $\sup_n \mathbb{E}\big[\big|X_i^{(n)}\big|^r\big] < \infty$ for some $r > p+2$, and that the non-degeneracy condition $\liminf_n \sigma_n^2/|T_n| > 0$ holds, where $\sigma_n^2 := \mathrm{Var}\Big(\sum_{i\in T_n} X_i^{(n)}\Big)$. Denote $W_n := \sigma_n^{-1}\sum_{i\in T_n} X_i$. Furthermore, suppose that the mixing coefficients satisfy that*

$$\sup_n \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_{\ell,n}^{(r-p-1)/r} \leq M < \infty.$$

For any $m \in \mathbb{N}_+$ and $\delta \in [0,1]$ (m and δ can depend on n), let

$$\begin{aligned} M_{2,m,\delta,n} :=& |T_n|^{-p/2}m^{2dp} + |T_n|^{-(p-1+\delta)/2}m^{dp}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}\alpha_{\ell,n}^{(r-p-1-\delta)/r} \\ &+ |T_n|^{-(p-1)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor} \ell^{dp-1}\alpha_{\ell,n}^{(r-p-1)/r}. \end{aligned} \tag{E.7}$$

If $M_{2,m,\delta,n} \to 0$ as $n \to \infty$, then for any $j \in [p-1]$, we have $\kappa_{j+2}(W_n) = \mathcal{O}\big(M_{2,m,\delta,n}^{j/p}\big)$, and that there exists $\widetilde{\kappa}_{p+1,n} = \mathcal{O}\big(M_{2,m,\delta,n}^{(p-1)/p}\big)$ depending on p and the joint distribution of $(X_i)_{i\in T_n}$ such that for any $h \in \Lambda_p$ the following holds

$$\begin{aligned} \mathbb{E}[h(W_n)] - \mathcal{N}h =& \sum_{(r,s_{1:r})\in\Gamma(p-1)} (-1)^r \prod_{j=1}^{r} \frac{\kappa_{s_j+2}(W_n)}{(s_j+1)!}\mathcal{N}\Big[\prod_{j=1}^{r}(\partial^{s_j+1}\Theta)\,h\Big] \\ &+ \frac{\widetilde{\kappa}_{p+1,n} - \kappa_{p+1}(W_n)}{p!}\mathcal{N}\left[\partial^p\Theta\, h\right] + \mathcal{O}(M_{2,m,\delta,n}). \end{aligned} \tag{E.8}$$

Lemma E.2 will also be proven in Appendix F.6. We remark that Lemma E.2 is different from Lemma E.1 in the following ways:

- p is required to be an integer (this is mainly due to the proof technique we use),
- The remainder is controlled using $M_{2,m,\delta,n}$ instead of $M_{1,n}$, which will lead to different convergence rates in the theorem,
- $\kappa_{p+1}(W_n)$ is replaced by $\widetilde{\kappa}_{p+1,n}$.

Note that in general $M_{1,n}$ does not dominate $M_{2,m,\delta,n}$, and vice versa, which leads to different conditions and convergence rates for the $\mathcal{W}_p$ bounds in Theorems 5.3 and 5.5.

Proof of Theorem 5.5. We follow techniques similar to the proof of Theorem 5.3. By Lemma E.2, we have that $\left|\kappa_{j+2}(W_n)\right| \lesssim M_{2,m,\delta,n}^{j/p} \to 0$ for any $j \in [p-1]$ and $\left|\widetilde{\kappa}_{p+1,n}\right| \lesssim M_{2,m,\delta,n}^{(p-1)/p} \to 0$ as $|T_n| \to \infty$, where $M_{2,m,\delta,n}$ is given in (E.7).

We will repeat all the derivation in the proof of Theorem 5.3 with $\kappa_{p+1}(W_n)$ replaced by $\widetilde{\kappa}_{p+1,n}$ and $M_{1,n}$ replaced by $M_{2,m,\delta,n}$. We now apply Lemma A.8 with $u_j^{(n)} = \kappa_{j+2}(W_n)$ where $j \in [p-2]$ and $u_{p-1}^{(n)} = \widetilde{\kappa}_{p+1,n}$. For any index set T_n with n large enough, there exist constants C_p and C_p' (that do not depend on n) and positive integers (q_n) and random variables $(\xi^{(n)})$ such that

(a) $\mathbb{E}[\xi^{(n)}] = 0, \quad \mathbb{E}[(\xi^{(n)})^2] = 1$;

(b) $\kappa_{j+2}(\xi^{(n)}) = q_n^{j/2}\kappa_{j+2}(W_n)$ for $j \in [p-2], \quad \kappa_{p+1}(\xi^{(n)}) = q_n^{(p-1)/2}\widetilde{\kappa}_{p+1,n}$;

(c) Either $\max_{1\le j\le p-1}\left|\kappa_{j+2}(\xi^{(n)})\right| = 0$ or $\max_{1\le j\le p-1}\left|\kappa_{j+2}(\xi^{(n)})\right| \ge C_p > 0$;

(d) $\mathbb{E}[|\xi^{(n)}|^{p+2}] \le C_p'$.

Furthermore, we know that $q_n \to \infty$ as $n \to \infty$.

Again we will bound the distance between the distance between the distribution of W_n to the one of an empirical average of at least q_n i.i.d. random variables, and will need the lower bounds on (q_n) for the convergence of the distribution of the empirical average to a standard normal. Thus, we introduce an alternative sequence $(\widetilde{q}_n)$ by setting $\widetilde{q}_n := |T_n|^{2(p+1)/p} \vee q_n$ if $\kappa_3(W_n) = \cdots = \kappa_p(W_n) = \widetilde{\kappa}_{p+1,n} = 0$, and $\widetilde{q}_n := q_n$ otherwise. Then we still have $(\widetilde{q}_n) \to \infty$ as $|T_n| \to \infty$.

Let $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ be i.i.d. copies of $\xi^{(n)}$. Define $V_n := \widetilde{q}_n^{-1/2}\sum_{i=1}^{\widetilde{q}_n}\xi_i^{(n)}$.

By construction, for any integer j such that $1 \le j \le p-1$, we have

$$\kappa_{j+2}(V_n) = \widetilde{q}_n^{-(j+2)/2}\sum_{i=1}^{\widetilde{q}_n}\kappa_{j+2}(\xi_i^{(n)}) = \widetilde{q}_n^{-j/2}\kappa_{j+2}(\xi^{(n)}) = \begin{cases}\kappa_{j+2}(W_n) & 1 \le j \le p-2\\ \widetilde{\kappa}_{p+1,n} & j = p-1\end{cases}.$$

Thus, by Lemma A.5 and Lemma A.7, for any $h \in \Lambda_p$, we have

$$\left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right| \lesssim M_{2,m,\delta,n} + \widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}[|\xi_i^{(n)}|^{p+2}]. \tag{E.9}$$

To be able to have this upper bound not depend on $\xi^{(n)}$ we will upper-bound

$$\widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}[|\xi_i^{(n)}|^{p+2}]$$

in terms of $M_{2,m,\delta,n}$. To do so we utilize the lower bounds on $(\widetilde{q}_n)$ implied by its choice.

If $|\widetilde{\kappa}_{p+1,n}| \vee \max_{1\le j\le p-2}\left|\kappa_{j+2}(W_n)\right| > 0$, item (c) implies that there exists

$$C_p \le \max_{1\le j\le p-1}\left|\kappa_{j+2}(\xi_1^{(n)})\right| \overset{(*)}{\lesssim} \max_{1\le j\le p-1}\left\{\widetilde{q}_n^{j/2} M_{2,m,\delta,n}^{j/p}\right\},$$

where we use item (b) and Lemma E.2 in $(*)$. Thus, we have the following inequality for some $j_0 \in [p-1]$

$$\widetilde{q}_n^{-p/2} = (\widetilde{q}_n^{-j_0/2})^{p/j_0} \lesssim M_{2,m,\delta,n}.$$

On the other hand, if $\kappa_3(W_n) = \cdots = \kappa_p(W_n) = \widetilde{\kappa}_{p+1,n} = 0$, then we get $\widetilde{q}_n \ge |T_n|^{2(p+1)/p}$ by definition of $\widetilde{q}_n$. Since $\sigma_n^2 = \mathbb{E}\Big[\Big(\sum_{i\in T_n} X_i^{(n)}\Big)^2\Big]$, we have

$$\begin{aligned}
\widetilde{q}_n^{-p/2} =& |T_n|^{-(p+1)}\sigma_n^{-(p+2)}\Big(\mathbb{E}\Big[\Big(\sum_{i\in T_n} X_i^{(n)}\Big)^2\Big]\Big)^{(p+2)/2} \\
\overset{(*)}{\le}& \sigma_n^{-(p+2)}|T_n|^{-p/2}\Big(\sum_{i\in T_n}\mathbb{E}\Big[\left|X_i^{(n)}\right|^2\Big]\Big)^{(p+2)/2} \\
\overset{(**)}{\le}& \sigma_n^{-(p+2)}\sum_{i\in T}\mathbb{E}\Big[\left|X_i^{(n)}\right|^{p+2}\Big] \lesssim |T_n|^{-p/2} \le M_{2,m,\delta,n},
\end{aligned}$$

where to obtain $(*)$ we use (E.5) and to obtain $(**)$ we use (E.4).

Thus, using item (d) and the fact that $\xi_1^{(n)}, \cdots, \xi_{\widetilde{q}_n}^{(n)}$ are i.i.d., we obtain

$$\widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}\Big[\left|\xi_i^{(n)}\right|^{p+2}\Big] \le C_p'\widetilde{q}_n^{-p/2} \lesssim M_{2,m,\delta,n}. \tag{E.10}$$

By taking supremum over $h \in \Lambda_p$ and by Lemma A.3, we obtain that

$$\begin{aligned}
\mathcal{W}_p(\mathcal{L}(W_n), \mathcal{L}(V_n)) \lesssim& \sup_{h\in\Lambda_p}\left|\mathbb{E}[h(W_n)] - \mathbb{E}[h(V_n)]\right|^{1/p} \\
\lesssim& \Big(M_{2,m,\delta,n} + \widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}\Big[\left|\xi_i^{(n)}\right|^{p+2}\Big]\Big)^{1/p} \lesssim M_{2,m,\delta,n}^{1/p}.
\end{aligned}$$

Moreover, by combining Lemma A.6 and (E.10) we have

$$\mathcal{W}_p(\mathcal{L}(V_n), \mathcal{N}(0,1)) \lesssim \Big(\widetilde{q}_n^{-(p+2)/2}\sum_{i=1}^{\widetilde{q}_n}\mathbb{E}\Big[\left|\xi_i^{(n)}\right|^{p+2}\Big]\Big)^{1/p} \lesssim M_{2,m,\delta,n}^{1/p}.$$

Therefore, as the Wasserstein distance $\mathcal{W}_p$ satisfies the triangle inequality we conclude that

$$\mathcal{W}_p\big(\mathcal{L}(W_n), \mathcal{N}(0,1)\big) \le \mathcal{W}_p\big(\mathcal{L}(W_n), \mathcal{L}(V_n)\big) + \mathcal{W}_p\big(\mathcal{L}(V_n), \mathcal{N}(0,1)\big) \lesssim M_{2,m,\delta,n}^{1/p}$$

$$\lesssim |T_n|^{-1/2} m^{2d} + |T_n|^{-1/2+(1-\delta)/(2p)} m^d \Big(\sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{d\delta-\delta} \alpha_{\ell,n}^{(r-p-1-\delta)/r} \Big)^{1/p}$$
$$+ |T_n|^{-1/2+1/(2p)} \Big(\sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_{\ell,n}^{(r-p-1)/r} \Big)^{1/p}.$$

■

As an application of Theorem 5.5, we show Corollary 5.6.

Proof of Corollary 5.6. We apply Theorem 5.5 with $p = 1$, $\delta = \epsilon$, $m \asymp |T_n|^{\frac{\epsilon \wedge (1/3)}{2d}}$. Then $m \to \infty$ as $n \to \infty$. Since

$$\sum_{\ell=1}^{\infty} \ell^{d-1} \alpha_{\ell,n}^{(r-2)/r} \le \sum_{\ell=1}^{\infty} \ell^{d-1} \alpha_{\ell,n}^{(r-2-\epsilon)/r},$$

the mixing condition I of Theorem 5.5 is satisfied. Now we check that

$$|T_n|^{-1/2} m^{2d} \lesssim |T_n|^{-1/2+\epsilon} \xrightarrow{n\to\infty} 0,$$
$$|T_n|^{-\epsilon/2} m^d \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{d\epsilon-\epsilon} \alpha_{\ell,n}^{(r-2-\epsilon)/r} \lesssim \sum_{\ell=m+1}^{\infty} \ell^{d-1} \alpha_{\ell,n}^{(r-2-\epsilon)/r},$$
$$\sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{d-1} \alpha_{\ell,n}^{(r-2)/r} \le \sum_{\ell=m+1}^{\infty} \ell^{d-1} \alpha_{\ell,n}^{(r-2-\epsilon)/r}.$$

Since $m \to \infty$ as $n \to \infty$, we have that by assumption $\sum_{\ell=m+1}^{\infty} \ell^{d-1} \alpha_{\ell,n}^{(r-2-\epsilon)/r}$ converges to zero. Thus, mixing condition II of Theorem 5.5 is also satisfied and the result follows. ■

Lastly, we prove Corollary 5.7 by applying Theorem 5.5 to random fields with strong mixing coefficients that converge at a polynomial rate, and combining the results of Corollary 5.4.

Proof of Corollary 5.7.

If $u \ge d(p+1)$, the results are directed implied by Corollary 5.4.

If $dp < u < d(p+1)$, on one hand, Corollary 5.4 gives that $\beta \ge 1/2 + u/(dp) - (p+1)/p$. On the other hand, we apply Theorem 5.5 with $\delta = 1$ and $m \asymp |T_n|^{\frac{1}{2(u+dp)}}$. Then we have

$$|T_n|^{-1/2} m^{2d} \asymp |T_n|^{-1/2+d/(u+dp)},$$
$$|T_n|^{-1/2+(1-\delta)/(2p)} m^d \Big(\sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{d\delta-\delta} \alpha_{\ell,n}^{(r-p-1-\delta)/r} \Big)^{1/p} \asymp |T_n|^{-\frac{1}{2}+\frac{d}{2(u+dp)}-\frac{u-d}{2(u+dp)p}}$$
$$\lesssim |T_n|^{-1/2+d/(u+dp)},$$
$$|T_n|^{-1/2+1/(2p)} \Big(\sum_{\ell=m+1}^{m+1+\lfloor \frac{|T_n|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_{\ell,n}^{(r-p-1)/r} \Big)^{1/p} \asymp |T_n|^{-\frac{1}{2}+\frac{1}{2p}-\frac{u-dp}{2(u+dp)p}}$$
$$= |T_n|^{-1/2+d/(u+dp)}.$$

Thus, by Theorem 5.5 we get $\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1)) = \mathcal{O}\big(|T_n|^{-1/2+d/(u+dp)}\big)$.

If $u = dp$, apply Theorem 5.5 with $\delta = m = 1$ and get that

$$
\begin{aligned}
&|T_n|^{-1/2}m^{2d} \asymp |T_n|^{-1/2},\\
&|T_n|^{-1/2+(1-\delta)/(2p)}m^d\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor}\ell^{d\delta-\delta}\alpha_{\ell,n}^{(r-p-1-\delta)/r}\Big)^{1/p} \le |T_n|^{-1/2}\Big(\sum_{\ell=1}^{\lceil|T_n|^{1/d}\rceil}\ell^{dp-1}\alpha_{\ell,n}^{(r-p-2)/r}\Big)^{1/p}\\
&\qquad\asymp |T_n|^{-1/2}\log|T_n|,\\
&|T_n|^{-1/2+1/(2p)}\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor}\ell^{dp-1}\alpha_{\ell,n}^{(r-p-1)/r}\Big)^{1/p} \le |T_n|^{-1/2+1/(2p)}\Big(\sum_{\ell=1}^{\lceil|T_n|^{1/d}\rceil}\ell^{dp-1}\alpha_{\ell,n}^{(r-p-2)/r}\Big)^{1/p}\\
&\qquad\asymp |T_n|^{-1/2+1/(2p)}\log|T_n|.
\end{aligned}
$$

Thus, we get

$$
\mathcal{W}_p(\mathcal{L}(W_n),\mathcal{N}(0,1)) = \mathcal{O}(|T_n|^{-1/2}) + \mathcal{O}\Big(|T_n|^{-1/2+1/(2p)}\sum_{\ell=1}^{\lfloor|T_n|^{1/d}\rfloor}\ell^{-1}\Big) = \mathcal{O}\big(|T_n|^{-1/2+1/(2p)}\log|T_n|\big).
$$

If $d(p+1)/2 < u < dp$, the results also follows from Theorem 5.5 as

$$
\begin{aligned}
&|T_n|^{-1/2}m^{2d} \asymp |T_n|^{-1/2},\\
&|T_n|^{-1/2+(1-\delta)/(2p)}m^d\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor}\ell^{d\delta-\delta}\alpha_{\ell,n}^{(r-p-1-\delta)/r}\Big)^{1/p} \le |T_n|^{-1/2}\Big(\sum_{\ell=1}^{\lfloor|T_n|^{1/d}\rfloor}\ell^{dp-u-1}\Big)^{1/p}\\
&\qquad\asymp |T_n|^{-\frac{1}{2}+\frac{dp-u}{dp}},\\
&|T_n|^{-1/2+1/(2p)}\Big(\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T_n|^{1/d}}{2}\rfloor}\ell^{dp-1}\alpha_{\ell,n}^{(r-p-1)/r}\Big)^{1/p} \lesssim |T_n|^{-\frac{1}{2}+\frac{1}{2p}+\frac{dp-u}{dp}}\\
&\qquad= |T_n|^{-\frac{1}{2}+\frac{2p+1}{2p}-\frac{u}{dp}}.
\end{aligned}
$$

Thus, Corollary 5.7 is proven. ∎

F Proofs of Lemmas E.1 and E.2

In this section, whenever it is not ambiguous we will drop the n notation and write α_ℓ, σ, W, X_i and T for respectively $\alpha_{\ell,n}$, σ_n, W_n, $X_i^{(n)}$ and T_n.

F.1 Example & Roadmap

Similar to the role Proposition B.1 plays in deriving Lemma A.7, the key step in proving Lemma E.1 is to obtain the following expansion of $\mathbb{E}[Wf(W)]$ for $f \in \mathcal{C}^{k,\omega}(\mathbb{R})$:

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \text{Remainders}, \tag{F.1}$$

where $\kappa_{j+1}(W)$ is the $(j+1)$-th cumulant of W.

To gain intuition, we first consider the simpler case of a stationary random sequence $(X_i)_{i=1}^n$ with $k=\omega=d=1$. We let $W := \sigma^{-1}\sum_{i=1}^n X_i$ where $\sigma^2 := \operatorname{Var}\left(\sum_{i=1}^n X_i\right)$ and require that $\mathbb{E}[X_1] = 0$ and $\mathbb{E}[|X_1|^r] < \infty$ for a given $r > 3$. For simplicity, we further assume $f \in \mathcal{C}^2(\mathbb{R}) \cap \mathcal{C}^{1,1}(\mathbb{R})$, i.e., f'' is continuous and bounded (see Definition A.1). We will see that (F.1) reduces to an upper bound on the absolute value of $\mathbb{E}[f'(W) - Wf(W)]$.

Fixing a positive integer $m \in \mathbb{N}$. For any positive integers i, j, we denote

$$W_{i,j} := \frac{1}{\sigma}\Big(\sum_{\ell=1}^{i-j-1} X_\ell + \sum_{\ell=i+j+1}^{n} X_\ell\Big), \quad W_{i,j}^* := \frac{1}{\sigma}\Big(\sum_{\ell=1}^{i-j-1} X_\ell + \sum_{\ell=i+j}^{n} X_\ell\Big),$$

where $X_\ell := 0$ if $\ell \le 0$ or $\ell \ge n+1$. Note that $W_{i,j}^* - W_{i,j} = X_{i+j}$ and $W_{i,j-1} - W_{i,j}^* = X_{i-j}$ if $j \ge 2$.

Now we have

$$\begin{aligned}
&\mathbb{E}\big[Wf(W) - f'(W)\big] \\
=&\frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}\big[X_i\big(f(W) - f(W_{i,m}) - f'(W)(W - W_{i,m})\big)\big] \\
&+ \frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}[X_i f(W_{i,m})] + \frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}\big[X_i(W - W_{i,m})f'(W)\big] - \mathbb{E}[f'(W)] \\
=:&E_1 + E_2 + E_3 - \mathbb{E}[f'(W)].
\end{aligned} \tag{F.2}$$

Intuitively, for each i, we split W into two parts, $W_{i,m}$ and $W - W_{i,m}$. The latter has limited number of X_j's and converges to 0 when n is relatively large compared to m. Although the first part, $W_{i,m}$, has a lot of X_j's in the sum, it is less dependent on X_i. As a result, the expectation terms can be controlled using the strong mixing conditions of the random sequence.

To study E_1 in (F.2), we apply the Taylor expansion and Young's inequality and obtain that

$$\begin{aligned}
|E_1| =&\left|\frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}\big[X_i\big(f(W) - f(W_{i,m}) - f'(W)(W - W_{i,m})\big)\big]\right| \\
\le&\frac{\|f''\|}{2\sigma}\sum_{i=1}^{n}\mathbb{E}\big[\big|X_i(W - W_{i,m})^2\big|\big] = \frac{\|f''\|}{2\sigma^3}\sum_{i=1}^{n}\mathbb{E}\Big[|X_i|\Big(\sum_{j=i-m}^{i+m} X_j\Big)^2\Big] \\
=&\frac{\|f''\|}{2\sigma^3}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\sum_{\ell=i-m}^{i+m}\mathbb{E}[|X_i X_j X_\ell|]
\end{aligned} \tag{F.3}$$

$$
\begin{aligned}
&\leq \frac{\|f''\|}{2\sigma^3} \sum_{i=1}^{n} \sum_{j=i-m}^{i+m} \sum_{\ell=i-m}^{i+m} \frac{1}{3}\Big(\mathbb{E}[|X_i|^3] + \mathbb{E}[|X_j|^3] + \mathbb{E}[|X_\ell|^3]\Big) \\
&\leq \frac{2(m+1)^2\|f''\|}{\sigma^3} \sum_{i=1}^{n} \mathbb{E}[|X_i|^3] \lesssim \|f''\| m^2 n^{-1/2}. \qquad \text{(F.4)}
\end{aligned}
$$

Next we consider E_2, and observe that

$$
E_2 = \frac{1}{\sigma} \sum_{i=1}^{n} \mathbb{E}[X_i f(W_{i,m})] = \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(\big(f(W_{i,j-1}) - f(W_{i,j}^*)\big) + \big(f(W_{i,j}^*) - f(W_{i,j})\big)\Big)\Big] \qquad \text{(F.5)}
$$

$$
\begin{aligned}
=& \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(f(W_{i,j-1}) - f(W_{i,j}) - f'(W_{i,j-1})(W_{i,j-1} - W_{i,j}^*) - f'(W_{i,j}^*)(W_{i,j}^* - W_{i,j})\Big)\Big] \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(X_{i-j} f'(W_{i,j}) + X_{i+j} f'(W_{i,j}^*)\Big)\Big] \\
=& \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(f(W_{i,j-1}) - f(W_{i,j}) - f'(W_{i,j-1})(W_{i,j-1} - W_{i,j}^*) - f'(W_{i,j}^*)(W_{i,j}^* - W_{i,j})\Big)\Big] \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(X_{i-j}\big(f'(W_{i,j}) - \mathbb{E}[f'(W_{i,j})]\big) + X_{i+j}\big(f'(W_{i,j}^*) - \mathbb{E}[f'(W_{i,j}^*)]\big)\Big)\Big] \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \Big(\mathbb{E}[X_i X_{i-j}]\,\mathbb{E}[f'(W_{i,j})] + \mathbb{E}[X_i X_{i+j}]\,\mathbb{E}[f'(W_{i,j}^*)]\Big) \\
=& \frac{1}{\sigma} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(f(W_{i,j-1}) - f(W_{i,j}) - f'(W_{i,j-1})(W_{i,j-1} - W_{i,j}^*) - f'(W_{i,j}^*)(W_{i,j}^* - W_{i,j})\Big)\Big] \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}\Big[X_i\Big(X_{i-j}\big(f'(W_{i,j}) - \mathbb{E}[f'(W_{i,j})]\big) + X_{i+j}\big(f'(W_{i,j}^*) - \mathbb{E}[f'(W_{i,j}^*)]\big)\Big)\Big] \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \Big(\mathbb{E}[X_i X_{i-j}]\big(\mathbb{E}[f'(W_{i,j})] - \mathbb{E}[f'(W)]\big) + \mathbb{E}[X_i X_{i+j}]\big(\mathbb{E}[f'(W_{i,j}^*)] - \mathbb{E}[f'(W)]\big)\Big) \\
&+ \frac{1}{\sigma^2} \sum_{i=1}^{n} \sum_{j=m+1}^{n-1} \mathbb{E}[X_i(X_{i-j} + X_{i+j})]\,\mathbb{E}[f'(W)] \\
=:& E_4 + E_5 + E_6 + E_7. \qquad \text{(F.6)}
\end{aligned}
$$

Intuitively, E_4 to E_6 can be controlled with the strong mixing conditions of $(X_i)_{i=1}^n$. For example in E_6, the strong mixing coefficient between the σ-algebra generated by X_i and the σ-algebra generated by X_{i-j} or X_{i+j} is no greater than α_j. We will illustrate how this helps get an upper bound later.

As for E_3 in (F.2), we have

$$\begin{aligned}
E_3 =& \frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}\big[X_i(W-W_{i,m})f'(W)\big] = \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}[X_iX_jf'(W)]\\
=& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}\big[X_iX_j\big(f'(W)-\mathbb{E}[f'(W)]\big)\big] + \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}[X_iX_j]\,\mathbb{E}[f'(W)]\\
=& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}\big[X_iX_j\big(f'(W)-f'(W_{i,j,m})\big)\big] + \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}[X_iX_j]\big(\mathbb{E}[f'(W_{i,j,m})]-\mathbb{E}[f'(W)]\big)\\
&+ \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}\big[X_iX_j\big(\mathbb{E}[f'(W_{i,j,m})]-f'(W_{i,j,m})\big)\big] + \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}[X_iX_j]\,\mathbb{E}[f'(W)]\\
=:& E_8+E_9+E_{10}+E_{11},
\end{aligned} \tag{F.7}$$

where we set

$$W_{i,j,m} := \frac{1}{\sigma}\Big(\sum_{\ell=1}^{i\wedge j-m-1} X_\ell + \sum_{\ell=i\vee j+m+1}^{n} X_\ell\Big).$$

Next we observe that

$$\begin{aligned}
E_7+E_{11} =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathbb{E}[X_i(X_{i-j}+X_{i+j})]\,\mathbb{E}[f'(W)] + \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}[X_iX_j]\,\mathbb{E}[f'(W)]\\
=& \frac{1}{\sigma^2}\mathbb{E}\Big[\Big(\sum_{i=1}^{n}X_i\Big)^2\Big]\,\mathbb{E}[f'(W)] = \mathbb{E}[f'(W)].
\end{aligned}$$

Thus, E_7+E_{11} cancels out with $-\mathbb{E}[f'(W)]$ in (F.6).

The terms E_8 and E_9 can be bounded by the Taylor expansion and Young's inequality in a way similar to (F.3). E_{10} can be controlled with the strong mixing conditions of $(X_i)_{i=1}^n$ by utilizing the covariance inequality as stated below.

Lemma F.1 (Theorem 3 in Chapter 1.2 of Doukhan [1994])**.** *Suppose X,Y are two random variables. X is measurable with respect to the σ-algebra $\mathcal{A}$ and Y is measurable with respect to the σ-algebra $\mathcal{B}$. Denoting $\|X\|_p = \big(\mathbb{E}[|X|^p]\big)^{1/p}$, we have*

$$|\mathrm{Cov}(X,Y)| \le 8\alpha^{1/r}(\mathcal{A},\mathcal{B})\|X\|_p\|Y\|_q, \tag{F.8}$$

for any $p,q,r\ge 1$ such that $1/p+1/q+1/r=1$.

To illustrate on how to use the strong mixing conditions, we consider a special case, where $f\in\mathcal{C}^2(\mathbb{R})\cap\mathcal{C}^{0,1}(\mathbb{R})\cap\mathcal{C}^{1,1}(\mathbb{R})$, i.e, both f' and f'' are continuous and bounded. Under this condition, the second term in (F.2) can be controlled more easily. By Lemma F.1, we have

$$\mathbb{E}[|X_iX_j|] \le 8\alpha_{|i-j|}^{(r-2)/r}\|X_i\|_r\|X_j\|_r = 8\alpha_{|i-j|}^{(r-2)/r}\|X_1\|_r^2.$$

Thus,

$$
\begin{aligned}
\frac{1}{\sigma}\left|\sum_{i=1}^{n}\mathbb{E}[X_i f(W_{i,m})]\right| \leq& \frac{1}{\sigma}\left|\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathbb{E}\big[X_i\big(f(W_{i,j})-f(W_{i,j+1})\big)\big]\right| \\
\leq& \frac{1}{\sigma}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\left|\mathbb{E}\big[X_i\big(f(W_{i,j})-f(W_{i,j+1})\big)\big]\right| \\
\leq& \frac{\|f'\|}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathbb{E}\big[|X_i|\big(|X_{i-j}|+|X_{i+j}|\big)\big] \\
=& \frac{2\|f'\|}{\sigma^2}\sum_{i,j:|i-j|\geq m+1}\mathbb{E}[|X_iX_j|] \\
\leq& \frac{2\|f'\|}{\sigma^2}\sum_{i,j:|i-j|\geq m+1}8\alpha_{|i-j|}^{(r-2)/r}\|X_1\|_r^2 \\
\lesssim& \|f'\|\sum_{\ell=m+1}^{n-1}\alpha_\ell^{(r-2)/r}.
\end{aligned}
\tag{F.9}
$$

Hence, this term vanishes at $n = m = \infty$ if $\sum_{\ell=1}^{\infty}\alpha_\ell^{(r-2)/r}$ converges. For this special case, we need $n \gg m^4 \to \infty$ so that both (F.3) and (F.9) will approach 0.

We omit the technical details on the rest of the derivation because the aim here is only to help build intuition. Please refer to Sunklodas [2007]; Bentkus & Sunklodas [2007] for more information on this case.

More generally, consider a mean-zero random field $(X_i)_{i\in T}$ indexed by a finite set $T \subsetneq \mathbb{Z}^d$ $(d \geq 1)$. To be able to bound $\mathbb{E}[f'(W)-Wf(W)]$, it is important to carefully keep track of which indexes in T are at a distance of less than m of each other as the corresponding random variables X_i's are non-negligibly dependent. Similarly, we will also want to keep track of which indexes are at distance of more than m from each other. Indeed if all the indexes in $U_1 \subsetneq T$ and $U_2 \subsetneq T$ are at distance of more than m from each other, then the dependence between $(X_{i_1})_{i_1\in U_1}$ and $(X_{i_2})_{i_2\in U_2}$ is negligible and we can control the correlation between those thanks to the mixing coefficients.

For $k \geq 1$, we would like to get an expansion (F.1) of $\mathbb{E}[Wf(W)]$ with controllable remainders instead of directly bounding $\mathbb{E}[Wf(W)-f'(W)]$. Following an idea similar to Appendix B, we can achieve this by encoding the structure of all possible sums appearing in the process and reformulate the expansion using a better representation called the "genogram". As we have seen in the example of the simple case $k = 1$, we expect to obtain an expansion, where each summand of the remainders (e.g. $E_1, E_4, E_5, E_6, E_8, E_9, E_{10}$) is an expectation or the product of expectations. We will use two different tools to control them, namely the Taylor expansion and the fact that covariances can be controlled by the mixing coefficients (see Lemma F.1).

For $k = 1$, we have shown that

$$
\begin{aligned}
\mathbb{E}\big[Wf(W)-f'(W)\big] &= E_1+E_2+E_3-\mathbb{E}[f'(W)] \\
&= E_1+(E_4+E_5+E_6+E_7)+(E_8+E_9+E_{10}+E_{11})-\mathbb{E}[f'(W)]
\end{aligned}
$$

$$= E_1 + E_4 + E_5 + E_6 + E_8 + E_9 + E_{10}. \tag{F.10}$$

Note that we use the word "summand" here to refer to the variable that is being summed. For example, E_7 is defined as

$$\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathbb{E}[X_i(X_{i-j}+X_{i+j})]\,\mathbb{E}[f'(W)].$$

Then the summand in E_7 refers to

$$\frac{1}{\sigma^2}\mathbb{E}[X_i(X_{i-j}+X_{i+j})]\,\mathbb{E}[f'(W)],$$

and it factorizes into two expectations

$$\mathbb{E}[X_i(X_{i-j}+X_{i+j})] \quad \text{and} \quad \mathbb{E}[f'(W)]$$

with a scaling constant σ^{-2}.

Re-examining the procedure, we see that what we actually have done is approximating $\mathbb{E}[Wf(W)]$ by $E_7 + E_{11}$ with error terms $E_1, E_4, E_5, E_6, E_8, E_9, E_{10}$. Then the "local" errors, E_1, E_8, E_9, are bounded directly by remainder estimation from the Taylor expansion, while to study the other terms we need to apply Lemma F.1. As we try to generalize this, we need to be careful that to apply Lemma F.1, the error terms need to have a factor that appears as a covariance rather than any arbitrary expectation. The idea to enforce this requirement is that for any random variables X, Y, we keep track of $\mathrm{Cov}(X,Y)$ a priori instead of writing out $\mathbb{E}[XY]$ and $\mathbb{E}[X]\,\mathbb{E}[Y]$ separately. To generalize, we will introduce a multilinear operator $\mathcal{D}^*$. In particular, for any random variables X, Y, Z, we let

$$\mathcal{D}^*(X) := \mathbb{E}[X], \quad \mathcal{D}^*(X,Y) := \mathrm{Cov}(X,Y),$$
$$\mathcal{D}^*(X,Y,Z) := \mathbb{E}[XYZ] - \mathbb{E}[XY]\,\mathbb{E}[Z] - \mathbb{E}[X]\,\mathbb{E}[YZ] + \mathbb{E}[X]\,\mathbb{E}[Y]\,\mathbb{E}[Z].$$

In the previous example, we can rewrite the expansion as

$$\begin{aligned}\mathbb{E}[Wf(W)] &= (E_1 + E_3) + E_2 \\ &= E_1 + (E_3 - E_{11}) + E_{11} + E_4 + E_5 + E_6 + E_7.\end{aligned} \tag{F.11}$$

Here

$$E_1 + E_3 = \frac{1}{\sigma}\sum_{i=1}^{n}\mathcal{D}^*\big(X_i\,,\, f(W) - f(W_{i,m})\big), \quad E_2 = \frac{1}{\sigma}\sum_{i=1}^{n}\mathcal{D}^*\big(X_i\,,\, f(W_{i,m})\big).$$

Noting that

$$\begin{aligned} f(W) - f(W_{i,m}) - f'(W)(W - W_{i,m}) &= (W - W_{i,m})\int_0^1 \Big(f'\big(\nu W + (1-\nu)W_{i,m}\big) - f'(W)\Big)\,\mathrm{d}\nu \\ &= \frac{1}{\sigma}\sum_{j=i-m}^{i+m} X_j \int_0^1 \Big(f'\big(\nu W + (1-\nu)W_{i,m}\big) - f'(W)\Big)\,\mathrm{d}\nu, \end{aligned}$$

we get

$$
\begin{aligned}
E_1 =& \frac{1}{\sigma}\sum_{i=1}^{n}\mathbb{E}\big[X_i\big(f(W)-f(W_{i,m})-f'(W)(W-W_{i,m})\big)\big] \\
=& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathbb{E}\Big[X_iX_j\int_0^1\Big(f'\big(\nu W+(1-\nu)W_{i,m}\big)-f'(W)\Big)\mathrm{d}\nu\Big] \\
=& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathcal{D}^*\Big(X_i\,,X_j\int_0^1\Big(f'\big(\nu W+(1-\nu)W_{i,m}\big)-f'(W)\Big)\mathrm{d}\nu\Big).
\end{aligned}
\tag{F.12}
$$

We can further check that

$$
\begin{aligned}
E_3-E_{11} =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\mathcal{D}^*\big(X_i\,,X_j\,,f'(W)\big), \\
E_{11} =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=i-m}^{i+m}\ \mathcal{D}^*(X_i,X_j)\ \mathcal{D}^*(f'(W)), \\
E_4 =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathcal{D}^*\Big(X_i\,,X_{i-j}\int_0^1\Big(f'\big(\nu W_{i,j-1}+(1-\nu)W^*_{i,j}\big)-f'(W_{i,j-1})\Big)\Big), \\
&+\frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathcal{D}^*\Big(X_i\,,X_{i+j}\int_0^1\Big(f'\big(\nu W^*_{i,j}+(1-\nu)W_{i,j}\big)-f'(W^*_{i,j})\Big)\Big), \\
E_5 =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\Big(\mathcal{D}^*\big(X_i\,,X_{i-j}\,,f'(W_{i,j-1})\big)+\mathcal{D}^*\big(X_i\,,X_{i+j}\,,f'(W^*_{i,j})\big)\Big), \\
E_6 =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\Big(\mathcal{D}^*\big(X_i\,,X_{i-j}\big)\ \mathcal{D}^*\big(f'(W_{i,j-1})-f'(W)\big) \\
&+\mathcal{D}^*\big(X_i\,,X_{i+j}\big)\ \mathcal{D}^*\big(f'(W^*_{i,j})-f'(W)\big)\Big), \\
E_7 =& \frac{1}{\sigma^2}\sum_{i=1}^{n}\sum_{j=m+1}^{n-1}\mathcal{D}^*(X_i\,,X_{i-j}+X_{i+j})\ \mathcal{D}^*(f'(W)).
\end{aligned}
$$

Thus, each summand in $E_1+E_3, E_2, E_1, E_3-E_{11}, E_{11}, E_4, \cdots, E_7$ is either a $\mathcal{D}^*$ term or the product of two $\mathcal{D}^*$ terms with some scaling constant.

Next we aim to encode the structure of these sums in a more efficient way. In general, we need to take into account the following issues:

- How each summand in the expansion factorizes into $\mathcal{D}^*$ terms;
- How each $\mathcal{D}^*$ term is constructed;
- Which values the running indexes in the summand are allowed to take.

To address all these issues, we introduce an abstract structure called a "genogram", consisting of a rooted tree and integers bigger or equal to -1 (called "identifiers"), each attached to a vertex of the rooted tree and satisfying certain requirements. Then we represent each sum (e.g. $E_1 + E_3, E_2, E_1, E_3 - E_{11}, E_{11}, E_4, \cdots, E_7$) with the help of genograms such that

- Each vertex of the rooted tree corresponds to a running index of summation (i.e., $i_1, i_2, \cdots$);
- Each branch (or each leaf) of the rooted tree corresponds to a $\mathcal{D}^*$ factor of the summand;
- The signs of identifiers control how each $\mathcal{D}^*$ term is contructed;
- The values of identifiers help determine the sets of values that running indexes take by encoding their distance structure, which reflects the dependency between corresponding random variables.

Interestingly, the process of expanding $\mathbb{E}[Wf(W)]$ precisely corresponds to growing a class of genograms. Instead of deriving the expansion solely for $\mathbb{E}[Wf(W)]$, we get a similar expansion for any genogram G and quantities $\mathcal{T}_f(G)$ (formally defined in (F.23)). In the expansion of $\mathcal{T}_f(G)$, the cumulants in (F.1) are replaced by other constants that depend on both G and the joint distribution of $(X_i)_{i\in T}$ but not on f.

As we will see later, $\mathbb{E}[Wf(W)]$ corresponds to $\mathcal{T}_f(G)$ with G being the order-1 genogram, which consists of only the root vertex. For this special case, directly calculation with f set to be polynomials helps recover the constants as the cumulants of W, and thus, (F.1) is obtained for general f by uniqueness of the constants. Finally, we carefully collect and control the remainders with the mixing coefficients.

The rest of the section is constructed as follows: In Appendix F.2, we formally define a genogram and related concepts. In Appendix F.3, we define three types of sums corresponding to a genogram, which will be used later in the expansion. In Appendix F.4, we show how to achieve the expansion by growing a class of genograms. In Appendix F.5, we control the remainders using the mixing coefficients. Finally, in Appendix F.6, we provide the proofs of Lemmas E.1 and E.2.

F.2 Genograms

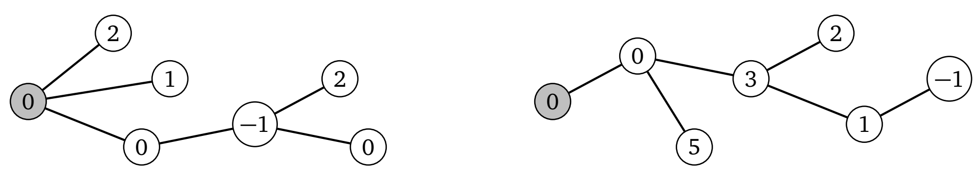
A **rooted tree** is a tree in which one vertex has been designated the root. In a rooted tree, the **parent** of a vertex v is the vertex connected to v on the path from the root to v; every vertex has a unique parent except the root, which has no parent. A **child** of a vertex v is a vertex of which v is the parent. An **ancestor** of a vertex v is any vertex other than v which is on the path from the root to v. A **sibling** to a vertex v is any other vertex on the tree which has the same parent as v. A **leaf** is a vertex with no children. See Bender & Williamson [2010] for a detailed exposition.

An order-k **genogram** is defined as the tuple $G := \big(V, E, \{s_v\}_{v\in V}\big)$, where (V, E) is a rooted tree with a vertex set V ($|G| := |V| = k$) and an edge set E, and s_v is an integer called the **identifier** associated to each $v \in V$ that satisfies the requirements below:

- $s_v = 0$ for the root v. $s_v \geq -1$ for any other vertex v;
- $s_v \geq 0$ if v is a child of the root;
- If v has more than one child, identifiers of v's children must be non-negative and mutually different.

Beware that the identifiers are part of the genograms by definition. We say that a vertex v is **negative** if and only if $s_v = -1$, **nil** if and only if $s_v = 0$, **positive** if and only if $s_v \geq 1$. The requirements above implies that the identifier of each child of v is different, and therefore, v's children can be uniquely identified by their identifiers. The last requirement also suggests that if v has more than one child, there is no negative and at most one nil among them. In other words, a negative vertex has no sibling, a nil vertex only has positive siblings, and any vertex must have an identifier different from all its siblings.

Furthermore, denote the set of all possible order-k genograms by $\mathcal{G}(k)$. Figure 1 depicts two examples of genograms $G_1, G_2 \in \mathcal{G}(7)$, where each circle represents a vertex, the one representing the root is filled with gray, and the identifiers are marked inside the circles.



(a) G_1. (b) G_2.

Figure 1: Examples of order-7 genograms.

We remark that the notion of genograms resembles the ordered trees in combinatorics. An **ordered (rooted) tree** $(V, E, \prec)$ is a rooted tree (V, E) where the children of every vertex are ordered (the order denoted by $\prec$) [Stanley 2011]. Note that $\prec$ is a strict partial order on the vertex set V. By definition every genogram induces a unique ordered tree if we set $v_1 \prec v_2 \iff s_{v_1} > s_{v_2}$ whenever $v_1, v_2 \in V$ are siblings. However, an ordered tree corresponds to infinitely many genograms since the largest identifier is allowed to take any sufficiently large value in $\mathbb{N} \cup \{-1\}$.

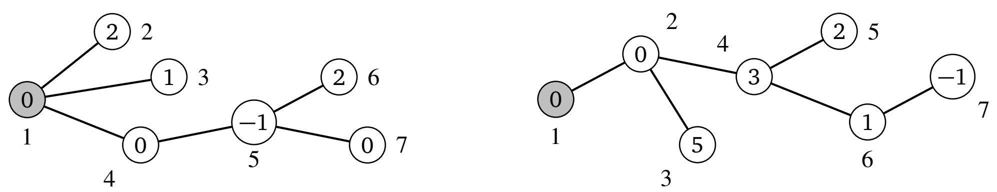
Compatible labeling, parent, progenitor, and ancestor

Next we consider a labeling of the vertices of a genogram (or the induced ordered tree of a genogram). We say a labeling $V = \{v[1], \cdots, v[k]\}$ is **compatible** with G (or $(V, E, \prec)$) if and only if

(a) It follows from a depth-first traversal: $v[1]$ is the root, and for any $1 \leq j \leq k-1$, the vertex $v[j+1]$ is chosen to be a children of the vertex with the largest label $\ell \leq j$ that has children. In particular, $v[j+1]$ is $v[j]$'s child as long as $v[j]$ has a child;

(b) It respects the partial order $\prec$ induced by G: If $v[j]$ and $v[h]$ ($2 \leq j, h \leq k$, $j \neq h$) are siblings, then we have $s_j > s_h \iff j < h$ (or equivalently, $v[j] \prec v[h] \iff j < h$). In other words, if a vertex has more than one child, a child with a larger identifier has a smaller label. In particular, if $s_j = 0$, then $v[j]$ has the largest label j.

Figure 2 shows the compatible labelings of G_1 and G_2, where the labels are marked outside the circles that represent the vertices.



(a) G_1. (b) G_2.

Figure 2: Examples of order-7 genograms with the compatible labeling.

We remark that there is a unique compatible labeling given any genogram G (or any ordered tree $(V, E, \prec)$).

Now we introduce more notations in order to express the compatible requirements for identifiers and the labeling in a more concise manner. Let $G = (V, E, s_{1:k})$ be an order $k-$genogram with vertices labelled as $V = \{v[1], \cdots, v[k]\}$and where s_j is the identifier of $v[j]$ for $1 \leq j \leq k$. We denote the label of $v[j]$'s **parent** by $p(j, G)$ for $2 \leq j \leq k+1$, and the label set of $v[j]$'s **ancestors** by $A(j, G)$ (we set $A(1, G) = \emptyset$). Moreover, we write

$$g(j, G) := \sup\{\ell : \ell = 1 \text{ or } \ell \in A(j, G) \;\&\; s_\ell \geq 1\}, \tag{F.13}$$

and call $v[g(j, G)]$ the **progenitor** of $v[j]$. In particular, we have that $g(1, G) = 1$. Intuitively, $v[g(j, G)]$ is the positive vertex closest to $v[j]$ in its ancestry if such vertex exists, in which case there is a path from $v[g(j, G)]$ to $v[p(j, G)]$, the parent of $v[j]$, such that $v[g(j, G)]$ is the only positive vertex along the path. Otherwise, $v[g(j, G)]$ is set to be the root. Note that $v[g(j, G)] \neq v[j]$ for $2 \leq j \leq k$. Take the genograms G_1 and G_2 in Figure 2 as examples, $g(j, G_1) = 1$ for all $1 \leq j \leq 7$ while in G_2, $g(1, G_2) = g(2, G_2) = g(3, G_2) = g(4, G_2) = 1$, $g(5, G_2) = g(6, G_2) = 4$, and $g(7, G_2) = 6$. We further denote

$$u(j, G) := \sup\{\ell \in \{j\} \cup A(j, G) : s_\ell \geq 0\}. \tag{F.14}$$

In other words, $v[u(j, G)]$ is the closest non-negative vertex in $v[j]$'s ancestry if $v[j]$ is negative, otherwise $u(j, G) = j$. In particular, $s_j = -1 \Leftrightarrow u(j, G) < j$, $s_j \geq 0 \Leftrightarrow u(j, G) = j$. For example, in the genogram G_1 shown in Figure 2a, $u(5, G_1) = 4$ and $u(j, G_1) = j$ for $j \neq 5$. For ease of notation, when there is no ambiguity, we will abuse notations and write $p(j), A(j), g(j), u(j)$ to mean $p(j, G), A(j, G), g(j, G), u(j, G)$.

We remark that the labeling has to respect the following properties

Proposition F.2. *Let k be a positive integer, (V, E) be a rooted tree with the vertex set $V = \{v[1], \cdots, v[k]\}$ and edge set E, and $s_1, \cdots, s_k$ be k integers. $(V, E, \{s_{1:k}\})$ is a genogram with the compatible labeling if and only if all the following statements are true:*

(a) $p(j+1) = \max\{p(\ell) : \ell \geq j+1, p(\ell) \leq j\}$ *for* $1 \leq j \leq k-1$*;*

(b) $s_1 = 0$. $s_j \geq -1$ *for* $2 \leq j \leq k$*;*

(c) If $s_j = -1$ $(2 \leq j \leq k)$*, then (i)* $p(j) \neq 1$*, (ii)* $p(j) = p(h) \Leftrightarrow j = h$ *for* $2 \leq h \leq k$*;*

(d) If $p(j) = p(h)$ $(2 \leq j, h \leq k)$*, then* $s_j > s_h \Leftrightarrow j < h$*,* $s_j = s_h \Leftrightarrow j = h$*.*

Induced sub-genograms

Lastly, given $G = (V, E, s_{1:k})$ and $1 \leq j \leq k$, we call an order-j genogram $G[j] := (V', E', s_{1:j})$ the induced **sub-genogram** of G the genograms by setting $V' := \{v[1], \cdots, v[j]\}$ and $E' \subseteq E$ be the set of all edges between the vertices V' in G. We further denote $H \subseteq G$ or $G \supseteq H$ if and only if a genogram H is a sub-genogram of G. If $j < k$, we say $G[j]$ is a **proper sub-genogram** of G and write $G[j] \subsetneq G$ or $G \supsetneq G[j]$.

F.3 Constructing sums from genograms

Consider a d-dimensional random field $(X_i)_{i \in T}$ with the index set T satisfying $T \subsetneq \mathbb{Z}^d$ and $|T| < \infty$. We write

$$\sigma^2 := \mathrm{Var}\Big(\sum_{i \in T} X_i\Big) \qquad W := \sigma^{-1} \sum_{i \in T} X_i.$$

For any index subset $J \subseteq T$, we denote

$$W(J) := \sigma^{-1} \sum_{i \in T \setminus J} X_i.$$

In this subsection, we build sums $\mathcal{S}(G), \mathcal{T}_f(G), \mathcal{U}_f(G)$ from a genogram G and a given function $f \in \mathcal{C}^{k-1}(\mathbb{R})$ in four steps:

- Use the genogram G to define the sets of values taken by running indexes;
- Introduce the generalized covariance operator $\mathcal{D}^*$;
- Construct an operator $\mathcal{E}_G$ from G, which leads to the summand;
- Define $\mathcal{S}(G), \mathcal{T}_f(G), \mathcal{U}_f(G)$.

Note that these sums will be used in the next subsection to track the expansion of the quantity $\mathbb{E}[Wf(W)]$.

Firstly, as we have pointed in the roadmap, we will construct from an order-k genogram G sums with k running indexes, where the $v[j]$ corresponds to the j-th running index, denoted by i_j. Since i_j will appear in the subscript of X_{i_j}, the value of i_j needs to be chosen from T. It is important to note that the vertices $v[1], \cdots, v[k]$ as well as the genogram do not represent specific values of

$i_1, \cdots, i_k$. The genogram reflects the dependency structure of random variables appearing in the sum by encoding the distance structure between the running indexes.

Setting $B_1 := T$ and $D_1 := \emptyset$, i_1 will be summed over $B_1 \backslash D_1 = T$. Next given the choice of the first $j-1$ running indexes ($j \geq 2$), we aim to define two index sets, B_j and D_j, using the chosen values $i_1, \cdots, i_{j-1}$ and the order-j sub-genogram $G[j]$. In the last step, we will take the sums over $i_k \in B_k \backslash D_k, T \backslash B_k$ or $T \backslash D_k$, and then $i_{k-1} \in B_{k-1} \backslash D_{k-1}, \cdots, i_1 \in B_1 \backslash D_1$ in turn. We call B_j the **outer constraint (set)** of the running index i_j, and D_j, the **inner constraint (set)** of i_j. For ease of notation, on most occasions we do not explicitly write out the dependencies on $G[j]$ and $i_1, \cdots, i_{j-1}$ when referring to the constraint sets B_j and D_j. However, if we are considering multiple genograms, we will use $B_j(G)$ and $D_j(G)$ to specify the constraint sets of i_j with respect to G to avoid ambiguity.

We will formally define B_j and D_j for $2 \leq j \leq k$ later by induction. Bur first we consider the case $j = 2$ to build intuition. When the first running index is set to be some specific element $i_1 \in T$, we define $B_2, D_2 \subseteq T$ using i_1 and s_2, and i_2 will be summed over $B_2 \backslash D_2$. If $s_2 = 0$, i_2 will be summed over all the indexes of distance no greater than m from i_1, in which case B_2 and D_2 are defined by

$$B_2 := \{i \in T : \|i - i_1\| \leq m\}, \quad D_2 := \emptyset,$$

where $\|\cdot\|$ is the maximum norm on $\mathbb{Z}^d$. Note that m is a positive integer that we have fixed earlier.

Otherwise, $s_2 \geq 1$, we let i_2 take the sum over a singleton $B_2 \backslash D_2$. In other words, the second running index has only one possible choice in the summation. Different values of $1 \leq s_2 \leq \left|\{i \in T : \|i - i_1\| \geq m+1\}\right| =: s^*$ will correspond to different singletons of elements in $\{i \in T : \|i - i_1\| \geq m+1\}$. Let $\prec$ be a strict total order on $\mathbb{Z}^d$. With the first level comparing the value of $\|i - i_1\|$ and the second level using the strict order $\prec$, we perform a two-level sorting of all elements i from $\{i \in T : \|i - i_1\| \geq m+1\}$ and obtain an ascending sequence, $z_1, \cdots, z_{s^*}$. Now i_2 is chosen to be z_{s_2}, and

$$B_2 := \{i \in T : \|i - i_1\| \leq m\} \cup \{z_j : 1 \leq j \leq s_2\}, \quad D_2 := \{i \in T : \|i - i_1\| \leq m\} \cup \{z_j : 1 \leq j \leq s_2 - 1\}.$$

The motivation of using singletons arises from deriving (F.5), where we have decomposed the quantity $\mathbb{E}[X_i f(W_{i,m})]$ into a telescoping sum:

$$\mathbb{E}[X_i f(W_{i,m})] = \sum_{j=m+1}^{n-1} \mathbb{E}\big[X_i\big((f(W_{i,j-1}) - f(W_{i,j}^*)) + (f(W_{i,j}^*) - f(W_{i,j}))\big)\big].$$

In order to accurately approximate the differences $f(W_{i,j-1}) - f(W_{i,j}^*)$ and $f(W_{i,j}^*) - f(W_{i,j})$ by the Taylor expansions, the differences between the inputs of f need to be small enough. The best we can do is to put exactly one random variable in each of such differences.

In general, we introduce some new notations in order to define B_j and D_j ($2 \leq j \leq k$) more conveniently. Denote the m-neighborhood of $J \subseteq T$ as

$$N(J) := \{i \in T : d(i, J) \leq m\}, \text{ where } d(i, J) := \min_{j \in J} \|i - j\|. \tag{F.15}$$

We will treat each element of $T \backslash N(J)$ sequentially starting from the closest elements to J. To make this precise, for any positive integer j, we write $A^{(j)}(J) := \{i \in T : d(i, J) = j\}$ to be the set of indexes in T that are at distance of j from J. Notably we remark that $T\backslash N(J) = \bigcup_{j=m+1}^{|T|} A^{(j)}(J)$. We write $\mathrm{rk}(i,J) = \sum_{\ell=m+1}^{j} |A^{(\ell)}(J)| + r$ if $d(i,J) = j+1$ ($j \geq m$) and i is the r-th smallest element of $A^{(j+1)}(J)$ with respect to the order $\prec$. Therefore, $\mathrm{rk}(\cdot,J)$ is a bijection between $T\backslash N(J)$ and $\{\ell \in \mathbb{Z} : 1 \leq \ell \leq |T\backslash N(J)|\}$. The smaller $\mathrm{rk}(i,J)$ is, the closer i is from J. The value of $\mathrm{rk}(\cdot,J)$ does not have an intrinsic mathematical significance but will allow us to determine the order in which we will treat the indexes in $T\backslash N(J)$. Now denote

$$N^{(s)}(J) := \{i \in T : i \in N(J) \text{ or } \mathrm{rk}(i,J) \leq s\}. \tag{F.16}$$

We note in particular that $N^{(0)}(J) = N(J)$, and $N^{(s_L)}(J) = \{i \in T : d(i,J) \leq L\}$ for $s_L := \sum_{\ell=0}^{L} |A^{(\ell)}(J)|$.

For any $2 \leq j \leq k$, fixing the genogram G and a sequence $i_1 \in B_1\backslash D_1, \cdots, i_{j-1} \in B_{j-1}\backslash D_{j-1}$, we define

$$B_j := \begin{cases} N^{(s_j)}\big(i_\ell : \ell \in A(j)\big) \cup D_{g(j)} & \text{if } s_j \geq 0 \\ B_{u(j)} & \text{if } s_j = -1 \end{cases}, \tag{F.17}$$

$$D_j := \begin{cases} N^{(s_j-1)}\big(i_\ell : \ell \in A(j)\big) \cup D_{g(j)} & \text{if } s_j \geq 1 \\ D_{g(j)} & \text{if } s_j \leq 0 \end{cases}. \tag{F.18}$$

Here (by abuse of notation) $A(j)$ is the label set of $v[j]$'s ancestors, and $u(j)$ and $g(j)$ are defined in (F.14) and (F.13). Note that B_j and D_j depend on $G[j]$ through $s_j, A(j), u(j)$, and $g(j)$. Moreover, by definition, we have that $D_j \subseteq B_j$ for any $1 \leq j \leq k$. We remark that when the identifier $s_j = 0$ is nill then $B_j\backslash D_j \subseteq N(i_\ell : \ell \in A(j))$, and when $s_j \geq 1$ then $B_j\backslash D_j$ is either empty or a singleton with element the unique i such that $\mathrm{rk}\big(i, \{i_\ell \in T : \ell \in A(j)\}\big) = s_j$. Finally, if $s_j = -1$, then $B_j\backslash D_j = B_{p(j)}\backslash D_{p(j)}$.

For instance, we consider the genograms G_1 and G_2 shown in Figure 2. The constraint sets of the running indexes are presented in Table 1.

Table 1: The constraint sets with respect to G_1 and G_2.

	j	1	2	3	4
G_1	$B_j(G_1)$	T	$N^{(2)}(i_1)$	$N^{(1)}(i_1)$	$N(i_1)$
	$D_j(G_1)$	$\emptyset$	$N^{(1)}(i_1)$	$N(i_1)$	$\emptyset$
G_2	$B_j(G_2)$	T	$N(i_1)$	$N^{(5)}(i_1,i_2)$	$N^{(3)}(i_1,i_2)$
	$D_j(G_2)$	$\emptyset$	$\emptyset$	$N^{(4)}(i_1,i_2)$	$N^{(2)}(i_1,i_2)$

	j	5	6	7
G_1	$B_j(G_1)$	$N(i_1)$	$N^{(2)}(i_1,i_4,i_5)$	$N(i_1,i_4,i_5)$
	$D_j(G_1)$	$\emptyset$	$N^{(1)}(i_1,i_4,i_5)$	$\emptyset$
G_2	$B_j(G_2)$	$N^{(2)}(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$	$N^{(1)}(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$	$N^{(1)}(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$
	$D_j(G_2)$	$N^{(1)}(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$	$N(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$	$N(i_1,i_2,i_4) \cup N^{(2)}(i_1,i_2)$

Secondly, as described in the roadmap, we define the **generalized covariance operator** $\mathcal{D}^*$ on a finite sequence of random variables $(Y_i)_{i\geq 1}$. To do so, we also need to inductively define another operator $\mathcal{D}$ that takes in a finite sequence of random variables and outputs a new random variable. For any random variable Y, define

$$\mathcal{D}^*(Y) := \mathbb{E}[Y], \quad \mathcal{D}(Y) := Y - \mathcal{D}^*(Y) = Y - \mathbb{E}[Y].$$

Suppose $\mathcal{D}$ is already defined for a random sequence of length $t-1$. Then for any random variables $Y_1, \cdots, Y_t$, let

$$\mathcal{D}(Y_1, Y_2, \cdots, Y_t) := \mathcal{D}\big(Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big) = Y_1 \mathcal{D}(Y_2, \cdots, Y_t) - \mathbb{E}\big[Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big],$$
$$\mathcal{D}^*(Y_1, Y_2, \cdots, Y_t) := \mathcal{D}^*\big(Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big) = \mathbb{E}\big[Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big].$$

In particular, for any two random variables Y_1 and Y_2, $\mathcal{D}^*(Y_1, Y_2) = \mathrm{Cov}(Y_1, Y_2)$ gives the covariance between Y_1 and Y_2. Here we remark that $\mathcal{D}(Y_1, \cdots, Y_t)$ and $\mathcal{D}^*(Y_1, \cdots, Y_t)$ are well-defined for a tuple of t random variables $(Y_i)_{i=1}^t$ supposing that for any $i, j \in \mathbb{N}_+$ such that $i \le j \le t$ we have $\mathbb{E}\big[|Y_i Y_{i+1} \cdots Y_j|\big] < \infty$. It is straightforward to see from the definition that both operators are multilinear. We will show more properties of them in Appendix H.

Thirdly, we construct from any genogram a new operator $\mathcal{E}_G$ that maps from $|G|$ random variables to a real number. Note that this $\mathcal{E}_G$ operator will provide us with the summands in $\mathcal{S}(G)$, $\mathcal{T}_f(G)$, and $\mathcal{U}_f(G)$. If $|G| = 1$, for any random variable Y, define $\mathcal{E}_G(Y) := \mathcal{D}^*(Y) = \mathbb{E}[Y]$. Suppose $\mathcal{E}_G$ is already defined for $|G| \le k-1$. Consider the case where $|G| = k$. Let

$$q_0 := \sup\{j : j = 1 \text{ or } p(j) \ne j-1 \text{ for } 2 \le j \le k\}, \tag{F.19}$$

In other words, either $v[q_0 - 1]$ is the leaf with the largest label smaller than k, or alternatively $v[k]$ is the only leaf and $q_0 = 1$. Intuitively, $v[q_0]$ is the starting vertex of the last branch of G. Next we set $w := \big|\{t : q_0 + 1 \le t \le k \ \& \ s_t \ge 0\}\big|$ to be the number of all indices $q_0 + 1 \le t \le k$ such that the identifier $s_t \ge 0$. If $w = 0$, define

$$\mathcal{E}_G(Y_1, \cdots, Y_k) := \begin{cases} \mathcal{D}^*\big(Y_1 Y_2 \cdots Y_k\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(Y_1, \cdots, Y_{q_0-1}\big) \cdot \mathcal{D}^*\big(Y_{q_0} Y_{q_0+1} \cdots Y_k\big) & \text{if } q_0 \ge 2 \end{cases}, \tag{F.20}$$

where $G[q_0 - 1] \subseteq G$ is the unique order-$(q_0 - 1)$ sub-genogram of G as defined in Appendix F.2.

Otherwise, we write $\{t : q_0 + 1 \le t \le k \ \& \ s_t \ge 0\} = \{q_1, \cdots, q_w\}$. Without loss of generality, we suppose that $q_0 + 1 \le q_1 < \cdots < q_w \le k$ is increasing. We define

$$\mathcal{E}_G(Y_1, \cdots, Y_k) := \begin{cases} \mathcal{D}^*\big(Y_1 \cdots Y_{q_1-1} \,,\, Y_{q_1} \cdots Y_{q_2-1} \,,\, \cdots \,,\, Y_{q_w} \cdots Y_k\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(Y_1, \cdots, Y_{q_0-1}\big) \cdot & \\ \quad \mathcal{D}^*\big(Y_{q_0} \cdots Y_{q_1-1} \,,\, Y_{q_1} \cdots Y_{q_2-1} \,,\, \cdots \,,\, Y_{q_w} \cdots Y_k\big) & \text{if } q_0 \ge 2 \end{cases}. \tag{F.21}$$

By definition, we can see that $\mathcal{E}_G(Y_1, \cdots, Y_k)$ is either a $\mathcal{D}^*$ term or the product of multiple $\mathcal{D}^*$ terms, each of which corresponds to a branch of the rooted tree (V, E).

Taking the genograms G_1 and G_2 shown in Figure 2 as examples, $\mathcal{E}_{G_1}$ and $\mathcal{E}_{G_2}$ are provided by

$$\mathcal{E}_{G_1}(Y_1, \cdots, Y_7) = \mathcal{D}^*(Y_1, Y_2)\, \mathcal{D}^*(Y_3)\, \mathcal{D}^*(Y_4 Y_5 \,,\, Y_6)\, \mathcal{D}^*(Y_7),$$

$$\mathcal{E}_{G_2}(Y_1,\cdots,Y_7) = \mathcal{D}^*(Y_1,Y_2,Y_3)\,\mathcal{D}^*(Y_4,Y_5)\,\mathcal{D}^*(Y_6Y_7).$$

Finally, we define $\mathcal{S}(G), \mathcal{T}_f(G), \mathcal{U}_f(G)$ with respect to any genogram $G = (V, E, s_{1:k}) \in \mathcal{G}(k)$ and function $f \in \mathcal{C}^{k-1}(\mathbb{R})$.

$$\mathcal{S}(G) := \sigma^{-k} \sum_{i_1\in B_1\backslash D_1} \sum_{i_2\in B_2\backslash D_2} \cdots \sum_{i_k\in B_k\backslash D_k} \mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{k-1}},\, X_{i_k}\big), \tag{F.22}$$

$$\mathcal{T}_f(G) := \sigma^{-k} \sum_{i_1\in B_1\backslash D_1} \sum_{i_2\in B_2\backslash D_2} \cdots \sum_{i_k\in B_k\backslash D_k} \mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{k-1}},\, X_{i_k}\partial^{k-1}f\big(W(D_k)\big)\big), \tag{F.23}$$

$$\mathcal{U}_f(G) := \sigma^{-(k-1)} \sum_{i_1\in B_1\backslash D_1} \sum_{i_2\in B_2\backslash D_2} \cdots \sum_{i_{k-1}\in B_{k-1}\backslash D_{k-1}} \mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{k-1}},\, \Delta_f(G)\big), \tag{F.24}$$

where

$$\Delta_f(G) := \begin{cases} \partial^{k-2}f\big(W(B_k)\big) - \partial^{k-2}f\big(W(D_k)\big) & \text{if } u(k) = k \\ \displaystyle\int_0^1 (k-u(k))v^{k-1-u(k)}\Big(\partial^{k-2}f\big(vW(D_k)+(1-v)W(B_k)\big) \\ \quad -\partial^{k-2}f\big(W(D_k)\big)\Big)\,\mathrm{d}v & \text{if } u(k) \le k-1 \end{cases}.$$

is called the **adjusted f-difference**. Note that $\Delta_f(G)$ depends on G and $i_1,\cdots,i_{k-1}$ through B_k, D_k and $u(k)$, where $k = |G|$. For ease of notation, we do not write out the other dependencies $i_1,\cdots,i_{k-1}$.

Intuitively, $\mathcal{S}(G)$ is analogous to the $\mathcal{S}$-sums defined for the local dependence case while $\mathcal{T}_f(G)$ is analogous to the $\mathcal{T}$-sums. And $\mathcal{U}_f(G)$ is defined in a similar spirit to the $\mathcal{R}$-sums as they are both used to handle the remainders. Δ_f is obtained from the integral-form remainders of the Taylor expansions (see (F.12) and Lemma F.5). Eventually, we would like to expand $\mathcal{T}_f(G)$ using $\mathcal{S}(H)$ and $\mathcal{U}_f(H)$ for some $H \supseteq G$ as shown in Theorem F.6.

Now we revisit the case $k = d = 1$ discussed in Appendix F.2. We rewrite the quantities that appear in (F.11), i.e. $\mathbb{E}[Wf(W)], E_1+E_3, E_2, E_1, E_3-E_{11}, E_{11}, E_4,\cdots,E_7$ using the notations we have developed so far. For example,

$$\mathbb{E}[Wf(W)] = \sum_{i=1}^n \mathbb{E}[X_if(W)] = \sum_{i_1\in B_1\backslash D_1} \mathcal{D}^*(X_{i_1}f(W(D_1))) = \mathcal{T}_f\left(\textcircled{0}\right), \tag{F.25}$$

and E_2 is written as

$$\begin{aligned} E_2 &= \frac{1}{\sigma}\sum_{i=1}^n \mathcal{D}^*\big(X_i\,,\, f(W_{i,m})\big) \\ &= \frac{1}{\sigma}\sum_{i=1}^n \sum_{j=m+1}^{n-1} \Big(\mathcal{D}^*\big(X_i, f(W_{i,j-1}) - f(W^*_{i,j})\big) + \mathcal{D}^*\big(X_i, f(W^*_{i,j}) - f(W_{i,j})\big)\Big) \\ &= -\sum_{j=m+1}^{n-1} \mathcal{U}_f\left(\textcircled{0}\!-\!\textcircled{\scriptsize 2(j-m)-1}\right) - \sum_{j=m+1}^{n-1} \mathcal{U}_f\left(\textcircled{0}\!-\!\textcircled{\scriptsize 2(j-m)}\right) \end{aligned}$$

$$= -\sum_{j=1}^{2(n-m-1)} \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{j}\right).$$

Furthermore, we can express all the other quantities in a similar way, and (F.11) will transform into the following equation (details omitted):

$$\mathcal{T}_f\left(\textcircled{0}\right) = -\mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{0}\right) - \sum_{j=1}^{2(n-m-1)} \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{j}\right)$$

$$= \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{0}\!-\!\!-\textcircled{-1}\right) - \sum_{j=0}^{2(n-m-1)} \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{0}\!-\!\!-\textcircled{j}\right) + \mathcal{S}\left(\textcircled{0}\!-\!\!-\textcircled{0}\right)\,\mathbb{E}[f'(W)]$$

$$+ \sum_{j=1}^{2(n-m-1)} \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{j}\!-\!\!-\textcircled{-1}\right) - \sum_{j=1}^{2(n-m-1)} \sum_{\ell=0}^{2(n-m-1)} \mathcal{U}_f\left(\textcircled{0}\!-\!\!-\textcircled{j}\!-\!\!-\textcircled{\ell}\right) + \sum_{j=1}^{2(n-m-1)} \sum_{\ell=0}^{j-1} \mathcal{U}_f\left(\textcircled{0}\begin{smallmatrix} \nearrow \textcircled{j} \\ \searrow \textcircled{\ell} \end{smallmatrix}\right)$$

$$+ \sum_{j=1}^{2(n-m-1)} \mathcal{S}\left(\textcircled{0}\!-\!\!-\textcircled{j}\right)\,\mathbb{E}[f'(W)].$$

F.4 Expansions

Firstly, we consider how to grow a genogram by following the compatible labeling order of the vertices. Initially, there is only the root and $|G| = 1$. We would like $|G|$ to increase from 1 to k after repeatedly choosing a "growing" vertex and adding a child to that vertex. In order to obtain a genogram with the compatible labeling at each step, we observe that the growing vertex needs to be $v[|G|]$ or an ancestor of $v[|G|]$, which is formally stated in Lemma F.3. Moreover, a negative vertex can only be added to $v[|G|]$ because negative vertices do not have siblings as required in the definition of genograms (see Proposition F.2c).

Lemma F.3. *Let (V, E) be a rooted tree with $V := \{v[1], \cdots, v[k]\}$, whose vertex labels satisfy Proposition F.2a. Then for any $1 \le j \le k-1$ either $p(j+1) = j$ or $p(j+1) \in A(j)$ and j is a leaf.*

The proof of Lemma F.3 is in Appendix G.

In general, a genogram can be constructed by repeating the following two operations (not necessarily consecutively):

(a) $G \rightsquigarrow \Omega[j, s_{|G|+1}](G)$: Fix the growing vertex $v[j]$ to be $v[|G|]$ or an ancestor of $v[|G|]$ that satisfies $\min_{t \le |G|: p(t)=j}\{s_t\} \ge 1$. Add a non-negative child $v[|G|+1]$ to $v[j]$ and choose $s_{|G|+1}$ to satisfy $0 \le s_{|G|+1} < \min_{t \le |G|: p(t)=j}\{s_t\}$;

(b) $G \rightsquigarrow \Lambda[h](G)$: Add a path of h negative vertices to $v[|G|]$. In other words, $v[|G|+t]$ is added as the single child of $v[|G|+t-1]$ for $t = 1, \cdots, h$, and we set $s_{|G|+1} = \cdots = s_{|G|+h} = -1$.

Take G_2 shown in Figure 2b as an example. A negative vertex can only be added as a child of $v[7]$ while a non-negative vertex can be added as a child of $v[4]$ with the identifier $0, 1$ or 2, a child of $v[6]$ with identifier 0, or a child of $v[7]$ with any non-negative identifier.

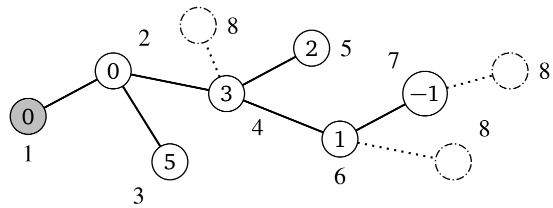


Figure 3: Adding a new vertex to G_2.

From the derivation of (F.5), we note that the expansion of $\mathcal{T}_f(G)$ is typically achieved by constructing a telescoping sum followed by the Taylor expansions. We will see later that the first operation of growing genograms corresponds to the telescoping sum argument, and the second one corresponds to the Taylor expansion, which will be formalized in Lemmas F.4 and F.5.

Lemma F.4. *Given an integer ℓ and an order-ℓ genogram G, let $\Omega[j,s](G)$ ($1 \le j \le \ell$, $s \ge 0$) be the genogram obtained by growing a child from the vertex $v[j]$, as defined in (a). Then we have*

$$\mathcal{T}_f(G) - \mathcal{S}(G)\,\mathbb{E}\big[\partial^{\ell-1} f(W)\big] = -\sum_{s\ge 0} \mathcal{U}_f\big(\Omega[\ell,s](G)\big) + \sum_{j\in A(\ell): s_j\ge 1} \sum_{s=0}^{s_j-1} \mathcal{U}_f\big(\Omega[p(j),s](G)\big), \tag{F.26}$$

$$= -\sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0,\\ p(\ell+1,H)=\ell}} \mathcal{U}_f(H) + \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0,\\ p(\ell+1,H)<\ell}} \mathcal{U}_f(H), \tag{F.27}$$

where $p(j)$ is the label of the parent of $v[j]$, s_j is the identifier, and $\mathcal{G}(k)$ is the set of all order-k genograms, all of which are defined in Appendix F.2.

Note that Lemma F.4 (proved in Appendix G) generalizes the idea of expanding to a telescoping sum while deriving (F.5).

Lemma F.5. *Given two integers $\ell \ge 1, k \ge 0$, and an order-ℓ genogram G, let $\Lambda[j](G)$ ($0 \le j \le k+1$) be the genogram obtained by gluing a path of j negative vertices to $v[\ell]$, as defined in (b) of Appendix F.2. Then we have*

$$\mathcal{U}_f(G) = \sum_{j=0}^{k} (-1)^{j+1} \frac{(\ell-u(\ell))!}{(j+1+\ell-u(\ell))!} \mathcal{T}_f\big(\Lambda[j](G)\big) + (-1)^{k+1} \frac{(\ell-u(\ell))!}{(k+1+\ell-u(\ell))!} \mathcal{U}_f\big(\Lambda[k+1](G)\big), \tag{F.28}$$

where $u(j)$ is given by (F.14).

As we have pointed out, Lemma F.5 is a direct consequence of the Taylor expansion, and the proof is also provided in Appendix G.

Theorem F.6. *Given a genogram G and an integer $k \ge |G|$, then the equation below holds for any $f \in \mathcal{C}^{k-1}(\mathbb{R})$*

$$\mathcal{T}_f(G) = \sum_{\substack{H\supseteq G:\\ |H|\le k,\\ s_{|G|+1}\ge 0}} a_{H,G}\,\mathcal{S}(H)\,\mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{|G|+1}\ge 0}} b_{H,G}\,\mathcal{U}_f(H), \tag{F.29}$$

where the coefficients $a_{H,G}$ and $b_{H,G}$ are provided by

$$a_{H,G} := \begin{cases} 1 & \text{if } |H| = |G| \\ (-1)^{\gamma_H-\gamma_G+\tau_H-\tau_G} \prod_{j=|G|+1}^{|H|} \frac{1}{j+1-u(j,H)} & \text{if } |H| \geq |G|+1 \end{cases}, \tag{F.30}$$

$$b_{H,G} := \begin{cases} (-1)^{\gamma_H-\gamma_G+\tau_H-\tau_G+1} & \text{if } |H| = |G|+1 \\ (-1)^{\gamma_H-\gamma_G+\tau_H-\tau_G+1} \prod_{j=|G|+1}^{|H|-1} \frac{1}{j+1-u(j,H)} & \text{if } |H| \geq |G|+2 \end{cases}. \tag{F.31}$$

Here γ_G denotes the number of leaves on G and τ_G is the number of negative vertices on G.

It will be useful in the future to note that for all genograms $H \supseteq G$ with $|H| \geq |G|+1$ we have $a_{H,G} = -\frac{b_{H,G}}{|H|+1-u(|H|,H)}$.

Proof. For convenience, denote $\ell := |G|$. We prove by performing induction on k ($k \geq \ell$).

If $k = \ell$, then we note that the set $\{H \supseteq G : |H| \leq k\} = \{G\}$ only contains the genogram G. Moreover, in (F.30) we set $a_{G,G} = 1$. Therefore, we obtain that

$$\begin{aligned}&\sum_{\substack{H\supseteq G:\\ |H|\leq k,\\ s_{|G|+1}\geq 0}} a_{H,G}\,\mathcal{S}(H)\,\mathbb{E}\big[\partial^{|H|-1}f(W)\big] + \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{|G|+1}\geq 0}} b_{H,G}\,\mathcal{U}_f(H)\\ =&\mathcal{S}(G)\,\mathbb{E}\big[\partial^{|H|-1}f(G)\big] + \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0}} b_{H,G}\,\mathcal{U}_f(H).\end{aligned}$$

Next let $H \in \mathcal{G}(\ell+1)$ be a genogram of order $\ell+1$ such that $H \supseteq G$ and such that $s_{\ell+1} \geq 0$. Note that $s_{\ell+1} \geq 0$ implies that $\tau_H = \tau_G$. To calculate $\gamma_H - \gamma_G$, we check that the parent of $v[\ell+1]$ is either $v[\ell]$ (i.e., $p(\ell+1,H) = \ell$) or an ancestor of $v[\ell]$. If its parent is $v[\ell]$ then the number of leaves in H, denoted by γ_H, is the same as that of G, and hence by (F.31) $b_{H,G} = -1$. Otherwise, the number of leaves increases by exactly one: $\gamma_H = \gamma_G + 1$, which implies $b_{H,G} = 1$. Thus, (F.29) reduces to

$$\sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0}} b_{H,G}\,\mathcal{U}_f(H) = -\sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1,H)=\ell}} \mathcal{U}_f(H) + \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1,H)<\ell}} \mathcal{U}_f(H),$$

Using Lemma F.4 we directly obtain that

$$\mathcal{T}_f(G) = \mathcal{S}(G)\,\mathbb{E}\big[\partial^{\ell-1}f(W)\big] - \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1,H)=\ell}} \mathcal{U}_f(H) + \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1,H)<\ell}} \mathcal{U}_f(H),$$

and the desired result is proven.

Now supposing the statement is true for k, we will establish that the desired result also holds for $k+1$.

By inductive hypothesis we have

$$\mathcal{T}_f(G) = \sum_{\substack{H\supseteq G:\\ |H|\le k,\\ s_{\ell+1}\ge 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0}} b_{H,G}\, \mathcal{U}_f(H). \tag{F.32}$$

For any $H \in \mathcal{G}(k+1)$, by Lemma F.5 we have

$$\mathcal{U}_f(H) = -\frac{1}{k+2-u(k+1,H)}\Big(\mathcal{T}_f(H) + \mathcal{U}_f\big(\Lambda[1](H)\big)\Big). \tag{F.33}$$

Combining (F.33) with (F.32), we get

$$\begin{aligned}
\mathcal{T}_f(G) &= \sum_{\substack{H\supseteq G:\\ |H|\le k,\\ s_{\ell+1}\ge 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] - \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0}} \frac{b_{H,G}}{k+2-u(k+1,H)} \mathcal{T}_f(H) \\
&\quad - \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0}} \frac{b_{H,G}}{k+2-u(k+1,H)} \mathcal{U}_f\big(\Lambda[1](H)\big) \\
&\overset{(a)}{=} \sum_{\substack{H\supseteq G:\\ |H|\le k,\\ s_{\ell+1}\ge 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0}} a_{H,G} \mathcal{T}_f(H) + \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq G,\\ s_{\ell+1}\ge 0,\\ s_{k+2}=-1}} b_{K,G} \mathcal{U}_f\big(K\big).
\end{aligned} \tag{F.34}$$

Note that $s_{k+2} = -1$ implies that $\tau_K = \tau_H + 1$ and $\gamma_K = \gamma_H$ (since $p(k+2,K) = k+1$). Moreover, equality (a) is due to the fact that for any $H \in \mathcal{G}(k+1)$ we have $a_{H,G} = -\frac{b_{H,G}}{k+2-u(k+1,H)}$, and that for any $K \in \mathcal{G}(k+2)$ with $s_{k+2} = -1$ we have $b_{K,G} = (-1)^{\gamma_K-\gamma_H+\tau_K-\tau_H} \frac{b_{H,G}}{k+2-u(k+1,H)} = -\frac{b_{H,G}}{k+2-u(k+1,H)}$.

Next by another application of Lemma F.4, we get that

$$\mathcal{T}_f(H) - \mathcal{S}(H)\, \mathbb{E}\big[\partial^k f(W)\big] = - \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq H,\\ s_{k+2}\ge 0,\\ p(k+2,K)=k+1}} \mathcal{U}_f(K) + \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq H,\\ s_{k+2}\ge 0,\\ p(k+2,K)<k+1}} \mathcal{U}_f(K). \tag{F.35}$$

Combining this with (F.34), we get

$$\begin{aligned}
\mathcal{T}_f(G) &= \sum_{\substack{H\supseteq G:\\ |H|\le k,\\ s_{\ell+1}\ge 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{H\in\mathcal{G}(k+1):\\ H\supseteq G,\\ s_{\ell+1}\ge 0}} a_{H,G} \mathcal{S}(H)\, \mathbb{E}[\partial^k f(W)] \\
&\quad - \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq H\supseteq G,\\ s_{\ell+1}\ge 0,\\ s_{k+2}\ge 0,\\ p(k+2,K)=k+1}} a_{H,G} \mathcal{U}_f(K) + \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq H\supseteq G,\\ s_{\ell+1}\ge 0,\\ s_{k+2}\ge 0,\\ p(k+2,K)<k+1}} a_{H,G} \mathcal{U}_f(K) + \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supseteq G,\\ s_{\ell+1}\ge 0,\\ s_{k+2}=-1}} b_{K,G} \mathcal{U}_f\big(K\big),
\end{aligned} \tag{F.36}$$

where H is the order-$(k+1)$ sub-genogram of K.

Now we simplify (F.36). For $K \in \mathcal{G}(k+2)$ such that $p(k+2,K) = k+1$, we have $\gamma_K = \gamma_H$. And $s_{k+2} \geq 0$ implies that $\tau_K = \tau_H$. Thus, $b_{K,G} = (-1)^{\gamma_K - \gamma_H + \tau_K - \tau_H} \frac{b_{H,G}}{k+2-u(k+1,H)} = \frac{b_{H,G}}{k+2-u(k+1,H)} = -a_{H,G}$. For $K \in \mathcal{G}(k+2)$ such that $p(k+2,K) < k+1$, we have $\gamma_K = \gamma_H + 1$. Thus, $b_{K,G} = (-1)^{\gamma_K - \gamma_H + \tau_K - \tau_H} \frac{b_{H,G}}{k+2-u(k+1,H)} = -\frac{b_{H,G}}{k+2-u(k+1,H)} = a_{H,G}$. (F.36) reduces to

$$\begin{aligned}
\mathcal{T}_f(G) = & \sum_{\substack{H \supseteq G: \\ |H| \leq k, \\ s_{\ell+1} \geq 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{H \in \mathcal{G}(k+1): \\ H \supseteq G, \\ s_{\ell+1} \geq 0}} a_{H,G} \mathcal{S}(H)\, \mathbb{E}[\partial^k f(W)] \\
& + \sum_{\substack{K \in \mathcal{G}(k+2): \\ K \supseteq G, \\ s_{\ell+1} \geq 0, \\ s_{k+2} \geq 0, \\ p(k+2,K) = k+1}} b_{K,G} \mathcal{U}_f(K) + \sum_{\substack{K \in \mathcal{G}(k+2): \\ K \supseteq G, \\ s_{\ell+1} \geq 0, \\ s_{k+2} \geq 0, \\ p(k+2,K) < k+1}} b_{K,G} \mathcal{U}_f(K) + \sum_{\substack{K \in \mathcal{G}(k+2): \\ K \supseteq G, \\ s_{\ell+1} \geq 0, \\ s_{k+2} = -1}} b_{K,G} \mathcal{U}_f\big(K\big) \\
= & \sum_{\substack{H \supseteq G: \\ |H| \leq k+1, \\ s_{\ell+1} \geq 0}} a_{H,G}\, \mathcal{S}(H)\, \mathbb{E}\big[\partial^{|H|-1} f(W)\big] + \sum_{\substack{K \in \mathcal{G}(k+2): \\ K \supseteq G, \\ s_{\ell+1} \geq 0}} b_{K,G}\, \mathcal{U}_f(K). \qquad \text{(F.37)}
\end{aligned}$$

The last equality is from the observation that

$$\begin{aligned}
& \{H \supseteq G : |H| \leq k+1, s_{\ell+1} \geq 0\} \\
& \quad = \{H \supseteq G : |H| \leq k, s_{\ell+1} \geq 0\} \sqcup \{H \in \mathcal{G}(k+1) : H \supseteq G, s_{\ell+1} \geq 0\}, \\
& \{K \in \mathcal{G}(k+2) : K \supseteq G, s_{\ell+1} \geq 0\} \\
& \quad = \{K \in \mathcal{G}(k+2) : K \supseteq G, s_{\ell+1} \geq 0, s_{k+2} \geq 0, p(k+2,K) = k+1\} \sqcup \\
& \qquad \{K \in \mathcal{G}(k+2) : K \supseteq G, s_{\ell+1} \geq 0, s_{k+2} \geq 0, p(k+2,K) < k+1\} \sqcup \\
& \qquad \{K \in \mathcal{G}(k+2) : K \supseteq G, s_{\ell+1} \geq 0, s_{k+2} = -1\},
\end{aligned}$$

where $\sqcup$ denotes the disjoint union of sets. Note that (F.37) is precisely (F.29) for the case $k+1$. By induction Theorem F.6 is proven. ■

Lemma F.7. *Given a genogram G and an integer $k \geq |G|$, suppose there exist two sets of constants that only depend on G and the joint distribution of $(X_i)_{i \in T}$, $(Q_{|G|}, \cdots, Q_k)$ and $(Q'_{|G|}, \cdots, Q'_k)$, which satisfy that for any polynomial f of degree at most $k-1$,*

$$\mathcal{T}_f(G) = \sum_{j=|G|}^{k} Q_j \mathbb{E}[\partial^{j-1} f(W)] = \sum_{j=|G|}^{k} Q'_j \mathbb{E}[\partial^{j-1} f(W)].$$

Then $Q_j = Q'_j$ for any $|G| \leq j \leq k$.

Proof. We prove the lemma by contradiction.

Let j be the smallest number such that $Q_j \neq Q'_j$. Since $Q_{|G|}, \cdots, Q_k$ does not depend on f, we choose $f(x) = cx^j$ such that $\partial^j f(x) = cj! \neq 0$. But $Q_{j+1}\mathbb{E}[\partial^{j+1} f(W)] = \cdots = Q_k \mathbb{E}[\partial^{k-1} f(W)] = 0$, which implies $cQ_j = cQ'_j$. This is a contradiction. Therefore, $Q_j = Q'_j$ for any $|G| \leq j \leq k$. ■

Let $\mathcal{P}_0(k) := \{G \in \mathcal{G}(k) : s_j \leq 0$, for any $1 \leq j \leq k\}$ denote the set of order-k genograms with no positive vertex. Let $\mathcal{G}_0(k) := \mathcal{G}(k) \backslash \mathcal{P}_0(k)$ denote the set of order-k genograms with at least one positive vertex. Let $\mathcal{P}_1(k) := \{G \in \mathcal{G}(k) : s_j \le 0, 1 \le j \le k-1, s_k \ge 1\}$ be the set of order-k genograms where $v[k]$ is the only positive vertex. Note that from the compatible conditions of identifiers, we know any genogram in $\mathcal{P}_0(k)$ or $\mathcal{P}_1(k)$ has only one branch.

Corollary F.8. *Given $k \ge 2$, the equation below holds for any $f \in \mathcal{C}^k(\mathbb{R})$*

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \sum_{H\in\mathcal{G}(k+2)} b_H \mathcal{U}_f(H) \tag{F.38}$$

$$= \sum_{j=1}^{k-1} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \frac{\widetilde{\kappa}_{k+1}}{k!}\mathbb{E}[\partial^k f(W)] + \sum_{\substack{H\in\mathcal{P}_0(k+2)\sqcup\\ \mathcal{P}_1(k+2)\sqcup\mathcal{G}_0(k+1)}} b_H \mathcal{U}_f(H), \tag{F.39}$$

where $\widetilde{\kappa}_{k+1}$ and b_H are defined as

$$\widetilde{\kappa}_{k+1} := \kappa_{k+1}(W) + \sum_{H\in\mathcal{G}_0(k+1)} \frac{k!\, b_H}{k+2-u(k+1)} \mathcal{S}(H), \tag{F.40}$$

$$b_H := \begin{cases} (-1)^{\gamma_H+\tau_H} & \text{if } |H| = 2 \\ (-1)^{\gamma_H+\tau_H} \prod_{j=2}^{|H|-1} \frac{1}{j+1-u(j)} & \text{if } |H| \ge 3 \end{cases}. \tag{F.41}$$

Here γ_H is the number of leaves on H, and τ_H is the number of negative vertices on H.

Proof. We write $G_0 = (1, \emptyset, 0)$ to be the genogram consisting only of the root. We remark that as already discussed in (F.25),

$$\mathcal{T}_f(G_0) = \mathbb{E}[Wf(W)].$$

As G_0 is a genogram of order-1, applying Theorem F.6 we have that for any $f \in \mathcal{C}^k(\mathbb{R})$,

$$\mathcal{T}_f(G_0) = \mathbb{E}[Wf(W)] = \sum_{j=1}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)] + \sum_{H\in\mathcal{G}(k+2)} b_{H,G_0} \mathcal{U}_f(H), \tag{F.42}$$

where $Q_1 = \mathcal{S}(G_0) = 0$ and for all $j \ge 2$ we take

$$Q_j = \sum_{H\in\mathcal{G}(j):H\supseteq G_0, s_2\ge 0} a_{H,G_0}\mathcal{S}(H) \overset{(a)}{=} -\sum_{H\in\mathcal{G}(j)} \frac{b_{H,G_0}}{j+1-u(j,H)} \mathcal{S}(H), \tag{F.43}$$

where a_{H,G_0} and b_{H,G_0} are defined in Theorem F.6, and where to obtain (a) we used the fact that for all genograms H of order bigger than 2, we have $a_{H,G_0} = -\frac{b_{H,G_0}}{|H|+1-u(|H|,H)}$. The expression above shows that Q_j only depends on the joint distribution of $(X_i)_{i\in T}$. Furthermore, we see that by definition b_{H,G_0} is identical to the b_H defined in the corollary.

For any $H \in \mathcal{G}(k+2)$ and polynomial f of degree at most k, $\partial^k f$ is a constant. Therefore, $\Delta_f(H) = 0$, which directly implies that

$$\mathcal{U}_f(H) = \sigma^{-(k+1)} \sum_{i_1\in B_1\setminus D_1} \sum_{i_2\in B_2\setminus D_2} \cdots \sum_{i_{k+1}\in B_{k+1}\setminus D_{k+1}} \mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_{k+1}}, \Delta_f(H)\big) = 0.$$

Therefore, by combining this with (F.42) we obtain that

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)]. \tag{F.44}$$

On the other hand, we note that for any random variable, the moments $(\mu_j)_{j\geq 0}$ and cumulants $(\kappa_j)_{j\geq 0}$ are connected through (B.43). If we choose $f(x) = x^j$ where $j \in [k]$, then by (B.43) we have

$$\mathbb{E}[Wf(W)] = \mu_{j+1}(W) = \sum_{h=1}^{j+1} \binom{j}{h-1} \kappa_h(W) \mu_{j+1-h}(W)$$
$$= \sum_{h=0}^{j} \binom{j}{h} \kappa_{h+1}(W) \mu_{j-h}(W) = \sum_{h=1}^{k+1} \frac{\kappa_h(W)}{h!} \mathbb{E}[\partial^{h-1} f(W)].$$

Any polynomial f of degree at most k can be written as $f(x) = \sum_{j=0}^{k} a_j x^j$. By linearity of expectations, we know

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k+1} \frac{\kappa_j(W)}{(j-1)!} \mathbb{E}[\partial^j f(W)].$$

Compare this to (B.42) and apply Lemma F.7. We conclude that $Q_j = \kappa_j(W)/(j-1)!$ for any $j \in [k+1]$. Thus, for any $f \in \mathcal{C}^k(\mathbb{R})$, we have shown

$$\begin{aligned}
\mathbb{E}[Wf(W)] &= \sum_{j=2}^{k+1} Q_j \mathbb{E}[\partial^{j-1} f(W)] + \sum_{H \in \mathcal{G}(k+2)} b_H \mathcal{U}_f(H) \\
&= \sum_{j=1}^{k} Q_{j+1} \mathbb{E}[\partial^j f(W)] + \sum_{H \in \mathcal{G}(k+2)} b_H \mathcal{U}_f(H) \\
&= \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!} \mathbb{E}[\partial^j f(W)] + \sum_{H \in \mathcal{G}(k+2)} b_H \mathcal{U}_f(H).
\end{aligned}$$

Since $f \in \mathcal{C}^k(\mathbb{R}) \subseteq \mathcal{C}^{k-1}(\mathbb{R})$, we also have

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k-1} \frac{\kappa_{j+1}(W)}{j!} \mathbb{E}[\partial^j f(W)] + \sum_{H \in \mathcal{G}(k+1)} b_H \mathcal{U}_f(H). \tag{F.45}$$

As $\mathcal{P}_0(k+1) \subseteq \mathcal{G}(k+1)$ we can decompose $\sum_{H \in \mathcal{G}(k+1)} b_H \mathcal{U}_f(H)$ into two sums as

$$\sum_{H \in \mathcal{G}(k+1)} b_H \mathcal{U}_f(H) = \sum_{H \in \mathcal{P}_0(k+1)} b_H \mathcal{U}_f(H) + \sum_{H \in \mathcal{G}_0(k+1) \setminus \mathcal{P}_0(k+1)} b_H \mathcal{U}_f(H). \tag{F.46}$$

For $H \in \mathcal{P}_0(k+1)$, applying Lemma F.5 we obtain

$$\sum_{H \in \mathcal{P}_0(k+1)} b_H \mathcal{U}_f(H) = -\sum_{H \in \mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)} \mathcal{T}_f(H) - \sum_{H \in \mathcal{P}_0(k+1)} \sum_{\substack{K \in \mathcal{G}(k+2): \\ K \supsetneq H, \\ s_{k+2} = -1}} \frac{b_H}{k+2-u(k+1)} \mathcal{U}_f(K)$$

$$\stackrel{(*)}{=} -\sum_{H\in\mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{T}_f(H) + \sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=-1}} b_K \mathcal{U}_f(K) \tag{F.47}$$

where in $(*)$ we use the fact that $b_K = (-1)^{\gamma_K-\gamma_H+\tau_K-\tau_H}\frac{b_H}{k+2-u(k+1)} = -\frac{b_H}{k+2-u(k+1)}$ since $s_{k+2}=-1$ implies that $\gamma_K=\gamma_H$ and $\tau_K=\tau_H+1$.

Noting that for any $H\in\mathcal{P}_0(k+1)$, if an order-$(k+2)$ genogram K satisfies that $K\supsetneq H$, then we have $p(k+2,K)=k+1$ since we have that $s_j\le 0$ for any $j\in[k+1]$. Thus, Lemma F.4 implies that

$$\sum_{H\in\mathcal{P}_0(k+1)} \big(\mathcal{T}_f(H)-\mathcal{S}(H)\,\mathbb{E}[\partial^k f(W)]\big) = -\sum_{H\in\mathcal{P}_0(k+1)} \sum_{\substack{K\in\mathcal{G}(k+2):\\ K\supsetneq H,\\ s_{k+2}\ge 0}} \mathcal{U}_f(K)$$

$$= -\sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=0}} \mathcal{U}_f(K) - \sum_{K\in\mathcal{P}_1(k+2)} \mathcal{U}_f(K).$$

Combining this with (F.47) we get

$$\begin{aligned}\sum_{H\in\mathcal{P}_0(k+1)} b_H\mathcal{U}_f(H) = &-\sum_{H\in\mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H)\,\mathbb{E}\big[\partial^k f(W)\big] + \sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=-1}} b_K\mathcal{U}_f(K)\\ &+\sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=0}} \frac{b_H}{k+2-u(k+1)}\mathcal{U}_f(K) + \sum_{K\in\mathcal{P}_1(k+2)} \frac{b_H}{k+2-u(k+1)}\mathcal{U}_f(K)\end{aligned} \tag{F.48}$$

$$\begin{aligned}&\stackrel{(a)}{=} -\sum_{H\in\mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H)\,\mathbb{E}\big[\partial^k f(W)\big] + \sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=-1}} b_K\mathcal{U}_f(K)\\ &\quad+\sum_{\substack{K\in\mathcal{P}_0(k+2):\\ s_{k+2}=0}} b_K\mathcal{U}_f(K) + \sum_{K\in\mathcal{P}_1(k+2)} b_K\mathcal{U}_f(K)\\ &= -\sum_{H\in\mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H)\,\mathbb{E}\big[\partial^k f(W)\big] + \sum_{K\in\mathcal{P}_0(k+2)\sqcup\mathcal{P}_1(k+2)} b_K\mathcal{U}_f(K).\end{aligned} \tag{F.49}$$

Again (a) is obtained by checking that $\gamma_K=\gamma_H$ and $\tau_K=\tau_H$, and thus, $b_K=\frac{b_H}{k+2-u(k+1)}$.

Since we have already established that $Q_j=\kappa_j(W)/(j-1)!$, by (F.43), the following equation holds that

$$\frac{\kappa_{k+1}(W)}{k!} = Q_{k+1} = -\sum_{H\in\mathcal{G}(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H).$$

Thus, we get

$$\frac{\widetilde{\kappa}_{k+1}}{k!} = \frac{\kappa_{k+1}(W)}{k!} + \sum_{H\in\mathcal{G}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H) = -\sum_{H\in\mathcal{P}_0(k+1)} \frac{b_H}{k+2-u(k+1)}\mathcal{S}(H). \tag{F.50}$$

Combining (F.46), (F.49), and (F.50) with (F.45), we obtain (F.39).

■

Properties of the coefficients b_H

We remark that from the definition of b_H, it is straightforward that for any H, we have the bound $|b_H| \leq 1$. Moreover, if two genograms H_1, H_2 share the same tree structure (V, E) and the set of negative vertices (i.e., $\{j : s_j = -1\}$), then $b_{H_1} = b_{H_2}$.

F.5 Controlling the remainders

From now on, we consider the case when $(X_i)_{i\in T}$ is a strongly mixing, stationary random field of mean-zero random variables (see Definition 5.2), and proceed to control the terms $\mathcal{S}(H)$ and $\mathcal{U}_f(H)$ in Corollary F.8.

Throughout this subsection for a genogram H we write

$$b_H := \begin{cases} (-1)^{\gamma_H+\tau_H} & \text{if } |H| = 2 \\ (-1)^{\gamma_H+\tau_H} \prod_{j=2}^{|H|-1} \frac{1}{j+1-u(j)} & \text{if } |H| \geq 3 \end{cases}$$

where γ_H designates the number of leaves on H and τ_H denotes the number of negative vertices on H.

Lemma F.9. *Let $T \subseteq \mathbb{Z}^d$ and $(X_i)_{i\in T}$ be a random field. Given $k \geq 1$, suppose that there is a real number $r > k+1$ such that $\mathbb{E}[X_1] = 0$, $\mathbb{E}[|X_1|^r] < \infty$. Assume that the non-degeneracy condition holds that $\liminf_{|T|\to\infty} \sigma^2/|T| > 0$. Then for any $m \in \mathbb{N}_+$ we have*

$$\sum_{H\in\mathcal{G}_0(k+1)} \left|\mathcal{S}(H)\right| \lesssim |T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}. \tag{F.51}$$

(Where the constant dropped from using the notation $\lesssim$ does not depend on m.)

Given $\omega \in (0,1]$, for any $f \in \mathcal{C}^{k-1,\omega}(\mathbb{R})$ and $m \in \mathbb{N}_+$, we have

$$\left|\sum_{H\in\mathcal{G}_0(k+1)} b_H\mathcal{U}_f(H)\right| \lesssim |f|_{k-1,\omega}|T|^{-(k+\omega-2)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d(k+\omega-1)-\omega}\alpha_\ell^{(r-k-\omega)/r}, \tag{F.52}$$

$$\left|\sum_{H\in\mathcal{P}_0(k+1)} b_H\mathcal{U}_f(H)\right| \lesssim |f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}m^{d(k+\omega-1)}. \tag{F.53}$$

Moreover, if $k \geq 2$, for any $f \in \mathcal{C}^{k-2,1}(\mathbb{R}) \cap \mathcal{C}^{k-1,1}(\mathbb{R})$, $m \in \mathbb{N}_+$ and real number $\delta \in [0,1]$, we have

$$\left|\sum_{H\in\mathcal{P}_1(k+1)} b_H\mathcal{U}_f(H)\right| \lesssim |f|_{k-2,1}^{1-\delta}|f|_{k-1,1}^{\delta}|T|^{-(k-2+\delta)/2}m^{d(k-1)} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}\alpha_\ell^{(r-k-\delta)/r}. \tag{F.54}$$

The proof of this lemma is long and technical, so we postpone it to Appendix H. Now combining Corollary F.8 and Lemma F.9, we have the following results.

Corollary F.10. *Under the same settings as Lemma F.9, for any $f \in \mathcal{C}^{k,\omega}(\mathbb{R})$ and $m \in \mathbb{N}_+$ we have*

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \mathcal{O}\Bigg(|f|_{k,\omega}|T|^{-(k+\omega-1)/2}\Bigg(m^{d(k+\omega)} + \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d(k+\omega)-\omega}\alpha_\ell^{(r-k-1-\omega)/r}\Bigg)\Bigg). \tag{F.55}$$

For any $f \in \mathcal{C}^{k-1,1}(\mathbb{R}) \cap \mathcal{C}^{k,1}(\mathbb{R})$, $m \in \mathbb{N}_+$ and real number $\delta \in [0,1]$ we have the following

$$\begin{aligned}\mathbb{E}[Wf(W)] = &\sum_{j=1}^{k-1} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \frac{\widetilde{\kappa}_{k+1}}{k!}\mathbb{E}[\partial^k f(W)] + \mathcal{O}\Bigg(|f|_{k,1}|T|^{-k/2}m^{d(k+1)} \\ &+ |f|_{k-1,1}^{1-\delta}|f|_{k,1}^{\delta}|T|^{-(k-1+\delta)/2}m^{dk}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}\alpha_\ell^{(r-k-1-\delta)/r} \\ &+ |f|_{k-1,1}|T|^{-(k-1)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}\Bigg),\end{aligned} \tag{F.56}$$

where $\widetilde{\kappa}_{k+1}$ is some constant that only depends on the joint distribution of $(X_i)_{i\in T}$, and it satisfies that

$$\left|\widetilde{\kappa}_{k+1} - \kappa_{k+1}(W)\right| \lesssim |T|^{-(k-1)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}.$$

Proof. By Corollary F.8, we have the expansion

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \sum_{H\in\mathcal{G}(k+2)} b_H\mathcal{U}_f(H).$$

By Lemma F.9 with $m \in \mathbb{N}_+$ we get the upper bound:

$$\begin{aligned}\left|\sum_{H\in\mathcal{G}(k+2)} b_H\mathcal{U}_f(H)\right| \le &\left|\sum_{H\in\mathcal{P}_0(k+2)} b_H\mathcal{U}_f(H)\right| + \left|\sum_{H\in\mathcal{G}_0(k+2)} \mathcal{U}_f(H)\right| \\ \lesssim &|f|_{k,\omega}|T|^{-(k+\omega-1)/2}\Bigg(m^{d(k+\omega)} + \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d(k+\omega)-\omega}\alpha_\ell^{(r-k-1-\omega)/r}\Bigg).\end{aligned}$$

Again by Corollary F.8, we have

$$\begin{aligned}\mathbb{E}[Wf(W)] = &\sum_{j=1}^{k-1} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] \\ &+ \frac{\widetilde{\kappa}_{k+1}}{k!}\mathbb{E}[\partial^k f(W)] + \sum_{H\in\mathcal{G}_0(k+1)} b_H\mathcal{U}_f(H) + \sum_{H\in\mathcal{P}_0(k+2)\cup\mathcal{P}_1(k+2)} b_H\mathcal{U}_f(H),\end{aligned}$$

where $\widetilde{\kappa}_{k+1}$ satisfies that

$$\widetilde{\kappa}_{k+1} := \kappa_{k+1}(W) - \sum_{H\in\mathcal{G}_0(k+1)} \frac{k!\, b_H}{k+2-u(k+1)}\mathcal{S}(H).$$

Note that $|b_H| \le 1$. By Lemma F.9 we get

$$\left|\widetilde{\kappa}_{k+1} - \kappa_{k+1}(W)\right| \le k! \sum_{H\in\mathcal{G}_0(k+1)} \left|\mathcal{S}(H)\right| \lesssim |T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}.$$

By Lemma F.9 with $\omega = 1$ we have

$$\begin{aligned}&\left|\sum_{H\in\mathcal{P}_0(k+2)} b_H\mathcal{U}_f(H)\right| + \left|\sum_{H\in\mathcal{P}_1(k+2)} b_H\mathcal{U}_f(H)\right| + \left|\sum_{H\in\mathcal{G}_0(k+1)} b_H\mathcal{U}_f(H)\right| \\ \lesssim& |f|_{k,1}|T|^{-k/2}m^{d(k+1)} + |f|_{k-1,1}^{1-\delta}|f|_{k,1}^{\delta}|T|^{-(k-1+\delta)/2}m^{dk} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{d\delta-\delta}\alpha_\ell^{(r-k-1-\delta)/r} \\ &+ |f|_{k-1,1}|T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}.\end{aligned}$$

Therefore, (F.56) is proven. ∎

Corollary F.11. *Under the same settings as Lemma F.9, the $(k+1)$-th cumulant of W $(k \ge 2)$ is upper-bounded by*

$$\left|\kappa_{k+1}(W)\right| \lesssim |T|^{-(k-1)/2}\Big(m^{dk} + \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}\Big). \tag{F.57}$$

Proof. Applying Corollary F.10 with $f(x) = x^k/k! \in \Lambda_k$ where $\Lambda_k := \big\{f \in \mathcal{C}^{k-1,1}(\mathbb{R}) : |f|_{k-1,1} \le 1\big\}$, we have

$$\mathbb{E}[Wf(W)] = \sum_{j=1}^{k-1} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \mathcal{O}\Big(|T|^{-(k-1)/2}\Big(m^{dk} + \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}\Big)\Big).$$

On the other hand, by (B.43) we have

$$\mathbb{E}[Wf(W)] = \frac{1}{k!}\mu_{k+1}(W) = \sum_{j=1}^{k}\binom{k}{j}\kappa_{j+1}(W)\mu_{k-j}(W) = \sum_{j=1}^{k-1}\frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \frac{\kappa_{k+1}(W)}{k!}.$$

Thus, we conclude that

$$\left|\kappa_{k+1}(W)\right| \lesssim |T|^{-(k-1)/2}\Big(m^{dk} + \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}\Big).$$

∎

F.6 Proofs of Lemmas E.1 and E.2

Proof of Lemma E.1. Let $k := \lceil p \rceil$. For convenience, for any $j \in [k-1]$, we denote

$$\widehat{R}_{j,\omega} := |T|^{-(j+\omega-1)/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{d(j+\omega)-\omega} \alpha_\ell^{(r-j-2)/r}\Big).$$

Then we have that $M_1 = \widehat{R}_{k,\omega}$, and that by Corollary F.11, $\left|\kappa_{j+2}(W)\right| \lesssim \widehat{R}_{j,1}$.

Firstly, we perform induction on k to prove that

$$\begin{aligned}\mathbb{E}[h(W)] - \mathcal{N}h = \sum_{(r,s_{1:r})\in\Gamma(k-1)} (-1)^r \prod_{j=1}^{r} \frac{\kappa_{s_j+2}(W)}{(s_j+1)!} \mathcal{N}\Big[\prod_{j=1}^{r} (\partial^{s_j+1}\Theta)\, h\Big] \\ + \mathcal{O}\Big(\sum_{(k)} \widehat{R}_{s_1,1}\widehat{R}_{s_2,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega}\Big),\end{aligned} \tag{F.58}$$

where $\Gamma(k-1) = \{r, s_{1:r} \in \mathbb{N}_+ : s_1 + \cdots s_r = k-1\}$.

For $p = 1$, by Lemma A.4, $f = \Theta h \in \mathcal{C}^{0,1}(\mathbb{R}) \cap \mathcal{C}^{1,1}(\mathbb{R})$. Both $|f|_{0,1}$ and $|f|_{1,1}$ is bounded by some constant. By the Stein equation and (F.55), we derive that

$$\mathbb{E}[h(W)] - \mathcal{N}h = \mathbb{E}[f'(W)] - \mathbb{E}[Wf(W)] = \mathcal{O}\Big(|T|^{-1/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{2d-1}\alpha_\ell^{(r-3)/r}\Big)\Big) = \mathcal{O}(\widehat{R}_{1,1}).$$

Suppose the proposition holds for $1 \le p \le k-1$ ($k \ge 2$), consider the case of $k-1 < p \le k$ ($p = k+\omega-1$). By Lemma A.4, $f = \Theta h \in \mathcal{C}^{k+1,\omega}(\mathbb{R})$ and $|f|_{k+1,\omega}$ is bounded by some constant that only depends on p. Thus, by (F.10), we have

$$\begin{aligned}&\mathbb{E}[h(W)] - \mathcal{N}h = \mathbb{E}[f'(W)] - \mathbb{E}[Wf(W)] \\ &= -\sum_{j=2}^{k} \frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)] + \mathcal{O}\Big(|T|^{-p/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{d(p+1)-\omega}\alpha_\ell^{(r-p-2)/r}\Big)\Big) \\ &= -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)] + \mathcal{O}(\widehat{R}_{k,\omega}).\end{aligned}$$

Noting that $\partial^{j+1}\Theta h \in \mathcal{C}^{k-j-1,\omega}(\mathbb{R})$ and $|\partial^{j+1}\Theta h|_{k-j-1,\omega}$ is bounded by a constant only depending on k, the inductive hypothesis shows that

$$\begin{aligned}\mathbb{E}[\partial^{j+1}\Theta h(W)] - \mathcal{N}[\partial^{j+1}\Theta h] = \sum_{(r,s_{1:r})\in\Gamma(k-j-1)} (-1)^r \prod_{\ell=1}^{r} \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!} \mathcal{N}\Big[\prod_{\ell=1}^{r} (\partial^{s_\ell+1}\Theta) \circ \partial^{j+1}\Theta\, h\Big] \\ + \mathcal{O}\Big(\sum_{(r,s_{1:r})\in\Gamma(k-j)} \widehat{R}_{s_1,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega}\Big).\end{aligned}$$

Here we use $\Gamma(k-j) = \left\{r, s_{1:r} \in \mathbb{N}_+ : \sum_{\ell=1}^r s_\ell \le k-j\right\}$.

Since $\left|\kappa_{j+2}(W)\right| \lesssim \widehat{R}_{j,1}$, we have

$$
\begin{aligned}
\mathbb{E}[h(W)] - \mathcal{N}h = & -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)] + \mathcal{O}(\widehat{R}_{k,1}) \\
= & -\sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!}\mathcal{N}[\partial^{j+1}\Theta h] \\
& + \sum_{j=1}^{k-1} \frac{\kappa_{j+2}(W)}{(j+1)!} \sum_{(r,s_{1:r})\in\Gamma(k-j-1)} (-1)^r \prod_{\ell=1}^r \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^r (\partial^{s_\ell+1}\Theta)\circ\partial^{j+1}\Theta\, h\right] \\
& + \mathcal{O}\Big(\widehat{R}_{k,\omega} + \sum_{j=1}^{k-1}\widehat{R}_{j,1} \sum_{(r,s_{1:r})\in\Gamma(k-j)} \widehat{R}_{s_1,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega}\Big) \\
= & \sum_{(r,s_{1:r})\in\Gamma(k-1)} (-1)^r \prod_{\ell=1}^r \frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^r (\partial^{s_\ell+1}\Theta)\, h\right] + \mathcal{O}\Big(\sum_{(r,s_{1:r})\in\Gamma(k)} \widehat{R}_{s_1,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega}\Big).
\end{aligned}
$$

By induction, (F.58) is true for any non-negative integer k.

Next we prove

$$
\widehat{R}_{s_1,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega} \le \widehat{R}_{k,\omega}(1+M)^k, \quad \text{for any } s_1+\cdots+s_r = k,\ s_j \ge 1, 1 \le j \le r. \tag{F.59}
$$

In fact, by Hölder's inequality, we get

$$
\begin{aligned}
\widehat{R}_{j,1} \le & |T|^{-j/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d(j+1)-1}\alpha_\ell^{(r-p-2)/r}\Big) \\
\le & |T|^{-j/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d(k+\omega)-\omega}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{jd}{kd-(d-1)(1-\omega)}}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d(j+1)-1}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{(k-j)d-(d-1)(1-\omega)}{kd-(d-1)(1-\omega)}}, \\
\widehat{R}_{j,\omega} \le & |T|^{-(j+\omega-1)/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d-1}\alpha_\ell^{(r-p-2)/r}\Big) \\
\le & |T|^{-(j+\omega-1)/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d(k+\omega)-\omega}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{jd-(d-1)(1-\omega)}{kd-(d-1)(1-\omega)}}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d-1}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{(k-j)d}{kd-(d-1)(1-\omega)}}.
\end{aligned}
$$

By substituting them into (F.59), we have

$$
\begin{aligned}
\widehat{R}_{s_1,1}\cdots\widehat{R}_{s_{r-1},1}\widehat{R}_{s_r,\omega} \le & |T|^{-(k+\omega-1)}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d(k+\omega)-\omega}\alpha_\ell^{(r-p-2)/r}\Big)\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d}\rfloor} \ell^{d-1}\alpha_\ell^{(r-p-2)/r}\Big)^k \\
\le & \widehat{R}_{k,\omega}\Big(1 + \sum_{\ell=1}^{\infty} \ell^{d-1}\alpha_\ell^{(r-p-2)/r}\Big)^k \le \widehat{R}_{k,\omega}(1+M)^k.
\end{aligned}
$$

Note that $M < \infty$ by assumption and $M_1 = \widehat{R}_{k,\omega}$. (E.2) is proven.

Finally, we prove $\left|\kappa_{j+2}(W)\right| \lesssim M_1^{j/p}$ for any $1 \le j \le p-1$. In fact, by Hölder's inequality again, we get

$$\begin{aligned}\widehat{R}_{j,1} \le& |T|^{-j/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{d(j+1)-1}\alpha_\ell^{(r-p-2)/r}\Big)\\ \le& |T|^{-j/2}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{d(k+\omega)-1}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{j}{k+\omega-1}}\Big(1 + \sum_{\ell=1}^{\lfloor |T|^{1/d} \rfloor} \ell^{d-1}\alpha_\ell^{(r-p-2)/r}\Big)^{\frac{k+\omega-1-j}{k+\omega-1}}\\ \le& M_1^{j/p}(1+M)^{\frac{k+\omega-1-j}{k+\omega-1}}.\end{aligned}$$

Thus, we get the upper bounds for the cumulants in terms of M_1:

$$\left|\kappa_{j+2}(W)\right| \lesssim \widehat{R}_{j,1} \lesssim M_1^{j/p}, \quad \text{for any } 1 \le j \le p-1.$$

Lemma E.1 is proven. ■

Now let's prove Lemma E.2.

Proof of Lemma E.2. Note that p is required to be an integer in this lemma. For convenience, for any $k \in [p]$, we denote

$$\widehat{R}_k := |T|^{-k/2}\Big(m^{d(k+1)} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d(k+1)-1}\alpha_\ell^{(r-k-2)/r}\Big).$$

Then by Corollary F.11, $\left|\kappa_{k+2}(W)\right| \lesssim \widehat{R}_k$.

Also denote

$$\begin{aligned}\widetilde{R}_p :=& |T|^{-p/2}m^{d(p+1)} + |T|^{-(p-1+\delta)/2}m^{d(p-1)} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d\delta-\delta}\alpha_\ell^{(r-p-1-\delta)/r}\\ &+ |T|^{-(p-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dp-1}\alpha_\ell^{(r-p-1)/r}.\end{aligned}$$

Then by definition $\widetilde{R}_p \le M_{2,m,\delta}$.

Firstly, we perform induction on p to prove that

$$\begin{aligned}&\mathbb{E}[h(W)] - \mathcal{N}h\\ =& \sum_{(r,s_{1:r})\in\Gamma(p-1)} (-1)^r \prod_{j=1}^r \frac{\kappa_{s_j+2}(W)}{(s_j+1)!}\mathcal{N}\left[\prod_{j=1}^r (\partial^{s_j+1}\Theta)\, h\right] + \mathcal{O}\Big(\sum_{j=1}^p \widehat{R}_j^{p/j}\Big) \qquad \text{(F.60)}\\ =& \sum_{(r,s_{1:r})\in\Gamma(p-1)} (-1)^r \prod_{j=1}^r \frac{\kappa_{s_j+2}(W)}{(s_j+1)!}\mathcal{N}\left[\prod_{j=1}^r (\partial^{s_j+1}\Theta)\, h\right]\end{aligned}$$

$$+\frac{\widetilde{\kappa}_{p+1}-\kappa_{p+1}(W)}{k!}\mathcal{N}\left[\partial^p\Theta\, h\right]+\mathcal{O}\Big(\sum_{j=1}^{p-1}\widehat{R}_j^{p/j}+\widetilde{R}_p\Big), \tag{F.61}$$

where $\Gamma(p-1)=\big\{r,s_{1:r}\in\mathbb{N}_+:\sum_{\ell=1}^r s_\ell\le p-1\big\}$.

For $p=1$, by Lemma A.4, $f=\Theta h\in\mathcal{C}^{0,1}(\mathbb{R})\cap\mathcal{C}^{1,1}(\mathbb{R})$. Both $|f|_{0,1}$ and $|f|_{1,1}$ is bounded by some constant. By Stein equation and (F.55), we get

$$\mathbb{E}[h(W)]-\mathcal{N}h=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]=\mathcal{O}\Big(|T|^{-1/2}\Big(m^{2d}+\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{2d-1}\alpha_\ell^{(r-3)/r}\Big)\Big)=\mathcal{O}(\widehat{R}_1).$$

By (F.56), we also have

$$\mathbb{E}[h(W)]-\mathcal{N}h=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]=\big(1-\widetilde{\kappa}_2\big)\,\mathbb{E}[f'(W)]+\mathcal{O}(\widetilde{R}_1).$$

Suppose the proposition holds for $1,\cdots,p-1$, consider the case of p. By Lemma A.4, $f=\Theta h\in\mathcal{C}^{p,1}(\mathbb{R})\cap\mathcal{C}^{p+1,1}(\mathbb{R})$. Both $|f|_{p,1}$ and $|f|_{p+1,1}$ are bounded by some constant that only depends on p. Thus, by (F.10), we have

$$\begin{aligned}
\mathbb{E}[h(W)]-\mathcal{N}h&=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]\\
&=-\sum_{j=2}^{p}\frac{\kappa_{j+1}(W)}{j!}\mathbb{E}[\partial^j f(W)]+\mathcal{O}\Big(|T|^{-p/2}\Big(m^{d(p+1)}+\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{d(p+1)-1}\alpha_\ell^{(r-p-2)/r}\Big)\Big)\\
&=-\sum_{j=1}^{p-1}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)]+\mathcal{O}(\widehat{R}_p),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}[h(W)]-\mathcal{N}h&=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]\\
&=-\sum_{j=1}^{p-2}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)]-\frac{\widetilde{\kappa}_{p+1}}{p!}\mathbb{E}[\partial^p\Theta h(W)]+\mathcal{O}(\widetilde{R}_p),
\end{aligned}$$

where $\widetilde{\kappa}_{p+1}$ is some constant that only depends on the joint distribution of $(X_i)_{i\in T}$, and it satisfies that

$$\big|\widetilde{\kappa}_{p+1}-\kappa_{p+1}(W)\big|\lesssim|T|^{-(p-1)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{dp-1}\alpha_\ell^{(r-p-1)/r}<\widetilde{R}_p.$$

Noting that $\partial^{j+1}\Theta h\in\mathcal{C}^{p-j-1,1}(\mathbb{R})$ and $|\partial^{j+1}\Theta h|_{p-j-1,1}$ is bounded by a constant only depending on k, the inductive hypothesis is given by

$$\begin{aligned}
&\mathbb{E}[\partial^{j+1}\Theta h(W)]-\mathcal{N}[\partial^{j+1}\Theta h]\\
&=\sum_{(r,s_{1:r})\in\Gamma(p-j-1)}(-1)^r\prod_{\ell=1}^r\frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\Big[\prod_{\ell=1}^r(\partial^{s_\ell+1}\Theta)\circ\partial^{j+1}\Theta\, h\Big]+\mathcal{O}\Big(\sum_{\ell=1}^{p-j}\widehat{R}_\ell^{(p-j)/\ell}\Big).
\end{aligned}$$

Here we use $\Gamma(p-j-1)=\{r,s_{1:r}\in\mathbb{N}_+:\sum_{\ell=1}^r s_\ell\le p-j-1\}$.

By Corollary B.5 and Young's inequality, we have the following bounds:

$$\begin{aligned}&\left|\kappa_{j+2}(W)\widehat{R}_\ell^{(p-j)/\ell}\right|\lesssim\widehat{R}_j\widehat{R}_\ell^{(p-j)/\ell}\le\widehat{R}_j^{p/j}+\widehat{R}_\ell^{p/\ell},\\&\left|\widetilde{\kappa}_{p+1}\widehat{R}_\ell^{1/\ell}\right|\lesssim(\widehat{R}_{p-1}+\widetilde{R}_p)\widehat{R}_\ell^{1/\ell}\lesssim\widehat{R}_{p-1}^{p/(p-1)}+\widetilde{R}_p^{p/(p-1)}+\widehat{R}_\ell^{p/\ell}.\end{aligned}$$

On one hand, we now have

$$\begin{aligned}\mathbb{E}[h(W)]-\mathcal{N}h&=-\sum_{j=1}^{p-1}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)]+\mathcal{O}(\widehat{R}_p)\\&=-\sum_{j=1}^{p-1}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathcal{N}[\partial^{j+1}\Theta h]\\&\quad+\sum_{j=1}^{p-1}\frac{\kappa_{j+2}(W)}{(j+1)!}\sum_{(r,s_{1:r})\in\Gamma(p-j-1)}(-1)^r\prod_{\ell=1}^r\frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^r(\partial^{s_\ell+1}\Theta)\circ\partial^{j+1}\Theta\, h\right]\\&\quad+\mathcal{O}\left(\widehat{R}_p+\sum_{j=1}^{p-1}\widehat{R}_j^{p/j}+\sum_{j=1}^{p-1}\sum_{\ell=1}^{p-j}\widehat{R}_\ell^{p/\ell}\right)\\&=\sum_{(r,s_{1:r})\in\Gamma(p-1)}(-1)^r\prod_{\ell=1}^r\frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^r(\partial^{s_\ell+1}\Theta)\, h\right]+\mathcal{O}\left(\sum_{\ell=1}^p\widehat{R}_\ell^{p/\ell}\right).\end{aligned}$$

Thus, (F.60) holds for the case p.

On the other hand, we derive that

$$\begin{aligned}\mathbb{E}[h(W)]-\mathcal{N}h&=\mathbb{E}[f'(W)]-\mathbb{E}[Wf(W)]\\&=-\sum_{j=1}^{p-2}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathbb{E}[\partial^{j+1}\Theta h(W)]-\frac{\widetilde{\kappa}_{p+1}}{p!}\mathbb{E}[\partial^p\Theta h(W)]+\mathcal{O}(\widetilde{R}_p)\\&=-\sum_{j=1}^{p-2}\frac{\kappa_{j+2}(W)}{(j+1)!}\mathcal{N}[\partial^{j+1}\Theta h]\\&\quad+\sum_{j=1}^{p-2}\frac{\kappa_{j+2}(W)}{(j+1)!}\sum_{(r,s_{1:r})\in\Gamma(p-j-1)}(-1)^r\prod_{\ell=1}^r\frac{\kappa_{s_\ell+2}(W)}{(s_\ell+1)!}\mathcal{N}\left[\prod_{\ell=1}^r(\partial^{s_\ell+1}\Theta)\circ\partial^{j+1}\Theta\, h\right]\\&\quad-\frac{\widetilde{\kappa}_{p+1}}{p!}\mathcal{N}[\partial^p\Theta h]+\mathcal{O}\left(\widetilde{R}_p+\widetilde{R}_p^{p/(p-1)}\sum_{j=1}^{p-1}\widehat{R}_j^{p/j}+\sum_{j=1}^{p-1}\sum_{\ell=1}^{p-j}\widehat{R}_\ell^{p/\ell}\right)\\&=\sum_{(r,s_{1:r})\in\Gamma(p-1)}(-1)^r\prod_{j=1}^r\frac{\kappa_{s_j+2}(W)}{(s_j+1)!}\mathcal{N}\left[\prod_{j=1}^r(\partial^{s_j+1}\Theta)\, h\right]\\&\quad+\frac{\widetilde{\kappa}_{p+1}-\kappa_{p+1}(W)}{k!}\mathcal{N}\left[\partial^p\Theta\, h\right]+\mathcal{O}\left(\sum_{j=1}^{p-1}\widehat{R}_j^{p/j}+\widetilde{R}_p\right).\end{aligned}$$

Thus, (F.61) also holds for the case p. By induction, we have established (F.60) and (F.61).

Next we prove that if $\sum_{\ell=1}^{\infty} \ell^{d-1} \alpha_\ell^{(r-p-1)/r} < \infty$, then for any $j \in [k]$, $\widehat{R}_j^{1/j}$ has the following bound:

$$\widehat{R}_j^{1/j} \lesssim |T|^{-1/2}\Big(m^{2dk} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d(k+1)-1} \alpha_\ell^{(r-k-1)/r}\Big)^{1/k}. \tag{F.62}$$

In fact, by Hölder's inequality, we get

$$\begin{aligned}
|T|^{jk/2} \cdot \widehat{R}_j^k \leq& \Big(m^{d(j+1)} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d(j+1)-1} \alpha_\ell^{(r-k-1)/r}\Big)^k \\
\leq& \Big(m^{d(j+1)k/j} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d(k+1)-1} \alpha_\ell^{(r-k-1)/r}\Big)^j \Big(1 + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d-1} \alpha_\ell^{(r-k-1)/r}\Big)^{k-j} \\
\lesssim& \Big(m^{2dk} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d(k+1)-1} \alpha_\ell^{(r-k-1)/r}\Big)^j.
\end{aligned}$$

By taking $1/j$-th power on both sides, (F.62) is proven.

For (F.61), we apply (F.62) with $k = p-1$ and get for $j \in [p-1]$

$$\begin{aligned}
\widehat{R}_j^{p/j} \lesssim& |T|^{-p/2}\Big(m^{2d(p-1)} + \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_\ell^{(r-p-1)/r}\Big)^{p/(p-1)} \\
\lesssim& |T|^{-p/2} m^{2dp} + \Big(|T|^{-(p-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_\ell^{(r-p-1)/r}\Big)^{p/(p-1)} \\
\lesssim& |T|^{-p/2} m^{2dp} + |T|^{-(p-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_\ell^{(r-p-1)/r} \leq M_{2,m,\delta},
\end{aligned}$$

given that $M_{2,m,\delta}$ converges to 0 as $|T| \to \infty$.

By substituting this into (F.61), we complete the proof of (E.8). Moreover, we have

$$\begin{aligned}
&\big|\kappa_{j+2}(W)\big| \lesssim \widehat{R}_j \lesssim M_{2,m,\delta}^{j/p}, \qquad \text{for any } 1 \leq j \leq p-1, \\
&\big|\widetilde{\kappa}_{p+1} - \kappa_{p+1}(W)\big| \lesssim |T|^{-(p-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dp-1} \alpha_\ell^{(r-p-1)/r} \leq M_{2,m,\delta}.
\end{aligned}$$

Thus, Lemma E.2 is proven. ∎

G Proofs of Lemmas F.4 and F.5

Before establishing Lemmas F.4 and F.5, we introduce the following lemma and prove Lemma F.3.

Lemma G.1. *Let (V,E) be a rooted tree whose vertices are ordered from a depth-first traversal (i.e., the labels satisfy Proposition F.2a). Suppose $i \notin A(j)$ and $j \notin A(i)$ for some $1 \le i < j \le k$. Then for any t such that $i \in A(t)$, we have $t < j$.*

Proof. First let $\prec$ be the strict total order on V such that for any $u,w \in V$, $u \prec w$ if and only if the label of u is smaller than the label of w. For any two vertex subsets $U,W \subseteq V$, we denote $U \prec W$ if and only if $u \prec w$ for any $u \in U$ and $w \in W$.

Claim. *Let $\prec$ be defined as above. Suppose $u,w \in V$ $(u \neq w)$ are siblings in (V,E). Let*

$$U := \{v : v = u \text{ or } u \text{ is an ancestor of } v\}, \quad W := \{v : v = w \text{ or } w \text{ is an ancestor of } v\}.$$

Then either $U \prec W$ or $W \prec U$.

To prove this we can perform induction on $k = |U| + |W|$. If $k = 2$, this is true because U and W each contain one element and $\prec$ is a strict total order on V. Now suppose the claim is true for k and consider the case for $k+1$. Without loss of generality, assume $|U| \ge 2$, and thus, u is not a leaf.

If there is only one leaf in U, denoted by u_0 $(u_0 \neq u)$, then restricting on $V' := V \backslash \{u_0\}$, we get a new rooted tree (V',E') and the order on V' is induced by that on V. Then (V',E') also satisfies Proposition F.2a and u,w are still siblings in (V',E'). By inductive hypothesis, either $U\backslash\{u_0\} \prec W$ or $W \prec U\backslash\{u_0\}$. Since u_0 is the only leaf in U, the parent of u_0, denoted by u_1 has only one child, which implies that $u_0 := \min_{\prec}\{v : u_1 \prec v\}$. Thus, we have $U \prec W$ or $W \prec U$.

If there are at least two leaves in U, two of which are denoted by u_1 and u_2 $(u,u_1,u_2$ are mutually different). Let $V_1' := V\backslash\{u_1\}$ and $V_2' := V\backslash\{u_2\}$. Restricting on V_1' or V_2', we get a new rooted tree (V_1',E_1') or (V_2',E_2') with the vertex order on V_1' or V_2' induced by $\prec$, respectively. By inductive hypothesis on (V_1',E_1'), we get $U\backslash\{u_1\} \prec W$ or $W \prec U\backslash\{u_1\}$. Similarly we have $U\backslash\{u_2\} \prec W$ or $W \prec U\backslash\{u_2\}$. Since $u \in (U\backslash\{u_1\}) \cap (U\backslash\{u_2\}) \neq \emptyset$ and $U = (U\backslash\{u_1\}) \cup (U\backslash\{u_2\})$, we conclude that $U \prec W$ or $W \prec U$.

By induction the claim is true. Let $h := \max A(i) \cap A(j)$, which is the closest common ancestor of $v[i]$ and $v[h]$. Since $i \notin A(j)$, we have $h < i$ and similarly $h < j$. Let $v[i_0]$ be the vertex such that $i_0 \in A(i)$ and $p(i_0) = h$, and $v[j_0]$ be the vertex such that $j_0 \in A(j)$ and $p(j_0) = h$. The definition of h implies that $i_0 \neq j_0$.

Let $U := \{v[t] : t = i_0 \text{ or } i_0 \in A(t)\}$ and $W := \{v[t] : t = j_0 \text{ or } j_0 \in A(t)\}$. Note that $v[i_0]$ and $v[j_0]$ are siblings. Using the claim above we get $U \prec W$ or $W \prec U$. Noticing that $v[i] \in U$, $v[j] \in W$ and $i < j$, we conclude that $U \prec W$. If $i \in A(t)$, then we have $v[t] \in U$ and $t < j$.

■

Proof of Lemma F.3. Firstly, if $p(j+1) \neq j$, then $v[j]$ is a leaf. Otherwise, suppose $v[j]$ has a child $v[h]$ where $h > j+1$. Then $p(h) > p(j+1)$ contradicts Proposition F.2a.

Suppose $p(j+1) \neq j$ and $p(j+1) \notin A(j)$. Proposition F.2a implies that $p(j+1) \le j$. Thus, we know $p(j+1) < j$. Since $v[j]$ is a leaf, we have $j \notin A(p(j+1))$. Thus, by Lemma G.1, for any t such that $p(j+1) \in A(t)$, we have $t < j$. In particular, let $t = j+1$ and we get $j+1 < j$. This is a contradiction. Thus, either $p(j+1) = j$ or $v[j]$ is a leaf and $p(j+1) \in A(j)$. ■

Proof of Lemma F.4. Consider the label set of all positive ancestors of $v[\ell]$, i.e., $\{j \in A(\ell) : s_j \geq 1\}$. Let $z := \left|\{j \in A(\ell) : s_j \geq 1\}\right|$. If $z \geq 1$, we write

$$\{j \in A(\ell) : s_j \geq 1\} = \{r_1, r_2, \cdots, r_z\},$$

where $2 \leq r_1 < r_2 < \cdots < r_z \leq \ell - 1$.

Then consider all possible genograms constructed by adding a non-negative child to a vertex of G. On one hand, Proposition F.2d implies that it is only possible to add such a child to $v[\ell]$ or $v[p(r_j)]$ for some $1 \leq j \leq z$ (provided that $z \geq 1$), and if a genogram is obtained by adding $v[\ell+1]$ as a non-negative child of $v[p(r_j)]$, then $s_{\ell+1} \leq s_{r_j} - 1$ because $\ell + 1 > r_j$.

On the other hand, we show that for $1 \leq j \leq z$,

$$s_{r_j} \leq s_t \text{ for any } t \leq \ell \text{ such that } p(t) = p(r_j).$$

Supposing this is true and $v[\ell]$ is added as a child of $v[p(r_j)]$ with $s_{\ell+1} \leq s_{r_j} - 1$, then $s_{\ell+1}$ is smaller than the identifiers of all $v[\ell+1]$'s siblings. Hence Proposition F.2 holds and $\Omega[p(r_j, s)](G)$ is indeed a genogram with the compatible labeling.

In fact, $p(t) = p(r_j)$ $(t \leq \ell)$ implies that $t \notin A(r_j)$ and $r_j \notin A(t)$. If $t > r_j$, by Lemma G.1, for any t' such that $r_j \in A(t')$, we have $t' < t$. In particular, let $t' = \ell$. Then $\ell < t$, contradicting $t \leq \ell$. Thus, $t \leq r_j$. Proposition F.2d implies that $s_t \geq s_{r_j}$.

Thus, we have shown

$$\begin{aligned}\mathcal{H} :=& \left\{H \in \mathcal{G}(\ell+1) : H \supseteq G, s_{\ell+1} \geq 0\right\} \\ =& \left\{\Omega[p(r_j), s](G) : 1 \leq j \leq w, 0 \leq s \leq s_{r_j} - 1\right\} \sqcup \left\{\Omega[\ell, s](G) : s \geq 0\right\},\end{aligned} \tag{G.1}$$

where $\sqcup$ denotes the disjoint union of sets. This directly implies that

$$-\sum_{s\geq 0} \mathcal{U}_f\big(\Omega[\ell,s](G)\big) + \sum_{j\in A(\ell): s_j\geq 1} \sum_{s=0}^{s_j-1} \mathcal{U}_f\big(\Omega[p(j),s](G)\big) = -\sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1)=\ell}} \mathcal{U}_f(H) + \sum_{\substack{H\in\mathcal{G}(\ell+1):\\ H\supseteq G,\\ s_{\ell+1}\geq 0,\\ p(\ell+1)<\ell}} \mathcal{U}_f(H). \tag{G.2}$$

Therefore, to conclude the proof we only need to show that this is also equal to

$$\mathcal{T}_f(G) - \mathcal{S}(G)\, \mathbb{E}\big[\partial^{\ell-1} f(W)\big].$$

For convenience, we list all elements of $\mathcal{H}$ (see (G.1)) in a sequence. If $z = 0$, we write

$$H_t := \Omega[\ell, t-1](G).$$

Otherwise, let

$$H_t := \begin{cases} \Omega[p(r_j), s-1](G) & \text{if } t = \sum_{i=1}^{j-1} r_i + s \\ \Omega[\ell, s-1](G) & \text{if } t = \sum_{i=1}^{z} r_i + s \end{cases}.$$

In other words, H_t is a genogram of order $\ell+1$ and the sequence $(H_t)_{t\geq 1}$ can be enumerated as

$$\begin{aligned}(H_t)_{t\geq 1}: \quad & \Omega[p(r_1),0](G)\,, \quad \Omega[p(r_1),1](G)\,, \quad \cdots, \quad \Omega[p(r_1),s_{r_1}-1](G)\,,\\ & \Omega[p(r_2),0](G)\,, \quad \Omega[p(r_2),1](G)\,, \quad \cdots, \quad \Omega[p(r_2),s_{r_2}-1](G)\,,\\ & \cdots\cdots,\\ & \Omega[p(r_z),0](G)\,, \quad \Omega[p(r_z),1](G)\,, \quad \cdots, \quad \Omega[p(r_z),s_{r_z}-1](G)\,,\\ & \Omega[\ell,0](G)\,, \quad \Omega[\ell,1](G)\,, \quad \cdots\end{aligned}$$

We write

$$c_0 := \begin{cases} 1 & \text{if } z=0 \\ s_{r_1}+\cdots+s_{r_z}+1 & \text{if } z\geq 1 \end{cases},$$

and remark that $H_t = \Omega[\ell, t-c_0](G)$ for $t\geq c_0$.

For any $t\geq 1$, we note that H_t is an order-$(\ell+1)$ genogram. We write $B_j(H_t)$ and $D_j(H_t)$ respectively the outer and inner constraints of i_j with respect to H_t. We remark that as $H_t[\ell]=G$, this directly implies that $B_j(H_t)=B_j(G)=B_j$ and $D_j(H_t)=D_j(G)=D_j$ for all $j\leq\ell$.

Let $i_1,\cdots,i_\ell$ be indexes in $i_j\in B_j\backslash D_j$ for $1\leq j\leq\ell$. We note that the sets $B_{\ell+1}(H_t)$ and $D_{\ell+1}(H_t)$ will depend on the value of $t\geq 1$.

Firstly, in H_1 we remark that by definition (F.13), if $z\geq 1$, the vertices $v[\ell+1]$ and $v[r_1]$ have the same parent and that the indentifier of $\ell+1$ is 0. This implies that if $z\geq 1$,

$$D_{\ell+1}(H_1) = D_{g(\ell+1,H_1)} = D_{g(r_1,H_1)} = D_{g(r_1,G)} = D_{g(r_1)} = D_1 = \emptyset.$$

If $z=0$, the progenitor of $v[\ell+1]$ is either $v[\ell]$ or $v[1]$. In both cases, the inner constraint is empty, i.e., $D_\ell = D_1 = \emptyset$. Thus,

$$D_{\ell+1}(H_1) = D_{g(\ell+1,H_1)} = \emptyset.$$

Note that here we have used $g(j,H)$ to denote the progenitor label of $v[j]$ with respect to the genogram H. Throughout the proof, H is omitted only if $H=G$.

Next we establish that for all $t\geq 1$ the following holds: $B_{\ell+1}(H_t) = D_{\ell+1}(H_{t+1})$.

If $z=0$, the result is directly implied by the definitions (F.17) and (F.18) as we have

$$\begin{aligned}B_{\ell+1}(H_t) &= N^{(t)}\big(i_t : t\in A(\ell+1,H_t)\big)\cup D_{g(\ell+1,H_t)}\\ &= N^{(t)}\big(i_t : t\in A(\ell+1,H_{t+1})\big)\cup D_{g(\ell+1,H_{t+1})}\\ &= D_{\ell+1}(H_{t+1}).\end{aligned}$$

Note that $A(j,H)$ is used to denote the label set of $v[j]$'s ancestors with respect to the genogram H. Again H is omitted only if $H=G$.

If $z\geq 1$, we remark that according to the values of t the relationship between the genograms H_{t+1} and H_t will be different. To make this clear, we distinguish 4 different cases according to the values of t:

(i) $t = \sum_{i=1}^{j-1} r_i + k + 1$ for $1 \le j \le z$ and $0 \le k \le s_{r_j} - 2$, in which case we observe that

$$B_{\ell+1}(H_t) = B_{\ell+1}\big(\Omega[p(r_j), k](G)\big), \quad D_{\ell+1}(H_{t+1}) = D_{\ell+1}\big(\Omega[p(r_j), k+1](G)\big);$$

(ii) $t = \sum_{i=1}^{z} r_i + k + 1$ for $k \ge 0$, where we have

$$B_{\ell+1}(H_t) = B_{\ell+1}\big(\Omega[\ell, k](G)\big), \quad D_{\ell+1}(H_{t+1}) = D_{\ell+1}\big(\Omega[\ell, k+1](G)\big);$$

(iii) $t = \sum_{i=1}^{j} r_i$ for $1 \le j \le z-1$, where we observe that

$$B_{\ell+1}(H_t) = B_{\ell+1}\big(\Omega[p(r_j), s_{r_j} - 1](G)\big), \quad D_{\ell+1}(H_t) = D_{\ell+1}\big(\Omega[p(r_{j+1}), 0](G)\big);$$

(iv) $t = \sum_{i=1}^{z} r_i$, in which case we have

$$B_{\ell+1}(H_t) = B_{\ell+1}\big(\Omega[p(r_z), s_{r_z} - 1](G)\big), \quad D_{\ell+1}(H_{t+1}) = D_{\ell+1}\big(\Omega[\ell, 0](G)\big).$$

Again cases (i) and (ii) are directly implied by the definitions (F.17) and (F.18). Indeed, for all $1 \le j \le z$, we have

$$\begin{aligned} B_{\ell+1}(H_t) &= N^{(k+1)}\big(i_t : t \in A(\ell+1, H_t)\big) \cup D_{g(\ell+1, H_t)} \\ &= N^{(k+1)}\big(i_t : t \in A(\ell+1, H_{t+1})\big) \cup D_{g(\ell+1, H_{t+1})} \\ &= D_{\ell+1}(H_{t+1}). \end{aligned}$$

We now prove cases (iii) and (iv). In this goal, let $t = \sum_{i=1}^{j} r_i$ for a given $j \le z-1$. Since in H_t, the vertices $v[\ell+1]$ and $v[r_j]$ are siblings, we have $A(\ell+1, H_t) = A(r_j, H_t) = A(r_j)$ and $g(\ell+1, H_t) = g(r_j, H_t) = g(r_j)$. On the other hand, for case (iii) we notice that as in H_{t+1} the vertices $v[\ell+1]$ and $v[r_{j+1}]$ are siblings, we have $g(\ell+1, H_{t+1}) = g(r_{j+1}) = r_j$. For case (iv), similarly we have $g(\ell+1, H_{t+1}) = g(\ell) = r_j$. Equipped with those equations, we remark that

$$B_{\ell+1}(H_t) = N^{(s_{r_j}-1)}\big(i_t : t \in A(\ell+1, H_t)\big) \cup D_{g(\ell+1, H_t)} = N^{(s_{r_j}-1)}\big(i_t : t \in A(r_j)\big) \cup D_{g(r_j)},$$

On the other hand, we also remark that as the indentifier of $\ell+1$ in H_{t+1} is 0, we have

$$D_{\ell+1}(H_{t+1}) = D_{g(\ell+1, H_{t+1})} = D_{r_j} = N^{(s_{r_j}-1)}\big(i_t : t \in A(r_j)\big) \cup D_{g(r_j)}.$$

Thus, $B_{\ell+1}(H_t) = D_{\ell+1}(H_{t+1})$.

Therefore, we have established that $B_{\ell+1}(H_t) = D_{\ell+1}(H_{t+1})$ for any $t \ge 1$.

Since the index set T of the random field is finite, there exists a finite number $c_1 \ge c_0$ such that $N^{(c_1-c_0)}\big(i_t : t \in A(\ell) \text{ or } t = \ell\big) = T$. Then

$$B_{\ell+1}(H_{c_1}) \supseteq N^{(c_1-c_0)}\big(i_t : t \in A(\ell+1, H_{c_1})\big) = N^{(c_1-c_0)}\big(i_t : t \in A(\ell) \text{ or } t = \ell\big) = T.$$

On the other hand, $B_{\ell+1}(H_{c_1}) \subseteq T$. Thus, we have $B_{\ell+1}(H_{c_1}) = T$. We remark that by definition of c_0 and c_1 we have

$$\emptyset = D_{\ell+1}(H_1) \subseteq B_{\ell+1}(H_1) \subseteq \cdots \subseteq B_{\ell+1}(H_{c_0}) \subseteq \cdots \subseteq B_{\ell+1}(H_{c_1}) = T.$$

We prove that for $c_0 \le t \le c_1 - 1$,

$$\begin{aligned} &-\mathcal{E}_{H_t}\big(X_{i_1},\cdots,X_{i_\ell}\ ,\ \Delta_f(H_t)\big)\\ =&\mathcal{E}_G\Big(X_{i_1},\cdots\ ,\ X_{i_\ell}\big(\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)-\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big)\Big)\\ &-\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_\ell}\big)\Big(\mathbb{E}\big[\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)\big]-\mathbb{E}\big[\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big]\Big); \end{aligned}$$

and similarly that for $1 \le t \le c_0 - 1$,

$$\begin{aligned} &\mathcal{E}_{H_t}\big(X_{i_1},\cdots,X_{i_\ell}\ ,\ \Delta_f(H_t)\big)\\ =&\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_\ell}\big)\Big(\mathbb{E}\big[\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big]-\mathbb{E}\big[\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)\big]\Big). \end{aligned} \tag{G.3}$$

Note that by definition $\mathcal{E}_G$ is the product of $\mathcal{D}^*$ factors. Let

$$q_0 := \sup\big\{j : j=1 \text{ or } p(j)\neq j-1 \text{ for } 2\le j\le \ell\big\}.$$

Intuitively, $v[q_0]$ is the starting vertex of the last branch of (V,E). Now set $w = \Big|\{t : q_0+1 \le t \le \ell \ \&\ s_t \ge 0\}\Big|$. If $w \ge 1$, we set $\{q_1,\cdots,q_w\} = \{t : q_0+1 \le t \le \ell \ \&\ s_t \ge 0\}$. Without loss of generality, we suppose that the sequence $q_0+1 \le q_1 < \cdots < q_w \le \ell$ is increasing. By definition, the last factor in $\mathcal{E}_G(X_{i_1},\cdots,X_{i_\ell})$ is given by

$$\begin{cases} \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_\ell}\big) & \text{if } w=0\\ \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_{q_1-1}}\ ,\ \cdots\ ,\ X_{i_{q_w}}\cdots X_{i_\ell}\big) & \text{if } w\ge 1 \end{cases}.$$

And the last factor in $\mathcal{E}_G(X_{i_1},\cdots,X_{i_{\ell-1}},X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big))$ is

$$\begin{cases} \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\big) & \text{if } w=0\\ \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_{q_1-1}}\ ,\ \cdots\ ,\ X_{i_{q_{(w-1)}}}\cdots X_{i_{q_w-1}}\ ,\ X_{i_{q_w}}\cdots X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\big) & \text{if } w\ge 1 \end{cases}.$$

For convenience, in this proof we temporarily denote

$$\mathfrak{D}^*(\cdot) := \begin{cases} \mathcal{D}^*(\cdot) & \text{if } w=0\\ \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_{q_1-1}}\ ,\ \cdots\ ,\ X_{i_{q_{(w-1)}}}\cdots X_{i_{q_w-1}}\ ,\ \cdot\ \big) & \text{if } w\ge 1 \end{cases}.$$

Since $s_{\ell+1} \ge 0$, we have

$$\Delta_f(H_t) = \partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big) - \partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big).$$

For $c_0 \le t \le c_1 - 1$, we derive that

$$\begin{aligned} &-\mathfrak{D}^*\Big(X_{i_{q_w}}\cdots X_{i_\ell}\ ,\ \Delta_f(H_t)\Big)\\ =&\mathfrak{D}^*\Big(X_{i_{q_w}}\cdots X_{i_\ell}\ ,\ \partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)-\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\Big) \end{aligned}$$

$$
\begin{aligned}
&\overset{(*)}{=}\mathfrak{D}^*\Big(X_{i_{q_w}}\cdots X_{i_\ell}\mathcal{D}\big(\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)-\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big)\Big)\\
&=\mathfrak{D}^*\Big(X_{i_{q_w}}\cdots X_{i_\ell}\big(\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)-\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big)\Big)\\
&\quad-\mathfrak{D}^*\big(X_{i_{q_w}}\cdots X_{i_\ell}\big)\Big(\mathbb{E}\big[\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)\big]-\mathbb{E}\big[\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big]\Big).
\end{aligned}
$$

Here the equality $(*)$ is implied by Lemma H.1. Noticing that $v[\ell+1]$ is the child of $v[\ell]$, we combine this with the definition of the $\mathcal{E}_G$ operator and obtain (G.3).

For $1\le t\le c_0-1$, we note that

$$
\begin{aligned}
&\mathfrak{D}^*\big(X_{i_{q_w}}\cdots X_{i_\ell}\big)\,\mathcal{D}^*(\Delta_f(H_t))\\
=&\mathfrak{D}^*\big(X_{i_{q_w}}\cdots X_{i_\ell}\big)\Big(\mathbb{E}\big[\partial^{\ell-1}f\big(W(B_{\ell+1}(H_t))\big)\big]-\mathbb{E}\big[\partial^{\ell-1}f\big(W(D_{\ell+1}(H_t))\big)\big]\Big).
\end{aligned}
$$

Since $v[\ell+1]$ is added as a child of $v[r_j]$ for some $1\le j\le z$ and $r_j<\ell$, $v[\ell]$ is a leaf in H_t. We obtain (G.3) by applying the definition of the $\mathcal{E}_G$ operator again.

Finally, using the definitions of $\mathcal{T}_f(G)$ and $\mathcal{S}(G)$, we derive

$$
\begin{aligned}
&\mathcal{T}_f(G)-\mathcal{S}(G)\,\mathbb{E}\big[\partial^{\ell-1}f(W)\big]=\\
&\quad\sigma^{-\ell}\sum_{i_1\in B_1\backslash D_1}\sum_{i_2\in B_2\backslash D_2}\cdots\sum_{i_\ell\in B_\ell\backslash D_\ell}\Big(\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{\ell-1}},\,X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\big)\\
&\qquad-\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{\ell-1}},\,X_{i_\ell}\big)\,\mathbb{E}\big[\partial^{\ell-1}f(W)\big]\Big).
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_{\ell-1}},X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\big)-\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_\ell}\big)\,\mathbb{E}\big[\partial^{\ell-1}f(W)\big]\\
=&\mathcal{E}_G\Big(X_{i_1},\cdots,X_{i_{\ell-1}}\,,\,X_{i_\ell}\partial^{\ell-1}f\big(W(B_{\ell+1}(H_{c_0-1}))\big)\Big)-\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_\ell}\big)\,\mathbb{E}\big[\partial^{\ell-1}f\big(W(D_{\ell+1}(H_1))\big)\big]\\
=&-\sum_{t=c_0}^{c_1-1}\mathcal{E}_{H_t}\big(X_{i_1},\cdots,X_{i_\ell}\,,\,\Delta_f(H_t)\big)+\sum_{t=1}^{c_0-1}\mathcal{E}_{H_t}\big(X_{i_1},\cdots,X_{i_\ell}\,,\,\Delta_f(H_t)\big).
\end{aligned}
$$

The last equality is due to a telecoping sum argument since $B_{\ell+1}(H_t)=D_{\ell+1}(H_{t+1})$ for $1\le t\le c_0-1$ and for $c_0\le t\le c_1-1$. Taking the sums over $i_j\in B_j\backslash D_j$ where $1\le j\le\ell$, we obtain

$$
\begin{aligned}
\mathcal{T}_f(G)-\mathcal{S}(G)\,\mathbb{E}\big[\partial^{\ell-1}f(W)\big]&=-\sum_{t=c_0}^{c_1-1}\mathcal{U}_f\big(H_t\big)+\sum_{t=1}^{c_0-1}\mathcal{U}_f\big(H_t\big)\\
&=-\sum_{s\ge 0}\mathcal{U}_f\big(\Omega[\ell,s](G)\big)+\sum_{j\in A(\ell):s_j\ge 1}\sum_{s=0}^{s_j-1}\mathcal{U}_f\big(\Omega[p(j),s](G)\big).
\end{aligned}
$$

■

Proof of Lemma F.5. If $u(\ell)=\ell$ then by definition,

$$
\Delta_f(G)=\partial^{\ell-2}f\big(W(B_\ell)\big)-\partial^{\ell-2}f\big(W(D_\ell)\big).
$$

Applying the Taylor expansion with integral-form remainders, we get

$$\begin{aligned}\Delta_f(G) =& \sum_{j=1}^{k+1}\frac{1}{j!}\big(W(B_\ell)-W(D_\ell)\big)^j\partial^{\ell-2+j}f\big(W(D_\ell)\big)+\frac{1}{(k+1)!}\big(W(B_\ell)-W(D_\ell)\big)^{k+1}\cdot \\ & \int_0^1(k+1)v^k\Big(\partial^{k+\ell-1}f\Big(vW(D_\ell)+(1-v)W(B_\ell)\Big)-\partial^{k+\ell-1}f\big(W(D_\ell)\big)\Big)\mathrm{d}v \\ =& \sum_{j=1}^{k+1}(-1)^j\sigma^{-j}\frac{1}{j!}\Big(\sum_{i\in B_\ell\setminus D_\ell}X_i\Big)^j\partial^{\ell-2+j}f\big(W(D_\ell)\big)+(-1)^{k+1}\sigma^{-(k+1)}\frac{1}{(k+1)!}\Big(\sum_{i\in B_\ell\setminus D_\ell}X_i\Big)^{k+1}\cdot \\ & \int_0^1(k+1)v^k\Big(\partial^{k+\ell-1}f\Big(vW(D_\ell)+(1-v)W(B_\ell)\Big)-\partial^{k+\ell-1}f\big(W(D_\ell)\big)\Big)\mathrm{d}v \\ \overset{(*)}{=}& \sum_{j=1}^{k+1}(-1)^j\sigma^{-j}\frac{1}{j!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+j-1}\in B_{\ell+j-1}\setminus D_{\ell+j-1}}X_{i_\ell}\cdots X_{i_{\ell+j-1}}\partial^{\ell+j-2}f\big(W(D_{\ell+j})\big) \\ & +(-1)^{k+1}\sigma^{-(k+1)}\frac{1}{(k+1)!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+k}\in B_{\ell+k}\setminus D_{\ell+k}}X_{i_\ell}\cdots X_{i_{\ell+k}}\cdot \\ & \int_0^1(k+1)v^k\Big(\partial^{\ell+k-1}f\Big(vW(D_{\ell+k+1})+(1-v)W(B_{\ell+k+1})\Big)-\partial^{\ell+k-1}f\big(W(D_{\ell+k+1})\big)\Big)\mathrm{d}v \\ =& -\sigma^{-1}\sum_{i_\ell\in B_\ell\setminus D_\ell}X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big) \\ & +\sum_{j=2}^{k+1}(-1)^j\sigma^{-j}\frac{1}{j!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+j-1}\in B_{\ell+j-1}\setminus D_{\ell+j-1}}X_{i_\ell}\cdots X_{i_{\ell+j-2}}X_{i_{\ell+j-1}}\partial^{\ell+j-2}f\big(W(D_{\ell+j-1})\big) \\ & +(-1)^{k+1}\sigma^{-(k+1)}\frac{1}{(k+1)!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+k}\in B_{\ell+k}\setminus D_{\ell+k}}X_{i_\ell}\cdots X_{i_{\ell+k}}\Delta_f\big(\Lambda[k+1](G)\big).\end{aligned} \tag{G.4}$$

where to obtain $(*)$ we have used the fact that $v[\ell+1],\cdots,v[\ell+k+1]$ are negative vertices in $\Lambda[k+1](G)$ and $\Lambda[j](G)\subseteq\Lambda[k+1](G)$ for $0\le j\le k$, which implies that $B_\ell=B_{\ell+1}=\cdots=B_{\ell+k+1}$ and $D_\ell=D_{\ell+1}=\cdots=D_{\ell+k+1}$ from the constructions of B_j's and D_j's.

In (F.19), we defined q_0 to be

$$q_0:=\sup\big\{j: j=1 \text{ or } p(j)\neq j-1 \text{ for } 2\le j\le\ell\big\}.$$

We write $w:=\big|\{t: q_0+1\le t\le\ell \,\&\, s_t\ge 0\}\big|$. Since $\ell=u(\ell)$, we know $s_\ell\ge 0$ and $w\ge 1$. We suppose without loss of generality that the elements of $\{t: q_0+1\le t\le\ell \,\&\, s_t\ge 0\}=\{q_1,\cdots,q_w\}$ are presented in increasing order: $q_0+1\le q_1<\cdots<q_w=\ell$. Moreover, by definition, the term $\mathcal{E}_G(X_{i_1},\cdots,X_{i_{\ell-1}},\Delta_f(G))$ is the product of $\mathcal{D}^*$ factors, and we remark that its last factor is

$$\begin{cases}\mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_{\ell-1}}\,,\ \Delta_f(G)\big) & \text{if } w=1\\ \mathcal{D}^*\big(X_{i_{q_0}}\cdots X_{i_{q_1-1}}\,,\ \cdots\,,\ X_{i_{q_{(w-1)}}}\cdots X_{i_{\ell-1}}\,,\ \Delta_f(G)\big) & \text{if } w\ge 2\end{cases}.$$

For convenience, in this proof we temporarily denote

$$\mathfrak{D}^*(\,\cdot\,) := \begin{cases} \mathcal{D}^*\big(X_{i_{q_0}} \cdots X_{i_{\ell-1}}\,,\,\cdot\,\big) & \text{if } w=1 \\ \mathcal{D}^*\big(X_{i_{q_0}} \cdots X_{i_{q_1-1}}\,,\,\cdots\,,\,X_{i_{q_{(w-1)}}} \cdots X_{i_{\ell-1}}\,,\,\cdot\,\big) & \text{if } w \geq 2 \end{cases}.$$

And write

$$\mathcal{E}_G(X_{i_1},\cdots,X_{i_{\ell-1}},\Delta_f(G)) = \mathfrak{E}\cdot\mathfrak{D}^*(\Delta_f(G)).$$

Combining this with (G.4) we obtain that

$$\begin{aligned}
&\mathcal{E}_G(X_{i_1},\cdots,X_{i_{\ell-1}},\Delta_f(G)) = \mathfrak{E}\cdot\mathfrak{D}^*(\Delta_f(G)) \\
&= -\sigma^{-1}\sum_{i_\ell\in B_\ell\setminus D_\ell}\mathfrak{E}\cdot\mathfrak{D}^*\Big(X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\Big) \\
&\quad+\sum_{j=2}^{k+1}(-1)^j\sigma^{-j}\frac{1}{j!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+j-1}\in B_{\ell+j-1}\setminus D_{\ell+j-1}}\mathfrak{E}\cdot\mathfrak{D}^*\Big(X_{i_\ell}\cdots X_{i_{\ell+j-2}}X_{i_{\ell+j}}\partial^{\ell+j-1}f\big(W(D_{\ell+j})\big)\Big) \\
&\quad+(-1)^{k+1}\sigma^{-(k+1)}\frac{1}{(k+1)!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+k}\in B_{\ell+k}\setminus D_{\ell+k}}\mathfrak{E}\cdot\mathfrak{D}^*\Big(X_{i_\ell}\cdots X_{i_{\ell+k}}\Delta_f\big(\Lambda[k+1](G)\big)\Big) \\
&\overset{(*)}{=} -\sigma^{-1}\sum_{i_\ell\in B_\ell\setminus D_\ell}\mathcal{E}_G\Big(X_{i_1},\cdots,X_{i_{\ell-1}},X_{i_\ell}\partial^{\ell-1}f\big(W(D_\ell)\big)\Big) \\
&\quad+\sum_{j=2}^{k+1}(-1)^j\sigma^{-j}\frac{1}{j!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+j-1}\in B_{\ell+j-1}\setminus D_{\ell+j-1}}\mathcal{E}_{\Lambda[j-1](G)}\Big(X_{i_1},\cdots,X_{i_{\ell+j-2}},X_{i_{\ell+j}}\partial^{\ell+j-1}f\big(W(D_{\ell+j})\big)\Big) \\
&\quad+(-1)^{k+1}\sigma^{-(k+1)}\frac{1}{(k+1)!}\sum_{i_\ell\in B_\ell\setminus D_\ell}\cdots\sum_{i_{\ell+k}\in B_{\ell+k}\setminus D_{\ell+k}}\mathcal{E}_{\Lambda[k+1](G)}\Big(X_{i_1},\cdots,X_{i_{\ell+k}},\Delta_f\big(\Lambda[k+1](G)\big)\Big).
\end{aligned}$$

where to get $(*)$ we have used the condition that $v[\ell+1],\cdots,v[\ell+k+1]$ are negative vertices in $\Lambda[k+1](G)$. Indeed, the factorization stay the same due to the fact that they are all added to the same branch, and $q_1,\cdots,q_w$ remain the same since $v[\ell+1],\cdots,v[\ell+k+1]$ are all negative. Taking the sum over $i_j\in B_j\setminus D_j$ for $1\le j\le \ell-1$, we have

$$\begin{aligned}
\mathcal{U}_f(G) &= \sum_{j=1}^{k+1}(-1)^j\frac{1}{j!}\mathcal{T}_f\big(\Lambda[j-1](G)\big)+(-1)^{k+1}\frac{1}{(k+1)!}\mathcal{U}_f\big(\Lambda[k+1](G)\big) \\
&= \sum_{j=0}^{k}(-1)^{j+1}\frac{1}{(j+1)!}\mathcal{T}_f\big(\Lambda[j](G)\big)+(-1)^{k+1}\frac{1}{(k+1)!}\mathcal{U}_f\big(\Lambda[k+1](G)\big). \qquad \text{(G.5)}
\end{aligned}$$

Now consider the case $u(\ell)<\ell$. Let $G[u(\ell)] := (V',E',s_{1:u(\ell)}) \subseteq G$ be the order-$u(\ell)$ sub-genogram of G as defined in the last paragraph of Appendix F.2. Now by (G.5), we have

$$\mathcal{U}_f(G[u(\ell)]) = \sum_{j=0}^{k}(-1)^{j+1}\frac{1}{(j+1)!}\mathcal{T}_f\big(\Lambda[j](G[u(\ell)])\big)+(-1)^{k+1}\frac{1}{(k+1)!}\mathcal{U}_f\big(\Lambda[k+1](G[u(\ell)])\big).$$

Replacing k by $\ell-u(\ell)-1$ and $\ell-u(\ell)+k$ respectively, we get that

$$\begin{aligned}\mathcal{U}_f(G[u(\ell)]) = &\sum_{j=0}^{\ell-u(\ell)-1}(-1)^{j+1}\frac{1}{(j+1)!}\mathcal{T}_f\big(\Lambda[j](G[u(\ell)])\big)\\ &+(-1)^{\ell-u(\ell)}\frac{1}{(\ell-u(\ell))!}\mathcal{U}_f\big(\Lambda[\ell-u(\ell)](G[u(\ell)])\big)\end{aligned}, \tag{G.6}$$

$$\begin{aligned}\mathcal{U}_f(G[u(\ell)]) = &\sum_{j=0}^{\ell-u(\ell)+k}(-1)^{j+1}\frac{1}{(j+1)!}\mathcal{T}_f\big(\Lambda[j](G[u(\ell)])\big)\\ &+(-1)^{\ell-u(\ell)+k+1}\frac{1}{(\ell-u(\ell)+k+1)!}\mathcal{U}_f\big(\Lambda[\ell-u(\ell)+k+1](G[u(\ell)])\big)\end{aligned}. \tag{G.7}$$

By taking the difference of (G.6) and (G.7) we obtain that

$$\begin{aligned}&(-1)^{\ell-u(\ell)}\frac{1}{(\ell-u(\ell))!}\mathcal{U}_f\big(\Lambda[\ell-u(\ell)](G[u(\ell)])\big)\\ &=\sum_{j=\ell-u(\ell)}^{\ell-u(\ell)+k}(-1)^{j+1}\frac{1}{(j+1)!}\mathcal{T}_f\big(\Lambda[j](G[u(\ell)])\big)\\ &\quad+(-1)^{\ell-u(\ell)+k+1}\frac{1}{(\ell-u(\ell)+k+1)!}\mathcal{U}_f\big(\Lambda[\ell-u(\ell)+k+1](G[u(\ell)])\big).\end{aligned}$$

Thus, we have

$$\begin{aligned}&\mathcal{U}_f(G)=\mathcal{U}_f\big(\Lambda[\ell-u(\ell)](G[u(\ell)])\big)\\ &=\sum_{j=0}^{k}(-1)^{j+1}\frac{(\ell-u(\ell))!}{(j+1+\ell-u(\ell))!}\mathcal{T}_f\big(\Lambda[j+\ell-u(\ell)](G[u(\ell)])\big)\\ &\quad+(-1)^{k+1}\frac{(\ell-u(\ell))!}{(k+1+\ell-u(\ell))!}\mathcal{U}_f\big(\Lambda[k+1+\ell-u(\ell)](G[u(\ell)])\big)\\ &=\sum_{j=0}^{k}(-1)^{j+1}\frac{(\ell-u(\ell))!}{(j+1+\ell-u(\ell))!}\mathcal{T}_f\big(\Lambda[j](G)\big)+(-1)^{k+1}\frac{(\ell-u(\ell))!}{(k+1+\ell-u(\ell))!}\mathcal{U}_f\big(\Lambda[k+1](G)\big).\end{aligned}$$

■

H Proof of Lemma F.9

Firstly, let's show some properties of the $\mathcal{D}$ and $\mathcal{D}^*$ operators that will be useful later.

Lemma H.1. *Let $(Y_i)_{i=1}^t$ be a sequences of random variables. Suppose for any $i, j \in \mathbb{N}_+$ such that $i \le j \le t$, we have $\mathbb{E}\big[|Y_i \cdots Y_j|\big] < \infty$. Then the following holds for all $t \in \mathbb{N}_+$ such that $t \ge 2$ and for any $j = 1, \cdots, t-1$:*

$$\mathcal{D}^*\big(Y_1, \cdots, Y_j\mathcal{D}(Y_{j+1}, \cdots, Y_t)\big) = \mathcal{D}^*(Y_1, \cdots, Y_t), \tag{H.1}$$

$$\mathcal{D}\big(Y_1, \cdots, Y_j \mathcal{D}(Y_{j+1}, \cdots, Y_t)\big) = \mathcal{D}(Y_1, \cdots, Y_t). \tag{H.2}$$

In particular,

$$\begin{aligned} &\mathcal{D}^*(Y_1, \cdots, Y_t) = \mathcal{D}^*(Y_1, \cdots, Y_{t-2}\ ,\ Y_{t-1}\mathcal{D}(Y_t)) \\ =&\mathcal{D}^*(Y_1, \cdots, Y_{t-2}\ ,\ Y_{t-1}Y_t) - \mathcal{D}^*(Y_1, \cdots, Y_{t-1})\, \mathbb{E}[Y_t]. \end{aligned} \tag{H.3}$$

Moreover, we know that

$$\mathbb{E}\big[\mathcal{D}(Y_1, \cdots, Y_t)\big] = 0.$$

Proof of Lemma H.1. We perform induction on j to prove that

$$\mathcal{D}^*\big(Y_1, \cdots, Y_j \mathcal{D}(Y_{j+1}, \cdots, Y_t)\big) = \mathcal{D}^*(Y_1, \cdots, Y_t).$$

If $j = 1$, this is precisely the definition. Supposing the lemma holds for j ($j \le t-1$), consider the case for $j+1$. By definition,

$$\begin{aligned} &\mathcal{D}\big(Y_1, \cdots, Y_{j+1}\mathcal{D}(Y_{j+2}, \cdots, Y_t)\big) \\ =&\mathcal{D}\Big(Y_1 \mathcal{D}\big(Y_2, \cdots, Y_{j+1}\mathcal{D}(Y_{j+2}, \cdots, Y_t)\big)\Big) \\ =&\mathcal{D}\big(Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big) = \mathcal{D}(Y_1, \cdots, Y_t). \end{aligned}$$

Note that we have used the inductive hypothesis in the second equation. By induction, (H.2) is proven.

Now for any $j = 1 \cdots t-1$,

$$\begin{aligned} &\mathcal{D}^*\big(Y_1, \cdots, Y_j \mathcal{D}(Y_{j+1}, \cdots, Y_t)\big) \\ =&\mathbb{E}\Big[Y_1 \mathcal{D}\big(Y_2, \cdots, Y_j \mathcal{D}(Y_{j+1}, \cdots, Y_t)\big)\Big] \\ =&\mathbb{E}\big[Y_1 \mathcal{D}(Y_2, \cdots, Y_t)\big] = \mathcal{D}^*(Y_1, \cdots, Y_t). \end{aligned}$$

Finally we remark that

$$\begin{aligned} \mathbb{E}\big[\mathcal{D}(Y_1, \ldots, Y_t)\big] &= \mathbb{E}\big[\mathcal{D}(Y_1 \mathcal{D}(Y_2, \ldots, Y_t))\big] \\ &= \mathbb{E}\big[Y_1 \mathcal{D}(Y_2, \ldots, Y_t)\big] - \mathbb{E}\big[Y_1 \mathcal{D}(Y_2, \ldots, Y_t)\big] = 0. \end{aligned}$$

■

Lemma H.2. *Let $(Y_i)_{i=1}^t$ be a sequences of random variables. Suppose for any $i, j \in \mathbb{N}_+$ such that $i \le j \le t$, we have $\mathbb{E}\big[|Y_i \cdots Y_j|\big] < \infty$. Then we have the following expression for $\mathcal{D}^*(Y_1, Y_2, \cdots, Y_t)$ and $\mathcal{D}(Y_1, Y_2, \cdots, Y_t)$:*

$$\mathcal{D}^*(Y_1, Y_2, \cdots, Y_t) = \sum_{(\ell, \eta_{1:\ell}) \in C(t)} (-1)^{\ell-1} [\eta_1, \cdots, \eta_\ell] \triangleright (Y_1, Y_2, \cdots, Y_t), \tag{H.4}$$

$$\mathcal{D}(Y_1, Y_2, \cdots, Y_t) = Y_1 Y_2 \cdots Y_t - \mathcal{D}^*(Y_1, \cdots, Y_t) - \sum_{j=1}^{\ell-1} Y_1 \cdots Y_j\ \mathcal{D}^*(Y_{j+1}, \cdots, Y_t). \tag{H.5}$$

where $C(t) = \{\ell, \eta_{1:\ell} \in \mathbb{N}_+ : \sum_{j=1}^{\ell} \eta_j = t\}$.

Proof of Lemma H.2. We perform induction on t.

If $t=1$, then by definition $\mathcal{D}^*(Y_1)=\mathbb{E}[Y_1]=[1]\triangleright(Y_1)$ and $\mathcal{D}(Y_1)=Y_1-\mathbb{E}[Y_1]=Y_1-\mathcal{D}^*(Y_1)$.

Supposing the results hold for $1,2,\cdots,t-1$, we consider the case t. Suppose that we have $\mathbb{E}[|Y_1\cdots Y_t|]<\infty$. By the inductive hypothesis, we have

$$\begin{aligned}
&\mathcal{D}^*(Y_1,Y_2,\cdots,Y_t)=\mathbb{E}[Y_1\mathcal{D}(Y_2,\cdots,Y_t)]\\
&\overset{(a)}{=}\mathbb{E}[Y_1Y_2\cdots Y_t]-\mathbb{E}(Y_1)\mathcal{D}^*(Y_2,\dots,Y_t)-\sum_{j=3}^{t}\mathbb{E}[Y_1\cdots Y_{j-1}\,\mathcal{D}^*(Y_j,\cdots,Y_t)]\\
&=\mathbb{E}[Y_1Y_2\cdots Y_t]-\sum_{j=2}^{t}\mathbb{E}[Y_1\cdots Y_{j-1}\,\mathcal{D}^*(Y_j,\cdots,Y_t)]\\
&\overset{(b)}{=}\mathbb{E}[Y_1Y_2\cdots Y_t]-\sum_{j=2}^{t}\sum_{C(t-j+1)}\mathbb{E}[Y_1\cdots Y_{j-1}\,(-1)^{\ell-1}[\eta_1,\cdots,\eta_\ell]\triangleright(Y_j,\cdots,Y_t)]\\
&=\mathbb{E}[Y_1Y_2\cdots Y_t]+\sum_{j=2}^{t}\sum_{C(t-j+1)}(-1)^{\ell}[j-1,\eta_1,\cdots,\eta_\ell]\triangleright(Y_1,\cdots,Y_t)\\
&\overset{(c)}{=}\mathbb{E}[Y_1Y_2\cdots Y_t]+\sum_{C(t)\setminus\{\ell=1,\ \eta_1=t\}}(-1)^{\ell-1}[\eta_1,\cdots,\eta_\ell]\triangleright(Y_1,\cdots,Y_t)\\
&=\sum_{C(t)}(-1)^{\ell-1}[\eta_1,\cdots,\eta_\ell]\triangleright(Y_1,\cdots,Y_t).
\end{aligned}$$

where to get (a) we have used the fact that by inductive hypothesis (H.5) holds for $t-1$, and to get (b) we have used the fact that we assumed that (H.4) hold for $t-1$. Finally to get (c) we have used the fact that

$$\begin{aligned}
C(t)=&\left\{\ell,\eta_{1:\ell}\in\mathbb{N}_+:\textstyle\sum_{j=1}^{\ell}\eta_j=t\right\}\\
=&\{\ell=1,\ \eta_1=t\}\cup\textstyle\bigcup_{i=2}^{t}\left\{\ell,\eta_{1:\ell}\in\mathbb{N}_+:\ell\geq 2,\ \eta_1=i,\ \sum_{j=1}^{\ell}\eta_j=t\right\}.
\end{aligned}$$

Moreover, we also have

$$\begin{aligned}
&\mathcal{D}(Y_1,Y_2,\cdots,Y_t)=Y_1\mathcal{D}(Y_2,\cdots,Y_t)-\mathcal{D}^*(Y_1,\cdots,Y_t)\\
=&Y_1Y_2\cdots Y_t-Y_1\mathcal{D}^*(Y_2,\cdots,Y_t)-\sum_{j=2}^{t-1}Y_1\cdots Y_j\mathcal{D}^*(Y_{j+1},\cdots,Y_t)-\mathcal{D}^*(Y_1,\cdots,Y_t)\\
=&Y_1Y_2\cdots Y_t-\mathcal{D}^*(Y_1,\cdots,Y_t)-\sum_{j=1}^{t-1}Y_1\cdots Y_j\mathcal{D}^*(Y_{j+1},\cdots,Y_t).
\end{aligned}$$

Thus, the results also hold for t. And the proof is complete by induction. ■

Next we need the following notions of **compositional** $\mathcal{D}^*$ and $\mathcal{D}$ **operators**.

$$[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t):=\mathcal{D}^*(Y_1\cdots Y_{\eta_1}\ ,\ Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\ ,\cdots,\ Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t),\quad\text{(H.6)}$$

$$[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}(Y_1,\cdots,Y_t):=\mathcal{D}(Y_1\cdots Y_{\eta_1}\ ,\ Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\ ,\cdots,\ Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t). \quad \text{(H.7)}$$

Note that a compositional $\mathcal{D}$ term is a random variable while a compositional $\mathcal{D}^*$ operator gives a deterministic value. We remark that

$$\mathbb{E}\big[[\eta_1,\ldots,\eta_l]\triangleright\mathcal{D}(Y_1,\ldots,Y_t)\big]=\mathbb{E}\Big[\mathcal{D}\big(Y_1\cdots Y_{\eta_1},\cdots,Y_{\eta_1+1}\cdots Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\big)\Big]=0. \quad \text{(H.8)}$$

Moreover, by definition and (H.1), we can directly check that

$$\begin{aligned}&[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(Y_1,\cdots,Y_t\big)\\=&[\eta_1,\cdots,\eta_{s-1},\eta_s+1]\triangleright\mathcal{D}^*\Big(Y_1,\cdots,Y_{\eta_1+\cdots+\eta_s}\ ,\ [\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(Y_{\eta_1+\cdots+\eta_s+1},\cdots,Y_t\big)\Big).\end{aligned} \quad \text{(H.9)}$$

The following lemma shows some upper bounds on their norms.

Lemma H.3. *Let $(Y_i)_{i=1}^t$ be random variable such that for all $i,j\in\mathbb{N}_+$ such that $i\le j\le t$ we have $\mathbb{E}\big[|Y_iY_{i+1}\cdots Y_j|\big]<\infty$. Then for any $q\ge 1$ the following holds*

$$\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\Big|\le\sum_{(s,\zeta_{1:s})\in C(t)}[\zeta_1,\cdots,\zeta_s]\triangleright\big(|Y_1|,\cdots,|Y_t|\big), \quad \text{(H.10)}$$

$$\begin{aligned}\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\Big|\le&\Big|\mathbb{E}[Y_1\cdots Y_t]\Big|\\&+\sum_{j=1}^{t-1}\sum_{(s,\zeta_{1:s})\in C(j)}[\zeta_1,\cdots,\zeta_s]\triangleright\big(|Y_1|,\cdots,|Y_j|\big)\cdot\Big|\mathbb{E}[Y_{j+1}\cdots Y_t]\Big|,\end{aligned} \quad \text{(H.11)}$$

$$\Big\|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}(Y_1,\cdots,Y_t)\Big\|_q\le 2\sum_{(s,\zeta_{1:s})\in C(t)}\Big([\zeta_1,\cdots,\zeta_s]\triangleright\big(|Y_1|^q,\cdots,|Y_t|^q\big)\Big)^{1/q}. \quad \text{(H.12)}$$

where $C(t):=\{s,\zeta_s\in\mathbb{N}_+:\sum_{j=1}^s\zeta_j=t\}$.

Proof. Applying Lemma H.2, we get

$$\begin{aligned}&\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\Big|\\=&\Big|\mathcal{D}^*\big(Y_1\cdots Y_{\eta_1}\ ,\ Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\ ,\cdots,\ Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\big)\Big|\\=&\Big|\sum_{(s,\zeta_{1:s})\in C(t)}(-1)^{s-1}[\zeta_1,\cdots,\zeta_s]\triangleright\big(Y_1\cdots Y_{\eta_1}\ ,\ Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\ ,\cdots,\ Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\big)\Big|\\\le&\sum_{(s,\zeta_{1:s})\in C(t)}[\zeta_1,\cdots,\zeta_s]\triangleright\big(|Y_1\cdots Y_{\eta_1}|\ ,\ |Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}|\ ,\cdots,\ |Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t|\big)\\\le&\sum_{(s,\lambda_{1:s})\in C(t)}[\lambda_1,\cdots,\lambda_s]\triangleright\big(|Y_1|\ ,\ |Y_2|\ ,\cdots,\ |Y_t|\big),\end{aligned}$$

where in the last inequality we have used the fact that for every $(s,\zeta_{1:s})\in C(t)$, if we write $\lambda_1:=\sum_{h=1}^{\zeta_1}\eta_h$ and $\lambda_j:=\sum_{h=\zeta_{j-1}}^{\zeta_j}\eta_h$ for all $j\le s$, we have that $(s,\lambda_{1:s})\in C(t)$ and

$$[\lambda_1,\cdots,\lambda_s]\triangleright(|Y_1|,|Y_2|,\cdots,|Y_t|)$$

$$=[\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_1\cdots Y_{\eta_1}|\,,\,|Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}|\,,\cdots,\,|Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t|\right).$$

Using similar ideas, we observe that

$$\begin{aligned}
&\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| \\
=\ &\left|\mathcal{D}^*\left(Y_1\cdots Y_{\eta_1}\,,\,Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\,,\cdots,\,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\right| \\
\overset{(H.4)}{=}&\left|\sum_{(s,\zeta_{1:s})\in C(t)}(-1)^{s-1}[\zeta_1,\cdots,\zeta_s]\triangleright\left(Y_1\cdots Y_{\eta_1}\,,\cdots,\,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\right| \\
=\ &\Bigg|\mathbb{E}[Y_1\cdots Y_t]+\sum_{j=1}^{\ell-1}\sum_{(s,\zeta_{1:s})\in C(j)}(-1)^s[\zeta_1,\cdots,\zeta_s]\triangleright \\
&\left(Y_1\cdots Y_{\eta_1},\cdots,\,Y_{\eta_1+\cdots+\eta_{j-1}+1}\cdots Y_{\eta_1+\cdots+\eta_j}\right)\cdot\mathbb{E}[Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_t]\Bigg| \\
\leq\ &\left|\mathbb{E}[Y_1\cdots Y_t]\right|+\sum_{j=1}^{\ell-1}\sum_{(s,\zeta_{1:s})\in C(j)}[\zeta_1,\cdots,\zeta_s]\triangleright \\
&\left(\left|Y_1\cdots Y_{\eta_1}\right|,\cdots,\left|Y_{\eta_1+\cdots+\eta_{j-1}+1}\cdots Y_{\eta_1+\cdots+\eta_j}\right|\right)\cdot\left|\mathbb{E}[Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_t]\right| \\
\overset{(*)}{\leq}&\left|\mathbb{E}[Y_1\cdots Y_t]\right|+\sum_{j=1}^{\ell-1}\sum_{(s,\lambda_{1:s})\in C(\eta_1+\cdots+\eta_j)}[\zeta_1,\cdots,\zeta_s]\triangleright \\
&\left(|Y_1|\,,\,|Y_2|\,,\cdots,\,\left|Y_{\eta_1+\cdots+\eta_j}\right|\right)\cdot\left|\mathbb{E}[Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_t]\right| \\
\leq\ &\left|\mathbb{E}[Y_1\cdots Y_t]\right|+\sum_{h=1}^{t-1}\sum_{(s,\lambda_{1:s})\in C(h)}[\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_1|\,,\,|Y_2|\,,\cdots,\,|Y_h|\right)\cdot\left|\mathbb{E}[Y_{h+1}\cdots Y_t]\right|.
\end{aligned}$$

where to obtain $(*)$ we have used the fact that

$$\begin{aligned}
&[\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_1|,\cdots,\left|Y_{\eta_1+\cdots+\eta_j}\right|\right) \\
=&[\zeta_1,\cdots,\zeta_s]\triangleright\left(\left|Y_1\cdots Y_{\eta_1}\right|,\cdots,\left|Y_{\eta_1+\cdots+\eta_{j-1}+1}\cdots Y_{\eta_1+\cdots+\eta_j}\right|\right).
\end{aligned}$$

Now let's prove (H.12). By Lemma H.2, we observe that

$$\begin{aligned}
&\left\|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}(Y_1,\cdots,Y_t)\right\|_q \\
=\ &\left\|\mathcal{D}\left(Y_1\cdots Y_{\eta_1}\,,\,Y_{\eta_1+1}\cdots Y_{\eta_1+\eta_2}\,,\cdots,\,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\right\|_q \\
\overset{(H.5)}{=}&\Bigg\|Y_1Y_2\cdots Y_t-\mathcal{D}^*\left(Y_1\cdots Y_{\eta_1},\cdots,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right) \\
&-\sum_{j=1}^{\ell-1}Y_1\cdots Y_{\eta_1+\cdots+\eta_j}\cdot\mathcal{D}^*\left(Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_{\eta_1+\cdots+\eta_{j+1}},\cdots,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\Bigg\|_q \\
=\ &\Bigg\|Y_1Y_2\cdots Y_t-[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)-\sum_{j=1}^{\ell-1}Y_1\cdots Y_{\eta_1+\cdots+\eta_j}\cdot \\
&\sum_{(s,\zeta_{1:s})\in C(\ell-j)}(-1)^{s-1}[\zeta_1,\cdots,\zeta_s]\triangleright\left(Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_{\eta_1+\cdots+\eta_{j+1}},\cdots,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\Bigg\|_q
\end{aligned}$$

We upper-bound this using the triangle inequality. Indeed we obtain that

$$
\begin{aligned}
&\left\|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}(Y_1,\cdots,Y_t)\right\|_q \\
\leq\;& \left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| + \|Y_1Y_2\cdots Y_t\|_q + \sum_{j=1}^{\ell-1}\sum_{(s,\zeta_{1:s})\in C(\ell-j)} \|Y_1\cdots Y_{\eta_1+\cdots+\eta_j}\|_q \cdot \\
&\quad \left|[\zeta_1,\cdots,\zeta_s]\triangleright\left(Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_{\eta_1+\cdots+\eta_{j+1}},\cdots,Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t\right)\right| \\
\overset{(*)}{\leq}\;& \left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| + \|Y_1Y_2\cdots Y_t\|_q + \sum_{j=1}^{\ell-1}\sum_{(s,\zeta_{1:s})\in C(\ell-j)} \|Y_1\cdots Y_{\eta_1+\cdots+\eta_j}\|_q \cdot \\
&\quad \left([\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_{\eta_1+\cdots+\eta_{j+1}}|^q,\cdots,|Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t|^q\right)\right)^{1/q} \\
\overset{(**)}{\leq}\;& \left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| + \|Y_1Y_2\cdots Y_t\|_q + \sum_{j=1}^{\ell-1}\sum_{(s,\lambda_{1:s})\in C(\eta_{j+1}+\cdots+\eta_\ell)} \|Y_1\cdots Y_{\eta_1+\cdots+\eta_j}\|_q \cdot \\
&\quad \left([\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_{\eta_1+\cdots+\eta_j+1}|^q,\cdots,|Y_t|^q\right)\right)^{1/q} \\
\leq\;& \left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| + \|Y_1Y_2\cdots Y_t\|_q + \sum_{h=1}^{t-1}\sum_{(s,\lambda_{1:s})\in C(t-h)} \|Y_1\cdots Y_h\|_q \cdot \\
&\quad \left([\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_{h+1}|^q,\cdots,|Y_t|^q\right)\right)^{1/q} \\
=\;& \left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(Y_1,\cdots,Y_t)\right| + \sum_{(s,\lambda_{1:s})\in C(t)} \left([\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_1|^q,\cdots,|Y_t|^q\right)\right)^{1/q} \\
\overset{(H.10)}{\leq}\;& \sum_{(s,\zeta_{1:s})\in C(t)} [\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_1|,\cdots,|Y_t|\right) + \sum_{(s,\zeta_{1:s})\in C(t)} \left([\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_1|^q,\cdots,|Y_t|^q\right)\right)^{1/q} \\
\leq\;& 2\sum_{(s,\zeta_{1:s})\in C(t)} \left([\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_1|^q,\cdots,|Y_t|^q\right)\right)^{1/q},
\end{aligned}
$$

where to obtain $(*)$ we have used the fact that by Jensen inequality for any random variable $X \in \mathcal{L}^q(\mathbb{R})$ we have $\left|\mathbb{E}[X]\right| \leq \left(\mathbb{E}[|X|^q]\right)^{1/q}$; and where to obtain $(**)$ we have used the fact that

$$
\begin{aligned}
&[\lambda_1,\cdots,\lambda_s]\triangleright\left(|Y_{\eta_1+\cdots+\eta_j+1}|^q,\cdots,|Y_t|^q\right) \\
=&[\zeta_1,\cdots,\zeta_s]\triangleright\left(|Y_{\eta_1+\cdots+\eta_j+1}\cdots Y_{\eta_1+\cdots+\eta_{j+1}}|^q,\cdots,|Y_{\eta_1+\cdots+\eta_{\ell-1}+1}\cdots Y_t|^q\right).
\end{aligned}
$$

■

Lemma H.4. *Let $(X_i)_{i\in T}$ be a stationary random field of random variables with a finite index set T and finite L^r-norms, i.e., $\|X_i\|_r < \infty$, where r is a real number such that $r > 2$. Let $S_{T_0,\omega}$ be a random variable that satisfies $|S_{T_0,\omega}| \leq \left|\sum_{i\in T_0} X_i\right|^\omega$ where $T_0 \subseteq T$ is an index set and $0 \leq \omega \leq 1$. Fix $t \in \mathbb{N}_+$ such that $1 \leq t < r-1$ and $i_{1:t} \in T$.*

For any $\ell, \eta_{1:\ell} \in \mathbb{N}_+$ such that $\eta_1 + \cdots + \eta_\ell = t+1$, we have

$$
\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(X_{i_1},\cdots,X_{i_t},X_{i_{t+1}})\right| \leq 2^t\|X_{i_1}\|_r^{t+1}, \tag{H.13}
$$

$$\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega})\Big|\le 2^t|T_0|^\omega\|X_{i_1}\|_r^{t+\omega}. \tag{H.14}$$

We further let j be an integer that satisfies $2\le j\le t+1$ and define the σ-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ by

$$\mathcal{F}_1:=\sigma\big(X_{i_1},\cdots,X_{i_j}\big),\quad \mathcal{F}_2:=\sigma\big(X_{i_{j+1}},\cdots,X_{i_{t+1}}\big),\quad \mathcal{F}_3:=\begin{cases}\sigma\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big) & \text{if } j\le t\\ \sigma(S_{T_0,\omega}) & \text{if } j=t+1\end{cases}.$$

For any $s,\ell,\eta_{1:\ell}\in\mathbb{N}_+$ such that $\eta_1+\cdots+\eta_s=j-1$ and $\eta_1+\cdots+\eta_\ell=t+1$, we have

$$\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(X_{i_1},\cdots,X_{i_t},X_{i_{t+1}})\Big|\le 2^{t+3}\big(\alpha(\mathcal{F}_1,\mathcal{F}_2)\big)^{(r-t-1)/r}\|X_{i_1}\|_r^{t+1}, \tag{H.15}$$

$$\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega})\Big|\le 2^{t+3}|T_0|^\omega\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t-\omega)/r}\|X_{i_1}\|_r^{t+\omega}, \tag{H.16}$$

where $\alpha(\mathcal{F}_1,\mathcal{F}_2)$ is the strong mixing coefficients between $\mathcal{F}_1$ and $\mathcal{F}_2$, and $\alpha(\mathcal{F}_1,\mathcal{F}_3)$ is the strong mixing coefficients between $\mathcal{F}_1$ and $\mathcal{F}_3$.

Proof. For ease of notation denote $M:=\|X_{i_1}\|_r$. To prove (H.13), we remark that

$$\begin{aligned}&\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_{t+1}}\big)\Big|\\ \overset{\text{(H.10)}}{\le}&\sum_{(\ell',\zeta_{1:\ell'})\in C(t+1)}[\zeta_1,\cdots,\zeta_{\ell'}]\triangleright\big(|X_{i_1}|,\cdots,|X_{i_{t+1}}|\big)\\ \overset{(*)}{\le}&\ 2^t\frac{1}{t+1}\big(\|X_{i_1}\|_{t+1}^{t+1}+\cdots+\|X_{i_{t+1}}\|_{t+1}^{t+1}\big)\le 2^tM^{t+1},\end{aligned}$$

where $(*)$ is implied by Lemma D.3 and the fact that $|C(t+1)|=2^t$ [Heubach & Mansour 2009]. For (H.14), we have

$$\begin{aligned}&\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big|\\ \overset{\text{(H.10)}}{\le}&\sum_{(\ell',\zeta_{1:\ell'})\in C(t+1)}[\zeta_1,\cdots,\zeta_{\ell'}]\triangleright\big(|X_{i_1}|,\cdots,|X_{i_t}|,|S_{T_0,\omega}|\big)\\ \le&\ |T_0|^\omega\sum_{(\ell',\zeta_{1:\ell'})\in C(t+1)}[\zeta_1,\cdots,\zeta_{\ell'}]\triangleright\left(|X_{i_1}|,\cdots,|X_{i_t}|,\left|\frac{1}{|T_0|}\sum_{i\in T_0}X_i\right|^\omega\right)\\ \overset{(*)}{\le}&\ |T_0|^\omega\sum_{(\ell',\zeta_{1:\ell'})\in C(t+1)}\left(\frac{1}{t+\omega}\big(\|X_{i_1}\|_{t+\omega}^{t+\omega}+\cdots+\|X_{i_t}\|_{t+\omega}^{t+\omega}\big)+\frac{\omega}{t+\omega}\left\|\frac{1}{|T_0|}\sum_{i\in T_0}X_i\right\|_{t+\omega}^{t+\omega}\right)\\ \overset{(**)}{\le}&\ 2^t|T_0|^\omega\left(\frac{1}{t+\omega}\big(\|X_{i_1}\|_{t+\omega}^{t+\omega}+\cdots+\|X_{i_t}\|_{t+\omega}^{t+\omega}\big)+\frac{\omega}{t+\omega}\frac{1}{|T_0|}\sum_{i\in T_0}\|X_i\|_{t+\omega}^{t+\omega}\right)\\ \le&\ 2^t|T_0|^\omega\cdot M^{t+\omega}.\end{aligned}$$

Here we have used Lemma D.3 in $(*)$, and $(**)$ is implied by $|C(t+1)|\le 2^t$ and Jensen's inequality as

$$\begin{aligned}&\Big\|\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}X_i\Big\|_{t+\omega}^{t+\omega}=\mathbb{E}\Big[\Big|\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}X_i\Big|^{t+\omega}\Big]\\ \le&\mathbb{E}\Big[\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}|X_i|^{t+\omega}\Big]=\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}\mathbb{E}[|X_i|^{t+\omega}]=\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}\|X_i\|_{t+\omega}^{t+\omega}.\end{aligned}$$

To show (H.15), we remark that assumption we have that s is such that $\eta_1 + \cdots + \eta_s = j-1$. Therefore, according to (H.9), we get that

$$\begin{aligned}
&[\eta_1, \cdots, \eta_\ell] \triangleright \mathcal{D}^*\big(X_{i_1}, \cdots, X_{i_{t+1}}\big) \\
=&[\eta_1, \cdots, \eta_s, \cdots, \eta_\ell] \triangleright \mathcal{D}^*\big(X_{i_1}, \cdots, X_{i_j}, \cdots, X_{i_{t+1}}\big) \\
=&[\eta_1, \cdots, \eta_{s-1}, \eta_s + 1] \triangleright \mathcal{D}^*\Big(X_{i_1}, \cdots, X_{i_{j-1}}, [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\Big).
\end{aligned} \tag{H.17}$$

Moreover, by exploiting (H.11), we obtain that

$$\begin{aligned}
&\Big|[\eta_1, \cdots, \eta_{s-1}, \eta_s + 1] \triangleright \mathcal{D}^*\Big(X_{i_1}, \cdots, X_{i_{j-1}}, [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\Big)\Big| \\
\leq&\Big|\mathbb{E}\big[X_{i_1} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big| \\
&+ \sum_{w=1}^{j-2} \sum_{(s', \zeta_{1:s'}) \in C(w)} [\zeta_1, \cdots, \zeta_{s'}] \triangleright \big(|X_{i_1}|, \cdots, |X_{i_w}|\big) \cdot \\
&\quad \Big|\mathbb{E}\big[X_{i_{w+1}} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big|
\end{aligned} \tag{H.18}$$

By Lemma D.3, we know that

$$[\zeta_1, \cdots, \zeta_{s'}] \triangleright \big(|X_{i_1}|, \cdots, |X_{i_w}|\big) \leq \frac{1}{w}\big(\|X_{i_1}\|_w^w + \cdots + \|X_{i_w}\|_w^w\big) \leq M^w. \tag{H.19}$$

Combining (H.17), (H.18), and (H.19), we get

$$\begin{aligned}
&\Big|[\eta_1, \cdots, \eta_\ell] \triangleright \mathcal{D}^*\big(X_{i_1}, \cdots, X_{i_{t+1}}\big)\Big| \\
\leq&\Big|\mathbb{E}\big[X_{i_1} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big| \\
&+ \sum_{w=1}^{j-2} \sum_{(s', \zeta_{1:s'}) \in C(w)} M^w \Big|\mathbb{E}\big[X_{i_{w+1}} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big| \\
\overset{(*)}{=}& \sum_{w=0}^{j-2} 2^{(w-1)\vee 0} M^w \Big|\mathbb{E}\big[X_{i_{w+1}} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big| \\
=& \sum_{w=1}^{j-1} 2^{(w-2)\vee 0} M^{w-1} \Big|\mathbb{E}\big[X_{i_w} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big|.
\end{aligned} \tag{H.20}$$

where $(*)$ is due to the fact that $|C(w)| = 2^{w-1}$.

As mentioned in (H.8), by definition of the compositional $\mathcal{D}$ operator we know that

$$\mathbb{E}\big[[\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big] = 0.$$

Thus, we can apply Lemma F.1:

$$\begin{aligned}
&\Big|\mathbb{E}\big[X_{i_w} \cdots X_{i_{j-1}} \cdot [\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big]\Big| \\
\leq& 8\big(\alpha(\mathcal{F}_1, \mathcal{F}_2)\big)^{(r-t+w-2)/r} \big\|X_{i_w} \cdots X_{i_{j-1}}\big\|_{r/(j-w)} \cdot \big\|[\eta_{s+1}, \cdots, \eta_\ell] \triangleright \mathcal{D}\big(X_{i_j}, \cdots, X_{i_{t+1}}\big)\big\|_{r/(t-j+2)}.
\end{aligned} \tag{H.21}$$

By Lemma D.3, we get

$$\begin{aligned}\left\|X_{i_w}\cdots X_{i_{j-1}}\right\|_{r/(j-w)} =&\Big(\mathbb{E}\Big[\left|X_{i_w}\cdots X_{i_{j-1}}\right|^{r/(j-w)}\Big]\Big)^{(j-w)/r} \\ \le&\Big(\frac{1}{j-w}\big(\|X_{i_w}\|_r^r+\cdots+\|X_{i_{j-1}}\|_r^r\big)\Big)^{(j-w)/r} = M^{j-w}.\end{aligned} \tag{H.22}$$

Moreover, remark that

$$\begin{aligned}&\left\|[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_{t+1}}\big)\right\|_{r/(t-j+2)} \\ \overset{\text{(H.12)}}{\le}&2\sum_{(\ell',\zeta_{1:\ell'})\in C(t-j+2)}\Big([\eta_{s+1},\cdots,\eta_\ell]\triangleright\big(|X_{i_j}|^{r/(t-j+2)},\cdots,|X_{i_{t+1}}|^{r/(t-j+2)}\big)\Big)^{(t-j+2)/r} \\ \overset{(*)}{\le}&\ 2^{t-j+2}\Big(\frac{1}{t-j+2}\big(\|X_{i_j}\|_r^r+\cdots+\|X_{i_{t+1}}\|_r^r\big)\Big)^{(t-j+2)/r} = 2^{t-j+2}M^{t-j+2}.\end{aligned} \tag{H.23}$$

Note that $(*)$ is implied by the fact that $|C(t-j+2)|=2^{t-j+1}$ and Lemma D.3.

Substituting (H.22) and (H.23) into (H.21), we get

$$\left|\mathbb{E}\Big[X_{i_w}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_{t+1}}\big)\Big]\right| \le 2^{t-j+5}\big(\alpha(\mathcal{F}_1,\mathcal{F}_2)\big)^{(r-t+w-2)/r}M^{t-w+2}.$$

Combining this and (H.20), we get

$$\begin{aligned}&\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_{t+1}}\big)\right| \\ \le&\sum_{w=1}^{j-1}2^{(w-2)\vee 0}\cdot 2^{t-j+5}\big(\alpha(\mathcal{F}_1,\mathcal{F}_2)\big)^{(r-t+w-2)/r}M^{t+1} \\ \le&2^{t+3}\big(\alpha(\mathcal{F}_1,\mathcal{F}_2)\big)^{(r-t-1)/r}M^{t+1}.\end{aligned}$$

Lastly, we prove (H.16). We consider two cases, $2\le j\le t$ and $j=t+1$.

If $2\le j\le t$, by (H.9), we have

$$\begin{aligned}&[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big) \\ =&[\eta_1,\cdots,\eta_s,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big) \\ =&[\eta_1,\cdots,\eta_{s-1},\eta_s+1]\triangleright\mathcal{D}^*\Big(X_{i_1},\cdots,X_{i_{j-1}}\ ,\ [\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big).\end{aligned} \tag{H.24}$$

By (H.11), we get

$$\begin{aligned}&\left|[\eta_1,\cdots,\eta_{s-1},\eta_s+1]\triangleright\mathcal{D}^*\Big(X_{i_1},\cdots,X_{i_{j-1}}\ ,\ [\eta_{s+1},\cdots,\eta_t]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big)\right| \\ \le&\left|\mathbb{E}\Big[X_{i_1}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big]\right| \\ &+\sum_{w=1}^{j-2}\sum_{(s',\zeta_{1:s'})\in C(w)}[\zeta_1,\cdots,\zeta_{s'}]\triangleright\big(|X_{i_1}|,\cdots,|X_{i_w}|\big)\cdot \\ &\qquad\left|\mathbb{E}\Big[X_{i_{w+1}}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big]\right|.\end{aligned} \tag{H.25}$$

Combining (H.24), (H.25), and (H.19), we have

$$\begin{aligned}
&\Big|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\Big|\\
\le&\Big|\mathbb{E}\big[X_{i_w}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big]\Big|\\
&+\sum_{w=1}^{j-2}\sum_{(s',\zeta_{1:s'})\in C(w)}M^w\Big|\mathbb{E}\big[X_{i_{w+1}}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big]\Big|\\
\overset{(*)}{=}&\sum_{w=1}^{j-1}2^{(w-2)\vee 0}M^{w-1}\Big|\mathbb{E}\big[X_{i_w}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big]\Big|,
\end{aligned}\tag{H.26}$$

where $(*)$ is due to the fact that $|C(w)|=2^{w-1}$.

We apply Lemma F.1 and obtain

$$\begin{aligned}
&\Big|\mathbb{E}\big[X_{i_w}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big]\Big|\\
\le&8\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}\big\|X_{i_w}\cdots X_{i_{j-1}}\big\|_{r/(j-w)}\cdot\big\|[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big\|_{r/(t+1+\omega-j)}.
\end{aligned}\tag{H.27}$$

We observe that

$$\begin{aligned}
&\big\|[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big\|_{r/(t+1+\omega-j)}\\
\overset{(H.12)}{\le}&2|T_0|^\omega\sum_{(\ell',\lambda_{1:\ell'})\in C(t+2-j)}\left([\lambda_1,\cdots,\lambda_{\ell'}]\triangleright\left(|X_{i_j}|^{\frac{r}{t+1+\omega-j}},\cdots,|X_{i_t}|^{\frac{r}{t+1+\omega-j}},\Big|\frac{1}{|T_0|}\sum_{i\in T_0}X_i\Big|^{\frac{r\omega}{t+1+\omega-j}}\right)\right)^{\frac{t+1+\omega-j}{r}}\\
\overset{(*)}{\le}&2|T_0|^\omega\sum_{(\ell',\lambda_{1:\ell'})\in C(t+2-j)}\left(\frac{1}{t+1+\omega-j}\big(\|X_{i_j}\|_r^r+\cdots+\|X_{i_t}\|_r^r\big)+\frac{\omega}{t+1+\omega-j}\Big\|\frac{1}{|T_0|}\sum_{i\in T_0}X_i\Big\|_r^r\right)^{\frac{t+1+\omega-j}{r}}\\
\overset{(**)}{\le}&2^{t+2-j}|T_0|^\omega\left(\frac{1}{t+1+\omega-j}\big(\|X_{i_j}\|_r^r+\cdots+\|X_{i_t}\|_r^r\big)+\frac{\omega}{t+1+\omega-j}\frac{1}{|T_0|}\sum_{i\in T_0}\|X_i\|_r^r\right)^{\frac{t+1+\omega-j}{r}}\\
\le&2^{t+2-j}|T_0|^\omega M^{t+1+\omega-j},
\end{aligned}\tag{H.28}$$

where we have used Lemma D.3 in $(*)$, and $(**)$ is implied by $|C(t+2-j)|=2^{t+1-j}$ and Jensen's inequality as

$$\Big\|\tfrac{1}{|T_0|}\textstyle\sum_{i\in T_0}X_i\Big\|_r^r=\mathbb{E}\Big[\Big|\tfrac{1}{|T_0|}\sum_{i\in T_0}X_i\Big|^r\Big]\le\mathbb{E}\Big[\tfrac{1}{|T_0|}\sum_{i\in T_0}|X_i|^r\Big]=\tfrac{1}{|T_0|}\sum_{i\in T_0}\mathbb{E}[|X_i|^r]=\tfrac{1}{|T_0|}\sum_{i\in T_0}\|X_i\|_r^r.$$

Substituting (H.22) and (H.28) into (H.27), we have

$$\begin{aligned}
&\Big|\mathbb{E}\big[X_{i_w}\cdots X_{i_{j-1}}\cdot[\eta_{s+1},\cdots,\eta_\ell]\triangleright\mathcal{D}\big(X_{i_j},\cdots,X_{i_t},S_{T_0,\omega}\big)\big]\Big|\\
\le&2^{t+5-j}|T_0|^\omega\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}M^{t+1+\omega-w}.
\end{aligned}$$

Combining this with (H.26), we obtain

$$\begin{aligned}&\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\right|\\ \le&\sum_{w=1}^{j-1}2^{(w-2)\vee 0}M^{w-1}\cdot 2^{t+5-j}|T_0|^{\omega}\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}M^{t+1+\omega-w}\\ \le&2^{t+3}|T_0|^{\omega}\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t-\omega)/r}M^{t+\omega}.\end{aligned}$$

If $j=t+1$, then $\eta_1+\cdots+\eta_s=j-1=t$ implies that $s=\ell-1$ and $\eta_\ell=1$. By (H.9) we have

$$\begin{aligned}&[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\\ =&[\eta_1,\cdots,\eta_s,1]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\\ =&[\eta_1,\cdots,\eta_{s-1},\eta_s+1]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},\mathcal{D}(S_{T_0,\omega})\big).\end{aligned}\tag{H.29}$$

By (H.11), we get

$$\begin{aligned}&\left|[\eta_1,\cdots,\eta_{s-1},\eta_s+1]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},\mathcal{D}(S_{T_0,\omega})\big)\right|\\ \le&\left|\mathbb{E}\big[X_{i_1}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\big]\right|\\ &+\sum_{w=1}^{t}\sum_{(s',\zeta_{1:s'})\in C(w)}[\zeta_1,\cdots,\zeta_{s'}]\triangleright\big(|X_{i_1}|,\cdots,|X_{i_w}|\big)\cdot\left|\mathbb{E}\big[X_{i_{w+1}}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\big]\right|\\ \overset{(*)}{\le}&\sum_{w=1}^{t}2^{(w-2)\vee 0}M^{w-1}\left|\mathbb{E}\big[X_{i_w}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\big]\right|,\end{aligned}\tag{H.30}$$

where $(*)$ is due to (H.19) and the fact that $|C(w)|=2^{w-1}$.

Combining (H.29) and (H.30), we have

$$\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\right|\le\sum_{w=1}^{t}2^{(w-2)\vee 0}M^{w-1}\left|\mathbb{E}\big[X_{i_w}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\big]\right|.\tag{H.31}$$

Again we apply Lemma F.1 and obtain

$$\left|\mathbb{E}\big[X_{i_w}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\big]\right|\le 8\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}\big\|X_{i_w}\cdots X_{i_t}\big\|_{r/(t+1-w)}\cdot\big\|\mathcal{D}(S_{T_0,\omega})\big\|_{r/\omega}.\tag{H.32}$$

Note that

$$\begin{aligned}&\big\|\mathcal{D}(S_{T_0,\omega})\big\|_{r/\omega}\le 2\big\|S_{T_0,\omega}\big\|_{r/\omega}\le 2\Big\|\sum_{i\in T_0}X_i\Big\|_r^{\omega}\\ \le&2|T_0|^{\omega}\Big\|\frac{1}{|T_0|}\sum_{i\in T_0}X_i\Big\|_r^{\omega}\overset{(*)}{\le}2|T_0|^{\omega}\Big(\frac{1}{|T_0|}\sum_{i\in T_0}\|X_i\|_r^r\Big)^{\omega/r}\le 2|T_0|^{\omega}M^{\omega}.\end{aligned}\tag{H.33}$$

Once again we have used Jensen's inequality to get $(*)$.

Substituting (H.22) and (H.33) into (H.32), we get

$$\left|\mathbb{E}\left[X_{i_w}\cdots X_{i_t}\cdot\mathcal{D}(S_{T_0,\omega})\right]\right|\le 16\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}|T_0|^{\omega}M^{t-w+\omega+1}.$$

Combining this and (H.31), we obtain

$$\begin{aligned}&\left|[\eta_1,\cdots,\eta_\ell]\triangleright\mathcal{D}^*\big(X_{i_1},\cdots,X_{i_t},S_{T_0,\omega}\big)\right|\\ \le&\sum_{w=1}^{t}2^{(w-2)\vee 0}M^{w-1}\cdot 16\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t+w-1-\omega)/r}|T_0|^{\omega}M^{t-w+\omega+1}\\ \le&2^{t+3}|T_0|^{\omega}\big(\alpha(\mathcal{F}_1,\mathcal{F}_3)\big)^{(r-t-\omega)/r}M^{t+\omega}.\end{aligned}$$

∎

Before providing the upper bound for the sums of $\mathcal{S}(H)$ and $\mathcal{U}_f(H)$, we first control the $\mathcal{E}_G$ operator defined in Appendix F.3 using strong mixing coefficients.

Lemma H.5. *Let $(X_i)_{i\in T}$ be a stationary random field of random variables with a finite index set T and finite L^r-norms, i.e., $\|X_i\|_r<\infty$. Given an order-$(k+1)$ genogram G, we have the following bounds:*

(a) For any real number $r>k+1$, we have

$$\left|\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},X_{i_{k+1}}\big)\right|\le 2^k\|X_{i_1}\|_r^{k+1}. \tag{H.34}$$

(b) For any $f\in\mathcal{C}^{k-1,\omega}(\mathbb{R})$, we have

$$\left|\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},\Delta_f(G)\big)\right|\le 2^k\sigma^{-\omega}\left|B_{k+1}\backslash D_{k+1}\right|^{\omega}\cdot|f|_{k-1,\omega}\|X_{i_1}\|_r^{k+\omega}, \tag{H.35}$$

where $\sigma^2:=\mathrm{Var}\big(\sum_{i\in T}X_i\big)$.

If $k\ge 2$, for any $f\in\mathcal{C}^{k-2,1}(\mathbb{R})\cap\mathcal{C}^{k-1,1}(\mathbb{R})$ and $\omega\in[0,1]$, we have

$$\left|\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},\Delta_f(G)\big)\right|\le 2^{k+1}\sigma^{-\omega}\left|B_{k+1}\backslash D_{k+1}\right|^{\omega}\cdot|f|_{k-2,1}^{1-\omega}|f|_{k-1,1}^{\omega}\|X_{i_1}\|_r^{k+\omega}. \tag{H.36}$$

(c) For any $f\in\mathcal{C}^{k-1,\omega}(\mathbb{R})$ and $c_1,c_2\in\mathbb{N}_+$ such that $c_1<c_2$, we have

$$\left|\sum_{c_1\le s_{k+1}<c_2}\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},\Delta_f(G)\big)\right|\le 2^k\sigma^{-\omega}(c_2-c_1)^{\omega}|f|_{k-1,\omega}\|X_{i_1}\|_r^{k+\omega},$$

where the sum is taken over genograms whose $c_1\le s_{k+1}<c_2$ with the vertex set, the edge set, and $s_{1:k}$ fixed.

If $k\ge 2$, for any $f\in\mathcal{C}^{k-2,1}(\mathbb{R})\cap\mathcal{C}^{k-1,1}(\mathbb{R})$, $\omega\in[0,1]$ and $c_1,c_2\in\mathbb{N}_+$ such that $c_1<c_2$, we have

$$\left|\sum_{c_1\le s_{k+1}<c_2}\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},\Delta_f(G)\big)\right|\le 2^{k+1}\sigma^{-\omega}(c_2-c_1)^{\omega}|f|_{k-2,1}^{1-\omega}|f|_{k-1,1}^{\omega}\|X_{i_1}\|_r^{k+\omega}.$$

Now suppose there exists $1 < j \le k+1$ such that $s_j \ge 1$. Then the following holds:

(d) *For any real number $r > k+1$, we have*

$$\left|\mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, X_{i_{k+1}}\big)\right| \le 2^{k+3} \alpha_{\ell_0}^{(r-k-1)/r} \|X_{i_1}\|_r^{k+1}, \tag{H.37}$$

where ℓ_0 is the smallest integer ℓ such that

$$k(2\ell+1)^d \ge \max_{1\le j\le k+1} s_j + k(2m+1)^d.$$

(e) *For any $f \in \mathcal{C}^{k-1,\omega}(\mathbb{R})$, we have*

$$\left|\mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \Delta_f(G)\big)\right| \le 2^{k+3} \sigma^{-\omega} \left|B_{k+1} \backslash D_{k+1}\right|^{\omega} \cdot |f|_{k-1,\omega} \alpha_{\ell_0}^{(r-k-\omega)/r} \|X_{i_1}\|_r^{k+\omega}, \tag{H.38}$$

where ℓ_0 is defined as above.

If $k \ge 2$, for any $f \in \mathcal{C}^{k-2,1}(\mathbb{R}) \cap \mathcal{C}^{k-1,1}(\mathbb{R})$ and $\omega \in [0,1]$, we have

$$\left|\mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \Delta_f(G)\big)\right| \le 2^{k+4} \sigma^{-\omega} \left|B_{k+1} \backslash D_{k+1}\right|^{\omega} \cdot |f|_{k-2,1}^{1-\omega} |f|_{k-1,1}^{\omega} \alpha_{\ell_0}^{(r-k-\omega)/r} \|X_{i_1}\|_r^{k+\omega}, \tag{H.39}$$

(f) *For any $f \in \mathcal{C}^{k-1,\omega}(\mathbb{R})$ and $c_1, c_2 \in \mathbb{N}_+$ such that $c_1 < c_2$, we have*

$$\left|\sum_{c_1 \le s_{k+1} < c_2} \mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \Delta_f(G)\big)\right| \le 2^{k+3} \sigma^{-\omega} (c_2 - c_1)^{\omega} |f|_{k-1,\omega} \alpha_{\ell_0}^{(r-k-\omega)/r} \|X_i\|_r^{k+\omega}, \tag{H.40}$$

where ℓ_0 is the smallest integer ℓ such that

$$k(2\ell+1)^d \ge c_1 \vee \max_{1\le j\le k} s_j + k(2m+1)^d.$$

If $k \ge 2$, for any $f \in \mathcal{C}^{k-2,1}(\mathbb{R}) \cap \mathcal{C}^{k-1,1}(\mathbb{R})$, $\omega \in [0,1]$ and $c_1, c_2 \in \mathbb{N}_+$ such that $c_1 < c_2$, we have

$$\left|\mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \Delta_f(G)\big)\right| \le 2^{k+4} \sigma^{-\omega} (c_2 - c_1)^{\omega} \cdot |f|_{k-2,1}^{1-\omega} |f|_{k-1,1}^{\omega} \alpha_{\ell_0}^{(r-k-\omega)/r} \|X_{i_1}\|_r^{k+\omega}. \tag{H.41}$$

Proof of Lemma H.5. We will perform induction on k to prove this lemma. But before that, we will present some prelimenary results.

Firstly, we observe that if $s_{k+1} \ge 1$, then $B_{k+1} \setminus D_{k+1}$ is a singleton and therefore, $\left|B_{k+1} \setminus D_{k+1}\right| = 1$ which implies that Lemma H.5b is a special case of Lemma H.5c and that Lemma H.5e is a special case of Lemma H.5f by setting $c_2 = c_1 + 1$. For notational convenience, we combine the two cases by denoting

$$\widetilde{\Delta}_f := \begin{cases} \Delta_f(G) & \text{if } s_{k+1} \le 0 \\ \sum_{c_1 \le s_{k+1} < c_2} \Delta_f(G) & \text{if } s_{k+1} \ge 1 \end{cases}.$$

Then by definition of $\mathcal{E}_G$, we have

$$\mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \widetilde{\Delta}_f\big) = \begin{cases} \mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_k}, \Delta_f(G)\big) & \text{if } s_{k+1} \le 0 \\ \sum_{c_1 \le s_{k+1} < c_2} \mathcal{E}_G\big(X_{i_h}, \cdots, X_{i_k}, \Delta_f(G)\big) & \text{if } s_{k+1} \ge 1 \end{cases}.$$

Further let

$$\widetilde{B}_{k+1} := \begin{cases} B_{k+1} & \text{if } s_{k+1} \le 0 \\ N^{(c_2-1)}\big(i_\ell : \ell \in A(k+1)\big) \cup D_{g(k+1)} & \text{if } s_{k+1} \ge 1 \end{cases},$$

$$\widetilde{D}_{k+1} := \begin{cases} D_{k+1} & \text{if } s_{k+1} \le 0 \\ N^{(c_1-1)}\big(i_\ell : \ell \in A(k+1)\big) \cup D_{g(k+1)} & \text{if } s_{k+1} \ge 1 \end{cases}$$

If $s_{k+1} \ge c_1 \ge 1$, we have $u(k+1) = k+1$. The following holds due to a telescoping sum argument:

$$\widetilde{\Delta}_f = \partial^{k-1} f\big(W(\widetilde{B}_{k+1})\big) - \partial^{k-1} f\big(W(\widetilde{D}_{k+1})\big).$$

Thus, we get that

$$\widetilde{\Delta}_f = \begin{cases} \partial^{k-1} f(W(\widetilde{B}_{k+1})) - \partial^{k-1} f(W(\widetilde{D}_{k+1})) & \text{if } u(k+1) = k+1 \\ \int_0^1 (k+1-u(k+1)) v^{k-u(k+1)} \cdot \\ \quad \big(\partial^{k-1} f(vW(\widetilde{D}_{k+1}) + (1-v)W(\widetilde{B}_{k+1})) - \partial^{k-1} f(W(\widetilde{D}_{k+1}))\big)\, \mathrm{d}v & \text{if } u(k+1) \le k \end{cases}.$$

If $u(k+1) = k+1$, we have

$$|\widetilde{\Delta}_f| \le \Big|\partial^{k-1} f\big(W(\widetilde{B}_{k+1})\big) - \partial^{k-1} f\big(W(\widetilde{D}_{k+1})\big)\Big| \le \sigma^{-\omega} |f|_{k-1,\omega} \Bigg| \sum_{i \in \widetilde{B}_{k+1} \setminus \widetilde{D}_{k+1}} X_i \Bigg|^{\omega}.$$

If $u(k+1) \le k$, we have

$$\begin{aligned} |\widetilde{\Delta}_f| \le & \int_0^1 (k+1-u(k+1)) v^{k-u(k+1)} \cdot \\ & \Big|\partial^{k-1} f\big(vW(\widetilde{D}_{k+1}) + (1-v)W(\widetilde{B}_{k+1})\big) - \partial^{k-1} f\big(W(\widetilde{D}_{k+1})\big)\Big| \,\mathrm{d}v \\ \le & \sigma^{-\omega} |f|_{k-1,\omega} \cdot \Bigg| \sum_{i \in \widetilde{B}_{k+1} \setminus \widetilde{D}_{k+1}} X_i \Bigg| \int_0^1 (k+1-u(k+1)) v^{k-u(k+1)} \,\mathrm{d}v \\ \le & \sigma^{-\omega} |f|_{k-1,\omega} \Bigg| \sum_{i \in \widetilde{B}_{k+1} \setminus \widetilde{D}_{k+1}} X_i \Bigg|^{\omega}. \end{aligned}$$

Therefore, in both cases, we can write

$$\widetilde{\Delta}_f = \sigma^{-\omega} |f|_{k-1,\omega} S_{T_0,\omega},$$

where $T_0 = \widetilde{B}_{k+1} \setminus \widetilde{D}_{k+1}$ and $S_{T_0,\omega}$ (which depends on f by definition) satisfies

$$|S_{T_0,\omega}| \le \Big|\textstyle\sum_{i \in T_0} X_i\Big|^{\omega}.$$

If $k \ge 2$ and $f \in \mathcal{C}^{k-2,1}(\mathbb{R}) \cap \mathcal{C}^{k-1,1}(\mathbb{R})$, $f \in \mathcal{C}^{k-1,1}(\mathbb{R})$ implies that

$$|\widetilde{\Delta}_f| \le \sigma^{-1} |f|_{k-1,1} \Big|\textstyle\sum_{i \in T_0} X_i\Big|.$$

On the other hand, $f \in \mathcal{C}^{k-2,1}(\mathbb{R})$ implies that

$$|\widetilde{\Delta}_f| \le 2|f|_{k-2,1}.$$

Thus, for any $\omega \in [0,1]$, we have

$$|\widetilde{\Delta}_f| \le 2\sigma^{-\omega}|f|_{k-2,1}^{1-\omega}|f|_{k-1,1}^{\omega}\Big|\sum\nolimits_{i\in T_0} X_i\Big|^{\omega}.$$

In this setting, we can write

$$\widetilde{\Delta}_f = 2\sigma^{-\omega}|f|_{k-2,1}^{1-\omega}|f|_{k-1,1}^{\omega} S_{T_0,\omega},$$

where $T_0 = \widetilde{B}_{k+1}\backslash\widetilde{D}_{k+1}$ and $S_{T_0,\omega}$ (which depends on f by definition) satisfies

$$|S_{T_0,\omega}| \le \Big|\sum\nolimits_{i\in T_0} X_i\Big|^{\omega}.$$

Then Lemmas H.5b and H.5c reduce to

(g)
$$\Big|\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},S_{T_0,\omega}\big)\Big| \le 2^k\Big|T_0\Big|^{\omega}\|X_{i_1}\|_r^{k+\omega}. \tag{H.42}$$

And Lemmas H.5e and H.5f reduce to

(h)
$$\Big|\mathcal{E}_G\big(X_{i_1},\cdots,X_{i_k},S_{T_0,\omega}\big)\Big| \le 2^{k+3}\Big|T_0\Big|^{\omega}\alpha_{\ell_0}^{(r-k-\omega)/r}\|X_{i_1}\|_r^{k+\omega}, \tag{H.43}$$

where ℓ_0 is the smallest integer ℓ such that

$$k(2\ell+1)^d \ge c_1 \vee \max_{1\le j\le k} s_j + k(2m+1)^d.$$

Secondly, if $s_j \ge 1$ for some $1 < j \le k+1$, we denote the σ-algebras $\mathcal{F}_{j-} := \sigma(X_{i_t} : t \in A(j))$ and

$$\mathcal{F}_{j+} := \begin{cases} \sigma(X_i : i \in T\backslash D_j) & \text{if } 2 \le j \le k \\ \sigma(X_i : i \in T\backslash \widetilde{D}_{k+1}) & \text{if } j = k+1 \end{cases}.$$

We will establish that $\alpha(\mathcal{F}_{j-},\mathcal{F}_{j+}) \le \alpha_{\ell_0}$ where $\alpha(\cdot,\cdot)$ is defined in Definition 5.1.

In this goal, we will write $\mathbb{B}_{\|\cdot\|}(i,b) := \{z \in \mathbb{Z}^d : \|i-z\| \le b\}$ for any $i \in \mathbb{Z}^d$, where $\|\cdot\|$ is the maximum norm on $\mathbb{Z}^d$. This is the set of elements at a distance at most b from i. Similarly if $I \subset \mathbb{Z}^d$ we write $\mathbb{B}_{\|\cdot\|}(I,b) := \{z \in \mathbb{Z}^d : \min_{i\in I}\|i-z\| \le b\}$. We denote by ℓ ($\ell \ge m+1$) the distance between i_j and $\{i_t : t \in A(j)\}$ in $\mathbb{Z}^d$, and by q the number of indices whose distance from $\{i_1,\cdots,i_{j-1}\}$ is at least $m+1$ and at most ℓ meaning that we set

$$q := \Big|\{s \in T : d(\{i_1,\ldots,i_{j-1}\},s) \in [m+1,\ell]\}\Big| = \Big|T \cap \mathbb{B}_{\|\cdot\|}\big(\{i_1,\ldots,i_{j-1}\},\ell\big)\backslash\mathbb{B}_{\|\cdot\|}\big(\{i_1,\ldots,i_{j-1}\},m\big)\Big|.$$

To bound q, we note that for any $i \in \mathbb{Z}^d$ and $b \in \mathbb{N}$ we have exactly $(2b+1)^d$ elements in $\mathbb{B}_{\|\cdot\|}(i,b)$. Thus, we have

$$q \le (j-1)\big((2\ell+1)^d - (2m+1)^d\big) \le k\big((2\ell+1)^d - (2m+1)^d\big).$$

Moreover, by definition, $N^{(s_j)}(i_t : t \in A(j)) \setminus N(i_t : t \in A(j))$ contains the smallest s_j indexes (with respect to the strict order on $\mathbb{Z}^d$) in $T \setminus N(i_t : t \in A(j))$. We remark that all the elements in $N^{(s_j)}(i_t : t \in A(j)) \setminus N(i_t : t \in A(j))$ have distance at least $m+1$ and at most ℓ from $\{i_t : t \in A(j)\}$ meaning that

$$N^{(s_j)}(i_t : t \in A(j)) \backslash N(i_t : t \in A(j)) \subseteq \mathbb{B}_{\|\cdot\|}\big(\{i_1, \cdots, i_{j-1}\}, \ell\big) \backslash \mathbb{B}_{\|\cdot\|}\big(\{i_1, \ldots, i_{j-1}\}, m\big).$$

Thus, we have $q \geq s_j$. As a result,

$$k(2\ell+1)^d \geq s_j + k(2m+1)^d.$$

As $\ell_0 := \min_\ell\{\ell : k(2\ell+1)^d \geq s_j + k(2m+1)^d\}$, we have $\ell \geq \ell_0$. Thus, we obtain that $\alpha(\mathcal{F}_{j-}, \mathcal{F}_{j+}) \leq \alpha_\ell \leq \alpha_{\ell_0}$.

Now we finish the proof of Lemmas H.5a, H.5d, H.5g and H.5h by performing induction on k.

Let $M := \|X_i\|_r$ for any $i \in T$.

If $k = 1$, by definition we have

$$\mathcal{E}_G(X_{i_1}, X_{i_2}) = \mathcal{D}^*(X_{i_1}, X_{i_2}), \quad \mathcal{E}_G(X_{i_1}, S_{T_0,\omega}) = \mathcal{D}^*(X_{i_1}, S_{T_0,\omega}).$$

By (H.13) and (H.14), we have

$$\left|\mathcal{D}^*(X_{i_1}, X_{i_2})\right| \leq 2M^2, \quad \left|\mathcal{D}^*(X_{i_1}, S_{T_0,\omega})\right| \leq 2|T_0|^\omega M^{1+\omega}.$$

Thus, Lemmas H.5a and H.5g hold for $k = 1$. Now supposing $s_2 \geq 1$, by (H.15) and (H.16), we get

$$\begin{aligned} \left|\mathcal{D}^*(X_{i_1}, X_{i_2})\right| &\leq 2^4\big(\alpha(\mathcal{F}_1, \mathcal{F}_2)\big)^{(r-2)/r} M^2, \\ \left|\mathcal{D}^*(X_{i_1}, S_{T_0,\omega})\right| &\leq 2^4 |T_0|^\omega \big(\alpha(\mathcal{F}_1, \mathcal{F}_3)\big)^{(r-1-\omega)/r} M^{1+\omega}, \end{aligned}$$

where $\mathcal{F}_1 := \sigma(X_{i_1}) = \mathcal{F}_{2-}$, $\mathcal{F}_2 := \sigma(X_{i_2}) \subseteq \mathcal{F}_{2+}$, and

$$\mathcal{F}_3 := \sigma(X_i : i \in T_0) = \sigma(X_i : i \in \widetilde{B}_2 \backslash \widetilde{D}_2) \subseteq \sigma(X_i : i \in T \backslash \widetilde{D}_2) = \mathcal{F}_{2+}.$$

As we have shown $\alpha(\mathcal{F}_{2-}, \mathcal{F}_{2+}) \leq \alpha_{\ell_0}$, we obtain

$$\alpha(\mathcal{F}_1, \mathcal{F}_2) \leq \alpha_{\ell_0}, \quad \alpha(\mathcal{F}_1, \mathcal{F}_3) \leq \alpha_{\ell_0}.$$

Thus, Lemmas H.5d and H.5h also hold for $k = 1$.

Suppose Lemmas H.5a, H.5d, H.5g and H.5h are true for $|G| \leq k$. Consider the case where $|G| = k+1$. Let

$$q_0 := \sup\{j : j = 1 \text{ or } p(j) \neq j-1 \text{ for } 2 \leq j \leq k+1\}, \tag{H.44}$$

We remark that q_0 is the first vertex in the branch of G with the highest indexes. We set $w := \big|\{t : q_0 + 1 \leq t \leq k+1 \;\&\; s_t \geq 0\}\big|$ to be the number of all indices $q_0 + 1 \leq t \leq k+1$ such that the identifier $s_t \geq 0$. If $\max_{1 \leq j \leq k+1} s_j \geq 1$, we let j_0 be an integer that satisfies $s_{j_0} = \max_{1 \leq j \leq k+1} s_j \geq 1$. We remark that such an index always exists.

We will first propose a simplified formulation for $\mathcal{E}_G$ that will hold irrespective of the value of w. Then we will distinguish two main cases in our analysis namely (i) when $q_0 = 1$ and (ii) when $q_0 \geq 2$.

In this goal, we first remark that if $w = 0$, by definition we know that for any random variables $Y_1, \cdots, Y_{k+1}$ the following holds

$$\mathcal{E}_G(Y_1, \cdots, Y_{k+1}) = \begin{cases} \mathcal{D}^*\big(Y_1 Y_2 \cdots Y_{k+1}\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(Y_1, \cdots, Y_{q_0-1}\big) \cdot \mathcal{D}^*\big(Y_{q_0} Y_{q_0+1} \cdots Y_{k+1}\big) & \text{if } q_0 \geq 2 \end{cases},$$

where $G[q_0 - 1] \subseteq G$ is the unique order-($q_0 - 1$) sub-genogram of G as defined in Appendix F.2.

For $w \geq 1$, we write $\{t : q_0 + 1 \leq t \leq k+1 \ \&\ s_t \geq 0\} = \{q_1, \cdots, q_w\}$. Without loss of generality, we suppose that the sequence $q_0 + 1 \leq q_1 < \cdots < q_w \leq k+1$ is increasing. By definition

$$\mathcal{E}_G(Y_1, \cdots, Y_{k+1}) = \begin{cases} \mathcal{D}^*\big(Y_1 \cdots Y_{q_1-1}\ ,\ Y_{q_1} \cdots Y_{q_2-1}\ ,\ \cdots\ ,\ Y_{q_w} \cdots Y_{k+1}\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(Y_1, \cdots, Y_{q_0-1}\big) \cdot \\ \qquad \mathcal{D}^*\big(Y_{q_0} \cdots Y_{q_1-1}\ ,\ Y_{q_1} \cdots Y_{q_2-1}\ ,\ \cdots\ ,\ Y_{q_w} \cdots Y_{k+1}\big) & \text{if } q_0 \geq 2 \end{cases}.$$

Set $q_{w+1} := k+2$, then by exploiting the definition of compositional $\mathcal{D}^*$ operators, we remark that $\mathcal{E}_G$ will take the following form irrespectively of the fact that $w \geq 1$ or not:

$$\mathcal{E}_G(Y_1, \cdots, Y_{k+1}) := \begin{cases} [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(Y_1, \cdots, Y_{k+1}\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(Y_1, \cdots, Y_{q_0-1}\big) \cdot \\ \qquad [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(Y_{q_0}, \cdots, Y_{k+1}\big) & \text{if } q_0 \geq 2 \end{cases}. \tag{H.45}$$

In particular, we know that

$$\begin{aligned} &\mathcal{E}_G(X_{i_1}, \cdots, X_{i_{k+1}}) = \\ &\quad \begin{cases} [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_1}, \cdots, X_{i_{k+1}}\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big) \cdot [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, \cdots, X_{i_{k+1}}\big) & \text{if } q_0 \geq 2 \end{cases}, \end{aligned} \tag{H.46}$$

$$\begin{aligned} &\mathcal{E}_G(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega}) = \\ &\quad \begin{cases} [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega}\big) & \text{if } q_0 = 1 \\ \mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big) \cdot [q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, X_{i_{q_0+1}}, \cdots, X_{i_k}, S_{T_0,\omega}\big) & \text{if } q_0 \geq 2 \end{cases}. \end{aligned} \tag{H.47}$$

We will use this simplified representation to prove the desired result. If $q_0 = 1$, by (H.13) and (H.14) we remark that

$$\begin{aligned} &\Big|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_{k+1}})\Big| \leq 2^{k+3} M^{k+1}, \\ &\Big|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega})\Big| \leq 2^{k+3} |T_0|^{\omega} M^{k+\omega}. \end{aligned}$$

Therefore, Lemmas H.5a and H.5g are true when $q_0 = 1$.

Supposing $s_{j_0} = \max_{1\le j\le k+1} s_j \ge 1$ ($j_0 \ge 2$ since $s_1 = 0$), by definition of $q_1, \cdots, q_w$ we know there is some $1 \le w' \le w$ such that $q_{w'} = j_0$. Hence

$$(q_1 - q_0) + \cdots + (q_{w'} - q_{w'-1}) = j_0 - 1.$$

By (H.15) and (H.16) we have

$$\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_{k+1}})\right| \le 2^{k+3}\big(\alpha(\mathcal{F}_1, \mathcal{F}_2)\big)^{(r-k-1)/r} M^{k+1},$$
$$\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega})\right| \le 2^{k+3}|T_0|^{\omega}\big(\alpha(\mathcal{F}_1, \mathcal{F}_3)\big)^{(r-k-\omega)/r} M^{k+\omega},$$

where

$$\mathcal{F}_1 := \sigma(X_{i_1}, \cdots, X_{i_{j_0-1}}) = \mathcal{F}_{j_0-},$$
$$\mathcal{F}_2 := \sigma(X_{i_{j_0}}, \cdots, X_{i_{k+1}}) \overset{(*)}{\subseteq} \begin{cases} \sigma(X_i : i \in T\backslash D_{j_0}) = \mathcal{F}_{j_0+} & \text{if } j_0 \le k \\ \sigma(X_i : i \in T\backslash \widetilde{D}_{k+1}) = \mathcal{F}_{j_0+} & \text{if } j_0 = k+1 \end{cases},$$
$$\mathcal{F}_3 := \begin{cases} \sigma(X_{i_{j_0}}, \cdots, X_{i_k}, S_{T_0,\omega}) \subseteq \sigma(X_i : i = j_0, \cdots k, \text{ or } i \in T\backslash \widetilde{D}_{k+1}) & \\ \qquad \overset{(**)}{\subseteq} \sigma(X_i : i \in T\backslash D_{j_0}) = \mathcal{F}_{j_0+} & \text{if } j_0 \le k \\ \sigma(S_{T_0,\omega}) \subseteq \sigma(X_i : i \in T\backslash \widetilde{D}_{k+1}) = \mathcal{F}_{j_0+} & \text{if } j_0 = k+1 \end{cases}.$$

Here $(*)$ and $(**)$ are due to the fact that $p(j) = j-1$ for any $2 \le j \le k+1$ since it implies that $1, \cdots, j_0 - 1 \in A(j)$ for any $j_0 \le j \le k+1$. Thus, we have

$$\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_{k+1}})\right| \le 2^{k+3}\alpha_{\ell_0}^{(r-k-1)/r} M^{k+1},$$
$$\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega})\right| \le 2^{k+3}|T_0|^{\omega}\alpha_{\ell_0}^{(r-k-\omega)/r} M^{k+\omega}.$$

Therefore, Lemmas H.5d and H.5h are true when $q_0 = 1$.

If $q_0 \ge 2$, note that

$$\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big)\right| \le 2^{q_0-2} M^{q_0-1}, \tag{H.48}$$

which is true for $q_0 = 2$ since $\mathcal{E}_{G[1]}(X_{i_1}) = \mathbb{E}[X_{i_1}]$ and is precisely the inductive hypothesis for $q_0 \ge 3$.

Thus, we have

$$\begin{aligned}&\left|\mathcal{E}_G(X_{i_1}, \cdots, X_{i_{k+1}})\right| \\ \le&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big)\right| \cdot \left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, X_{i_{q_0+1}}, \cdots, X_{i_{k+1}}\big)\right| \\ \overset{(*)}{\le}& 2^{q_0-2} M^{q_0-1} \cdot 2^{k+1-q_0} M^{k+2-q_0} \le 2^k M^{k+1},\end{aligned}$$

where $(*)$ is due to (H.48) and (H.13). Similarly we have

$$\begin{aligned}&\left|\mathcal{E}_G(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega})\right| \\ \le&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big)\right| \cdot \left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, X_{i_{q_0+1}}, \cdots, X_{i_k}, S_{T_0,\omega}\big)\right| \\ \le& 2^{q_0-2} M^{q_0-1} \cdot 2^{k+1-q_0}|T_0|^{\omega} M^{k+1+\omega-q_0} \le 2^k |T_0|^{\omega} M^{k+\omega}.\end{aligned}$$

Therefore, Lemmas H.5a and H.5g are true when $q_0 \geq 2$.

Supposing $s_{j_0} = \max_{1\leq j\leq k+1} s_j \geq 1$, we claim that $p(j_0) = j_0 - 1$. In fact, if $p(j_0) < j_0 - 1$, set $j_0' = p(j_0)+1$. Since $j_0' < j_0$ and $v[j_0']$ and $v[j_0]$ are siblings, by Proposition F.2d we have $s_{j_0'} > s_{j_0}$, which contradicts the definition of j_0. Therefore, we have shown $p(j_0) = j_0 - 1$, and thus, $j_0 \neq q_0$ by definition of q_0.

If $j_0 \geq q_0 + 1$, by definition of $q_1, \cdots, q_w$ we know there is some $1 \leq w' \leq w$ such that $q_{w'} = j_0$. Hence

$$(q_1 - q_0) + \cdots + (q_{w'} - q_{w'-1}) = j_0 - q_0.$$

Thus, by (H.15) and (H.16) we have

$$\begin{aligned}
&\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_{q_0}}, \cdots, X_{i_{k+1}})\right| \\
&\qquad \leq 2^{k-q_0+4}\big(\alpha(\mathcal{F}_1, \mathcal{F}_2)\big)^{(r-k+q_0-2)/r} M^{k-q_0+2}, \\
&\left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*(X_{i_{q_0}}, \cdots, X_{i_k}, S_{T_0,\omega})\right| \\
&\qquad \leq 2^{k-q_0+4}|T_0|^{\omega}\big(\alpha(\mathcal{F}_1, \mathcal{F}_3)\big)^{(r-k+q_0-1-\omega)/r} M^{k-q_0+1+\omega},
\end{aligned}$$

where

$$\begin{aligned}
&\mathcal{F}_1 := \sigma(X_{i_{q_0}}, \cdots, X_{i_{j_0-1}}) = \mathcal{F}_{j_0-}, \\
&\mathcal{F}_2 := \sigma(X_{i_{j_0}}, \cdots, X_{i_{k+1}}) \overset{(*)}{\subseteq} \begin{cases} \sigma(X_i : i \in T\backslash D_{j_0}) = \mathcal{F}_{j_0+} & \text{if } j_0 \leq k \\ \sigma(X_i : i \in T\backslash \widetilde{D}_{k+1}) = \mathcal{F}_{j_0+} & \text{if } j_0 = k+1 \end{cases}, \\
&\mathcal{F}_3 := \begin{cases} \sigma(X_{i_{j_0}}, \cdots, X_{i_k}, S_{T_0,\omega}) \subseteq \sigma(X_i : i = j_0, \cdots k, \text{ or } i \in T\backslash \widetilde{D}_{k+1}) & \\ \qquad\qquad \overset{(**)}{\subseteq} \sigma(X_i : i \in T\backslash D_{j_0}) = \mathcal{F}_{j_0+} & \text{if } j_0 \leq k \\ \sigma(S_{T_0,\omega}) \subseteq \sigma(X_i : i \in T\backslash \widetilde{D}_{k+1}) = \mathcal{F}_{j_0+} & \text{if } j_0 = k+1 \end{cases}.
\end{aligned}$$

Here $(*)$ and $(**)$ are due to the fact that $p(j) = j - 1$ for any $q_0 + 1 \leq j \leq k + 1$ since it implies that $q_0, \cdots, j_0 - 1 \in A(j)$ for any $j_0 \leq j \leq k + 1$. Thus, we have

$$\begin{aligned}
&\left|\mathcal{E}_G(X_{i_1}, \cdots, X_{i_{k+1}})\right| \\
\leq&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big)\right| \cdot \left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, X_{i_{q_0+1}}, \cdots, X_{i_{k+1}}\big)\right| \\
\leq&2^{q_0-2}M^{q_0-1} \cdot 2^{k-q_0+4}\alpha_{\ell_0}^{(r-k+q_0-2)/r}M^{k-q_0+2} \leq 2^{k+3}\alpha_{\ell_0}^{(r-k-1)/r}M^{k+1},
\end{aligned}$$

and

$$\begin{aligned}
&\left|\mathcal{E}_G(X_{i_1}, \cdots, X_{i_k}, S_{T_0,\omega})\right| \\
\leq&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1}, \cdots, X_{i_{q_0-1}}\big)\right| \cdot \left|[q_1 - q_0, \cdots, q_{w+1} - q_w] \triangleright \mathcal{D}^*\big(X_{i_{q_0}}, X_{i_{q_0+1}}, \cdots, X_{i_k}, S_{T_0,\omega}\big)\right| \\
\leq&2^{q_0-2}M^{q_0-1} \cdot 2^{k-q_0+4}|T_0|^{\omega}\alpha_{\ell_0}^{(r-k+q_0-1-\omega)/r}M^{k-q_0+1+\omega} \leq 2^{k+3}|T_0|^{\omega}\alpha_{\ell_0}^{(r-k-\omega)/r}M^{k+\omega}.
\end{aligned}$$

If $2 \leq j_0 \leq q_0 - 1$, by inductive hypothesis we have

$$\left|\mathcal{E}_{G[q_0-1]}(X_{i_1}, \cdots, X_{i_{q_0-1}})\right| \leq 2^{q_0+1}\alpha_{\ell_0}^{r-q_0+1}M^{q_0-1}. \tag{H.49}$$

Thus, we have

$$\begin{aligned}
&\left|\mathcal{E}_G(X_{i_1},\cdots,X_{i_{k+1}})\right| \\
\leq&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1},\cdots,X_{i_{q_0-1}}\big)\right|\cdot\left|[q_1-q_0,\cdots,q_{w+1}-q_w]\triangleright\mathcal{D}^*\big(X_{i_{q_0}},X_{i_{q_0+1}},\cdots,X_{i_{k+1}}\big)\right| \\
\overset{(*)}{\leq}&2^{q_0+1}\alpha_{\ell_0}^{(r-q_0+1)/r}M^{q_0-1}\cdot 2^{k-q_0+1}M^{k-q_0+2}\leq 2^{k+3}\alpha_{\ell_0}^{(r-k-1)/r}M^{k+1},
\end{aligned}$$

where $(*)$ is implied by (H.49) and (H.34). Similarly

$$\begin{aligned}
&\left|\mathcal{E}_G(X_{i_1},\cdots,X_{i_k},S_{T_0,\omega})\right| \\
\leq&\left|\mathcal{E}_{G[q_0-1]}\big(X_{i_1},\cdots,X_{i_{q_0-1}}\big)\right|\cdot\left|[q_1-q_0,\cdots,q_{w+1}-q_w]\triangleright\mathcal{D}^*\big(X_{i_{q_0}},X_{i_{q_0+1}},\cdots,X_{i_k},S_{T_0,\omega}\big)\right| \\
\leq&2^{q_0+1}\alpha_{\ell_0}^{r-q_0+1}M^{q_0-1}\cdot 2^{k-q_0+1}|T_0|^{\omega}M^{k-q_0+1+\omega}\leq 2^{k+3}|T_0|^{\omega}\alpha_{\ell_0}^{(r-k-\omega)/r}M^{k+\omega}.
\end{aligned}$$

Therefore, Lemmas H.5d and H.5h are true when $q_0\geq 2$.

By induction the proof is complete. ■

Equipped with the tools in Lemma H.5, we are able to show the proof of Lemma F.9.

Proof of Lemma F.9. For ease of notation, let $M := \|X_i\|_r$. For each of (F.51)–(F.54), we conduct the sum in two steps:

1. Fixing an ordered tree $(V,E,\prec)$ with the compatible labeling. Here $V=\{v[1],\cdots,v[k+1]\}$ denotes the vertex set and E denotes the edge set. We take the sum of $\mathcal{S}(H)$ or $\mathcal{U}_f(H)$ over all possible values of $s_1,\cdots,s_{k+1}$ such that H is a genogram that induces $(V,E,\prec)$;
2. Sum over all possible ordered trees $(V,E,\prec)$ of order-$(k+1)$.

Note that an ordered tree corresponds to infinitely many genograms (as there is an infinite number of possible identifiers). However, when the index set T of the random field is finite, only finitely many genograms give non-zero values of $\mathcal{S}(H)$ and $\mathcal{U}_f(H)$.

For the second step, we observe that the total number of ordered trees of order-$(k+1)$ solely depends on k. (In fact, this is exactly the k-th Catalan number [Roman 2015].) Hence summing over all such trees only contributes to the constant in the bounds. As for the first step, the following statement will be crucial to our proof.

Claim. *Fix a positive integer $s\geq 1$. For any $2\leq t\leq k+1$, given a sequence $i_1,\cdots,i_{t-1}$, the sum of $\left|B_t\backslash D_t\right|$ over $-1\leq s_t\leq s$ is smaller or equal to $2(k(2m+1)^d+s)$.*

To see this we will consider the following three cases:

(i) When $s_t=-1$ and $s_{u(t)}=0$;

(ii) When $s_t=-1$ and $s_{u(t)}\geq 1$;

(iii) When $0\leq s_t\leq s$.

Firstly, if $s_t = -1$ and $s_{u(t)} = 0$, then we note that

$$B_t \backslash D_t = B_{u(t)} \backslash D_{u(t)} \subseteq N(i_h : h \in A(u(t))) \subseteq N(i_h : h \in A(t)) \subseteq N^{(s)}(i_h : h \in A(t)).$$

If $s_t = -1$ and $s_{u(t)} \geq 1$, then by definition, $B_t \backslash D_t = B_{u(t)} \backslash D_{u(t)}$ has at most one element namely $i_{u(t)}$. Thus, $B_t \backslash D_t \in N(i_h : h \in A(t)) \subseteq N^{(s)}(i_h : h \in A(t))$.

Finally if $0 \leq s_t \leq s$, by definition, $B_t \backslash D_t \subseteq N^{(s)}(i_h : h \in A(t))$.

To bound $\sum_{s_t \leq s} \left| B_t \backslash D_t \right|$ we remark that the sets $B_t \backslash D_t$ are disjoints for different values of $0 \leq s_t \leq s$. Thus, this implies that

$$\sum_{s_t \leq s} \left| B_t \backslash D_t \right| \leq 2 \left| N^{(s)}(i_h : h \in A(t)) \right|.$$

To further bound this, note that for any index i, the indices with distance from i at most m lie in the d-dimensional hypercube centered at i with the sides of length $2m+1$. Thus, $\left| N(i_h : h \in A(t)) \right| \leq k(2m+1)^d$. By noticing that for any subset J (by definition of $N^{(s)}(\cdot)$) there is at most s elements in $N^{(s)}(J) \setminus N(J)$ we obtain that

$$\left| N^{(s)}(i_h : h \in A(t)) \right| = \left| N(i_h : h \in A(t)) \right| + s \leq k(2m+1)^d + s.$$

Next we establish (F.51).

Suppose $v[j_0]$ is a vertex with the largest identifier among all vertices and $s_{j_0} = s$. By (H.37) of Lemma H.5, we obtain that

$$\left| \mathcal{E}_H\big(X_{i_1}, \cdots, X_{i_{k+1}}\big) \right| \lesssim \alpha_{\ell_s}^{(r-k-1)/r} M^{k+1},$$

where ℓ_s is the smallest integer ℓ that satisfies

$$k(2\ell+1)^d \geq s + k(2m+1)^d. \tag{H.50}$$

Thus, we obtain that

$$\begin{aligned} \sum_{\substack{s_{1:(k+1)}: \\ s_{j_0}=s, \\ s_h \leq s, \ \forall h \neq j_0}} \left| \mathcal{S}(H) \right| \leq & \sigma^{-(k+1)} \sum_{\substack{s_{1:(k+1)}: \\ s_{j_0}=s, \\ s_h \leq s, \forall h \neq j_0}} \sum_{i_1 \in B_1 \backslash D_1} \sum_{i_2 \in B_2 \backslash D_2} \cdots \sum_{i_{k+1} \in B_{k+1} \backslash D_{k+1}} \left| \mathcal{E}_G\big(X_{i_1}, \cdots, X_{i_{k+1}}\big) \right| \\ \leq & 2^{2k+3} \sigma^{-(k+1)} |T| \big(k(2m+1)^d + s\big)^{k-2} \alpha_{\ell_s}^{(r-k-1)/r} M^{k+1}. \end{aligned}$$

Since the set $\{s_1, \cdots, s_{k+1} : \max_{1 \leq h \leq k+1} s_h = s\}$ is the union (not necessarily disjoint) of $\{s_1, \cdots, s_{k+1} : s_j = s, s_h \leq s \ \forall 1 \leq h \leq k+1\}$ over $2 \leq j \leq k+1$, we have

$$\begin{aligned} \sum_{\substack{s_{1:(k+1)}: \\ \max_{1 \leq h \leq k+1} s_h = s}} \left| \mathcal{S}(H) \right| \leq & \sum_{j_0=2}^{k+1} \sum_{\substack{s_{1:(k+1)}: \\ s_{j_0}=s, \\ s_h \leq s, \forall h \neq j_0}} \left| \mathcal{S}(H) \right| \\ \leq & 2^{2k+3} k \sigma^{-(k+1)} |T| \big(k(2m+1)^d + s\big)^{k-1} \alpha_{\ell_s}^{(r-k-1)/r} M^{k+1}. \end{aligned}$$

Next we take the sum over all possible $|T| \geq s \geq 1$ and obtain that

$$\sum_{\substack{s_{1:(k+1)}:\\ \max_{1\leq h\leq k+1}\geq 1}} \left|\mathcal{S}(H)\right| \leq \sum_{s=1}^{|T|} \sum_{\substack{s_{1:(k+1)}:\\ \max_{1\leq h\leq k+1} s_h = s}} \left|\mathcal{S}(H)\right| \tag{H.51}$$
$$\leq 2^{2k+3} k\sigma^{-(k+1)}|T| \sum_{s=1}^{|T|} \left(k(2m+1)^d + s\right)^{k-1} \alpha_{\ell_s}^{(r-k-1)/r} M^{k+1}.$$

To further bound this it will be important to know what is the number of different possible values of s will have the same $\ell_s = \ell$. To do so, we note that by definition (H.50) of ℓ_s any such s much satisfy

$$k(2\ell_s+1)^d \geq s + k(2m+1)^d \geq k(2\ell_s-1)^d + 1,$$

which implies that for any $\ell \geq m+1$

$$\left|\{s : \ell_s = \ell\}\right| \leq k(2\ell+1)^d - k(2\ell-1)^d \leq 2kd(2\ell+1)^{d-1}.$$

On the other hand, $s \leq |T|-1$ implies that

$$k(2\ell_s-1)^d + 1 \leq |T| + k(2m+1)^d + 1 \leq \left(|T|^{1/d} + k(2m+1)\right)^d + 1.$$

Thus, we have $\ell_s \leq m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor$ for any s. And we conclude that

$$\left|\{s : \ell_s = \ell\}\right| \leq \begin{cases} 2kd(2\ell+1)^{d-1} & \text{if } m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor \geq \ell \geq m+1 \\ 0 & \text{otherwise} \end{cases}.$$

Therefore, by combining this with (H.51) we obtain that

$$\sum_{\substack{s_{1:(k+1)}:\\ \max_{1\leq h\leq k+1}\geq 1}} \left|\mathcal{S}(H)\right| \leq 2^{2k+3} k\sigma^{-(k+1)}|T| \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} 2kd(2\ell+1)^{d-1}\left(k(2\ell+1)^d\right)^{k-1} \alpha_\ell^{(r-k-1)/r} M^{k+1}$$
$$\overset{(a)}{\lesssim} |T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{dk-1} \alpha_\ell^{(r-k-1)/r}.$$

where to obtain (a) we used the assumption that the asymptotic variance does not degenerate:

$$\liminf_{|T|\to\infty} \sigma^2/|T| > 0.$$

For any $G \in \mathcal{G}_0(k+1)$, there exists at least one positive vertex. Thus, $\max_{1\leq h\leq k+1} s_h \geq 1$. Now that the number of labeled rooted trees on $k+1$ vertices only depends on k, we conclude

$$\begin{aligned}
\sum_{H\in\mathcal{G}_0(k+1)}\left|\mathcal{S}(H)\right| &= \sum_{\substack{(V,E,\prec):\\|V|=k+1}} \sum_{\substack{s_{1:(k+1)}:\\H=(V,E,s_{1:(k+1)})\\\in\mathcal{G}_0(k+1)}} \left|\mathcal{S}(H)\right| \\
&\le \sum_{\substack{(V,E,\prec):\\|V|=k+1}} \sum_{\substack{s_{1:(k+1)}:\\\max_{1\le h\le k+1}s_h\ge 1}} \left|\mathcal{S}(H)\right| \\
&\lesssim \sum_{\substack{(V,E,\prec):\\|V|=k+1}} |T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r} \\
&\lesssim |T|^{-(k-1)/2} \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor} \ell^{dk-1}\alpha_\ell^{(r-k-1)/r}.
\end{aligned}$$

Next we prove (F.52).

Again suppose $v[j_0]$ is a vertex with the largest identifier among all vertices and $s_{j_0}=s$. In other words, $\max\{s_t : 1\le t\le k+1\}=s=s_{j_0}$. We discuss the following two cases (i) when $j_0\le k$ (whose analysis will be further split depending on the fact that $s_{k+1}\ge 1$ or not), and (ii) when $j_0=k+1$.

First consider the case where $j_0\le k$. From the claim above, we know that for any $2\le t\le k$, $t\ne j_0$, given a sequence $i_1,\cdots,i_{t-1}$, we have that $\sum_{s_t\le s}\left|B_t\setminus D_t\right|\le 2(k(2m+1)^d+s)$.

Suppose that $s_{k+1}\le 0$, then by (H.38) of Lemma H.5, we know that

$$\left|\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right| \lesssim |f|_{k-1,\omega}\sigma^{-\omega}|B_{k+1}\backslash D_{k+1}|^{\omega}\alpha_{\ell_s}^{(r-k-\omega)/r},$$

where ℓ_s is the smallest integer ℓ that satisfies

$$k(2\ell+1)^d\ge s+k(2m+1)^d.$$

If $s_{k+1}\ge 1$ then by (H.40) of Lemma H.5 we have

$$\left|\sum_{1\le s_{k+1}\le s}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right| \lesssim |f|_{k-1,\omega}\sigma^{-\omega}s^{\omega}\alpha_{\ell_s}^{(r-k-\omega)/r}.$$

Thus, as we have shown that $|B_{k+1}\backslash D_{k+1}|\le k(2m+1)^d$ since $B_{k+1}\backslash D_{k+1}\subseteq N(i_h : h\in A(k+1))$ we obtain that

$$\begin{aligned}
&\left|\sum_{s_{k+1}\le s}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right| \\
\le&\left|\sum_{s_{k+1}=0,-1}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right| + \left|\sum_{1\le s_{k+1}\le s}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right|
\end{aligned}$$

$$\lesssim |f|_{k-1,\omega}\sigma^{-\omega}(k(2m+1)^d+s)^{\omega}\alpha_{\ell_s}^{(r-k-\omega)/r}.$$

Noting that b_H is the same for genograms with the same $(V, E, \prec)$ and negative vertices, and that $|b_H| \le 1$ (see the remark following Corollary F.8), we have

$$\begin{aligned}
&\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ \exists 1\le j_0\le k \text{ s.t.}\\ s_{j_0}=\max_{1\le h\le k+1} s_h=s}} b_H \mathcal{U}_f(H)\Bigg| \\
&\le \sum_{j_0=1}^{k}\sum_{\substack{s_{1:k}:\\ s_{j_0}=s,\\ s_j\le s, 1\le j\le k}}\Bigg|\sum_{s_{k+1}\le s} b_H\mathcal{U}_f(H)\Bigg| = \sum_{j_0=1}^{k}\sum_{\substack{s_{1:k}:\\ s_{j_0}=s,\\ s_j\le s, 1\le j\le k}} |b_H|\Bigg|\sum_{s_{k+1}\le s}\mathcal{U}_f(H)\Bigg| \\
&\le \sigma^{-k}\sum_{j_0=1}^{k}\sum_{\substack{s_{1:k}:\\ s_{j_0}=s,\\ s_j\le s, \forall 1\le j\le k}}\sum_{i_1\in B_1\setminus D_1}\sum_{i_2\in B_2\setminus D_2}\cdots\sum_{i_k\in B_k\setminus D_k}\Bigg|\sum_{s_{k+1}\le s}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\Bigg| \\
&\lesssim k|f|_{k-1,\omega}\sigma^{-(k+\omega)}|T|\big(k(2m+1)^d+s\big)^{k-2+\omega}\alpha_{\ell_s}^{(r-k-\omega)/r}M^{k+\omega}.
\end{aligned}$$

Thus, we get that

$$\begin{aligned}
&\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ \exists 1\le j_0\le k \text{ s.t.}\\ s_{j_0}=\max_{1\le h\le k+1} s_h\ge 1}} b_H\mathcal{U}_f(H)\Bigg| \le \sum_{s=1}^{|T|}\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ \exists 1\le j_0\le k \text{ s.t.}\\ s_{j_0}=\max_{1\le h\le k+1} s_h=s}} b_H\mathcal{U}_f(H)\Bigg| \\
&\lesssim k|f|_{k-1,\omega}\sigma^{-(k+\omega)}|T|\sum_{s=1}^{|T|}\big(k(2m+1)^d+s\big)^{k-2+\omega}\alpha_{\ell_s}^{(r-k-\omega)/r}M^{k+\omega} \\
&\overset{(a)}{\lesssim} |f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{d(k+\omega-1)-1}\alpha_{\ell}^{(r-k-\omega)/r},
\end{aligned}$$

where once again to obtain (a) we used the fact that by assumption $\limsup |T|/\sigma^2 < \infty$.

We now consider the case where $j_0 = k+1$. To do so we first note that by (H.40) of Lemma H.5 for any $\ell \ge 1$ we have

$$\begin{aligned}
&\Bigg|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\ -k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\ -k(2m+1)^d}}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\Bigg| \\
&\lesssim |f|_{k-1,\omega}\sigma^{-\omega}\big(k(2\ell+1)^d-k(2\ell-1)^d\big)^{\omega}\alpha_{\ell}^{(r-k-\omega)/r} \\
&\lesssim |f|_{k-1,\omega}\sigma^{-\omega}\ell^{d\omega-\omega}\alpha_{\ell}^{(r-k-\omega)/r}.
\end{aligned} \tag{H.52}$$

Taking the sum over ℓ and $s_{1:k}$, we get

$$
\begin{aligned}
&\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ s_j\le s_{k+1},\forall 1\le j\le k,\\ s_{k+1}\ge 1}} b_H\mathcal{U}_f(H)\Bigg| \\
\le&\sum_{\substack{s_{1:k}:\\ s_j\le s_{k+1},\forall 1\le j\le k,\\ s_{k+1}\ge 1}}\ \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\Bigg|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\ -k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\ -k(2m+1)^d}} b_H\mathcal{U}_f(H)\Bigg| \qquad (H.53)\\
=&\sum_{\substack{s_{1:k}:\\ s_j\le s_{k+1},\forall 1\le j\le k,\\ s_{k+1}\ge 1}}\ \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}|b_H|\Bigg|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\ -k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\ -k(2m+1)^d}} \mathcal{U}_f(H)\Bigg|\\
\le&\sum_{\substack{s_{1:k}:\\ s_j\le s_{k+1},\forall 1\le j\le k,\\ s_{k+1}\ge 1}}\ \sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ \sum_{i_1\in B_1\setminus D_1}\cdots\sum_{i_k\in B_k\setminus D_k}\Bigg|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\ -k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\ -k(2m+1)^d}} \mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\Bigg|\\
\lesssim&|f|_{k-1,\omega}\sigma^{-(k+\omega)}|T|\big(k(2m+1)^d+s_{k+1}\big)^{k-1}\ell^{d\omega-\omega}\alpha_\ell^{(r-k-\omega)/r}\\
\lesssim&|f|_{k-1,\omega}\sigma^{-(k+\omega)}|T|(k(2\ell+1)^d)^{k-1}\ell^{d\omega-\omega}\alpha_\ell^{(r-k-\omega)/r}\\
\lesssim&|f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{d(k+\omega-1)-\omega}\alpha_\ell^{(r-k-\omega)/r}.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\Bigg|\sum_{H\in\mathcal{G}_0(k+1)} b_H\mathcal{U}_f(H)\Bigg| &\le \sum_{\substack{(V,E,\prec):\\ |V|=k+1}}\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ H=(V,E,s_{1:(k+1)})\\ \in\mathcal{G}_0(k+1)}} b_H\mathcal{U}_f(H)\Bigg| \le \sum_{\substack{(V,E,\prec):\\ |V|=k+1}}\Bigg|\sum_{\substack{s_{1:k+1}:\\ \max_{1\le h\le k+1}s_h\ge 1}} b_H\mathcal{U}_f(H)\Bigg|\\
&\le \sum_{\substack{(V,E,\prec):\\ |V|=k+1}}\Bigg|\sum_{\substack{s_{1:k}:\\ s_j\le s_{k+1},\forall 1\le j\le k\\ s_{k+1}\ge 1}} b_H\mathcal{U}_f(H)\Bigg| + \sum_{\substack{(V,E,\prec):\\ |V|=k+1}}\Bigg|\sum_{\substack{s_{1:(k+1)}:\\ \exists j_0\le k \text{ s.t.}\\ s_{j_0}=\max_{1\le h\le k+1}s_h\ge 1}} b_H\mathcal{U}_f(H)\Bigg|\\
&\lesssim \sum_{\substack{(V,E,\prec):\\ |V|=k+1}}|f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{d(k+\omega-1)-\omega}\alpha_\ell^{(r-k-\omega)/r}\\
&\lesssim |f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\ell^{d(k+\omega-1)-\omega}\alpha_\ell^{(r-k-\omega)/r}.
\end{aligned}
$$

Next we prove (F.53). If $H\in\mathcal{P}_0(k+1)$, then for any $t\le k$ we know that $i_{t+1}\in N(i_{1:t})$. In other words, new indexes lie in the m-neighborhood of previous ones. By (H.35), we have

$$\left|\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right| \le 2^k\sigma^{-\omega}\left|B_{k+1}\backslash D_{k+1}\right|^{\omega}\cdot|f|_{k-1,\omega}M^{k+\omega},$$

Taking the sums over $i_j \in B_j\backslash D_j$ for all $1\le j\le k$, we get

$$\begin{aligned}\left|\mathcal{U}_f(H)\right| \le& 2^k\sigma^{-\omega}\left|B_{k+1}\backslash D_{k+1}\right|^{\omega}\cdot|f|_{k-1,\omega}M^{k+\omega}\prod_{j=1}^{k}\left|B_j\backslash D_j\right|\\ \lesssim&|f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}m^{d(k+\omega-1)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left|\sum_{H\in\mathcal{P}_0(k+1)}b_H\mathcal{U}_f(H)\right| \le& \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\left|\sum_{\substack{s_{1:(k+1)}:\\H=(V,E,s_{1:(k+1)})\\\in\mathcal{P}_0(k+1)}}b_H\mathcal{U}_f(H)\right| \le \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\left|\sum_{\substack{s_{1:k+1}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k+1}}b_H\mathcal{U}_f(H)\right|\\ \le& \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\sum_{\substack{s_{1:k+1}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k+1}}|b_H|\left|\mathcal{U}_f(H)\right| \le \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\sum_{\substack{s_{1:k+1}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k+1}}\left|\mathcal{U}_f(H)\right|\\ \lesssim& \sum_{\substack{(V,E,\prec):\\|V|=k+1}}|f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}m^{d(k+\omega-1)}\\ \lesssim&|f|_{k-1,\omega}|T|^{-(k+\omega-2)/2}m^{d(k+\omega-1)}.\end{aligned}$$

Finally, to prove (F.54), we follow the derivation similar to (H.53) to obtain that

$$\begin{aligned}\left|\sum_{H\in\mathcal{P}_1(k+1)}b_H\mathcal{U}_f(H)\right| \le& \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\left|\sum_{\substack{s_{1:(k+1)}:\\H=(V,E,s_{1:(k+1)})\\\in\mathcal{P}_1(k+1)}}b_H\mathcal{U}_f(H)\right| \le \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\left|\sum_{\substack{s_{1:(k+1)}:\\s_j\le 0,\forall 1\le j\le k,\\s_{k+1}\ge 1}}b_H\mathcal{U}_f(H)\right|\\ =& \sum_{\substack{(V,E,\prec):\\|V|=k+1}}|b_H|\left|\sum_{\substack{s_{1:(k+1)}:\\s_j\le 0,\forall 1\le j\le k,\\s_{k+1}\ge 1}}\mathcal{U}_f(H)\right| \le \sum_{\substack{(V,E,\prec):\\|V|=k+1}}\left|\sum_{\substack{s_{1:(k+1)}:\\s_j\le 0,\forall 1\le j\le k,\\s_{k+1}\ge 1}}\mathcal{U}_f(H)\right|\\ \le&\sigma^{-k}\sum_{\substack{(V,E,\prec):\\|V|=k+1}}\sum_{\substack{s_{1:k}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k}}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\left|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\-k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\-k(2m+1)^d}}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right|\\ \le&\sigma^{-k}\sum_{\substack{(V,E,\prec):\\|V|=k+1}}\sum_{\substack{s_{1:k}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k}}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\sum_{i_1\in B_1\backslash D_1}\cdots\sum_{i_k\in B_k\backslash D_k}\left|\sum_{\substack{s_{k+1}=k(2\ell-1)^d\\-k(2m+1)^d+1}}^{\substack{k(2\ell+1)^d\\-k(2m+1)^d}}\mathcal{E}_H\big(X_{i_1},\cdots,X_{i_k},\Delta_f(H)\big)\right|\\ \overset{(*)}{\lesssim}&|f|_{k-2,\delta}^{1-\delta}|f|_{k-1,\delta}^{\delta}\sigma^{-(k+\delta)}\sum_{\substack{(V,E,\prec):\\|V|=k+1}}\sum_{\substack{s_{1:k}:\\s_j=0\text{ or }-1,\\\forall 1\le j\le k}}\sum_{\ell=m+1}^{m+1+\lfloor\frac{|T|^{1/d}}{2}\rfloor}\sum_{i_1\in B_1\backslash D_1}\cdots\sum_{i_k\in B_k\backslash D_k}\ell^{d\delta-\delta}\alpha_{\ell}^{(r-k-\delta)/r}\end{aligned}$$

$$\overset{(**)}{\lesssim} |f|_{k-2,1}^{1-\delta} |f|_{k-1,1}^{\delta} \sigma^{-(k+\delta)} 2^{k-1} |T| \big(k(2m+1)^d\big)^{k-1} \ell^{d\delta-\delta} \alpha_\ell^{(r-k-\delta)/r}$$

$$\lesssim |f|_{k-2,1}^{1-\delta} |f|_{k-1,1}^{\delta} |T|^{-(k+\delta-2)/2} m^{d(k-1)} \sum_{\ell=m+1}^{m+1+\lfloor \frac{|T|^{1/d}}{2} \rfloor} \ell^{d\delta-\delta} \alpha_\ell^{(r-k-\delta)/r}.$$

Note that the inequality $(*)$ is due to (H.41). For $(**)$, we note that $s_j = 0$ or -1 for $2 \le j \le k$, and that the number of choices for i_j $(2 \le j \le k)$ is upper-bounded by $k(2m+1)^d$ since i_j lies in the m-neighborhood of $i_1, \cdots, i_{j-1}$. ∎