Pseudocycles for Borel-Moore Homology

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September 22, 2022

Abstract

Pseudocycles are geometric representatives for integral homology classes on smooth manifolds that have proved useful in particular for defining gauge-theoretic invariants. The Borel-Moore homology is often a more natural object to work with in the case of non-compact manifolds than the usual homology. We define weaker versions of the standard notions of pseudocycle and pseudocycle equivalence and then describe a natural isomorphism between the set of equivalence classes of these weaker pseudocycles and the Borel-Moore homology. We also include a direct proof of a Poincaré Duality between the singular cohomology of an oriented manifold and its Borel-Moore homology.

Contents

1 Introduction

1.1 Main theorem

Constructions of some important gauge-theoretic invariants involve representing cohomology classes on a smooth manifold X geometrically. As the submanifolds in X and embedded cobordisms between them do not generally suffice for representing the singular homology $H_*(X;\mathbb{Z})$ of X, pseudocycles have been used as a suitable replacement in the case of compact manifolds. For example, pseudocycles are central to the constructions of Gromov-Witten invariants for compact semi-positive symplectic manifolds in [7, Section 7.1] and [10, Section 1].

The Borel-Moore homology $H_*^{\mathrm{lf}}(X;\mathbb{Z})$ of a topological space X, also known as the homology with closed supports and the homology based on locally finite chains, is introduced from a sheaf-theoretic perspective in [1]. If X is compact, $H_*^{\mathrm{lf}}(X;\mathbb{Z})$ is just the usual singular homology $H_*(X;\mathbb{Z})$. On the other hand, a closed oriented k-submanifold M in a manifold X determines a class $[M]_X$ in $H_k^{\mathrm{lf}}(X;\mathbb{Z})$, even if M is not compact. If X is an oriented n-manifold, $H_k^{\mathrm{lf}}(X;\mathbb{Z})$ is Poincaré dual to the singular cohomology $H^{n-k}(X;\mathbb{Z})$ with respect to their pairing with the compactly supported cohomology $H_c^k(X;\mathbb{Z})$. The purpose of the present paper is to provide an analogue of pseudocycles for the Borel-Moore homology of a non-compact manifold X and a geometric way of representing all cohomology of X. As indicated in [2, Section 1] and [3, Section 1], classes on non-compact manifolds can be relevant even if one is interested only in compact manifolds.

A subset Z of a manifold X is of dimension at most k, which we write as

$$\dim Z \leq k\,,$$

if there exists a k-dimensional manifold Y and a smooth map $h\colon Y \longrightarrow X$ such that $Z \subset h(Y)$. If $f\colon M \longrightarrow X$ is a continuous map between topological spaces, the boundary of f is the subspace

$$\mathrm{Bd}\, f \equiv \bigcap_{K \subset M \text{ cmpt}} \overline{f(M-K)} \subset X,$$

where the overline $\overline{\cdot}$ denotes the closure in X. If $A \subset X$ is a closed compact subset disjoint from $\mathrm{Bd}\, f$, then $f^{-1}(A) \subset M$ is compact. A continuous map f as above is called proper if $f^{-1}(A) \subset M$ is compact for every compact subset $A \subset X$. If $\mathrm{Bd}\, f = \emptyset$, then f is proper. If f is proper and X is locally compact, i.e. every point of X has an arbitrarily small precompact open neighborhood, then $\mathrm{Bd}\, f = \emptyset$. If $f\colon M \longrightarrow X$ is a continuous map from a compact space to a Hausdorff one, then f is proper.

Definition 1.1. Let X be a smooth manifold.

(a) A smooth map $f\colon M \longrightarrow X$ is a Borel-Moore k-pseudocycle if M is an oriented k-manifold and $\dim \mathrm{Bd}\, f \leq k-2$.

(b) Two Borel-Moore k-pseudocycles $f_0\colon M_0 \longrightarrow X$ and $f_1\colon M_1 \longrightarrow X$ are equivalent if there exist a smooth oriented manifold $\widetilde{M}$, a smooth map $\widetilde{f}\colon \widetilde{M} \longrightarrow X$, and closed subsets $Y_0 \subset M_0$ and $Y_1 \subset M_1$ such that

$$\dim Y_0, \dim Y_1 \leq k-1, \quad \partial\widetilde{M} = (M_1-Y_1)\sqcup -(M_0-Y_0),$$
$$\dim \mathrm{Bd}\, \widetilde{f} \leq k-1, \quad \widetilde{f}|_{M_0-Y_0} = f_0|_{M_0-Y_0}, \quad \widetilde{f}|_{M_1-Y_1} = f_1|_{M_1-Y_1}.$$

(c) The k-th Borel-Moore pseudocycle group is the set $\mathcal{H}^{\mathrm{cl}}_*(X)$ of equivalence classes of Borel-Moore k-pseudocycles to X with the addition induced by the disjoint union.

Example 1.2. If $f\colon M \longrightarrow X$ is a Borel-Moore k-pseudocycle and $Z \subset X$ is a closed subset of dimension at most $k-2$, then $f|_{M-f^{-1}(Z)}$ is also a Borel-Moore k-pseudocycle (with $\mathrm{Bd}\, f|_{M-f^{-1}(Z)}$ contained in $(\mathrm{Bd}\, f)\cup Z$) and

$$\widetilde{f}\colon \widetilde{M} \equiv [0,1]\times M - \{1\}\times f^{-1}(Z) \longrightarrow X, \quad \widetilde{f}(t,p) = f(p),$$

is a Borel-Moore pseudocycle equivalence between f and $f|_{M-f^{-1}(Z)}$.

Theorem 1.3. *Let X be a smooth manifold. There exist homomorphisms of graded abelian groups*

$$\Psi_*\colon H^{\mathrm{lf}}_*(X;\mathbb{Z}) \longrightarrow \mathcal{H}^{\mathrm{cl}}_*(X) \quad and \quad \Phi_*\colon \mathcal{H}^{\mathrm{cl}}_*(X) \longrightarrow H^{\mathrm{lf}}_*(X;\mathbb{Z}) \tag{1.1}$$

that are natural with respect to proper maps such that $\Phi_*\circ\Psi_* = \mathrm{Id}$ and $\Psi_*\circ\Phi_* = \mathrm{Id}$.

A pseudocycle is a Borel-Moore pseudocycle f as in Definition 1.1 such that the closure $\overline{f(M)}$ of $f(M)$ in X is compact. Two pseudocycles f_0 and f_1 are equivalent if there exists a Borel-Moore pseudocycle equivalence $\widetilde{f}$ as in Definition 1.1 such that $\overline{\widetilde{f}(M)}$ is a compact subset of X. The set $\mathcal{H}_k(X)$ of equivalence classes of k-pseudocycles to X with the addition induced by the disjoint union is also an abelian group. The analogue of Theorem 1.3 for pseudocycles and the standard singular homology is [13, Theorem 1.1].

Remark 1.4. Let X, f_0, f_1, $\widetilde{f}$, $Y_0 \subset M_0$, and $Y_1 \subset M_1$ be as in Definition 1.1(b). Identify a neighborhood W of $\partial\widetilde{M}$ in $\widetilde{M}$ with $[0,1)\times\partial\widetilde{M}$. The space

$$\widehat{M} \equiv \big(\widetilde{M}\sqcup[0,1]\times(M_0\sqcup M_1)-\{1\}\times Y_0-\{0\}\times Y_1\big)\big/\!\sim, \qquad \text{where}$$
$$\widetilde{M}\ni p_0 \sim (1,p_0)\in[0,1]\times(M_0-Y_0), \quad \widetilde{M}\ni p_1 \sim (0,p_1)\in[0,1]\times(M_1-Y_1),$$

is then a smooth oriented manifold with boundary $M_1\sqcup(-M_0)$. We can deform $\widetilde{F}$, while keeping it fixed on $\partial\widetilde{M}$, so that it is constant on the fibers of the projection $W \longrightarrow \partial\widetilde{M}$. The map

$$\widehat{f}\colon \widehat{M} \longrightarrow X, \qquad \widehat{f}(p) = \widetilde{F}(p) \ \ \forall\, p\in\widetilde{M}, \quad \widehat{f}(t,p) = f_r(p) \ \ \forall\, p\in M_r,\, r=0,1,$$

is then well-defined and smooth. It satisfies the conditions in Definition 1.1(b) with $\widetilde{f}$ replaced by $\widehat{f}$ and $Y_0, Y_1 = \emptyset$. Thus, D. McDuff's idea of attaching two collars, which is used in the proof of [13, Theorem 1.1], leads to a more relaxed, but equivalent, formulation of pseudocycle equivalence than the traditional one, with $Y_0, Y_1 = \emptyset$.

Remark 1.5. As with [13, Theorem 1.1], it is sufficient for the purposes of Theorem 1.3 to require Borel-Moore pseudocycles and equivalences to be just continuous. All constructions in this paper would go through; Lemma 2.1 would no longer be needed. On the other hand, smooth pseudocycles are more advantageous for transversality considerations.

The constructions in this paper and in [13] are direct and geometric; both are motivated by the outline proposed in [6, Section 7.1]. The proof of Theorem 1.3 is conceptually similar to the proof of [13, Theorem 1.1], but the specifics are different because the Borel-Moore homology does not behave like the standard singular homology. Inspired by [11], we use the chain complex $S^{\mathrm{lf}}_{\{U\};*}(X;\mathbb{Z})$ of singular chains that are locally finite in X and lie in a subspace $U\subset X$ to adjust the construction

in [13] to the setting of Theorem 1.3. The homologies $H^{\mathrm{lf}}_{\{U\};*}(X;\mathbb{Z})$ of this complex, $H^{\mathrm{lf}}_*(X;\mathbb{Z})$ of $S^{\mathrm{lf}}_*(X;\mathbb{Z})$, and $H^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$ of the quotient complex

$$S^{\mathrm{lf}}_*\big(X,\{U\};\mathbb{Z}\big) \equiv S^{\mathrm{lf}}_*(X;\mathbb{Z})/S^{\mathrm{lf}}_{\{U\};*}(X;\mathbb{Z}) \tag{1.2}$$

form an exact triangle. Given a Borel-Moore k-pseudocycle f to X, we construct an arbitrarily small neighborhood U of $\mathrm{Bd}\, f$ with $H^{\mathrm{lf}}_{\{U\};*}(X;\mathbb{Z})$ vanishing for $l > k-2$ and define an element $[f]_{X;U}$ in $H^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$. Via the aforementioned exact triangle, $[f]_{X;U}$ corresponds to an element $[f]$ in $H^{\mathrm{lf}}_*(X;\mathbb{Z})$. It is shown in [13] that for each k-pseudocycle f there is an arbitrarily small neighborhood U of $\mathrm{Bd}\, f$ with $H_l(U;\mathbb{Z})$ vanishing for $l > k-2$; a class $[f]_{X;U}$ is then constructed in $H_k(X,U;\mathbb{Z})$. Our neighborhoods U are more carefully chosen versions of the neighborhoods U of [13]; see the proof of Proposition 3.1.

Section 1.2 outlines the proof of Theorem 1.3 in Section 3. This outline is nearly identical to [13, Section 1.2], with the standard homology theory replaced by an *appropriate* homology theory of locally finite singular chains. However, care needs to be exercised in actually implementing this outline as we are now dealing with infinite chains. Section 2 thoroughly reviews the relevant background on the Borel-Moore homology in a straightforward manner readily accessible to a broad mathematical audience and provides the necessary tools to adapt the approach of [13]. In order to show that the Borel-Moore pseudocycles represent all of the cohomology of an oriented manifold, we also give a relatively simple proof of a Poincaré Duality between the singular cohomology of such a manifold and its Borel-Moore homology. Our proof is motivated by the approach of [8, Appendix A], which shows that the compactly supported cohomology of an oriented manifold is dual to its standard singular homology. Throughout the remainder of this paper, a manifold will always mean a smooth manifold.

1.2 Outline of Section 3

An oriented k-manifold is equipped with a fundamental class $[M] \in H^{\mathrm{lf}}_k(X;\mathbb{Z})$; see Proposition 2.12. A smooth proper map $f\colon M \longrightarrow X$ from such a manifold determines an element

$$[f] \equiv f_*[M] \in H^{\mathrm{lf}}_k(X;\mathbb{Z}) \,.$$

A Borel-Moore k-pseudocycle $f\colon M \longrightarrow X$ need not be a proper map. However, one can choose a closed k-submanifold with boundary, $\overline{V} \subset M$, so that $f|_{\overline{V}}$ is proper and $f(M-\overline{V})$ lies in a small neighborhood U of $\mathrm{Bd}\, f$. This implies that $f|_{\overline{V}}$ determines an element

$$[f]_{X;U} \equiv \big[f|_{\overline{V}}\big] \equiv f_*[\overline{V}] \in H^{\mathrm{lf}}_k\big(X,\{U\};\mathbb{Z}\big).$$

By Proposition 3.1, U can be chosen so that $H_k(X,\{U\};\mathbb{Z})$ is naturally isomorphic to $H^{\mathrm{lf}}_k(X;\mathbb{Z})$.

In order to show that the image $[f]$ of $f_*[\overline{V}]$ in $H^{\mathrm{lf}}_k(X;\mathbb{Z})$ depends only on f, we replace the chain complex (1.2) by a quotient complex $\overline{S}^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$. The latter is the direct adaptation of the complex $\overline{S}_*(X)$ of [13] from the standard singular chains to the locally finite singular chains. The homology $\overline{H}^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$ of $\overline{S}^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$ is naturally isomorphic to $H^{\mathrm{lf}}_*(X,\{U\};\mathbb{Z})$, but cycles and boundaries in this chain complex can be constructed more easily; see the last paragraph of [13, Section 2.3].

A Borel-Moore pseudocycle equivalence $\widetilde{f}\colon \widetilde{M} \longrightarrow X$ between two Borel-Moore pseudocycles

$$f_r\colon M_r \longrightarrow X, \qquad r=0,1,$$

gives rise to a chain equivalence between the corresponding cycles in $\overline{S}^{\mathrm{lf}}_*(X,\{W\};\mathbb{Z})$, for a small neighborhood W of $\mathrm{Bd}\,\widetilde{f}$. This implies that

$$[f_0]_{X;W} = [f_1]_{X;W} \in \overline{H}^{\mathrm{lf}}_k(X,\{W\};\mathbb{Z}) \approx H^{\mathrm{lf}}_k(X,\{W\};\mathbb{Z}).$$

By Proposition 3.1, W can be chosen so that $H^{\mathrm{lf}}_k(X;\mathbb{Z})$ naturally injects into $H^{\mathrm{lf}}_k(X,\{W\};\mathbb{Z})$. Thus,

$$[f_0] = [f_1] \in H^{\mathrm{lf}}_k(X;\mathbb{Z})$$

and the homomorphism Φ_* is well-defined; see Section 3.4 for details.

The homomorphism Ψ_* is constructed by first showing that all codimension 1 faces of the simplices of a cycle in $\overline{S}^{\mathrm{lf}}_k(X;\mathbb{Z})$ come in pairs with opposite orientations; see Lemma 3.4. By gluing the k-simplices along the codimension 1 faces paired up in this way, we obtain a proper map from a simplicial complex M' to X. The complement of the codimension 2 simplices in M' is a manifold; the continuous map from it can be smoothed out in a standardized manner via Lemma 2.1. This systematic procedure produces a Borel-Moore pseudocycle from a cycle in $\overline{S}^{\mathrm{lf}}_k(X;\mathbb{Z})$. A chain equivalence between two k-cycles in $\overline{S}^{\mathrm{lf}}_k(X;\mathbb{Z})$, $\{c_0\}$ and $\{c_1\}$, similarly determines a Borel-Moore pseudocycle equivalence between the pseudocycles obtained from $\{c_0\}$ and $\{c_1\}$.

In Section 3.5, we conclude by confirming that the homomorphisms Ψ_* and Φ_* are mutual inverses. As in [13], it is fairly straightforward to show that the map $\Phi_* \circ \Psi_*$ is the identity on $H^{\mathrm{lf}}_*(X;\mathbb{Z})$. Following the approach in [13], we then show that the homomorphism Φ_* is injective.

We now note some basic facts concerning proper maps that will be used in the proof of Theorem 1.3.

Lemma 1.6. *Let $f\colon M \longrightarrow X$ be a continuous map.*

(1) If $U \subset X$ is an open neighborhood of $\mathrm{Bd}\,f$, then $f|_{M-f^{-1}(U)}$ is a proper map.

(2) If X is Hausdorff and locally compact, then

$$\mathrm{Bd}\,f|_{M-B} \subset (\mathrm{Bd}\,f) \cup \overline{f(B)} \qquad \forall\, B \subset M\,.$$

(3) If f is proper, $B \subset M$ is closed, and either M or X is Hausdorff, then $f|_B$ is also proper.

(4) If f is proper and X is Hausdorff and locally compact, then f is a closed map.

(5) If X is Hausdorff and admits a locally finite cover $\{A_i\}_{i \in \mathcal{I}}$ by compact subsets, M is normal and locally compact, and $B \subset M$ is a closed subset such that $f|_B$ is proper, then there exists an open neighborhood $W \subset M$ of B so that $f|_{\overline{W}}$ is still proper.

Proof. We give a proof of the last statement; the remaining ones are straightforward. Since the cover $\{A_i\}_{i \in \mathcal{I}}$ of X is locally finite, every compact subset $A \subset M$ is covered by finitely many elements of this collection. It is thus sufficient to construct a neighborhood $W \subset M$ of B so that

$\overline{W} \cap f^{-1}(A_i)$ is compact for every $i \in \mathcal{I}$.

The cover $\{f^{-1}(A_i)\}_{i \in \mathcal{I}}$ of M is locally finite and consists of closed subsets of M. For each $i \in \mathcal{I}$, let

$$\mathcal{I}_i = \{j \in \mathcal{I} \colon A_i \cap A_j \neq \emptyset\} \quad \text{and} \quad B_i^c = \bigcup_{j \in \mathcal{I} - \mathcal{I}_i} f^{-1}(A_j) \subset M.$$

By the compactness of A_i, the collection $\mathcal{I}_i$ is finite. Since $\{f^{-1}(A_i)\}_{i \in \mathcal{I} - \mathcal{I}_i}$ is a locally finite collection of closed subsets of M, B_i^c is a closed subset of M disjoint from the closed subset $f^{-1}(A_i)$. Let $U_i \subset M$ be an open neighborhood of $f^{-1}(A_i)$ disjoint from B_i^c. Since

$$\{j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\} \subset \bigcup_{k \in \mathcal{I}_i} \mathcal{I}_k,$$

the open cover $\{U_i\}_{i \in \mathcal{I}}$ is locally finite.

For each $i \in \mathcal{I}$, $B \cap f^{-1}(A_i) \subset M$ is a compact subset. Let $V_i \subset M$ be an open neighborhood of $B \cap f^{-1}(A_i)$ so that $\overline{V}_i \subset M$ is compact and contained in U_i. Let

$$W = \bigcup_{i \in \mathcal{I}} V_i \subset M.$$

Since the collection $\{\overline{V}_i\}_{i \in \mathcal{I}}$ is locally finite,

$$\overline{W} = \bigcup_{i \in \mathcal{I}} \overline{V}_i \subset M.$$

For any $i \in \mathcal{I}$,

$$\overline{W} \cap f^{-1}(A_i) = \bigcup_{j \in \mathcal{I}_i} \left(\overline{V}_j \cap f^{-1}(A_i) \right) \subset M.$$

The above finite union of compact subsets of M is compact, as needed. $\qquad\square$

2 Borel-Moore homology

2.1 Standard simplicies

In order to set up notation for the standard simplices, their subsets, and maps between them consistent with [13], we reproduce most of [13, Section 2.1]. The present section can be skipped at first and referred to as needed later.

For $k \in \mathbb{Z}^{\geq 0}$, let

$$[k] = \{0, 1, 2, \ldots, k\}.$$

For a finite subset $A \subset \mathbb{R}^k$, we denote by $\mathrm{CH}(A)$ and $\mathrm{CH}^0(A)$ the (closed) convex hull of A and the open convex hull of A, respectively, i.e.

$$CH(A) = \Big\{ \sum_{v \in A} t_v v \colon t_v \in [0, 1]; \ \sum_{v \in A} t_v = 1 \Big\} \qquad \text{and}$$

$$CH^0(A) = \Big\{ \sum_{v \in A} t_v v \colon t_v \in (0, 1); \ \sum_{v \in A} t_v = 1 \Big\}.$$

If $B \subset \mathbb{R}^m$ is also a finite set, a map $f \colon CH(A) \longrightarrow CH(B)$ is linear if

$$f\left(\sum_{v \in A} t_v v\right) = \sum_{v \in A} t_v f(v) \qquad \forall\, t_v \in [0,1]^A \ \text{ s.t. } \ \sum_{v \in A} t_v = 1.$$

Such a map is determined by its values on A.

For each $p = 1, \ldots, k$, let e_p be the p-th coordinate vector in $\mathbb{R}^k$. Put $e_0 = 0 \in \mathbb{R}^k$. Denote by

$$\Delta^k = \mathrm{CH}(e_0, e_1, \ldots, e_k) \qquad \text{and} \qquad \mathrm{Int}\,\Delta^k = \mathrm{CH}^0(e_0, e_1, \ldots, e_k)$$

the standard k-simplex and its interior. Let

$$b_k = \frac{1}{k+1}\left(\sum_{p=0}^{p=k} e_p\right) = \left(\frac{1}{k+1}, \ldots, \frac{1}{k+1}\right) \in \mathbb{R}^k$$

be the barycenter of Δ^k.

For each $p = 0, 1, \ldots, k$, let

$$\Delta_p^k = \mathrm{CH}\left(\{e_q \colon q \in [k] - \{p\}\}\right) \qquad \text{and} \qquad \mathrm{Int}\,\Delta_p^k = \mathrm{CH}^0\left(\{e_q \colon q \in [k] - \{p\}\}\right)$$

denote the p-th face of Δ^k and its interior. Define a linear map

$$\iota_{k;p} \colon \Delta^{k-1} \longrightarrow \Delta_p^k \subset \Delta^k \qquad \text{by} \qquad \iota_{k;p}(e_q) = \begin{cases} e_q, & \text{if } q < p; \\ e_{q+1}, & \text{if } q \geq p. \end{cases}$$

We also define a projection map

$$\widetilde{\pi}_p^k \colon \Delta^k - \{e_p\} \longrightarrow \Delta_p^k \qquad \text{by} \qquad \widetilde{\pi}_p^k\left(\sum_{q=0}^{q=k} t_q e_q\right) = \frac{1}{1-t_p}\left(\sum_{\substack{0 \leq q \leq k \\ q \neq p}} t_q e_q\right).$$

Put

$$b_{k;p} = \iota_{k;p}(b_{k-1}), \qquad b'_{k;p} = \frac{1}{k+1}\left(b_k + \sum_{\substack{0 \leq q \leq k \\ q \neq p}} e_q\right).$$

The points $b_{k;p}$ and $b'_{k;p}$ are the barycenters of the $(k-1)$-simplex Δ_p^k and of the k-simplex spanned by b_k and the vertices of Δ_p^k. Define a neighborhood of $\mathrm{Int}\,\Delta_p^k$ in Δ^k by

$$U_p^k = \left\{t_p b'_{k;p} + \sum_{\substack{0 \leq q \leq k \\ q \neq p}} t_q e_q \colon t_p \geq 0, \ t_q > 0 \ \forall q \neq p; \ \sum_{q=0}^{q=k} t_q = 1\right\}$$

$$= \left(\mathrm{Int}\,\Delta_p^k\right) \cup \mathrm{CH}^0\left(\{e_q \colon q \in [k] - \{p\}\} \cup \{b'_{k;p}\}\right);$$

see Figure 1. These disjoint neighborhoods are used to construct Borel-Moore pseudocycles out of Borel-Moore homology cycles via Lemma 2.1.
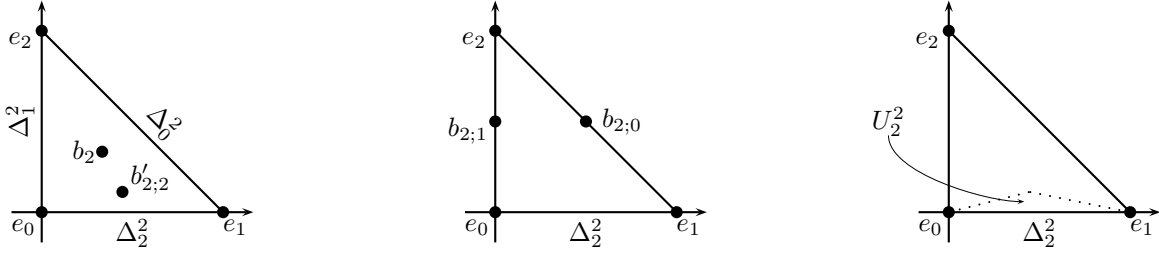
Figure 1: The standard 2-simplex and some of its distinguished subsets

If $p, q = 0, 1, \ldots, k$ and $p \neq q$, let

$$\Delta_{p,q}^k \equiv \Delta_p^k \cap \Delta_q^k$$

be the corresponding codimension 2 simplex. Define a projection map

$$\widetilde{\pi}_{p,q}^k : \Delta^k - \mathrm{CH}(e_p, e_q) \longrightarrow \Delta_{p,q}^k \qquad \text{by} \qquad \widetilde{\pi}_{p,q}^k \Big(\sum_{r=0}^{r=k} t_r e_r \Big) = \frac{1}{1 - t_p - t_q} \Big(\sum_{\substack{0 \leq r \leq k \\ r \neq p,q}} t_r e_r \Big).$$

We define a neighborhood of $\mathrm{Int}\, \Delta_{p,q}^k$ in Δ^k by

$$U_{p,q}^k = \Big\{ t_p \iota_{k;p}(b_{k-1; \iota_{k;p}^{-1}(q)}') + t_q \iota_{k;q}(b_{k-1; \iota_{k;q}^{-1}(p)}') + \sum_{\substack{0 \leq r \leq k \\ r \neq p,q}} t_r e_r : t_p, t_q \geq 0,\, t_r > 0\, \forall r \neq p, q;\, \sum_{r=0}^{r=k} t_r = 1 \Big\}$$

$$= \Big(\mathrm{Int}\, \Delta_{p,q}^k \Big) \cup \mathrm{CH}^0 \Big(\{ e_r : r \in [k] - \{p, q\} \} \cup \{ \iota_{k;p}(b_{k-1; \iota_{k;p}^{-1}(q)}'),\, \iota_{k;q}(b_{k-1; \iota_{k;q}^{-1}(p)}') \} \Big);$$

see Figure 2. These disjoint neighborhoods are used to construct Borel-Moore pseudocycle equivalences out of Borel-Moore bounding chains via Lemma 2.1.

Denote by $\mathcal{S}_k$ the group of permutations of the set $[k]$. We view the set $\mathcal{S}_k$ as a subset of $\mathcal{S}_{k+1}$ by setting $\tau(k+1) = k+1$ for each $\tau \in \mathcal{S}_k$. For any $\tau \in \mathcal{S}_k$, let

$$\tau : \Delta^k \longrightarrow \Delta^k$$

be the linear map defined by

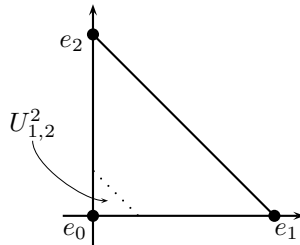$$\tau(e_q) = e_{\tau(q)} \qquad \forall q = 0, 1, \ldots, k.$$

Figure 2: The standard 2-simplex and a distinguished neighborhood of a codimension 2 simplex

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Lemma 2.1 ([13, Lemma 2.1]). *Let $k \in \mathbb{Z}^+$, $Y \subset \Delta^k$ be the $(k{-}2)$-skeleton of Δ^k, and $\widetilde{Y} \subset \Delta^{k+1}$ be the $(k{-}2)$-skeleton of Δ^{k+1}. There exist continuous functions*

$$\varphi_k \colon \Delta^k \longrightarrow \Delta^k \qquad and \qquad \widetilde{\varphi}_{k+1} \colon \Delta^{k+1} \longrightarrow \Delta^{k+1}$$

such that

(a) *φ_k is smooth outside of Y and $\widetilde{\varphi}_{k+1}$ is smooth outside of $\widetilde{Y}$;*

(b) *for all $p = 0, \ldots, k$ and $\tau \in \mathcal{S}_k$,*

$$\varphi_k|_{U_p^k} = \widetilde{\pi}_p^k\big|_{U_p^k}, \qquad \varphi_k \circ \tau = \tau \circ \varphi_k; \tag{2.1}$$

(c) *for all $p, q = 0, \ldots, k{+}1$ with $p \neq q$ and $\widetilde{\tau} \in \mathcal{S}_{k+1}$,*

$$\widetilde{\varphi}_{k+1}\big|_{U_{p,q}^{k+1}} = \widetilde{\pi}_{p,q}^{k+1}\big|_{U_{p,q}^{k+1}}, \quad \widetilde{\varphi}_{k+1} \circ \widetilde{\tau} = \widetilde{\tau} \circ \widetilde{\varphi}_{k+1}, \quad \widetilde{\varphi}_{k+1} \circ \iota_{k+1;p} = \iota_{k+1;p} \circ \varphi_k. \tag{2.2}$$

2.2 Basic definitions

Let R be a commutative ring with unity 1 and X be a topological space. For $k \in \mathbb{Z}^{\geq 0}$, denote by $\mathrm{Hom}(\Delta^k, X)$ the set of singular k-simplicies on X, i.e. of continuous maps from Δ^k to X. An singular chain on X with coefficients in R, i.e. a map

$$c \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow R, \tag{2.3}$$

can be written as a formal sum

$$c = \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} a_\sigma \sigma, \qquad a_\sigma \in R. \tag{2.4}$$

We identify $\mathrm{Hom}(\Delta^k, X)$ with a subset of such maps by defining

$$\sigma \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow R, \qquad \sigma(\tau) = \begin{cases} 1, & \text{if } \tau = \sigma; \\ 0, & \text{if } \tau \neq \sigma; \end{cases} \quad \forall\, \sigma, \tau \in \mathrm{Hom}(\Delta^k, X).$$

We say that a singular k-simplex σ appears in a singular chain c as in (2.3) and (2.4) if $c(\sigma) \equiv a_\sigma$ is not zero.

For a singular chain c as in (2.3) and (2.4), define

$$\mathrm{supp}(c) = \bigcup_{\substack{\sigma \in \mathrm{Hom}(\Delta^k, X) \\ c(\sigma) \neq 0}} \sigma(\Delta^k) = \bigcup_{\substack{\sigma \in \mathrm{Hom}(\Delta^k, X) \\ a_\sigma \neq 0}} \sigma(\Delta^k) \subset X \tag{2.5}$$

to be the support of c. If a k-simplex σ appears in c, then $\sigma(\Delta^k) \subset \mathrm{supp}(c)$. For $U \subset X$, let

$$\begin{aligned}
\aleph_c(U) &= \big\{ \sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \sigma(\Delta^k) \cap U \neq \emptyset \big\} \\
&= \big\{ \sigma \in \mathrm{Hom}(\Delta^k, X) \colon a_\sigma \neq 0,\ \sigma(\Delta^k) \cap U \neq \emptyset \big\}.
\end{aligned} \tag{2.6}$$

A finite singular k-chain on X with coefficients in R is a map c as in (2.3) such that the set $\aleph_c(X)$ is finite. The R-module of such chains is the k-th module of the usual chain complex $S_*(X; R)$

determining the standard singular homology $H_*(X; R)$ of X.

A Borel-Moore k-chain on X is a map c as in (2.3) such that for every $x \in X$ there exists an open neighborhood $U_x \subset X$ of x so that the set $\aleph_c(U_x)$ defined by (2.6) is finite. If X is second countable, at most countably many simplicies appear in a Borel-Moore k-chain on X. If X is Hausdorff, the support (2.5) of a Borel-Moore chain c is closed in X. The set $S_k^{\mathrm{lf}}(X; R)$ of Borel-Moore k-chains on X with coefficients in R is an R-module under the addition and scalar multiplication of the values of the chains on the k-simplices. This set contains $\mathrm{Hom}(\Delta^k, X)$. We call a map

$$h \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow S_p^{\mathrm{lf}}(X; R) \tag{2.7}$$

rigid if

$$\mathrm{supp}\big(h(\sigma)\big) \subset \sigma(\Delta^k) \qquad \forall\, \sigma \in \mathrm{Hom}(\Delta^k, X). \tag{2.8}$$

Lemma 2.2. *Let X be a topological space. A rigid map h as in (2.7) induces a homomorphism*

$$h \colon S_k^{\mathrm{lf}}(X; R) \longrightarrow S_p^{\mathrm{lf}}(X; R),$$
$$\big\{h(c)\big\}(\tau) = \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} c(\sigma)\big\{h(\sigma)\big\}(\tau) \in R \quad \forall\, \tau \in \mathrm{Hom}(\Delta^p, X),\ c \in S_k^{\mathrm{lf}}(X; R), \tag{2.9}$$

extending (2.7) such that

$$\mathrm{supp}\big(h(c)\big) \subset \mathrm{supp}(c) \qquad \forall\, c \in S_k^{\mathrm{lf}}(X; R). \tag{2.10}$$

Proof. Let $c \in S_k^{\mathrm{lf}}(X; R)$ and $\tau \in \mathrm{Hom}(\Delta^p, X)$. By the compactness of $\tau(\Delta^p) \subset X$, there exists an open neighborhood U_τ of $\tau(\Delta^p)$ in X such that the set $\aleph_c(U_\tau)$ is finite. By (2.5) and (2.8),

$$\big\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma)\big\{h(\sigma)\big\}(\tau) \neq 0\big\} \subset \big\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \tau(\Delta^p) \subset \mathrm{supp}\big(h(\sigma)\big)\big\}$$
$$\subset \big\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \tau(\Delta^p) \subset \sigma(\Delta^k)\big\}$$
$$\subset \big\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \sigma(\Delta^k) \cap U_\tau \neq \emptyset\big\} = \aleph_c(U_\tau).$$

Thus, the sum in (2.9) is finite.

Let $c \in S_k^{\mathrm{lf}}(X; \mathbb{Z})$, $x \in X$, and U_c be an open neighborhood of x in X such that the set $\aleph_c(U_c)$ is finite. For each $\sigma \in \mathrm{Hom}(\Delta^k, X)$, let U_σ be an open neighborhood of x in X such that the set

$$\aleph_{h(\sigma)}(U_\sigma) \equiv \big\{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \big\{h(\sigma)\big\}(\tau) \neq 0,\ \tau(\Delta^p) \cap U_\sigma \neq \emptyset\big\}$$

is finite. The subset

$$U_x \equiv U_c \cap \bigcap_{\sigma \in \aleph_c(U_c)} U_\sigma \subset X$$

is also an open neighborhood of x in X. By (2.8),

$$\aleph_{h(\sigma)}(U_c) \subset \big\{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \tau(\Delta^p) \subset \sigma(\Delta^k),\ \tau(\Delta^p) \cap U_c \neq \emptyset\big\} = \emptyset \quad \forall\, \sigma \in \aleph_c(X) - \aleph_c(U_c).$$

Combining this with (2.9), we obtain

$$\aleph_{h(c)}(U_x) \subset \bigcup_{\sigma \in \aleph_c(X)} \aleph_{h(\sigma)}(U_x) = \bigcup_{\sigma \in \aleph_c(U_c)} \aleph_{h(\sigma)}(U_x) \subset \bigcup_{\sigma \in \aleph_c(U_c)} \aleph_{h(\sigma)}(U_\sigma).$$

Since the last set above is finite, we conclude that $h(c) \in S_p^{\mathrm{lf}}(X; R)$.

It is immediate that the map h in (2.9) is a homomorphism of R-modules and restricts to (2.7). By (2.9) and (2.8),

$$\{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \{h(c)\}(\tau) \neq 0\} \subset \bigcup_{\sigma \in \aleph_c(X)} \{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \{h(\sigma)\}(\tau) \neq 0\}$$

$$\subset \bigcup_{\substack{\sigma \in \mathrm{Hom}(\Delta^k, X) \\ \sigma(\Delta^k) \subset \mathrm{supp}(c)}} \{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \tau(\Delta^p) \subset \mathrm{supp}(h(\sigma))\} \subset \{\tau \in \mathrm{Hom}(\Delta^p, X) \colon \tau(\Delta^p) \subset \mathrm{supp}(c)\}.$$

This establishes (2.10). $\qquad\square$

In the notation of (2.4),

$$h(c) = \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} a_\sigma h(\sigma) \,.$$

Each $h(\sigma)$ is a formal sum. By the first part of the proof of Lemma 2.2, each p-simplex τ appears in only finitely many chains $h(\sigma)$. Thus, the implicit double sum above can be reduced to a single sum as in (2.4). By the second part of the proof of Lemma 2.2, $h(c)$ satisfies the required local finiteness condition.

A map

$$\hbar \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow S_p^{\mathrm{lf}}(\Delta^k; R) = S_p(\Delta^k; R) \tag{2.11}$$

induces a rigid map

$$h \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow S_p(X; R), \qquad h(\sigma) = \sigma_\# \big(\hbar(\sigma)\big), \tag{2.12}$$

and thus a homomorphism

$$h \equiv \hbar_\# \colon S_k^{\mathrm{lf}}(X; R) \longrightarrow S_p^{\mathrm{lf}}(X; R).$$

If $k \in \mathbb{Z}^+$, the **boundary homomorphism**

$$\partial_X \colon S_k^{\mathrm{lf}}(X; R) \longrightarrow S_{k-1}^{\mathrm{lf}}(X; R), \quad \partial_X \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} a_\sigma \sigma = \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} \sum_{p=0}^{k} (-1)^p a_\sigma \big(\sigma \circ \iota_{k;p}\big) \tag{2.13}$$

is induced by the constant map

$$\hbar \colon \mathrm{Hom}(\Delta^k, X) \longrightarrow S_{k-1}^{\mathrm{lf}}(\Delta^k; R), \quad \hbar(\sigma) = \partial_{\Delta^k} \mathrm{id}_{\Delta^k} \equiv \sum_{p=0}^{k} (-1)^p \iota_{k;p} \,.$$

By Lemma 2.2, the homomorphism (2.13) is thus well-defined. We define ∂_X on $S_0^{\mathrm{lf}}(X; R)$ to be the zero homomorphism. It is immediate that $\partial_X^2 = 0$. The quotient

$$H_k^{\mathrm{lf}}(X; R) = \frac{\ker(\partial_X \colon S_k^{\mathrm{lf}}(X; R) \longrightarrow S_{k-1}^{\mathrm{lf}}(X; R))}{\mathrm{Im}\,(\partial_X \colon S_{k+1}^{\mathrm{lf}}(X; R) \longrightarrow S_k^{\mathrm{lf}}(X; R))}$$

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is the k-th Borel-Moore homology module of X with coefficients in R. If X is compact, $(S^{\mathrm{lf}}_*(X;\mathbb{R}),\partial_X)$ is the usual singular chain complex $(S_*(X;R),\partial_X)$ and the Borel-Moore homology modules are the standard homology modules with coefficients in R.

For $q \in \mathbb{Z}^{\geq 0}$, let
$$S^q(X;R) \equiv \mathrm{Hom}_{\mathbb{R}}\big(S_q(X;R),R\big)$$
denote the usual R-module of the R-valued p-cochains on X. For each $\alpha \in S^q(X;R)$, the map
$$\alpha\cap\colon \mathrm{Hom}(\Delta^{p+q},X) \longrightarrow S^{\mathrm{lf}}_p(X;R), \quad \alpha\cap\sigma = \alpha(\sigma^q)\,{}^p\sigma,$$
where ${}^p\sigma$ and σ^q are the p-th front and q-th back faces, respectively, of a singular $(p+q)$-simplex σ, is rigid. By Lemma 2.2, this map thus induces a homomorphism
$$\cap\colon S^q(X;R)\otimes_R S^{\mathrm{lf}}_{p+q}(X;R) \longrightarrow S^{\mathrm{lf}}_p(X;R), \quad \alpha\otimes\mu \mapsto \alpha\cap\mu. \tag{2.14}$$
This $\mathsf{cap\ product}$ restricts to the cap product on $S^q(X;R)\otimes_R S_{p+q}(X;R)$ in the standard singular theory defined in [9, Section 66]. The homomorphism (2.14) satisfies
$$\partial_X\big(\alpha\cap\mu\big) = (-1)^p(\delta_X\alpha)\cap\mu + \alpha\cap(\partial_X\mu) \qquad \forall\,\alpha\in S^p(X;R),\ \mu\in S^{\mathrm{lf}}_{p+q}(X;R), \tag{2.15}$$
where $\delta_X = \partial_X^*$. Thus, (2.14) descends to a homomorphism
$$\cap\colon H^q(X;R)\otimes_R H^{\mathrm{lf}}_{p+q}(X;R) \longrightarrow H^{\mathrm{lf}}_p(X;R).$$

2.3   Basic properties

Let X be a topological space. We call a collection of maps
$$\hbar\colon \mathrm{Hom}(\Delta^k,X) \longrightarrow S_*(\Delta^k;R), \quad k\in\mathbb{Z}^{\geq 0}, \tag{2.16}$$
a $\mathsf{pre\text{-}chain\ map}$ if
$$\partial_{\Delta^k}\big(\hbar(\sigma)\big) = \sum_{p=0}^k (-1)^p\big\{\iota_{k;p}\big\}_{\#}\big(\hbar(\sigma\circ\iota_{k;p})\big) \quad \forall\,\sigma\in\mathrm{Hom}(\Delta^k,X),\ k\in\mathbb{Z}^{\geq 0}. \tag{2.17}$$

A pre-chain map $\hbar$ determines a chain map
$$\hbar_{\#}\colon S^{\mathrm{lf}}_*(X;R) \longrightarrow S^{\mathrm{lf}}_*(X;R), \tag{2.18}$$
not necessarily preserving the grading, via (2.12) and Lemma 2.2. A linear combination of pre-chain maps is a pre-chain map.

Let $\hbar$ be a collection of maps as in (2.16). A $\mathsf{null\text{-}homotopy}$ for $\hbar$ is a collection of maps
$$D_{\hbar}\colon \mathrm{Hom}(\Delta^k,X) \longrightarrow S_{*+1}(\Delta^k;R), \quad k\in\mathbb{Z}^{\geq 0},$$
such that
$$\partial_{\Delta^k}\big(D_{\hbar}(\sigma)\big) = \hbar(\sigma) - \sum_{p=0}^k (-1)^p\big\{\iota_{k;p}\big\}_{\#}\big(D_{\hbar}(\sigma\circ\iota_{k;p})\big) \quad \forall\,\sigma\in\mathrm{Hom}(\Delta^k,X),\ k\in\mathbb{Z}^{\geq 0}. \tag{2.19}$$

In such a case,
$$\hbar_{\#} = \partial_X D_{\hbar\,\#} + D_{\hbar\,\#}\partial_X\colon S^{\mathrm{lf}}_*(X;R) \longrightarrow S^{\mathrm{lf}}_*(X;R),$$
i.e. $D_{\hbar\#}$ is a chain homotopy from $\hbar_{\#}$ to the zero homomorphism.

Lemma 2.3. *Let X be a topological space and*

$$\hbar \colon \operatorname{Hom}(\Delta^k, X) \longrightarrow S_k(\Delta^k; R), \qquad k \in \mathbb{Z}^{\geq 0}, \tag{2.20}$$

be a pre-chain map. If $\hbar$ vanishes on $\operatorname{Hom}(\Delta^0, X)$, then there exists a null-homotopy

$$D_\hbar \colon \operatorname{Hom}(\Delta^k, X) \longrightarrow S_{k+1}(\Delta^k; R), \qquad k \in \mathbb{Z}^{\geq 0},$$

for $\hbar$.

Proof. We take $D_\hbar = 0$ on $\operatorname{Hom}(\Delta^0, X)$. Suppose $k \in \mathbb{Z}^+$ and we have constructed $D_\hbar$ on $\operatorname{Hom}(\Delta^l, X)$ with $l < k$ so that it satisfies (2.19) on $\operatorname{Hom}(\Delta^l, X)$ with $l < k$. Let $\sigma \in \operatorname{Hom}(\Delta^k, X)$ and

$$c_\sigma = \hbar(\sigma) - \sum_{p=0}^{k} (-1)^p \iota_{k;p\,\#}\big(D_\hbar(\sigma \circ \iota_{k;p})\big).$$

For $k \geq 2$, the inductive assumption gives

$$\partial_{\Delta^k}(c_\sigma) = \partial_{\Delta^k}\big(\hbar(\sigma)\big) - \sum_{p=0}^{k} (-1)^p \iota_{k;p\,\#}\big(\partial_{\Delta^{k-1}} D_\hbar(\sigma \circ \iota_{k;p})\big)$$

$$= \partial_{\Delta^k}\big(\hbar(\sigma)\big) - \sum_{p=0}^{k} (-1)^p \iota_{k;p\,\#}\left(\hbar(\sigma \circ \iota_{k;p}) - \sum_{q=0}^{k-1} (-1)^q \iota_{k-1;q\,\#} D_\hbar\big(\sigma \circ \iota_{k;p} \circ \iota_{k-1;q}\big)\right).$$

The terms in the double sum cancel in pairs, while the remaining difference vanishes by (2.17). For $k = 1$, (2.17) and the vanishing of $\hbar$ and $D_\hbar$ on $\operatorname{Hom}(\Delta^0, X)$ imply that

$$\partial_{\Delta^k} c_\sigma = 0$$

in this case as well. Since $H_k(\Delta^k; R)$ is trivial, there exists

$$D_\hbar \sigma \in S_{k+1}(\Delta^k; R) \qquad \text{s.t.} \quad \partial_{\Delta^k}\big(D_\hbar(\sigma)\big) = c_\sigma.$$

This completes the inductive step. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A Hausdorff topological space X' is **locally compact** if for every point $x \in X'$ there exists an open neighborhood U_x of x in X' such that the closure $\overline{U_x}$ of U_x in X' is compact (if X' is not necessarily Hausdorff, there are various versions of this definition that are equivalent for Hausdorff spaces).

Lemma 2.4. *Let $f \colon X \longrightarrow X'$ be a proper map between topological spaces. If either X is compact or X' is locally compact, then the map*

$$f_\# \colon S_*^{\mathrm{lf}}(X; R) \longrightarrow S_*^{\mathrm{lf}}(X'; R),$$

$$\big\{f_\#(c)\big\}(\tau) = \sum_{\sigma \in \operatorname{Hom}(\Delta^k, X)} c(\sigma)\big\{f \circ \sigma\big\}(\tau) \in R \qquad \forall\, \tau \in \operatorname{Hom}(\Delta^p, X'), \tag{2.21}$$

is a well-defined homomorphism of chain complexes and

$$\operatorname{supp}\big(f_\#(c)\big) \subset f\big(\operatorname{supp}(c)\big) \qquad \forall\, c \in S_k^{\mathrm{lf}}(X; R). \tag{2.22}$$

If $g \colon X' \longrightarrow X''$ is another proper continuous map and either X' is compact or X'' is locally compact, then

$$\big(g \circ f\big)_\# = g_\# \circ f_\# \colon S_*^{\mathrm{lf}}(X; R) \longrightarrow S_*^{\mathrm{lf}}(X''; R). \tag{2.23}$$

Proof. If X is compact, the map (2.21) is the composition

$$S_*^{\mathrm{lf}}(X;R) = S_*(X;R) \longrightarrow S_*(X';R) \longrightarrow S_*^{\mathrm{lf}}(X';R).$$

The first arrow above is the pushforward homomorphism of the standard singular homology theory.

For all $c \in S_k^{\mathrm{lf}}(X;R)$ and $\tau \in \mathrm{Hom}(\Delta^p, X')$,

$$\begin{aligned}
\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma)\{f \circ \sigma\}(\tau) \neq 0\} &= \{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \tau = f \circ \sigma\} \\
&\subset \{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0,\ \sigma(\Delta^k) \cap f^{-1}(\tau(\Delta^p)) \neq \emptyset\} \subset \aleph_c\big(f^{-1}(\tau(\Delta^p))\big).
\end{aligned}$$

If f is a proper map, then $f^{-1}(\tau(\Delta^p))$ is a compact subset of X and thus the last set above is finite. This implies that the sum in (2.21) is finite.

For all $c \in S_k^{\mathrm{lf}}(X;R)$ and $U \subset X'$,

$$\aleph_{f_\#(c)}(U) \subset \{f \circ \sigma \colon \sigma \in \mathrm{Hom}(\Delta^k, X),\ c(\sigma) \neq 0,\ f(\sigma(\Delta^k)) \cap \overline{U} \neq \emptyset\} = \{f \circ \sigma \colon \sigma \in \aleph_c(f^{-1}(\overline{U}))\}.$$

If f is a proper map and $\overline{U} \subset X'$ is compact, then $f^{-1}(\overline{U})$ is a compact subset of X and thus the last set above is finite. This implies that $f_\#(c) \in S_k^{\mathrm{lf}}(X;R)$ if in addition X' is locally compact.

It is immediate that the map $f_\#$ in (2.21) is a homomorphism of R-modules intertwining ∂_X and $\partial_{X'}$ and that (2.23) holds. Furthermore,

$$\{\tau \in \mathrm{Hom}(\Delta^p, X') \colon \{f_\#(c)\}(\tau) \neq 0\} \subset \{f \circ \sigma \colon \sigma \in \mathrm{Hom}(\Delta^k, X),\ c(\sigma) \neq 0\} \quad \forall c \in S_k^{\mathrm{lf}}(X;R).$$

This establishes (2.22). $\qquad\square$

In the notation of (2.4),

$$f_\#(c) = \sum_{\sigma \in \mathrm{Hom}(\Delta^k, X)} a_\sigma (f \circ \sigma).$$

By the second paragraph in the proof of Lemma 2.4, each p-simplex τ in X' appears only finitely many times in this sum. Thus, the sum above can be reduced to a sum as in (2.4). By the third paragraph in the proof of Lemma 2.4, $f_\#(c)$ satisfies the required local finiteness condition. The corollary below is an immediate consequence of Lemma 2.4.

Corollary 2.5. *Let $f : X \longrightarrow X'$ be a proper map between topological spaces. If either X is compact or X' is locally compact, then the composition of the k-simplicies to X with f induces a homomorphism*

$$f_* \colon H_k^{\mathrm{lf}}(X;R) \longrightarrow H_k^{\mathrm{lf}}(X';R).$$

If $g : X' \longrightarrow X''$ is another proper continuous map and either X' is compact or X'' is locally compact, then

$$(g \circ f)_* = g_* \circ f_* \colon H_*^{\mathrm{lf}}(X;R) \longrightarrow H_*^{\mathrm{lf}}(X'';R).$$

2.4 Subcomplexes and quotients

For a collection $\mathcal{A}$ of subsets of a topological space X, let

$$S_{\mathcal{A};*}^{\mathrm{lf}}(X;R) \subset S_*^{\mathrm{lf}}(X;R)$$

denote the subset of chains c such that

$$\big\{\sigma \in \mathrm{Hom}(\Delta^k, X) \colon c(\sigma) \neq 0\big\} \subset \bigcup_{U \in \mathcal{A}} \mathrm{Hom}(\Delta^k, U) \quad \forall\, k \in \mathbb{Z}^{\geq 0}\,.$$

This subset is a chain sub-complex of $S_*^{\mathrm{lf}}(X;R)$. We denote its homology by $H_{\mathcal{A};*}^{\mathrm{lf}}(X;R)$. Let

$$S_*^{\mathrm{lf}}\big(X,\mathcal{A};R\big) = \frac{S_*^{\mathrm{lf}}(X;R)}{S_{\mathcal{A};*}^{\mathrm{lf}}(X;R)}$$

be the quotient complex and $H_*^{\mathrm{lf}}(X,\mathcal{A};R)$ be its homology. If $W \subset X$ contains every $U \in \mathcal{A}$, then $S_{\mathcal{A};*}^{\mathrm{lf}}(X;R)$ is a sub-complex of $S_{\{W\};*}^{\mathrm{lf}}(X;R)$. In such a case, let

$$S_{\{W\};*}^{\mathrm{lf}}\big(X,\mathcal{A};R\big) = \frac{S_{\{W\};*}^{\mathrm{lf}}(X;R)}{S_{\mathcal{A};*}^{\mathrm{lf}}(X;R)}$$

be the quotient complex and $H_{\{W\};*}^{\mathrm{lf}}\big(X,\mathcal{A};R\big)$ be its homology.

By definition, $S_{\{X\};*}^{\mathrm{lf}}(X;R) = S_*^{\mathrm{lf}}(X;R)$. If $U \subset W \subset X$ and $\overline{W} \subset X$ is compact, then

$$S_{\{W\};*}^{\mathrm{lf}}(X,\{U\};R) = S_*(W,U;R) \tag{2.24}$$

is the standard relative simplicial complex for the pair (W,U). If $\mathcal{A}$ is a collection of subsets of X and $\{W_U \colon U \in \mathcal{A}\}$ is a locally finite collection of disjoint subsets of X with union W so that $U \subset W_U$ for every $U \in \mathcal{A}$, then

$$S_{\{W\};*}^{\mathrm{lf}}(X,\mathcal{A};R) = \prod_{U \in \mathcal{A}} S_{\{W_U\};*}^{\mathrm{lf}}\big(X,\{U\};R\big). \tag{2.25}$$

Lemma 2.6. *Let X be a topological space and $\mathcal{A}$ be a collection of subsets of X with union $W \subset X$. If*

$$W = \bigcup_{U \in \mathcal{A}} \big(\mathrm{Int}_W U\big)\,,$$

there exists a pre-chain map (2.20) such that

$$\sigma_\#\big(\hbar(\sigma)\big) \in S_{\mathcal{A};*}^{\mathrm{lf}}(X;\mathbb{R}) \; \forall\, \sigma \in \mathrm{Hom}(\Delta^k, W), \quad \hbar(\sigma) = \mathrm{id}_{\Delta^k} \; \forall\, \sigma \in \mathrm{Hom}(\Delta^k, U),\, U \in \mathcal{A}. \tag{2.26}$$

Proof. This lemma is established in [12, Appendix I] in different terminology. For any topological space Y, let

$$\mathrm{sd}_Y \colon S_*(Y;R) \longrightarrow S_*(Y;R) \quad \text{and} \quad D_Y \colon S_*(Y;R) \longrightarrow S_{*+1}(Y;R)$$

be the barycentric subdivision operator and a natural chain homotopy from sd_Y to the identity on $S_*(Y;R)$; see [9, Section 31]. In particular,

$$\mathrm{sd}_Y - \mathrm{id}_{S_*(Y;R)} = \partial_Y D_Y + D_Y \partial_Y \colon S_*(Y;R) \longrightarrow S_*(Y;R). \tag{2.27}$$

By [9, Theorem 31.3],

$$m(\sigma) \equiv \min\{m\in\mathbb{Z}^{\geq 0}\colon \mathrm{sd}_X^m\sigma\in S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)\} < \infty \qquad \forall\,\sigma\in\mathrm{Hom}(\Delta^k,W)\,.$$

In particular, $m(\sigma)\!=\!0$ if $\sigma\in S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)$ and $m(\sigma\circ\iota_{k;q})\!\leq\! m(\sigma)$ for all $q\!=\!0,1,\ldots,k$. Define (2.20) by

$$\hbar(\sigma) = \mathrm{sd}_{\Delta^k}^{m(\sigma)}\mathrm{id}_{\Delta^k} - D_{\Delta^k}\sum_{q=0}^{k}(-1)^q\sum_{r=m(\sigma\circ\iota_{k;q})}^{m(\sigma)-1}\mathrm{sd}_{\Delta^k}^r\iota_{k;q} \in S_k(\Delta^k;R)\,.$$

By (2.27) and the naturality of sd_Y and D_Y, the collection of maps $\hbar$ with $k\!\in\!\mathbb{Z}^{\geq 0}$ defined in this way is a pre-chain map. By construction, this collection satisfies (2.26). $\qquad\square$

Remark 2.7. The proof of Lemma 2.6 defines a pre-chain map $\hbar$ as in (2.20) only on

$$\mathrm{Hom}(\Delta^k,W)\subset\mathrm{Hom}(\Delta^k,X),$$

which suffices for our purposes below. We can define $\hbar(\sigma)$ for σ in $\mathrm{Hom}(\Delta^k,X)-\mathrm{Hom}(\Delta^k,W)$ by taking $m(\sigma)\!=\!0$ if σ does not map any of the simplicies of Δ^k to W and the largest value of $m(\sigma|_{\Delta'})$ taken over the simplicies $\Delta'\subset\Delta$ such that $\sigma(\Delta')\subset W$ if such a simplex Δ' exists.

Corollary 2.8. *Let X be a topological space and $\mathcal{A}$ be a collection of subsets of X with union $W\subset X$. If*

$$W = \bigcup_{U\in\mathcal{A}}\bigl(\mathrm{Int}_W U\bigr)\,,$$

then the inclusion of $S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)$ into $S^{\mathrm{lf}}_{\{W\};*}(X;R)$ is a chain homotopy equivalence. If in addition $W\subset Y\subset X$, then the homomorphism

$$H^{\mathrm{lf}}_{\{Y\};*}(X,\mathcal{A};R)\longrightarrow H^{\mathrm{lf}}_{\{Y\};*}(X,\{W\};R)$$

induced by this inclusion is an isomorphism.

Proof. Let $\hbar$ be the pre-chain map of Lemma 2.6 (and Remark 2.7). By Lemma 2.3 applied to the pre-chain map

$$\mathrm{Hom}(\Delta^k,X)\longrightarrow S_k(\Delta^k;R),\quad \sigma\mapsto\hbar(\sigma)-\mathrm{id}_{\Delta^k},\qquad k\in\mathbb{Z}^{\geq 0},$$

the homomorphism

$$\hbar_\#\colon S^{\mathrm{lf}}_{\{W\};*}(X;R)\longrightarrow S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)\subset S^{\mathrm{lf}}_{\{W\};*}(X;R)$$

induced by $\hbar$ is a chain homotopy inverse for the inclusion of $S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)$ into $S^{\mathrm{lf}}_{\{W\};*}(X;R)$.

The second claim follows from the commutativity of the diagram

$$
\begin{array}{ccccccccc}
\cdots\longrightarrow & H^{\mathrm{lf}}_{\mathcal{A};k}(X) & \longrightarrow & H^{\mathrm{lf}}_{\{Y\};*}(X) & \longrightarrow & H^{\mathrm{lf}}_{\{Y\};k}(X,\mathcal{A}) & \longrightarrow & H^{\mathrm{lf}}_{\mathcal{A};k-1}(X) & \longrightarrow\cdots \\
\downarrow & \downarrow{\scriptstyle\approx} & & \downarrow{\scriptstyle\mathrm{id}} & & \downarrow & & \downarrow{\scriptstyle\approx} & \downarrow{\scriptstyle\mathrm{id}} \\
\cdots\longrightarrow & H^{\mathrm{lf}}_{\{W\};k}(X) & \longrightarrow & H^{\mathrm{lf}}_{\{Y\};*}(X) & \longrightarrow & H^{\mathrm{lf}}_{\{Y\};k}(X,\{W\}) & \longrightarrow & H^{\mathrm{lf}}_{\{W\};k-1}(X) & \longrightarrow\cdots
\end{array}
$$

where the rows are the long exact sequences for the pairs

$$S^{\mathrm{lf}}_{\mathcal{A};*}(X;R)\subset S^{\mathrm{lf}}_{\{Y\};*}(X;R)\quad\text{and}\quad S^{\mathrm{lf}}_{\{W\};*}(X;R)\subset S^{\mathrm{lf}}_{\{Y\};*}(X;R)$$

with the coefficient ring R omitted, the first claim, and the Five Lemma. $\qquad\square$

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For $U \subset W \subset X$, denote by

$$\iota_{W,U} \colon S^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow S^{\mathrm{lf}}_{\{W\};*}(X;R) \quad \text{and} \quad \iota_{W,U*} \colon H^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow H^{\mathrm{lf}}_{\{W\};*}(X;R)$$

the inclusion homomorphism and the induced homomorphism on homology. If in addition $W \subset Y \subset X$, denote by

$$j^Y_{W,U} \colon S^{\mathrm{lf}}_{\{Y\};*}(X,\{U\};R) \longrightarrow S^{\mathrm{lf}}_{\{Y\};*}(X,\{W\};R) \qquad \text{and}$$
$$j^Y_{W,U*} \colon H^{\mathrm{lf}}_{\{Y\};*}(X,\{U\};R) \longrightarrow H^{\mathrm{lf}}_{\{Y\};*}(X,\{W\};R)$$

the homomorphisms induced by the inclusion $U \subset W$ and the induced homomorphism on homology.

Corollary 2.9 (Mayer-Vietoris). *Let X be a topological space and $U, V \subset X$ be subsets such that*

$$U \cup V = \left(\mathrm{Int}_{U \cup V} U \right) \cup \left(\mathrm{Int}_{U \cup V} V \right).$$

Then there is a homomorphism

$$\partial \colon H^{\mathrm{lf}}_{\{U \cup V\};*}(X;R) \longrightarrow H^{\mathrm{lf}}_{\{U \cap V\};*-1}(X;R),$$

which is natural with respect to the homomorphisms induced by the admissible inclusions $U \subset U'$ and $V \subset V'$, so that the sequence

$$\ldots \xrightarrow{\partial} H^{\mathrm{lf}}_{\{U \cap V\};k}(X;R) \xrightarrow{(\iota_{U,U \cap V*}, \iota_{V,U \cap V*})} H^{\mathrm{lf}}_{\{U\};k}(X;R) \oplus H^{\mathrm{lf}}_{\{V\};k}(X;R)$$
$$\xrightarrow{\iota_{U \cup V,U*} - \iota_{U \cup V,V*}} H^{\mathrm{lf}}_{\{U \cup V\};k}(X;R) \xrightarrow{\partial} H^{\mathrm{lf}}_{\{U \cap V\};k-1}(X;R) \longrightarrow \ldots$$

of R-modules is exact.

Proof. For $A = U, V$, let $\iota_A \colon S^{\mathrm{lf}}_{\{A\};*}(X;R) \longrightarrow S^{\mathrm{lf}}_{\{U,V\};*}(X;R)$ denote the inclusion. The short sequence

$$0 \longrightarrow S^{\mathrm{lf}}_{\{U \cap V\};*}(X;R) \xrightarrow{(\iota_{U,U \cap V}, \iota_{V;U \cap V})} S^{\mathrm{lf}}_{\{U\};*}(X;R) \oplus H^{\mathrm{lf}}_{\{V\};*}(X;R)$$
$$\xrightarrow{\iota_U - \iota_V} S^{\mathrm{lf}}_{\{U,V\};k}(X;R) \longrightarrow 0$$

of chain complexes is exact. Thus, the claim follows from the Snake Lemma and the first claim of Corollary 2.8 with $\mathcal{A} = \{U, V\}$. $\qquad\square$

Corollary 2.10 (Relative Mayer-Vietoris). *Let $U, V \subset X$ be as in Corollary 2.9 and $W \subset X$ be such that $U \cup V \subset W$. Then there is a homomorphism*

$$\partial \colon H^{\mathrm{lf}}_{\{W\};*}\left(X, \{U \cup V\}; R\right) \longrightarrow H^{\mathrm{lf}}_{\{W\};*-1}\left(X, \{U \cap V\}; R\right),$$

which is natural with respect to the homomorphisms induced by the admissible inclusions $U \subset U'$, $V \subset V'$, and $W \subset W'$, so that the sequence

$$\ldots \xrightarrow{\partial} H^{\mathrm{lf}}_{\{W\};k}(X, \{U \cap V\}; R) \xrightarrow{(j^W_{U,U \cap V*}, j^W_{V;U \cap V*})} H^{\mathrm{lf}}_{\{W\};k}(X, \{U\}; R) \oplus H^{\mathrm{lf}}_{\{W\};k}(X, \{V\}; R)$$
$$\xrightarrow{j^W_{U \cup V,U*} - j^W_{U \cup V,V*}} H^{\mathrm{lf}}_{\{W\};k}(X, \{U \cup V\}; R) \xrightarrow{\partial} H^{\mathrm{lf}}_{\{W\};k-1}(X, \{U \cap V\}; R) \longrightarrow \ldots$$

of R-modules is exact.

Proof. For $A = U, V$, let

$$j_A^W : S_{\{W\};*}^{\mathrm{lf}}(X, \{A\}; R) \longrightarrow S_{\{W\};*}^{\mathrm{lf}}(X, \{U, V\}; R)$$

denote the homomorphism induced by the inclusion ι_A in the proof of Corollary 2.9. The short sequence

$$0 \longrightarrow S_{\{W\};*}^{\mathrm{lf}}(X, \{U \cap V\}; R) \xrightarrow{(j_{U, U \cap V}^W, j_{V; U \cap V}^W)} S_{\{W\};*}^{\mathrm{lf}}(X, \{U\}; \mathbb{Z}) \oplus H_{\{W\};*}^{\mathrm{lf}}(X, \{U\}; R)$$

$$\xrightarrow{j_U^W - j_V^W} S_{\{W\};*}^{\mathrm{lf}}(X, \{U, V\}; R) \longrightarrow 0$$

of chain complexes is then exact. Thus, the claim follows from the Snake Lemma and the second claim of Corollary 2.8 with $\mathcal{A} = \{U, V\}$ and $Y = W$. $\square$

Corollary 2.11 (Excision). *Let X be a topological space and $U, W \subset X$ be subspaces such that the closure of $X - U$ in X is contained in* Int W. *Then the homomorphism*

$$\iota_* : H_{\{W\};*}^{\mathrm{lf}}(X, \{U \cap W\}; R) \longrightarrow H_*^{\mathrm{lf}}(X, \{U\}; R) \tag{2.28}$$

induced by the inclusion $(W, U \cap W) \longrightarrow (X, U)$ is an isomorphism.

Proof. Let $\mathcal{A} = \{U, W\}$. The homomorphism (2.28) is induced by the composition

$$\frac{S_{\{W\};*}^{\mathrm{lf}}(X; R)}{S_{\{U \cap W\};*}^{\mathrm{lf}}(X; R)} \longrightarrow \frac{S_{\mathcal{A};*}^{\mathrm{lf}}(X; R)}{S_{\{U\};*}^{\mathrm{lf}}(X; R)} \longrightarrow \frac{S_*^{\mathrm{lf}}(X; R)}{S_{\{U\};*}^{\mathrm{lf}}(X; R)} \tag{2.29}$$

of homomorphisms of chain complexes. The first homomorphism above is an isomorphism. By the assumptions, the interiors of U and W cover X. By the first claim of Corollary 2.8 and the Five Lemma, the second homomorphism in (2.29) thus also induces an isomorphism in homology. $\square$

2.5 Fundamental class

For a topological space X, subsets $A \subset B \subset X$ and $W \subset X$, and a class $\mu \in H_{\{W\};*}^{\mathrm{lf}}(X, \{W - B\}; R)$, we denote by

$$\mu|_A \in H_{\{W\};*}^{\mathrm{lf}}(X, \{W - A\}; R)$$

the image of μ under the homomorphism

$$H_{\{W\};*}^{\mathrm{lf}}(X, \{W - B\}; R) \longrightarrow H_{\{W\};*}^{\mathrm{lf}}(X, \{W - A\}; R) \tag{2.30}$$

induced by the inclusion $(W, W - B) \longrightarrow (W, W - A)$.

Let X be an n-manifold and $B \subset X$ be a ball (open or closed) around a point $x \in X$. By Corollary 2.11 with $W = B$, (2.24), and the Kunneth formula,

$$H_k^{\mathrm{lf}}(X, \{X - \{x\}\}; R) \approx H_{\{B\};k}^{\mathrm{lf}}(X, \{B - \{x\}\}; R) = H_k(B, B - \{x\}; R) \approx \begin{cases} R, & \text{if } k = n; \\ \{0\}, & \text{otherwise.} \end{cases}$$

An R-orientation for X at $x \in X$ is a choice of generator $\mu_x \in H_n^{\mathrm{lf}}(X, X - \{x\}; R)$. An R-orientation for X is a collection $(\mu_x)_{x \in X}$ of R-orientations for X at x so that for every $x \in X$ there exist a neighborhood $U \subset X$ of x and $\mu_U \in H_k^{\mathrm{lf}}(X, \{X - U\}; R)$ such that

$$\mu_U\big|_y = \mu_y \in H_n^{\mathrm{lf}}\big(X, \{X - \{y\}\}; R\big) \qquad \forall\, y \in U.$$

An R-oriented manifold is a pair $(X, (\mu_x)_{x \in X})$ consisting of a manifold X and an orientation $(\mu_x)_{x \in X}$ for X. By Proposition 2.12(3) below with $A = X$, an R-oriented n-manifold $(X, (\mu_x)_{x \in X})$ carries a fundamental class

$$[X] \equiv \mu_X \in H_n^{\mathrm{lf}}(X, \emptyset; R) \equiv H_n^{\mathrm{lf}}(X; R).$$

Proposition 2.12 (Fundamental Class)**.** *Let X be an n-manifold and $A \subset X$ be a closed subset.*

(1) For every $k > n$, $H_k^{\mathrm{lf}}(X, \{X - A\}; R) = 0$.

(2) An element $\mu_A \in H_n^{\mathrm{lf}}(X, \{X - A\}; R)$ is zero if and only if

$$\mu_A\big|_x = 0 \in H_n^{\mathrm{lf}}\big(X, \{X - \{x\}\}; R\big) \qquad \forall\, x \in A.$$

(3) If $(\mu_x)_{x \in X}$ is an R-orientation on X, there exists a unique $\mu_A \in H_n^{\mathrm{lf}}(X, \{X - A\}; R)$ such that

$$\mu_A\big|_x = \mu_x \in H_n^{\mathrm{lf}}\big(X, \{X - \{x\}\}; R\big) \qquad \forall\, x \in A. \tag{2.31}$$

Proof of Proposition 2.12(1),(2). The proof is divided into four steps.

Case 1. Suppose A is compact. Let $U \subset X$ be a precompact open neighborhood of A. By Corollary 2.11 with $W = U$ and (2.24),

$$H_*^{\mathrm{lf}}\big(X, \{X - A\}; R\big) \approx H_{\{U\};*}^{\mathrm{lf}}\big(X, \{U - A\}; R\big) = H_*\big(U, U - A; R\big). \tag{2.32}$$

The two claims in this case thus follow from [8, Lemma A.7].

Case 2. Suppose A is the union of a locally finite collection $\mathcal{A}$ of disjoint compact subsets of X. Let $\{U_B : B \in \mathcal{A}\}$ be a locally finite collection of disjoint precompact open subsets of X so that $B \subset U_B$ for every $B \in \mathcal{A}$. Let $U \subset X$ be the union of the subsets U_B. By Corollary 2.11 with $W = U$ and (2.25),

$$\begin{aligned}
H_*^{\mathrm{lf}}\big(X, \{X - A\}; R\big) &\approx H_{\{U\};*}^{\mathrm{lf}}\big(X, \{U - A\}; R\big) \\
&\approx \prod_{B \in \mathcal{A}} H_{\{U_B\};*}^{\mathrm{lf}}\big(X, \{U_B - B\}; R\big) \approx \prod_{B \in \mathcal{A}} H_*^{\mathrm{lf}}\big(X, \{X - B\}; R\big).
\end{aligned} \tag{2.33}$$

The composition of the above isomorphism with the projection to the B-th component of the product is the restriction homomorphism

$$H_*^{\mathrm{lf}}\big(X, \{X - A\}; R\big) \longrightarrow H_*^{\mathrm{lf}}\big(X, \{X - B\}; R\big). \tag{2.34}$$

The two claims in this case thus follow from Case 1.

Case 3. Suppose $A_1, A_2 \subset X$ are closed, $A = A_1 \cup A_2$, and the two claims hold for the subsets $A_1, A_2, A_1 \cap A_2$ of X. By Corollary 2.10 with $W = X$, $U = X - A_1$, and $V = X - A_2$, there is an exact sequence

$$\ldots \longrightarrow H_{k+1}^{\mathrm{lf}}(X, \{X - A_1 \cap A_2\}; R) \longrightarrow H_k^{\mathrm{lf}}(X, \{X - A\}; R)$$
$$\longrightarrow H_k^{\mathrm{lf}}(X, \{X - A_1\}; R) \oplus H_k^{\mathrm{lf}}(X, \{X - A_2\}; R) \longrightarrow \ldots \tag{2.35}$$

Thus, the two claims also hold for A.

Case 4. A is arbitrary. Let $\{A_i\}_{i \in \mathbb{Z}}$ be a locally finite collection of compact subsets of X such that

$$A = \bigcup_{i \in \mathbb{Z}} A_i \qquad \text{and} \qquad A_i \cap A_j = \emptyset \ \text{ if } |i - j| > 1.$$

By Case 2, the two claims hold for the subsets

$$A_{odd} \equiv \bigcup_{i \in \mathbb{Z}} A_{2i-1}, \qquad A_{even} \equiv \bigcup_{i \in \mathbb{Z}} A_{2i}, \quad \text{and} \quad A_{odd} \cap A_{even} = \bigcup_{i \in \mathbb{Z}} A_i \cap A_{i+1}$$

of X. By Case 3, the two claims hold for $A \equiv A_{odd} \cup A_{even}$ as well. $\qquad \square$

Proof of Proposition 2.12(3). The uniqueness of μ_A follows immediately from the second claim of the proposition. The uniqueness property implies that

$$\mu_{A'} = \mu_A|_{A'} \in H_n^{\mathrm{lf}}(X, \{X - A'\}; R) \tag{2.36}$$

whenever $A' \subset A$ and an element $\mu_A \in H_n^{\mathrm{lf}}(X, \{X - A\}; R)$ satisfying (2.31) exists. The existence proof is again divided into four steps.

Case 1. Suppose A is compact. Let $U \subset X$ be a precompact open neighborhood of A. The claim in this case follows from (2.32) with $* = n$ and [8, Theorem A.8].

Case 2. Suppose A is the union of a locally finite collection $\mathcal{A}$ of disjoint compact subsets of X. Let $\{U_B : B \in \mathcal{A}\}$ and $U \subset X$ be as in Case 2 in the proof of Proposition 2.12(1),(2). Since the composition of the isomorphism (2.33) with the projection to the B-th component of the product is the restriction homomorphism (2.34), the preimage μ_A of the element $(\mu_B)_{B \in \mathcal{A}}$ under this isomorphism satisfies (2.31).

Case 3. Suppose $A_1, A_2 \subset X$ are closed, $A = A_1 \cup A_2$, and the claim holds for the subsets $A_1, A_2, A_1 \cap A_2$ of X. By the first claim of the proposition, the long exact sequence (2.35) becomes

$$0 \longrightarrow H_n^{\mathrm{lf}}(X, \{X - A\}; R) \longrightarrow H_n^{\mathrm{lf}}(X, \{X - A_1\}; R) \oplus H_n^{\mathrm{lf}}(X, \{X - A_2\}; R)$$
$$\longrightarrow H_n^{\mathrm{lf}}(X, \{X - A_1 \cap A_2\}; R) \longrightarrow \ldots$$

By (2.36), $\mu_{A_1}|_{A_1 \cap A_2} = \mu_{A_1 \cap A_2} = \mu_{A_2}|_{A_1 \cap A_2}$. Thus, there exists

$$\mu_A \in H_n^{\mathrm{lf}}(X, \{X - A\}; R) \qquad \text{s.t.} \quad \mu_A|_{A_1} = \mu_{A_1}, \quad \mu_A|_{A_2} = \mu_{A_2}.$$

Since $\mu_A|_x = \mu_{A_i}|_x$ for all $x \in A_i$, μ_A satisfies (2.31).

Case 4. A is arbitrary. Let $\{A_i\}_{i\in\mathbb{Z}}$ be as in Case 4 in the proof of Proposition 2.12(1),(2). By Case 2, the claim holds for the subsets

$$A_{odd} \equiv \bigcup_{i\in\mathbb{Z}} A_{2i-1} \qquad \text{and} \qquad A_{even} \equiv \bigcup_{i\in\mathbb{Z}} A_{2i}.$$

By Case 3, the claims holds for $A \equiv A_{odd} \cup A_{even}$ as well. $\qquad\square$

2.6 Poincaré Duality

For a collection $\mathcal{A}$ of subsets of a topological space X and a subset $W \subset X$ containing every $U \in \mathcal{A}$, the homomorphism (2.14) induces a homomorphism

$$\cap \colon S^q(W;R) \otimes_R S^{\mathrm{lf}}_{\{W\};p+q}(X,\mathcal{A};R) \longrightarrow S^{\mathrm{lf}}_{\{W\};p}(X,\mathcal{A};R).$$

The latter in turn induces a natural homomorphism

$$\cap \colon H^q(W;R) \otimes_R H^{\mathrm{lf}}_{\{W\};p+q}(X,\mathcal{A};R) \longrightarrow H^{\mathrm{lf}}_{\{W\};p}(X,\mathcal{A};R). \tag{2.37}$$

For $U, W' \subset W$, let

$$\{\iota_{W,W'}\}_* \colon H^{\mathrm{lf}}_{\{W'\};p}\big(X,\{U\cap W'\};R\big) \longrightarrow H^{\mathrm{lf}}_{\{W\};p}\big(X,\{U\};R\big)$$

be the homomorphism induced by the inclusion $(W', U\cap W') \longrightarrow (W,U)$. By the naturality of (2.37),

$$\begin{aligned}
\{\iota_{W,W'}\}_*\big((\alpha|_{W'})\cap\mu\big) = \alpha\cap\big(\{\iota_{W,W'}\}_*(\mu)\big) &\in H^{\mathrm{lf}}_{\{W\};p}\big(X,\{U\};R\big) \\
&\forall\, \alpha \in H^q(W;R),\ \mu \in H^{\mathrm{lf}}_{\{W'\};p+q}\big(X,\{U\cap W'\};R\big).
\end{aligned} \tag{2.38}$$

For subsets U, W of a topological space X such that the closure of $X-U$ in X is contained in Int W and $\mu \in H^{\mathrm{lf}}_*(X,\{U\};R)$, we denote by

$$\mu|_W \in H^{\mathrm{lf}}_{\{W\};*}\big(X,\{U\cap W\};R\big)$$

the preimage of μ under the excision isomorphism (2.28). If $W' \subset W$ is another subset such that the closure of $X-U$ in X is contained in Int W', then

$$\begin{aligned}
\{\iota_{X,W'}\}_* = \{\iota_{X,W}\}_* \circ \{\iota_{W,W'}\}_* \colon H^{\mathrm{lf}}_{\{W'\};*}\big(X,\{U\cap W'\};R\big) &\longrightarrow H^{\mathrm{lf}}_{\{W\};*}\big(X,\{U\cap W\};R\big) \\
&\longrightarrow H^{\mathrm{lf}}_*\big(X,\{U\};R\big)
\end{aligned} \tag{2.39}$$

and thus

$$\{\iota_{W,W'}\}_*\big(\mu|_{W'}\big) = \mu|_W \in H^{\mathrm{lf}}_{\{W\};*}\big(X,\{U\cap W\};R\big) \quad \forall \mu \in H^{\mathrm{lf}}_*\big(X,\{U\};R\big). \tag{2.40}$$

Let $(X,(\mu_x)_{x\in X})$ be an R-oriented n-manifold, $A \subset X$ a closed subset, and

$$\mu_A \in H^{\mathrm{lf}}_n\big(X,\{X-A\};R\big)$$

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the fundamental class provided by Proposition 2.12(3). Suppose $U_A \subset X$ is an open neighborhood of A that deformation retracts onto A. Thus, the restriction homomorphism

$$H^*(U_A; R) \longrightarrow H^*(A; R), \qquad \alpha \longmapsto \alpha|_A,$$

is an isomorphism. It follows that the homomorphism

$$\begin{aligned} \mathrm{PD}_{A;U_A} \colon H^k(A; R) &\longrightarrow H^{\mathrm{lf}}_{n-k}(X, \{X-A\}; R), \\ \mathrm{PD}_{A;U_A}(\alpha|_A) &= \{\iota_{X,U_A}\}_*(\alpha \cap (\mu_A|_{U_A})) \quad \forall\, \alpha \in H^k(U_A; R), \end{aligned} \tag{2.41}$$

is well-defined.

If $B \supset A$ is another closed subset of X and $U_B \subset X$ is an open neighborhood of B that deformation retracts onto B and contains U_A,

$$\mu_A = \mu_B|_A \in H^{\mathrm{lf}}_n(X, \{X-A\}; R),$$
$$\{\iota_{U_B,U_A}\}_*(\mu_A|_{U_A}) = \mu_A|_{U_B} = (\mu_B|_{U_B})\big|_A \in H^{\mathrm{lf}}_{\{U_B\};n}(X, \{U_B-A\}; R) \tag{2.42}$$

by the uniqueness part of Proposition 2.12(3), (2.40), and the commutativity of the diagram

$$\begin{array}{ccc} H^{\mathrm{lf}}_{\{U_B\};*}(X, \{U_B-B\}; R) & \xrightarrow{\ \cdot|_A\ } & H^{\mathrm{lf}}_{\{U_B\};*}(X, \{U_B-A\}; R) \\ {\scriptstyle\approx}\big\downarrow & & \big\downarrow{\scriptstyle\approx} \\ H^{\mathrm{lf}}_*(X, \{X-B\}; R) & \xrightarrow{\ \cdot|_A\ } & H^{\mathrm{lf}}_*(X, \{X-A\}; R)\,. \end{array} \tag{2.43}$$

Along with (2.38), (2.42) gives

$$\begin{aligned} \{\iota_{U_B,U_A}\}_*(\alpha|_{U_A} \cap (\mu_A|_{U_A})) &= \alpha \cap (\mu_A|_{U_B}) = \alpha \cap ((\mu_B|_{U_B})|_A) \\ &= (\alpha \cap (\mu_B|_{U_B}))\big|_A \in H^{\mathrm{lf}}_{\{U_B\};n-k}(X, \{U_B-A\}; R) \quad \forall\, \alpha \in H^k(U_B; R). \end{aligned}$$

Combining this with (2.39) and the commutativity of (2.43), we conclude that the diagram

$$\begin{array}{ccc} H^k(B; R) & \xrightarrow{\ \cdot|_A\ } & H^k(A; R) \\ {\scriptstyle\mathrm{PD}_{B;U_B}}\big\downarrow & & \big\downarrow{\scriptstyle\mathrm{PD}_{A;U_A}} \\ H^{\mathrm{lf}}_{n-k}(X, \{X-B\}; R) & \xrightarrow{\ \cdot|_A\ } & H^{\mathrm{lf}}_{n-k}(X, \{X-A\}; R) \end{array} \tag{2.44}$$

commutes.

By the commutativity of (2.44) with $A = B$, the homomorphism (2.41) does not depend on the choice of U_A if A is a neighborhood retract, i.e. every open neighborhood $W \subset X$ of A contains an open neighborhood U_A of A that deformation retracts onto A. This is in particular the case if $A \subset X$ is a closed submanifold with corners. If $A \subset X$ is a closed neighborhood retract, we denote by

$$\mathrm{PD}_A \colon H^k(A; R) \longrightarrow H^{\mathrm{lf}}_{n-k}(X, \{X-A\}; R) \tag{2.45}$$

the homomorphism (2.41) for any admissible neighborhood U_A of A. For $A = X$, this homomorphism is given by

$$\mathrm{PD}_X \colon H^k(X;R) \longrightarrow H^{\mathrm{lf}}_{n-k}(X;R), \qquad \mathrm{PD}_X(\alpha) = \alpha \cap [X].$$

If $A \subset B \subset X$ are closed neighborhood retracts, the commutativity of (2.44) implies that the diagram

$$
\begin{array}{ccc}
H^k(B;R) & \xrightarrow{\ \cdot|_A\ } & H^k(A;R) \\
{\scriptstyle \mathrm{PD}_B}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{PD}_A} \\
H^{\mathrm{lf}}_{n-k}(X,\{X-B\};R) & \xrightarrow{\ \cdot|_A\ } & H^{\mathrm{lf}}_{n-k}(X,\{X-A\};R)
\end{array}
\tag{2.46}
$$

commutes as well.

Proposition 2.13 (Poincaré Duality). *Let $(X,(\mu_x)_{x\in X})$ be an R-oriented n-manifold. If $A \subset X$ is a closed n-submanifold with corners, the homomorphism (2.45) is an isomorphism.*

Proof. The proof is again divided into four steps.

Case 1. Suppose A is compact. Let $U \subset X$ be a precompact open neighborhood of A that deformation retracts onto A. Combining the isomorphism (2.32) with the homotopy invariance of the standard singular homology for $(U, U-A) \approx (A, \partial A)$, we obtain

$$H^{\mathrm{lf}}_*(X,\{X-A\};R) \approx H_*(A,\partial A;R). \tag{2.47}$$

Since μ_A corresponds to the standard fundamental class $[A,\partial A] \in H_n(A,\partial A;R)$ under this isomorphism, the diagram

$$
\begin{array}{ccccc}
H^k(A;R) & \xrightarrow{\ \mathrm{id}\ } & H^k(A;R) & & \alpha \\
{\scriptstyle \mathrm{PD}_{A;U}}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{PD}_{(A,\partial A)}} & & \Big\downarrow{\scriptstyle \mathrm{PD}_{(A,\partial A)}} \\
H^{\mathrm{lf}}_*(X,\{X-A\};R) & \xrightarrow[\approx]{(2.47)} & H_*(A,\partial A;R) & & \alpha\cap[A,\partial A]
\end{array}
$$

commutes. Since $(A,\partial A)$ is a compact topological manifold with boundary, $\mathrm{PD}_{(A,\partial A)}$ is an isomorphism by the compact case of [8, Exercise A.1] and the $(M,A,B) = (A,\emptyset,\partial A)$ case of [4, Theorem 3.43]. Thus, $\mathrm{PD}_{A;U}$ is an isomorphism as well.

Case 2. Suppose A is the union of a locally finite collection $\mathcal{A}$ of disjoint compact subsets of X so that each $B \in \mathcal{A}$ is an n-submanifold with corners. Let $\{U_B \colon B \in \mathcal{A}\}$ and $U \subset X$ be as in Case 2 in the proof of Proposition 2.12(1),(2) so that each U_B deformation retracts onto B. In particular, the restriction homomorphisms

$$H^*(A;R) \longrightarrow H^*(B;R) \quad \text{and} \quad H^*(U;R) \longrightarrow H^*(U_B;R)$$

induce isomorphisms

$$H^*(A;R) \approx \prod_{B\in\mathcal{A}} H^*(B;R) \quad \text{and} \quad H^*(U;R) \approx \prod_{B\in\mathcal{A}} H^*(U_B;R), \tag{2.48}$$

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respectively. Since μ_A corresponds to $(\mu_B)_{B\in\mathcal{A}}$ under the isomorphism (2.33), the diagram

$$
\begin{array}{ccc}
H^k(A;R) & \xrightarrow[\approx]{\text{(2.48)}} & \prod_{B\in\mathcal{A}} H_*(B;R) \\
{\scriptstyle \mathrm{PD}_{A;U}}\Big\downarrow & & \Big\downarrow{\scriptstyle \prod_{B\in\mathcal{A}}\mathrm{PD}_{B;U_B}} \\
H^{\mathrm{lf}}_*(X,\{X-A\};R) & \xrightarrow[\approx]{\text{(2.33)}} & \prod_{B\in\mathcal{A}} H^{\mathrm{lf}}_*(X,\{X-B\};R)
\end{array}
$$

commutes. Thus, $\mathrm{PD}_{A;U}$ is an isomorphism by Case 1.

Case 3. Suppose $A_1, A_2, A_1\cap A_2\subset X$ are closed n-submanifolds with corners which satisfy the claim and $A=A_1\cup A_2$. For a subspace $B\subset X$, let

$$
\mathscr{H}_*(B) = H^{\mathrm{lf}}_*\big(X,\{X-B\};R\big) \qquad \text{and} \qquad \mathscr{H}^*(B) = H^*(B;R).
$$

Let $A_{12}=A_1\cap A_2$. For $i=1,2$, define

$$
\begin{aligned}
_i\colon \mathscr{H}_*(A) &\longrightarrow \mathscr{H}_*(A_i), & j_i\colon \mathscr{H}_*(A_i) &\longrightarrow \mathscr{H}_*(A_{12}), \\
\iota_i^*\colon \mathscr{H}^*(A) &\longrightarrow \mathscr{H}^*(A_i), & j_i^*\colon \mathscr{H}^*(A_i) &\longrightarrow \mathscr{H}^*(A_{12})
\end{aligned}
$$

to be the homology homomorphisms as in (2.30) and the usual cohomology restriction homomorphisms. By Mayer-Vietoris for the standard singular cohomology and Corollary 2.10 with $W=X$, $U=X-A_1$, and $V=X-A_2$, the rows in the diagram

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & \mathscr{H}^{k-1}(A_{12}) & \xrightarrow{\delta} & \mathscr{H}^k(A) & \xrightarrow{(\iota_1^*,\iota_2^*)} & \mathscr{H}^k(A_1)\oplus\mathscr{H}^k(A_2) & \xrightarrow{j_1^*-j_2^*} & \mathscr{H}^k(A_{12}) & \xrightarrow{\delta} \cdots \\
& {\scriptstyle \mathrm{PD}_{A_{12}}}\Big\downarrow & & {\scriptstyle \mathrm{PD}_A}\Big\downarrow & & {\scriptstyle \mathrm{PD}_{A_1}\oplus\mathrm{PD}_{A_2}}\Big\downarrow & & {\scriptstyle \mathrm{PD}_{A_{12}}}\Big\downarrow & \\
\cdots \longrightarrow & \mathscr{H}_{n-k+1}(A_{12}) & \xrightarrow{\partial} & \mathscr{H}_{n-k}(A) & \xrightarrow{(\iota_1,\iota_2)} & \mathscr{H}_{n-k}(A_1)\oplus\mathscr{H}_{n-k}(A_2) & \xrightarrow{j_1-j_2} & \mathscr{H}_{n-k}(A_{12}) & \xrightarrow{\partial} \cdots
\end{array}
$$

are exact. The second and third squares above commute by the commutativity of (2.46). By (2.42) and (2.15), the first square commutes up to the multiplication by $(-1)^{n-k+1}$. Since the homomorphisms $\mathrm{PD}_{A_{12}}, \mathrm{PD}_{A_1}, \mathrm{PD}_{A_2}$ are isomorphisms, the Five Lemma implies that so are the homomorphisms PD_A.

Case 4. A is arbitrary. Let $\{A_i\}_{i\in\mathbb{Z}}$ be as in Case 4 in the proof of Proposition 2.12(1),(2) so that all $A_i, A_i\cap A_j\subset X$ are compact n-submanifolds with corners. By Case 2, the claim holds for the subsets

$$
A_{odd} \equiv \bigcup_{i\in\mathbb{Z}} A_{2i-1}, \qquad A_{even} \equiv \bigcup_{i\in\mathbb{Z}} A_{2i}, \quad \text{and} \quad A_{odd}\cap A_{even} = \bigcup_{i\in\mathbb{Z}} A_i\cap A_{i+1}
$$

of X. By Case 3, the claims holds for $A\equiv A_{odd}\cup A_{even}$ as well. $\qquad\square$

Remark 2.14. Let $A\subset X$ be a closed submanifold with corners and $f\colon X\longrightarrow\mathbb{R}$ a proper smooth function. Choose a collection $(a_i)_{i\in\mathbb{Z}}$ of regular values of f and its restrictions to the strata of A so that

$$
a_i < a_j \ \ \forall\, i<j, \qquad \lim_{i\to-\infty} a_i = -\infty, \qquad \lim_{i\to\infty} a_i = \infty.
$$

A decomposition as in Case 4 can then be obtained by taking $A_i=f^{-1}([a_{2i-1}, a_{2i+2}])$.

3 Proof of Theorem 1.3

3.1 Homology of neighborhoods of smooth maps

The next proposition is an analogue of Proposition 2.2 in [13] for the Borel-Moore homology groups used in this paper.

Proposition 3.1. *Let $h\colon Y \longrightarrow X$ be a smooth map between manifolds, $A \subset X$ be a closed subset so that $A \subset h(Y)$, and $W \subset X$ be an open neighborhood of A. There exists an open neighborhood $U \subset W$ of A such that*

$$H^{\mathrm{lf}}_{\{U\};l}(X; R) = 0 \qquad \text{if } l > \dim Y.$$

If $h\colon Y \longrightarrow X$ is a smooth map and k is a nonnegative integer, put

$$N_k(h) = \big\{ y \in Y : \mathrm{rk}\, \mathrm{d}_y h \le k \big\}.$$

Proposition 3.1 follows from Lemma 3.2 applied with $k = \dim Y$.

For a simplicial complex K, we denote by $|K|$ a geometric realization of K in a Euclidean space in the sense of [9, Section 3] and by $\mathrm{sd}\, K$ the barycentric subdivision of K. The simplicies of $\mathrm{sd}\, K$ are the sets

$$\tau = b_{\sigma_1} \ldots b_{\sigma_j} \equiv \big\{ b_{\sigma_1}, \ldots, b_{\sigma_j} \big\} \qquad \text{with} \quad \sigma_1, \ldots, \sigma_j \in K, \quad \sigma_1 \subsetneq \ldots \subsetneq \sigma_j.$$

In a geometric realization $|K| = |\mathrm{sd}\, K|$, b_σ corresponds to the barycenter of the simplex $\sigma \in K$. For $l \in \mathbb{Z}^{\ge 0}$, denote by $K_l \subset K$ the l-skeleton of K.

For a simplex $\sigma \in K$, let

$$\mathrm{St}(\sigma, K) = \bigcup_{\substack{\sigma' \in K \\ \sigma \subset \sigma'}} \mathrm{Int}\, \sigma' \subset |K|$$

be the (open) star of σ in K; see [9, Section 62]. Its closure in $|K|$ is the closed star

$$\overline{\mathrm{St}}(\sigma, K) \equiv \bigcup_{\substack{\sigma' \in K \\ \sigma \subset \sigma'}} |\sigma'| \subset |K|$$

of σ in K and is in particular compact.

A triangulation of a manifold X is a pair $T = (K, \eta)$ consisting of a simplicial complex and a homeomorphism $\eta\colon |K| \longrightarrow X$ such that $\eta|_{\mathrm{Int}\,\sigma}$ is smooth for every simplex $\sigma \in K$.

Lemma 3.2. *Let $h\colon Y \longrightarrow X$ be a smooth map and $k \in \mathbb{Z}^{\ge 0}$. For every closed subset $A \subset X$ such that $A \subset h(N_k(h))$ and an open neighborhood $W \subset X$ of A, there exists an open neighborhood $U \subset W$ of A such that*

$$H^{\mathrm{lf}}_{\{U\};l}(X; R) = 0 \qquad \text{if } l > k. \tag{3.1}$$

Proof. Let $n = \dim X$. Since the open subsets $X - A, W \subset X$ cover X, there exists a triangulation $T = (K, \eta)$ of X such that the image of every simplex $\sigma \in K$ is contained either in $X - A$ or in W.

By the proof of [14, Theorem 1], we can also assume that the smooth map h is transverse to $\eta|_{\mathrm{Int}\,\sigma}$ for every $\sigma \in K$. In particular,

$$h\big(N_k(h)\big) \subset \eta\big(|K| - |K_{n-1-k}|\big) = \bigcup_{\substack{\sigma \in K \\ \dim \sigma \geq n-k}} \eta(\mathrm{Int}\,\sigma).$$

Since $A \subset h(N_k(h))$, it follows that

$$A \subset U \equiv \bigcup_{\substack{\sigma \in K \\ \dim \sigma \geq n-k \\ \eta(|\sigma|) \cap A \neq \emptyset}} \eta(\mathrm{Int}\,\sigma) = \bigcup_{\substack{\sigma \in K \\ \dim \sigma \geq n-k \\ \eta(|\sigma|) \cap A \neq \emptyset}} \eta\big(\mathrm{St}(b_\sigma, \mathrm{sd}\,K)\big) \subset \bigcup_{\substack{\sigma \in K \\ \dim \sigma \geq n-k \\ \eta(|\sigma|) \cap A \neq \emptyset}} \eta\big(|\sigma|\big) \subset W.$$

We show below that the open neighborhood $U \subset W$ of A satisfies (3.1), adapting the proof of [13, Lemma 2.4].

For each $m \in [n]$, let

$$U_m = \bigcup_{\substack{\sigma \in K \\ \dim \sigma = m \\ \eta(\sigma) \cap A \neq \emptyset}} \eta\big(\mathrm{St}(b_\sigma, \mathrm{sd}\,K)\big) \subset W.$$

For $m_1, \ldots, m_j \in [n]$ with $m_1 < \ldots < m_j$, let

$$\mathcal{A}_{m_1 \ldots m_j} = \big\{(\sigma_1, \ldots, \sigma_j) \in K^j : \sigma_1 \subset \ldots \subset \sigma_j, \ \dim \sigma_1 = m_1, \ \ldots, \ \dim \sigma_j = m_j, \ \eta(\sigma_1) \cap A \neq \emptyset\big\}.$$

We note that

$$\mathrm{St}(b_\sigma, \mathrm{sd}\,K) \cap \mathrm{St}(b_{\sigma'}, \mathrm{sd}\,K) = \emptyset \qquad\qquad \text{if } \sigma \not\subset \sigma' \text{ and } \sigma \not\supset \sigma',$$
$$\mathrm{St}(b_{\sigma_1}, \mathrm{sd}\,K) \cap \ldots \cap \mathrm{St}(b_{\sigma_j}, \mathrm{sd}\,K) = \mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K) \qquad \text{if } \sigma_1 \subset \ldots \subset \sigma_j.$$

Thus, every intersection $U_{m_1} \cap \ldots \cap U_{m_j}$ with $m_1 < \ldots < m_j$ is a disjoint union of the open stars $\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K))$ with $(\sigma_1, \ldots, \sigma_j) \in \mathcal{A}_{m_1 \ldots m_j}$.

Since the collection $\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K))$ with $(\sigma_1, \ldots, \sigma_j) \in \mathcal{A}_{m_1 \ldots m_j}$ is locally finite in X and consists of disjoint subsets, (2.25) gives

$$H^{\mathrm{lf}}_{\{U_{m_1} \cap \ldots \cap U_{m_j}\};*}(X; R) = \prod_{(\sigma_1, \ldots, \sigma_j) \in \mathcal{A}_{m_1 \ldots m_j}} H^{\mathrm{lf}}_{\{\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K))\};*}(X; R) \tag{3.2}$$

for all $m_1, \ldots, m_j \in [n]$ with $m_1 < \ldots < m_j$. Since the closure of each contractible subset $\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K))$ in X is compact, (2.24) gives

$$H^{\mathrm{lf}}_{\{\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K))\};l}(X; R) = H_l\big(\eta(\mathrm{St}(b_{\sigma_1} \ldots b_{\sigma_j}, \mathrm{sd}\,K)); R\big) = 0 \ \ \forall\, l \neq 0, \ (\sigma_1, \ldots, \sigma_j) \in \mathcal{A}_{m_1 \ldots m_j}.$$

Combining this with (3.2), we obtain

$$H^{\mathrm{lf}}_{\{U_{m_1} \cap \ldots \cap U_{m_j}\};l}(X; R) = 0 \quad \forall\, l \geq 1. \tag{3.3}$$

By induction on $j = 1, 2, \ldots$, Corollary 2.9 (Mayer-Vietoris) and (3.3) give

$$H^{\mathrm{lf}}_{\{U_{m_1} \cup \ldots \cup U_{m_j}\};l}(X; R) = 0 \quad \forall\, l \geq j.$$

Since $U = U_{n-k} \cup \ldots \cup U_n$, this gives (3.1). $\qquad\square$

3.2 Oriented Borel-Moore homology

The construction of the oriented singular chain complex $\overline{S}_*(X;\mathbb{Z})$ in [13, Section 2.3] readily extends to locally finite chains. Cycles are much easier to construct in the resulting quotient chain complexes $\overline{S}_*^{\mathrm{lf}}(X;R)$ and $\overline{S}_*^{\mathrm{lf}}(X,\{U\};R)$. By Proposition 3.3 below, the homologies $\overline{H}_*^{\mathrm{lf}}(X;R)$ of $\overline{S}_*^{\mathrm{lf}}(X;R)$ and $\overline{H}_*^{\mathrm{lf}}(X,\{U\};R)$ of $\overline{S}_*^{\mathrm{lf}}(X,\{U\};R)$ are naturally isomorphic to $H_*^{\mathrm{lf}}(X;R)$ and $H_*^{\mathrm{lf}}(X,\{U\};R)$, respectively.

For $k\in\mathbb{Z}^{\geq 0}$ and $\tau\in\mathcal{S}_k$, let
$$\widetilde{\tau} = \mathrm{Id}_{\Delta^k} - (\mathrm{sign}\,\tau)\tau \in S_k(\Delta^k;R).$$
For a topological space X, let
$$S_k'(X;R) \subset S_k(X;R)$$
be the R-submodule generated by the chains $\sigma_\#(\widetilde{\tau})\in S_k(X;R)$ with $\sigma\in\mathrm{Hom}(\Delta^k,X)$ and $\tau\in\mathcal{S}_k$. In the notation (2.4), define
$$S_k'^{\mathrm{lf}}(X;R) = \Big\{ \sum_{\sigma\in\mathrm{Hom}(\Delta^k,X)} \sum_{\tau\in\mathcal{S}_k} a_{\sigma,\tau}\sigma_\#(\widetilde{\tau})\in S_k^{\mathrm{lf}}(X;R)\colon a_{\sigma,\tau}\in R\Big\}.$$

In the perspective of (2.3), $S_k'^{\mathrm{lf}}(X;R)$ consists of the singular chains $c\in S_k^{\mathrm{lf}}(X;R)$ such that
$$c\big|_{\mathcal{S}_k\sigma} \in \big\{\sigma_\#(c')|_{\mathcal{S}_k\sigma}\colon c'\in S_k'(\Delta^k;R)\big\} \quad \forall\,\sigma\in\mathrm{Hom}(\Delta^k,X), \quad \text{where}\quad \mathcal{S}_k\sigma \equiv \{\sigma\circ\tau\colon \tau\in\mathcal{S}_k\}.$$
If in addition $U\subset X$, let
$$S_{\{U\};*}'^{\mathrm{lf}}(X;R) = S_{\{U\};*}^{\mathrm{lf}}(X;R)\cap S_*'^{\mathrm{lf}}(X;R).$$

By [13, Lemma 2.6], $\partial\widetilde{\tau}\in S_{k-1}(\Delta^k)$ for all $\tau\in\mathcal{S}_k$ and $k\in\mathbb{Z}^{\geq 0}$. Thus, $S_*'^{\mathrm{lf}}(X;R)$ is a subcomplex of $(S_*^{\mathrm{lf}}(X;R),\partial_X)$ and $S_{\{U\};*}'^{\mathrm{lf}}(X;R)$ is a subcomplex of $(S_{\{U\};*}^{\mathrm{lf}}(X;R),\partial_X)$. Let
$$\overline{S}_*^{\mathrm{lf}}(X;R) = \frac{S_*^{\mathrm{lf}}(X;R)}{S_*'^{\mathrm{lf}}(X;R)}, \qquad \overline{S}_{\{U\};*}^{\mathrm{lf}}(X;R) = \frac{S_{\{U\};*}^{\mathrm{lf}}(X;R)}{S_{\{U\};*}'^{\mathrm{lf}}(X;R)} \subset \overline{S}_*^{\mathrm{lf}}(X;R),$$
$$\overline{S}_*^{\mathrm{lf}}(X,\{U\};R) = \frac{\overline{S}_*^{\mathrm{lf}}(X;R)}{\overline{S}_{\{U\};*}^{\mathrm{lf}}(X;R)}.$$

We denote the image of a Borel-Moore singular chain $c\in S_k^{\mathrm{lf}}(X;R)$ in $\overline{S}_k^{\mathrm{lf}}(X;R)$ by $\{c\}$, the induced boundary operator on $\overline{S}_k^{\mathrm{lf}}(X;R)$ by $\overline{\partial}_X$, and the homologies of the above three chain complexes by $\overline{H}_*^{\mathrm{lf}}(X;R)$, $\overline{H}_{\{U\};*}^{\mathrm{lf}}(X;R)$, and $\overline{H}_*^{\mathrm{lf}}(X,\{U\};R)$, respectively. The quotient projection maps on the chain complexes induce homomorphisms
$$H_*^{\mathrm{lf}}(X;R) \longrightarrow \overline{H}_*^{\mathrm{lf}}(X;R), \qquad H_{\{U\};*}^{\mathrm{lf}}(X;R) \longrightarrow \overline{H}_{\{U\};*}^{\mathrm{lf}}(X;R),$$
$$H_*^{\mathrm{lf}}(X,\{U\};R) \longrightarrow \overline{H}_*^{\mathrm{lf}}(X,\{U\};R). \tag{3.4}$$

If $h\colon X\longrightarrow Y$ is a proper continuous map between topological spaces and $f(U)\subset W\subset Y$, the induced homomorphism
$$h_\#\colon S_*^{\mathrm{lf}}(X;R) \longrightarrow S_*(Y;R)$$

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takes $S'^{\mathrm{lf}}_{\{U\};*}(X;R)$ into $S'^{\mathrm{lf}}_{\{W\};*}(Y;R)$. Thus, $h_\#$ induces homomorphisms

$$h_*\colon \overline{H}^{\mathrm{lf}}_*(X;R) \longrightarrow \overline{H}^{\mathrm{lf}}_*(Y;R), \qquad h_*\colon \overline{H}^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow \overline{H}^{\mathrm{lf}}_{\{W\};*}(Y;R),$$
$$h_*\colon \overline{H}^{\mathrm{lf}}_*(X,\{U\};R) \longrightarrow \overline{H}^{\mathrm{lf}}_*(Y,\{W\};R).$$

Proposition 3.3. *For any topological space X, the homomorphisms (3.4) are isomorphisms.*

Proof. The natural transformation of functors $D_X \colon S_* \longrightarrow S_{*+1}$ provided by [13, Lemma 2.7] satisfies

$$D_X\big(S'_k(X;R)\big) \subset S'_{k+1}(X;R) \quad\text{and}\quad \partial_X D_X\big|_{S'_k(X;R)} = \big\{(-1)^{k+1}\mathrm{Id} + D_X\partial_X\big\}\big|_{S'_k(X;R)}. \qquad (3.5)$$

Define
$$\hbar\colon \mathrm{Hom}(\Delta^k,X) \longrightarrow S_{k+1}(\Delta^k;R), \quad \hbar(\sigma) = D_{\Delta^k}\big(\mathrm{id}_{\Delta^k}\big).$$

By the naturality of D_X (or [13, (2.11)]),

$$D_X = \hbar_\#\colon S_k(X;R) \longrightarrow S_{k+1}(X;R).$$

By Lemma 2.2, D_X thus extends to a homomorphism

$$D_X = \hbar_\#\colon S^{\mathrm{lf}}_k(X;R) \longrightarrow S^{\mathrm{lf}}_{k+1}(X;R),$$

which is natural with respect to proper continuous maps. By (3.5),

$$D_X\big(S'^{\mathrm{lf}}_k(X;R)\big) \subset S'^{\mathrm{lf}}_{k+1}(X;R) \quad\text{and}\quad \partial_X D_X\big|_{S'^{\mathrm{lf}}_k(X;R)} = \big\{(-1)^{k+1}\mathrm{Id} + D_X\partial_X\big\}\big|_{S'^{\mathrm{lf}}_k(X;R)}. \qquad (3.6)$$

Thus, all homology groups of the chain complex $(S'^{\mathrm{lf}}_*(X;R), \partial_X|_{S'^{\mathrm{lf}}_*(X;R)})$ vanish. Combining this with the homology long exact sequence for the exact sequence of chain complexes

$$0 \longrightarrow S'^{\mathrm{lf}}_*(X;R) \longrightarrow S^{\mathrm{lf}}_*(X;R) \longrightarrow \overline{S}^{\mathrm{lf}}_*(X;R) \longrightarrow 0,$$

we conclude that the first homomorphism in (3.4) is an isomorphism.

Since $D_X(S^{\mathrm{lf}}_{\{U\};k}(X;R)) \subset S^{\mathrm{lf}}_{\{U\};k+1}(X;R)$,

$$D_X\big(S'^{\mathrm{lf}}_{\{U\};k}(X;R)\big) \subset S'^{\mathrm{lf}}_{\{U\};k+1}(X;R) \quad\text{and}\quad \partial_X D_X\big|_{S'^{\mathrm{lf}}_{\{U\};k}(X;R)} = \big\{(-1)^{k+1}\mathrm{Id} + D_X\partial_X\big\}\big|_{S'^{\mathrm{lf}}_{\{U\};k}(X;R)}.$$

Along with the second statement in (3.6) and the homology long exact sequence for the exact sequence of chain complexes

$$0 \longrightarrow S'^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow S^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow \overline{S}^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow 0,$$

this implies that the second homomorphism in (3.4) is an isomorphism. The claim for the third homomorphism in (3.4) follows from the homology long exact sequence for the exact sequence of chain complexes

$$0 \longrightarrow \overline{S}^{\mathrm{lf}}_{\{U\};*}(X;R) \longrightarrow \overline{S}^{\mathrm{lf}}_*(X;R) \longrightarrow \overline{S}^{\mathrm{lf}}_*(X,\{U\};R) \longrightarrow 0,$$

the claims for the first two homomorphisms, and the Five Lemma. $\qquad\square$

If X is a manifold, the operator D_X of [13, Lemma 2.7] sends smooth maps into linear combinations of smooth maps. Thus, the above constructions go through for the chain complexes based on elements in $C^\infty(\Delta^k, X)$ instead of $\mathrm{Hom}(\Delta^k, X)$. The two chain complexes define the same homology groups of X by Whitney Approximation Theorem [5, Theorem 6.21]. In Sections 3.3-3.5, all chain complexes and homology groups are based on smooth maps.

From now on, we restrict the coefficient ring R to $\mathbb{Z}$. We call a tuple $(\sigma_i)_{i \in \mathcal{I}}$ of elements of $\mathrm{Hom}(\Delta^k, X)$ locally finite if for every $x \in X$ there exists an open neighborhood $U_x \subset X$ so that the set

$$\aleph_{(\sigma_i)_{i \in \mathcal{I}}}(U_x) \equiv \left\{ i \in \mathcal{I} \colon \sigma_i(\Delta^k) \cap U_x \neq \emptyset \right\}$$

is finite. For any such collection,

$$c \equiv \sum_{i \in \mathcal{I}} \sigma_i \in S_k^{\mathrm{lf}}(X; \mathbb{Z}). \tag{3.7}$$

If $k \in \mathbb{Z}^+$, every element of $\overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z})$ can be represented by a chain as in (3.7) for some locally finite tuple $(\sigma_i)_{i \in \mathcal{I}}$ of elements of $\mathrm{Hom}(\Delta^k, X)$.

For c in (3.7), let

$$\mathcal{B}_c = \left\{ (i, p) \colon i \in \mathcal{I}, \ p \in [k] \right\}.$$

Lemmas 3.4 and 3.5 below will be used to glue the summands in chains c as in (3.7) that represent cycles and bounding chains in $\overline{S}_*^{\mathrm{lf}}(X; \mathbb{Z})$ into smooth maps from manifolds. The two lemmas are the direct extensions of Lemmas 2.10 and 2.11 in [13] to the Borel-Moore chains. They hold for the same reasons because the local finiteness conditions implies that each boundary simplex $\sigma_i \circ \iota_{k;p}$ with $(i, p) \in \mathcal{B}_c$ appears only finitely many times in $\partial_X c$.

Lemma 3.4. *If $k \in \mathbb{Z}^+$ and the chain (3.7) determines a cycle in $\overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z})$, there exist a subset $\mathcal{D}_c \subset \mathcal{B}_c \times \mathcal{B}_c$ disjoint from the diagonal and a map*

$$\tau \colon \mathcal{D}_c \longrightarrow \mathcal{S}_{k-1}, \qquad \left((i_1, p_1), (i_2, p_2) \right) \longrightarrow \tau_{(i_1, p_1), (i_2, p_2)},$$

with the following properties:

(i) if $\left((i_1, p_1), (i_2, p_2) \right) \in \mathcal{D}_c$, then $\left((i_2, p_2), (i_1, p_1) \right) \in \mathcal{D}_c$;

(ii) the projection $\mathcal{D}_c \longrightarrow \mathcal{B}_c$ on either coordinate is a bijection;

(iii) for all $\left((i_1, p_1), (i_2, p_2) \right) \in \mathcal{D}_c$,

$$\tau_{(i_1, p_1), (i_2, p_2)}^{-1} = \tau_{(i_2, p_2), (i_1, p_1)}, \qquad \sigma_{i_1} \circ \iota_{k; p_1} \circ \tau_{(i_1, p_1), (i_2, p_2)} = \sigma_{i_2} \circ \iota_{k; p_2}, \tag{3.8}$$

$$\text{and} \qquad \mathrm{sign}\, \tau_{(i_1, p_1), (i_2, p_2)} = -(-1)^{p_1 + p_2}. \tag{3.9}$$

Lemma 3.5. *Suppose $k \geq 1$, $(\sigma_{0;i})_{i \in \mathcal{I}_0}$ and $(\sigma_{1;i})_{i \in \mathcal{I}_1}$ are locally finite tuples of elements of $\mathrm{Hom}(\Delta^k, X)$, $(\widetilde{\sigma}_i)_{i \in \widetilde{\mathcal{I}}}$ is a locally finite tuple of elements of $\mathrm{Hom}(\Delta^{k+1}, X)$, and*

$$c_0 \equiv \sum_{i \in \mathcal{I}_0} \sigma_{0;i}, \quad c_1 \equiv \sum_{i \in \mathcal{I}_1} \sigma_{1;i}, \quad \widetilde{c} \equiv \sum_{i \in \widetilde{\mathcal{I}}} \widetilde{\sigma}_i, \quad \overline{\partial}\{\widetilde{c}\} = \{c_1\} - \{c_0\} \in \overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z}). \tag{3.10}$$

Then there exist a subset $\mathcal{D}_{\widetilde{c}} \subset \mathcal{B}_{\widetilde{c}} \times \mathcal{B}_{\widetilde{c}}$ disjoint from the diagonal, disjoint subsets $\mathcal{B}_{\widetilde{c}}^{(0)}, \mathcal{B}_{\widetilde{c}}^{(1)} \subset \mathcal{B}_{\widetilde{c}}$, and maps

$$\widetilde{\tau} \colon \mathcal{D}_{\widetilde{c}} \longrightarrow \mathcal{S}_k, \quad \big((i_1, p_1), (i_2, p_2)\big) \longrightarrow \widetilde{\tau}_{(i_1, p_1), (i_2, p_2)},$$

$$(\widetilde{\iota}_r, \widetilde{p}_r) \colon \mathcal{I}_r \longrightarrow \mathcal{B}_{\widetilde{c}}^{(r)}, \quad \text{and} \quad \widetilde{\tau}_r \colon \mathcal{I}_r \longrightarrow \mathcal{S}_k, \quad i \longrightarrow \widetilde{\tau}_{(r,i)}, \quad r = 0, 1,$$

with the following properties:

(i) if $\big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_{\widetilde{c}}$, then $\big((i_2, p_2), (i_1, p_1)\big) \in \mathcal{D}_{\widetilde{c}}$;

(ii) the projection $\mathcal{D}_{\widetilde{c}} \longrightarrow \mathcal{B}_{\widetilde{c}}$ on either coordinate is a bijection onto the complement of $\mathcal{B}_{\widetilde{c}}^{(0)} \cup \mathcal{B}_{\widetilde{c}}^{(1)}$;

(iii) for all $\big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_{\widetilde{c}}$,

$$\widetilde{\tau}_{(i_1, p_1), (i_2, p_2)}^{-1} = \widetilde{\tau}_{(i_2, p_2), (i_1, p_1)}, \qquad \widetilde{\sigma}_{i_1} \circ \iota_{k+1; p_1} \circ \widetilde{\tau}_{(i_1, p_1), (i_2, p_2)} = \widetilde{\sigma}_{i_2} \circ \iota_{k+1; p_2}, \tag{3.11}$$

$$\text{and} \qquad \operatorname{sign} \widetilde{\tau}_{(i_1, p_1), (i_2, p_2)} = -(-1)^{p_1 + p_2}; \tag{3.12}$$

(iv) for all $r = 0, 1$ and $i \in \mathcal{A}_r$,

$$\widetilde{\sigma}_{\widetilde{\iota}_r(i)} \circ \iota_{k+1; \widetilde{p}_r(i)} \circ \widetilde{\tau}_{(r,i)} = \sigma_{r;i} \qquad \text{and} \qquad \operatorname{sign} \widetilde{\tau}_{(r,i)} = -(-1)^{r + \widetilde{p}_r(i)}; \tag{3.13}$$

(v) $(\widetilde{\iota}_r, \widetilde{p}_r)$ is a bijection onto $\mathcal{B}_{\widetilde{c}}^{(r)}$ for $r = 0, 1$.

Suppose $\overline{V}$ is an oriented k-manifold with boundary and (K, η) is a triangulation of $\overline{V}$ that restricts to a triangulation of $\partial\overline{V}$. Let

$$K^{\mathrm{top}} = \big\{\sigma \in K : \dim \sigma = k\big\}.$$

For each k-dimensional simplex $\sigma \in K$, let

$$l_\sigma \colon \Delta^k \longrightarrow \sigma \subset |K| \subset \mathbb{R}^\infty \tag{3.14}$$

be a linear map such that the composition $\eta \circ l_\sigma$ is orientation-preserving. The fundamental class $[\overline{V}] \in \overline{H}_k^{\mathrm{lf}}(\overline{V}, \partial\overline{V}; \mathbb{Z})$ of M is then represented by

$$\sum_{\sigma \in K^{\mathrm{top}}} \big\{\eta \circ l_\sigma\big\} \in \overline{S}_k^{\mathrm{lf}}\big(\overline{V}, \{\partial\overline{V}\}; \mathbb{Z}\big).$$

The corresponding sum

$$\sum_{\sigma \in K^{\mathrm{top}}} \eta \circ l_\sigma \in S_k^{\mathrm{lf}}\big(\overline{V}, \{\partial\overline{V}\}; \mathbb{Z}\big)$$

may not be a cycle. If $f \colon \overline{V} \longrightarrow X$ is a proper map and $U \subset X$ is a subset containing $f(\partial\overline{V})$, then $f_*([\overline{V}]) \in \overline{H}_k^{\mathrm{lf}}(X, \{U\}; \mathbb{Z})$ is represented by

$$\sum_{\sigma \in K^{\mathrm{top}}} \big\{f \circ \eta \circ l_\sigma\big\} \in \overline{S}_k^{\mathrm{lf}}\big(X, \{U\}; \mathbb{Z}\big);$$

by the properness of f, the collection $\{f \circ \eta \circ l_\sigma\}_{\sigma \in K^{\mathrm{top}}}$ is locally finite in X.

3.3 From integral cycles to pseudocycles

In this section, we extend the constructions of [13, Section 3.1] from finite to locally finite singular chains and obtain the first homomorphism in (1.1). We start with a cycle $\{c\} \in \overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z})$ as in Lemma 3.4 and replace each singular simplex σ_i by its composition with the self-map φ_k of Δ^k provided by Lemma 2.1. The functions $\sigma \circ \varphi_k$ still satisfy the second equation in (3.8), i.e.

$$\sigma_{i_1} \circ \varphi_k \circ \iota_{k;p_1} \circ \tau_{(i_1,p_1),(i_2,p_2)} = \sigma_{i_2} \circ \varphi_k \circ \iota_{k;p_2} \quad \forall \; \big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_c, \tag{3.15}$$

because φ_k restricts to the identity on the boundary of Δ^k. This allows us to glue the maps $\sigma_i \circ \varphi_k$ into a proper map F from a k-dimensional simplicial complex M to X. Removing the codimension 2 simplicies, we obtain a Borel-Moore pseudocycle in the proof of Lemma 3.6. In the proof of Lemma 3.7, we use a similar procedure to turn a bounding chain $\{\widetilde{c}\} \in \overline{S}_{k+1}^{\mathrm{lf}}(X; \mathbb{Z})$ into a Borel-Moore pseudocycle equivalence between the Borel-Moore pseudocycles determined by its boundaries.

Lemma 3.6. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. Every integer locally finite singular k-chain c as in (3.7) with $\sigma_i \in C^\infty(\Delta^k; X)$ for all $i \in \mathcal{I}$ representing a cycle in $\overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z})$ determines an element of $\mathcal{H}_k^{\mathrm{cl}}(X)$.*

Proof. If $k = 0$, $(\sigma_i)_{i \in \mathcal{I}}$ is a discrete collection of points of X. Thus,

$$F \colon M \equiv M' \equiv \mathcal{I} \longrightarrow X, \qquad F(i) = \sigma_i(0),$$

is a Borel-Moore 0-pseudocycle in X.

Suppose $k \geq 1$. Let

$$\mathcal{D}_c \subset \mathcal{B}_c \times \mathcal{B}_c \qquad \text{and} \qquad \tau \colon \mathcal{D}_c \longrightarrow \mathcal{S}_{k-1}$$

be the subset and map corresponding to c as in Lemma 3.4. Define

$$M' = \Big(\bigsqcup_{i \in \mathcal{I}} \{i\} \times \Delta^k \Big) \Big/ \sim, \qquad \text{where} \tag{3.16}$$

$$\big(i_1, \iota_{k;p_1}(\tau_{(i_1,p_1),(i_2,p_2)}(t))\big) \sim \big(i_2, \iota_{k;p_2}(t)\big) \quad \forall \; \big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_c, \; t \in \Delta^{k-1}.$$

Let π be the quotient map and

$$F \colon M' \longrightarrow X, \qquad F\big([i, t]\big) = \sigma_i\big(\varphi_k(t)\big) \quad \forall \; i \in \mathcal{I}, \; t \in \Delta^{k+1}. \tag{3.17}$$

This map is well-defined by (3.15) and continuous by the universal property of the quotient topology.

Since the maps $\tau_{(i_1,p_1),(i_2,p_2)}$ are linear automorphisms of Δ^{k-1}, M' is homeomorphic to a geometric realization of a simplicial complex. Thus, M' is a Hausdorff topological space, and π is a closed map. By the local finiteness of $(\sigma_i)_{i \in \mathcal{I}}$, the set

$$\big\{i \in \mathcal{I} \colon F\big(\pi(\{i\} \times \Delta^k)\big) \cap A \neq \emptyset\big\} = \aleph_{(\sigma_i)_{i \in \mathcal{I}}}(A)$$

is finite for every compact subset $A \subset X$. Since $\pi(\{i\} \times \Delta^k) \subset M'$ is compact as well, it follows that F is a proper map. Since X is second countable, $\mathcal{I}$ is countable, and thus M' is second countable.

With $Y \subset \Delta^k$ denoting the $(k{-}2)$-skeleton, let $M \subset M'$ be the complement of the subset

$$Y_c \equiv \pi\left(\bigsqcup_{i \in \mathcal{I}} \{i\} \times Y\right) \subset M'. \tag{3.18}$$

Since M' is Hausdorff, $Y_c \subset M'$ is closed, and F is a proper map,

$$\mathrm{Bd}\, F|_M = F(Y_c) = \bigcup_{i \in \mathcal{I}} \sigma_i\big(\varphi_k(Y)\big) = \bigcup_{i \in \mathcal{I}} \sigma_i(Y); \tag{3.19}$$

the last equality holds by the first equation in (2.1). Since $\sigma_i|_{\mathrm{Int}\,\Delta'}$ is smooth for all $i \in \mathcal{I}$ and all simplices $\Delta' \subset \Delta^k$, $\mathrm{Bd}\, F|_M$ has dimension at most $k{-}2$ by (3.19).

By the above, $F|_M$ is a Borel-Moore k-pseudocycle, provided M is an oriented manifold and $F|_M$ is a smooth map. These are local statements, and (2) in the proof of [13, Lemma 3.2] applies *verbatim*. $\qquad\square$

Lemma 3.7. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. Suppose c_0, c_1 are integer locally finite singular k-chains as in (3.10) with $\sigma_{r;i} \in C^\infty(\Delta^k; X)$ for all $i \in \mathcal{I}_r$ representing cycles in $\overline{S}_k^{\mathrm{lf}}(X; \mathbb{Z})$ and (M'_r, M_r, F_r) with $r{=}0, 1$ are the triples corresponding to c_0, c_1 via the construction of Lemma 3.6. Every integer locally finite singular $(k{+}1)$-chain $\widetilde{c}$ as in (3.10) with $\widetilde{\sigma}_i \in C^\infty(\Delta^{k+1}; X)$ for all $i \in \widetilde{\mathcal{I}}$ determines a Borel-Moore pseudocycle equivalence between the pseudocycles $F_0|_{M_0}$ and $F_1|_{M_1}$.*

Proof. If $k{=}0$, there are subsets $\mathcal{D}_{\widetilde{c}} \subset \widetilde{\mathcal{I}} \times \widetilde{\mathcal{I}}$ and $\mathcal{I}_{\widetilde{c}}^{(0)}, \mathcal{I}_{\widetilde{c}}^{(1)} \subset \widetilde{\mathcal{I}}$ and bijections

$$\widetilde{\iota}_r : \mathcal{I}_r \longrightarrow \mathcal{I}_{\widetilde{c}}^{(r)}, \qquad r = 0, 1,$$

such that the projections

$$\mathcal{D}_{\widetilde{c}} \longrightarrow \widetilde{\mathcal{I}} - \mathcal{I}_{\widetilde{c}}^{(1)} \quad \text{and} \quad \mathcal{D}_{\widetilde{c}} \longrightarrow \widetilde{\mathcal{I}} - \mathcal{I}_{\widetilde{c}}^{(0)}$$

on the first and second component, respectively, are bijections,

$$\widetilde{\sigma}_{i_1}(1) = \widetilde{\sigma}_{i_2}(0) \ \ \forall \, (i_1, i_2) \in \mathcal{D}_{\widetilde{c}}, \qquad \widetilde{\sigma}_{\widetilde{\iota}_r(i)}(r) = \sigma_{r;i}(0) \ \ \forall \, i \in \mathcal{I}_r,\, r = 0, 1.$$

The space

$$\widetilde{M} = \left(\bigsqcup_{i \in \widetilde{\mathcal{I}}} \{i\} \times \Delta^1\right)\Big/\!\sim, \quad \text{where} \quad (i_1, 1) \sim (i_2, 0) \ \forall \, (i_1, i_2) \in \mathcal{D}_{\widetilde{c}},$$

is then an oriented one-dimensional manifold with boundary $\partial \widetilde{M} = M_1 - M_0$. Similarly to the proof of Lemma 3.6, the map

$$\widetilde{F} : \widetilde{M} \longrightarrow X, \qquad \widetilde{F}\big([i, t]\big) = \widetilde{\sigma}_i\big(\varphi_1(t)\big),$$

is well-defined, continuous, proper, and smooth. Since $\widetilde{F}|_{M_r} = F_r$, $\widetilde{F}$ is a pseudocycle equivalence between $F_0 = F_0|_{M_0}$ and $F_1 = F_1|_{M_1}$.

Suppose $k \geq 1$. Let

$$\mathcal{D}_{\widetilde{c}} \subset \mathcal{B}_{\widetilde{c}} \times \mathcal{B}_{\widetilde{c}}, \quad \mathcal{B}_{\widetilde{c}}^{(0)}, \mathcal{B}_{\widetilde{c}}^{(1)} \subset \mathcal{B}_{\widetilde{c}}, \quad \widetilde{\tau} : \mathcal{D}_{\widetilde{c}} \longrightarrow \mathcal{S}_k, \quad (\widetilde{\iota}_r, \widetilde{p}_r) : \mathcal{I}_r \longrightarrow \mathcal{B}_{\widetilde{c}}^{(r)}, \quad \widetilde{\tau}_r : \mathcal{I}_r \longrightarrow \mathcal{S}_k$$

be the subsets and maps corresponding to $\widetilde{c}$ as in Lemmas 3.5. As detailed in [13, Section 3.1], $\varphi_{k+1} = \text{id}$ on $\partial\Delta^{k+1}$, the third equation in (2.2), the second equation in (2.1), and the first equation in (3.13) give

$$\widetilde{\sigma}_{\widetilde{\iota}_r(i)} \circ \widetilde{\varphi}_{k+1} \circ \varphi_{k+1} \circ \iota_{k+1;\widetilde{p}_r(i)} \circ \widetilde{\tau}_{(r,i)} = \sigma_{r;i} \circ \varphi_k \qquad \forall\, i \in \mathcal{I}_r, \ r = 0, 1. \tag{3.20}$$

Furthermore, $\varphi_{k+1} = \text{id}$ on $\partial\Delta^{k+1}$, the third equation in (2.2) used twice, the second equation in (2.1), and the second equation in (3.11) give

$$\widetilde{\sigma}_{i_1} \circ \widetilde{\varphi}_{k+1} \circ \varphi_{k+1} \circ \iota_{k+1;p_1} \circ \widetilde{\tau}_{(i_1,p_1),(i_2,p_2)} = \widetilde{\sigma}_{i_2} \circ \widetilde{\varphi}_{k+1} \circ \varphi_{k+1} \circ \iota_{k+1;p_2} \quad \forall\, \big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_{\widetilde{c}}. \tag{3.21}$$

Define

$$\widetilde{M}' = \Big(\bigsqcup_{i \in \widetilde{\mathcal{I}}} \{i\} \times \Delta^{k+1} \Big) \Big/ \sim, \qquad \text{where}$$

$$\big(i_1, \iota_{k+1;p_1}(\widetilde{\tau}_{(i_1,p_1),(i_2,p_2)}(t))\big) \sim \big(i_2, \iota_{k+1;p_2}(t)\big) \quad \forall\, \big((i_1, p_1), (i_2, p_2)\big) \in \widetilde{\mathcal{D}}_{\widetilde{c}}, \ t \in \Delta^k.$$

Let $\widetilde{\pi}$ be the quotient map and

$$\widetilde{F} \colon \widetilde{M}' \longrightarrow X, \qquad \widetilde{F}\big([i, t]\big) = \widetilde{\sigma}_i\big(\widetilde{\varphi}_{k+1}(\varphi_{k+1}(t))\big) \quad \forall\, i \in \widetilde{\mathcal{I}}, \ t \in \Delta^{k+1}.$$

This map is well-defined by (3.21) and is continuous by the universal property of the quotient topology. By the same reasoning as in the proof of Lemma 3.6, $\widetilde{M}$ is a second countable, Hausdorff topological space, $\widetilde{\pi}$ is a closed map, and $\widetilde{F}$ is a proper map.

With $\widetilde{Y} \subset \Delta^{k+1}$ denoting the $(k-1)$-skeleton, let $\widetilde{M} \subset \widetilde{M}'$ be the complement of the subset

$$Y_{\widetilde{c}} \equiv \widetilde{\pi}\Big(\bigsqcup_{i \in \widetilde{\mathcal{I}}} \{i\} \times \widetilde{Y} \Big) \subset \widetilde{M}'.$$

Since $\widetilde{M}'$ is Hausdorff, $Y_{\widetilde{c}} \subset \widetilde{M}'$ is closed, and $\widetilde{F}$ is a proper map,

$$\text{Bd}\, \widetilde{F}\big|_{\widetilde{M}} = \widetilde{F}(Y_{\widetilde{c}}) = \bigcup_{i \in \widetilde{\mathcal{I}}} \widetilde{\sigma}_i\big(\widetilde{\varphi}_{k+1}(\varphi_{k+1}(\widetilde{Y}))\big) = \bigcup_{i \in \widetilde{\mathcal{I}}} \widetilde{\sigma}_i(\widetilde{Y}). \tag{3.22}$$

Since $\widetilde{\sigma}_i|_{\text{Int}\,\Delta'}$ is smooth for all $i \in \widetilde{\mathcal{I}}$ and all simplices $\Delta' \subset \Delta^{k+1}$, $\text{Bd}\, \widetilde{F}\big|_{\widetilde{M}}$ has dimension at most $k-1$ by (3.22).

For $r = 0, 1$, let $Y_r \subset M_r$ denote the union of the images of the open $(k-1)$-simplices of Δ^k under the quotient map π in the proof of Lemma 3.6 (this is also the intersection of M_r with the union of the images of the closed $(k-1)$-simplices of Δ^k under π). The maps

$$\iota_r \colon M_r - Y_r \longrightarrow \widetilde{M}, \qquad \iota_r\big([i, t]\big) = \big[\widetilde{\iota}_r(i), \iota_{k+1;\widetilde{p}_r(i)}\big(\widetilde{\tau}_{r;i}(t)\big)\big] \quad \forall\, i \in \mathcal{I}_r, \ t \in \text{Int}\,\Delta^k,$$

are well-defined embeddings with disjoint images. By (3.20) and (3.17),

$$\widetilde{F} \circ \iota_r = F\big|_{M_r - Y_r}.$$

Thus, $\widetilde{F}\big|_{\widetilde{M}}$ is a Borel-Moore pseudocycle equivalence between the Borel-Moore k-pseudocycles $F_0|_{M_0}$ and $F_1|_{M_1}$, provided $\widetilde{M}$ is an oriented manifold, $\widetilde{F}\big|_{\widetilde{M}}$ is a smooth map, ι_0, ι_1 are smooth embeddings, and

$$\partial\widetilde{M} = \iota_1\big(M_1 - Y_1)\big) \sqcup -\iota_0(M_0 - Y_0).$$

These are straightforward local statements, which are established as in (3) in the proof of [13, Lemma 3.3]. $\qquad\square$

3.4 From pseudocycles to integral cycles

We next adapt the constructions of [13, Section 3.2] from pseudocycles to Borel-Moore pseudocycles and obtain the second homomorphism in (1.1). As indicated in Section 1.2, we first define a homology class $[f]_{X;U}$ of a pseudocycle f relative to a nice neighborhood U provided by Proposition 3.1 and then pull it back to the absolute Borel-Moore homology of the target.

Lemma 3.8. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. Every Borel-Moore k-pseudocycle $f \colon M \longrightarrow X$ determines an element of $H_k^{\mathrm{lf}}(X;\mathbb{Z})$.*

Proof. By Proposition 3.1, there exists an open neighborhood $U \subset X$ of Bd f such that

$$\overline{H}_{\{U\};l}^{\mathrm{lf}}(X;\mathbb{Z}) = 0 \qquad \forall\, l > k-2.$$

Thus, $f|_{M-f^{-1}(U)}$ is a proper map and the homomorphism

$$\overline{H}_k^{\mathrm{lf}}(X;\mathbb{Z}) \longrightarrow \overline{H}_k^{\mathrm{lf}}(X,\{U\};\mathbb{Z}) \tag{3.23}$$

induced by inclusion is an isomorphism. Let $V \subset M$ be an open neighborhood of $M-f^{-1}(U)$ so that $f|_{\overline{V}}$ is still proper and $\overline{V}$ is a manifold with boundary. This manifold inherits an orientation from M and thus defines a homology class

$$[\overline{V}] \in \overline{H}_k^{\mathrm{lf}}(\overline{V},\{\partial\overline{V}\};\mathbb{Z}).$$

Put

$$[f]_{X;U} = f_*\big([\overline{V}]\big) \in \overline{H}_k^{\mathrm{lf}}\big(X,\{U\};\mathbb{Z}\big) \overset{(3.23)}{\approx} \overline{H}_k^{\mathrm{lf}}(X;\mathbb{Z}), \tag{3.24}$$

where

$$f_* \colon \overline{H}_k^{\mathrm{lf}}\big(\overline{V},\{\partial\overline{V}\};\mathbb{Z}\big) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X,\{U\};\mathbb{Z}\big) \tag{3.25}$$

is the homology homomorphism induced by the proper map $f|_{\overline{V}}$.

Suppose $V' \subset X$ is an open neighborhood of $\overline{V}$ so that $f|_{\overline{V}'}$ is also proper and $\overline{V}'$ is a manifold with boundary. Choose a triangulation of $\overline{V}'$ extending some triangulation of $(\partial\overline{V}) \cup (\partial\overline{V}')$; such a triangulation exists by [9, Section 16]. Since $f(\overline{V}'-V) \subset U$, the classes

$$f_*([\overline{V}]), f_*([\overline{V}']) \in \overline{H}_k^{\mathrm{lf}}\big(X,\{U\};\mathbb{Z}\big)$$

are represented by cycles that differ by singular simplices lying in U; see the last paragraph of Section 3.2. It follows that

$$f_*\big([\overline{V}]\big) = f_*\big([\overline{V}']\big) \in \overline{H}_k^{\mathrm{lf}}\big(X,\{U\};\mathbb{Z}\big).$$

Thus, the homology class $[f]_{X;U}$ is independent of the choice of V.

Suppose $U' \subset U$ is an another open neighborhood of Bd f. By the previous paragraph, we can choose V for U and V' for U' to be the same. Since the isomorphism (3.23) is the composition of the isomorphisms

$$\overline{H}_k^{\mathrm{lf}}(X;\mathbb{Z}) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X,\{U'\};\mathbb{Z}\big) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X,\{U\};\mathbb{Z}\big)$$

induced by inclusions and the homomorphism (3.25) is the composition

$$\overline{H}_k^{\mathrm{lf}}\big(\overline{V}, \{\partial\overline{V}\}; \mathbb{Z}\big) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X, \{U'\}; \mathbb{Z}\big) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X, \{U\}; \mathbb{Z}\big),$$

the homology classes in $\overline{H}_k^{\mathrm{lf}}(X; \mathbb{Z})$ corresponding to $[f]_{X;U'}$ and $[f]_{X;U}$ are the same. Thus, the homology class $[f]$ in $\overline{H}_k^{\mathrm{lf}}(X; \mathbb{Z})$ corresponding to $[f]_{X;U}$ under the isomorphism (3.23) is independent of the choice of U as well. $\qquad\square$

Lemma 3.9. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. If Borel-Moore k-pseudocycles $f_0 \colon M_0 \longrightarrow X$ and $f_1 \colon M_1 \longrightarrow X$ are equivalent, then*

$$[f_0] = [f_1] \in H_k^{\mathrm{lf}}(X; \mathbb{Z}).$$

Proof. Let $\widetilde{f} \colon \widetilde{M} \longrightarrow X$ be a Borel-Moore pseudocycle equivalence between f_0 and f_1 as in Definition 1.1(b). By Remark 1.4, we can assume that $Y_0, Y_1 = \emptyset$. By Proposition 3.1, there exists an open neighborhood $\widetilde{U} \subset X$ of Bd $\widetilde{f}$ such that

$$\overline{H}_{\{\widetilde{U}\};l}^{\mathrm{lf}}(X; \mathbb{Z}) = 0 \qquad \forall\, l > k-1.$$

Thus, $\widetilde{f}|_{\widetilde{M} - \widetilde{f}^{-1}(\widetilde{U})}$ is a proper map and the homomorphism

$$\overline{H}_k^{\mathrm{lf}}\big(X; \mathbb{Z}\big) \longrightarrow \overline{H}_k^{\mathrm{lf}}\big(X, \{\widetilde{U}\}; \mathbb{Z}\big) \tag{3.26}$$

induced by inclusion is injective.

For $r = 0, 1$, let $U_r \subset \widetilde{U}_r$ be an open neighborhood of Bd $f_r \subset$ Bd $\widetilde{f}$ such that

$$\overline{H}_{\{U_r\};l}^{\mathrm{lf}}(X; \mathbb{Z}) = 0 \qquad \forall\, l > k-2. \tag{3.27}$$

Let $V_r \subset M_r$ be a choice of an open subset for (f_r, U_r) as in the proof of Lemma 3.8. Since the restriction of $\widetilde{f}$ to the closed subset

$$B \equiv \big(\widetilde{M} - \widetilde{U}_r\big) \cup \overline{V}_0 \cup \overline{V}_1 \subset \widetilde{M}$$

is proper, Lemma 1.6(5) implies that there exists a neighborhood $W \subset \widetilde{M}$ of B so that $\widetilde{f}|_{\overline{W}}$ is still proper and $\overline{W}$ is a manifold with boundary and corners (with the corners contained in $\partial\widetilde{M} - \overline{V}_0 - \overline{V}_1$). We note that

$$\widetilde{f}\big(\partial\overline{W} - V_0 \cup V_1\big) = \widetilde{f}\big((\overline{W} - W) \cup (W \cap (M_0 \cup M_1)) - V_0 \cup V_1\big) \subset \widetilde{U} \cup U_0 \cup U_1 = \widetilde{U}. \tag{3.28}$$

For $r = 0, 1$, let

$$\iota_{X;r*} \colon H_k^{\mathrm{lf}}\big(X, \{U_r\}; \mathbb{Z}\big) \longrightarrow H_k^{\mathrm{lf}}\big(X, \{\widetilde{U}\}; \mathbb{Z}\big) \qquad \text{and}$$
$$\iota_{\widetilde{M};r*} \colon H_k^{\mathrm{lf}}\big(\overline{V}_r, \{\partial\overline{V}_r\}; \mathbb{Z}\big) \longrightarrow H_k^{\mathrm{lf}}\big(\overline{W}, \{\partial\overline{W} - V_0 \cup V_1\}; \mathbb{Z}\big)$$

be the homomorphisms induced by inclusions.

Choose a triangulation $\widetilde{T} = (\widetilde{K}, \widetilde{\eta})$ of $\overline{W}$ that restricts to triangulations of $\overline{V}_0, \partial\overline{V}_0, \overline{V}_1, \partial\overline{V}_1$ and $\partial\overline{W}$. Let
$$K^{\text{top}} = \{\sigma \in K \colon \dim\sigma = k+1\}.$$
For $r = 0, 1$, put
$$K_r = \{\sigma \in K \colon \eta(\sigma) \subset \overline{V}_r\}, \qquad K_r^{\text{top}} = \{\sigma \in K_r \colon \dim\sigma = k\}.$$
For each $\sigma \in K^{\text{top}}$ and $\sigma \in K_r^{\text{top}}$, let
$$l_\sigma \colon \Delta^{k+1} \longrightarrow \sigma \subset |K| \qquad \text{and} \qquad l_\sigma \colon \Delta^k \longrightarrow \sigma \subset |K_r|,$$
respectively, be as in (3.14). By our assumptions,
$$\partial\sum_{\sigma \in K^{\text{top}}}\{\eta \circ l_\sigma\} + \sum_{r=0,1}(-1)^r\sum_{\sigma \in K_r^{\text{top}}}\{\eta \circ l_\sigma\} \in S^{\text{lf}}_{\{\partial\overline{W} - V_0 \cup V_1\}; k}(\widetilde{M}; \mathbb{Z}).$$

Along with (3.28), this gives
$$\partial\sum_{\sigma \in K^{\text{top}}}\{\widetilde{f} \circ \eta \circ l_\sigma\} = \sum_{\sigma \in K_1^{\text{top}}}\{f_1 \circ \eta \circ l_\sigma\} - \sum_{\sigma \in K_0^{\text{top}}}\{f_0 \circ \eta \circ l_\sigma\} \in \overline{S}^{\text{lf}}_l(X, \{\widetilde{U}\}; \mathbb{Z}). \qquad (3.29)$$

For $r = 0, 1$, let $[f_r]_{X;U_r} \in H^{\text{lf}}_k(X, \{U_r\}; \mathbb{Z})$ be as in the proof of Lemma 3.8 and
$$[f_r]_{X;\widetilde{U}} = \iota_{X;r*}\big([f_r]_{X;U_r}\big) \in H^{\text{lf}}_k(X, \widetilde{U}; \mathbb{Z}).$$
Since the diagram

$$
\begin{CD}
H^{\text{lf}}_k(\overline{V}_r, \{\partial\overline{V}_r\}; \mathbb{Z}) @>{\{\widetilde{f}|_{\overline{V}_r}\}_*}>> H^{\text{lf}}_k(X, \{U_r\}; \mathbb{Z}) \\
@V{\iota_{\widetilde{V};r*}}VV @VV{\iota_{X;r*}}V \\
H^{\text{lf}}_k(\overline{W}, \{\partial\overline{W} - V_0 \cup V_1\}; \mathbb{Z}) @>{\{\widetilde{f}|_{\overline{W}}\}_*}>> H^{\text{lf}}_k(X, \{\widetilde{U}\}; \mathbb{Z})
\end{CD}
$$

commutes,
$$[f_r]_{X;\widetilde{U}} = \{\widetilde{f}|_{\overline{W}}\}_*\big(\iota_{X;r*}([\overline{V}_r])\big) \in H^{\text{lf}}_k(X, \widetilde{U}; \mathbb{Z}).$$
By the last paragraph of Section 3.2, the first term and the second term on the right-hand side of (3.29) represent $[f_1]_{X;\widetilde{U}}$ and $[f_0]_{X;\widetilde{U}}$, respectively. Thus,
$$\iota_{X;0*}\big([f_0]_{X;U_0}\big) = \iota_{X;1*}\big([f_1]_{X;U_1}\big) \in H^{\text{lf}}_k(X, \widetilde{U}; \mathbb{Z}).$$
Since the diagram

$$
\begin{CD}
H^{\text{lf}}_k(X; \mathbb{Z}) @>{\underset{\approx}{(3.27)}}>> H^{\text{lf}}_k(X, \{U_0\}; \mathbb{Z}) \\
@V{(3.27)}V{\approx}V @VV{\iota_{X;0*}}V \\
H^{\text{lf}}_k(X, \{U_1\}; \mathbb{Z}) @>{\{\iota_{X;1*}\}_*}>> H^{\text{lf}}_k(X, \{\widetilde{U}\}; \mathbb{Z})
\end{CD}
$$

of homomorphisms induced by inclusions commutes and the diagonal homomorphism is injective, the classes $[f_0], [f_1] \in H^{\text{lf}}_k(X; \mathbb{Z})$ corresponding to $[f_0]_{X;U_0}$ and $[f_1]_{X;U_1}$ are the same. $\qquad\square$

3.5 Isomorphisms of homology theories

In order to establish that the homomorphisms of Theorem 1.3 as constructed in Section 3.3 and 3.4 are isomorphisms and mutual inverses, we first show that

$$\Phi_* \circ \Psi_* = \mathrm{id} : H^{\mathrm{lf}}_*(X; \mathbb{Z}) \longrightarrow H^{\mathrm{lf}}_*(X; \mathbb{Z}).$$

We then show that the homomorphism Φ_* is injective.

Lemma 3.10. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. Suppose c is an integer locally finite singular k-chain c as in (3.7) with $\sigma_i \in C^\infty(\Delta^k; X)$ for all $i \in \mathcal{I}$ representing a cycle in $\overline{S}^{\mathrm{lf}}_k(X; \mathbb{Z})$ and (M', M, F) is the triple corresponding to c via the construction of Lemma 3.6. The homology class $[F|_M]$ obtained via the construction of Lemma 3.8 then satisfies*

$$\left[F|_M\right] = [c] \in H^{\mathrm{lf}}_k(X; \mathbb{Z}). \tag{3.30}$$

Proof. For $k = 0$, the claim clearly holds on the chain level. Thus, suppose $k \geq 1$. Since the self-map φ_k of Lemma 2.1 restricts to the identity on $\partial \Delta^k$,

$$\varphi_k - \mathrm{id}_k = \partial_{\Delta^k} s_k \in S_k(\Delta^k; \mathbb{Z}) \tag{3.31}$$

for some $s_k \in S_{k+1}(\Delta^k; \mathbb{Z})$. Define

$$\hbar : \mathrm{Hom}(\Delta^k, X) \longrightarrow S_k(\Delta^k; \mathbb{Z}), \qquad \hbar(\sigma) = \varphi_k,$$
$$\widetilde{\hbar} : \mathrm{Hom}(\Delta^k, X) \longrightarrow S_{k+1}(\Delta^k; \mathbb{Z}), \qquad \widetilde{\hbar}(\sigma) = s_k.$$

By Lemma 2.2 and (3.31), the homomorphisms

$$\hbar_\# : S^{\mathrm{lf}}_k(X; \mathbb{Z}) \longrightarrow S^{\mathrm{lf}}_k(X; \mathbb{Z}) \qquad \text{and} \qquad \widetilde{\hbar}_\# : S^{\mathrm{lf}}_k(X; \mathbb{Z}) \longrightarrow S^{\mathrm{lf}}_{k+1}(X; \mathbb{Z})$$

induced via (2.9) and (2.12) are well-defined and satisfy

$$\hbar_\#(c') - c' = \partial_X\!\left(\widetilde{\hbar}_\#(c')\right) \in S^{\mathrm{lf}}_k(X; \mathbb{Z}) \qquad \forall \, c' \in S^{\mathrm{lf}}_k(X; \mathbb{Z}).$$

In particular,

$$\sum_{i \in \mathcal{I}} \sigma_i \circ \varphi_k - \sum_{i \in \mathcal{I}} \sigma_i \equiv \hbar_\#(c) - c \in \partial S^{\mathrm{lf}}_{k+1}(X; \mathbb{Z}). \tag{3.32}$$

Let π be the quotient map of the proof of Lemma 3.6 and $U \subset X$ be a neighborhood of $\mathrm{Bd}\, F|_M$ as in the proof of Lemma 3.8. Choose a manifold with boundary $\overline{V} \subset M$ containing $M - F^{-1}(U)$ as in the latter proof so that $(\overline{V}, \partial\overline{V})$ admits a triangulation $T \equiv (K, \eta)$ with each k-simplex of T contained in $\pi(\{i\} \times \Delta^k)$ for some $i \in \mathcal{I}$. Let

$$K^{\mathrm{top}} = \left\{\sigma : \dim \sigma = k\right\}.$$

For each $\sigma \in K^{\mathrm{top}}$, choose a linear map

$$l_\sigma : \Delta^k \longrightarrow \sigma \subset |K| \tag{3.33}$$

so that the map $\eta \circ l_\sigma : \Delta^k \longrightarrow M$ is orientation-preserving. For each $i \in \mathcal{I}$, let

$$K_i = \left\{\sigma \in K : \eta(\sigma) \subset \pi(\{i\} \times \Delta^k)\right\}, \qquad K^{\mathrm{top}}_i = \left\{\sigma \in K_i : \dim \sigma = k\right\}.$$

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Let $\widetilde{T}_i \equiv (\widetilde{K}_i, \widetilde{\eta}_i)$ be a triangulation of a subset of Δ^k that along with K_i gives a triangulation of Δ^k. Put

$$\widetilde{K}_i^{\text{top}} = \{\sigma \in \widetilde{K}_i : \dim \sigma = k\}.$$

By definition of T and F,

$$\widetilde{\eta}_i(\sigma) \subset F^{-1}(U), \quad \{\sigma_i \circ \varphi_k\}(\eta_i(\sigma)) \subset U \qquad \forall\, \sigma \in \widetilde{K}_i^{\text{top}},\ i \in \mathcal{I}. \tag{3.34}$$

Furthermore, by (3.32)

$$\{c\} = \sum_{i \in \mathcal{I}} \{\sigma_i \circ \varphi_k\} = \sum_{i \in \mathcal{I}} \sum_{\sigma \in K_i^{\text{top}}} \{\sigma_i \circ \varphi_k \circ \eta \circ l_\sigma\} + \sum_{i \in \mathcal{I}} \sum_{\sigma \in \widetilde{K}_i^{\text{top}}} \{\sigma_i \circ \varphi_k \circ \widetilde{\eta}_i \circ l_\sigma\} \in \overline{S}_k^{\text{lf}}(X; \mathbb{Z}); \tag{3.35}$$

the second equality above holds because subdivisions of cycles do not change the homology class. By the proof of Lemma 3.8, the first sum on the right-hand side of (3.35) represents the image $[F|_M]_{X;U}$ of $[F|_M]$ under the isomorphism (3.23). By (3.34), the second sum lies in $\overline{S}_{\{U\};k}^{\text{lf}}(X; \mathbb{Z})$. Since the sum of these two sums represents a cycle in $\overline{S}_k^{\text{lf}}(X)$, it must represent $[F|_M]$ in $\overline{S}_k^{\text{lf}}(X; \mathbb{Z})$. This gives (3.30). $\qquad\square$

Lemma 3.11. *Let X be a manifold and $k \in \mathbb{Z}^{\geq 0}$. Suppose $f \colon M \longrightarrow X$ is a Borel-Moore k-pseudocycle such that the homology class $[f]$ provided by Lemma 3.8 vanishes. Then f represents the zero element of $\mathcal{H}_k^{\text{cl}}(X)$.*

Proof. The case $k=0$ is straightforward and very similar to the $k=0$ case of the proof of Lemma 3.7. Thus, we assume that $k \geq 1$. By Example 1.2, we can also assume that $f^{-1}(\operatorname{Bd} f) = \emptyset$.

By the first countability of the topology of X and Proposition 3.1, there exists a sequence $\{U_r\}_{r \in \mathbb{Z}^+}$ of open neighborhoods of $\operatorname{Bd} f$ in X such that

$$U_r \supset \overline{U}_{r+1} \ \forall\, r \in \mathbb{Z}^+, \qquad \bigcap_{r=1}^{\infty} U_r = \operatorname{Bd} f, \quad \text{and} \quad H_{\{U_r\};l}^{\text{lf}}(X; \mathbb{Z}) = 0 \quad \forall\, l > k-2. \tag{3.36}$$

By the first condition above, the closed subset $M - f^{-1}(U_r) \subset M$ is contained in the open subset $M - f^{-1}(\overline{U}_{r+1})$. Thus, we can choose submanifolds with boundary $\overline{V}_r \subset M$ as in the proof of Lemma 3.8 so that

$$M - f^{-1}(U_r) \subset V_r \subset \overline{V}_r \subset M - f^{-1}(\overline{U}_{r+1}) \quad \forall\, r \in \mathbb{Z}^+.$$

By the second condition in (3.36),

$$\bigcup_{r=1}^{\infty} V_r \supset \bigcup_{r=1}^{\infty} \big(M - f^{-1}(\overline{U}_{r+1})\big) = M - f^{-1}(\operatorname{Bd} f) = M,$$

i.e. the open collection $\{V_r\}_{r \in \mathbb{Z}^+}$ covers M.

Choose a triangulation $T = (K, \eta)$ of M that extends triangulations of all $\partial \overline{V}_r$ (which are pairwise disjoint). Let

$$K^{\text{top}} = \{\sigma \in K : \dim \sigma = k\}, \qquad \mathcal{B}_\eta = \{(\sigma, p) : \sigma \in K^{\text{top}},\ p = 0, 1, \ldots, k\}.$$

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For each $\sigma \in K^{\mathrm{top}}$, let l_σ be as in (3.33). Put

$$f_\sigma = f \circ \eta \circ l_\sigma : \Delta^k \longrightarrow X \quad \forall \, \sigma \in K^{\mathrm{top}} \qquad \text{and} \tag{3.37}$$

$$\mathcal{D}_\eta = \left\{ \big((\sigma_1, p_1), (\sigma_2, p_2)\big) \in \mathcal{B}_\eta \times \mathcal{B}_\eta : (\sigma_1, p_1) \neq (\sigma_2, p_2), \; l_{\sigma_1}(\Delta^k_{p_1}) = l_{\sigma_2}(\Delta^k_{p_2}) \subset |K| \right\}.$$

For each $\big((\sigma_1, p_1), (\sigma_2, p_2)\big) \in \mathcal{D}_\eta$, define

$$\tau_{(\sigma_1, p_1), (\sigma_2, p_2)} \in \mathcal{S}_{k-1} \qquad \text{by} \qquad l_{\sigma_1} \circ \iota_{k; p_1} \circ \tau_{(\sigma_1, p_1), (\sigma_2, p_2)} = l_{\sigma_2} \circ \iota_{k; p_2}.$$

Since M is an oriented manifold,

$$\mathcal{D}_\eta \subset \mathcal{B}_\eta \times \mathcal{B}_\eta \qquad \text{and} \qquad \tau : \mathcal{D}_\eta \longrightarrow \mathcal{S}_{k-1}$$

satisfy (i)-(iii) of Lemma 3.4 with the subscript $c = \eta$ and the maps σ replaced by f_σ. Furthermore, the geometric realization $|K|$ of K is the topological space (3.16) with $(\mathcal{I}, c) = (\eta, K^{\mathrm{top}})$ and

$$f \circ \eta \circ \pi|_{\sigma \times \Delta^k} = f_\sigma \qquad \forall \, \sigma \in K^{\mathrm{top}},$$

where π is the quotient map as in the proof of Lemma 3.6.

For each $r \in \mathbb{Z}^+$, let

$$K^{\mathrm{top}}_r = \left\{ \sigma \in K^{\mathrm{top}} : \eta(\sigma) \subset \bar{V}_r \right\}, \quad \mathcal{B}_{\eta; r} = \left\{ (\sigma, p) \in \mathcal{B}_\eta : \sigma \in K^{\mathrm{top}}_r \right\}, \quad \mathcal{D}_{\eta; r} = \mathcal{D}_\eta \cap (\mathcal{B}_{\eta; r} \times \mathcal{B}_{\eta; r}).$$

By the construction of $[f]$ in the proof of Lemma 3.6 and by the last paragraph of Section 3.2, there exists a Borel-Moore singular chain

$$c_r \equiv \sum_{i \in \mathcal{I}_r} f_{r; i} \in S^{\mathrm{lf}}_{\{U_r\}; k}(X; \mathbb{Z})$$

such that

$$\sum_{\sigma \in K^{\mathrm{top}}_r} \{f_\sigma\} + \{c_r\} \in \overline{S}^{\mathrm{lf}}_k(X; \mathbb{Z}) \tag{3.38}$$

is a cycle representing $[f]$. Similarly to Lemma 3.4, there exist a symmetric subset

$$\mathcal{D}_r \subset (\mathcal{B}_{\eta; r} \sqcup \mathcal{B}_{c_r}) \times (\mathcal{B}_{\eta; r} \sqcup \mathcal{B}_{c_r})$$

disjoint from the diagonal and a map

$$\tau_r : \mathcal{D}_r \longrightarrow \mathcal{S}_{k-1}$$

such that

(i) $\mathcal{D}_{\eta; r} \subset \mathcal{D}_r$ and $\tau_r|_{\mathcal{D}_{\eta; r}} = \tau|_{\mathcal{D}_{\eta; r}}$;

(ii) the projection map $\mathcal{D}_r \longrightarrow \mathcal{B}_{\eta; r} \sqcup \mathcal{B}_{c_r}$ on either coordinate is a bijection;

(iii) for all $\big((i_1, p_1), (i_2, p_2)\big) \in \mathcal{D}_r$,

$$\tau^{-1}_{r; (i_1, p_1), (i_2, p_2)} = \tau_{r; (i_2, p_2), (i_1, p_1)}, \qquad f_{r; i_1} \circ \iota_{k; p_1} \circ \tau_{(i_1, p_1), (i_2, p_2)} = f_{r; i_2} \circ \iota_{k; p_2},$$

$$\text{and} \qquad \operatorname{sign} \tau_{r; (i_1, p_1), (i_2, p_2)} = -(-1)^{p_1 + p_2},$$

where $f_{r; \sigma} \equiv f_\sigma$ for all $\sigma \in K^{\mathrm{top}}_r$.

Since every Borel-Moore singular chain (3.38) is a cycle,

$$\sum_{\sigma\in K_r^{\mathrm{top}}-K_{r-1}^{\mathrm{top}}}\{f_\sigma\}+\{c_r\}-\{c_{r-1}\}\in \overline{S}^{\mathrm{lf}}_{\{U_{r-1}\};k}(X;\mathbb{Z})$$

is a cycle as well. By the third condition in (3.36), this cycle is a boundary. Since $[f]=0$ by assumption, this conclusion also holds for $r=1$ with $U_0\equiv X$, $K_0^{\mathrm{top}}=\emptyset$, and $c_0=0$. Let

$$\widetilde{c}_r\equiv\sum_{i\in\widetilde{\mathcal{I}}_r}\widetilde{f}_{r;i}\in \overline{S}^{\mathrm{lf}}_{\{U_{r-1}\};k+1}(X;\mathbb{Z}) \tag{3.39}$$

be a Borel-Moore singular chain such that

$$\sum_{\sigma\in K_r^{\mathrm{top}}-K_{r-1}^{\mathrm{top}}}\{f_\sigma\}+\{c_r\}-\{c_{r-1}\}=\overline{\partial}_X\{\widetilde{c}_r\}\in \overline{S}^{\mathrm{lf}}_{\{U_{r-1}\};k}(X;\mathbb{Z}).$$

Summing this equation with r replaced by r' from 1 to r, we obtain

$$\sum_{\sigma\in K_r^{\mathrm{top}}}\{f_\sigma\}+\{c_r\}=\overline{\partial}_X\sum_{r'=1}^{r}\{\widetilde{c}_{r'}\}\in \overline{S}^{\mathrm{lf}}_k(X;\mathbb{Z})\quad\forall\,r\in\mathbb{Z}^+. \tag{3.40}$$

Similarly to Lemma 3.5, (3.40) implies that there exist a subset

$$\widetilde{\mathcal{B}}_r^f\subset\widetilde{\mathcal{B}}_r\equiv\bigsqcup_{r'=1}^{r}\mathcal{B}_{\widetilde{c}_{r'}},$$

a symmetric subset $\widetilde{\mathcal{D}}_r\subset\widetilde{\mathcal{B}}_r\times\widetilde{\mathcal{B}}_r$ disjoint from the diagonal, and maps

$$\widetilde{\tau}_r\colon\widetilde{\mathcal{D}}_r\longrightarrow\mathcal{S}_k,\quad \big((i_1,p_1),(i_2,p_2)\big)\longrightarrow\widetilde{\tau}_{r;((i_1,p_1),(i_2,p_2))},$$
$$(\widetilde{\iota}_r,\widetilde{p}_r)\colon K_r^{\mathrm{top}}\sqcup\mathcal{I}_r\longrightarrow\widetilde{\mathcal{B}}_r^f,\quad\text{and}\quad\widetilde{\tau}_r\colon K_r^{\mathrm{top}}\sqcup\mathcal{I}_r\longrightarrow\mathcal{S}_k,\quad i\longrightarrow\widetilde{\tau}_{(r,i)},$$

such that

(i) $\widetilde{\mathcal{D}}_{r-1}\subset\widetilde{\mathcal{D}}_r$, $\widetilde{\tau}_r|_{\widetilde{\mathcal{D}}_{r-1}}=\widetilde{\tau}_{r-1}$, and $(\widetilde{\iota}_r,\widetilde{p}_r,\widetilde{\tau}_r)|_{K_{r-1}^{\mathrm{top}}}=(\widetilde{\iota}_{r-1},\widetilde{p}_{r-1},\widetilde{\tau}_{r-1})|_{K_{r-1}^{\mathrm{top}}}$ if $r\geq 2$;

(ii) the projection $\widetilde{\mathcal{D}}_r\longrightarrow\widetilde{\mathcal{B}}_r$ on either coordinate is a bijection onto the complement of $\widetilde{\mathcal{B}}_r^f$;

(iii) for all $\big((i_1,p_1),(i_2,p_2)\big)\in\widetilde{\mathcal{D}}_r\cap(\mathcal{B}_{\widetilde{c}_{r_1}}\times\mathcal{B}_{\widetilde{c}_{r_2}})$ with $r_1,r_2\in[r]$,

$$\widetilde{\tau}^{-1}_{r;((i_1,p_1),(i_2,p_2))}=\widetilde{\tau}_{r;((i_2,p_2),(i_1,p_1))},\quad \widetilde{f}_{r_1;i_1}\circ\iota_{k+1;p_1}\circ\widetilde{\tau}_{r;((i_1,p_1),(i_2,p_2))}=\widetilde{f}_{r_2;i_2}\circ\iota_{k+1;p_2},$$
$$\text{and}\qquad\mathrm{sign}\,\widetilde{\tau}_{r;((i_1,p_1),(i_2,p_2))}=-(-1)^{p_1+p_2}; \tag{3.41}$$

(iv) for all $\sigma\in K_r^{\mathrm{top}}-K_{r-1}^{\mathrm{top}}$,

$$\widetilde{f}_{r;\widetilde{\iota}_r(\sigma)}\circ\iota_{k+1;\widetilde{p}_r(\sigma)}\circ\widetilde{\tau}_{(r,\sigma)}=f_\sigma\qquad\text{and}\qquad\mathrm{sign}\,\widetilde{\tau}_{(r,i)}=-(-1)^{\widetilde{p}_r(\sigma)}; \tag{3.42}$$

(v) $(\tilde{\iota}_r, \tilde{p}_r)$ is a bijection onto $\widetilde{\mathcal{B}}_r^f$.

Put

$$\widetilde{M}' = \Big(\coprod_{r=1}^{\infty} \coprod_{i \in \widetilde{\mathcal{I}}_r} \{r\} \times \{i\} \times \Delta^{k+1} \Big) \Big/ \sim, \qquad \text{where}$$

$$\big(r_1, i_1, \iota_{k+1;p_1}(\tilde{\tau}_{r;((i_1,p_1),(i_2,p_2))}(t)) \big) \sim \big(r_2, i_2, \iota_{k+1;p_2}(t) \big)$$
$$\forall \, \big((i_1, p_1), (i_2, p_2) \big) \in \widetilde{\mathcal{D}}_r \cap (\mathcal{B}_{\tilde{c}_{r_1}} \times \mathcal{B}_{\tilde{c}_{r_2}}), \, r_1, r_2, r \in \mathbb{Z}^+, \, t \in \Delta^k.$$

Let $\tilde{\pi}$ be the quotient map. Define

$$\tilde{f} : \widetilde{M}' \longrightarrow X, \qquad \tilde{f}([r, i, t]) = \tilde{f}_{r;i}(\varphi_{k+1}(t)) \quad \forall \, t \in \Delta^{k+1}, \, i \in \widetilde{\mathcal{I}}_r, \, r \in \mathbb{Z}^+,$$

where φ_{k+1} is the self-map of Δ^{k+1} provided by Lemma 2.1. Since φ_{k+1} restricts to the identity on $\partial \Delta^{k+1}$, the map $\tilde{f}$ is well-defined by the second condition in (3.41) and continuous by the universal property of the quotient topology. Similarly to the proof of Lemma 3.6, the restriction of $\tilde{f}$ to

$$\tilde{\pi} \Big(\coprod_{i \in \widetilde{\mathcal{I}}_r} \{r\} \times \{i\} \times \Delta^{k+1} \Big) \subset \widetilde{M}'$$

is proper for every $r \in \mathbb{Z}^+$. By (3.39), $\tilde{f}_{r';i}(\Delta^{k+1}) \subset U_r$ for all $r' > r$. Thus,

$$\mathrm{Bd}\, \tilde{f} \subset \bigcap_{r=1}^{\infty} \overline{U}_r = \mathrm{Bd}\, f. \tag{3.43}$$

Let $\widetilde{M} \subset \widetilde{M}'$ be the complement of the subset

$$\tilde{\pi} \Big(\coprod_{r=0}^{\infty} \coprod_{i \in \widetilde{\mathcal{I}}_r} \{r\} \times \{i\} \times \widetilde{Y} \Big) \subset \widetilde{M}',$$

where $\widetilde{Y} \subset \Delta^{k+1}$ is the $(k-1)$-skeleton as before. By Lemma 1.6(2) and (3.43),

$$\mathrm{Bd}\, \tilde{f}|_{\widetilde{M}} \subset \big(\mathrm{Bd}\, \tilde{f} \big) \cup \bigcup_{r=1}^{\infty} \bigcup_{i \in \widetilde{\mathcal{I}}_r} \tilde{f}_{r;i}\big(\varphi_{k+1}(\widetilde{Y}) \big) = \big(\mathrm{Bd}\, \tilde{f} \big) \cup \bigcup_{r=1}^{\infty} \bigcup_{i \in \widetilde{\mathcal{I}}_r} \tilde{f}_{r;i}(\widetilde{Y}). \tag{3.44}$$

Since $\tilde{f}_{r;i}|_{\mathrm{Int}\, \Delta'}$ is smooth for all $i \in \widetilde{\mathcal{I}}_r$, $r \in \mathbb{Z}^+$, and all simplices $\Delta' \subset \Delta^{k+1}$, $\mathrm{Bd}\, \tilde{f}|_{\widetilde{M}}$ has dimension at most $k-1$ by (3.44).

Let $Y_f \subset M$ denote the image of the $(k-1)$-skeleton of $|K|$ under η. The map

$$\iota_f : M - Y_f \longrightarrow \widetilde{M},$$
$$\iota_f\big(\eta(l_\sigma(t)) \big) = \big[r, \tilde{\iota}_r(\sigma), \iota_{k+1;\tilde{p}_r(\sigma)}(\tilde{\tau}_{(r,\sigma)}(t)) \big] \quad \forall \, \sigma \in K_r^{\mathrm{top}} - K_{r-1}^{\mathrm{top}}, \, r \in \mathbb{Z}^+, \, t \in \mathrm{Int}\, \Delta^k,$$

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is a well-defined embedding. By the first condition in (3.42) and (3.37),

$$\widetilde{f} \circ \iota_f = f|_{M-Y_f}.$$

Thus, $\widetilde{f}|_{\widetilde{M}}$ is a Borel-Moore pseudocycle equivalence between the Borel-Moore k-pseudocycles f and $\emptyset$, provided $\widetilde{M}$ is an oriented manifold, $\widetilde{F}|_{\widetilde{M}}$ is a smooth map, ι is a smooth embedding, and

$$\partial \widetilde{M} = \iota_f(M-Y_f).$$

These are again straightforward local statements, which are established as in (3) in the proof of [13, Lemma 3.3]. □

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