Vector-valued orthogonal modular forms

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2020 *Mathematics Subject Classification.* 11F55, 11F50

Key words and phrases. orthogonal modular forms, vector-valued modular forms, Siegel operator, Fourier-Jacobi expansion, vector-valued Jacobi forms, Petersson metric, vanishing theorem

Abstract. This monograph is devoted to the theory of vector-valued modular forms for orthogonal groups of signature $(2, n)$. Our purpose is multi-layered: (1) to lay a foundation of the theory of vector-valued orthogonal modular forms; (2) to develop some aspects of the theory in more depth such as geometry of the Siegel operators, filtrations associated to 1-dimensional cusps, decomposition of vector-valued Jacobi forms, square integrability etc; and (3) as applications derive several types of vanishing theorems for vector-valued modular forms of small weight. Our vanishing theorems imply in particular vanishing of holomorphic tensors of degree $< n/2 - 1$ on orthogonal modular varieties, which is optimal as a general bound.

The fundamental ingredients of the theory are the two Hodge bundles. The first is the Hodge line bundle which already appears in the theory of scalar-valued modular forms. The second Hodge bundle emerges in the vector-valued theory and plays a central role. It corresponds to the non-abelian part $\mathrm{O}(n, \mathbb{R})$ of the maximal compact subgroup of $\mathrm{O}(2, n)$. The main focus of this monograph is centered around the properties and the role of the second Hodge bundle in the theory of vector-valued orthogonal modular forms.

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CHAPTER 1

Introduction

In the theory of modular forms of several variables, it is natural and also necessary to study vector-valued modular forms. One way to account for this is that scalar-valued modular forms are concerned only with the 1-dimensional abelian quotient of the maximal compact subgroup K of the Lie group, while the contribution from the whole K emerges if we consider vector-valued modular forms. In more concrete levels, the significance of vector-valued modular forms appears in the study of the cohomology of modular varieties, holomorphic tensors on modular varieties, and constructions of Galois representations etc. The passage from scalar-valued to vector-valued modular forms is an intrinsic non-abelianization step.

This subject has been well-developed for Siegel modular forms since the pioneering work of Freitag, Weissauer and others around the early 1980's (see, e.g., [**19**] for a survey). In particular, a lot of detailed study have been done in the case of Siegel modular forms of genus 2.

By contrast, despite its potential and expected applications, no systematic theory of vector-valued modular forms for orthogonal groups of signature $(2, n)$ seems to have been developed so far. Only recently its application to holomorphic tensors on modular varieties started to be investigated ([**36**]). The observation that some aspects of the theory of scalar-valued Siegel modular forms of genus 2 have been generalized to orthogonal modular forms, rather than to Siegel modular forms of higher genus, also suggests a promising theory.

Vector-valued orthogonal modular forms will have applications to the geometry and arithmetic of orthogonal modular varieties, and so especially to the moduli spaces of $K3$ surfaces and holomorphic symplectic varieties. Moreover, from the geometric viewpoint of $K3$ surfaces, vector-valued modular forms on a period domain of (lattice-polarized) $K3$ surfaces are considered as holomorphic invariants related to the family that can be captured by the variation of the Hodge structures on $H^2(K3)$ but typically not by the Hodge line bundle $H^0(K_{K3})$ alone. For example in this direction, the infinitesimal invariants of normal functions for higher Chow cycles in $CH^2(K3, 1)$ give vector-valued modular forms with singularities (§3.8).

This geometric viewpoint offers another motivation to develop the theory of vector-valued orthogonal modular forms.

The purpose of this monograph is multi-layered:

(1) to lay a foundation of the theory of vector-valued orthogonal modular forms,
(2) to investigate some aspects of the theory in more depth, and
(3) as applications to establish several types of vanishing theorems for vector-valued modular forms of small weight.

Our theory is developed in a full generality in the sense that we work with general arithmetic groups $\Gamma < \mathrm{O}^+(L)$ for general integral quadratic forms L of signature $(2, n)$. The facts that unimodular lattices are rare even up to $\mathbb{Q}$-equivalence (unlike the symplectic case) and that various types of groups Γ appear in the moduli examples urge us to work in this generality.

Our approach is geometric in the sense that we define modular forms as sections of the automorphic vector bundles. Trivializations of the automorphic vector bundles, and thus passage from sections of vector bundles to vector-valued functions, are provided for each 0-dimensional cusp. This approach is suitable for working with general Γ, without losing connection with the more classical style.

In the rest of this introduction, we give a summary of the theory developed in this monograph.

The two Hodge bundles (§2). Let L be an integral quadratic lattice of signature $(2, n)$. We assume $n \geq 3$ for simplicity. The Hermitian symmetric domain $\mathcal{D} = \mathcal{D}_L$ attached to L is defined as an open subset of the isotropic quadric in $\mathbb{P}L_{\mathbb{C}}$. It parametrizes polarized Hodge structures $0 \subset F^2 \subset F^1 \subset L_{\mathbb{C}}$ of weight 2 on L with $\dim F^2 = 1$ and $F^1 = (F^2)^{\perp}$. Over $\mathcal{D}$ we have two fundamental Hodge bundles. The first is the Hodge line bundle

$$\mathcal{L} = \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)|_{\mathcal{D}},$$

which geometrically consists of the period lines F^2 in the Hodge filtrations. In terms of representation theory, $\mathcal{L}$ is the homogeneous line bundle associated to the standard character of $\mathbb{C}^* \subset \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$, where $\mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$ is the reductive part of a standard parabolic subgroup of $\mathrm{O}(L_{\mathbb{C}}) \simeq \mathrm{O}(n+2, \mathbb{C})$. Invariant sections of powers of $\mathcal{L}$ are scalar-valued modular forms on $\mathcal{D}$, which have been classically studied.

The Hodge line bundle $\mathcal{L}$ is naturally embedded in $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ as an isotropic sub line bundle. The second Hodge bundle is defined as

$$\mathcal{E} = \mathcal{L}^{\perp}/\mathcal{L}.$$

Geometrically this vector bundle consists of the middle graded quotients F^1/F^2 of the Hodge filtrations. In terms of representation theory, $\mathcal{E}$ is the homogeneous vector bundle associated to the standard representation of $\mathrm{O}(n,\mathbb{C}) \subset \mathbb{C}^* \times \mathrm{O}(n,\mathbb{C})$. It is this second Hodge bundle $\mathcal{E}$ that emerges in the theory of vector-valued modular forms on $\mathcal{D}$ and plays a central role in this monograph.

While $\mathcal{L}$ is concerned with scalar-valued modular forms, $\mathcal{E}$ is responsible for the higher rank aspect of the theory of vector-valued modular forms. While $\mathcal{L}$ provides a polarization, $\mathcal{E}$ is an orthogonal vector bundle, and in particular self-dual (but not trivial). Thus $\mathcal{L}$ and $\mathcal{E}$ are rather contrastive.

Vector-valued modular forms (§3). Weights of vector-valued modular forms on $\mathcal{D}$ are expressed by pairs (λ, k), where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ is a partition which corresponds to an irreducible representation V_λ of $\mathrm{O}(n,\mathbb{C})$, and k is an integer which corresponds to a character of $\mathbb{C}^*$. The partition λ satisfies ${}^t\lambda_1 + {}^t\lambda_2 \leq n$ where ${}^t\lambda$ is the transpose of λ. To such a pair (λ, k) we associate the automorphic vector bundle

$$\mathcal{E}_{\lambda,k} = \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k},$$

where $\mathcal{E}_\lambda$ is the vector bundle constructed from $\mathcal{E}$ by applying the orthogonal Schur functor associated to λ. Modular forms of weight (λ, k) are defined as holomorphic sections of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$ invariant under a finite-index subgroup Γ of $\mathrm{O}^+(L)$ (with cusp conditions when $n \leq 2$). We denote by $M_{\lambda,k}(\Gamma)$ the space of Γ-modular forms of weight (λ, k).

Sometimes it is more appropriate to work with irreducible representations of $\mathrm{SO}(n,\mathbb{C})$ rather than $\mathrm{O}(n,\mathbb{C})$, but in that way we obtain only $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundles. Since in some applications we encounter subgroups Γ of $\mathrm{O}^+(L)$ not contained in $\mathrm{SO}^+(L)$, we decided to work with $\mathrm{O}(n,\mathbb{C})$ at the outset. It is not difficult to switch to $\mathrm{SO}(n,\mathbb{C})$ (see §3.6).

Fourier expansion (§3). A first basic point is that $\mathcal{E}_{\lambda,k}$ can be trivialized for each 0-dimensional cusp of $\mathcal{D}$ in a natural way. Let I be a rank 1 primitive isotropic sublattice of L, which corresponds to a 0-dimensional cusp of $\mathcal{D}$. The quotient lattice $I^\perp/I$ is naturally endowed with a hyperbolic quadratic form. Then we have isomorphisms

$$I_\mathbb{C}^\vee \otimes \mathcal{O}_\mathcal{D} \to \mathcal{L}, \qquad (I^\perp/I)_\mathbb{C} \otimes \mathcal{O}_\mathcal{D} \to \mathcal{E},$$

canonically associated to I. If we write $V(I)_{\lambda,k} = ((I^\perp/I)_\mathbb{C})_\lambda \otimes (I_\mathbb{C}^\vee)^{\otimes k}$, these induce an isomorphism

$$V(I)_{\lambda,k} \otimes \mathcal{O}_\mathcal{D} \to \mathcal{E}_{\lambda,k},$$

which we call the *I-trivialization* of $\mathcal{E}_{\lambda,k}$. Via this trivialization, modular forms of weight (λ, k) are identified with $V(I)_{\lambda,k}$-valued holomorphic functions f on $\mathcal{D}$ satisfying invariance with the factor of automorphy. Then, after taking the tube domain realization of $\mathcal{D}$ associated to I ([**40**]), we obtain the Fourier expansion of f of the form

$$f(Z) = \sum_{l \in U(I)_{\mathbb{Z}}^{\vee}} a(l) \exp(2\pi i (l, Z))), \qquad Z \in \mathcal{D}_I, \tag{1.1}$$

where $\mathcal{D}_I$ is the tube domain in $(I^{\perp}/I)_{\mathbb{C}} \otimes I_{\mathbb{C}}$, $U(I)_{\mathbb{Z}}^{\vee}$ is a certain lattice in $(I^{\perp}/I)_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$, and $a(l) \in V(I)_{\lambda,k}$. By the Koecher principle, the index vectors l range only over the intersection of $U(I)_{\mathbb{Z}}^{\vee}$ with the closure of the positive cone (a connected component of the locus of vectors of positive norm). We prove that the constant term $a(0)$ always vanishes unless $\lambda = (0), (1^n)$, which correspond to the trivial and the determinant characters respectively. (In what follows, we write $\lambda = 1, \det$ instead.) Therefore the Siegel operators are interesting only at the 1-dimensional cusps. We can speak of rationality of the Fourier coefficients $a(l)$ because $V(I)_{\lambda,k}$ has a natural $\mathbb{Q}$-structure.

In this way, the choice of a 0-dimensional cusp I determines a passage to a more classical style of defining modular forms. Since there is no distinguished 0-dimensional cusp for a general arithmetic group Γ, we need to treat all 0-dimensional cusps equally. Even after the I-trivialization, it is more suitable to have $V(I)_{\lambda,k}$ as the *canonical* space of values, rather than identifying it with $\mathbb{C}^N$ by choosing a basis. This approach enables us to develop various later constructions in an intrinsic and coherent way (and so in a full generality) without sacrificing the classical style.

These most basic parts of the theory are developed in §2 and §3. In §4, as a functorial aspect of the theory, we study pullback and quasi-pullback of vector-valued modular forms to sub orthogonal modular varieties. This type of operations are sometimes called the *Witt operators*. The consideration of pullbacks leads to an elementary vanishing theorem for $M_{\lambda,k}(\Gamma)$ in $k \leq 0$ (Proposition 4.4). We prove that the quasi-pullback produces *cusp* forms (Proposition 4.10), generalizing a result of Gritsenko-Hulek-Sankaran [**23**] in the scalar-valued case.

After these foundational parts, this monograph is developed in the following two directions:

(1) Geometric treatment of the Siegel operators and the Fourier-Jacobi expansions at 1-dimensional cusps (§5 – §9).
(2) Square integrability of modular forms (§10 – §11).

Both lead, as applications, to vanishing theorems of respective type for modular forms of small weight.

Siegel operator (§6). Let J be a rank 2 primitive isotropic sublattice of L. This corresponds to a 1-dimensional cusp $\mathbb{H}_J$ of $\mathcal{D}$, which is isomorphic to the upper half plane. We take a geometric approach for introducing and studying the Siegel operator and the Fourier-Jacobi expansion at the cusp $\mathbb{H}_J$, by using the partial toroidal compactification over $\mathbb{H}_J$. The Siegel operator is the restriction to the boundary divisor, and the Fourier-Jacobi expansion is the Taylor expansion along it.

The Siegel domain realization of $\mathcal{D}$ with respect to J ([**40**]) is a two-step fibration

$$\mathcal{D} \xrightarrow{\pi_1} \mathcal{V}_J \xrightarrow{\pi_2} \mathbb{H}_J,$$

where π_1 is a fibration of upper half planes and π_2 is an affine space bundle. Dividing $\mathcal{D}$ by a rank 1 abelian group $U(J)_{\mathbb{Z}} < \Gamma$, the quotient $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$ is a fibration of punctured discs over $\mathcal{V}_J$. The partial toroidal compactification $\mathcal{X}(J) \hookrightarrow \overline{\mathcal{X}(J)}$ is obtained by filling the origins of the punctured discs ([**2**]). Its boundary divisor Δ_J is naturally identified with $\mathcal{V}_J$. We can extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\overline{\mathcal{X}(J)}$ via the I-trivialization for an arbitrary 0-dimensional cusp $I \subset J$, the result being independent of I (§5.4). This is an explicit form of Mumford's canonical extension [**37**] which is suitable for dealing with the Fourier-Jacobi expansion. If f is a Γ-modular form of weight (λ, k), it extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$.

Intuitively (and more traditionally), the Siegel operator should be an operation of "restriction to $\mathbb{H}_J$" which produces vector-valued modular forms of some reduced weight on $\mathbb{H}_J$. Geometrically this requires some justification because of the complicated structure around the boundary of the Baily-Borel compactification. We take a somewhat indirect but more geometrically tractable approach, working with the automorphic vector bundle $\mathcal{E}_{\lambda,k}$ over the partial toroidal compactification $\overline{\mathcal{X}(J)}$.

Let $\mathcal{L}_J$ be the Hodge line bundle on $\mathbb{H}_J$. We write $V(J) = (J^{\perp}/J)_{\mathbb{C}}$. For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ we denote by $V(J)_{\lambda'}$ the irreducible representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$ for the partition $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$.

Theorem 1.1 (Theorem 6.1). *Let $\lambda \neq 1, \det$. There exists a sub vector bundle $\mathcal{E}^J_{\lambda,k}$ of $\mathcal{E}_{\lambda,k}$ such that $\mathcal{E}^J_{\lambda,k}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J^{\otimes k+\lambda_1} \otimes V(J)_{\lambda'}$ and that the restriction of every modular form f of weight (λ, k) to Δ_J takes values in $\mathcal{E}^J_{\lambda,k}|_{\Delta_J}$. In particular, there exists a $V(J)_{\lambda'}$-valued cusp form $\Phi_J f$ of weight $k + \lambda_1$ on $\mathbb{H}_J$ such that $f|_{\Delta_J} = \pi_2^*(\Phi_J f)$.*

The map

$$M_{\lambda,k}(\Gamma) \to S_{k+\lambda_1}(\Gamma_J) \otimes V(J)_{\lambda'}, \qquad f \mapsto \Phi_J f,$$

is the Siegel operator at the J-cusp, where Γ_J is a suitable subgroup of $\mathrm{SL}(J) \simeq \mathrm{SL}(2, \mathbb{Z})$. If we take the I-trivialization for a 0-dimensional cusp $I \subset J$ and introduce suitable coordinates (τ, z, w) on the tube domain in which the Siegel domain realization is given by $(\tau, z, w) \mapsto (\tau, z) \mapsto \tau$, the

Siegel operator can be expressed as

$$(\Phi_J f)(\tau) = \lim_{t\to\infty} f(\tau, 0, it), \qquad \tau \in \mathbb{H}.$$

In this way, the naive "restriction to $\mathbb{H}_J$" can be geometrically justified at the level of automorphic vector bundles as the combined operation

$$\text{restrict to } \Delta_J \;\; + \;\; \text{reduce to } \mathcal{E}^J_{\lambda,k} \;\; + \;\; \text{descend to } \mathbb{H}_J.$$

This a priori tells us the modularity of $\Phi_J f$ with its weight. When $n = 3$, the weight calculation in Theorem 1.1 agrees with the corresponding result for Siegel modular forms of genus 2 ([**47**], [**1**]). The sub vector bundle $\mathcal{E}^J_{\lambda,k}$ will be taken up in §8 again from the viewpoint of a filtration on $\mathcal{E}_{\lambda,k}$.

Fourier-Jacobi expansion (§7). Next we explain the Fourier-Jacobi expansion at the J-cusp. Let Θ_J be the conormal bundle of Δ_J in $\overline{\mathcal{X}(J)}$. After certain choices, we have a special generator ω_J of the ideal sheaf of Δ_J. With this normal coordinate, we can take the Taylor expansion of a modular form $f \in M_{\lambda,k}(\Gamma)$ along Δ_J as a section of the extended bundle $\mathcal{E}_{\lambda,k}$:

$$f = \sum_{m\geq 0} \phi_m \omega_J^m. \tag{1.2}$$

The m-th Taylor coefficient ϕ_m, or rather $\phi_m \otimes \omega_J^{\otimes m}$, is essentially a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. We call (1.2) the *Fourier-Jacobi expansion* of f at the J-cusp, and call the section $\phi_m \otimes \omega_J^{\otimes m}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ for $m > 0$ the m-th *Fourier-Jacobi coefficient* of f. (ϕ_0 is just $f|_{\Delta_J}$ considered above.) Although the choice of ω_J is needed for defining the Fourier-Jacobi expansion, the resulting expansion and the sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ are independent of this choice, thus canonically determined by J (§7.2). This geometric definition of Fourier-Jacobi expansion, whose advantage is its canonicity, agrees with the more familiar style of defining Fourier-Jacobi expansion by slicing the Fourier expansion (§7.1) if we take the (I, ω_J)-trivialization.

In general, we define *vector-valued Jacobi forms* of weight (λ, k) and index $m > 0$ as holomorphic sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over $\Delta_J = \mathcal{V}_J$ which is invariant under the integral Jacobi group and satisfies a certain cusp condition (Definition 7.10). The m-th Fourier-Jacobi coefficient of a modular form of weight (λ, k) is such a vector-valued Jacobi form (Proposition 7.12). In the scalar-valued case, our geometric definition agrees with the classical definition of Jacobi forms ([**44**], [**21**]) after introducing suitable coordinates and trivialization (§7.4). When $n = 3$, our vector-valued Jacobi forms essentially agree with those considered by Ibukiyama-Kyomura [**28**] for Siegel modular forms of genus 2.

Filtrations associated to 1-dimension cusps (§8). While a 0-dimensional cusp of $\mathcal{D}$ provides a trivialization of $\mathcal{E}_{\lambda,k}$ which enables the Fourier expansion, we will show that a 1-dimensional cusp introduces a filtration on $\mathcal{E}_{\lambda,k}$ which is useful when studying the Fourier-Jacobi expansion. To start with, we observe that for each 1-dimensional cusp J, the second Hodge bundle $\mathcal{E}$ has an isotropic sub line bundle $\mathcal{E}_J$ canonically determined by J. This defines the filtration

$$0 \subset \mathcal{E}_J \subset \mathcal{E}_J^{\perp} \subset \mathcal{E}$$

associated to the J-cusp, which we call the *J-filtration*. Its graded quotients are respectively isomorphic to

$$\mathcal{E}_J \simeq \pi^*\mathcal{L}_J, \qquad \mathcal{E}_J^{\perp}/\mathcal{E}_J \simeq (J^{\perp}/J)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}, \qquad \mathcal{E}/\mathcal{E}_J^{\perp} \simeq \pi^*\mathcal{L}_J^{-1},$$

where $\pi = \pi_2 \circ \pi_1$ is the projection from $\mathcal{D}$ to $\mathbb{H}_J$. The J-filtration is translated to a constant filtration on $V(I) \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization for every adjacent 0-dimensional cusp $I \subset J$ (Proposition 8.3).

The J-filtration on $\mathcal{E}$ induces a (decreasing) filtration on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$, also called the J-filtration, whose graded quotient in level r is isomorphic to a direct sum of copies of $\pi^*\mathcal{L}_J^{\otimes k+r}$. Representation-theoretic calculations show that the J-filtration on $\mathcal{E}_{\lambda,k}$ has length $\leq 2\lambda_1 + 1$ (from level $-\lambda_1$ to λ_1), and that the sub vector bundle $\mathcal{E}_{\lambda,k}^J$ of $\mathcal{E}_{\lambda,k}$ in Theorem 1.1 is exactly the last (= level λ_1) sub vector bundle in the J-filtration (Proposition 8.13). Moreover, we have a duality between the graded quotients in level r and $-r$.

We give two applications of the J-filtration. The first is decomposition of vector-valued Jacobi forms. We prove that a vector-valued Jacobi form of weight (λ, k) decomposes, in a certain sense, into a tuple of scalar-valued Jacobi forms of various weights in the range $[k - \lambda_1, k + \lambda_1]$ (Proposition 8.15). More precisely, what is proved is that certain graded pieces are scalar-valued Jacobi forms, so this result does not mean that the theory of vector-valued Jacobi forms reduces to the scalar-valued theory. Nevertheless this decomposition theorem enables us to derive some basic results for vector-valued Jacobi forms from those for scalar-valued ones. For example, we deduce that vector-valued Jacobi forms of weight (λ, k) with $k + \lambda_1 < n/2 - 1$ vanish (Corollary 8.18). In the case of Siegel modular forms of genus 2 (namely $n = 3$), the fact that vector-valued Jacobi forms decompose into scalar-valued Jacobi forms was first found by Ibukiyama and Kyomura [**28**]. Their method is different, using differential operators, but it might be plausible that their decomposition agrees with that of us.

Vanishing theorem I (§9). It is a classical fact that there is no nonzero scalar-valued modular form of weight $0 < k < n/2 - 1$ on $\mathcal{D}$. Two proofs of this fact are well-known. The first uses vanishing of Jacobi forms (cf. [**21**], [**44**]), and the second uses classification of unitary representations. We give two generalizations of this classical vanishing theorem to the vector-valued case, corresponding to these two approaches.

Our first vanishing theorem belongs to the Jacobi form approach, and is obtained as the second application of the J-filtration. We assume that L has Witt index 2, i.e., $\mathcal{D}$ has a 1-dimensional cusp. This is always satisfied when $n \geq 5$.

THEOREM 1.2 (Theorem 9.1). *Let* $\lambda \neq 1, \det$. *If* $k < \lambda_1 + n/2 - 1$*, then* $M_{\lambda,k}(\Gamma) = 0$*. In particular,* $M_{\lambda,k}(\Gamma) = 0$ *whenever* $k < n/2$*.*

As a consequence, we obtain the following vanishing theorem for holomorphic tensors on the modular variety $\Gamma\backslash\mathcal{D}$.

COROLLARY 1.3 (Theorem 9.5). *Let X be the regular locus of* $\Gamma\backslash\mathcal{D}$*. Then we have* $H^0(X, (\Omega^1_X)^{\otimes k}) = 0$ *for all* $0 < k < n/2 - 1$*.*

Moreover, we obtain a classification of possible types of holomorphic tensors of the next few degrees up to $n/2$ (Proposition 9.6). The vanishing bound $k < n/2 - 1$ is optimal as a general bound.

The proof of Theorem 1.2 is built on the results of §7 and §8, and proceeds as follows. We apply the classical vanishing theorem of scalar-valued Jacobi forms of weight $< n/2 - 1$ ([**44**], [**21**]) to the first graded quotient of the J-filtration on $\mathcal{E}_{\lambda,k}$. This implies that the Fourier-Jacobi coefficients of $f \in M_{\lambda,k}(\Gamma)$ take values in a certain sub vector bundle of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. Passing to the Fourier expansion at $I \subset J$, we see that the Fourier coefficients of f are contained in a proper subspace of $V(I)_{\lambda,k}$. Finally, running J over all 1-dimensional cusps containing I, we conclude that the Fourier coefficients are zero.

In the case of Siegel modular forms of genus 2, the idea to use Jacobi forms to deduce a vanishing theorem for vector-valued modular forms seems to go back to Ibukiyama ([**26**] Section 6). Our proof of Theorem 1.2 can be regarded as a generalization of his argument.

In this way, we have the unified viewpoint that the Siegel operator is concerned with the last sub vector bundle in the J-filtration, while the proof of Theorem 1.2 makes use of the first graded quotient. We expect that a closer look at the intermediate pieces of the J-filtration would tell us more.

Square integrability (§10). We now turn to our second line of investigation. We can explicitly define and calculate an invariant Hermitian metric on $\mathcal{E}$ (and on $\mathcal{L}$, which is well-known). They are essentially the Hodge metrics. They induce an invariant Hermitian metric $(\ ,\ )_{\lambda,k}$ on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$. Apart from the matter of convergence, this defines the Petersson inner product on $M_{\lambda,k}(\Gamma)$:

$$(f,g) = \int_{\Gamma\backslash\mathcal{D}} (f,g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}, \qquad f,g \in M_{\lambda,k}(\Gamma),$$

where $\mathrm{vol}_{\mathcal{D}}$ is the invariant volume form on $\mathcal{D}$. When f or g is a cusp form, this integral converges as usual. Conversely, we prove the following. Let

$$\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_{[n/2]}) = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \cdots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

be the highest weight for $\mathrm{SO}(n,\mathbb{C})$ associated to λ. We write $|\bar{\lambda}| = \sum_i \bar{\lambda}_i$.

Theorem 1.4 (Theorem 10.1). *Let $\lambda \neq 1, \det$ and assume that $k \geq n + |\bar{\lambda}| - 1$. Then a modular form f of weight (λ, k) is a cusp form if and only if $(f,f) < \infty$.*

This holds also for $\lambda = 1, \det$ at least when L has Witt index 2 (Remark 10.13). In fact, Theorem 10.1 contains one more result that any modular form of weight (λ, k) with $k < n - |\bar{\lambda}| - 1$ and $\lambda \neq 1, \det$ is square integrable, but this is rather an intermediate step in the proof of our second vanishing theorem.

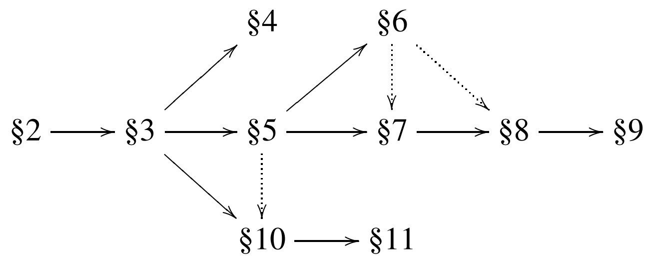
Vanishing theorem II (§11). Our study of square integrability is partly motivated by the following vanishing theorem. Let $\mathrm{corank}(\lambda)$ be the maximal index $1 \leq i \leq [n/2]$ such that $\bar{\lambda}_1 = \bar{\lambda}_2 = \cdots = \bar{\lambda}_i$. Let $S_{\lambda,k}(\Gamma) \subset M_{\lambda,k}(\Gamma)$ be the subspace of cusp forms.

Theorem 1.5 (Theorem 11.1). *Let $\lambda \neq 1, \det$. If $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$, there is no nonzero square integrable modular form of weight (λ, k). In particular,*
(1) $S_{\lambda,k}(\Gamma) = 0$ if $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$.
(2) $M_{\lambda,k}(\Gamma) = 0$ if $k < n - |\bar{\lambda}| - 1$.

Although $\lambda_1 + n/2 - 1 < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$, Theorem 1.5 does not supersede Theorem 1.2 because it is about square integrable modular forms. It depends on (λ, k) which bound in Theorem 1.2 or Theorem 1.5 (2) is larger. The two vanishing theorems are rather complementary.

The proof of Theorem 1.5 is parallel to Weissauer's vanishing theorem [**47**] for Siegel modular forms. If we have a square integrable modular form, we can construct a unitary highest weight module for $\mathrm{SO}^+(L_{\mathbb{R}})$ by a standard procedure. Then the bound $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$ is derived from the classification of unitary highest weight modules ([**13**], [**12**], [**29**]). The more specific conclusions (1), (2) are consequences of the square integrability theorem (Theorem 10.1).

Organization. The logical dependence between the chapters is as follows. A dotted arrow means that the dependence is weak.



Notations. Let us summarize some standing terminologies and notations.

(1) By a *lattice* we mean a free $\mathbb{Z}$-module L of finite rank equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$. (Sometimes we still use the word "lattice" when the bilinear form is only $\mathbb{Q}$-valued.) The dual lattice $\mathrm{Hom}(L, \mathbb{Z})$ of L is written as $L^\vee$. A sublattice M of L is called *primitive* if L/M is free. We denote by $M^\perp$ the orthogonal complement of M in L. A sublattice I of L is called *isotropic* if $(I, I) \equiv 0$. The lattice L is called an *even lattice* if $(l,l) \in 2\mathbb{Z}$ for every $l \in L$. The orthogonal group of a lattice L is denoted by $\mathrm{O}(L)$. For $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ we write $L_F = L \otimes_{\mathbb{Z}} F$. This is a quadratic space over F. Its orthogonal group is denoted by $\mathrm{O}(L_F)$. The special orthogonal group, namely the subgroup of $\mathrm{O}(L_F)$ of determinant 1, is denoted by $\mathrm{SO}(L_F)$. A lattice L in a $\mathbb{Q}$-quadratic space V is called a *full lattice* in V if $V = L_{\mathbb{Q}}$. For a rational number $\alpha \neq 0$ we write $L(\alpha)$ for the α-scaling of L, namely the same underlying $\mathbb{Z}$-module with the bilinear form multiplied by α. In the context of lattices, the symbol U will stand for the integral hyperbolic plane, namely the even unimodular lattice of signature $(1, 1)$.

(2) Let G be a group acting on a set X and let Y be a subset of X. By the *stabilizer* of Y in G, we mean the subgroup of G consisting of elements g such that $g(Y) = Y$.

(3) Let V be a nondegenerate quadratic space over $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Let I be an isotropic line in V, and $P(I)$ be the stabilizer of I in $\mathrm{O}(V)$. Then we have the canonical exact sequence

$$0 \to (I^\perp/I) \otimes_F I \to P(I) \to \mathrm{GL}(I) \times \mathrm{O}(I^\perp/I) \to 1. \tag{1.3}$$

Here $P(I) \to \mathrm{GL}(I)$ and $P(I) \to \mathrm{O}(I^\perp/I)$ are the natural maps, and the map $(I^\perp/I) \otimes_F I \to P(I)$ sends a vector $m \otimes l$ of $(I^\perp/I) \otimes_F I$ to the isometry $E_{m\otimes l}$ of V defined by

$$E_{m\otimes l}(v) = v - (\tilde{m}, v)l + (l, v)\tilde{m} - \frac{1}{2}(m,m)(l,v)l, \qquad v \in V. \tag{1.4}$$

Here $\tilde{m} \in I^{\perp}$ is a lift of $m \in I^{\perp}/I$. In particular, when $v \in I^{\perp}$, (1.4) is simplified to

$$E_{m\otimes l}(v) = v - (m, v)l.$$

The isometries $E_{m\otimes l}$ are sometimes called the *Eichler transvections*. If we take a basis $e_1, \cdots, e_n$ of V such that $I = \langle e_1 \rangle$, $I^{\perp} = \langle e_1, \cdots, e_{n-1} \rangle$ and $(e_1, e_n) = 1$, $(e_i, e_n) = 0$ for $i > 1$, then $E_{m\otimes e_1}$ is expressed by the matrix

$$\begin{pmatrix} 1 & -m^{\vee} & -(m,m)/2 \\ 0 & I_{n-2} & m \\ 0 & 0 & 1 \end{pmatrix}$$

where we regard $m \in \langle e_2, \cdots, e_{n-1} \rangle \simeq I^{\perp}/I$. The group $(I^{\perp}/I)\otimes_F I$ of Eichler transvections is the unipotent radical of $P(I)$.

(4) We will not distinguish between vector bundles and locally free sheaves on a complex manifold X. The fiber of a vector bundle $\mathcal{F}$ over a point $x \in X$ is denoted by $\mathcal{F}_x$ (not the germ of the sheaf). A collection of sections of a vector bundle $\mathcal{F}$ is called a *frame* of $\mathcal{F}$ when it defines an isomorphism $\mathcal{O}_X^{\oplus r} \simeq \mathcal{F}$, i.e., it forms a basis in every fiber. The dual vector bundle of $\mathcal{F}$ is denoted by $\mathcal{F}^{\vee}$.

(5) Let X be a complex manifold and G be a group acting on X. Let $\mathcal{F}$ be a G-equivariant vector bundle on X. Suppose that $\mathcal{F}$ is endowed with an isomorphism $\iota \colon V \otimes \mathcal{O}_X \to \mathcal{F}$ for a $\mathbb{C}$-linear space V. Then the *factor of automorphy* of the G-action on $\mathcal{F}$ with respect to the trivialization ι is the $\mathrm{GL}(V)$-valued function on $G \times X$ defined by

$$j(g, x) = \iota_{gx}^{-1} \circ g \circ \iota_x : \ V \to \mathcal{F}_x \to \mathcal{F}_{gx} \to V \tag{1.5}$$

for $g \in G$, $x \in X$. Here the middle map is the equivariant action by g. If Γ is a subgroup of G, a Γ-invariant section of $\mathcal{F}$ over X is identified via ι with a V-valued holomorphic function f on X satisfying $f(\gamma x) = j(\gamma, x)f(x)$ for every $\gamma \in \Gamma$ and $x \in X$.

(6) We write $e(z) = \exp(2\pi i z)$ for $z \in \mathbb{C}/\mathbb{Z}$. We use the symbol $\mathbb{H}$ for the upper half plane $\{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$.

Acknowledgements. It is our pleasure to thank Eberhard Freitag and Tomoyoshi Ibukiyama for sharing with us their perspectives on vector-valued modular forms. This work is supported by KAKENHI 21H00971 and 20H00112.

CHAPTER 2

The two Hodge bundles

In this chapter we study some basic properties of the Hodge bundles $\mathcal{L}$ and $\mathcal{E}$. In §2.1 we recall basic facts on the Hermitian symmetric domains of type IV. The Hodge line bundle $\mathcal{L}$ is well-known, and we recall it in §2.2. In §2.3 and §2.4 we study the second Hodge bundle $\mathcal{E}$. In §2.5 we describe $\mathcal{E}$ and $\mathcal{L}$ in the case $n \leq 4$ under the accidental isomorphisms.

2.1. The domain

Let L be a lattice of signature $(2, n)$. Let $Q = Q_L$ be the isotropic quadric in $\mathbb{P}L_{\mathbb{C}}$ defined by the equation $(\omega, \omega) = 0$ for $\omega \in L_{\mathbb{C}}$. We express a point of Q as $[\omega] = \mathbb{C}\omega$. The open set of Q defined by the inequality $(\omega, \bar{\omega}) > 0$ has two connected components. They are interchanged by the complex conjugation $\omega \mapsto \bar{\omega}$. We choose one of them and denote it by $\mathcal{D} = \mathcal{D}_L$. This is the Hermitian symmetric domain attached to L. In Cartan's classification, $\mathcal{D}$ is a Hermitian symmetric domain of type IV. The isotropic quadric Q is the compact dual of $\mathcal{D}$. Points of $\mathcal{D}$ are in one-to-one correspondence with positive-definite planes in $L_{\mathbb{R}}$, by associating

$$\mathcal{D} \ni [\omega] \quad \mapsto \quad H_\omega = \langle \mathrm{Re}(\omega), \mathrm{Im}(\omega) \rangle.$$

The choice of the component $\mathcal{D}$ determines orientation on the positive-definite planes. Note that $(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) = 0$ and $(\mathrm{Re}(\omega), \mathrm{Re}(\omega)) = (\mathrm{Im}(\omega), \mathrm{Im}(\omega))$ by the isotropicity condition $(\omega, \omega) = 0$.

We denote by $\mathrm{O}^+(L_{\mathbb{R}})$ the index 2 subgroup of $\mathrm{O}(L_{\mathbb{R}})$ preserving the component $\mathcal{D}$. Then $\mathrm{O}^+(L_{\mathbb{R}})$ consists of two connected components, the identity component being $\mathrm{SO}^+(L_{\mathbb{R}}) = \mathrm{O}^+(L_{\mathbb{R}}) \cap \mathrm{SO}(L_{\mathbb{R}})$. The stabilizer K of a point $[\omega] \in \mathcal{D}$ in $\mathrm{O}^+(L_{\mathbb{R}})$ is the same as the stabilizer of the oriented plane H_ω, and is described as

$$K = \mathrm{SO}(H_\omega) \times \mathrm{O}(H_\omega^\perp) \simeq \mathrm{SO}(2, \mathbb{R}) \times \mathrm{O}(n, \mathbb{R}).$$

This is a maximal compact subgroup of $\mathrm{O}^+(L_{\mathbb{R}})$. We have $\mathcal{D} \simeq \mathrm{O}^+(L_{\mathbb{R}})/K$. On the other hand, as explained in (1.3), the stabilizer P of $[\omega]$ in $\mathrm{O}(L_{\mathbb{C}})$ sits in the canonical exact sequence

$$0 \to (\omega^\perp/\mathbb{C}\omega) \otimes \mathbb{C}\omega \to P \to \mathrm{GL}(\mathbb{C}\omega) \times \mathrm{O}(\omega^\perp/\mathbb{C}\omega) \to 1. \tag{2.1}$$

The reductive part

$$\mathrm{GL}(\mathbb{C}\omega) \times \mathrm{O}(\omega^{\perp}/\mathbb{C}\omega) \simeq \mathbb{C}^* \times \mathrm{O}(n, \mathbb{C})$$

is the complexification of K.

The domain $\mathcal{D}$ has two types of rational boundary components (cusps): 0-dimensional and 1-dimensional cusps. The 0-dimensional cusps correspond to rational isotropic lines in $L_{\mathbb{Q}}$, or equivalently, rank 1 primitive isotropic sublattices I of L. The point $p_I = [I_{\mathbb{C}}]$ of Q is in the closure of $\mathcal{D}$, and this is the 0-dimensional cusp corresponding to I. The 1-dimensional cusps correspond to rational isotropic planes in $L_{\mathbb{Q}}$, or equivalently, rank 2 primitive isotropic sublattices J of L. Each such J determines the line $\mathbb{P}J_{\mathbb{C}}$ on Q. If we remove $\mathbb{P}J_{\mathbb{R}}$ from $\mathbb{P}J_{\mathbb{C}}$, then $\mathbb{P}J_{\mathbb{C}} - \mathbb{P}J_{\mathbb{R}}$ consists of two copies of the upper half plane, one in the closure of $\mathcal{D}$. This component, say $\mathbb{H}_J$, is the 1-dimensional cusp corresponding to J. A 0-dimensional cusp p_I is in the closure of a 1-dimensional cusp $\mathbb{H}_J$ if and only if $I \subset J$.

Let $\mathrm{O}^+(L) = \mathrm{O}(L) \cap \mathrm{O}^+(L_{\mathbb{R}})$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. By Baily-Borel [**3**], the quotient space

$$\mathcal{F}(\Gamma)^{bb} = \Gamma\backslash\left(\mathcal{D} \cup \bigcup_J \mathbb{H}_J \cup \bigcup_I p_I\right)$$

has the structure of a normal projective variety of dimension n. Here the union of $\mathcal{D}$ and the cusps is equipped with the so-called Satake topology. In particular, the quotient

$$\mathcal{F}(\Gamma) = \Gamma\backslash\mathcal{D}$$

is a normal quasi-projective variety. The variety $\mathcal{F}(\Gamma)^{bb}$ is called the *Baily-Borel compactification* of $\mathcal{F}(\Gamma)$.

2.2. The Hodge line bundle

In this section we recall the first Hodge bundle. Let $\mathcal{O}_Q(-1)$ be the tautological line bundle over $Q \subset \mathbb{P}L_{\mathbb{C}}$. The Hodge line bundle over $\mathcal{D}$ is defined as

$$\mathcal{L} = \mathcal{O}_Q(-1)|_{\mathcal{D}}.$$

This is an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant sub line bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$. The fiber of $\mathcal{L}$ over $[\omega] \in \mathcal{D}$ is the line $\mathbb{C}\omega$. By definition $\mathcal{L}$ extends over Q naturally, and we sometimes write $\mathcal{L} = \mathcal{O}_Q(-1)$ when no confusion is likely to occur. A holomorphic section of $\mathcal{L}^{\otimes k}$ over $\mathcal{D}$ invariant under a finite-index subgroup of $\mathrm{O}^+(L)$ and holomorphic at the cusps (in the sense explained later) is called a (scalar-valued) modular form of weight k.

The stabilizer $K \subset \mathrm{O}^+(L_{\mathbb{R}})$ of a point $[\omega] \in \mathcal{D}$ acts on the fiber $\mathcal{L}_{[\omega]}$ of $\mathcal{L}$ as the weight 1 character of $\mathrm{SO}(2, \mathbb{R}) \subset K$. Therefore, if we denote by

$W \simeq \mathbb{C}$ the representation space of the weight 1 character of $\mathrm{SO}(2,\mathbb{R})$, we have an $\mathrm{O}^+(L_\mathbb{R})$-equivariant isomorphism

$$\mathcal{L} \simeq \mathrm{O}^+(L_\mathbb{R}) \times_K \mathcal{L}_{[\omega]} \simeq \mathrm{O}^+(L_\mathbb{R}) \times_K W.$$

Similarly, the extension $\mathcal{O}_Q(-1)$ over Q is the homogeneous line bundle corresponding to the weight 1 character of $\mathbb{C}^* \subset \mathbb{C}^* \times \mathrm{O}(n,\mathbb{C})$.

A trivialization of $\mathcal{L}$ can be defined for each 0-dimensional cusp of $\mathcal{D}$ as follows. Let I be a rank 1 primitive isotropic sublattice of L. For later use, it is useful to work over the following enlargement of $\mathcal{D}$:

$$Q(I) = Q - Q \cap \mathbb{P}I_\mathbb{C}^\perp.$$

This is a Zariski open set of Q containing $\mathcal{D}$. Its complement $Q \cap \mathbb{P}I_\mathbb{C}^\perp$ is the cone over the isotropic quadric in $\mathbb{P}(I^\perp/I)_\mathbb{C}$ with vertex $[I_\mathbb{C}]$. If $[\omega] \in Q(I)$, the pairing between $I_\mathbb{C}$ and $\mathbb{C}\omega$ is nonzero. This defines an isomorphism $\mathbb{C}\omega \to I_\mathbb{C}^\vee$. Since $\mathbb{C}\omega$ is the fiber of $\mathcal{L} = \mathcal{O}_Q(-1)$ over $[\omega]$, by varying $[\omega]$ we obtain an isomorphism

$$I_\mathbb{C}^\vee \otimes \mathcal{O}_{Q(I)} \to \mathcal{L} \tag{2.2}$$

of line bundles on $Q(I)$. We call this isomorphism the *I-trivialization* of $\mathcal{L}$. This is equivariant with respect to the stabilizer of $I_\mathbb{C}$ in $\mathrm{O}(L_\mathbb{C})$. Over Q the I-trivialization has pole of order 1 at the divisor $Q \cap \mathbb{P}I_\mathbb{C}^\perp$, and hence extends to an isomorphism

$$I_\mathbb{C}^\vee \otimes \mathcal{O}_Q \to \mathcal{L}(Q \cap \mathbb{P}I_\mathbb{C}^\perp).$$

In what follows, we work over $\mathcal{D}$. We call the restriction of (2.2) to $\mathcal{D}$ the *I-trivialization* of $\mathcal{L}$ too. If we choose a nonzero vector of $I_\mathbb{C}^\vee$, it defines a nowhere vanishing section of $\mathcal{L}$ via the I-trivialization. To be more specific, we choose a vector $l \neq 0 \in I$ and let s_l be the section of $\mathcal{L}$ corresponding to the dual vector of l. This section is determined by the condition that the vector $s_l([\omega]) \in \mathcal{L}_{[\omega]} = \mathbb{C}\omega$ has pairing 1 with l. The factor of automorphy of the $\mathrm{O}^+(L_\mathbb{R})$-action on $\mathcal{L}$ with respect to the I-trivialization is a function on $\mathrm{O}^+(L_\mathbb{R}) \times \mathcal{D}$ which can be written as

$$j(g,[\omega]) = \frac{g \cdot s_l([\omega])}{s_l([g\omega])} = \frac{(g\omega, l)}{(\omega, l)}, \qquad g \in \mathrm{O}^+(L_\mathbb{R}),\ [\omega] \in \mathcal{D}. \tag{2.3}$$

This gives a more classical style of defining scalar-valued modular forms. Note that if g acts trivially on $I_\mathbb{R}$, then $j(g,[\omega]) \equiv 1$.

2.3. The second Hodge bundle

In this section we define the second Hodge bundle. We have a natural quadratic form on the vector bundle $L_\mathbb{C} \otimes \mathcal{O}_\mathcal{D}$. By the definition of Q, $\mathcal{L}$ is an isotropic sub line bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$, so we have $\mathcal{L} \subset \mathcal{L}^{\perp}$. The second Hodge bundle is defined by

$$\mathcal{E} = \mathcal{L}^{\perp}/\mathcal{L}.$$

This is an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant vector bundle of rank n over $\mathcal{D}$. The fiber of $\mathcal{E}$ over $[\omega] \in \mathcal{D}$ is $\omega^{\perp}/\mathbb{C}\omega$. The quadratic form on $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ induces a nondegenerate $\mathrm{O}^{+}(L_{\mathbb{R}})$-invariant quadratic form on $\mathcal{E}$. In other words, $\mathcal{E}$ is an orthogonal vector bundle. In particular, we have $\mathcal{E}^{\vee} \simeq \mathcal{E}$. Since $\mathcal{L}$ is naturally defined on Q, $\mathcal{E}$ is also naturally defined on Q. This is an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle. By abuse of notation, we often use the same notation $\mathcal{E}$ for this extended vector bundle.

The stabilizer $K \subset \mathrm{O}^{+}(L_{\mathbb{R}})$ of a point $[\omega] \in \mathcal{D}$ acts on the fiber $\mathcal{E}_{[\omega]}$ of $\mathcal{E}$ as the standard $\mathbb{C}$-representation of $\mathrm{O}(n,\mathbb{R}) \subset K$, because we have a natural isomorphism $H_{\omega}^{\perp} \otimes_{\mathbb{R}} \mathbb{C} \simeq \omega^{\perp}/\mathbb{C}\omega$. Therefore, if we denote by $V = \mathbb{C}^{n}$ the standard representation space of $\mathrm{O}(n,\mathbb{C})$, we have an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E} \simeq \mathrm{O}^{+}(L_{\mathbb{R}}) \times_{K} \mathcal{E}_{[\omega]} \simeq \mathrm{O}^{+}(L_{\mathbb{R}}) \times_{K} V. \tag{2.4}$$

Similarly, the extension of $\mathcal{E}$ over Q is the homogeneous vector bundle corresponding to the standard representation of $\mathrm{O}(n,\mathbb{C}) \subset \mathbb{C}^{*} \times \mathrm{O}(n,\mathbb{C})$.

We present some examples where $\mathcal{E}$ and $\mathcal{L}$ appear naturally.

Example 2.1. The “third” Hodge bundle $(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}})/\mathcal{L}^{\perp}$ is isomorphic to $\mathcal{L}^{-1}$ by the natural pairing with $\mathcal{L}$.

Example 2.2. The determinant line bundle $\det \mathcal{E} = \wedge^{n}\mathcal{E}$ of $\mathcal{E}$ is isomorphic, as an $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant bundle, to the line bundle $\det \otimes \mathcal{O}_{\mathcal{D}}$ associated to the determinant character $\det\colon \mathrm{O}^{+}(L_{\mathbb{R}}) \to \{\pm 1\}$ of $\mathrm{O}^{+}(L_{\mathbb{R}})$. Indeed, by Example 2.1, we have the $\mathrm{O}^{+}(L_{\mathbb{R}})$-equivariant isomorphism

$$\det \mathcal{E} \simeq \det(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}) \otimes \mathcal{L} \otimes \mathcal{L}^{-1} \simeq \det(L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}) \simeq \det \otimes \mathcal{O}_{\mathcal{D}}.$$

The line bundle $\det \otimes \mathcal{O}_{\mathcal{D}}$ appears in the study of scalar-valued modular forms with determinant character.

Example 2.3. Let $T_{\mathcal{D}}$ and $\Omega^{1}_{\mathcal{D}}$ be the tangent and cotangent bundles of $\mathcal{D}$ respectively. Then we have the canonical isomorphisms

$$T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}^{-1}, \qquad \Omega^{1}_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}. \tag{2.5}$$

Indeed, by the Euler sequence for $\mathbb{P}L_{\mathbb{C}}$, we have

$$T_{\mathbb{P}L_{\mathbb{C}}} \simeq \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(1) \otimes ((L_{\mathbb{C}} \otimes \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}})/\mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)).$$

As a sub vector bundle of $T_{\mathbb{P}L_{\mathbb{C}}}|_{Q}$, we have

$$T_{Q} \simeq \mathcal{O}_{Q}(1) \otimes (\mathcal{O}_{Q}(-1)^{\perp}/\mathcal{O}_{Q}(-1)) = \mathcal{L}^{-1} \otimes \mathcal{E}.$$

The isomorphism for Ω^{1}_{Q} is obtained by taking the dual.

Tautologically, the identity of $\mathcal{D}$ can be regarded as the period map $[\omega] \mapsto \mathcal{L}_{[\omega]}$ for the universal variation $0 \subset \mathcal{L} \subset \mathcal{L}^{\perp} \subset L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ of Hodge structures on $\mathcal{D}$. Then the isomorphism $T_{\mathcal{D}} \simeq \mathcal{L}^{-1} \otimes \mathcal{E}$ is nothing but the differential of this tautological period map (cf. [**46**] §10.1). By taking the adjunctions of $T_{\mathcal{D}} \simeq \mathcal{L}^{-1} \otimes \mathcal{E}$, we obtain the homomorphisms

$$\mathcal{L} \otimes T_{\mathcal{D}} \xrightarrow{\simeq} \mathcal{E}, \qquad \mathcal{E} \otimes T_{\mathcal{D}} \to \mathcal{L}^{-1}. \tag{2.6}$$

These are familiar forms in the context of variation of Hodge structures. Here the second homomorphism is given by the pairing on $\mathcal{E}$:

$$\mathcal{E} \otimes T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{E} \otimes \mathcal{L}^{-1} \to \mathcal{L}^{-1}.$$

Example 2.4. Adjunctions of (2.6) induce the following complex of vector bundles on $\mathcal{D}$ (the *Koszul complex*):

$$\mathcal{L} \to \mathcal{E} \otimes \Omega^1_{\mathcal{D}} \to \mathcal{L}^{-1} \otimes \Omega^2_{\mathcal{D}}. \tag{2.7}$$

Here the second homomorphism is the composition

$$\mathcal{E} \otimes \Omega^1_{\mathcal{D}} \simeq \mathcal{L}^{-1} \otimes \Omega^1_{\mathcal{D}} \otimes \Omega^1_{\mathcal{D}} \xrightarrow{\wedge} \mathcal{L}^{-1} \otimes \Omega^2_{\mathcal{D}}.$$

By (2.5), the Koszul complex is identified with the complex

$$\mathcal{L} \otimes (\mathcal{O}_{\mathcal{D}} \to \mathcal{E}^{\otimes 2} \xrightarrow{\wedge} \wedge^2 \mathcal{E}),$$

where $\mathcal{O}_{\mathcal{D}} \to \mathcal{E}^{\otimes 2}$ is the embedding defined by the quadratic form on $\mathcal{E}$. This shows that (2.7) is indeed a complex, and its middle cohomology sheaf is isomorphic to

$$(\mathrm{Sym}^2\mathcal{E}/\mathcal{O}_{\mathcal{D}}) \otimes \mathcal{L} \simeq \mathcal{E}_{(2)} \otimes \mathcal{L},$$

where $\mathcal{E}_{(2)}$ is the automorphic vector bundle associated to the representation $\mathrm{Sym}^2\mathbb{C}^n/\mathbb{C}$ of $\mathrm{O}(n, \mathbb{C})$ (see §3.2). The Koszul complex will be taken up in §3.8.

2.4. I-trivialization of the second Hodge bundle

In this section we define a trivialization of $\mathcal{E}$ associated to each 0-dimensional cusp. This is the starting point of various later constructions.

Let I be a rank 1 primitive isotropic sublattice of L. The quadratic form on L induces a hyperbolic quadratic form on the $\mathbb{Z}$-module $I^{\perp}/I$. We write $V(I)_F = (I^{\perp}/I) \otimes_{\mathbb{Z}} F$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. This is a quadratic space over F. We especially abbreviate $V(I) = V(I)_{\mathbb{C}}$. We consider the following sub vector bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{Q(I)}$:

$$I^{\perp} \cap \mathcal{L}^{\perp} = (I^{\perp}_{\mathbb{C}} \otimes \mathcal{O}_{Q(I)}) \cap \mathcal{L}^{\perp}.$$

Th fiber of $I^{\perp} \cap \mathcal{L}^{\perp}$ over $[\omega] \in Q(I)$ is the subspace $I^{\perp}_{\mathbb{C}} \cap \omega^{\perp}$ of $L_{\mathbb{C}}$. The projection $\mathcal{L}^{\perp} \to \mathcal{E}$ induces a homomorphism $I^{\perp} \cap \mathcal{L}^{\perp} \to \mathcal{E}$, and the projection $I^{\perp}_{\mathbb{C}} \to V(I)$ induces a homomorphism $I^{\perp} \cap \mathcal{L}^{\perp} \to V(I) \otimes \mathcal{O}_{Q(I)}$.

Lemma 2.5. *The homomorphisms* $I^\perp \cap \mathcal{L}^\perp \to \mathcal{E}$ *and* $I^\perp \cap \mathcal{L}^\perp \to V(I) \otimes \mathcal{O}_{Q(I)}$ *are isomorphisms. Therefore we obtain an isomorphism*

$$V(I) \otimes \mathcal{O}_{Q(I)} \to \mathcal{E} \tag{2.8}$$

of vector bundles on $Q(I)$*. This is equivariant with respect to the stabilizer of* $I_{\mathbb{C}}$ *in* $\mathrm{O}(L_{\mathbb{C}})$*, and preserves the quadratic forms on both sides.*

Proof. At the fibers over a point $[\omega] \in Q(I)$, the two homomorphisms are given by the linear maps $I_{\mathbb{C}}^\perp \cap \omega^\perp \to \omega^\perp/\mathbb{C}\omega$ and $I_{\mathbb{C}}^\perp \cap \omega^\perp \to (I^\perp/I)_{\mathbb{C}}$ respectively. The source and the target have the same dimension ($= n$) for both maps, so it suffices to check the injectivity of these two maps. This is equivalent to $I_{\mathbb{C}}^\perp \cap \mathbb{C}\omega = 0$ and $\omega^\perp \cap I_{\mathbb{C}} = 0$ respectively, and both follow from the nondegeneracy $(I_{\mathbb{C}}, \mathbb{C}\omega) \neq 0$ for $[\omega] \in Q(I)$.

Since both $I_{\mathbb{C}}^\perp \cap \omega^\perp \to \omega^\perp/\mathbb{C}\omega$ and $I_{\mathbb{C}}^\perp \cap \omega^\perp \to (I^\perp/I)_{\mathbb{C}}$ preserve the quadratic forms, so does the composition $\omega^\perp/\mathbb{C}\omega \to (I^\perp/I)_{\mathbb{C}}$. Hence (2.8) preserves the quadratic forms. The equivariance of (2.8) can be verified similarly. □

We call the isomorphism (2.8) and its restriction to $\mathcal{D}$ the *I-trivialization* of $\mathcal{E}$. This is a trivialization as an orthogonal vector bundle. See Claim 6.10 for the boundary behavior of this isomorphism at a Zariski open set of the divisor $Q \cap \mathbb{P}I_{\mathbb{C}}^\perp$.

For later use, we calculate the sections of $\mathcal{E}$ corresponding to vectors of $V(I)$. We choose a vector $l \neq 0$ of I and let s_l be the corresponding section of $\mathcal{L}$ as defined in §2.2.

Lemma 2.6. *Let* v *be a vector of* $V(I)$*. We define a section of* $I^\perp \cap \mathcal{L}^\perp$ *by*

$$s_v([\omega]) = \tilde{v} - (\tilde{v}, s_l([\omega]))l, \qquad [\omega] \in Q(I),$$

where $\tilde{v} \in I_{\mathbb{C}}^\perp$ *is a lift of* $v \in V(I)$ *and we regard* $s_l([\omega]) \in \mathbb{C}\omega \subset L_{\mathbb{C}}$*. Then the image of* s_v *in* $\mathcal{E}$ *is the section of* $\mathcal{E}$ *which corresponds by the I-trivialization to the constant section of* $V(I) \otimes \mathcal{O}_{Q(I)}$ *with value* v*.*

Proof. It is straightforward to check that $s_v([\omega])$ does not depend on the choice of the lift $\tilde{v}$ and that $(s_v([\omega]), \omega) = (s_v([\omega]), l) = 0$. Thus s_v is indeed a section of $I^\perp \cap \mathcal{L}^\perp$. Since $s_v([\omega]) \equiv \tilde{v} \bmod I_{\mathbb{C}}$ as a vector of $I_{\mathbb{C}}^\perp$, the image of $s_v([\omega])$ in $V(I)$ is v. This proves our assertion. □

2.5. Accidental isomorphisms

When $n \leq 4$, orthogonal modular varieties are isomorphic to other types of classical modular varieties by the so-called accidental isomorphisms. In this section we explain how the second Hodge bundle $\mathcal{E}$ in $n \leq 4$ is translated under the accidental isomorphism. (This is well-known for $\mathcal{L}$; we also include it for completeness.) This correspondence is the basis of comparing vector-valued orthogonal modular forms in $n = 3, 4$ with vector-valued Siegel and Hermitian modular forms respectively. We explain the translation from both algebro-geometric and representation-theoretic viewpoints. Since the contents of this section will be used only sporadically in the rest of this monograph, the reader may skip it for the moment.

2.5.1. Modular curves. When $n = 1$, the accidental isomorphism between the real Lie groups is $\mathrm{PSL}(2, \mathbb{R}) \simeq \mathrm{SO}^+(1, 2)$. Its complexification is $\mathrm{PSL}(2, \mathbb{C}) \simeq \mathrm{SO}(3, \mathbb{C})$. This lifts to $\mathrm{SL}(2, \mathbb{C}) \simeq \mathrm{Spin}(3, \mathbb{C})$. The isomorphism between the compact duals is provided by the anti-canonical embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ of $\mathbb{P}^1$, which maps $\mathbb{P}^1$ to a conic $Q \subset \mathbb{P}^2$. This gives an isomorphism between the upper half plane and the type IV domain in $n = 1$. The line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\mathcal{O}_{\mathbb{P}^1}(-2)$ on $\mathbb{P}^1$. This means that orthogonal modular forms of weight k correspond to elliptic modular forms of weight $2k$.

The reductive part of a standard parabolic subgroup of $\mathrm{SL}(2, \mathbb{C})$ is the 1-dimensional torus T consisting of diagonal matrices $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ of determinant 1. The corresponding group in $\mathrm{PSL}(2, \mathbb{C})$ is $T/-1$. The weight 2 character $\alpha \mapsto \alpha^2$ of T defines an isomorphism $T/-1 \simeq \mathbb{C}^*$. This explains $\mathcal{O}_Q(-1) \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$ from representation theory.

The full orthogonal group $\mathrm{O}(3, \mathbb{C})$ is $\mathrm{SO}(3, \mathbb{C}) \times \{\pm \mathrm{id}\}$. By Example 2.2, the second Hodge bundle $\mathcal{E}$ is the line bundle associated to the determinant character $\det\colon \mathrm{O}(3, \mathbb{C}) \to \{\pm 1\}$. This is nontrivial as an $\mathrm{O}(3, \mathbb{C})$-line bundle, but trivial as an $\mathrm{SO}(3, \mathbb{C})$-line bundle. Therefore $\mathcal{E}$ cannot be detected at the side of $\mathrm{SL}(2, \mathbb{C})$.

2.5.2. Hilbert modular surfaces. When $n = 2$, the accidental isomorphism between the real Lie groups is

$$\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/(-1, -1) \simeq \mathrm{SO}^+(2, 2).$$

Its complexification is

$$\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})/(-1, -1) \simeq \mathrm{SO}(4, \mathbb{C}).$$

This lifts to $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C}) \simeq \mathrm{Spin}(4, \mathbb{C})$. The isomorphism between the compact duals is provided by the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ of $\mathbb{P}^1 \times \mathbb{P}^1$, which maps $\mathbb{P}^1 \times \mathbb{P}^1$ to a quadric surface $Q \subset \mathbb{P}^3$. This gives an isomorphism between the product of two upper half planes and the type IV domain in $n = 2$. Since the Segre embedding is defined by $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. This means that orthogonal modular forms of weight k correspond to Hilbert modular forms of weight (k, k).

We explain the representation-theoretic aspect. The reductive part of a standard parabolic subgroup of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})$ is the 2-dimensional torus $T_1\times T_2\simeq\mathbb{C}^*\times\mathbb{C}^*$ consisting of pairs (α,β) of diagonal matrices in each $\mathrm{SL}(2,\mathbb{C})$. The corresponding group in $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})/(-1,-1)$ is $T_1\times T_2/(-1,-1)$. We have natural isomorphisms

$$T_1\times T_2/(-1,-1)\simeq\mathbb{C}^*\times\mathbb{C}^*\simeq\mathbb{C}^*\times\mathrm{SO}(2,\mathbb{C}), \tag{2.9}$$

where the first isomorphism is induced by

$$T_1\times T_2\to\mathbb{C}^*\times\mathbb{C}^*,\quad(\alpha,\beta)\mapsto(\alpha\beta,\alpha^{-1}\beta).$$

This is the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{SL}(2,\mathbb{C})\times\mathrm{SL}(2,\mathbb{C})/(-1,-1)$ and $\mathrm{SO}(4,\mathbb{C})$. The pullback of the weight 1 character of $\mathbb{C}^*\subset\mathbb{C}^*\times\mathrm{SO}(2,\mathbb{C})$ to $T_1\times T_2$ by (2.9) is the tensor product $\chi_1\boxtimes\chi_2$ of the weight 1 characters χ_1,χ_2 of T_1, T_2. This explains $\mathcal{O}_Q(-1)\simeq\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$ from representation theory.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.7. *We have an* $\mathrm{O}(4,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E}\simeq\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1). \tag{2.10}$$

Proof. Let $\pi_i\colon\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ be the i-th projection. Then

$$\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}\simeq\pi_1^*\Omega^1_{\mathbb{P}^1}\oplus\pi_2^*\Omega^1_{\mathbb{P}^1}\simeq\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,0)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(0,-2).$$

By (2.5) and $\mathcal{L}^{-1}\simeq\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$, we have

$$\begin{aligned}\mathcal{E}&\simeq\Omega^1_{\mathbb{P}^1\times\mathbb{P}^1}\otimes\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\\&\simeq\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,1)\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1).\end{aligned}$$

This proves (2.10). □

Note that $\mathrm{O}(4,\mathbb{C})$ is the semi-product $\mathfrak{S}_2\ltimes\mathrm{SO}(4,\mathbb{C})$, where $\mathfrak{S}_2$ switches the two $\mathrm{SL}(2,\mathbb{C})$. This involution switches the two rulings of $Q\simeq\mathbb{P}^1\times\mathbb{P}^1$, and acts on the right hand side of (2.10) by switching the two components.

At the level of representations, the isomorphism (2.10) comes from the following correspondence. Let χ be the weight 1 character of $\mathrm{SO}(2,\mathbb{C})\simeq\mathbb{C}^*$. The 2-dimensional standard representation of $\mathrm{SO}(2,\mathbb{C})$ is $\chi\oplus\chi^{-1}$. The pullback of χ to $T_1\times T_2$ by (2.9) is the character $\chi_1^{-1}\boxtimes\chi_2$. Hence the pullback of the standard representation of $\mathrm{SO}(2,\mathbb{C})$ to $T_1\times T_2$ is $(\chi_1^{-1}\boxtimes\chi_2)\oplus(\chi_1\boxtimes\chi_2^{-1})$. This explains (2.10) from representation theory.

By Lemma 2.7, a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$ on Q decomposes into a direct sum of various line bundles $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a,b)$. This means that vector-valued orthogonal modular forms in $n=2$ decompose into tuples of scalar-valued Hilbert modular forms of various weights, so we have nothing new here.

2.5.3. Siegel modular 3-folds. When $n = 3$, the accidental isomorphism between the real Lie groups is $\mathrm{PSp}(4,\mathbb{R}) \simeq \mathrm{SO}^+(2,3)$. Its complexification is $\mathrm{PSp}(4,\mathbb{C}) \simeq \mathrm{SO}(5,\mathbb{C})$, which lifts to $\mathrm{Sp}(4,\mathbb{C}) \simeq \mathrm{Spin}(5,\mathbb{C})$. The isomorphism between the compact duals is provided by the Plücker embedding $\mathrm{LG}(2,4) \hookrightarrow \mathbb{P}V = \mathbb{P}^4$ of the Lagrangian Grassmannian $\mathrm{LG}(2,4)$. Here V is the 5-dimensional irreducible representation of $\mathrm{Sp}(4,\mathbb{C})$ appearing in $\wedge^2\mathbb{C}^4$. The Plücker embedding maps $\mathrm{LG}(2,4)$ to a 3-dimensional quadric $Q \subset \mathbb{P}^4$, and hence gives an isomorphism between the Siegel upper half space of genus 2 and the type IV domain in $n = 3$.

Let $\mathcal{F}$ be the rank 2 universal sub vector bundle over $\mathrm{LG}(2,4)$. (This is the weight 1 Hodge bundle for Siegel modular 3-folds.) Since the Plücker embedding is defined by $\mathcal{O}_{\mathrm{LG}}(1) = \det\mathcal{F}^\vee$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ on Q is identified with $\det\mathcal{F}$ on $\mathrm{LG}(2,4)$. This means that orthogonal modular forms of weight k correspond to Siegel modular forms of weight k.

We explain the representation-theoretic aspect. The reductive part of a standard parabolic subgroup of $\mathrm{Sp}(4,\mathbb{C})$ is isomorphic to $\mathrm{GL}(2,\mathbb{C})$. The corresponding group in $\mathrm{PSp}(4,\mathbb{C})$ is $\mathrm{GL}(2,\mathbb{C})/-1$. We have a natural isomorphism

$$\mathrm{GL}(2,\mathbb{C})/-1 \simeq \mathbb{C}^* \times \mathrm{PGL}(2,\mathbb{C}) \simeq \mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C}), \tag{2.11}$$

where $\mathrm{GL}(2,\mathbb{C}) \to \mathbb{C}^*$ in the first isomorphism is the determinant character, and $\mathrm{PGL}(2,\mathbb{C}) \simeq \mathrm{SO}(3,\mathbb{C})$ in the second isomorphism is the accidental isomorphism in $n = 1$. This gives the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{PSp}(4,\mathbb{C})$ and $\mathrm{SO}(5,\mathbb{C})$. By construction, the pullback of the weight 1 character of $\mathbb{C}^*$ to $\mathrm{GL}(2,\mathbb{C})$ by (2.11) is the determinant character of $\mathrm{GL}(2,\mathbb{C})$. This explains $\mathcal{L} \simeq \det\mathcal{F}$ from representation theory.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.8. *We have an* $\mathrm{SO}(5,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E} \simeq \mathrm{Sym}^2\mathcal{F} \otimes \mathcal{L}^{-1}. \tag{2.12}$$

Proof. As it is well-known, we have an $\mathrm{Sp}(4,\mathbb{C})$-equivariant isomorphism $\Omega^1_{\mathrm{LG}} \simeq \mathrm{Sym}^2\mathcal{F}$ (see, e.g., [**19**] §14). Then (2.12) follows from the isomorphism $\mathcal{E} \simeq \Omega^1_{\mathrm{LG}} \otimes \mathcal{L}^{-1}$ in (2.5). □

Note that $\mathcal{F}$ is not $\mathrm{SO}(5,\mathbb{C})$-linearized but $\mathrm{Sym}^2\mathcal{F}$ is. At the level of representations, the isomorphism (2.12) comes from the following fact: the symmetric square of the standard representation of $\mathrm{GL}(2,\mathbb{C})$, when viewed as a representation of $\mathbb{C}^* \times \mathrm{SO}(3,\mathbb{C})$ via (2.11), is isomorphic to the tensor product of the weight 1 character of $\mathbb{C}^*$ and the standard representation of $\mathrm{SO}(3,\mathbb{C})$.

The full orthogonal group $\mathrm{O}(5,\mathbb{C})$ is $\mathrm{SO}(5,\mathbb{C}) \times \{\pm\mathrm{id}\}$. As an $\mathrm{O}(5,\mathbb{C})$-vector bundle, we have

$$\mathcal{E} \simeq \mathrm{Sym}^2\mathcal{F} \otimes \mathcal{L}^{-1} \otimes \det .$$

The twist by det cannot be detected at the side of $\mathrm{Sp}(4,\mathbb{C})$.

2.5.4. Hermitian modular 4-folds. When $n = 4$, the accidental isomorphism between the real Lie groups is $\mathrm{SU}(2,2)/-1 \simeq \mathrm{SO}^+(2,4)$. The complexification is $\mathrm{SL}(4,\mathbb{C})/-1 \simeq \mathrm{SO}(6,\mathbb{C})$. This lifts to $\mathrm{SL}(4,\mathbb{C}) \simeq \mathrm{Spin}(6,\mathbb{C})$. The isomorphism between the compact duals is provided by the Plücker embedding $\mathrm{G}(2,4) \hookrightarrow \mathbb{P}(\wedge^2\mathbb{C}^4) = \mathbb{P}^5$ of the Grassmannian $\mathrm{G}(2,4)$. This maps $\mathrm{G}(2,4)$ to a 4-dimensional quadric $Q \subset \mathbb{P}^5$, and gives an isomorphism between the Hermitian upper half space of degree 2 and the type IV domain in $n = 4$.

The reductive part of a standard parabolic subgroup of $\mathrm{SL}(4,\mathbb{C})$ is the group

$$G = \left\{ \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \;\middle|\; g_1, g_2 \in \mathrm{GL}(2,\mathbb{C}),\ \det g_2 = \det g_1^{-1} \right\}.$$

The corresponding group in $\mathrm{SL}(4,\mathbb{C})/-1$ is $G/-1$. We have a natural isomorphism

$$G/-1 \simeq \mathbb{C}^* \times (\mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C})/(-1,-1)) \simeq \mathbb{C}^* \times \mathrm{SO}(4,\mathbb{C}). \tag{2.13}$$

Here the first isomorphism sends $(g_1, g_2) \in G$ to $(\det g_1, \pm\alpha^{-1}g_1, \pm\alpha g_2)$ where α is one of the square roots of $\det g_1$, and the second isomorphism is given by the accidental isomorphism in $n = 2$. This is the isomorphism between the reductive parts of standard parabolic subgroups of $\mathrm{SL}(4,\mathbb{C})/-1$ and $\mathrm{SO}(6,\mathbb{C})$.

Let $\mathcal{F}, \mathcal{G}$ be the universal sub and quotient vector bundles on $\mathrm{G}(2,4)$ respectively. Since the Plücker embedding is defined by $\mathcal{O}_{\mathrm{G}(2,4)}(1) = \det\mathcal{G} = (\det\mathcal{F})^{-1}$, the Hodge line bundle $\mathcal{L} = \mathcal{O}_Q(-1)$ is isomorphic to $\det\mathcal{F}$. Thus orthogonal modular forms of weight k correspond to Hermitian modular forms of weight k. At the level of representations, this comes from the fact that the pullback of the weight 1 character of $\mathbb{C}^*$ to G by (2.13) is the character of G given by $(g_1, g_2) \mapsto \det g_1$.

The second Hodge bundle $\mathcal{E}$ is described as follows.

Lemma 2.9. *We have an* $\mathrm{SO}(6,\mathbb{C})$*-equivariant isomorphism*

$$\mathcal{E} \simeq \mathcal{F} \otimes \mathcal{G}. \tag{2.14}$$

Proof. We have a canonical isomorphism $T_{\mathrm{G}(2,4)} \simeq \mathcal{F}^\vee \otimes \mathcal{G}$. The natural symplectic form $\mathcal{F} \otimes \mathcal{F} \to \det\mathcal{F}$ induces an isomorphism $\mathcal{F}^\vee \simeq \mathcal{F} \otimes \mathcal{L}^{-1}$.

Therefore, by (2.5), we have

$$\mathcal{E} \simeq T_{\mathrm{G}(2,4)} \otimes \mathcal{L} \simeq \mathcal{F}^{\vee} \otimes \mathcal{G} \otimes \mathcal{L} \simeq \mathcal{F} \otimes \mathcal{G}.$$

This proves (2.14). □

Note that each $\mathcal{F}$, $\mathcal{G}$ is not $\mathrm{SO}(6,\mathbb{C})$-linearized, but $\mathcal{F} \otimes \mathcal{G}$ is. At the level of representations, the isomorphism (2.14) comes from the following correspondence. Let V_i, $i = 1, 2$, be the representation of G obtained as the pullback of the standard representation of $\mathrm{GL}(2,\mathbb{C})$ by the i-th projection $G \to \mathrm{GL}(2,\mathbb{C})$, $(g_1, g_2) \mapsto g_i$. Then V_1, V_2 correspond to the homogeneous vector bundles $\mathcal{F}, \mathcal{G}$ respectively. Each V_1, V_2 is not a representation of $G/-1$, but $V_1 \otimes V_2$ is. Then, as a representation of $\mathbb{C}^* \times (\mathrm{SL}(2,\mathbb{C})^2/(-1,-1))$ via the first isomorphism in (2.13), $V_1 \otimes V_2$ is isomorphic to the external tensor product of the standard representations of the two $\mathrm{SL}(2,\mathbb{C})$ (with weight 0 on $\mathbb{C}^*$). This in turn is the standard representation of $\mathrm{SO}(4,\mathbb{C})$ via the second isomorphism in (2.13). This explains the isomorphism (2.14) from representation theory.

Finally, $\mathrm{O}(6,\mathbb{C})$ is the semi-product $\mathfrak{S}_2 \ltimes \mathrm{SO}(6,\mathbb{C})$ where $\mathfrak{S}_2 = \langle \iota \rangle$ acts on $\mathrm{G}(2,4)$ by the following involution: choose a symplectic form on $\mathbb{C}^4$ (say the standard one), and sends 2-dimensional subspaces $W \subset \mathbb{C}^4$ to $W^{\perp} \subset \mathbb{C}^4$. This involution exchanges the two $\mathbb{P}^3$-families of planes on $\mathrm{G}(2,4) = Q$. (This is essentially the involution $Z \mapsto Z'$ in [**16**] §1 on the Hermitian upper half space.) The involution ι acts on the vector bundle $\mathcal{F} \otimes \mathcal{G} \simeq \mathcal{F}^{\vee} \otimes \mathcal{G}^{\vee}$ by $\iota^*\mathcal{F} \simeq \mathcal{G}^{\vee}$ and $\iota^*\mathcal{G} \simeq \mathcal{F}^{\vee}$. Then (2.14) is an $\mathrm{O}(6,\mathbb{C})$-equivariant isomorphism.

CHAPTER 3

Vector-valued modular forms

In this chapter we define vector-valued orthogonal modular forms (§3.2) and explain their Fourier expansions at 0-dimensional cusps (§3.3 – §3.5). These are the most fundamental parts of this monograph. The rest of this chapter (§3.6 – §3.8) is devoted to supplementary materials: the passage from O to SO, an example of explicit construction, and an interaction with algebraic cycles.

3.1. Representations of $\mathrm{O}(n, \mathbb{C})$

We begin by recollection of some basic facts from the representation theory for $\mathrm{O}(n, \mathbb{C})$. Our main reference for representation theory is [**39**] §8 (whose main contents are more or less covered by [**18**] §19 and [**20**] §5.5.5, §10.2). In what follows and in §3.6, all representations are assumed to be finite-dimensional over $\mathbb{C}$.

Irreducible representations of $\mathrm{O}(n, \mathbb{C})$ are labelled by partitions $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ such that ${}^t\lambda_1 + {}^t\lambda_2 \leq n$, where ${}^t\lambda$ is the transpose of λ. The irreducible representation corresponding to such a partition λ is constructed as follows. Let $V = \mathbb{C}^n$ be the standard representation of $\mathrm{O}(n, \mathbb{C})$. Let $d = |\lambda| = \sum_i \lambda_i$ be the size of λ. We denote by $V^{[d]}$ the intersection of the kernels of the contraction maps $V^{\otimes d} \to V^{\otimes d-2}$ for all pairs of indices. Vectors in $V^{[d]}$ are called *traceless tensors* or *harmonic tensors* in the literature. The symmetric group $\mathfrak{S}_d$ acts on $V^{\otimes d}$ naturally and preserves $V^{[d]}$. Let $T = T_\lambda^{\downarrow}$ be the column canonical tableau on λ (namely $1, 2, \cdots, {}^t\lambda_1$ on the first column, ${}^t\lambda_1 + 1, \cdots, {}^t\lambda_1 + {}^t\lambda_2$ on the second column, $\cdots$). Let $c_\lambda = b_\lambda a_\lambda \in \mathbb{C}\mathfrak{S}_d$ be the Young symmetrizer associated to T, where

$$a_\lambda = \sum_{\sigma \in H_T} \sigma, \qquad b_\lambda = \sum_{\tau \in V_T} \mathrm{sgn}(\tau)\tau$$

as usual. (H_T and V_T are the row and the column Young subgroups of $\mathfrak{S}_d$ for the tableau T respectively.) We apply the orthogonal Schur functor for λ to V:

$$V_\lambda = c_\lambda \cdot V^{[d]} = V^{[d]} \cap (c_\lambda \cdot V^{\otimes d}).$$

This space V_λ is the irreducible representation of $\mathrm{O}(n, \mathbb{C})$ labelled by the partition λ. Since b_λ maps $V^{\otimes d}$ to $\wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V$, we have

$$V_\lambda \subset \wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V \subset V^{\otimes d}. \tag{3.1}$$

If we take a basis $e_1, \cdots, e_n$ of V such that $(e_i, e_j) = 1$ when $i + j = n + 1$ and $(e_i, e_j) = 0$ otherwise, V_λ especially contains the vector

$$(e_1 \wedge \cdots \wedge e_{{}^t\lambda_1}) \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_{\lambda_1}}) \tag{3.2}$$

(see [**39**] §8.3.1).

Example 3.1. (1) The exterior tensor $\wedge^d V$ for $0 \leq d \leq n$ corresponds to the partition $\lambda = (1^d) = (1, \cdots, 1)$. By abuse of notation, we sometimes write $\lambda = 1, \mathrm{St}, \wedge^d, \det$ instead of $\lambda = (0), (1), (1^d), (1^n)$ respectively.

(2) The symmetric tensor $\mathrm{Sym}^d V$ is reducible and decomposes as

$$\begin{aligned} \mathrm{Sym}^d V &= V_{(d)} \oplus \mathrm{Sym}^{d-2} V = \cdots \\ &= V_{(d)} \oplus V_{(d-2)} \oplus \cdots \oplus V_{(1)\mathrm{or}(0)}. \end{aligned}$$

Geometrically, $V_{(d)}$ is the cohomology $H^0(\mathcal{O}_{Q_{n-2}}(d))$ for the isotropic quadric $Q_{n-2} \subset \mathbb{P}V$ of dimension $n - 2$.

3.2. Automorphic vector bundles

In this section we define automorphic vector bundles and vector-valued modular forms. Let L be a lattice of signature $(2, n)$. For simplicity of exposition we assume $n \geq 3$ so that the Koecher principle holds. (This assumption can be somewhat justified by our calculation of $\mathcal{E}$ in the case $n \leq 2$ in §2.5.) Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition as in §3.1 and let $d = |\lambda|$. Recall that the second Hodge bundle $\mathcal{E}$ is endowed with a canonical quadratic form. Let $\mathcal{E}^{[d]} \subset \mathcal{E}^{\otimes d}$ be the intersection of the kernels of the contractions $\mathcal{E}^{\otimes d} \to \mathcal{E}^{\otimes d-2}$ for all pairs of indices. The fibers of $\mathcal{E}^{[d]}$ consist of traceless tensors in the fibers of $\mathcal{E}^{\otimes d}$. The symmetric group $\mathfrak{S}_d$ acts on $\mathcal{E}^{\otimes d}$ fiberwisely and preserves $\mathcal{E}^{[d]}$. We define the vector bundle $\mathcal{E}_\lambda$ by applying the orthogonal Schur functor for λ relatively to $\mathcal{E}$:

$$\mathcal{E}_\lambda = c_\lambda \cdot \mathcal{E}^{[d]} = \mathcal{E}^{[d]} \cap (c_\lambda \cdot \mathcal{E}^{\otimes d}).$$

By construction $\mathcal{E}_\lambda$ is a sub vector bundle of $\mathcal{E}^{\otimes d}$, naturally defined over Q and is $\mathrm{O}(L_\mathbb{C})$-invariant.

Let I be a rank 1 primitive isotropic sublattice of L. Recall from §2.4 that we have the I-trivialization $\mathcal{E} \simeq V(I) \otimes \mathcal{O}_{Q(I)}$ over $Q(I) = Q - Q \cap \mathbb{P}I_\mathbb{C}^\perp$. Let $V(I)_\lambda$ be the irreducible representation of $\mathrm{O}(V(I)) \simeq \mathrm{O}(n, \mathbb{C})$ obtained by applying the orthogonal Schur functor for λ to $V(I)$. Since the I-trivialization of $\mathcal{E}$ preserves the quadratic forms, it induces an isomorphism

$$\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_{Q(I)}$$

over $Q(I)$. We call this isomorphism the *I-trivialization* of $\mathcal{E}_\lambda$.

Next for $k \in \mathbb{Z}$ we consider the tensor product

$$\mathcal{E}_{\lambda,k} = \mathcal{E}_\lambda \otimes \mathcal{L}^{\otimes k}.$$

This is an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle on Q. If we write

$$V(I)_{\lambda,k} = V(I)_\lambda \otimes (I_{\mathbb{C}}^\vee)^{\otimes k},$$

the I-trivializations of $\mathcal{E}_\lambda$ and $\mathcal{L}^{\otimes k}$ induce an isomorphism

$$\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{Q(I)}$$

over $Q(I)$. This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$. We call this isomorphism the *I-trivialization* of $\mathcal{E}_{\lambda,k}$.

In what follows, we work over $\mathcal{D}$. We use the same notations $\mathcal{E}_\lambda, \mathcal{E}_{\lambda,k}$ for the restriction of $\mathcal{E}_\lambda, \mathcal{E}_{\lambda,k}$ to $\mathcal{D}$. These are $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant vector bundles on $\mathcal{D}$. Like (2.4), we have an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E}_\lambda \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K (\mathcal{E}_\lambda)_{[\omega]} \simeq \mathrm{O}^+(L_{\mathbb{R}}) \times_K V_\lambda, \tag{3.3}$$

where K is the stabilizer of $[\omega]$ in $\mathrm{O}^+(L_{\mathbb{R}})$. The I-trivialization of $\mathcal{E}_{\lambda,k}$ is defined over $\mathcal{D}$. Let $j(g,[\omega])$ be the factor of automorphy for the $\mathrm{O}^+(L_{\mathbb{R}})$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization. This is a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\mathrm{O}^+(L_{\mathbb{R}}) \times \mathcal{D}$. Since the I-trivialization is equivariant with respect to the stabilizer of $I_{\mathbb{R}}$ in $\mathrm{O}^+(L_{\mathbb{R}})$, we especially have the following.

Lemma 3.2. *When $g \in \mathrm{O}^+(L_{\mathbb{R}})$ stabilizes $I_{\mathbb{R}}$, the value of $j(g,[\omega])$ is constant over $\mathcal{D}$, given by the natural action of g on $V(I)_{\lambda,k}$.*

Now let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We call a Γ-invariant holomorphic section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$ a *modular form of weight* (λ,k) with respect to Γ. By the I-trivialization, a modular form of weight (λ,k) is identified with a $V(I)_{\lambda,k}$-valued holomorphic function f on $\mathcal{D}$ such that

$$f(\gamma[\omega]) = j(\gamma,[\omega])f([\omega])$$

for every $\gamma \in \Gamma$ and $[\omega] \in \mathcal{D}$. We denote by $M_{\lambda,k}(\Gamma)$ the space of modular forms of weight (λ,k) with respect to Γ. When $\lambda = (0)$, we especially write $M_{(0),k}(\Gamma) = M_k(\Gamma)$ as usual.

When $-\mathrm{id} \in \Gamma$, the weight (λ,k) satisfies a parity condition. We state it in a slightly generalized form.

Lemma 3.3. *Let $[\omega] \in \mathcal{D}$ and $\Gamma_{[\omega]}$ be the stabilizer of $[\omega]$ in Γ. The value of a Γ-modular form of weight (λ,k) at $[\omega]$ is contained in the $\Gamma_{[\omega]}$-invariant part of $(\mathcal{E}_{\lambda,k})_{[\omega]}$. In particular, when $-\mathrm{id} \in \Gamma$ and $k+|\lambda|$ is odd, we have $M_{\lambda,k}(\Gamma) = 0$.*

Proof. The first assertion follows from the $\Gamma_{[\omega]}$-invariance of the section. As for the second assertion, we note that $-\mathrm{id}$ acts on both $\mathcal{L}$ and $\mathcal{E}$ as the scalar multiplication by -1. Since $\mathcal{E}_\lambda$ is a sub vector bundle of $\mathcal{E}^{\otimes|\lambda|}$, $-\mathrm{id}$ acts on $\mathcal{E}_{\lambda,k}$ as the scalar multiplication by $(-1)^{k+|\lambda|}$. Therefore, when $k+|\lambda|$ is odd, $-\mathrm{id}$ has no nonzero invariant part in every fiber of $\mathcal{E}_{\lambda,k}$. □

Product of vector-valued modular forms can be given as follows. Suppose that we have a nonzero $\mathrm{O}(n,\mathbb{C})$-homomorphism

$$\varphi : V_{\lambda_1} \otimes V_{\lambda_2} \to V_{\lambda_3} \tag{3.4}$$

for partitions $\lambda_1, \lambda_2, \lambda_3$ for $\mathrm{O}(n,\mathbb{C})$. This uniquely induces an $\mathrm{O}^+(L_\mathbb{R})$-equivariant homomorphism

$$\varphi : \mathcal{E}_{\lambda_1,k_1} \otimes \mathcal{E}_{\lambda_2,k_2} \to \mathcal{E}_{\lambda_3,k_1+k_2}.$$

If f_1, f_2 are Γ-modular forms of weight (λ_1,k_1), (λ_2,k_2) respectively, then

$$f_1 \times_\varphi f_2 := \varphi(f_1 \otimes f_2)$$

is a Γ-modular form of weight (λ_3, k_1+k_2). This is the "φ-product" of f_1 and f_2. Note that a homomorphism (3.4) exists exactly when V_{λ_3} appears in the irreducible decomposition of $V_{\lambda_1} \otimes V_{\lambda_2}$, and it is unique up to constant when the multiplicity is 1. This information can be read off from the Littlewood-Richardson numbers ([**30**], [**32**], see also [**39**] §12).

The map (3.4) also uniquely induces an $\mathrm{O}(V(I))$-homomorphism

$$\varphi_I : V(I)_{\lambda_1,k_1} \otimes V(I)_{\lambda_2,k_2} \to V(I)_{\lambda_3,k_1+k_2}. \tag{3.5}$$

If we denote by ι the relevant I-trivialization maps, then we have

$$\iota(f_1) \times_{\varphi_I} \iota(f_2) = \iota(f_1 \times_\varphi f_2). \tag{3.6}$$

In this sense, φ-product and I-trivialization are compatible.

It will be useful to know how orthogonal weights (λ,k) in $n = 3, 4$ are translated by the accidental isomorphisms. For simplicity we assume ${}^t\lambda_1 < n/2$, namely ${}^t\lambda_1 = 1$. See §3.6 for some justification of this assumption. (There is no essential loss of generality when $n = 3$.) Henceforth we write $\lambda = (d)$ with d a natural number.

Example 3.4. Let $n = 3$. Let $\mathcal{F}$ be the rank 2 Hodge bundle considered in §2.5.3. Automorphic vector bundles on Siegel modular 3-folds can be expressed as $\mathrm{Sym}^j\mathcal{F} \otimes \mathcal{L}^{\otimes l}$ with $j \in \mathbb{Z}_{\geq 0}$ and $l \in \mathbb{Z}$. In the literature this is often referred to as weight $(\mathrm{Sym}^j, \det^l)$. This corresponds to the highest weight $(\rho_1, \rho_2) = (j+l, l)$ of $\mathrm{GL}(2,\mathbb{C})$. When $j = 2d$ is even, we have

$$\mathrm{Sym}^{2d}\mathcal{F} \simeq (\mathrm{Sym}^2\mathcal{F})_{(d)} \simeq \mathcal{E}_{(d)} \otimes \mathcal{L}^{\otimes d}$$

by Lemma 2.8. Therefore

$$\mathrm{Sym}^{2d}\mathcal{F} \otimes \mathcal{L}^{\otimes l} \simeq \mathcal{E}_{(d)} \otimes \mathcal{L}^{\otimes l+d}.$$

Thus we have the following correspondence of weights:

$$\begin{aligned} & \text{orthogonal weight } ((d),k) \\ \leftrightarrow\ & \text{Siegel weight } (\mathrm{Sym}^j, \det{}^l) \text{ with } (j,l)=(2d,k-d) \\ \leftrightarrow\ & \mathrm{GL}(2,\mathbb{C})\text{-weight } (\rho_1,\rho_2)=(k+d,k-d) \end{aligned}$$

Example 3.5. Let $n = 4$. Let $\mathcal{F}$ and $\mathcal{G}$ be the rank 2 Hodge bundles considered in §2.5.4. Automorphic vector bundles on Hermitian modular 4-folds can be expressed as

$$\mathcal{L}^{\otimes k} \otimes \mathrm{Sym}^{j_1}\mathcal{F} \otimes \mathrm{Sym}^{j_2}\mathcal{G}, \qquad k \in \mathbb{Z}, \ \ j_1, j_2 \in \mathbb{Z}_{\geq 0}. \tag{3.7}$$

On the other hand, in [**16**] §2, weights of vector-valued Hermitian modular forms of degree 2 are expressed as $(r, \rho_1 \boxtimes \rho_2)$ where $r \in \mathbb{Z}$ and ρ_1, ρ_2 are symmetric tensors of the standard representation of $\mathrm{GL}(2,\mathbb{C})$. (We are working with $\mathrm{SU}(2,2)$ rather than $\mathrm{U}(2,2)$, and we do not consider twist by a character as in [**16**].) Then $\mathcal{L}$ corresponds to the weight $r = 1$, $\mathcal{F}$ corresponds to the weight $\rho_1 = \mathrm{St}$, and $\mathcal{L} \otimes \mathcal{G} \simeq \mathcal{G}^\vee \simeq \iota^*\mathcal{F}$ corresponds to the weight $\rho_2 = \mathrm{St}$. Thus the vector bundle (3.7) corresponds to the Hermitian weight $(r, \rho_1 \boxtimes \rho_2)$ with $r = k - j_2$, $\rho_1 = \mathrm{Sym}^{j_1}$ and $\rho_2 = \mathrm{Sym}^{j_2}$.

Now, by Lemma 2.9, we have

$$\mathcal{E}_{(d)} \simeq \mathrm{Sym}^d\mathcal{F} \otimes \mathrm{Sym}^d\mathcal{G}.$$

Therefore the orthogonal weight $((d), k)$ corresponds to the Hermitian weight $(r, \rho_1 \boxtimes \rho_2)$ with $r = k - d$ and $\rho_1 = \rho_2 = \mathrm{Sym}^d$. In [**16**] §3 and §4, some examples in the case $d = 1$ are studied in detail.

3.3. Tube domain realization

In this section we recall the tube domain realization of $\mathcal{D}$ associated to a 0-dimensional cusp. We refer the reader to [**22**], [**34**], [**35**] for some more details. This section is preliminaries for the Fourier expansion (§3.4).

We choose a rank 1 primitive isotropic sublattice I of L, which is fixed throughout §3.3 – §3.5. Recall that this corresponds to the 0-dimensional cusp $[I_{\mathbb{C}}]$ of $\mathcal{D}$. The $\mathbb{Z}$-module $(I^\perp/I) \otimes_{\mathbb{Z}} I$ is canonically endowed with the structure of a hyperbolic lattice, from the quadratic form on $I^\perp/I$ and the standard quadratic form $I \times I \to I^{\otimes 2} \simeq \mathbb{Z}$ on I in which the generators of I have norm 1. For $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ we write

$$U(I)_F = (I^\perp/I)_F \otimes_F I_F = V(I)_F \otimes_F I_F.$$

This is a quadratic space over F, hyperbolic when $F = \mathbb{Q}, \mathbb{R}$.

3.3.1. Tube domain realization. The linear projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}}$ from the point $[I_{\mathbb{C}}] \in Q$ defines an isomorphism

$$Q(I) \to \mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I). \tag{3.8}$$

We choose, as an auxiliary data, a rank 1 sublattice $I' \subset L$ such that $(I, I') \neq 0$. This determines a base point of the affine space $\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$ and hence an isomorphism

$$\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I) \to V(I) \otimes_{\mathbb{C}} I_{\mathbb{C}} = U(I)_{\mathbb{C}}. \tag{3.9}$$

Since the quadratic form on $U(I)_{\mathbb{R}}$ is hyperbolic, the set of vectors $v \in U(I)_{\mathbb{R}}$ with $(v, v) > 0$ consists of two connected components. The choice of the component $\mathcal{D}$ determines one of them, which we denote by C_I (the positive cone). Let

$$\mathcal{D}_I = \{Z \in U(I)_{\mathbb{C}} \mid \mathrm{Im}(Z) \in C_I\}$$

be the tube domain associated to C_I. Then the composition of (3.8) and (3.9) gives an isomorphism

$$\mathcal{D} \xrightarrow{\simeq} \mathcal{D}_I \subset U(I)_{\mathbb{C}}. \tag{3.10}$$

This is the tube domain realization of $\mathcal{D}$ associated to I. If we change I', this isomorphism is shifted by the translation by a vector of $U(I)_{\mathbb{Q}}$.

3.3.2. Stabilizer. Next we recall the structure of the stabilizer of the I-cusp. Let $F = \mathbb{Q}, \mathbb{R}$. We denote by $\Gamma(I)_F$ the stabilizer of I in $\mathrm{O}^+(L_F)$ (not the stabilizer of I_F). Elements of $\Gamma(I)_F$ act on $U(I)_F$ as isometries. Let $\mathrm{O}^+(U(I)_F)$ be the subgroup of $\mathrm{O}(U(I)_F)$ preserving the positive cone C_I. By (1.3), $\Gamma(I)_F$ sits in the canonical exact sequence

$$0 \to U(I)_F \to \Gamma(I)_F \to \mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I) \to 1. \tag{3.11}$$

Here the subgroup $U(I)_F$ consists of the Eichler transvections of L_F with respect to the isotropic line I_F. The adjoint action of $\Gamma(I)_F$ on $U(I)_F$ via (3.11) coincides with the natural action of $\Gamma(I)_F$ on $(I^{\perp}/I)_F \otimes I_F$.

The choice of I' determines the lift $V(I)_F \simeq (I_F \oplus I'_F)^{\perp}$ of $V(I)_F$ in $I_F^{\perp}$, and thus a splitting $L_F \simeq U_F \oplus V(I)_F$. This determines a section of (3.11)

$$\mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I) \hookrightarrow \Gamma(I)_F,$$

by letting $\mathrm{O}^+(U(I)_F) \simeq \mathrm{O}^+(V(I)_F)$ act on the lifted component $V(I)_F \subset L_F$ and mapping $\mathrm{GL}(I) = \{\pm 1\}$ to $\{\pm \mathrm{id}\}$. In this way, from the choice of I', we obtain a splitting of (3.11):

$$\Gamma(I)_F \simeq (\mathrm{O}^+(U(I)_F) \times \mathrm{GL}(I)) \ltimes U(I)_F, \tag{3.12}$$

where $\mathrm{O}^+(U(I)_F)$ acts on $U(I)_F$ in the natural way and $\mathrm{GL}(I)$ acts on $U(I)_F$ trivially. This splitting is compatible with the tube domain realization in the following sense. We translate the $\Gamma(I)_F$-action on $\mathcal{D}$ to action of $\Gamma(I)_F$ on $\mathcal{D}_I$ via the tube domain realization (3.10) defined by (the same) I'. Then,

- the unipotent radical $U(I)_F \subset \Gamma(I)_F$ acts on $\mathcal{D}_I$ as the translation by $U(I)_F$ on $U(I)_{\mathbb{C}}$,
- the lifted group $\mathrm{O}^+(U(I)_F)$ in (3.12) acts on $\mathcal{D}_I$ by its linear action on $U(I)_{\mathbb{C}}$,
- the lifted group $\mathrm{GL}(I) = \{\pm \mathrm{id}\}$ acts trivially.

Now let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We write

$$\Gamma(I)_{\mathbb{Z}} = \Gamma(I)_{\mathbb{Q}} \cap \Gamma, \quad U(I)_{\mathbb{Z}} = U(I)_{\mathbb{Q}} \cap \Gamma, \quad \overline{\Gamma(I)}_{\mathbb{Z}} = \Gamma(I)_{\mathbb{Z}}/U(I)_{\mathbb{Z}}.$$

The group $\Gamma(I)_{\mathbb{Z}}$ is the stabilizer of I in Γ. The exact sequence

$$0 \to U(I)_{\mathbb{Z}} \to \Gamma(I)_{\mathbb{Z}} \to \overline{\Gamma(I)}_{\mathbb{Z}} \to 1 \tag{3.13}$$

is naturally embedded in (3.11). The group $U(I)_{\mathbb{Z}}$ is a full lattice in $U(I)_{\mathbb{Q}}$. It defines the algebraic torus

$$T(I) = U(I)_{\mathbb{C}}/U(I)_{\mathbb{Z}}.$$

Then the tube domain realization (3.10) induces an isomorphism

$$\mathcal{D}/U(I)_{\mathbb{Z}} \xrightarrow{\simeq} \mathcal{D}_I/U(I)_{\mathbb{Z}} \subset T(I).$$

The group $\overline{\Gamma(I)}_{\mathbb{Z}}$ acts on $\mathcal{D}/U(I)_{\mathbb{Z}} \simeq \mathcal{D}_I/U(I)_{\mathbb{Z}}$. Let $\bar{\gamma} \in \overline{\Gamma(I)}_{\mathbb{Z}}$ and let $\gamma \in \Gamma(I)_{\mathbb{Z}}$ be its lift. According to the splitting (3.12), we express γ as

$$\gamma = (\gamma_1, \varepsilon, \alpha), \qquad \gamma_1 \in \mathrm{O}^+(U(I)_{\mathbb{Z}}), \ \ \varepsilon = \pm \mathrm{id}, \ \ \alpha \in U(I)_{\mathbb{Q}}. \tag{3.14}$$

Here γ_1, a priori an element of $\mathrm{O}^+(U(I)_{\mathbb{Q}})$, is contained in $\mathrm{O}^+(U(I)_{\mathbb{Z}})$ because the adjoint action of $\Gamma(I)_{\mathbb{Z}}$ on $U(I)_{\mathbb{Q}}$ preserves the lattice $U(I)_{\mathbb{Z}}$. Then the action of $\bar{\gamma}$ on $\mathcal{D}_I/U(I)_{\mathbb{Z}}$ is given by the linear action by γ_1 plus the translation by $[\alpha] \in U(I)_{\mathbb{Q}}/U(I)_{\mathbb{Z}}$. Note that $\bar{\gamma}$ is determined by (γ_1, ε) because the projection $\overline{\Gamma(I)}_{\mathbb{Z}} \to \mathrm{O}^+(U(I)_{\mathbb{Q}}) \times \mathrm{GL}(I)$ is injective. Nevertheless the translation component $[\alpha]$ could be nontrivial because (3.13) may not necessarily split.

3.4. Fourier expansion

Let I and I' be as in §3.3. Let f be a modular form of weight (λ, k) on $\mathcal{D}$ with respect to a finite-index subgroup Γ of $\mathrm{O}^+(L)$. By the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$ and the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$, we regard f as a $V(I)_{\lambda,k}$-valued holomorphic function on the tube domain $\mathcal{D}_I$ (again denoted by f). The subgroup $U(I)_{\mathbb{Z}}$ of $\Gamma(I)_{\mathbb{Z}}$ acts on $\mathcal{D}_I$ by translation, and acts on $V(I)_{\lambda,k}$ trivially . By Lemma 3.2, this shows that the function f is invariant under the translation by the lattice $U(I)_{\mathbb{Z}}$. Therefore it admits a Fourier expansion of the form

$$f(Z) = \sum_{l \in U(I)_{\mathbb{Z}}^{\vee}} a(l) q^l, \qquad q^l = e((l, Z)), \tag{3.15}$$

for $Z \in \mathcal{D}_I$, where $a(l) \in V(I)_{\lambda,k}$ and $U(I)_{\mathbb{Z}}^{\vee} \subset U(I)_{\mathbb{Q}}$ is the dual lattice of $U(I)_{\mathbb{Z}}$. This series is convergent when $\mathrm{Im}(Z)$ is sufficiently large. The Fourier coefficients $a(l)$ can be expressed as

$$a(l) = \int_{U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}} f(Z_0 + v)\, e(-(Z_0 + v, l))\, dv, \tag{3.16}$$

where dv is the flat volume form on $U(I)_{\mathbb{R}}$ normalized so that $U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}$ has volume 1.

The Koecher principle says that we have $a(l) \neq 0$ only when l is in the closure of the positive cone C_I, which is the dual cone of C_I. See, e.g., [**19**] p.191 for a proof of the Koecher principle in the vector-valued Siegel modular case. The present case can be proved similarly by using (3.16) and Proposition 3.6 below. See also [**8**] Proposition 4.15 for the scalar-valued case. In general, when $n \leq 2$, the condition $a(l) \neq 0 \Rightarrow l \in \overline{C_I}$ is the holomorphicity condition required around the I-cusp. The modular form f is called a *cusp form* if $a(l) \neq 0$ only when $l \in C_I$ at every 0-dimensional cusp I. We denote by $S_{\lambda,k}(\Gamma) \subset M_{\lambda,k}(\Gamma)$ the subspace of cusp forms.

It should be noted that the Fourier expansion depends on the choice of I'. If we change I', the tube domain realization is shifted by the translation by a vector of $U(I)_{\mathbb{Q}}$, say v_0. Then we need to replace $f(Z)$ by $f(Z+v_0)$, and the Fourier coefficient $a(l)$ is replaced by $e((l, v_0)) \cdot a(l)$. In what follows, when we speak of Fourier expansion at the I-cusp, the choice of I' (and hence of the tube domain realization $\mathcal{D} \to \mathcal{D}_I$) is subsumed.

The Fourier coefficients satisfy the following symmetry under $\overline{\Gamma(I)_{\mathbb{Z}}}$.

Proposition 3.6. *Let $\bar{\gamma} \in \overline{\Gamma(I)_{\mathbb{Z}}}$. Let $\gamma = (\gamma_1, \varepsilon, \alpha)$ be its lift in $\Gamma(I)_{\mathbb{Z}}$ expressed as in* (3.14). *Then we have*

$$a(\gamma_1 l) = e(-(\gamma_1 l, \alpha)) \cdot \gamma(a(l)) \tag{3.17}$$

for every $l \in U(I)_{\mathbb{Z}}^{\vee}$.

Proof. By Lemma 3.2, the factor of automorphy for γ is given by its natural action on $V(I)_{\lambda,k}$. Therefore we have

$$f(\gamma(Z)) = \gamma(f(Z)), \quad Z \in \mathcal{D}_I,$$

where γ acts on $\mathcal{D}_I$ via the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$. We compute the Fourier expansion of both sides. Since $\gamma(Z) = \gamma_1 Z + \alpha$, we have

$$\begin{aligned} f(\gamma(Z)) &= \sum_l a(l)e((l, \gamma_1 Z + \alpha)) \\ &= \sum_l a(l)e((l, \alpha))e((\gamma_1^{-1} l, Z)) \\ &= \sum_l a(\gamma_1 l)e((\gamma_1 l, \alpha))e((l, Z)). \end{aligned}$$

In the last equality we rewrote l as $\gamma_1 l$. Comparing this with

$$\gamma(f(Z)) = \sum_l \gamma(a(l))e((l, Z)),$$

we obtain $\gamma(a(l)) = e((\gamma_1 l, \alpha))a(\gamma_1 l)$. □

In the right hand side of (3.17), the action of γ on $a(l) \in V(I)_{\lambda,k}$ is determined by (γ_1, ε). More precisely, γ acts on $I_{\mathbb{C}}$ by $\varepsilon \in \{\pm 1\}$, and on $V(I) = U(I)_{\mathbb{C}} \otimes I_{\mathbb{C}}^{\vee}$ by $\gamma_1 \otimes \varepsilon$.

Proposition 3.6 implies the vanishing of the constant term $a(0)$ in most cases.

Proposition 3.7. *Assume that* $\lambda \neq 1, \det$. *Then* $a(0) = 0$.

Proof. We apply Proposition 3.6 to $l = 0$ and elements $\bar{\gamma}$ in the subgroup

$$\{\, \bar{\gamma} \in \overline{\Gamma(I)}_{\mathbb{Z}} \mid \varepsilon = 1, \det \gamma_1 = 1 \,\} \tag{3.18}$$

of $\overline{\Gamma(I)}_{\mathbb{Z}}$. By trivializing $I \simeq \mathbb{Z}$, we identify $V(I)_{\lambda,k} = V(I)_{\lambda}$. We also identify $\mathrm{SO}(U(I)_{\mathbb{Q}}) = \mathrm{SO}(V(I)_{\mathbb{Q}})$ naturally. Then elements $\bar{\gamma}$ of the group (3.18) act on $V(I)_{\lambda,k}$ by the action of $\gamma_1 \in \mathrm{SO}(V(I)_{\mathbb{Q}})$ on $V(I)_{\lambda}$. Therefore, by Proposition 3.6, we find that $a(0) = \gamma_1(a(0)) \in V(I)_{\lambda}$ for every such $\bar{\gamma}$. The mapping $\bar{\gamma} \mapsto \gamma_1$ embeds the group (3.18) into $\mathrm{SO}(V(I)_{\mathbb{Q}})$, and the image is an arithmetic subgroup of $\mathrm{SO}(V(I)_{\mathbb{Q}})$. By the density theorem of Borel [**5**] (see also [**42**] Corollary 5.15), it is Zariski dense in $\mathrm{SO}(V(I))$. Therefore the vector $a(0) \in V(I)_{\lambda}$ is invariant under the action of $\mathrm{SO}(V(I))$ on $V(I)_{\lambda}$. However, by our assumption $\lambda \neq 1, \det$, the $\mathrm{SO}(n, \mathbb{C})$-representation V_{λ} contains no nonzero invariant vector (cf. §3.6). Therefore $a(0) = 0$. □

Remark 3.8. Since $V(I)$ and $I_{\mathbb{C}}$ have the natural $\mathbb{Q}$-structures $V(I)_{\mathbb{Q}}$ and $I_{\mathbb{Q}}$ respectively, $V(I)_{\lambda,k}$ has the natural $\mathbb{Q}$-structure $V(I)_{\mathbb{Q},\lambda} \otimes (I_{\mathbb{Q}}^{\vee})^{\otimes k}$ where $V(I)_{\mathbb{Q},\lambda} = c_{\lambda} \cdot V(I)_{\mathbb{Q}}^{[d]}$ is the $\mathbb{Q}$-representation of $\mathrm{O}(V(I)_{\mathbb{Q}})$ obtained by applying the orthogonal Schur functor to $V(I)_{\mathbb{Q}}$. Thus we can speak of rationality and algebraicity of the Fourier coefficients $a(l)$. (Rationality depends on the choice of I', but algebraicity does not because the transition constant $e((l, v_0))$ is a root of unity.) When the homomorphism φ_I in (3.5) is defined over $\mathbb{Q}$, the φ-product of two modular forms with rational Fourier coefficients at the I-cusp again has rational Fourier coefficients by (3.6).

3.5. Geometry of Fourier expansion

Let I and I' be as in §3.3 and §3.4. In this section we recall the partial toroidal compactifications of $\mathcal{D}/U(I)_{\mathbb{Z}}$ following [**2**] and explain the Fourier expansion from that point of view.

3.5.1. Partial toroidal compactification. We write $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$. The tube domain realization identifies $\mathcal{X}(I)$ with the open set $\mathcal{D}_I/U(I)_{\mathbb{Z}}$ of the torus $T(I)$. Let $C_I^+ = C_I \cup \bigcup_v \mathbb{R}_{\geq 0} v$ be the union of the positive cone C_I and the rays $\mathbb{R}_{\geq 0} v$ generated by rational isotropic vectors v in $\overline{C_I}$. Let $\Sigma_I = (\sigma_\alpha)$ be a rational polyhedral cone decomposition of C_I^+, namely a fan in $U(I)_{\mathbb{R}}$ whose support is C_I^+. Note that every rational isotropic ray in C_I^+ must be included in Σ_I. We will often abbreviate $\Sigma_I = \Sigma$ when I is clear from the context. The fan Σ is said to be $\Gamma(I)_{\mathbb{Z}}$-*admissible* if it is preserved by the $\Gamma(I)_{\mathbb{Z}}$-action on $U(I)_{\mathbb{R}}$ and there are only finitely many cones up to the $\Gamma(I)_{\mathbb{Z}}$-action. The fan Σ is called *regular* if each cone σ_α is generated by a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$. It is possible to choose Σ to be $\Gamma(I)_{\mathbb{Z}}$-admissible and regular ([**2**], [**14**]).

Let Σ be $\Gamma(I)_{\mathbb{Z}}$-admissible. It determines a $\overline{\Gamma(I)}_{\mathbb{Z}}$-equivariant torus embedding $T(I) \hookrightarrow T(I)^\Sigma$. The toric variety $T(I)^\Sigma$ is normal; it is nonsingular if Σ is regular. The cones σ in Σ correspond to the boundary strata of $T(I)^\Sigma$, say Δ_σ. A stratum Δ_σ is in the closure of another stratum Δ_τ if and only if τ is a face of σ. Each stratum Δ_σ is isomorphic to the quotient torus of $T(I)$ defined by the quotient lattice $U(I)_{\mathbb{Z}}/U(I)_{\mathbb{Z}} \cap \langle \sigma \rangle$, where $\langle \sigma \rangle$ is the $\mathbb{R}$-span of σ. In particular, the rays $\mathbb{R}_{\geq 0} v$ in Σ correspond to the boundary strata of codimension 1, say Δ_v. If we take v to be a primitive vector of $U(I)_{\mathbb{Z}}$, the stratum Δ_v is isomorphic to the quotient torus of $T(I)$ defined by $U(I)_{\mathbb{Z}}/\mathbb{Z}v$. The variety $T(I)^\Sigma$ is nonsingular along Δ_v. If we take a vector $l \in U(I)_{\mathbb{Z}}^\vee$ with $(v, l) = 1$, then $q^l = e((l, Z))$ is a character of $T(I)$ and extends holomorphically over Δ_v. The divisor Δ_v is defined by $q^l = 0$. More generally, a character q^l of $T(I)$ where $l \in U(I)_{\mathbb{Z}}^\vee$ extends holomorphically around a boundary stratum Δ_σ (i.e., extends over Δ_σ and the strata Δ_τ which contains Δ_σ in its closure) if and only if $(l, \sigma) \geq 0$, or in other words, l is in the dual cone of σ. If moreover l has positive pairing with the relative interior of σ, the extended function vanishes identically at Δ_σ.

Now let $\mathcal{X}(I)^\Sigma$ be the interior of the closure of $\mathcal{X}(I)$ in $T(I)^\Sigma$. We call $\mathcal{X}(I)^\Sigma$ the *partial toroidal compactification* of $\mathcal{X}(I)$ defined by the fan Σ. As a partial compactification of $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$, this does not depend on the choice of I'. When a cone $\sigma \in \Sigma$ is not an isotropic ray, its relative interior is contained in C_I, and the corresponding boundary stratum Δ_σ of

$T(I)^\Sigma$ is totally contained in $\mathcal{X}(I)^\Sigma$. On the other hand, when $\sigma = \mathbb{R}_{\geq 0}v$ is an isotropic ray, only an open subset of Δ_v is contained in $\mathcal{X}(I)^\Sigma$. (This will be glued with the boundary of the partial toroidal compactification over the corresponding 1-dimensional cusp: see §5.3.) By abuse of notation, we still write Δ_v for the boundary stratum in $\mathcal{X}(I)^\Sigma$ in this case.

3.5.2. Fourier expansion and Taylor expansion. Let $f(Z) = \sum_l a(l)q^l$ be the Fourier expansion of a Γ-modular form of weight (λ, k) at the I-cusp. This can be viewed as the expansion of the $V(I)_{\lambda,k}$-valued function f on $\mathcal{X}(I)$ by the characters of $T(I)$.

Lemma 3.9. *The function f on $\mathcal{X}(I)$ extends holomorphically over $\mathcal{X}(I)^\Sigma$. When $\lambda \neq 1, \det$ and σ is not an isotropic ray, f vanishes at the corresponding boundary stratum Δ_σ. When f is a cusp form, it vanishes at every boundary stratum Δ_σ.*

Proof. Since the dual cone of $\overline{C_I}$ is $\overline{C_I}$ itself, $\overline{C_I}$ is contained in the dual cone of every cone σ in Σ. Therefore, if $l \in U(I)^\vee_{\mathbb{Z}} \cap \overline{C_I}$, then l is contained in the dual cone of every σ, which implies that the function q^l extends holomorphically over $\mathcal{X}(I)^\Sigma$. By the cusp condition in the Fourier expansion, this shows that the function f extends holomorphically over $\mathcal{X}(I)^\Sigma$.

When σ is not an isotropic ray, its relative interior is contained in C_I. Hence any nonzero vector $l \in U(I)^\vee_{\mathbb{Z}} \cap \overline{C_I}$ has positive pairing with the relative interior of σ. This shows that the corresponding character q^l vanishes at the boundary stratum Δ_σ. It follows that $f|_{\Delta_\sigma}$ is the constant $a(0)$. By Proposition 3.7, this vanishes when $\lambda \neq 1, \det$.

Finally, if f is a cusp form, we have $a(l) \neq 0$ only when $l \in C_I$. Such a vector l has positive pairing with the relative interior of every cone $\sigma \in \Sigma$, and so q^l vanishes at Δ_σ. Therefore f vanishes at the boundary of $\mathcal{X}(I)^\Sigma$. □

Let us explain that the Fourier expansion gives Taylor expansion along each boundary divisor. Let $\sigma = \mathbb{R}_{\geq 0}v$ be a ray in Σ with $v \in U(I)_{\mathbb{Z}}$ primitive. We can rewrite the Fourier expansion of f as

$$f(Z) = \sum_{m \geq 0} \sum_{\substack{l \in U(I)^\vee_{\mathbb{Z}} \\ (l,v)=m}} a(l)q^l. \tag{3.19}$$

We choose a vector $l_0 \in U(I)^\vee_{\mathbb{Z}}$ with $(l_0, v) = 1$ and put $q_0 = q^{l_0}$. The boundary divisor Δ_v is defined by $q_0 = 0$. We put

$$\phi_m = \sum_{\substack{l \in U(I)^\vee_{\mathbb{Z}} \\ (l,v)=m}} a(l)q^{l-ml_0} = \sum_{l \in v^\perp \cap U(I)^\vee_{\mathbb{Z}}} a(l+ml_0)q^l.$$

Note that $v^{\perp} \cap U(I)^{\vee}_{\mathbb{Z}}$ is the dual lattice of $U(I)_{\mathbb{Z}}/\mathbb{Z}v$ and hence is the character lattice of the quotient torus Δ_v. Therefore ϕ_m is (the pullback of) a $V(I)_{\lambda,k}$-valued function on Δ_v. Then (3.19) can be rewritten as

$$f(Z) = \sum_{m \geq 0} \phi_m q_0^m.$$

This is the Taylor expansion of f along the divisor Δ_v with normal parameter q_0, and ϕ_m (as a function on Δ_v) is the m-th Taylor coefficient. In particular, the restriction of f to Δ_v is given by ϕ_0:

$$f|_{\Delta_v} = \phi_0 = \sum_{l \in v^{\perp} \cap U(I)^{\vee}_{\mathbb{Z}}} a(l) q^l.$$

When $(v, v) \neq 0$, this reduces to $a(0)$ because $v^{\perp} \cap \overline{C_I} = \{0\}$ holds (cf. the proof of Lemma 3.9). On the other hand, when $(v, v) = 0$, this reduces to

$$f|_{\Delta_v} = \sum_{l \in \mathbb{Q}v \cap U(I)^{\vee}_{\mathbb{Z}}} a(l) q^l \tag{3.20}$$

because $v^{\perp} \cap \overline{C_I} = \mathbb{R}_{\geq 0} v$.

Remark 3.10. Sometimes it is useful to allow l_0 from an overlattice of $U(I)^{\vee}_{\mathbb{Z}}$, e.g., when considering the Fourier-Jacobi expansion (§7). Then q_0 and ϕ_m are still defined, as functions on a finite cover of $T(I)$.

3.5.3. Canonical extension. In §3.4 and §3.5, we regarded modular forms as $V(I)_{\lambda,k}$-valued functions via the I-trivialization. Let us go back to the viewpoint of sections of $\mathcal{E}_{\lambda,k}$. The vector bundle $\mathcal{E}_{\lambda,k}$ on $\mathcal{D}$ descends to a vector bundle on $\mathcal{X}(I) = \mathcal{D}/U(I)_{\mathbb{Z}}$, which we again denote by $\mathcal{E}_{\lambda,k}$. We extend it over $\mathcal{X}(I)^{\Sigma}$ as follows.

Since the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$ is equivariant with respect to $U(I)_{\mathbb{Z}}$ which acts on $V(I)_{\lambda,k}$ trivially, it descends to an isomorphism $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(I)}$ over $\mathcal{X}(I)$. Then we can extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\mathcal{X}(I)^{\Sigma}$ (still denoted by $\mathcal{E}_{\lambda,k}$) so that this isomorphism extends to $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(I)^{\Sigma}}$ over $\mathcal{X}(I)^{\Sigma}$. In other words, the extension is defined so that the frame of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ corresponding to a basis of $V(I)_{\lambda,k}$ by the I-trivialization extends to a frame of the extended bundle $\mathcal{E}_{\lambda,k}$. This is an explicit form of Mumford's canonical extension [**37**]. By construction, a section f of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma}$ if and only if f viewed as a $V(I)_{\lambda,k}$-valued function via the I-trivialization extends holomorphically over $\mathcal{X}(I)^{\Sigma}$. Then Lemma 3.9 can be restated as follows.

Lemma 3.11. *A modular form $f \in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$ extends to a holomorphic section of the extended bundle $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma}$.*

When $\lambda \neq 1, \det$ and σ is not an isotropic ray, this extended section vanishes at Δ_σ. When f is a cusp form, this section vanishes at every Δ_σ.

3.6. Special orthogonal groups

In the theory of orthogonal modular forms, there is an option at the outset: which Lie group to mainly work with. The full orthogonal group O, or the special orthogonal group SO, or the spin group Spin, or even the pin group Pin. We decided to start with O for two reasons: (1) in some applications we need to consider subgroups Γ of $O^+(L)$ not contained in $SO^+(L)$, and (2) the explicit construction by the orthogonal Schur functor for $\mathcal{E}$ will be useful at some points.

On the other hand, it is sometimes more convenient to work with SO. In this section we explain the switch from O to SO. The contents of this section will be used only in §6.1, §10 and §11, so the reader may skip it for the moment.

3.6.1. Representations of $SO(n,\mathbb{C})$**.** We first recall some basic facts from the representation theory of $SO(n,\mathbb{C})$ following [**39**] §4, §8 and [**18**] §19. Irreducible representations of $SO(n,\mathbb{C})$ are labelled by their highest weights. When $n = 2m$ is even, the highest weights are expressed by m-tuples $\rho = (\rho_1, \cdots, \rho_m)$ of integers, nonnegative for $i < m$, such that $\rho_1 \geq \cdots \geq \rho_{m-1} \geq |\rho_m|$. We write $\rho^\dagger = (\rho_1, \cdots, \rho_{m-1}, -\rho_m)$ for such ρ. When $n = 2m+1$ is odd, the highest weights are expressed by m-tuples $\rho = (\rho_1, \cdots, \rho_m)$ of nonnegative integers such that $\rho_1 \geq \cdots \geq \rho_m \geq 0$. We denote by W_ρ the irreducible representation of $SO(n,\mathbb{C})$ with highest weight ρ. The dual representation $W_\rho^\vee$ is isomorphic to W_ρ itself when n is odd or $4|n$, while it is isomorphic to $W_{\rho^\dagger}$ in the case $n \equiv 2 \bmod 4$.

By the Weyl unitary trick, W_ρ remains irreducible as a representation of $SO(n,\mathbb{R}) \subset SO(n,\mathbb{C})$, and the above classification is the same as the classification of irreducible $\mathbb{C}$-representations of $SO(n,\mathbb{R})$.

The restriction rule from $O(n,\mathbb{C})$ to $SO(n,\mathbb{C})$ is as follows ([**39**] Proposition 8.24). Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ be a partition expressing an irreducible representation V_λ of $O(n,\mathbb{C})$. We define a highest weight $\bar{\lambda}$ for $SO(n,\mathbb{C})$ by

$$\bar{\lambda} = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \cdots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]}).$$

Note that $\bar{\lambda}$ itself can be viewed as a partition for $O(n,\mathbb{C})$. When n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$, the $O(n,\mathbb{C})$-representation V_λ remains irreducible as a representation of $SO(n,\mathbb{C})$, with highest weight $\bar{\lambda}$. The vector defined in (3.2) is a highest weight vector. Thus $V_\lambda \simeq W_{\bar{\lambda}}$ as a representation of $SO(n,\mathbb{C})$ in this case. In particular, since the highest weight for the partition $\bar{\lambda}$ is $\bar{\lambda}$ itself, we have $V_\lambda \simeq V_{\bar{\lambda}}$ as $SO(n,\mathbb{C})$-representations. More specifically, when ${}^t\lambda_1 < n/2$ we have $\bar{\lambda} = \lambda$, while when ${}^t\lambda_1 > n/2$ we have $V_\lambda \simeq V_{\bar{\lambda}} \otimes \det$ as $\mathrm{O}(n, \mathbb{C})$-representations. (In the latter case, the partitions λ and $\bar{\lambda}$ are called *associated* in [**39**] and [**18**].)

In the remaining case, namely when $n = 2m$ is even and ${}^t\lambda_1 = m$, V_λ gets reducible when restricted to $\mathrm{SO}(n, \mathbb{C})$. More precisely,

$$V_\lambda \simeq W_{\bar{\lambda}} \oplus W_{\bar{\lambda}^\dagger} \tag{3.21}$$

as a representation of $\mathrm{SO}(n, \mathbb{C})$. Note that $\bar{\lambda} = \lambda$ and $\lambda_m \neq 0$ in this case. Since $\bar{\lambda} \neq \bar{\lambda}^\dagger$, this decomposition is unique. In this case, V_λ is the induced representation from the representation $W_{\bar{\lambda}}$ of $\mathrm{SO}(n, \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$.

3.6.2. Automorphic vector bundles. We go back to the automorphic vector bundles on $\mathcal{D}$. We choose a base point $[\omega_0] \in \mathcal{D}$. Let $K \simeq \mathrm{SO}(2, \mathbb{R}) \times \mathrm{O}(n, \mathbb{R})$ and $SK \simeq \mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(n, \mathbb{R})$ be the stabilizers of $[\omega_0]$ in $\mathrm{O}^+(L_\mathbb{R})$ and in $\mathrm{SO}^+(L_\mathbb{R})$ respectively (cf. §2.1).

Proposition 3.12. *The following holds.*

(1) If either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$, then $\mathcal{E}_\lambda$ remains irreducible as an $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundle, and we have $\mathcal{E}_\lambda \simeq \mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\bar{\lambda}}$. In particular, we have $\mathcal{E}_\lambda \simeq \mathcal{E}_{\bar{\lambda}}$ as $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundles.

(2) If n is even and ${}^t\lambda_1 = n/2$, then $\mathcal{E}_\lambda$ as an $\mathrm{SO}^+(L_\mathbb{R})$-vector bundle decomposes into the direct sum of two non-isomorphic vector bundles:

$$\mathcal{E}_\lambda \simeq \mathcal{E}_\lambda^+ \oplus \mathcal{E}_\lambda^- \tag{3.22}$$

with each component isomorphic to $\mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\bar{\lambda}}$ and $\mathrm{SO}^+(L_\mathbb{R}) \times_{SK} W_{\bar{\lambda}^\dagger}$ respectively.

Proof. By (3.3), we have $\mathcal{E}_\lambda \simeq \mathrm{O}^+(L_\mathbb{R}) \times_K V_\lambda$ as an $\mathrm{O}^+(L_\mathbb{R})$-equivariant vector bundle. Therefore, as an $\mathrm{SO}^+(L_\mathbb{R})$-equivariant vector bundle, we have $\mathcal{E}_\lambda \simeq \mathrm{SO}^+(L_\mathbb{R}) \times_{SK} V_\lambda$. Note that the representation of $\mathrm{O}(n, \mathbb{R}) \simeq \mathrm{O}(H_{\omega_0}^\perp) \subset K$ on $V_\lambda = (\omega_0^\perp/\mathbb{C}\omega_0)_\lambda \simeq (H_{\omega_0}^\perp \otimes_\mathbb{R} \mathbb{C})_\lambda$ extends to a representation of $\mathrm{O}(n, \mathbb{C}) \simeq \mathrm{O}(H_{\omega_0}^\perp \otimes_\mathbb{R} \mathbb{C})$ naturally. Then our assertions follow from the restriction rule for $\mathrm{SO}(n, \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$. □

At each fiber, the decomposition (3.22) is the irreducible decomposition of $(\omega^\perp/\mathbb{C}\omega)_\lambda$ as a representation of $\mathrm{SO}(\omega^\perp/\mathbb{C}\omega)$. The I-trivialization respects the decomposition (3.22) in the following sense. As a representation of $\mathrm{SO}(V(I))$, $V(I)_\lambda$ decomposes according to (3.21), which we denote by $V(I)_\lambda = W(I)_{\bar{\lambda}} \oplus W(I)_{\bar{\lambda}^\dagger}$. By the uniqueness of the decomposition (3.21), the I-trivialization $\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_\mathcal{D}$ sends the decomposition (3.22) of $\mathcal{E}_\lambda$ to the decomposition

$$V(I)_\lambda \otimes \mathcal{O}_\mathcal{D} = (W(I)_{\bar{\lambda}} \otimes \mathcal{O}_\mathcal{D}) \oplus (W(I)_{\bar{\lambda}^\dagger} \otimes \mathcal{O}_\mathcal{D})$$

of $V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$. Thus we have the I-trivializations

$$\mathcal{E}_\lambda^+ \simeq W(I)_{\bar{\lambda}} \otimes \mathcal{O}_{\mathcal{D}}, \qquad \mathcal{E}_\lambda^- \simeq W(I)_{\bar{\lambda}^\dagger} \otimes \mathcal{O}_{\mathcal{D}} \tag{3.23}$$

of each component $\mathcal{E}_\lambda^+, \mathcal{E}_\lambda^-$.

3.7. Rankin-Cohen brackets

In this section, as an example of explicit construction of vector-valued modular forms, we define the Rankin-Cohen bracket of two scalar-valued modular forms. This is a general method: see, e.g., [**43**], [**27**], [**9**], [**16**], [**17**] for the case of some other types of modular forms, where Rankin-Cohen bracket is a successful technique for explicitly describing some modules of vector-valued modular forms.

Let f, g be nonzero scalar-valued modular forms of weight k, l respectively for $\Gamma < \mathrm{O}^+(L)$. We define the Rankin-Cohen bracket of f and g by

$$\{f, g\} = (g^{k+1}/f^{l-1}) \otimes d(f^l/g^k).$$

Here g^{k+1}/f^{l-1} is a meromorphic section of $\mathcal{L}^{\otimes l(k+1)-k(l-1)} = \mathcal{L}^{\otimes k+l}$, and $d(f^l/g^k)$ is the exterior differential of the meromorphic function f^l/g^k on $\mathcal{D}$. Thus $d(f^l/g^k)$ is a meromorphic 1-form on $\mathcal{D}$. It is immediate to see that $\{g, f\} = -\{f, g\}$. When $k = l$, the Rankin-Cohen bracket reduces to the more simple expression

$$\begin{aligned} \{f, g\} &= (g^{k+1}/f^{k-1}) \otimes k\,(f/g)^{k-1} \cdot d(f/g) \\ &= k\,g^2 \otimes d(f/g). \end{aligned}$$

Proposition 3.13. *The Rankin-Cohen bracket* $\{f, g\}$ *is a modular form of weight* $(\mathrm{St}, k+l+1)$ *for* Γ. *We have* $\{f, g\} \neq 0$ *unless when* f^l *is a constant multiple of* g^k.

Proof. Since g^{k+1}/f^{l-1} and $d(f^l/g^k)$ are meromorphic sections of $\mathcal{L}^{\otimes k+l}$ and $\Omega^1_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}$ respectively, $\{f, g\}$ is a meromorphic section of $\mathcal{E} \otimes \mathcal{L}^{\otimes k+l+1}$, i.e., has weight $(\mathrm{St}, k+l+1)$. The Γ-invariance is obvious from the definition. It remains to check the holomorphicity over $\mathcal{D}$. We take a frame s of $\mathcal{L}$ and write $f = \tilde{f} s^{\otimes k}$, $g = \tilde{g} s^{\otimes l}$ with $\tilde{f}, \tilde{g}$ holomorphic functions on $\mathcal{D}$. Then

$$\begin{aligned} \{f, g\} &= (\tilde{g}^{k+1}/\tilde{f}^{l-1}) s^{\otimes k+l} \otimes d(\tilde{f}^l/\tilde{g}^k) \\ &= s^{\otimes k+l} \otimes (l\,(d\tilde{f})\tilde{g} - k\,(d\tilde{g})\tilde{f}). \end{aligned}$$

From this expression, we find that $\{f, g\}$ is holomorphic. The nonvanishing assertion is apparent. □

When $f = 0$ or $g = 0$, we simply set $\{f, g\} = 0$. Then the Rankin-Cohen bracket defines a bilinear map

$$M_k(\Gamma) \times M_l(\Gamma) \to M_{\mathrm{St},k+l+1}(\Gamma).$$

When $k = l$, this induces $\wedge^2 M_k(\Gamma) \to M_{\mathrm{St},2k+1}(\Gamma)$ by the anti-commutativity.

3.8. Higher Chow cycles on $K3$ surfaces

One of the geometric significance of vector-valued modular forms on $\mathcal{D}$ is the appearance of the middle graded piece of the Hodge filtration, while scalar-valued modular forms are concerned only with the first piece. Thus the connection between modular forms and geometry related to the variation of Hodge structures on $\mathcal{D}$ shows up fully. In this section we present such an example of geometric construction of vector-valued modular forms with singularities. This section is independent of the rest of the monograph.

Let $\pi\colon X \to B$ be a smooth family of $K3$ surfaces. We say that $\pi\colon X \to B$ is *lattice-polarized* with period lattice L if we have a sub local system Λ_{NS} of $R^2\pi_*\mathbb{Z}$ whose fibers are primitive hyperbolic sublattices of the Néron-Severi lattices of the π-fibers X_b and the fibers of $\Lambda_T = \Lambda_{NS}^{\perp}$ are isometric to L. Let $\tilde{B}$ be an unramified cover of B where the local system Λ_T can be trivialized (e.g., the universal cover) and let $\tilde{X} = X \times_B \tilde{B}$. After choosing a base point $o \in \tilde{B}$ and an isometry $(\Lambda_T)_o \simeq L$, we have the period map

$$\tilde{\mathcal{P}} : \tilde{B} \to \mathcal{D}, \quad b \mapsto [H^{2,0}(\tilde{X}_b) \subset L_{\mathbb{C}}].$$

If Γ is a finite-index subgroup of $\mathrm{O}^+(L)$ which contains the monodromy group of Λ_T, $\tilde{\mathcal{P}}$ descends to a holomorphic map

$$\mathcal{P} : B \to \mathcal{F}(\Gamma).$$

When B is algebraic, $\mathcal{P}$ is a morphism of algebraic varieties by Borel's extension theorem.

Let $Z = (Z_b)$ be a family of higher Chow cycles in $CH^2(X_b, 1)$. By this, we mean that

- Z is a higher Chow cycle of type $(2, 1)$ on the total space X, i.e., a codimension 2 cycle on $X \times \mathbb{A}^1$ which meets $X \times \{0\}$ and $X \times \{1\}$ properly and satisfies $Z|_{X\times\{0\}} = Z|_{X\times\{1\}}$, and
- the restriction $Z_b = Z|_{X_b}$ to each fiber X_b is well-defined, i.e., without using the moving lemma, Z already intersects with $X_b \times \mathbb{A}^1$ properly and gives a higher Chow cycle on X_b.

The normal function ν_Z of Z is defined as a holomorphic section of the fibration of the generalized intemediate Jacobians $\mathcal{H}/(F^2\mathcal{H} + R^2\pi_*\mathbb{Z})$. Here $\mathcal{H} = R^2\pi_*\mathbb{C}\otimes\mathcal{O}_B$ and $(F^p\mathcal{H})_p$ is the Hodge filtration on $\mathcal{H}$. The infinitesimal invariant $\delta\nu_Z$ of ν_Z is defined as a section of the middle cohomology sheaf of the Koszul complex

$$F^2\mathcal{H} \to (F^1\mathcal{H}/F^2\mathcal{H}) \otimes \Omega_B^1 \to (\mathcal{H}/F^1\mathcal{H}) \otimes \Omega_B^2 \tag{3.24}$$

over B. See [**45**], [**11**] for more details and examples.

We explain the connection with vector-valued modular forms. We first consider the case where $\tilde{B} = B$ is an analytic open set of $\mathcal{D}$ and the period map $B \to \mathcal{D}$ coincides with the inclusion map. Then we can identify

$$F^2\mathcal{H} = \mathcal{L}|_B, \quad F^1\mathcal{H}/F^2\mathcal{H} = \mathcal{E}|_B \oplus (\Lambda_{NS} \otimes_{\mathbb{Z}} \mathcal{O}_B), \quad \mathcal{H}/F^1\mathcal{H} = \mathcal{L}^{-1}|_B.$$

The Koszul complex (3.24) is the direct sum of the complex

$$0 \to \Lambda_{NS} \otimes \Omega_B^1 \to 0$$

and the modular Koszul complex (2.7) restricted to B:

$$\mathcal{L} \to \mathcal{E} \otimes \Omega_B^1 \to \mathcal{L}^{-1} \otimes \Omega_B^2.$$

According to this decomposition, we can write $\delta\nu_Z$ as $((\delta\nu_Z)_{pol}, (\delta\nu_Z)_{prim})$ where $(\delta\nu_Z)_{pol}$ is a section of $\Lambda_{NS} \otimes \Omega_B^1$ and $(\delta\nu_Z)_{prim}$ is a section of the middle cohomology sheaf of the modular Koszul complex over B. By the calculation in Example 2.4, we see that

$$(\delta\nu_Z)_{prim} \in H^0(B, \mathcal{E}_{(2)} \otimes \mathcal{L}),$$

namely $(\delta\nu_Z)_{prim}$ is a local modular form of weight $(\lambda, k) = ((2), 1)$ over B.

Now we consider the case where the family $\pi\colon X \to B$ is algebraic, $-\mathrm{id} \notin \Gamma$, and the algebraic period map $\mathcal{P}\colon B \to \mathcal{F}(\Gamma)$ is birational. By removing some divisors from B if necessary, we may assume that $\mathcal{P}$ is an open immersion and $\mathcal{D} \to \mathcal{F}(\Gamma)$ is unramified over $B \subset \mathcal{F}(\Gamma)$. Then we may take $\tilde{B}$ to be a Γ-invariant Zariski open set of $\mathcal{D}$. In this case, the Koszul complex (3.24) over B is the direct sum of $0 \to \Lambda_{NS} \otimes \Omega_B^1 \to 0$ and the descent of the modular Koszul complex (2.7) from $\tilde{B} \subset \mathcal{D}$ to $B \subset \mathcal{F}(\Gamma)$. Let Z be a family of higher Chow cycles on $X \to B$ as above. According to the decomposition of the Koszul complex over B, we can write

$$\delta\nu_Z = ((\delta\nu_Z)_{pol},\ (\delta\nu_Z)_{prim})$$

as in the local case. Then the pullback of the primitive part $(\delta\nu_Z)_{prim}$ to $\tilde{B}$ is a Γ-invariant holomorphic section of $\mathcal{E}_{(2)} \otimes \mathcal{L}$ over $\tilde{B}$. By a vanishing theorem proved later (Theorem 9.1), there is no nonzero holomorphic modular form of weight $((2), 1)$ on $\mathcal{D}$. Hence, if $(\delta\nu_Z)_{prim}$ does not vanish identically, it must have a singularity at some component of the complement of $\tilde{B}$ in $\mathcal{D}$. In other words, the primitive part $(\delta\nu_Z)_{prim}$ of the infinitesimal invariant $\delta\nu_Z$ of Z is a modular form of weight $((2), 1)$ with singularities.

CHAPTER 4

Witt operators

In this chapter, as a functorial aspect of the theory, we study pullback of vector-valued modular forms to sub orthogonal modular varieties, an operation sometimes called the *Witt operator*. Let L be a lattice of signature $(2, n)$ and L' be a primitive sublattice of L of signature $(2, n')$. We put $K = (L')^{\perp} \cap L$ and $r = \mathrm{rank}(K) = n - n'$. If we write $\mathcal{D}' = \mathcal{D}_{L'}$, then $\mathcal{D}' = \mathbb{P}L'_{\mathbb{C}} \cap \mathcal{D}$. Let f be a vector-valued modular form on $\mathcal{D}$. In §4.1 we study the restriction of f to $\mathcal{D}'$. This produces a vector-valued modular form on $\mathcal{D}'$, whose weight (in general reducible) can be known from the branching rule for $\mathrm{O}(n', \mathbb{C}) \subset \mathrm{O}(n, \mathbb{C})$. An immediate consequence is the vanishing of $M_{\lambda,k}(\Gamma)$ in $k \leq 0$ (Proposition 4.4). A more interesting situation is the case when f vanishes identically at $\mathcal{D}'$, which we study in §4.2. In that case, we can define the so-called *quasi-pullback* of f, which produces a *cusp* form on $\mathcal{D}'$ (Proposition 4.10). These operations will be useful when studying concrete examples.

Restriction of modular forms to sub modular varieties has been considered classically for scalar-valued Siegel modular forms, going back to Witt [**48**]. Quasi-pullback has been also considered in this case: see [**10**] §2 for a general treatment.

Quasi-pullback of orthogonal modular forms was first considered for Borcherds products by Borcherds ([**6**], [**7**]), and later for general scalar-valued modular forms by Gritsenko-Hulek-Sankaran ([**23**] §8.4) in the case $r = 1$. Our terminology "quasi-pullback" comes from this series of work. The cuspidality of quasi-pullback was first proved in [**22**], [**23**] in the scalar-valued case. Our Proposition 4.10 is the vector-valued generalization.

4.1. Ordinary pullback

We embed $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$ in $\mathrm{O}^+(L_{\mathbb{R}})$ naturally. This is the stabilizer of $L'_{\mathbb{R}}$ in $\mathrm{O}^+(L_{\mathbb{R}})$. Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Then $\Gamma' = \Gamma \cap \mathrm{O}^+(L')$ is a finite-index subgroup of $\mathrm{O}^+(L')$, and $G = \Gamma \cap \mathrm{O}(K)$ is a finite group. The product $\Gamma' \times G$ is a finite-index subgroup of the stabilizer of L' in Γ.

Let $\mathcal{L}', \mathcal{E}'$ be the Hodge bundles on $\mathcal{D}'$. Since $\mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)|_{\mathbb{P}L'_{\mathbb{C}}} = \mathcal{O}_{\mathbb{P}L'_{\mathbb{C}}}(-1)$, we have $\mathcal{L}|_{\mathcal{D}'} = \mathcal{L}'$. We also have a natural isomorphism

$$\mathcal{E}|_{\mathcal{D}'} \simeq \mathcal{E}' \oplus (K_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}'}), \tag{4.1}$$

which at each fiber is the decomposition

$$(\omega^{\perp} \cap L_{\mathbb{C}})/\mathbb{C}\omega = ((\omega^{\perp} \cap L'_{\mathbb{C}})/\mathbb{C}\omega) \oplus K_{\mathbb{C}}.$$

This corresponds to the decomposition $\mathrm{St} = \mathrm{St}' \oplus \mathrm{St}''$ of the standard representation of $\mathrm{O}(n, \mathbb{C})$ when restricted to the subgroup $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, where St' and St'' are the standard representations of $\mathrm{O}(n', \mathbb{C})$ and $\mathrm{O}(r, \mathbb{C})$ respectively.

Let λ be a partition expressing an irreducible representation V_λ of $\mathrm{O}(n, \mathbb{C})$. We denote by

$$V_\lambda \simeq \bigoplus_{\alpha} V'_{\lambda'(\alpha)} \boxtimes V''_{\lambda''(\alpha)} \tag{4.2}$$

the irreducible decomposition as a representation of $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, where $V'_{\lambda'(\alpha)}$ (resp. $V''_{\lambda''(\alpha)}$) is the irreducible representation of $\mathrm{O}(n', \mathbb{C})$ (resp. $\mathrm{O}(r, \mathbb{C})$) with partition $\lambda'(\alpha)$ (resp. $\lambda''(\alpha)$). See [**31**], [**33**] for an explicit description of this restriction rule in terms of the Littlewood-Richardson numbers. Let k be an integer.

PROPOSITION 4.1. *Restriction of modular forms to $\mathcal{D}' \subset \mathcal{D}$ defines a linear map*

$$M_{\lambda,k}(\Gamma) \to \bigoplus_{\alpha} M_{\lambda'(\alpha),k}(\Gamma') \otimes (K_{\mathbb{C}})^{G}_{\lambda''(\alpha)}, \quad f \mapsto f|_{\mathcal{D}'}.$$

This maps cusp forms to cusp forms.

For the proof of Proposition 4.1, we need to calculate the Fourier expansion of $f|_{\mathcal{D}'}$. We take a rank 1 primitive isotropic sublattice I of L'. Let $U(I)_{\mathbb{Z}} \subset U(I)_{\mathbb{Q}}$ be as in §3.3 and we define $U(I)'_{\mathbb{Z}} \subset U(I)'_{\mathbb{Q}}$ similarly for (L', Γ'). Then $U(I)'_{\mathbb{Q}} \subset U(I)_{\mathbb{Q}}$ and $U(I)'_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$. If we write $K'_{\mathbb{Q}} = K_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$, we have $U(I)_{\mathbb{Q}} = U(I)'_{\mathbb{Q}} \oplus K'_{\mathbb{Q}}$. The tube domain realization with respect to I (with I' also taken from L') identifies $\mathcal{D}' \subset \mathcal{D}$ with $\mathcal{D}'_I = \mathcal{D}_I \cap U(I)'_{\mathbb{C}} \subset \mathcal{D}_I$.

LEMMA 4.2. *Let $f(Z) = \sum_{l \in U(I)^{\vee}_{\mathbb{Z}}} a(l) q^l$ be the Fourier expansion of $f \in M_{\lambda,k}(\Gamma)$ at the I-cusp of $\mathcal{D}$. Then we have*

$$f|_{\mathcal{D}'_I}(Z') = \sum_{l' \in (U(I)'_{\mathbb{Z}})^{\vee}} b(l')(q')^{l'}, \quad (q')^{l'} = e((l', Z')), \tag{4.3}$$

for $Z' \in \mathcal{D}'_I$, where

$$b(l') = \sum_{\substack{l'' \in K'_{\mathbb{Q}} \\ l' + l'' \in U(I)^{\vee}_{\mathbb{Z}}}} a(l' + l'').$$

Proof. Let $\pi\colon U(I)_{\mathbb{Q}} \to U(I)'_{\mathbb{Q}}$ be the orthogonal projection. This maps $U(I)^{\vee}_{\mathbb{Z}}$ to a sublattice of $(U(I)'_{\mathbb{Z}})^{\vee}$. For $l \in U(I)^{\vee}_{\mathbb{Z}}$, the restriction of the function $q^l = e((l,Z))$ to $\mathcal{D}'_I \subset \mathcal{D}_I$ is $(q')^{\pi(l)} = e((\pi(l), Z'))$. Then our assertion follows by substituting $q^l = (q')^{\pi(l)}$ in $f = \sum_l a(l)q^l$. Note that the sum defining $b(l')$ is actually a finite sum by the condition $l' + l'' \in \overline{C_I}$ (the cusp condition for f) and the fact that $K'_{\mathbb{Q}}$ is negative-definite. □

Now we prove Proposition 4.1.

(Proof of Proposition 4.1). From the expression (3.3) and the decomposition (4.2), we see that

$$\mathcal{E}_{\lambda}|_{\mathcal{D}'} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha)} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)} \tag{4.4}$$

as an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant vector bundle on $\mathcal{D}'$. With the isomorphism $\mathcal{L}|_{\mathcal{D}'} = \mathcal{L}'$, we obtain

$$\mathcal{E}_{\lambda,k}|_{\mathcal{D}'} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)}.$$

If f is a Γ-invariant section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}$, this shows that $f|_{\mathcal{D}'}$ is a $\Gamma' \times G$-invariant section of $\bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)}$ over $\mathcal{D}'$. Hence it is a Γ'-invariant section of $\bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k} \otimes (K_{\mathbb{C}})^{G}_{\lambda''(\alpha)}$ over $\mathcal{D}'$.

Holomorphicity of $f|_{\mathcal{D}'}$ at the cusps of $\mathcal{D}'$ holds automatically when $n' \geq 3$ by the Koecher principle. In general, this can be seen from Lemma 4.2 as follows. Let I and $K'_{\mathbb{Q}}$ be as in Lemma 4.2. Since $K'_{\mathbb{Q}}$ is negative-definite, the orthogonal projection $U(I)_{\mathbb{R}} \to U(I)'_{\mathbb{R}}$ maps the positive cone C_I of $U(I)_{\mathbb{R}}$ to the positive cone C'_I of $U(I)'_{\mathbb{R}}$, and maps $\overline{C_I}$ to $\overline{C'_I}$. Hence the vectors l' in (4.3) actually range over $(U(I)'_{\mathbb{Z}})^{\vee} \cap \overline{C'_I}$. This proves the holomorphicity of $f|_{\mathcal{D}'}$ around the I-cusp of $\mathcal{D}'$. Since I is arbitrary, f is holomorphic at all cusps of $\mathcal{D}'$. When f is a cusp form, the vectors l' range over $(U(I)'_{\mathbb{Z}})^{\vee} \cap C'_I$ for the same reason. This means that $f|_{\mathcal{D}'}$ is a cusp form. This proves Proposition 4.1. □

Example 4.3. Let us look at a typical example. Let $\lambda = \mathrm{St}$. As noticed before, this decomposes as $\mathrm{St} = \mathrm{St}' \oplus \mathrm{St}''$ when restricted to $\mathrm{O}(n', \mathbb{C}) \times \mathrm{O}(r, \mathbb{C})$, which corresponds to the decomposition (4.1). Therefore restriction to $\mathcal{D}'$ gives a linear map

$$M_{\mathrm{St},k}(\Gamma) \to M_{\mathrm{St}',k}(\Gamma') \oplus (M_k(\Gamma') \otimes K^G_{\mathbb{C}}).$$

The first component $M_{\mathrm{St},k}(\Gamma) \to M_{\mathrm{St}',k}(\Gamma')$ can be considered as the main component of the restriction, but we also obtain some scalar-valued modular forms in $M_k(\Gamma') \otimes K^G_{\mathbb{C}}$ as "extra" components. When G fixes no nonzero vector of K, these extra components vanish. For example, this happens when Γ contains a reflection and L' is the fixed lattice of this reflection.

As an application of Proposition 4.1, we obtain the following elementary vanishing theorem. Although this will be superseded later (§9), we present it here because it can be proved easily and is already informative.

Proposition 4.4. *When $k < 0$, we have $M_{\lambda,k}(\Gamma) = 0$. Moreover, we have $M_{\lambda,0}(\Gamma) = 0$ when $\lambda \neq 1, \det$.*

Proof. Let $f \in M_{\lambda,k}(\Gamma)$ with $k < 0$. We consider restriction of f to 1-dimensional domains $\mathcal{D}_{L'} \subset \mathcal{D}$ for sublattices $L' \subset L$ of signature $(2, 1)$. As a representation of $\mathrm{O}(1, \mathbb{C}) = \{\pm \mathrm{id}\}$, V_λ is a direct sum of copies of the trivial character and the determinant character. By Proposition 4.1 and the calculation in §2.5.1, we see that $f|_{\mathcal{D}_{L'}}$ is a tuple of scalar-valued modular forms of weight $2k < 0$ on the upper half plane $\mathcal{D}_{L'}$. Since there is no nonzero elliptic modular form of negative weight, we find that f vanishes identically at $\mathcal{D}_{L'}$. Now, if we vary L', then $\mathcal{D}_{L'}$ run over a dense subset of $\mathcal{D}$. Therefore $f \equiv 0$.

When $f \in M_{\lambda,0}(\Gamma)$ with $\lambda \neq 1, \det$, by combining Proposition 3.7 and Lemma 4.2, we see that $f|_{\mathcal{D}_{L'}}$ is a tuple of scalar-valued cusp forms of weight 0 on $\mathcal{D}_{L'}$, which vanish identically. Therefore $f \equiv 0$ similarly. □

The idea to deduce a vanishing theorem by considering restriction to sub modular varieties is classical. In the case of Siegel modular forms, this goes back to Freitag [**15**].

Proposition 4.4 in particular implies the following.

Proposition 4.5. *Let $n \geq 3$. Assume that $\langle \Gamma, -\mathrm{id} \rangle$ does not contain a reflection. Let X be the regular locus of $\mathcal{F}(\Gamma) = \Gamma \backslash \mathcal{D}$. Then $H^0(X, T_X^{\otimes k}) = 0$ for every $k > 0$.*

Proof. Let $\pi \colon \mathcal{D} \to \mathcal{F}(\Gamma)$ be the projection and $X' \subset X$ be the locus where π is unramified. By [**22**], the absence of reflection in $\langle \Gamma, -\mathrm{id} \rangle$ implies that π is unramified in codimension 1, so the complement of $\pi^{-1}(X')$ in $\mathcal{D}$ has codimension ≥ 2. Since we can pullback sections of $T_{X'}^{\otimes k}$ by the étale map $\pi^{-1}(X') \to X'$, we see that

$$H^0(X, T_X^{\otimes k}) = H^0(X', T_{X'}^{\otimes k}) = H^0(\pi^{-1}(X'), T_{\pi^{-1}(X')}^{\otimes k})^\Gamma = H^0(\mathcal{D}, T_{\mathcal{D}}^{\otimes k})^\Gamma.$$

Since $T_{\mathcal{D}} \simeq \mathcal{E} \otimes \mathcal{L}^{-1}$ by (2.5), we find that

$$H^0(X, T_X^{\otimes k}) = H^0(\mathcal{D}, \mathcal{E}^{\otimes k} \otimes \mathcal{L}^{\otimes -k})^\Gamma = \bigoplus_i M_{\lambda(i),-k}(\Gamma),$$

where $\lambda(i)$ run over the irreducible summands of $\mathrm{St}^{\otimes k}$. By Proposition 4.4, the last space vanishes when $-k < 0$. □

4.2. Quasi-pullback

In this section we show that when $f|_{\mathcal{D}'} \equiv 0$, we can still obtain a nonzero *cusp* form on $\mathcal{D}'$ by considering the Taylor expansion of f along $\mathcal{D}'$. We assume $n' \geq 3$ for simplicity of exposition, but the results below hold also when $n' \leq 2$ (see the proof of Proposition 4.10).

We first describe the normal bundle $\mathcal{N} = \mathcal{N}_{\mathcal{D}'/\mathcal{D}}$ of $\mathcal{D}'$ in $\mathcal{D}$.

Lemma 4.6. *We have* $\mathcal{N} \simeq (\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}$ *as an* $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$*-equivariant vector bundle on* $\mathcal{D}'$.

Proof. By (2.5) and (4.1), we have natural isomorphisms

$$\begin{aligned} T_{\mathcal{D}}|_{\mathcal{D}'} &\simeq (\mathcal{E} \otimes \mathcal{L}^{-1})|_{\mathcal{D}'} \simeq (\mathcal{E}' \oplus (K_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}'})) \otimes (\mathcal{L}')^{-1} \\ &\simeq T_{\mathcal{D}'} \oplus ((\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}). \end{aligned}$$

This implies $\mathcal{N} \simeq (\mathcal{L}')^{-1} \otimes K_{\mathbb{C}}$. □

Let $\mathcal{I}$ be the ideal sheaf of $\mathcal{D}' \subset \mathcal{D}$ and $\nu \geq 0$. By Lemma 4.6 we have

$$\mathcal{I}^{\nu}/\mathcal{I}^{\nu+1}|_{\mathcal{D}'} \simeq \mathrm{Sym}^{\nu} \mathcal{N}^{\vee} \simeq (\mathcal{L}')^{\otimes \nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \tag{4.5}$$

as an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant vector bundle on $\mathcal{D}'$. Therefore we have the exact sequence

$$0 \to \mathcal{I}^{\nu+1}\mathcal{E}_{\lambda,k} \to \mathcal{I}^{\nu}\mathcal{E}_{\lambda,k} \to \mathcal{E}_{\lambda}|_{\mathcal{D}'} \otimes (\mathcal{L}')^{\otimes k+\nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \to 0 \tag{4.6}$$

of sheaves on $\mathcal{D}$. By (4.4) we have an $\mathrm{O}^+(L'_{\mathbb{R}}) \times \mathrm{O}(K_{\mathbb{R}})$-equivariant isomorphism

$$\mathcal{E}_{\lambda}|_{\mathcal{D}'} \otimes (\mathcal{L}')^{\otimes k+\nu} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee} \simeq \bigoplus_{\alpha} \mathcal{E}'_{\lambda'(\alpha),k+\nu} \otimes (K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}}^{\vee}.$$

Note that $K_{\mathbb{C}}^{\vee} \simeq K_{\mathbb{C}}$ canonically by the pairing on K. Taking global sections in (4.6), and then the $\Gamma' \times G$-invariant part, we obtain the exact sequence

$$\begin{aligned} 0 \to H^0(\mathcal{D}, \mathcal{I}^{\nu+1}\mathcal{E}_{\lambda,k})^{\Gamma' \times G} &\to H^0(\mathcal{D}, \mathcal{I}^{\nu}\mathcal{E}_{\lambda,k})^{\Gamma' \times G} \\ &\to \bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}})^G. \end{aligned} \tag{4.7}$$

By definition, a modular form $f \in M_{\lambda,k}(\Gamma)$ vanishes to order $\geq \nu$ along $\mathcal{D}'$ if it is a section of the subsheaf $\mathcal{I}^{\nu}\mathcal{E}_{\lambda,k}$ of $\mathcal{E}_{\lambda,k}$. The *vanishing order* of f along $\mathcal{D}'$ is the largest ν for which f is a section of $\mathcal{I}^{\nu}\mathcal{E}_{\lambda,k}$.

Definition 4.7. Let $f \in M_{\lambda,k}(\Gamma)$ and ν be the vanishing order of f at $\mathcal{D}'$. We define the *quasi-pullback* of f

$$f\|_{\mathcal{D}'} \in \bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu} K_{\mathbb{C}})^G$$

as the image of f by the last map in (4.7).

By the exactness of (4.7) and the definition of the vanishing order, we have $f\|_{\mathcal{D}'} \not\equiv 0$. Note that the vanishing order ν contributes to the increase $k \rightsquigarrow k+\nu$ of the scalar weight. When $\nu = 0$, the quasi-pullback is just the ordinary pullback considered in §4.1.

EXAMPLE 4.8. When $r = 1$, ignoring the symmetry by $G \subset \{\pm \mathrm{id}\}$, the quasi-pullback $f\|_{\mathcal{D}'}$ belongs to $\bigoplus_{\alpha} M_{\lambda'(\alpha),k+\nu}(\Gamma')$. Explicitly, $f\|_{\mathcal{D}'}$ is given by the restriction of $f/(\cdot,\delta)^{\nu}$ to $\mathcal{D}'$, where δ is a nonzero vector of K and $(\cdot,\delta)$ is the section of $\mathcal{O}(1)$ defined by the pairing with δ.

EXAMPLE 4.9. The quasi-pullback of a Borcherds product f considered by Borcherds ([**6**], [**7**]) is defined as $f/\prod_{\delta}(\delta,\cdot)|_{\mathcal{D}'}$, where δ run over primitive vectors in K (with multiplicity) such that f vanishes at $\delta^{\perp} \cap \mathcal{D}$. This is a single scalar-valued modular form (again a Borcherds product), while our quasi-pullback produces a tuple of scalar-valued modular forms, or more canonically, a $\mathrm{Sym}^{\nu}K_{\mathbb{C}}$-valued modular form. The relationship is as follows.

The denominator $\prod_{\delta}(\delta,\cdot)$ is a section of $\mathcal{I}^{\nu} \cdot \mathcal{O}(\nu)$ over $\mathcal{D}$. This corresponds to a sheaf homomorphism $\iota\colon \mathcal{L}^{\otimes \nu} \to \mathcal{I}^{\nu}$. By a property of Borcherds products, f is a section of the subsheaf $\iota(\mathcal{L}^{\otimes\nu}) \cdot \mathcal{L}^{\otimes k}$ of $\mathcal{I}^{\nu} \cdot \mathcal{L}^{\otimes k}$. Let $\bar{\iota}\colon (\mathcal{L}')^{\otimes\nu} \to \mathrm{Sym}^{\nu}\mathcal{N}^{\vee}$ be the embedding induced by $\iota|_{\mathcal{D}'}$ and (4.5). Under the isomorphism $\mathrm{Sym}^{\nu}\mathcal{N}^{\vee} \simeq (\mathcal{L}')^{\otimes\nu} \otimes \mathrm{Sym}^{\nu}K_{\mathbb{C}}^{\vee}$, this corresponds to the vector $\prod_{\delta}(\cdot,\delta)$ of $\mathrm{Sym}^{\nu}K_{\mathbb{C}}^{\vee}$, which in turn corresponds to the vector $\prod_{\delta}\delta$ of $\mathrm{Sym}^{\nu}K_{\mathbb{C}}$. Then $f\|_{\mathcal{D}'}$ as a section of $\mathrm{Sym}^{\nu}\mathcal{N}^{\vee} \otimes (\mathcal{L}')^{\otimes k}$ takes values in the sub line bundle $\bar{\iota}((\mathcal{L}')^{\otimes\nu}) \otimes (\mathcal{L}')^{\otimes k} \simeq (\mathcal{L}')^{\otimes k+\nu}$. This section of $(\mathcal{L}')^{\otimes k+\nu}$ is the quasi-pullback in [**6**] and [**7**].

Next we prove the cuspidality of quasi-pullback. In the case $\lambda = 0$ and $r = 1$, this is due to Gritsenko-Hulek-Sankaran ([**23**] Theorem 8.18).

PROPOSITION 4.10. *Let $f \in M_{\lambda,k}(\Gamma)$ and ν be the vanishing order of f at $\mathcal{D}'$. Suppose that $\nu > 0$. Then $f\|_{\mathcal{D}'}$ is a cusp form. Thus*

$$f\|_{\mathcal{D}'} \quad \in \quad \bigoplus_{\alpha} S_{\lambda'(\alpha),k+\nu}(\Gamma') \otimes ((K_{\mathbb{C}})_{\lambda''(\alpha)} \otimes \mathrm{Sym}^{\nu}K_{\mathbb{C}})^{G}.$$

For the proof of Proposition 4.10, we calculate the Fourier expansion of $f\|_{\mathcal{D}'}$. We work under the same setting and notation as in the proof of Lemma 4.2. We choose a basis of $K'_{\mathbb{Q}}$. According to the decomposition $U(I)_{\mathbb{Q}} = U(I)'_{\mathbb{Q}} \oplus K'_{\mathbb{Q}}$, we express a point of $U(I)_{\mathbb{C}}$ as $Z = (Z', z_1, \cdots, z_r)$ with $Z' \in U(I)'_{\mathbb{C}}$ and $z_i \in \mathbb{C}$. Then $\mathcal{D}'_I \subset \mathcal{D}_I$ is defined by $z_1 = \cdots = z_r = 0$. The coordinates $z_1, \cdots, z_r$ give a trivialization of the conormal bundle $\mathcal{N}^{\vee}$ of $\mathcal{D}'_I$. The quasi-pullback $f\|_{\mathcal{D}'}$ as a $V(I)_{\lambda,k} \otimes \mathrm{Sym}^{\nu}\mathbb{C}^r$-valued function on $\mathcal{D}'_I$ is given, up to constants, by the Taylor coefficients of f along $\mathcal{D}'_I$ in degree

ν:

$$f\|_{\mathcal{D}'}(Z') = \left(\frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', 0) \right)_{\nu_1 + \cdots + \nu_r = \nu}$$

We calculate the Fourier expansion of the Taylor coefficients. In what follows, we identify $(K'_{\mathbb{Q}})^\vee \simeq \mathbb{Q}^r$ by the dual basis of the chosen basis of $K'_{\mathbb{Q}}$ and express vectors of $(K'_{\mathbb{Q}})^\vee$ as $(n_1, \cdots, n_r)$, $n_i \in \mathbb{Q}$.

Lemma 4.11. *Let $f(Z) = \sum_l a(l) q^l$ be the Fourier expansion of f. Let $(\nu_1, \cdots, \nu_r)$ be an index with $\nu_1 + \cdots + \nu_r = \nu$. Then we have*

$$\frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', 0) = (2\pi\sqrt{-1})^\nu \sum_{l' \in (U(I)'_{\mathbb{Z}})^\vee} b(l')(q')^{l'},$$

where $(q')^{l'} = e((l', Z'))$ and

$$b(l') = \sum_{\substack{(n_1, \cdots, n_r) \in \mathbb{Q}^r \\ l' + (n_1, \cdots, n_r) \in U(I)_{\mathbb{Z}}^\vee}} n_1^{\nu_1} \cdots n_r^{\nu_r} \cdot a(l' + (n_1, \cdots, n_r)).$$

Here, by convention, $0^0 = 1$ but $0^m = 0$ when $m > 0$.

Note that the sum defining $b(l')$ is actually a finite sum for the same reason as in Lemma 4.2.

Proof. We can rewrite the Fourier expansion of f as

$$\begin{aligned} & f(Z', z_1, \cdots, z_r) \\ = & \sum_{l'} \sum_{(n_1, \cdots, n_r)} a(l' + (n_1, \cdots, n_r)) \cdot e((l' + (n_1, \cdots, n_r), (Z', z_1, \cdots, z_r))) \\ = & \sum_{l'} \sum_{(n_1, \cdots, n_r)} a(l' + (n_1, \cdots, n_r)) \cdot e((l', Z')) \cdot \prod_{i=1}^r e(n_i z_i). \end{aligned}$$

Here l' ranges over $(U(I)'_{\mathbb{Z}})^\vee$ and $(n_1, \cdots, n_r)$ ranges over vectors in $\mathbb{Q}^r = (K'_{\mathbb{Q}})^\vee$ such that $l' + (n_1, \cdots, n_r) \in U(I)_{\mathbb{Z}}^\vee$. Since we have

$$\frac{\partial^\nu \prod_i e(n_i z_i)}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}} = (2\pi\sqrt{-1})^\nu \prod_i n_i^{\nu_i} \cdot e(n_i z_i),$$

we see that

$$\begin{aligned} & \frac{\partial^\nu f}{\partial z_1^{\nu_1} \cdots \partial z_r^{\nu_r}}(Z', z_1, \cdots, z_r) \\ = & (2\pi\sqrt{-1})^\nu \sum_{l'} \sum_{(n_1, \cdots, n_r)} a(l' + (n_1, \cdots, n_r)) \cdot (q')^{l'} \cdot \prod_i n_i^{\nu_i} \cdot e(n_i z_i). \end{aligned}$$

Substituting $z_1 = \cdots = z_r = 0$, this proves Lemma 4.11. □

Now we complete the proof of Proposition 4.10.

(PROOF OF PROPOSITION 4.10). Let l' be a vector in $\overline{C'_I} \cap (U(I)'_{\mathbb{Z}})^\vee$ with $(l', l') = 0$. For $(n_1, \cdots, n_r) \in \mathbb{Q}^r$, we have $l' + (n_1, \cdots, n_r) \in \overline{C_I}$ only when $(n_1, \cdots, n_r) = (0, \cdots, 0)$ because $K'_{\mathbb{Q}}$ is negative-definite and perpendicular to $U(I)'_{\mathbb{Q}}$. By Lemma 4.11, this shows that

$$b(l') = 0^{\nu_1} \cdots 0^{\nu_r} \cdot a(l') = 0$$

because $(\nu_1, \cdots, \nu_r) \neq (0, \cdots, 0)$ by the assumption $\nu > 0$. This proves Proposition 4.10. □

CHAPTER 5

Canonical extension over 1-dimensional cusps

In this chapter we recall the partial toroidal compactification over a 1-dimensional cusp and the canonical extension of the automorphic vector bundles over it. This provides a geometric basis for the Siegel operator (§6) and the Fourier-Jacobi expansion (§7). Except for a few calculations in §5.4 and §5.5, most contents of this chapter are essentially expository. We refer the reader to [**2**] for the general theory of toroidal compactification, to [**22**], [**34**], [**35**] for its specialization to the case of orthogonal modular varieties (especially for more details on the contents of §5.1 – §5.3), and to [**37**] for the general theory of canonical extension. Nevertheless, since this chapter is the basis of many later chapters, we tried to keep the presentation as self-contained, explicit, and coherent as possible.

Throughout this chapter, L is a lattice of signature $(2, n)$ with $n \geq 3$. We fix a rank 2 primitive isotropic sublattice J of L, which corresponds to a 1-dimensional cusp of $\mathcal{D} = \mathcal{D}_L$. We write $V(J)_F = (J^{\perp}/J)_F$ for $F = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. This is a quadratic space over F, negative-definite when $F = \mathbb{Q}, \mathbb{R}$. We especially abbreviate $V(J) = V(J)_{\mathbb{C}}$. We also write $U(J)_F = \wedge^2 J_F$. The choice of the component $\mathcal{D}$ determines an orientation of J so that the $\mathbb{R}$-isomorphism $(\omega, \cdot) \colon J_{\mathbb{R}} \to \mathbb{C}$ preserves the orientation for any $[\omega] \in \mathcal{D}$. This determines the positive part of $U(J)_{\mathbb{R}}$.

For $2U = U \oplus U$, where U is the integral hyperbolic plane, we will denote by e_1, f_1 and e_2, f_2 the standard hyperbolic basis of the first and the second components respectively. We say that an embedding $\iota \colon 2U_F \hookrightarrow L_F$ is compatible with J if $\iota(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) = J$. This defines a lift $V(J)_F \simeq \iota(2U_F)^{\perp} \cap L_F$ of $V(J)_F$ in $J_F^{\perp}$ and hence a splitting

$$(5.1) \qquad L_F \simeq 2U_F \oplus V(J)_F = (J_F \oplus J_F^{\vee}) \oplus V(J)_F,$$

where we identify $\iota(\langle f_1, f_2 \rangle)$ with $J_F^{\vee}$. We often choose a rank 1 primitive sublattice I of J. We say that $\iota \colon 2U_F \hookrightarrow L_F$ is compatible with $I \subset J$ if $\iota(\mathbb{Z}e_1 \oplus \mathbb{Z}e_2) = J$ and $\iota(\mathbb{Z}e_1) = I$.

5.1. Siegel domain realization

In this section we recall the Siegel domain realization of $\mathcal{D}$ with respect to the J-cusp and explain its relation with the tube domain realization.

5.1.1. Siegel domain realization. The filtration $J \subset J^{\perp} \subset L$ on L determines the two-step linear projection

$$\mathbb{P}L_{\mathbb{C}} \overset{\pi_1}{\dashrightarrow} \mathbb{P}(L/J)_{\mathbb{C}} \overset{\pi_2}{\dashrightarrow} \mathbb{P}(L/J^{\perp})_{\mathbb{C}}. \tag{5.2}$$

Via the pairing on $L_{\mathbb{C}}$, this is identified with the dual projection

$$\mathbb{P}L_{\mathbb{C}}^{\vee} \dashrightarrow \mathbb{P}(J_{\mathbb{C}}^{\perp})^{\vee} \dashrightarrow \mathbb{P}J_{\mathbb{C}}^{\vee}.$$

The center of π_1 is $\mathbb{P}J_{\mathbb{C}}$, and the center of π_2 is $\mathbb{P}V(J)$. The projection π_2 identifies $\mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$ with an affine space bundle over $\mathbb{P}(L/J^{\perp})_{\mathbb{C}}$. If we choose a lift $V(J) \hookrightarrow J_{\mathbb{C}}^{\perp}$ of $V(J)$, it defines a splitting $(L/J)_{\mathbb{C}} = V(J) \oplus (L/J^{\perp})_{\mathbb{C}}$, and so defines an isomorphism between the affine space bundle $\mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$ with the vector bundle $V(J) \otimes \mathcal{O}(1)$ over $\mathbb{P}(L/J^{\perp})_{\mathbb{C}}$.

We restrict (5.2) to the isotropic quadric $Q \subset \mathbb{P}L_{\mathbb{C}}$. The closure of a π_1-fiber is a plane containing $\mathbb{P}J_{\mathbb{C}}$. When this plane is not contained in $\mathbb{P}J_{\mathbb{C}}^{\perp}$, it intersects properly with Q at two distinct lines, one being $\mathbb{P}J_{\mathbb{C}}$. This shows that

$$\pi_1|_Q : Q - Q \cap \mathbb{P}J_{\mathbb{C}}^{\perp} \;\to\; \mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}V(J)$$

is an affine line bundle.

Next we restrict (5.2) further to an enlargement of the domain $\mathcal{D} \subset Q$. Let $\mathbb{H}_J$ be the connected component of $\mathbb{P}J_{\mathbb{C}}^{\vee} - \mathbb{P}J_{\mathbb{R}}^{\vee}$ consisting of $\mathbb{C}$-linear maps $\phi\colon J_{\mathbb{C}} \to \mathbb{C}$ such that $\phi|_{J_{\mathbb{R}}} : J_{\mathbb{R}} \to \mathbb{C}$ is an orientation-preserving $\mathbb{R}$-isomorphism. By the canonical isomorphism $\mathbb{P}J_{\mathbb{C}}^{\vee} \simeq \mathbb{P}J_{\mathbb{C}}$, $\mathbb{H}_J$ corresponds to the J-cusp. We put $\mathcal{V}_J = \pi_2^{-1}(\mathbb{H}_J)$ and $\mathcal{D}(J) = (\pi_1|_Q)^{-1}(\mathcal{V}_J)$. Then $\mathcal{D} \subset \mathcal{D}(J)$. We thus have the extended two-step fibration

$$\mathcal{D} \subset \mathcal{D}(J) \overset{\pi_1}{\to} \mathcal{V}_J \overset{\pi_2}{\to} \mathbb{H}_J, \tag{5.3}$$

where $\mathcal{V}_J \to \mathbb{H}_J$ is an affine space bundle isomorphic to $V(J) \otimes \mathcal{O}_{\mathbb{H}_J}(1)$, $\mathcal{D}(J) \to \mathcal{V}_J$ is an affine line bundle, and $\mathcal{D} \to \mathcal{V}_J$ is an upper half plane bundle inside $\mathcal{D}(J) \to \mathcal{V}_J$. This is the Siegel domain realization of $\mathcal{D}$ with respect to J. (Up to this point, canonically determined by J.)

5.1.2. Relation with tube domain realization. We choose a rank 1 primitive sublattice I of J. Recall from §3.3 that the tube domain realization at the I-cusp (before choosing a base point) is the canonical embedding

$$\mathcal{D} \subset Q(I) \overset{\simeq}{\to} \mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$$

induced by the projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}}$. Note that $\mathcal{D}(J) \subset Q(I)$. We can factor the projection π_1 in (5.2) as:

$$\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/I)_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J)_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J^{\perp})_{\mathbb{C}}.$$

Hence we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I) & \longrightarrow & \mathbb{P}(L/J)_{\mathbb{C}} - \mathbb{P}(I^{\perp}/J)_{\mathbb{C}} & \longrightarrow & \mathbb{P}(L/J^{\perp})_{\mathbb{C}} - \mathbb{P}(I^{\perp}/J^{\perp})_{\mathbb{C}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{D} \subset \mathcal{D}(J) & \xrightarrow{\pi_1} & \mathcal{V}_J & \xrightarrow{\pi_2} & \mathbb{H}_J.
\end{array}$$

Here the upper row is projections of affine spaces, the left vertical map is the tube domain realization at I, and other vertical maps are natural inclusions. The two squares are cartesian, i.e., $\mathcal{D}(J) \to \mathcal{V}_J \to \mathbb{H}_J$ is the restriction of the upper row over $\mathbb{H}_J$. Thus the Siegel domain realization at J can be given by a decomposition of the tube domain realization at $I \subset J$.

Next we choose a rank 1 isotropic sublattice $I' \subset L$ with $(I, I') \neq 0$ and accordingly a base point of the affine space $\mathbb{P}(L/I)_{\mathbb{C}} - \mathbb{P}V(I)$. This identifies the upper row of the above diagram with the linear maps

$$U(I)_{\mathbb{C}} = (I^{\perp}/I)_{\mathbb{C}} \otimes I_{\mathbb{C}} \to (I^{\perp}/J)_{\mathbb{C}} \otimes I_{\mathbb{C}} \to (I^{\perp}/J^{\perp})_{\mathbb{C}} \otimes I_{\mathbb{C}}.$$

We identify $U(J)_{\mathbb{C}} = \wedge^2 J_{\mathbb{C}}$ with the isotropic line $(J/I)_{\mathbb{C}} \otimes I_{\mathbb{C}}$ in $U(I)_{\mathbb{C}}$. Then this is written as the quotient maps

$$U(I)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)^{\perp}_{\mathbb{C}}. \tag{5.4}$$

Therefore, after choosing the base point I', the above commutative diagram can be rewritten as

$$\begin{array}{ccccc}
U(I)_{\mathbb{C}} & \xrightarrow{\pi_1} & U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} & \xrightarrow{\pi_2} & U(I)_{\mathbb{C}}/U(J)^{\perp}_{\mathbb{C}} \\
\uparrow & & \uparrow & & \uparrow \\
\mathcal{D} \subset \mathcal{D}(J) & \xrightarrow{\pi_1} & \mathcal{V}_J & \xrightarrow{\pi_2} & \mathbb{H}_J
\end{array}$$

where the vertical embeddings are defined by I' and the two squares are cartesian. This gives a simpler (but depending on I, I') expression of the Siegel domain realization.

Finally, we introduce coordinates. Let v_J be the positive generator of $\wedge^2 J \simeq \mathbb{Z}$. We choose an isotropic vector $l_J \in U(I)_{\mathbb{Q}}$ with $(v_J, l_J) = 1$. This defines a splitting $U(I)_{\mathbb{Q}} \simeq U_{\mathbb{Q}} \oplus K_{\mathbb{Q}}$ where $K_{\mathbb{Q}} = V(J)_{\mathbb{Q}} \otimes I_{\mathbb{Q}}$, which determines a splitting of (5.4). Accordingly, we express a point of $U(I)_{\mathbb{C}} \simeq \mathbb{C}l_J \times K_{\mathbb{C}} \times \mathbb{C}v_J$ as

$$Z = (\tau, z, w) = \tau l_J + z + w v_J, \qquad z \in K_{\mathbb{C}}, \ \tau, w \in \mathbb{C}. \tag{5.5}$$

In this coordinates, the I-directed Siegel domain realization (5.4) is expressed by

$$(\tau, z, w) \mapsto (\tau, z) \mapsto \tau.$$

The w-component gives coordinates on the π_1-fibers ($\simeq U(J)_{\mathbb{C}}$), and τ gives coordinates on the base $U(I)_{\mathbb{C}}/U(J)^{\perp}_{\mathbb{C}} \simeq U(J)^{\vee}_{\mathbb{C}}$. The images of the embeddings

$$\mathcal{D}(J) \hookrightarrow U(I)_{\mathbb{C}}, \quad \mathcal{V}_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}, \quad \mathbb{H}_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)^{\perp}_{\mathbb{C}}$$

are all defined by the inequality $\mathrm{Im}(\tau) > 0$, and the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ is defined by the inequalities

$$-(\mathrm{Im}(z), \mathrm{Im}(z)) \ < \ 2\,\mathrm{Im}(\tau)\cdot \mathrm{Im}(w), \qquad \mathrm{Im}(\tau) > 0.$$

Thus the choice of I, I', l_J defines a passage from the canonical presentation (5.3) to a more classical presentation of the Siegel domain realization.

Remark 5.1. The choice of I' and l_J is almost equivalent to the choice of an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ compatible with $I_{\mathbb{Q}} \subset J_{\mathbb{Q}}$. More precisely, we choose one of the two generators of $I \simeq \mathbb{Z}$, say v_I. Let $v_I' \in I'_{\mathbb{Q}}$ be the dual vector of v_I in $I'_{\mathbb{Q}}$. We can write $v_J = \tilde{v}_J \otimes v_I$ and $l_J = \tilde{l}_J \otimes v_I$ for some vectors $\tilde{v}_J \in (I'_{\mathbb{Q}})^{\perp} \cap J_{\mathbb{Q}}$ and $\tilde{l}_J \in (I'_{\mathbb{Q}})^{\perp} \cap I^{\perp}_{\mathbb{Q}}$. This defines an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ compatible with $I_{\mathbb{Q}} \subset J_{\mathbb{Q}}$ by sending

$$e_1 \mapsto v_I, \quad f_1 \mapsto v_I', \quad e_2 \mapsto \tilde{v}_J, \quad f_2 \mapsto \tilde{l}_J.$$

5.2. Jacobi group

In this section we describe the rational/real Jacobi group of the J-cusp and its action on the Siegel domain realization.

Let $F = \mathbb{Q}, \mathbb{R}$. Let $\Gamma(J)_F$ be the subgroup of the stabilizer of J_F in $\mathrm{O}(L_F)$ acting trivially on $\wedge^2 J_F$ and $V(J)_F$. We call $\Gamma(J)_F$ the *Jacobi group* for J over F. (It is certainly useful to take into account the action on $V(J)_F$, but here we refrain from doing so for simplicity of exposition.) The Jacobi group has the canonical filtration

$$U(J)_F \subset W(J)_F \subset \Gamma(J)_F$$

defined by

$$W(J)_F = \mathrm{Ker}(\Gamma(J)_F \to \mathrm{SL}(J_F)),$$
$$U(J)_F = \mathrm{Ker}(\Gamma(J)_F \to \mathrm{GL}(J_F^{\perp})).$$

The group $U(J)_F$ consists of the Eichler transvections $E_{l\otimes l'}$ for $l, l' \in J_F$. Since $E_{l'\otimes l} = E_{-l\otimes l'}$, $U(J)_F$ is canonically isomorphic to $\wedge^2 J_F$. This justifies our use of the notation $U(J)_F$. We also have the canonical isomorphism

$$V(J)_F \otimes J_F \to W(J)_F/U(J)_F, \qquad m\otimes l \mapsto E_{\tilde{m}\otimes l} \ \mathrm{mod}\ U(J)_F,$$

where $\tilde{m} \in J_F^{\perp}$ is a lift of $m \in V(J)_F$. The linear space $V(J)_F \otimes J_F$ has a canonical $U(J)_F$-valued symplectic form as the tensor product of the quadratic form on $V(J)_F$ and the canonical $\wedge^2 J_F$-valued symplectic form on J_F.

We thus have the canonical exact sequences

$$0 \to W(J)_F \to \Gamma(J)_F \to \mathrm{SL}(J_F) \to 1, \tag{5.6}$$

$$0 \to U(J)_F \to W(J)_F \to V(J)_F \otimes J_F \to 0.$$

The group $U(J)_F$ is the center of $\Gamma(J)_F$, and $W(J)_F$ is the unipotent radical of $\Gamma(J)_F$. The first sequence (5.6) splits if we choose an embedding $2U_F \hookrightarrow L_F$ compatible with J_F and hence a splitting $L_F \simeq (J_F \oplus J_F^\vee) \oplus V(J)_F$ as in (5.1):

$$\Gamma(J)_F \simeq \mathrm{SL}(J_F) \ltimes W(J)_F. \tag{5.7}$$

Here the lifted group $\mathrm{SL}(J_F) \subset \Gamma(J)_F$ acts on the component $J_F \oplus J_F^\vee$ in the natural way. The adjoint action of $\mathrm{SL}(J_F)$ on $W(J)_F/U(J)_F \simeq V(J)_F \otimes J_F$ is the tensor product of the natural action of $\mathrm{SL}(J_F)$ on J_F and the trivial action on $V(J)_F$. The group $W(J)_F$ is isomorphic to the Heisenberg group for the symplectic space $V(J)_F \otimes J_F$ with center $U(J)_F$. We call $W(J)_F$ the *Heisenberg group* for J over F.

If I is a rank 1 primitive sublattice of J, we have

$$U(J)_F \subset U(I)_F \subset \Gamma(J)_F, \tag{5.8}$$

as can be seen from the definitions. In $U(I)_F = (I^\perp/I)_F \otimes I_F$, $U(J)_F$ corresponds to the isotropic line $(J/I)_F \otimes I_F$. We also have $W(J)_F \subset \Gamma(I)_F$ and

$$U(I)_F \cap W(J)_F = U(J)_F^\perp = (J^\perp/I)_F \otimes I_F.$$

The image of $W(J)_F$ in $\mathrm{O}(V(I)_F)$ is the group of Eichler transvections of $V(I)_F$ with respect to the isotropic line $(J/I)_F$.

The Jacobi group $\Gamma(J)_F$ preserves the Siegel domain realization (5.3) by definition. The actions of the factors $U(J)_F$, $W(J)_F/U(J)_F$, $\mathrm{SL}(J_F)$ of $\Gamma(J)_F$ on the spaces in (5.3) are described as follows.

(1) The group $U(J)_F$ acts on $\mathcal{V}_J$ trivially. The projection $\mathcal{D}(J) \to \mathcal{V}_J$ is a principal $U(J)_{\mathbb{C}}$-bundle, where $U(J)_{\mathbb{C}} = \wedge^2 J_{\mathbb{C}}$ is the group of Eichler transvections $E_{l\otimes l'}$ with $l, l' \in J_{\mathbb{C}}$.

(2) The Heisenberg group $W(J)_F$ acts on $\mathbb{H}_J$ trivially. The quotient $W(J)_F/U(J)_F \simeq V(J)_F \otimes J_F$ acts on the fibers of $\mathcal{V}_J \to \mathbb{H}_J$ by translation. More precisely, if τ is a point of $\mathbb{H}_J \subset \mathbb{P}J_{\mathbb{C}}^\vee$ and $J_{\mathbb{C}} = J^{1,0} \oplus J^{0,1}$ is the corresponding Hodge decomposition of $J_{\mathbb{C}}$ (where $J^{1,0}$ is the kernel), the fiber of $\mathcal{O}_{\mathbb{H}_J}(1)$ over τ is $J_{\mathbb{C}}/J^{1,0}$. So the fiber $(\mathcal{V}_J)_\tau$ of $\mathcal{V}_J$ over τ is an affine space for $V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$. On the other hand, we have a natural projection $V(J)_{\mathbb{R}} \otimes_{\mathbb{R}} J_{\mathbb{R}} \to V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$ which is an $\mathbb{R}$-isomorphism. Then the action of an element of $W(J)_{\mathbb{R}}/U(J)_{\mathbb{R}} \simeq V(J)_{\mathbb{R}} \otimes_{\mathbb{R}} J_{\mathbb{R}}$ on the affine space $(\mathcal{V}_J)_\tau$ is the translation by its projection image in $V(J) \otimes_{\mathbb{C}} (J_{\mathbb{C}}/J^{1,0})$.

(3) To describe the action of $\mathrm{SL}(J_F)$, we take an embedding $2U_F \hookrightarrow L_F$ compatible with J_F. As explained before, this induces an isomorphism

$\mathcal{V}_J \simeq V(J) \otimes \mathcal{O}_{\mathbb{H}_J}(1)$ and a lift $\mathrm{SL}(J_F) \hookrightarrow \Gamma(J)_F$. Then the lifted group $\mathrm{SL}(J_F)$ acts on $\mathcal{V}_J$ by its equivariant action on $\mathcal{O}_{\mathbb{H}_J}(1)$.

5.3. Partial toroidal compactification

Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. We take the intersection of $\Gamma(J)_{\mathbb{Q}}$, $W(J)_{\mathbb{Q}}$, $U(J)_{\mathbb{Q}}$ with Γ and denote them by

$$\Gamma(J)_{\mathbb{Z}} = \Gamma(J)_{\mathbb{Q}} \cap \Gamma, \quad W(J)_{\mathbb{Z}} = W(J)_{\mathbb{Q}} \cap \Gamma, \quad U(J)_{\mathbb{Z}} = U(J)_{\mathbb{Q}} \cap \Gamma.$$

By the orientation on J, we have a distinguished isomorphism $U(J)_{\mathbb{Z}} \simeq \mathbb{Z}$. We also denote by $\Gamma(J)_{\mathbb{Z}}^*$ the stabilizer of J in Γ. The integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$ is of finite index in $\Gamma(J)_{\mathbb{Z}}^*$ because

$$\Gamma(J)_{\mathbb{Z}}^*/\Gamma(J)_{\mathbb{Z}} \hookrightarrow \mathrm{O}(J^\perp/J)$$

and $\mathrm{O}(J^\perp/J)$ is a finite group. If Γ is neat, we have $\Gamma(J)_{\mathbb{Z}}^* = \Gamma(J)_{\mathbb{Z}}$.

We put

$$\overline{\Gamma(J)}_{\mathbb{Z}} = \Gamma(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}, \quad \overline{\Gamma(J)}_F = \Gamma(J)_F/U(J)_{\mathbb{Z}}$$

for $F = \mathbb{Q}, \mathbb{R}$. These quotients make sense because $U(J)_F$ is the center of $\Gamma(J)_F$. By definition we have the canonical exact sequence

$$0 \to W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}} \to \overline{\Gamma(J)}_{\mathbb{Z}} \to \Gamma(J)_{\mathbb{Z}}/W(J)_{\mathbb{Z}} \to 1,$$

which is canonically embedded in the quotient of (5.6) by $U(J)_F$: more specifically, $\Gamma(J)_{\mathbb{Z}}/W(J)_{\mathbb{Z}}$ is embedded in $\mathrm{SL}(J)$ as a finite-index subgroup, and $W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ is embedded in $V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}$ as a full lattice.

Let $T(J) = U(J)_{\mathbb{C}}/U(J)_{\mathbb{Z}} \simeq \mathbb{C}^*$ be the 1-dimensional torus defined by $U(J)_{\mathbb{Z}}$. We denote by $\overline{T(J)} \simeq \mathbb{C}$ the natural partial compactification of $T(J)$. We take the quotient of $\mathcal{D} \subset \mathcal{D}(J)$ by $U(J)_{\mathbb{Z}}$:

$$\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}, \quad \mathcal{T}(J) = \mathcal{D}(J)/U(J)_{\mathbb{Z}}.$$

Then $\mathcal{T}(J)$ is a principal $T(J)$-bundle over $\mathcal{V}_J$, which contains $\mathcal{X}(J)$ as a fibration of punctured discs. Let $\overline{\mathcal{T}(J)} = \mathcal{T}(J) \times_{T(J)} \overline{T(J)}$ be the relative torus embedding. This has the structure of a line bundle on $\mathcal{V}_J$: the scalar multiplication on each fiber is given by the action of $T(J) \simeq \mathbb{C}^*$, and the sum is determined by the scalar multiplication because the fiber is 1-dimensional. The group $\overline{\Gamma(J)}_{\mathbb{R}}$ acts on $\mathcal{T}(J)$ naturally, and this extends to an action on $\overline{\mathcal{T}(J)}$. The fact that $\Gamma(J)_{\mathbb{R}}$ commutes with $U(J)_{\mathbb{C}}$ implies that the action of $\overline{\Gamma(J)}_{\mathbb{R}}$ on $\overline{\mathcal{T}(J)}$ is an equivariant action on the line bundle.

Let $\overline{\mathcal{X}(J)}$ be the interior of the closure of $\mathcal{X}(J)$ in $\overline{\mathcal{T}(J)}$. We call $\overline{\mathcal{X}(J)}$ the *partial toroidal compactification* of $\mathcal{X}(J)$. This is a disc bundle over $\mathcal{V}_J$ obtained by filling the origins in the punctured disc bundle $\mathcal{X}(J) \to \mathcal{V}_J$. Let Δ_J be the boundary divisor of $\overline{\mathcal{X}(J)}$. This is naturally isomorphic to $\mathcal{V}_J$. We denote by Θ_J the conormal bundle of Δ_J in $\overline{\mathcal{X}(J)}$. This is a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant line bundle on Δ_J. (Although the subgroup $U(J)_{\mathbb{R}}/U(J)_{\mathbb{Z}}$ of $\overline{\Gamma(J)}_{\mathbb{R}}$ acts on Δ_J trivially, it acts on the fibers of Θ_J by rotations.)

LEMMA 5.2. *We have a natural $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism $\Theta^{\vee} \simeq \overline{\mathcal{T}(J)}$ of line bundles on Δ_J.*

PROOF. Since Δ_J is the zero section of the line bundle $\overline{\mathcal{T}(J)}$, its normal bundle in $\overline{\mathcal{X}(J)}$ is the same as the normal bundle in $\overline{\mathcal{T}(J)}$, which is isomorphic to $\overline{\mathcal{T}(J)}$ itself. □

The partial compactification $\overline{\mathcal{X}(J)}$ already appears in essence in the partial compactifications $\mathcal{X}(I)^{\Sigma}$ for $I \subset J$ considered in §3.5.1. Recall that the isotropic ray $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ appears in every $\Gamma(I)_{\mathbb{Z}}$-admissible fan Σ as in §3.5.1. Since $U(J)_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$, we have a natural étale map $\mathcal{X}(J) \to \mathcal{X}(I)$ which is a free quotient map by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$.

LEMMA 5.3. *The map $\mathcal{X}(J) \to \mathcal{X}(I)$ extends to an étale map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma}$. The image of Δ_J is a Zariski open set of the boundary divisor of $\mathcal{X}(I)^{\Sigma}$ associated to the isotropic ray σ_J.*

PROOF. Since $\mathcal{D}(J) \subset Q(I)$, we have the following commutative diagram (cf. §5.1.2):

$$\begin{array}{ccccc} \mathcal{T}(J) & \hookrightarrow & Q(I)/U(J)_{\mathbb{Z}} & \longrightarrow & T(I) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{V}_J & \hookrightarrow & Q(I)/U(J)_{\mathbb{C}} & \longrightarrow & T(I)/T(J). \end{array}$$

Here the vertical maps are principal $T(J)$-bundles, and the two right horizontal maps are free quotients by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. The two squares are cartesian: the right is the pullback of a principal $T(J)$-bundle to a $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$-cover, and the left is the restriction to an open set. Since the upper row is $T(J)$-equivariant, it extends to

$$\overline{\mathcal{T}(J)} \hookrightarrow (Q(I)/U(J)_{\mathbb{Z}}) \times_{T(J)} \overline{T(J)} \to T(I) \times_{T(J)} \overline{T(J)}.$$

The second map is still a free quotient by $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. The image of $\Delta_J \subset \overline{\mathcal{T}(J)}$ by this map is an open set of the (unique) boundary divisor of $T(I) \times_{T(J)} \overline{T(J)}$. Since $T(I) \times_{T(J)} \overline{T(J)}$ is the torus embedding of $T(I)$ associated to the ray σ_J, it is a Zariski open set of $T(I)^{\Sigma}$. Thus we obtain an étale map $\overline{\mathcal{T}(J)} \to T(I)^{\Sigma}$ which maps Δ_J to an open set of the boundary divisor of $T(I)^{\Sigma}$ corresponding to σ_J. □

5.4. Canonical extension

In this section, which is the central part of §5, we extend the automorphic vector bundles $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$. This is an explicit form of Mumford's canonical extension [**37**] which is suitable for dealing with the Fourier-Jacobi expansion. We use the same notations $\mathcal{L}, \mathcal{E}, \mathcal{E}_{\lambda}, \mathcal{E}_{\lambda,k}$ for the descends of these vector bundles to $\mathcal{X}(J)$. They are $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant vector bundles on $\mathcal{X}(J)$.

We choose an adjacent 0-dimensional cusp $I \subset J$. Since $U(J)_{\mathbb{Z}} \subset \Gamma(I)_{\mathbb{R}}$, the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$ over $\mathcal{D}$ descends to an isomorphism $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)}$ over $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$. Thus we still have the I-trivialization over $\mathcal{X}(J)$. This is equivariant with respect to $(\Gamma(I)_{\mathbb{R}} \cap \Gamma(J)_{\mathbb{R}})/U(J)_{\mathbb{Z}}$. We extend $\mathcal{E}_{\lambda,k}$ to a vector bundle over $\overline{\mathcal{X}(J)}$ (still use the same notation) by requiring that this isomorphism extends to $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$. We call it the *canonical extension* of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$. This is the pullback of the canonical extension over $\mathcal{X}(I)^{\Sigma}$ defined in §3.5.3 by the gluing map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma}$ in Lemma 5.3. By construction, the frame of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(J)$ corresponding to a basis of $V(I)_{\lambda,k}$ via the I-trivialization extends to a frame of the extended bundle over $\overline{\mathcal{X}(J)}$.

Proposition 5.4. *The canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ defined above does not depend on the choice of I. The action of $\overline{\Gamma(J)_{\mathbb{R}}}$ on $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(J)$ extends to action on the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$.*

The proof of this proposition amounts to the following assertion.

Lemma 5.5. *The factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization is constant on each fiber of $\pi_1 : \mathcal{D} \to \mathcal{V}_J$. In particular, if I' is another $\mathbb{R}$-line in $J_{\mathbb{R}}$, the difference of the I-trivialization and the I'-trivialization at $[\omega] \in \mathcal{D}$ as the composition map*

$$V(I)_{\lambda,k} \to (\mathcal{E}_{\lambda,k})_{[\omega]} \to V(I')_{\lambda,k} \tag{5.9}$$

is constant on each π_1-fiber.

Proof. Let $j(\gamma, [\omega])$ be the factor of automorphy in question. This is a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\Gamma(J)_{\mathbb{R}} \times \mathcal{D}$. What has to be shown is that $j(\gamma, [\omega]) = j(\gamma, [\omega'])$ if $\pi_1([\omega]) = \pi_1([\omega'])$. We consider the natural extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}(J)$, on which the group $U(J)_{\mathbb{C}} \cdot \Gamma(J)_{\mathbb{R}}$ acts equivariantly. Note that $U(J)_{\mathbb{C}}$ commutes with $\Gamma(J)_{\mathbb{R}}$. We can write $[\omega'] = g[\omega]$ for some $g \in U(J)_{\mathbb{C}}$. Since $U(J)_{\mathbb{C}}$ acts trivially on $I_{\mathbb{C}}$ and $V(I)$, we have $j(g, \cdot) \equiv \mathrm{id}$. Therefore

$$j(\gamma, g[\omega]) = j(\gamma g, [\omega]) = j(g\gamma, [\omega]) = j(\gamma, [\omega]).$$

As for the second assertion, we choose $\gamma \in \Gamma(J)_{\mathbb{R}}$ with $\gamma(I_{\mathbb{R}}) = I'$. Then (5.9) coincides with the isomorphism

$$\gamma \circ j(\gamma^{-1}, [\omega]) : V(I)_{\lambda,k} \to V(I)_{\lambda,k} \to V(I')_{\lambda,k}.$$

Hence the constancy of $j(\gamma^{-1}, [\omega])$ over π_1-fibers implies that of (5.9). □

Now we can prove Proposition 5.4.

(Proof of Proposition 5.4). Let I, I' be two rank 1 primitive sublattices of J. By the second assertion of Lemma 5.5, the difference of the I-trivialization and the I'-trivialization

$$V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)} \to \mathcal{E}_{\lambda,k} \to V(I')_{\lambda,k} \otimes \mathcal{O}_{\mathcal{X}(J)}, \tag{5.10}$$

viewed as a $\mathrm{GL}(n, \mathbb{C})$-valued holomorphic function on $\mathcal{X}(J)$ via basis of $V(I)_{\lambda,k}$ and $V(I')_{\lambda,k}$, is constant on each fiber of $\mathcal{X}(J) \to \mathcal{V}_J$. Therefore it extends to a $\mathrm{GL}(n, \mathbb{C})$-valued holomorphic function over $\overline{\mathcal{X}(J)}$. This implies that (5.10) extends to an isomorphism

$$V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}} \to V(I')_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$$

over $\overline{\mathcal{X}(J)}$. Thus the two extensions agree.

Extendability of the $\overline{\Gamma(J)}_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ can be verified as follows. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$. The γ-action on $\mathcal{E}_{\lambda,k}$ sends a frame corresponding to a basis of $V(I)_{\lambda,k}$ via the I-trivialization to a frame corresponding to a basis of $V(\gamma I)_{\lambda,k}$ via the γI-trivialization. By Lemma 5.5 again, the latter extends to a frame over $\overline{\mathcal{X}(J)}$ also in the I-trivialization. Thus γ sends an extendable frame to an extendable frame. This means that the γ-action extends over $\overline{\mathcal{X}(J)}$. □

The fact that the canonical extension comes with an I-trivialization (but independent of it) enables us to develop the theory of Fourier-Jacobi expansion (§7) in an intrinsic but still explicit way. The following property will play a fundamental role in §7.

Proposition 5.6. *Let $\pi_1 : \overline{\mathcal{X}(J)} \to \mathcal{V}_J \simeq \Delta_J$ be the projection. Then we have a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism $\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$ over $\overline{\mathcal{X}(J)}$.*

Proof. We fix a rank 1 primitive sublattice $I \subset J$ and let $j(\gamma, [\omega])$ be the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}$ with respect to the I-trivialization. By Lemma 5.5, the $\mathrm{GL}(V(I)_{\lambda,k})$-valued function $j(\gamma, [\omega])$ on $\Gamma(J)_{\mathbb{R}} \times \mathcal{X}(J)$ descends to a $\mathrm{GL}(V(I)_{\lambda,k})$-valued function on $\Gamma(J)_{\mathbb{R}} \times \Delta_J$. This gives the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ with respect to the I-trivialization $\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$. The fact that its pullback agrees with the factor of automorphy of $\mathcal{E}_{\lambda,k}$ implies that the composition

$$\pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \to \pi_1^*(V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}) \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}} \to \mathcal{E}_{\lambda,k}$$

gives a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \to \mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$, where the first isomorphism is the pullback of the I-trivialization over Δ_J, and the last isomorphism is the I-trivialization over $\overline{\mathcal{X}(J)}$. □

Remark 5.7. By the proof, we have the following commutative diagram:

$$\begin{array}{ccc} \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J}) & \longrightarrow & \mathcal{E}_{\lambda,k} \\ \downarrow & & \downarrow \\ \pi_1^*(V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}) & \longrightarrow & V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}. \end{array}$$

Here the upper arrow is the isomorphism in Proposition 5.6, the vertical arrows are the I-trivializations, and the lower arrow is the natural isomorphism.

Remark 5.8. Although the canonical extension at the level of $\overline{\mathcal{X}(J)}$ still has a trivialization (by construction), this no longer holds when passing to the full toroidal compactifications (§5.6). Around Δ_J we need to further take the quotient by $\overline{\Gamma(J)_{\mathbb{Z}}}$, which does not preserve the trivialization.

5.5. The Hodge line bundle at the boundary

In this section we study the Hodge line bundle $\mathcal{L}$ relative to the J-cusp and show that its canonical extension can be understood more directly. Let

$$\mathcal{L}_J = \mathcal{O}_{\mathbb{H}_J}(-1) = \mathcal{O}_{\mathbb{P}(L/J^\perp)_{\mathbb{C}}}(-1)|_{\mathbb{H}_J}$$

be the Hodge bundle over the upper half plane $\mathbb{H}_J$. The group $\Gamma(J)_{\mathbb{R}}$ acts on $\mathcal{L}_J$ equivariantly via the natural map $\Gamma(J)_{\mathbb{R}} \to \mathrm{SL}(J_{\mathbb{R}})$. Let $\pi = \pi_2 \circ \pi_1 : \mathcal{D} \to \mathbb{H}_J$ be the projection from $\mathcal{D}$ to $\mathbb{H}_J$.

Lemma 5.9. *We have a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J$ over $\mathcal{D}$.*

Proof. Recall that π is restriction of the projection $\mathbb{P}L_{\mathbb{C}} \dashrightarrow \mathbb{P}(L/J^\perp)_{\mathbb{C}}$. Since this is induced by the linear map $L_{\mathbb{C}} \to (L/J^\perp)_{\mathbb{C}}$, we have a natural isomorphism $\pi^*\mathcal{O}_{\mathbb{P}(L/J^\perp)_{\mathbb{C}}}(-1) \simeq \mathcal{O}_{\mathbb{P}L_{\mathbb{C}}}(-1)$ over $\mathbb{P}L_{\mathbb{C}} - \mathbb{P}J_{\mathbb{C}}^\perp$. By restricting this isomorphism to $\mathcal{D}$, we obtain $\mathcal{L} \simeq \pi^*\mathcal{L}_J$. Since the projection $L_{\mathbb{C}} \to (L/J^\perp)_{\mathbb{C}}$ is $\Gamma(J)_{\mathbb{R}}$-equivariant, so is the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J$. □

The fiber of $\pi^*\mathcal{L}_J$ over $[\omega] \in \mathcal{D}$ is the image of the projection $\mathbb{C}\omega \to (L/J^\perp)_{\mathbb{C}}$, and the isomorphism $\mathcal{L} \to \pi^*\mathcal{L}_J$ over $[\omega]$ is identified with the natural map $\mathbb{C}\omega \to \mathrm{Im}(\mathbb{C}\omega \to (L/J^\perp)_{\mathbb{C}})$.

The projection $\mathcal{D} \to \mathbb{H}_J$ descends to $\mathcal{X}(J) \to \mathbb{H}_J$ and extends to $\overline{\mathcal{X}(J)} \to \mathbb{H}_J$ naturally. We denote it again by $\pi\colon \overline{\mathcal{X}(J)} \to \mathbb{H}_J$. The isomorphism in Lemma 5.9 descends to a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ over $\mathcal{X}(J)$. We have respective extension of both sides over $\overline{\mathcal{X}(J)}$: for $\mathcal{L}$ the canonical extension constructed in §5.4, and for $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ the natural extension $\pi^*\mathcal{L}_J$. It turns out that these two extensions agree:

Proposition 5.10. *The isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ over $\mathcal{X}(J)$ extends to a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism between the canonical extension of $\mathcal{L}$ and $\pi^*\mathcal{L}_J$ over $\overline{\mathcal{X}(J)}$. In particular, we have $\mathcal{L}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J$ over Δ_J.*

Proof. We choose a rank 1 primitive sublattice $I \subset J$. Recall that the canonical extension of $\mathcal{L}$ is defined via the I-trivialization of $\mathcal{L}$, which we denote by $\iota_I : \mathcal{L} \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}$. On the other hand, we also have a trivialization $\iota_I' : \mathcal{L}_J \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J}$ of $\mathcal{L}_J = \mathcal{O}_{\mathbb{H}_J}(-1)$ over $\mathbb{H}_J \subset \mathbb{P}(L/J^{\perp})_{\mathbb{C}}$ induced by the pairing between $(L/J^{\perp})_{\mathbb{C}}$ and $I_{\mathbb{C}}$. The natural extension $\pi^*\mathcal{L}_J$ of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ over $\overline{\mathcal{X}(J)}$ coincides with the extension via the trivialization

$$\pi^*\mathcal{L}_J|_{\mathcal{X}(J)} \stackrel{\pi^*\iota_I'}{\to} \pi^*(I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J})|_{\mathcal{X}(J)} = I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}, \tag{5.11}$$

because $\pi^*\iota_I'$ is defined over $\overline{\mathcal{X}(J)}$.

We observe that the composition of (5.11) with the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ in Lemma 5.9 coincides with the I-trivialization ι_I of $\mathcal{L}$: this is just the remark that taking the pairing of a vector $\omega \in L_{\mathbb{C}}$ with $I_{\mathbb{C}}$ (this is ι_I) is the same as projecting ω to $(L/J^{\perp})_{\mathbb{C}}$ (this is $\mathcal{L} \to \pi^*\mathcal{L}_J$) and then taking pairing with $I_{\mathbb{C}}$ (this is $\pi^*\iota_I'$). From this coincidence, we see that the isomorphism in Lemma 5.9 extends to an isomorphism over $\overline{\mathcal{X}(J)}$ from the extension of $\mathcal{L}$ via ι_I (this is the canonical extension of $\mathcal{L}$) to the extension of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$ via $\pi^*\iota_I'$ (this is $\pi^*\mathcal{L}_J$). The $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariance holds by continuity. □

Thus the canonical extension of $\mathcal{L}$ defined in §5.4 via the I-trivialization can be understood more directly as the *canonical* (verbatim) extension $\pi^*\mathcal{L}_J$ of $\pi^*\mathcal{L}_J|_{\mathcal{X}(J)}$.

Remark 5.11. By the proof of Proposition 5.10, $\mathcal{L}_J$ is endowed with the I-trivialization $I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathbb{H}_J} \to \mathcal{L}_J$ induced by the pairing between $(L/J^{\perp})_{\mathbb{C}}$ and $I_{\mathbb{C}}$, and its pullback by π agrees with the I-trivialization of $\mathcal{L}$ via the isomorphism $\mathcal{L} \simeq \pi^*\mathcal{L}_J$.

5.6. Toroidal compactification

In this section we recall the (full) toroidal compactifications of the modular variety $\mathcal{F}(\Gamma) = \Gamma\backslash\mathcal{D}$ following [**2**]. While this provides a background for our geometric approach, logically it will be used only in §10 in a rather auxiliary way, so the reader may skip it for the moment.

The data for constructing a toroidal compactification of $\mathcal{F}(\Gamma)$ is a collection $\Sigma = (\Sigma_I)$ of $\Gamma(I)_{\mathbb{Z}}$-admissible rational polyhedral cone decomposition of $\mathcal{C}_I^+ \subset U(I)_{\mathbb{R}}$ in the sense of §3.5.1, one for each Γ-equivalence class of rank 1 primitive isotropic sublattices I of L. Two fans $\Sigma_I, \Sigma_{I'}$ for different Γ-equivalence classes I, I' are independent, and no choice is required for rank

2 isotropic sublattices J (it is canonical). Then the toroidal compactification is defined by

$$\mathcal{F}(\Gamma)^{\Sigma} = \left(\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_{J} \overline{\mathcal{X}(J)} \right) / \sim,$$

where I (resp. J) run over all primitive isotropic sublattices of L of rank 1 (resp. rank 2), and $\sim$ is the equivalence relation generated by the following étale maps:

(1) The γ-action $\mathcal{D} \to \mathcal{D}$, $\mathcal{X}(I)^{\Sigma_I} \to \mathcal{X}(\gamma I)^{\Sigma_{\gamma I}}$, $\overline{\mathcal{X}(J)} \to \overline{\mathcal{X}(\gamma J)}$ for $\gamma \in \Gamma$.
(2) The gluing maps $\mathcal{D} \to \mathcal{X}(I)^{\Sigma_I}$, $\mathcal{D} \to \overline{\mathcal{X}(J)}$ and $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma_I}$ for $I \subset J$ as in Lemma 5.3.

By [**2**] §III.5, $\mathcal{F}(\Gamma)^{\Sigma}$ is a compact Moishezon space which contains $\mathcal{F}(\Gamma)$ as a Zariski open set and has a morphism $\mathcal{F}(\Gamma)^{\Sigma} \to \mathcal{F}(\Gamma)^{bb}$ to the Baily-Borel compactification. We have natural maps

$$\mathcal{X}(I)^{\Sigma_I}/\overline{\Gamma(I)}_{\mathbb{Z}} \to \mathcal{F}(\Gamma)^{\Sigma}, \quad \overline{\mathcal{X}(J)}/(\Gamma(J)^{*}_{\mathbb{Z}}/U(J)_{\mathbb{Z}}) \to \mathcal{F}(\Gamma)^{\Sigma}. \tag{5.12}$$

These maps are isomorphims in a neighborhood of the locus of boundary points lying over the I-cusp and the J-cusp respectively (see [**2**] p.175). We may choose Σ so that $\mathcal{F}(\Gamma)^{\Sigma}$ is projective. When Γ is neat and each fan Σ_I is regular, i.e., every cone is generated by a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$, then $\mathcal{F}(\Gamma)^{\Sigma}$ is nonsingular ([**2**] §III.7).

Next we explain the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{F}(\Gamma)^{\Sigma}$ (cf. [**37**]). We assume that Γ is neat and Σ is regular. Then not only Γ itself but also $\overline{\Gamma(I)}_{\mathbb{Z}}$ and $\Gamma(J)^{*}_{\mathbb{Z}}/U(J)_{\mathbb{Z}} = \overline{\Gamma(J)}_{\mathbb{Z}}$ are torsion-free, so the quotient map

$$\mathcal{D} \sqcup \bigsqcup_{I} \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_{J} \overline{\mathcal{X}(J)} \to \mathcal{F}(\Gamma)^{\Sigma}$$

is étale. The vector bundle $\mathcal{E}_{\lambda,k}$ is initially defined over $\mathcal{D}$ and hence over $\mathcal{D} \sqcup \bigsqcup_I \mathcal{X}(I) \sqcup \bigsqcup_J \mathcal{X}(J)$. In §3.5.3 and §5.4, we constructed the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)^{\Sigma_I}$ and $\overline{\mathcal{X}(J)}$ respectively. By construction we have a natural isomorphism $p^{*}\mathcal{E}_{\lambda,k} \simeq \mathcal{E}_{\lambda,k}$ for a gluing map p in (2) above. Moreover, we have a natural isomorphism $\gamma^{*}\mathcal{E}_{\lambda,k} \simeq \mathcal{E}_{\lambda,k}$ for the action of $\gamma \in \Gamma$: this is evident for $\mathcal{D}$ and $\mathcal{X}(I)^{\Sigma_I}$, while it is assured by Proposition 5.4 for $\overline{\mathcal{X}(J)}$. Since these isomorphisms are compatible with each other, the extended vector bundle $\mathcal{E}_{\lambda,k}$ on $\mathcal{D} \sqcup \bigsqcup_I \mathcal{X}(I)^{\Sigma_I} \sqcup \bigsqcup_J \overline{\mathcal{X}(J)}$ descends to a vector bundle on $\mathcal{F}(\Gamma)^{\Sigma}$. We denote it again by $\mathcal{E}_{\lambda,k}$. This is the same as extending $\mathcal{E}_{\lambda,k}$ on $\mathcal{F}(\Gamma)$ over the boundary of $\mathcal{F}(\Gamma)^{\Sigma}$ by using the local charts (5.12).

Proposition 5.12. *For Γ neat, we have $M_{\lambda,k}(\Gamma) = H^0(\mathcal{F}(\Gamma)^{\Sigma}, \mathcal{E}_{\lambda,k})$.*

Proof. We have the natural inclusion

$$H^0(\mathcal{F}(\Gamma)^\Sigma, \mathcal{E}_{\lambda,k}) \hookrightarrow H^0(\mathcal{F}(\Gamma), \mathcal{E}_{\lambda,k}) = M_{\lambda,k}(\Gamma).$$

It is sufficient to see that this is surjective. Let $f \in M_{\lambda,k}(\Gamma)$. As a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{X}(I)$, f extends holomorphically over $\mathcal{X}(I)^{\Sigma_I}$ by Lemma 3.11. By the gluing, f extends holomorphically over $\overline{\mathcal{X}(J)}$. Therefore, as a section of $\mathcal{E}_{\lambda,k}$ over $\mathcal{F}(\Gamma)$, f extends holomorphically over $\mathcal{F}(\Gamma)^\Sigma$. □

Let us remark an immediate consequence of this interpretation. We go back to a general finite-index subgroup Γ of $\mathrm{O}^+(L)$. For a fixed λ, the direct sum $\bigoplus_{k\geq 0} M_{\lambda,k}(\Gamma)$ is a module over the ring $\bigoplus_{k\geq 0} M_k(\Gamma)$ of scalar-valued modular forms.

Proposition 5.13. *For each λ, the module $\bigoplus_k M_{\lambda,k}(\Gamma)$ is finitely generated over the ring $\bigoplus_k M_k(\Gamma)$.*

Proof. We may assume that Γ is neat by replacing the given Γ by its neat subgroup of finite index. We take a smooth toroidal compactification $\mathcal{F}(\Gamma)^\Sigma$ as above and let $\pi \colon \mathcal{F}(\Gamma)^\Sigma \to \mathcal{F}(\Gamma)^{bb}$ be the projection to the Baily-Borel compactification. Then $\mathcal{L}^{\otimes n} = \pi^*\mathcal{O}(1)$ for an ample line bundle $\mathcal{O}(1)$ on $\mathcal{F}(\Gamma)^{bb}$ by [**37**] Proposition 3.4 (b). (In fact, $\mathcal{L}$ itself descends, but we do not need that.) It suffices to show that for each $0 \leq k_0 < n$, the module $\bigoplus_k M_{\lambda,k_0+nk}(\Gamma)$ is finitely generated over $\bigoplus_k M_{nk}(\Gamma)$. By Proposition 5.12, we have

$$\begin{aligned} \bigoplus_{k\geq 0} M_{\lambda,k_0+nk}(\Gamma) &= \bigoplus_{k\geq 0} H^0(\mathcal{F}(\Gamma)^\Sigma, \ \mathcal{E}_{\lambda,k_0} \otimes \pi^*\mathcal{O}(k)) \\ &\simeq \bigoplus_{k\geq 0} H^0(\mathcal{F}(\Gamma)^{bb}, \ \pi_*\mathcal{E}_{\lambda,k_0} \otimes \mathcal{O}(k)) \end{aligned}$$

where the second isomorphism follows from the projection formula for π. Since $\mathcal{F}(\Gamma)^{bb}$ is projective, the last module is finitely generated over $\bigoplus_k H^0(\mathcal{F}(\Gamma)^{bb}, \mathcal{O}(k)) = \bigoplus_k M_{nk}(\Gamma)$ by a general theorem of Serre (see, e.g., [**38**] p.128). □

CHAPTER 6

Geometry of Siegel operators

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(n, \mathbb{C})$. We assume $\lambda \neq 1, \det$. This in particular implies $\lambda_n = 0$ and so ${}^t\lambda_1 < n$. In Proposition 3.7, we proved that a modular form $f \in M_{\lambda,k}(\Gamma)$ always vanishes at all 0-dimensional cusps. In this chapter we study the restriction of f to a 1-dimensional cusp, an operation usually called the *Siegel operator*.

Let J be a rank 2 primitive isotropic sublattice of L, which we fix throughout this chapter. A traditional way to define the Siegel operator Φ_J at the J-cusp is to choose a 0-dimensional cusp $I \subset J$, take the I-trivialization and the coordinates (τ, z, w) as in §5.1.2, and set

$$(\Phi_J f)(\tau) = \lim_{t \to \infty} f(\tau, 0, it), \qquad \tau \in \mathbb{H}. \tag{6.1}$$

In this way it is easy to define the Siegel operator, but we have to check the modularity of $\Phi_J f$ and calculate its reduced weight *after* defining it.

In this chapter we take a more geometric approach working directly with the automorphic vector bundle $\mathcal{E}_{\lambda,k}$. This improves the geometric understanding of the Siegel operator, and tells us a priori the modularity of $\Phi_J f$ and its weight. We work with the partial toroidal compactification $\overline{\mathcal{X}(J)}$, rather than with the Baily-Borel compactification, because the boundary structure of $\overline{\mathcal{X}(J)}$ is easier to handle and $\mathcal{E}_{\lambda,k}$ extends to a vector bundle over $\overline{\mathcal{X}(J)}$ as we have seen in §5. We also wanted to put the Siegel operator on the same ground as the Fourier-Jacobi expansion (§7). Understanding the Siegel operator at the level of toroidal compactification will be useful in some geometric applications.

Let Δ_J be the boundary divisor of $\overline{\mathcal{X}(J)}$ and $\pi_2 \colon \Delta_J \to \mathbb{H}_J$ be the projection to the J-cusp. Let $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$. For $V(J) = (J^\perp/J)_{\mathbb{C}}$ we denote by $V(J)_{\lambda'}$ the irreducible representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$ with partition $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$. Our result is summarized as follows.

Theorem 6.1. *Let $\lambda \neq 1, \det$. There exists a $\Gamma(J)_{\mathbb{R}}$-invariant sub vector bundle $\mathcal{E}_\lambda^J$ of $\mathcal{E}_\lambda$ with the following properties.*

(1) $\mathcal{E}_\lambda^J$ extends to a sub vector bundle of the canonical extension of $\mathcal{E}_\lambda$ over $\overline{\mathcal{X}(J)}$.

(2) We have a $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant isomorphism $\mathcal{E}_{\lambda}^{J}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J^{\otimes\lambda_1} \otimes V(J)_{\lambda'}$.

(3) If f is a Γ-modular form of weight (λ, k), its restriction to Δ_J as a section of $\mathcal{E}_{\lambda,k}$ takes values in the sub vector bundle $\mathcal{E}_{\lambda}^{J}\otimes\mathcal{L}^{\otimes k}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J^{\otimes k+\lambda_1}\otimes V(J)_{\lambda'}$ of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$.

In particular, we have

$$f|_{\Delta_J} = \pi_2^*(\Phi_J f)$$

for a $V(J)_{\lambda'}$-valued cusp form $\Phi_J f$ of weight $k + \lambda_1$ on $\mathbb{H}_J$ with respect to the image of $\Gamma(J)_{\mathbb{Z}} \to \mathrm{SL}(J)$. If $f = \sum_l a(l)q^l$ is the Fourier expansion of f at a 0-dimensional cusp $I \subset J$, the Fourier expansion of $\Phi_J f$ at the I-cusp of $\mathbb{H}_J$ is given by

$$(6.2) \qquad (\Phi_J f)(\tau) = \sum_{l\in\sigma_J\cap U(I)_{\mathbb{Z}}^{\vee}} a(l)e((l,\tau)), \qquad \tau \in \mathbb{H}_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp},$$

where $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ is the isotropic ray in $U(I)_{\mathbb{R}}$ corresponding to J.

In (6.2), the pairing (l, τ) for $l \in \sigma_J$ and $\tau \in \mathbb{H}_J$ is the natural pairing between $U(J)_{\mathbb{C}}$ and $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^{\perp}$. (This $\tau \in \mathbb{H}_J$ is different from the coordinate $\tau \in \mathbb{H}$ in §5.1.2, but rather is identified with the point τl_J there.)

A point here is that the vector bundle $\mathcal{E}_{\lambda,k}$ "reduces" to the sub vector bundle $\mathcal{E}_{\lambda}^{J} \otimes \mathcal{L}^{\otimes k}$ at the boundary divisor Δ_J. This is the difference with the Siegel operator in the scalar-valued case. This reduction corresponds to the reduction $\lambda \rightsquigarrow \lambda_1 \boxtimes \lambda'$ of the weight, and makes it possible to descend $f|_{\Delta_J}$ to $\mathbb{H}_J$. Roughly speaking, this reduction occurs as a result of taking the direct image of $\mathcal{E}_{\lambda,k}$ to the Baily-Borel compactification. In this way, the naive Siegel operator (6.1) can be more geometrically understood as

$$\text{restriction to } \Delta_J \ + \ \text{reduction to } \mathcal{E}_{\lambda}^{J}\otimes\mathcal{L}^{\otimes k} \ + \ \text{descend to } \mathbb{H}_J.$$

The sub vector bundle $\mathcal{E}_{\lambda}^{J}$ will be taken up again in §8.3 from the viewpoint of a filtration on $\mathcal{E}_{\lambda}$.

In §6.1 we prepare some calculations related to $\mathcal{E}_{\lambda}^{J}$. In §6.2 we define $\mathcal{E}_{\lambda}^{J}$ and prove the properties (1), (2) in Theorem 6.1. The Siegel operator Φ_J is defined in §6.3, and the remaining assertions of Theorem 6.1 are proved there.

6.1. Invariant part for a unipotent group

This section is preliminaries for introducing the Siegel operator. We prove that the Fourier coefficients of a modular form in the J-ray are contained in the invariant subspace for a certain unipotent subgroup of $\mathrm{O}(n, \mathbb{C})$, and study this space as a representation of $\mathbb{C}^* \times \mathrm{O}(n-2, \mathbb{C})$.

Let $F = \mathbb{Q}, \mathbb{R}$. Let $W(J)_F \subset \Gamma(J)_F$ be the Heisenberg group and the Jacobi group for J over F defined in §5.2. We choose a rank 1 primitive sublattice I of J, and also a rank 1 sublattice I' of L with $(I, I') \neq 0$. Let

$$\Gamma(I, J)_F = \Gamma(J)_F \cap \operatorname{Ker}(\Gamma(I)_F \to \mathrm{GL}(I)).$$

By definition $\Gamma(I, J)_F$ consists of isometries of L_F which act trivially on I_F, J_F/I_F and $V(J)_F = (J^\perp/J)_F$. As a subgroup of $\Gamma(J)_F$, $\Gamma(I, J)_F$ contains $W(J)_F$, and the quotient $\Gamma(I, J)_F/W(J)_F \simeq F$ is the subgroup of $\Gamma(J)_F/W(J)_F \simeq \mathrm{SL}(J_F)$ which acts trivially on I_F.

As a subgroup of $\Gamma(I)_F$, $\Gamma(I, J)_F$ contains the unipotent radical $U(I)_F$ of $\Gamma(I)_F$ by (5.8). Let $U(J/I)_F$ be the subgroup of $\mathrm{O}(V(I)_F)$ acting trivially on J_F/I_F and $V(J)_F$. Then $U(J/I)_F$ is the image of $\Gamma(I, J)_F$ in $\mathrm{O}(V(I)_F)$. This is also the image of $W(J)_F$ in $\mathrm{O}(V(I)_F)$. From (3.11), we have the exact sequence

$$(6.3) \qquad 0 \to U(I)_F \to \Gamma(I, J)_F \to U(J/I)_F \to 0.$$

By (1.3), the group $U(J/I)_F$ is the unipotent radical of the stabilizer of J_F/I_F in $\mathrm{O}(V(I)_F)$ and consists of the Eichler transvections of $V(I)_F$ with respect to J_F/I_F. We have a canonical isomorphism

$$U(J/I)_F \simeq V(J)_F \otimes_F (J_F/I_F).$$

We define $U(J/I)_{\mathbb{C}} < \mathrm{O}(V(I))$ similarly.

Now let f be a modular form of weight (λ, k) with respect to Γ, and $f = \sum_l a(l)q^l$ be its Fourier expansion at I. We are interested in the Fourier coefficients $a(l) \in V(I)_{\lambda,k}$ for l in the isotropic ray $\sigma_J = ((J/I)_{\mathbb{R}} \otimes I_{\mathbb{R}})_{\geq 0}$ corresponding to J. We denote by

$$V(I)_\lambda^U = V(I)_\lambda^{U(J/I)_{\mathbb{C}}}$$

the invariant subspace of $V(I)_\lambda$ for the unipotent subgroup $U(J/I)_{\mathbb{C}}$ of $\mathrm{O}(V(I))$, and put

$$V(I)_{\lambda,k}^U = V(I)_\lambda^U \otimes (I_{\mathbb{C}}^\vee)^{\otimes k} \subset V(I)_{\lambda,k}.$$

Lemma 6.2. *If $l \in U(I)_{\mathbb{Z}}^\vee \cap \sigma_J$, then $a(l) \in V(I)_{\lambda,k}^U$.*

Proof. We take the splitting of (6.3) for $F = \mathbb{Q}$ following (3.12), and accordingly express elements of $\Gamma(I, J)_{\mathbb{Q}}$ as (γ_1, α) where $\gamma_1 \in U(J/I)_{\mathbb{Q}} \subset \mathrm{O}(V(I)_{\mathbb{Q}})$ and $\alpha \in U(I)_{\mathbb{Q}}$. (In the notation (3.14), this is $(\gamma_1 \otimes \mathrm{id}_I, 1, \alpha)$.) There exists a finite-index subgroup H of $\Gamma(I, J)_{\mathbb{Q}} \cap \Gamma$ such that $\alpha \in U(I)_{\mathbb{Z}}$ for every element (γ_1, α) of H. The group $\Gamma(I, J)_{\mathbb{Q}}$ acts trivially on the isotropic ray σ_J. Therefore, if $l \in U(I)_{\mathbb{Z}}^\vee \cap \sigma_J$, we see from Proposition 3.6 that

$$a(l) = a(\gamma_1 l) = \gamma_1(a(l))$$

for every element (γ_1, α) of H. Here $\gamma_1 \in U(J/I)_{\mathbb{Q}}$ acts on $V(I)_{\lambda,k}$ by its natural action on $V(I)_\lambda$. This equality means that $a(l)$ is contained in the H-invariant subspace $V(I)_{\lambda,k}^H = V(I)_\lambda^H \otimes (I_{\mathbb{C}}^\vee)^{\otimes k}$ of $V(I)_{\lambda,k}$. The image of H by the projection $\Gamma(I,J)_{\mathbb{Q}} \to U(J/I)_{\mathbb{Q}}$, $(\gamma_1,\alpha)\mapsto\gamma_1$, is a full lattice in $U(J/I)_{\mathbb{Q}}$. In particular, it is Zariski dense in $U(J/I)_{\mathbb{C}}$. This shows that $V(I)_\lambda^H = V(I)_\lambda^U$, and so $a(l)\in V(I)_{\lambda,k}^U$. □

Let $P(J/I)_{\mathbb{C}}$ be the stabilizer of the isotropic line $J_{\mathbb{C}}/I_{\mathbb{C}} \subset V(I)$ in $\mathrm{O}(V(I))$. Then $U(J/I)_{\mathbb{C}}$ is the unipotent radical of $P(J/I)_{\mathbb{C}}$ and sits in the exact sequence (cf. (1.3))

$$0 \to U(J/I)_{\mathbb{C}} \to P(J/I)_{\mathbb{C}} \to \mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}})\times \mathrm{O}(V(J)) \to 1. \tag{6.4}$$

Therefore $V(I)_\lambda^U$ is a representation of

$$\mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}})\times\mathrm{O}(V(J)) \;\simeq\; \mathbb{C}^*\times\mathrm{O}(n-2,\mathbb{C}) \;\simeq\; \mathrm{SO}(2,\mathbb{C})\times\mathrm{O}(n-2,\mathbb{C}).$$

Proposition 6.3. *Let $\lambda\neq\det$. As a representation of $\mathbb{C}^*\times\mathrm{O}(V(J))$ we have*

$$V(I)_\lambda^U \simeq \chi_{\lambda_1}\boxtimes V(J)_{\lambda'},$$

where χ_{λ_1} is the character of $\mathbb{C}^*$ of weight λ_1 and $V(J)_{\lambda'}$ is the irreducible representation of $\mathrm{O}(V(J))$ associated to the partition $\lambda'=(\lambda_2\geq\cdots\geq\lambda_{n-1})$.

Proof. This is purely a representation-theoretic calculation. Let us first rewrite the setting. Let $V=\mathbb{C}^n$ be an n-dimensional quadratic space over $\mathbb{C}$ with a basis $e_1,\cdots,e_n$ such that $(e_i,e_j)=1$ if $i+j=n+1$ and $(e_i,e_j)=0$ otherwise. Let P be the stabilizer of the isotropic line $\mathbb{C}e_1$ in $\mathrm{O}(V)$ and let $V'=\langle e_2,\cdots,e_{n-1}\rangle$. Then

$$P=(\mathbb{C}^*\times\mathrm{O}(V'))\ltimes U,$$

where $\mathbb{C}^*=\mathrm{SO}(\langle e_1,e_n\rangle)\simeq\mathrm{GL}(\mathbb{C}e_1)$ and U is the group of unipotent matrices

$$\begin{pmatrix} 1 & -v^\vee & -(v,v)/2 \\ 0 & I_{n-2} & v \\ 0 & 0 & 1\end{pmatrix} \qquad v\in V'.$$

The problem is to calculate the U-invariant part V_λ^U of V_λ as a representation of $\mathbb{C}^*\times\mathrm{O}(V')$.

Step 1. *There exists a $\mathbb{C}^*\times\mathrm{O}(V')$-equivariant embedding $\chi_{\lambda_1}\boxtimes V'_{\lambda'}\hookrightarrow V_\lambda^U$.*

Proof. We write

$$W_0=\wedge^{{}^t\lambda_1}V\otimes\cdots\otimes\wedge^{{}^t\lambda_{\lambda_1}}V,$$
$$W_0'=\wedge^{{}^t\lambda_1-1}V'\otimes\cdots\otimes\wedge^{{}^t\lambda_{\lambda_1}-1}V',$$
$$W_1=(\mathbb{C}e_1\wedge\wedge^{{}^t\lambda_1-1}V')\otimes\cdots\otimes(\mathbb{C}e_1\wedge\wedge^{{}^t\lambda_{\lambda_1}-1}V').$$

We have a natural $\mathbb{C}^*\times\mathrm{O}(V')$-equivariant isomorphism

$$\iota:\mathbb{C}e_1^{\otimes\lambda_1}\otimes W_0'\xrightarrow{\simeq}W_1 \;\subset W_0.$$

Recall from (3.1) that $V_\lambda \subset W_0$ and $V'_{\lambda'} \subset W'_0$. (Note that the transpose of λ' is $({}^t\lambda_1 - 1, \cdots, {}^t\lambda_{\lambda_1} - 1)$.) We shall show that the image of $\mathbb{C}e_1^{\otimes\lambda_1} \otimes V'_{\lambda'}$ by ι is contained in V_λ^U. Since $\mathbb{C}e_1^{\otimes\lambda_1} \simeq \chi_{\lambda_1}$ as a representation of $\mathbb{C}^*$, this would imply our assertion.

Since U acts on W_1 trivially, it does so on $\iota(\mathbb{C}e_1^{\otimes\lambda_1} \otimes V'_{\lambda'})$. Thus it suffices to see that $\iota(\mathbb{C}e_1^{\otimes\lambda_1} \otimes V'_{\lambda'})$ is contained in V_λ. Recall from (3.2) that V_λ and $V'_{\lambda'}$ respectively contain the vectors

$$v_\lambda = \bigotimes_{i=1}^{\lambda_1}(e_1 \wedge \cdots \wedge e_{{}^t\lambda_i}), \quad v'_{\lambda'} = \bigotimes_{i=1}^{\lambda_1}(e_2 \wedge \cdots \wedge e_{{}^t\lambda_i}).$$

Since $\iota(e_1^{\otimes\lambda_1} \otimes v'_{\lambda'}) = v_\lambda$, we have $\mathrm{O}(V') \cdot \iota(e_1^{\otimes\lambda_1} \otimes v'_{\lambda'}) \subset V_\lambda$. Taking the linear hull and using the irreducibility of $V'_{\lambda'}$, we see that $\iota(\mathbb{C}e_1^{\otimes\lambda_1} \otimes V'_{\lambda'}) \subset V_\lambda$. □

For the proof of Proposition 6.3, it thus suffices to prove $\dim V'_{\lambda'} = \dim V_\lambda^U$. We use the restriction to $\mathrm{SO}(V) \subset \mathrm{O}(V)$. We first consider the case when V_λ remains irreducible as a representation of $\mathrm{SO}(V)$. As recalled in §3.6.1, this occurs exactly when n is odd or n is even with ${}^t\lambda_1 \neq n/2$, and V_λ has highest weight

$$\bar\lambda = (\bar\lambda_1, \cdots, \bar\lambda_{[n/2]}) = (\lambda_1, \lambda_2 - \lambda_{n-1}, \cdots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

in this case.

Step 2. *When V_λ is irreducible as a representation of* $\mathrm{SO}(V)$, V_λ^U *is irreducible as a representation of* $\mathrm{SO}(V')$ *with highest weight* $\bar\lambda' = (\bar\lambda_2, \cdots, \bar\lambda_{[n/2]})$. *In particular, we have* $\dim V'_{\lambda'} = \dim V_\lambda^U$.

Proof. Let B and B' be the groups of upper triangular matrices in $\mathrm{SO}(V)$ and $\mathrm{SO}(V')$ respectively (the standard Borel subgroups). Let U_0 and U'_0 be the groups of unipotent matrices in B and B' respectively. Then U and U'_0 generate U_0. Therefore we have

$$V_\lambda^{U_0} = (V_\lambda^U)^{U'_0}. \tag{6.5}$$

The space $V_\lambda^{U_0}$ is the highest weight space for the $\mathrm{SO}(V)$-representation V_λ, while $(V_\lambda^U)^{U'_0}$ is the highest weight space for the $\mathrm{SO}(V')$-representation V_λ^U. The irreducibility of V_λ as an $\mathrm{SO}(V)$-representation implies $\dim V_\lambda^{U_0} = 1$, which in turn implies by (6.5) the irreducibility of V_λ^U as a representation of $\mathrm{SO}(V')$.

We shall calculate the highest weight of V_λ^U for $\mathrm{SO}(V')$. Let T and T' be the groups of diagonal matrices in B and B' respectively. Then $T = \mathbb{C}^* \times T'$. The highest weight $\bar\lambda = (\bar\lambda_1, \cdots, \bar\lambda_{[n/2]})$ of the $\mathrm{SO}(V)$-representation V_λ is the weight of the T-action on the highest weight space $V_\lambda^{U_0}$. Therefore T' acts by weight $\bar\lambda' = (\bar\lambda_2, \cdots, \bar\lambda_{[n/2]})$ on $V_\lambda^{U_0}$. By (6.5), this means that the highest weight of V_λ^U for $\mathrm{SO}(V')$ is $\bar\lambda'$. □

It remains to cover the exceptional case where V_λ gets reducible when restricted to $\mathrm{SO}(V)$, namely n is even and ${}^t\lambda_1 = n/2$.

STEP 3. *We have* $\dim V'_{\lambda'} = \dim V_\lambda^U$ *even when* V_λ *is reducible as a representation of* $\mathrm{SO}(V)$.

PROOF. In this case, the irreducible summands of V_λ have highest weight $\bar{\lambda} = \lambda$ and $\lambda^\dagger$ respectively. We can argue similarly for each irreducible summand. This shows that V_λ^U as a representation of $\mathrm{SO}(V')$ has two irreducible summands, of highest weight $\lambda' = (\lambda_2, \cdots, \lambda_{n/2})$ and $(\lambda^\dagger)' = (\lambda_2, \cdots, -\lambda_{n/2})$. On the other hand, $V'_{\lambda'}$ is also reducible as a representation of $\mathrm{SO}(V')$ with highest weight λ' and $(\lambda')^\dagger = (\lambda^\dagger)'$ by §3.6.1. This implies that $V_\lambda^U \simeq V'_{\lambda'}$ as $\mathrm{SO}(V')$-representations. □

The proof of Proposition 6.3 is now complete. □

6.2. The sub vector bundle $\mathcal{E}_\lambda^J$

Let $\lambda \neq \det$. We define the sub vector bundle $\mathcal{E}_\lambda^J$ of $\mathcal{E}_\lambda$ as the image of $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization $\iota_I \colon V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \to \mathcal{E}_\lambda$.

LEMMA 6.4. *The sub vector bundle* $\mathcal{E}_\lambda^J$ *of* $\mathcal{E}_\lambda$ *is* $\Gamma(J)_{\mathbb{R}}$*-invariant. In particular, it does not depend on the choice of* I.

PROOF. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$. What has to be shown is that the image of $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ by the composition homomorphism

$$V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\iota_I} \mathcal{E}_\lambda \xrightarrow{\gamma} \mathcal{E}_\lambda \xrightarrow{\iota_I^{-1}} V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$$

is again $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$. This homomorphism coincides with

$$V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\gamma} V(\gamma I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \xrightarrow{\iota_{\gamma I}} \mathcal{E}_\lambda \xrightarrow{\iota_I^{-1}} V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}, \tag{6.6}$$

where $\iota_{\gamma I}$ is the γI-trivialization. The image of $V(I)_\lambda^U$ by $\gamma \colon V(I)_\lambda \to V(\gamma I)_\lambda$ is $V(\gamma I)_\lambda^U$, the invariant subspace of $V(\gamma I)_\lambda$ for the unipotent radical $U(\gamma J_{\mathbb{C}}/\gamma I_{\mathbb{C}}) = U(J_{\mathbb{C}}/\gamma I_{\mathbb{C}})$ of the stabilizer of $J_{\mathbb{C}}/\gamma I_{\mathbb{C}}$ in $\mathrm{O}(V(\gamma I))$. Therefore it suffices to show that the homomorphism

$$\iota_{\gamma I}^{-1} \circ \iota_I : V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}} \to V(\gamma I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$$

sends $V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$ to $V(\gamma I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$.

The problem is pointwise. Let $[\omega] \in \mathcal{D}$. At the fiber of $\mathcal{E}$ over $[\omega]$, the difference of the I-trivialization and the γI-trivialization is the isometry

$$I_{\mathbb{C}}^\perp/I_{\mathbb{C}} \to I_{\mathbb{C}}^\perp \cap \omega^\perp \to \omega^\perp/\mathbb{C}\omega \to \gamma I_{\mathbb{C}}^\perp \cap \omega^\perp \to \gamma I_{\mathbb{C}}^\perp/\gamma I_{\mathbb{C}}.$$

This sends the isotropic line $J_{\mathbb{C}}/I_{\mathbb{C}}$ of $I_{\mathbb{C}}^\perp/I_{\mathbb{C}}$ as

$$J_{\mathbb{C}}/I_{\mathbb{C}} \to J_{\mathbb{C}} \cap \omega^\perp = J_{\mathbb{C}} \cap \omega^\perp \to J_{\mathbb{C}}/\gamma I_{\mathbb{C}}.$$

Therefore the induced isomorphism $\mathrm{O}(V(I)) \to \mathrm{O}(V(\gamma I))$ sends the subgroup $U(J/I)_{\mathbb{C}}$ to $U(J_{\mathbb{C}}/\gamma I_{\mathbb{C}})$. It follows that the induced isomorphism

$$(\iota_{\gamma I}^{-1} \circ \iota_I)_{[\omega]} : V(I)_\lambda \to V(\gamma I)_\lambda$$

sends $V(I)_\lambda^U$ to $V(\gamma I)_\lambda^U$. □

Recall that the canonical extension of $\mathcal{E}_\lambda$ over the partial toroidal compactification $\overline{\mathcal{X}(J)}$ is defined via the I-trivialization $V(I)_\lambda \otimes \mathcal{O}_{\mathcal{X}(J)} \to \mathcal{E}_\lambda$. Therefore, by construction, $\mathcal{E}_\lambda^J$ extends to a sub vector bundle of the canonical extension of $\mathcal{E}_\lambda$ (again denoted by $\mathcal{E}_\lambda^J$). The I-trivialization $\mathcal{E}_\lambda \to V(I)_\lambda \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ over $\overline{\mathcal{X}(J)}$ sends $\mathcal{E}_\lambda^J$ to $V(I)_\lambda^U \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$.

Proposition 6.5. *There exists an* $\mathrm{SL}(J_{\mathbb{R}})$*-equivariant vector bundle* $\Phi_J\mathcal{E}_\lambda$ *on* $\mathbb{H}_J$ *such that we have a* $\overline{\Gamma(J)_{\mathbb{R}}}$*-equivariant isomorphism*

$$\mathcal{E}_\lambda^J|_{\Delta_J} \simeq \pi_2^*(\Phi_J\mathcal{E}_\lambda)$$

of vector bundles on Δ_J.

Proof. Let $j(\gamma, [\omega])$ be the factor of automorphy of the $\Gamma(J)_{\mathbb{R}}$-action on $\mathcal{E}_\lambda^J$ with respect to the I-trivialization $\mathcal{E}_\lambda^J \simeq V(I)_\lambda^U \otimes \mathcal{O}_{\mathcal{D}}$. This is a $\mathrm{GL}(V(I)_\lambda^U)$-valued function on $\Gamma(J)_{\mathbb{R}} \times \mathcal{D}$. We shall prove the following.

(1) For fixed γ, the function $j(\gamma, [\omega])$ of $[\omega]$ is constant on each fiber of $\mathcal{D} \to \mathbb{H}_J$.
(2) $j(\gamma, [\omega]) = \mathrm{id}$ if $\gamma \in W(J)_{\mathbb{R}}$.

Since $\Gamma(J)_{\mathbb{R}}/W(J)_{\mathbb{R}} \simeq \mathrm{SL}(J_{\mathbb{R}})$, these properties ensure that $j(\gamma, [\omega])$ descends to a $\mathrm{GL}(V(I)_\lambda^U)$-valued function on $\mathrm{SL}(J_{\mathbb{R}}) \times \mathbb{H}_J$. This function defines the factor of automorphy of an $\mathrm{SL}(J_{\mathbb{R}})$-equivariant vector bundle $\Phi_J\mathcal{E}_\lambda$ on $\mathbb{H}_J$ such that $\mathcal{E}_\lambda^J \simeq \pi^*(\Phi_J\mathcal{E}_\lambda)$ as $\Gamma(J)_{\mathbb{R}}$-equivariant vector bundles on $\mathcal{D}$. This gives an isomorphism $\mathcal{E}_\lambda^J|_{\Delta_J} \simeq \pi_2^*(\Phi_J\mathcal{E}_\lambda)$ over Δ_J.

We first check the property (2). Since $W(J)_{\mathbb{R}}$ acts on $I_{\mathbb{R}}$ trivially, we see from Lemma 3.2 that the factor of automorphy of the $W(J)_{\mathbb{R}}$-action on $\mathcal{E}_\lambda$ with respect to the I-trivialization is given by the natural action of $W(J)_{\mathbb{R}}$ on $V(I)_\lambda$. Since the image of $W(J)_{\mathbb{R}}$ in $\mathrm{O}(V(I)_{\mathbb{R}})$ is equal to $U(J/I)_{\mathbb{R}}$, $W(J)_{\mathbb{R}}$ acts on $V(I)_\lambda^U$ trivially by definition. This implies (2).

Next we verify the property (1). The fibers of $\mathcal{D} \to \mathcal{V}_J$ are contained in $U(J)_{\mathbb{C}}$-orbits in $\mathcal{D}(J) \supset \mathcal{D}$, and the fibers of $\Delta_J \to \mathbb{H}_J$ are $W(J)_{\mathbb{R}}/U(J)_{\mathbb{R}}$-orbits. In particular, the constancy on the fibers of $\mathcal{D} \to \mathcal{V}_J$ would follow from the constancy on $U(J)_{\mathbb{R}}$-orbits and the identity theorem in complex analysis for $U(J)_{\mathbb{R}} \subset U(J)_{\mathbb{C}}$. Thus we are reduced to checking the constancy on $W(J)_{\mathbb{R}}$-orbits. Let $\gamma \in \Gamma(J)_{\mathbb{R}}$ and $g \in W(J)_{\mathbb{R}}$. Then we can calculate

$$\begin{aligned} j(\gamma, g([\omega])) &= j(\gamma g, [\omega]) \circ j(g, [\omega])^{-1} = j(\gamma g, [\omega]) \\ &= j(\gamma g \gamma^{-1}, \gamma([\omega])) \circ j(\gamma, [\omega]) = j(\gamma, [\omega]). \end{aligned}$$

In the second and the last equalities we used the property (2) proved above, with the normality of $W(J)_{\mathbb{R}}$ in $\Gamma(J)_{\mathbb{R}}$ in the last equality. The property (1) is thus proved. □

Remark 6.6. By construction, $\Phi_J\mathcal{E}_\lambda$ is endowed with a trivialization $V(I)_\lambda^U \otimes \mathcal{O}_{\mathbb{H}_J} \simeq \Phi_J\mathcal{E}_\lambda$, whose pullback agrees with the I-trivialization $V(I)_\lambda^U \otimes \mathcal{O}_{\Delta_J} \simeq \mathcal{E}_\lambda^J|_{\Delta_J}$ of $\mathcal{E}_\lambda^J$ over Δ_J.

We can calculate the weights of $\Phi_J\mathcal{E}_\lambda$ by using Proposition 6.3. Let $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$.

Proposition 6.7. *There exists an* $\mathrm{SL}(J_{\mathbb{R}})$*-equivariant isomorphism*

$$\Phi_J\mathcal{E}_\lambda \simeq \mathcal{L}_J^{\otimes \lambda_1} \otimes V(J)_{\lambda'}, \tag{6.7}$$

of vector bundles on $\mathbb{H}_J$.

The proof of Proposition 6.7 is divided into several steps. Let us formulate the first half as preparatory lemmas as follows. Let $P(J)$ be the stabilizer of $J_{\mathbb{C}}$ in $\mathrm{O}(L_{\mathbb{C}})$. We write $Q(J) = Q - Q \cap \mathbb{P}J_{\mathbb{C}}^{\perp}$. Recall that $\mathcal{E}_\lambda$ is naturally defined over Q as an $\mathrm{O}(L_{\mathbb{C}})$-equivariant vector bundle.

Lemma 6.8. *The vector bundle* $\mathcal{E}_\lambda^J$ *extends to a* $P(J)$*-invariant sub vector bundle of* $\mathcal{E}_\lambda$ *over* $Q(J)$ *(again denoted by* $\mathcal{E}_\lambda^J$*).*

Proof. For each $\mathbb{C}$-line $I' \subset J_{\mathbb{C}}$, the I'-trivialization $\iota_{I'} : V(I')_\lambda \otimes \mathcal{O} \to \mathcal{E}_\lambda$ is defined over $Q(I') = Q - Q \cap \mathbb{P}(I')^{\perp}$. The same argument as the second half of the proof of Lemma 6.4 shows that for two $\mathbb{C}$-lines $I_1, I_2 \subset J_{\mathbb{C}}$, we have

$$\iota_{I_1}(V(I_1)_\lambda^U \otimes \mathcal{O}) = \iota_{I_2}(V(I_2)_\lambda^U \otimes \mathcal{O})$$

over $Q(I_1) \cap Q(I_2)$. Therefore, by gluing the image of $\iota_{I'}$ for all $\mathbb{C}$-lines $I' \subset J_{\mathbb{C}}$, we obtain a sub vector bundle of $\mathcal{E}_\lambda$ over $Q(J) = \cup_{I'} Q(I')$ which extends $\mathcal{E}_\lambda^J$. Since $\gamma \in P(J)$ sends $\iota_{I'}(V(I')_\lambda^U \otimes \mathcal{O})$ to $\iota_{\gamma I'}(V(\gamma I')_\lambda^U \otimes \mathcal{O})$ (cf. the proof of Lemma 6.4), this sub vector bundle is $P(J)$-invariant. □

Lemma 6.9. *Let* $D_J = Q(J) \cap \mathbb{P}I_{\mathbb{C}}^{\perp}$. *The* I*-trivialization* $V(I)_\lambda^U \otimes \mathcal{O}_{Q(I)} \to \mathcal{E}_\lambda^J$ *over* $Q(I)$ *extends to an isomorphism*

$$V(I)_\lambda^U \otimes \mathcal{O}_{Q(J)} \to \mathcal{E}_\lambda^J \otimes \mathcal{O}_{Q(J)}(\lambda_1 D_J) \tag{6.8}$$

over $Q(J)$*, which is equivariant with respect to the stabilizer of* $I_{\mathbb{C}}$ *in* $P(J)$*.*

Proof. We choose an arbitrary embedding $2U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$ compatible with $I_{\mathbb{C}} \subset J_{\mathbb{C}}$ in the sense of §5 and accordingly take a lift $\mathrm{GL}(J_{\mathbb{C}}) \hookrightarrow P(J)$ of $\mathrm{GL}(J_{\mathbb{C}})$. Let $T \simeq \mathbb{C}^*$ be the subgroup of $\mathrm{GL}(J_{\mathbb{C}})$ consisting of matrices $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha \in \mathbb{C}^*$, with respect to the basis e_1, e_2 of $J_{\mathbb{C}}$. (e_1 spans $I_{\mathbb{C}}$ and e_2

spans $J_{\mathbb{C}}/I_{\mathbb{C}}$.) The image of T in $P(J/I)_{\mathbb{C}}$ is a lift of $\mathrm{GL}(J_{\mathbb{C}}/I_{\mathbb{C}})$ in (6.4). Then

$$V(I) = \mathbb{C}e_2 \oplus V(J) \oplus \mathbb{C}f_2$$

is the weight decomposition for T, where $\mathbb{C}e_2$, $V(J)$, $\mathbb{C}f_2$ have weight 1, 0, -1 respectively. A general T-orbit $C^\circ = T[\omega]$ in $Q(I)$ gives a flow converging to the point $p = [f_2]$ of D_J as $\alpha \to 0$ from a normal direction. Let $C = C^\circ \cup p \simeq \mathbb{C}$ be the closure of such a T-orbit in $Q(J)$. The proof of Lemma 6.9 is based on the following assertion.

Claim 6.10. The I-trivialization $\mathcal{E}|_{C^\circ} \simeq V(I) \otimes \mathcal{O}_{C^\circ}$ over C° extends to an isomorphism

$$\mathcal{E}|_C \;\simeq\; \mathbb{C}e_2 \otimes \mathcal{O}_C(-p) \;\oplus\; V(J) \otimes \mathcal{O}_C \;\oplus\; \mathbb{C}f_2 \otimes \mathcal{O}_C(p)$$

over C.

We postpone the proof of this claim for a while and continue the proof of Lemma 6.9. From Claim 6.10, we see that if $V(I)_\lambda = \bigoplus_r V(r)$ is the weight decomposition for T with $V(r)$ the weight r subspace, the I-trivialization of $\mathcal{E}_\lambda$ over C° extends to an isomorphism

$$\mathcal{E}_\lambda|_C \simeq \bigoplus_r V(r) \otimes \mathcal{O}_C(-rp)$$

over C. Since $V(I)_\lambda^U \subset V(\lambda_1)$ by Proposition 6.3, we obtain

$$\mathcal{E}_\lambda^J|_C \simeq V(I)_\lambda^U \otimes \mathcal{O}_C(-\lambda_1 p).$$

Finally, if we vary the embedding $2U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$, then the point $p = [f_2]$ runs over D_J. This implies the assertion of Lemma 6.9. □

We give the postponed proof of Claim 6.10.

(Proof of Claim 6.10). Let $v \in V(I)$ be a weight vector for T with weight $r \in \{-1, 0, 1\}$ and let s_v be the corresponding section of $\mathcal{E}$. We calculate the limit behavior of s_v on the T-orbit $C^\circ = T[\omega]$ as $\alpha \to 0$. We write $\gamma_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \in T$. We lift $V(I) \hookrightarrow I_{\mathbb{C}}^\perp$ by the given embedding $U_{\mathbb{C}} \hookrightarrow L_{\mathbb{C}}$. By Lemma 2.6 for $l = e_1 \in I$, we have

$$\begin{aligned} s_v(\gamma_\alpha[\omega]) &= v - (v, s_{e_1}(\gamma_\alpha[\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - (v, \gamma_\alpha(s_{e_1}([\omega])))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - (\gamma_\alpha^{-1}(v), s_{e_1}([\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega) \\ &= v - \alpha^{-r}(v, s_{e_1}([\omega]))e_1 \mod \mathbb{C}\gamma_\alpha(\omega). \end{aligned} \tag{6.9}$$

We take the $\mathbb{C}e_2$-trivialization of $\mathcal{E}$ and express $s_v(\gamma_\alpha[\omega])$ as a $V(\mathbb{C}e_2)$-valued function. We identify $V(\mathbb{C}e_2) = \mathbb{C}e_1 \oplus V(J) \oplus \mathbb{C}f_1$ naturally. Then, according to the weight r of v, we have

$$s_v(\gamma_\alpha[\omega]) = \begin{cases} v + C_1(v)e_1 & v \in V(J) \\ \alpha^{-1}C_2 e_1 & v = e_2 \\ \alpha C_2^{-1} f_1 + \alpha C_3 e_1 + \alpha v_0 & v = f_2 \end{cases}$$

as a $V(\mathbb{C}e_2)$-valued function. Here $C_1(v)$ is a linear function on $V(J)$, $C_2 \neq 0$ and C_3 are constants, and $v_0 \in V(J)$ is some constant vector. These expressions in the cases $v \in V(J)$ and $v = e_2$ are apparent from (6.9), because the vector in (6.9) is already perpendicular to e_2 in these cases. The case $v = f_2$ follows from the conditions

$$(s_{f_2}, s_{e_2}) = 1, \quad (s_{f_2}, s_{f_2}) = 0, \quad (s_{f_2}, s_w) = 0 \text{ for } w \in V(J).$$

(This can also be calculated by using the coordinates (τ, z, w) in §5.1.2.) The assertion of Claim 6.10 now follows from these expressions. □

Now we can complete the proof of Proposition 6.7.

(Proof of Proposition 6.7). We pass from $Q(J)$ to $\mathbb{P}J^\vee_{\mathbb{C}}$. By the same argument as the proof of Proposition 6.5 with $\Gamma(J)_{\mathbb{R}}$ replaced by $P(J)$ and $W(J)_{\mathbb{R}}$ replaced by the kernel of $P(J) \to \mathrm{GL}(J_{\mathbb{C}}) \times \mathrm{O}(V(J))$, we find that the $P(J)$-equivariant vector bundle $\mathcal{E}^J_\lambda$ on $Q(J)$ descends to a $\mathrm{GL}(J_{\mathbb{C}}) \times \mathrm{O}(V(J))$-equivariant vector bundle on $\mathbb{P}J^\vee_{\mathbb{C}}$. This is an extension of $\Phi_J\mathcal{E}_\lambda$, and we denote it again by $\Phi_J\mathcal{E}_\lambda$. Let $p_I = I^\perp \cap \mathbb{P}J^\vee_{\mathbb{C}}$ be the I-cusp of $\mathbb{H}_J$. Since D_J is the fiber of $Q(J) \to \mathbb{P}J^\vee_{\mathbb{C}}$ over p_I, we find that the isomorphism (6.8) descends to an isomorphism

$$V(I)^U_\lambda \otimes \mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}} \to \Phi_J\mathcal{E}_\lambda \otimes \mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}}(\lambda_1 p_I). \tag{6.10}$$

This is equivariant with respect to the stabilizer of $I_{\mathbb{C}}$ in $\mathrm{GL}(J_{\mathbb{C}})$ and $\mathrm{O}(V(J))$. Note that these groups act on $V(I)^U_\lambda$ by the representation in Proposition 6.3.

Claim 6.11. The element $g(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}$ of $\mathrm{GL}(J_{\mathbb{C}})$, $\alpha \in \mathbb{C}^*$, acts on the fiber of $\Phi_J\mathcal{E}_\lambda$ over p_I as the scalar multiplication by α^{λ_1}.

We prove Claim 6.11. By Proposition 6.3, the matrices $\begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ in $\mathrm{GL}(J_{\mathbb{C}})$ act on $V(I)^U_\lambda$ as the scalar multiplication by α^{λ_1}. Moreover, the matrices $\begin{pmatrix} \beta & 0 \\ 0 & 1 \end{pmatrix}$ act on $V(I)$ trivially. It follows that $g(\alpha)$ acts on $V(I)^U_\lambda$ as the scalar multiplication by $\alpha^{-\lambda_1}$. On the other hand, since the tangent space of $p_I \in \mathbb{P}J^\vee_{\mathbb{C}}$ is

$$\mathrm{Hom}(I^\perp \cap J^\vee_{\mathbb{C}},\ J^\vee_{\mathbb{C}}/(I^\perp \cap J^\vee_{\mathbb{C}})) \simeq \mathrm{Hom}((J/I)^\vee_{\mathbb{C}}, I^\vee_{\mathbb{C}}),$$

the element $g(\alpha)$ acts on it by the multiplication by α^{-2}. Hence $g(\alpha)$ acts on the fiber of $\mathcal{O}_{\mathbb{P}J^\vee_{\mathbb{C}}}(-\lambda_1 p_I)$ over p_I as the multiplication by $\alpha^{2\lambda_1}$. By the isomorphism (6.10), we find that $g(\alpha)$ acts on the fiber of $\Phi_J\mathcal{E}_\lambda$ over p_I as the scalar multiplication by $\alpha^{-\lambda_1}\cdot\alpha^{2\lambda_1}=\alpha^{\lambda_1}$. This proves Claim 6.11.

We go back to the proof of Proposition 6.7. The torus consisting of the matrices $g(\alpha)$ is the reductive part of the stabilizer of p_I in $\mathrm{SL}(J_{\mathbb{C}})$. Therefore Claim 6.11 implies that $\Phi_J\mathcal{E}_\lambda$ is isomorphic to a direct sum of copies of $\mathcal{L}_J^{\otimes\lambda_1}$ as an $\mathrm{SL}(J_{\mathbb{C}})$-equivariant vector bundle on $\mathbb{P}J^\vee_{\mathbb{C}}$. Moreover, by the isomorphism (6.10) and Proposition 6.3, the action of $\mathrm{O}(V(J))$ on the fibers of $\Phi_J\mathcal{E}_\lambda$ is isomorphic to the representation $V(J)_{\lambda'}$. Therefore $\Phi_J\mathcal{E}_\lambda\simeq\mathcal{L}_J^{\otimes\lambda_1}\otimes V(J)_{\lambda'}$ as $\mathrm{SL}(J_{\mathbb{C}})\times\mathrm{O}(V(J))$-equivariant vector bundles on $\mathbb{P}J^\vee_{\mathbb{C}}$. This finishes the proof of Proposition 6.7. □

Remark 6.12. By the proof, the vector bundle $\Phi_J\mathcal{E}_\lambda$ over $\mathbb{H}_J$ is in fact $\mathrm{SL}(J_{\mathbb{R}})\times\mathrm{O}(V(J)_{\mathbb{R}})$-linearized, and the isomorphism $\Phi_J\mathcal{E}_\lambda\simeq\mathcal{L}_J^{\otimes\lambda_1}\otimes V(J)_{\lambda'}$ over $\mathbb{H}_J$ is $\mathrm{SL}(J_{\mathbb{R}})\times\mathrm{O}(V(J)_{\mathbb{R}})$-equivariant.

6.3. The Siegel operator

Combining the arguments so far, we can now define the Siegel operator at the J-cusp.

Proposition 6.13. *Let $f\in M_{\lambda,k}(\Gamma)$ with $\lambda\neq 1,\det$. There exists a cusp form $\Phi_J f$ on $\mathbb{H}_J$ with values in $\Phi_J\mathcal{E}_\lambda\otimes\mathcal{L}_J^{\otimes k}\simeq\mathcal{L}_J^{\otimes\lambda_1+k}\otimes V(J)_{\lambda'}$ and invariant under the image of $\Gamma(J)_{\mathbb{Z}}\to\mathrm{SL}(J)$ such that $f|_{\Delta_J}=\pi_2^*(\Phi_J f)$. If $f=\sum_l a(l)q^l$ is the Fourier expansion of f at a 0-dimensional cusp $I\subset J$, the Fourier expansion of $\Phi_J f$ at the I-cusp of $\mathbb{H}_J$ is given by*

$$(\Phi_J f)(\tau)=\sum_{l\in\sigma_J\cap U(I)^\vee_{\mathbb{Z}}}a(l)e((l,\tau)),\qquad \tau\in\mathbb{H}_J\subset U(I)_{\mathbb{C}}/U(J)^\perp_{\mathbb{C}}.$$

Here we recall that $\mathcal{L}_J$ and $\Phi_J\mathcal{E}_\lambda$ on $\mathbb{H}_J$ are endowed with I-trivializations whose pullback agree with the I-trivializations of $\mathcal{L}$ and $\mathcal{E}_\lambda^J$ respectively (Remarks 5.11 and 6.6). These define the I-trivialization $\Phi_J\mathcal{E}_\lambda\otimes\mathcal{L}_J^{\otimes k}\simeq V(I)^U_{\lambda,k}\otimes\mathcal{O}_{\mathbb{H}_J}$ of $\Phi_J\mathcal{E}_\lambda\otimes\mathcal{L}_J^{\otimes k}$ whose pullback agrees with the I-trivialization of $\mathcal{E}_\lambda^J\otimes\mathcal{L}^{\otimes k}$. The Fourier expansion of $\Phi_J f$ is done with respect to this trivialization.

Proof. We choose a rank 1 primitive sublattice $I\subset J$ and let $f=\sum_l a(l)q^l$ be the Fourier expansion of f at I. By (3.20) and the gluing map $\overline{\mathcal{X}(J)}\to\mathcal{X}(I)^\Sigma$ in Lemma 5.3, we see that

$$f|_{\Delta_J}=\sum_{l\in\sigma_J\cap U(I)^\vee_{\mathbb{Z}}}a(l)q^l \tag{6.11}$$

as a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. By Lemma 6.2, the function $f|_{\Delta_J}$ takes values in $V(I)_{\lambda,k}^U$. This in turn implies that $f|_{\Delta_J}$ as a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ takes values in the sub vector bundle

$$\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}|_{\Delta_J} \simeq \pi_2^*(\Phi_J\mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k}) \simeq \pi_2^*\mathcal{L}_J^{\otimes \lambda_1+k} \otimes V(J)_{\lambda'}.$$

Since the section $f|_{\Delta_J}$ is $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant, it is in particular $W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$-invariant, and so it descends to a section of $\bar{\pi}_2^*\mathcal{L}_J^{\otimes \lambda_1+k} \otimes V(J)_{\lambda'}$ over $\Delta_J/(W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}})$ where

$$\bar{\pi}_2 : \Delta_J/(W(J)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}) \to \mathbb{H}_J$$

is the projection. Since $\bar{\pi}_2$ is a proper map (family of abelian varieties), we find that $f|_{\Delta_J}$ is constant on each π_2-fiber. Therefore $f|_{\Delta_J} = \pi_2^*(\Phi_J f)$ for a section $\Phi_J f$ of $\mathcal{L}_J^{\otimes \lambda_1+k} \otimes V(J)_{\lambda'}$ over $\mathbb{H}_J$. Since $f|_{\Delta_J}$ is $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant, $\Phi_J f$ is invariant under the image of $\Gamma(J)_{\mathbb{Z}} \to \mathrm{SL}(J)$.

The fact that the pullback of the I-trivialization of $\Phi_J\mathcal{E}_\lambda \otimes \mathcal{L}_J^{\otimes k}$ agrees with the I-trivialization of $\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}$ implies that the pullback of $\Phi_J f$ as a $V(I)_{\lambda,k}^U$-valued function by $\Delta_J \to \mathbb{H}_J$ equals to $f|_{\Delta_J}$ as a $V(I)_{\lambda,k}^U$-valued function. Therefore $\Phi_J f$ as a $V(I)_{\lambda,k}^U$-valued function on $\mathbb{H}_J$ is given by the right hand side of (6.11):

$$\Phi_J f = \sum_{l \in \sigma_J \cap U(I)_{\mathbb{Z}}^\vee} a(l) q^l.$$

Here q^l for $l \in \sigma_J \cap U(I)_{\mathbb{Z}}^\vee$ is naturally viewed as a function on $\mathbb{H}_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^\perp$ by the pairing between $U(J)_{\mathbb{C}}$ and $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}^\perp$. This gives the Fourier expansion of $\Phi_J f$ at the I-cusp of $\mathbb{H}_J$. By Proposition 3.7, $\Phi_J f$ vanishes at the I-cusp. Since this holds at every cusp of $\mathbb{H}_J$, we see that $\Phi_J f$ is a cusp form. □

Let Γ_J be the image of $\Gamma(J)_{\mathbb{Z}}$ in $\mathrm{SL}(J) \simeq \mathrm{SL}(2,\mathbb{Z})$. We call the map

$$M_{\lambda,k}(\Gamma) \to S_{\lambda_1+k}(\Gamma_J) \otimes V(J)_{\lambda'}, \quad f \mapsto \Phi_J f, \tag{6.12}$$

the *Siegel operator* at the J-cusp.

We look at some examples. We use the same notation as in the proof of Proposition 6.3.

Example 6.14. Let $\lambda = (1^d)$ for $0 < d < n$, namely $V_\lambda = \wedge^d V$. Then

$$(\wedge^d V)^U = \mathbb{C}e_1 \wedge (\wedge^{d-1}\langle e_1, \cdots, e_{n-1}\rangle) \simeq \mathbb{C}e_1 \otimes \wedge^{d-1}\langle e_2, \cdots, e_{n-1}\rangle.$$

In this case, $\Phi_J f$ is a $\binom{n-2}{d-1}$-tuple of scalar-valued cusp forms of weight $k+1$.

Example 6.15. Let $\lambda = (d)$, namely $V_{(d)}$ is the main irreducible component of $\mathrm{Sym}^d V$ (see Example 3.1 (2)). We have

$$V_{(d)}^U = \mathbb{C}e_1^{\otimes d} \subset \mathrm{Sym}^d V.$$

In this case, $\Phi_J f$ is a single scalar-valued cusp form of weight $k + d$.

The Siegel operator for vector-valued Siegel modular forms is studied in [**47**] §2. The case of genus 2 is also studied in [**1**] §1. Let us observe that the weight calculation in Example 6.15 in the case $n = 3$ agrees with the results of [**1**] and [**47**] for Siegel modular forms of genus 2.

Example 6.16. Let $n = 3$. In [**1**] and [**47**], it is proved that the Siegel operator for a Siegel modular form of genus 2 and weight $(\mathrm{Sym}^j, \det^l)$ produces a scalar-valued cusp form of weight $j + l$ on the 1-dimensional cusp.

On the other hand, when $j = 2d$ is even, we saw in Example 3.4 that the Siegel weight $(\mathrm{Sym}^{2d}, \det^l)$ corresponds to the orthogonal weight $(\lambda, k) = ((d), d + l)$. According to Example 6.15, $\Phi_J f$ is a cusp form of weight $d + (d + l) = j + l$. This agrees with the above results of [**1**] and [**47**].

In general, the Siegel operator in the form of (6.12) is still not surjective for the following obvious reason. Let $\Gamma(J)^*_{\mathbb{Z}}$ be the stabilizer of J in Γ, and let Γ^*_J be the image of $\Gamma(J)^*_{\mathbb{Z}}$ in $\mathrm{SL}(J) \times \mathrm{O}(J^\perp/J)$. Then $\Gamma_J = \Gamma^*_J \cap \mathrm{SL}(J)$ is of finite index and normal in Γ^*_J. Let

$$G = \Gamma^*_J/\Gamma_J \simeq \Gamma(J)^*_{\mathbb{Z}}/\Gamma(J)_{\mathbb{Z}}.$$

The modular forms are not only $\Gamma(J)_{\mathbb{Z}}$-invariant but also $\Gamma(J)^*_{\mathbb{Z}}$-invariant. Therefore, in view of Remark 6.12, we see that the image of the map (6.12) is contained in the G-invariant part of $S_{\lambda_1+k}(\Gamma_J) \otimes V(J)_{\lambda'}$.

CHAPTER 7

Fourier-Jacobi expansion

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $O^+(L)$. We fix a rank 2 primitive isotropic sublattice J of L. In this chapter we study the Fourier-Jacobi expansion of vector-valued modular forms at the J-cusp. From a geometric point of view, the Fourier-Jacobi expansion is the Taylor expansion along the boundary divisor Δ_J of the partial toroidal compactification $\overline{\mathcal{X}(J)}$. The m-th Fourier-Jacobi coefficient is the m-th Taylor coefficient, and is essentially a section of the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J where Θ_J is the conormal bundle of Δ_J. Here we have some special properties beyond general Taylor expansion:

- existence of the projection $\pi_1 \colon \overline{\mathcal{X}(J)} \to \Delta_J$ and the isomorphism $\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$ (Proposition 5.6), and
- existence of a special generator ω_J of the ideal sheaf of Δ_J which is a linear map on each fiber of π_1.

These properties ensure that the m-th Fourier-Jacobi coefficient as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J is canonically defined (Corollary 7.5) and is invariant under $\overline{\Gamma(J)}_{\mathbb{Z}}$. If we take the (I, ω_J)-trivialization for $I \subset J$, we can pass to a more familiar definition of the Fourier-Jacobi coefficient as a slice in the Fourier expansion at I.

In general, we define vector-valued Jacobi forms as $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J with cusp condition (Definition 7.10). Thus the Fourier-Jacobi coefficients are vector-valued Jacobi forms (Proposition 7.12). Although our approach is geometric, our Jacobi forms in the scalar-valued case are indeed classical Jacobi forms in the sense of Skoruppa [**44**] if we introduce suitable coordinates and the (I, ω_J)-trivialization (§7.4). When $n = 3$, our vector-valued Jacobi forms essentially agree with those considered by Ibukiyama-Kyomura [**28**] for Siegel modular forms of genus 2.

When J comes from an integral embedding $2U \hookrightarrow L$ and Γ is the so-called stable orthogonal group, the Fourier-Jacobi expansion of scalar-valued modular forms is well-understood through the work of Gritsenko [**21**]. A large part of this chapter can be regarded as a geometric reformulation and a generalization of the calculation in [**21**] §2. A lot of effort will be paid for keeping introduction of coordinates as minimal as possible (though never zero), or in other words, for describing what is canonical in a canonical way. We believe that this style would be suitable even in the scalar-valued case when working with general (Γ, J), for which simple expression by coordinates is no longer available.

7.1. Fourier-Jacobi and Fourier expansion

We begin with the familiar (but non-canonical) way to define Fourier-Jacobi expansion: slicing the Fourier expansion. The passage to a canonical formulation will be given in §7.2.

We choose a rank 1 primitive sublattice I of J, and also a rank 1 sublattice $I' \subset L$ with $(I, I') \neq 0$. Recall from §5.1.2 that $U(J)_{\mathbb{R}} = \wedge^2 J_{\mathbb{R}}$ is identified with the isotropic line $(J/I)_{\mathbb{R}} \otimes I_{\mathbb{R}}$ in $U(I)_{\mathbb{R}} = (I^{\perp}/I)_{\mathbb{R}} \otimes I_{\mathbb{R}}$, and that the Siegel domain realization of $\mathcal{D}$ with respect to J can be identified with the restriction of the projection

$$U(I)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(I)_{\mathbb{C}}/U(J)^{\perp}_{\mathbb{C}}$$

to the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ after the tube domain realization $\mathcal{D} \simeq \mathcal{D}_I$. The orientation of J determines the nonnegative part $\sigma_J = (U(J)_{\mathbb{R}})_{\geq 0}$ of $U(J)_{\mathbb{R}}$. Let $v_{J,\Gamma}$ be the positive generator of $U(J)_{\mathbb{Z}} = U(J)_{\mathbb{Q}} \cap \Gamma$. We choose a rational isotropic vector $l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ such that $(v_{J,\Gamma}, l_{J,\Gamma}) = 1$. Then $v_{J,\Gamma}, l_{J,\Gamma}$ span a rational hyperbolic plane in $U(I)_{\mathbb{Q}}$. We put

$$\omega_J = q^{l_{J,\Gamma}} = e((l_{J,\Gamma}, Z)), \qquad Z \in U(I)_{\mathbb{C}}.$$

This is a holomorphic function on $U(I)_{\mathbb{C}}$ invariant under the translation by $U(J)_{\mathbb{Z}}$. Thus we have chosen the auxiliary datum I, I', $l_{J,\Gamma}$. These will be fixed until Lemma 7.4.

Let f be a Γ-modular form of weight (λ, k). We identify f with a $V(I)_{\lambda,k}$-valued holomorphic function on $\mathcal{D}_I$ via the I-trivialization and the tube domain realization, and let $f(Z) = \sum_l a(l) q^l$ be its Fourier expansion. Like the calculation in §3.5.2 (see also Remark 3.10), we can rewrite the Fourier expansion as

$$f(Z) = \sum_{m \geq 0} \left(\sum_{l \in U(J)^{\perp}_{\mathbb{Q}}} a(l + m l_{J,\Gamma}) q^l \right) \omega_J^m. \tag{7.1}$$

Here l ranges over vectors in $U(J)^{\perp}_{\mathbb{Q}}$ such that $l + m l_{J,\Gamma} \in U(I)^{\vee}_{\mathbb{Z}}$. They form a translation of a full lattice in $U(J)^{\perp}_{\mathbb{Q}}$. Although $l_{J,\Gamma}$ is not necessarily a vector in $U(I)^{\vee}_{\mathbb{Z}}$, this expression still makes sense over the tube domain $\mathcal{D}_I$. We call (7.1) the *Fourier-Jacobi expansion* of f at the J-cusp relative to I,

I', $l_{J,\Gamma}$, and usually write it as

$$f = \sum_{m\geq 0} \phi_m \omega_J^m \tag{7.2}$$

with

$$\phi_m = \sum_{l\in U(J)^{\perp}_{\mathbb{Q}}} a(l + ml_{J,\Gamma})q^l. \tag{7.3}$$

We call ϕ_m the m-th *Fourier-Jacobi coefficient* of f at the J-cusp relative to I, I', $l_{J,\Gamma}$. This is a $V(I)_{\lambda,k}$-valued function on $\mathcal{D}_I$. Since $l \in U(J)^{\perp}_{\mathbb{Q}}$ in (7.3), ϕ_m actually descends to a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. We often do not specify the precise index lattice in (7.3); it is convenient to allow enlarging it as necessary by putting $a(l + ml_{J,\Gamma}) = 0$ when $l + ml_{J,\Gamma} \notin U(I)^{\vee}_{\mathbb{Z}}$. When $m = 0$, ϕ_0 is the restriction of f to Δ_J and was studied in §6. In this chapter we study the case $m > 0$.

7.2. Geometric approach to Fourier-Jacobi expansion

In §7.2 and §7.3 we give a geometric reformulation of the Fourier-Jacobi expansion (7.2). Our starting observation is (compare with §3.5.2):

Lemma 7.1. *The Fourier-Jacobi expansion* (7.2) *gives the Taylor expansion of the $V(I)_{\lambda,k}$-valued holomorphic function f on $\overline{\mathcal{X}(J)}$ along the boundary divisor Δ_J with respect to the normal parameter ω_J, where ϕ_m is the m-th Taylor coefficient as a $V(I)_{\lambda,k}$-valued function on Δ_J.*

Proof. Since the function f is invariant under the translation by $U(J)_{\mathbb{Z}} \subset U(I)_{\mathbb{Z}}$, it descends to a function on $\mathcal{X}(J) \simeq \mathcal{D}_I/U(J)_{\mathbb{Z}}$. Since $(l_{J,\Gamma}, v_{J,\Gamma}) = 1$ for the positive generator $v_{J,\Gamma}$ of $U(J)_{\mathbb{Z}}$, the function $\omega_J = e((l_{J,\Gamma}, Z))$ descends to a function on $\mathcal{X}(J)$ and extends holomorphically over $\overline{\mathcal{X}(J)}$, with the boundary divisor Δ_J defined by $\omega_J = 0$. In particular, ω_J generates the ideal sheaf of Δ_J. On the other hand, as explained above, the Fourier-Jacobi coefficient ϕ_m is the pullback of a $V(I)_{\lambda,k}$-valued function on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$ (again denoted by ϕ_m). Thus $f = \sum_m (\pi_1^* \phi_m)\omega_J^m$ gives the Taylor expansion of f along Δ_J with respect to the normal parameter ω_J, in which the $V(I)_{\lambda,k}$-valued function ϕ_m on Δ_J is the m-th Taylor coefficient. □

Recall from §5.3 that $\overline{\mathcal{X}(J)}$ is an open set of the relative torus embedding $\overline{\mathcal{T}(J)} = \mathcal{T}(J) \times_{T(J)} \overline{T(J)}$ which has the structure of a line bundle on Δ_J. Since $\mathcal{D}(J) \subset Q(I) \simeq U(I)_{\mathbb{C}}$, the function ω_J on $\overline{\mathcal{X}(J)}$ extends over $\overline{\mathcal{T}(J)}$ naturally. It is a linear map on each fiber of $\overline{\mathcal{T}(J)} \to \Delta_J$. Indeed, the fact that ω_J preserves the scalar multiplication follows from the equality

$$e((l_{J,\Gamma}, \alpha v_{J,\Gamma} + Z)) = e(\alpha) \cdot e((l_{J,\Gamma}, Z)), \quad \alpha \in \mathbb{C},$$

and similarly for the sum. The following property will be used in §7.3.

Lemma 7.2. *For each $\gamma \in \overline{\Gamma(J)}_{\mathbb{Z}}$ we have $\gamma^*\omega_J = (\pi_1^* j_\gamma)\cdot\omega_J$ for a nowhere vanishing function j_γ on Δ_J.*

Proof. Since γ acts on $\overline{\mathcal{T}(J)} \to \Delta_J$ as an equivariant action on the line bundle (see §5.3), $\gamma^*\omega_J$ is also linear on each fiber. Therefore $\gamma^*\omega_J/\omega_J$ is the pullback of a function on Δ_J. See also Corollary 7.15 for a computational proof. □

Let us reformulate Lemma 7.1 by passing from vector-valued functions to sections of vector bundles. Let $\mathcal{I} = \mathcal{I}_{\Delta_J}$ be the ideal sheaf of Δ_J and $\Theta_J = \mathcal{I}/\mathcal{I}^2$ be the conormal bundle of Δ_J. As explained above, ω_J generates $\mathcal{I}$ over $\overline{\mathcal{X}(J)}$. In particular, it generates Θ_J over Δ_J. We have

$$\mathcal{I}^m/\mathcal{I}^{m+1} \simeq \Theta_J^{\otimes m} = \mathcal{O}_{\Delta_J}\omega_J^{\otimes m}$$

for every $m \geq 0$. In what follows, we write $\mathcal{E}_{\lambda,k}|_{\Delta_J} \otimes \Theta_J^{\otimes m} = \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ for simplicity. The I-trivialization $\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ and the trivialization of $\Theta_J^{\otimes m}$ by $\omega_J^{\otimes m}$ define an isomorphism $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$. We call it the (I, ω_J)-*trivialization* of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$. Via this isomorphism, we regard the $V(I)_{\lambda,k}$-valued function ϕ_m over Δ_J as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. Specifically, the process is to multiply the function ϕ_m by $\omega_J^{\otimes m}$, and then regard $\phi_m \otimes \omega_J^{\otimes m}$ as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ by the I-trivialization.

Proposition 7.3. *The Taylor expansion of sections of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ along the boundary divisor Δ_J with respect to the normal parameter ω_J and with the pullback $\pi_1^* \colon H^0(\Delta_J, \mathcal{E}_{\lambda,k}|_{\Delta_J}) \hookrightarrow H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k})$ defines an embedding*

$$H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k}) \hookrightarrow \prod_{m\geq 0} H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}), \quad f \mapsto (\phi_m \otimes \omega_J^{\otimes m})_m, \tag{7.4}$$

where ϕ_m are the sections of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ with $f = \sum_m (\pi_1^*\phi_m)\omega_J^m$. If we send a modular form $f \in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k}$ by this map, its image is the Fourier-Jacobi coefficients of f regarded as sections of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ via the (I, ω_J)-trivialization.

Here the pullback $\pi_1^* \colon H^0(\Delta_J, \mathcal{E}_{\lambda,k}|_{\Delta_J}) \to H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k})$ is defined by the isomorphism $\mathcal{E}_{\lambda,k} \simeq \pi_1^*(\mathcal{E}_{\lambda,k}|_{\Delta_J})$ in Proposition 5.6. Via the I-trivialization, this is just the pullback of $V(I)_{\lambda,k}$-valued functions by $\pi_1 \colon \overline{\mathcal{X}(J)} \to \Delta_J$ (see Remark 5.7). The existence of this pullback map is one of key properties in the Fourier-Jacobi expansion.

Proof. The exact sequence of sheaves

$$0 \to \mathcal{I}^{m+1}\mathcal{E}_{\lambda,k} \to \mathcal{I}^m\mathcal{E}_{\lambda,k} \to \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \to 0$$

on $\overline{\mathcal{X}(J)}$ defines the canonical exact sequence
(7.5)
$$0 \to H^0(\overline{\mathcal{X}(J)},\, \mathcal{I}^{m+1}\mathcal{E}_{\lambda,k}) \to H^0(\overline{\mathcal{X}(J)},\, \mathcal{I}^{m}\mathcal{E}_{\lambda,k}) \to H^0(\Delta_J,\, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}).$$

The generator ω_J^m of $\mathcal{I}^m$ and the pullback map

$$\pi_1^* \colon H^0(\Delta_J,\, \mathcal{E}_{\lambda,k}|_{\Delta_J}) \to H^0(\overline{\mathcal{X}(J)},\, \mathcal{E}_{\lambda,k})$$

define the splitting map

$$H^0(\Delta_J,\, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}) \hookrightarrow H^0(\overline{\mathcal{X}(J)},\, \mathcal{I}^m\mathcal{E}_{\lambda,k}), \quad \phi \otimes \omega_J^{\otimes m} \mapsto \omega_J^m \cdot \pi_1^*\phi \tag{7.6}$$

of (7.5). Here $\omega_J^{\otimes m}$ in the source is a section of $\Theta_J^{\otimes m}$ over Δ_J, while ω_J^m in the target is a section of $\mathcal{I}^m$ over $\overline{\mathcal{X}(J)}$. This defines a splitting of the filtration $(H^0(\overline{\mathcal{X}(J)}, \mathcal{I}^m\mathcal{E}_{\lambda,k}))_m$ on $H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k})$ and thus an embedding

$$H^0(\overline{\mathcal{X}(J)}, \mathcal{E}_{\lambda,k}) \hookrightarrow \prod_{m\geq 0} H^0(\Delta_J, \mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}).$$

Explicitly, this is given by writing a section f of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ as $f = \sum_m (\pi_1^*\phi_m)\omega_J^m$ with ϕ_m a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$, and sending f to the collection $(\phi_m \otimes \omega_J^{\otimes m})_m$ of sections.

Since π_1^* is just the ordinary pullback after the I-trivialization, the equation $f = \sum_m (\pi_1^*\phi_m)\omega_J^m$ when f is a modular form coincides with the Fourier-Jacobi expansion (7.2) of f after the I-trivialization. Thus the I-trivialization of ϕ_m is the m-th Fourier-Jacobi coefficient (7.3). It follows that the section $\phi_m \otimes \omega_J^{\otimes m}$ is identified with the Fourier-Jacobi coefficient by the (I, ω_J)-trivialization. □

At first glance, the Taylor expansion (7.4) may seem non-canonical because the lifting map (7.6) uses the special normal parameter ω_J, which as a function on $\overline{\mathcal{X}(J)}$ depends on the choice of $l_{J,\Gamma}, I', I$. In fact, it *is* canonical:

Lemma 7.4. *The map* (7.6)*, and hence the Taylor expansion* (7.4)*, does not depend on the choice of* $l_{J,\Gamma}, I', I$.

Proof. Let $\tilde{\omega}_J$ be the special normal parameter constructed from another such data $(\tilde{I}, \tilde{I}', \tilde{l}_{J,\Gamma})$. Both ω_J and $\tilde{\omega}_J$ extend over $\overline{\mathcal{T}(J)}$ and are linear at each fiber of $\pi_1 \colon \overline{\mathcal{T}(J)} \to \Delta_J$. Therefore we have $\tilde{\omega}_J/\omega_J = \pi_1^*\xi$ for a nowhere vanishing holomorphic function ξ on Δ_J. Then the map (7.6) defined by using $\tilde{\omega}_J$ in place of ω_J sends $\phi \otimes \omega_J^{\otimes m}$ as

$$\phi \otimes \omega_J^{\otimes m} = (\xi^{-m}\phi) \otimes \tilde{\omega}_J^{\otimes m} \;\mapsto\; \tilde{\omega}_J^m \cdot \pi_1^*(\xi^{-m}\phi) = \omega_J^m \cdot \pi_1^*\phi.$$

This coincides with the map using ω_J. □

This in particular implies the following.

Corollary 7.5. *The m-th Fourier-Jacobi coefficient of a modular form, viewed as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J via the (I, ω_J)-trivialization, does not depend on the choice of $l_{J,\Gamma}, I', I$.*

This means that we obtain the same section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ even if we start from the Fourier expansion at another 0-dimensional cusp $\tilde{I} \subset J$.

To summarize, the Fourier-Jacobi expansion of a modular form f as a section of $\mathcal{E}_{\lambda,k}$ is a canonical Taylor expansion along Δ_J which uses but does not depend on the choice of a special normal parameter ω_J. The m-th Fourier-Jacobi coefficient is canonically determined as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. If we take the (I, ω_J)-trivialization, this section is identified with the $V(I)_{\lambda,k}$-valued function (7.3) defined as a slice in the Fourier expansion of f at the I-cusp.

7.3. Vector-valued Jacobi forms

We want to refine Proposition 7.3 by taking the invariant part for the integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$ and imposing cusp condition. This leads us to define vector-valued Jacobi forms in a geometric style. In what follows, we let $m > 0$ and consider the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J, leaving modular forms on $\mathcal{D}$ for a while.

As in §7.1 and §7.2, we choose a rank 1 primitive sublattice I of J, a rank 1 sublattice $I' \subset L$ with $(I, I') \neq 0$, and an isotropic vector $l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ with $(l_{J,\Gamma}, v_{J,\Gamma}) = 1$. ($I$ will be fixed until Definition 7.10, and $I', l_{J,\Gamma}$ will be fixed until Lemma 7.9.) We keep the same notation as in §7.1. Since $U(I)_{\mathbb{Z}} \subset \Gamma(J)_{\mathbb{Z}}$ by (5.8), the group $\overline{\Gamma(J)}_{\mathbb{R}}$ contains $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ as a subgroup. As recalled in §7.1, I' determines an embedding $\Delta_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$. The action of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ on Δ_J is given by the translation on $U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.

We consider the action of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ on the vector bundle $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. The I-trivialization $\mathcal{E}_{\lambda,k}|_{\Delta_J} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ over Δ_J is equivariant with respect to the subgroup $(\Gamma(I)_{\mathbb{R}} \cap \Gamma(J)_{\mathbb{R}})/U(J)_{\mathbb{Z}}$ of $\overline{\Gamma(J)}_{\mathbb{R}}$. In particular, it is equivariant with respect to $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. Since $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ acts trivially on $V(I)_{\lambda,k}$, the factor of automorphy for the I-trivialization of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ is trivial on this group. On the other hand, as for the ω_J-trivialization of Θ_J, we note the following.

Lemma 7.6. *There exists a finite-index sublattice Λ_0 of $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$ such that $\gamma^*\omega_J = \omega_J$ for every $\gamma \in \Lambda_0$. In particular, the factor of automorphy for the (I, ω_J)-trivialization $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\Delta_J}$ of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ is trivial on the group Λ_0.*

Proof. Recall that $v \in U(I)_{\mathbb{R}}$ acts on the tube domain $\mathcal{D}_I \simeq \mathcal{D}$ as the translation by v, say t_v. Then

$$t_v^*\omega_J = e((l_{J,\Gamma}, Z + v)) = e((l_{J,\Gamma}, v)) \cdot \omega_J.$$

Therefore, if we put

$$\Lambda_0 = \{ v \in U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}} \mid (l_{J,\Gamma}, v + U(J)_{\mathbb{Z}}) \subset \mathbb{Z} \}, \tag{7.7}$$

we have $t_v^*\omega_J = \omega_J$ for every $v \in \Lambda_0$. Since $(U(I)_{\mathbb{Z}}, l_{J,\Gamma}) \subset \mathbb{Q}$ and $U(I)_{\mathbb{Z}}$ is finitely generated, we have $(U(I)_{\mathbb{Z}}, l_{J,\Gamma}) \subset N^{-1}\mathbb{Z}$ for some natural number N. This shows that Λ_0 is of finite index in $U(I)_{\mathbb{Z}}/U(J)_{\mathbb{Z}}$. □

Let ϕ be a $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ over Δ_J. By the (I, ω_J)-trivialization of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$, we regard ϕ as a $V(I)_{\lambda,k}$-valued holomorphic function on Δ_J. By Lemma 7.6, the function ϕ is invariant under the translation by the lattice Λ_0. Therefore it admits a Fourier expansion of the form

$$\phi(Z) = \sum_{l \in \Lambda} a(l) q^l, \qquad Z \in \Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}, \tag{7.8}$$

where $a(l) \in V(I)_{\lambda,k}$, $q^l = e((l, Z))$, and Λ is a full lattice in $U(J)_{\mathbb{Q}}^{\perp}$ (which is the dual space of $U(I)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$).

At this point, Λ can be taken to be the dual lattice of Λ_0 defined by (7.7), but we can replace Λ by its arbitrary overlattice (or even the whole $U(J)_{\mathbb{Q}}^{\perp}$) by setting $a(l) = 0$ if $l \notin \Lambda_0^{\vee}$. It is sometimes convenient to enlarge Λ in this way. For this reason, we do not specify the lattice Λ in (7.8).

Remark 7.7. The dual lattice of Λ_0 in $U(J)_{\mathbb{Q}}^{\perp}$ can be explicitly written as

$$\Lambda_0^{\vee} = \langle U(I)_{\mathbb{Z}}^{\vee}, \mathbb{Z}l_{J,\Gamma} \rangle \cap U(J)_{\mathbb{Q}}^{\perp}.$$

We do not use this information.

Replacing Λ by its overlattice, we assume that Λ is of the split form

$$\Lambda = \mathbb{Z}(\beta_1 v_{J,\Gamma}) \oplus K,$$

where $\beta_1 > 0$ is a rational number and K is a full lattice in $l_{J,\Gamma}^{\perp} \cap U(J)_{\mathbb{Q}}^{\perp}$. Note that K is negative-definite. Accordingly, we can rewrite the Fourier expansion of ϕ as

$$\phi(Z) = \sum_{n \in \beta_1\mathbb{Z}} \sum_{l \in K} a(n, l) q^l q_{J,\Gamma}^n, \qquad q_{J,\Gamma} = e((v_{J,\Gamma}, Z)), \tag{7.9}$$

for $Z \in \Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.

Definition 7.8. We say that ϕ is *holomorphic at the I-cusp* of $\mathbb{H}_J$ if $a(n, l) \neq 0$ only when $2nm \geq |(l, l)|$. We say that ϕ *vanishes at the I-cusp* if $a(n, l) \neq 0$ only when $2nm > |(l, l)|$.

The expression (7.9) of the Fourier expansion of ϕ depends on the choice of I', $l_{J,\Gamma}$, Λ. Specifically,

- I' determines the embedding $\Delta_J \hookrightarrow U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$.
- $l_{J,\Gamma}$ determines the normal parameter ω_J which determines the trivialization of $\Theta_J^{\otimes m}$. The vector $l_{J,\Gamma}$ also determines the splitting $U(J)^{\perp}_{\mathbb{Q}} = U(J)_{\mathbb{Q}} \oplus K_{\mathbb{Q}}$ of the index space $U(J)^{\perp}_{\mathbb{Q}}$.
- Λ is the index lattice in the Fourier expansion which is taken to be a split form.

However, we can prove the following.

Lemma 7.9. *Definition 7.8 does not depend on the choice of I', $l_{J,\Gamma}$, Λ.*

Proof. We verify this for the holomorphicity condition. The case of vanishing condition is similar.

(1) If we change I', its effect is the translation on $\Delta_J \subset U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$ by a vector of $U(I)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. This multiplies each Fourier coefficient $a(n,l)$ by a nonzero constant, so its vanishing/nonvanishing does not change.

(2) The condition $2nm \geq |(l,l)|$ is the same as the condition

$$(ml_{J,\Gamma} + v,\ ml_{J,\Gamma} + v) \geq 0 \tag{7.10}$$

for the vector $v = nv_{J,\Gamma} + l$ of $U(J)^{\perp}_{\mathbb{Q}}$ which corresponds to the index (n,l). With $l_{J,\Gamma}$ fixed, this condition does not depend on the lattice Λ.

(3) Finally, if we change $l_{J,\Gamma}$, the new vector can be written as

$$l'_{J,\Gamma} = l_{J,\Gamma} + l_0 - 2^{-1}(l_0,l_0)v_{J,\Gamma}$$

for some vector $l_0 \in K_{\mathbb{Q}}$. Since the normal parameter $\omega_J = e((l_{J,\Gamma}, Z))$ is replaced by

$$\omega'_J = e((l'_{J,\Gamma}, Z)) = q^{l_0} \cdot q_{J,\Gamma}^{-(l_0,l_0)/2} \cdot \omega_J,$$

we have to multiply the function ϕ by $q^{-ml_0} \cdot q_{J,\Gamma}^{m(l_0,l_0)/2}$ when passing from the ω_J-trivialization to the ω'_J-trivialization of $\Theta_J^{\otimes m}$. Also $K_{\mathbb{Q}} = l_{J,\Gamma}^{\perp} \cap U(J)^{\perp}_{\mathbb{Q}}$ is replaced by $K'_{\mathbb{Q}} = (l'_{J,\Gamma})^{\perp} \cap U(J)^{\perp}_{\mathbb{Q}}$, for which we have the natural isometry

$$K_{\mathbb{Q}} \to K'_{\mathbb{Q}}, \qquad l \mapsto l' := l - (l,l_0)v_{J,\Gamma}.$$

Therefore the new Fourier expansion is

$$\begin{aligned} \phi' &:= \phi \cdot q^{-ml_0} \cdot q_{J,\Gamma}^{m(l_0,l_0)/2} \\ &= \sum_{n\in\mathbb{Q}} \sum_{l \in K_{\mathbb{Q}}} a(n,l) q^{l-ml_0} q_{J,\Gamma}^{n+m(l_0,l_0)/2} \\ &= \sum_{n\in\mathbb{Q}} \sum_{l \in K_{\mathbb{Q}}} a(n,l) q^{l'-ml'_0} q_{J,\Gamma}^{n+(l,l_0)-m(l_0,l_0)/2}. \end{aligned}$$

In the last equality we used

$$l - ml_0 = (l-ml_0)' + (l-ml_0, l_0)v_{J,\Gamma}.$$

This means that $a(n,l)$ is equal to the Fourier coefficient of ϕ' of index

$$(n+(l,l_0)-m(l_0,l_0)/2,\ l'-ml_0') \quad \in \mathbb{Q}\oplus K'_{\mathbb{Q}}.$$

The holomorphicity condition $2nm \geq -(l,l)$ for ϕ can be rewritten as

$$2m(n+(l,l_0)-m(l_0,l_0)/2) \geq -(l'-ml_0', l'-ml_0').$$

This is the holomorphicity condition for ϕ'. □

Lemma 7.9 ensures that Definition 7.8 is well-defined for a $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant section of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$.

Definition 7.10. We denote by

$$J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) \subset H^0(\Delta_J, \mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m})$$

the space of $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant sections ϕ of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ over Δ_J which are holomorphic at every cusp $I\subset J$ of $\mathbb{H}_J$ in the sense of Definition 7.8. We call such a section ϕ a *Jacobi form* of weight (λ,k) and index m for the integral Jacobi group $\Gamma(J)_{\mathbb{Z}}$. We call ϕ a *Jacobi cusp form* if it vanishes at every cusp $I\subset J$. When $\lambda=0$, we especially write $J_{0,k,m}(\Gamma(J)_{\mathbb{Z}}) = J_{k,m}(\Gamma(J)_{\mathbb{Z}})$.

For later use (§7.4), we note the following.

Lemma 7.11. *Let γ be an element of $\Gamma(J)_{\mathbb{Q}}$ which stabilizes J. A $\overline{\Gamma(J)_{\mathbb{Z}}}$-invariant section ϕ of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ over Δ_J is holomorphic at the $\gamma(I)$-cusp of $\mathbb{H}_J$ if and only if the $\gamma^{-1}\overline{\Gamma(J)_{\mathbb{Z}}}\gamma$-invariant section $\gamma^*\phi$ of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ is holomorphic at the I-cusp of $\mathbb{H}_J$.*

Proof. This holds because the pullback of a Fourier expansion of ϕ at the $\gamma(I)$-cusp by the γ-action

$$\gamma : U(I)_{\mathbb{C}}/U(J)_{\mathbb{C}} \to U(\gamma I)_{\mathbb{C}}/U(J)_{\mathbb{C}}$$

and the isomorphism $\gamma\colon V(I)_{\lambda,k} \to V(\gamma I)_{\lambda,k}$ gives a Fourier expansion of $\gamma^*\phi$ at the I-cusp. □

Now we go back to modular forms on $\mathcal{D}$ and refine Proposition 7.3 for $M_{\lambda,k}(\Gamma)$. Recall that the m-th Fourier-Jacobi coefficient of a modular form was initially defined as a $V(I)_{\lambda,k}$-valued function on Δ_J by (7.3), and then regarded as a section of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ by the (I,ω_J)-trivialization. By Corollary 7.5, this section is independent of I.

Proposition 7.12. *For $m>0$ the m-th Fourier-Jacobi coefficient of a modular form $f\in M_{\lambda,k}(\Gamma)$ as a section of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ is a Jacobi form of weight (λ,k) and index m in the sense of Definition 7.10. When f is a cusp form, the Fourier-Jacobi coefficient is a Jacobi cusp form.*

Proof. In what follows, $\tilde{\phi}_m$ stands for the m-th Fourier-Jacobi coefficient of f as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. What has to be shown is that $\tilde{\phi}_m$ is $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant and is holomorphic at every cusp of $\mathbb{H}_J$. We first check the cusp condition. Let $I \subset J$ be an arbitrary cusp (not necessarily the initial one). Corollary 7.5 ensures that the Fourier expansion of $\tilde{\phi}_m$ at the I-cusp of $\mathbb{H}_J$ is given by the series (7.3) obtained from the Fourier expansion of f at the I-cusp of $\mathcal{D}$. Then the holomorphicity condition for $\tilde{\phi}_m$ at I, written in the form (7.10), follows from the cusp condition in the Fourier expansion of f at I. The assertion for cusp forms follows similarly.

It remains to check the $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariance of $\tilde{\phi}_m$. Let $\phi_m = \tilde{\phi}_m \otimes \omega_J^{\otimes -m}$. This is a section of $\mathcal{E}_{\lambda,k}|_{\Delta_J}$ whose I-trivialization is the (I, ω_J)-trivialized form (7.3) of $\tilde{\phi}_m$. By Proposition 7.3, we have the expansion

$$f = \sum_m (\pi_1^* \phi_m) \omega_J^m \tag{7.11}$$

as a section of $\mathcal{E}_{\lambda,k}$, where we view ω_J as a generator of the ideal sheaf $\mathcal{I}$ of Δ_J. We let $\gamma \in \overline{\Gamma(J)}_{\mathbb{Z}}$ act on this equality. Then we have

$$\gamma^* f = \sum_m \gamma^*(\pi_1^* \phi_m)(\gamma^* \omega_J)^m = \sum_m \pi_1^*(\gamma^* \phi_m)(\gamma^* \omega_J)^m \tag{7.12}$$

by Proposition 5.6. By Lemma 7.2, we have $\gamma^* \omega_J = (\pi_1^* j_\gamma) \cdot \omega_J$ for a holomorphic function j_γ on Δ_J. Therefore we have

$$\gamma^* f = \sum_m \pi_1^*(j_\gamma^m \cdot \gamma^* \phi_m) \omega_J^m. \tag{7.13}$$

Since f is Γ-invariant, we have $\gamma^* f = f$. Comparing (7.11) and (7.13), we obtain $\phi_m = j_\gamma^m \cdot \gamma^* \phi_m$ for every m. This means that $\tilde{\phi}_m = \phi_m \otimes \omega_J^{\otimes m}$ is γ-invariant. This proves Proposition 7.12. □

For the sake of completeness, for $m = 0$ with $\lambda \neq \det$, let us denote by $J_{\lambda,k,0}(\Gamma(J)_{\mathbb{Z}})$ the space of $\overline{\Gamma(J)}_{\mathbb{Z}}$-invariant sections of $\mathcal{E}_\lambda^J \otimes \mathcal{L}^{\otimes k}|_{\Delta_J} \simeq \pi_2^* \mathcal{L}_J^{\otimes k + \lambda_1} \otimes V(J)_{\lambda'}$ over Δ_J which is holomorphic at every cusp of $\mathbb{H}_J$. By the result of §6, the 0-th Fourier-Jacobi coefficient $\phi_0 = f|_{\Delta_J}$ of a modular form $f \in M_{\lambda,k}(\Gamma)$ belongs to this space (cuspidal when $\lambda \neq 1$). Then, as a refinement of Proposition 7.3 for $M_{\lambda,k}(\Gamma)$, we see that the Fourier-Jacobi expansion gives the embedding

$$M_{\lambda,k}(\Gamma) \hookrightarrow \prod_{m \geq 0} J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}), \quad f = \sum_m (\pi_1^* \phi_m) \omega_J^m \mapsto (\phi_m \otimes \omega_J^m)_m,$$

which is canonically determined by J.

7.4. Classical Jacobi forms

In this section we introduce coordinates and translate Jacobi forms with $\lambda = 0$ in the sense of Definition 7.10 to classical scalar-valued Jacobi forms à la [**21**] and [**44**]. The result is stated in Proposition 7.18. Our purpose is to deduce a vanishing theorem in the present setting (Proposition 7.19) from the one for classical Jacobi forms.

7.4.1. Coordinates. We begin by setting some notations. In $U(J)_{\mathbb{Q}} \simeq \wedge^2 J_{\mathbb{Q}}$ we have two natural lattices: $\wedge^2 J$ and $U(J)_{\mathbb{Z}}$. The former depends on L, and the latter depends on Γ. Recall that the positive generator of $U(J)_{\mathbb{Z}}$ is denoted by $v_{J,\Gamma}$ (§7.1), and the positive generator of $\wedge^2 J$ is denoted by v_J (§5.1.2). Then $v_J = \beta_0 v_{J,\Gamma}$ for some rational number $\beta_0 > 0$. This constant β_0 depends only on L and Γ. We choose an isotropic plane in $L_{\mathbb{Q}}$ whose pairing with $J_{\mathbb{Q}}$ is nondegenerate, and denote it by $J_{\mathbb{Q}}^{\vee}$ for the obvious reason. This is fixed throughout §7.4. We identify $V(J)_{\mathbb{Q}} = (J^{\perp}/J)_{\mathbb{Q}}$ with the subspace $(J_{\mathbb{Q}} \oplus J_{\mathbb{Q}}^{\vee})^{\perp}$ of $L_{\mathbb{Q}}$.

Next we choose a rank 1 primitive sublattice I of J. Let e_1, f_1, e_2, f_2 be the standard hyperbolic basis of $2U$. We take an embedding $2U_{\mathbb{Q}} \hookrightarrow L_{\mathbb{Q}}$ which sends

$$\mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \to J, \quad \mathbb{Z}e_1 \to I, \quad \mathbb{Q}f_1 \oplus \mathbb{Q}f_2 \to J_{\mathbb{Q}}^{\vee}$$

isomorphically. Thus it is compatible with $I \subset J$ in the sense of §5. We identify e_1, f_1, e_2, f_2 with their image in $L_{\mathbb{Q}}$. Then $v_J = e_2 \otimes e_1$. We define vectors $l_J, l_{J,\Gamma} \in U(I)_{\mathbb{Q}}$ (as in §5.1.2 and §7.1) by $l_J = f_2 \otimes e_1$ and $l_{J,\Gamma} = \beta_0 l_J$. We also put $I' = \mathbb{Z}f_1$. The choice of these data has two effects: it introduces coordinates on $\mathcal{D}$ and on the Jacobi group.

The coordinates on $\mathcal{D}$ are introduced following §5.1.2. The choice of $I' = \mathbb{Z}f_1$ defines the tube domain realization $\mathcal{D} \to \mathcal{D}_I \subset U(I)_{\mathbb{C}}$. According to the decomposition

$$U(I)_{\mathbb{C}} = (U_{\mathbb{C}} \oplus V(J)_{\mathbb{C}}) \otimes I_{\mathbb{C}} = \mathbb{C}l_J \times (V(J) \otimes \mathbb{C}e_1) \times \mathbb{C}v_J,$$

we express a point of $U(I)_{\mathbb{C}}$ as

$$Z = \tau l_J + z \otimes e_1 + w v_J = (\tau, z, w), \qquad \tau, w \in \mathbb{C},\ z \in V(J)_{\mathbb{C}}.$$

These are the same coordinates as in (5.5) except that z in (5.5) is $z \otimes e_1$ here. When $Z \in \mathcal{D}_I$, the corresponding point of $\mathcal{D}$ is $\mathbb{C}\omega(Z)$ where

$$\omega(Z) = f_1 + \tau f_2 + z + w e_2 - ((z,z)/2 + \tau w) e_1 \quad \in L_{\mathbb{C}}. \tag{7.14}$$

Note that this vector is normalized so as to have pairing 1 with e_1. In this coordinates, the Siegel domain realization $\mathcal{D} \to \mathcal{V}_J \to \mathbb{H}_J$ with respect to J is the restriction of the projection

$$\mathbb{C}l_J \times V(J) \times \mathbb{C}v_J \to \mathbb{C}l_J \times V(J) \to \mathbb{C}l_J, \qquad (\tau, z, w) \mapsto (\tau, z) \mapsto \tau$$

to the tube domain $\mathcal{D}_I$. The coordinates introduced on $\mathbb{H}_J \subset \mathbb{P}(L/J^\perp)_{\mathbb{C}}$ and $\mathcal{V}_J \subset \mathbb{P}(L/J)_{\mathbb{C}}$ are written as

$$\mathbb{H} \xrightarrow{\simeq} \mathbb{H}_J, \quad \tau \mapsto \tau l_J = \mathbb{C}(f_1 + \tau f_2), \tag{7.15}$$

$$\mathbb{H} \times V(J) \xrightarrow{\simeq} \mathcal{V}_J, \quad (\tau, z) \mapsto \tau l_J + z \otimes e_1 = \mathbb{C}(f_1 + \tau f_2 + z). \tag{7.16}$$

Note that the isomorphism (7.15) maps the cusps $\mathbb{P}^1_{\mathbb{Q}} = \{i\infty\} \cup \mathbb{Q}$ of $\mathbb{H} \subset \mathbb{P}^1$ to the cusps $\mathbb{P}J^\vee_{\mathbb{Q}}$ of $\mathbb{H}_J \subset \mathbb{P}J^\vee_{\mathbb{C}}$, and especially maps the cusp $i\infty$ to the I-cusp $I^\perp \cap \mathbb{P}J^\vee_{\mathbb{C}}$ of $\mathbb{H}_J$.

Next we consider the Jacobi group $\Gamma(J)_F$, $F = \mathbb{Q}, \mathbb{R}$. Recall from (5.7) that the splitting $L_F = (J_F \oplus J_F^\vee) \oplus V(J)_F$ defines an isomorphism

$$\Gamma(J)_F \simeq \mathrm{SL}(J_F) \ltimes W(J)_F, \tag{7.17}$$

which we fix below. (This splitting depends on $J_F^\vee$, but not on I.) We identify

$$\mathrm{SL}(J_F) = \mathrm{SL}(J_F^\vee) = \mathrm{SL}(2, F)$$

by the basis f_2, f_1 of $J_F^\vee$, or equivalently, by the basis $e_1, -e_2$ of J_F. Thus an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, F)$ acts on $J_F \oplus J_F^\vee$ by

$$e_1 \mapsto ae_1 - ce_2, \quad e_2 \mapsto -be_1 + de_2, \quad f_1 \mapsto df_1 + bf_2, \quad f_2 \mapsto cf_1 + af_2.$$

Finally, we have a splitting of the Heisenberg group $W(J)_F$ *as a set*:

$$\begin{aligned} W(J)_F &\simeq U(J)_F \times (V(J)_F \otimes Fe_1) \times (V(J)_F \otimes Fe_2) \\ &\simeq F \times V(J)_F \times V(J)_F, \end{aligned} \tag{7.18}$$

where we take v_J as the basis of $U(J)_F$. Accordingly, we write an element of $W(J)_F$ as (α, v_1, v_2) where $\alpha \in F$ and $v_1, v_2 \in V(J)_F \subset L_F$. In this expression, $(\alpha, 0, 0) = \alpha v_J$ corresponds to $E_{\alpha e_2 \wedge e_1} \in U(J)_F$, $(0, v_1, 0)$ to $E_{v_1 \otimes e_1}$, and $(0, 0, v_2)$ to $E_{v_2 \otimes e_2}$. Note that each $V(J)_F \otimes Fe_1$ and $V(J)_F \otimes Fe_2$ are respectively subgroups of $W(J)_F$, but they do not commute.

Proposition 7.13. *The action of* $\Gamma(J)_F$ *on* $\mathcal{D}$ *is described as follows.*
(1) $(\alpha, 0, 0) \in U(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z, w + \alpha).$$

(2) $(0, v_1, 0) \in W(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z + v_1, w).$$

(3) $(0, 0, v_2) \in W(J)_F$ *acts by*

$$(\tau, z, w) \mapsto (\tau, z + \tau v_2, w - (v_2, z) - 2^{-1}(v_2, v_2)\tau).$$

(4) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, F)$ *acts by*

$$(\tau,\, z,\, w) \mapsto \left(\frac{a\tau + b}{c\tau + b},\; \frac{z}{c\tau + d},\; w + \frac{c(z,z)}{2(c\tau + d)} \right).$$

Proof. Let $\omega(Z) \in L_{\mathbb{C}}$ be as in (7.14). By direct calculation using the definition (1.4) of Eichler transvections, we see that

$$\begin{aligned} E_{\alpha e_2 \wedge e_1}(\omega(Z)) &= f_1 + \tau f_2 + z + (w + \alpha) e_2 + A e_1 \\ &= \omega(Z + (0, 0, \alpha)), \end{aligned}$$

$$\begin{aligned} E_{v_1 \otimes e_1}(\omega(Z)) &= f_1 + \tau f_2 + (z + v_1) + w e_2 + A e_1 \\ &= \omega(Z + (0, v_1, 0)), \end{aligned}$$

$$\begin{aligned} E_{v_2 \otimes e_2}(\omega(Z)) &= f_1 + \tau f_2 + (z + \tau v_2) + (w - (z, v_2) - (\tau/2)(v_2, v_2)) e_2 + A e_1 \\ &= \omega(Z + (0, \tau v_2, -(z, v_2) - (\tau/2)(v_2, v_2))), \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\omega(Z)) = (c\tau + d) f_1 + (a\tau + b) f_2 + z + ((c\tau + d) w + (c/2)(z, z)) e_2 + A e_1.$$

Here the constant A in each equation is an unspecified constant determined by the isotropicity condition. This proves (1) – (4). □

Proposition 7.13 agrees with the classical description of the action of Jacobi group in [**21**] p.1185. (α, v_1, v_2 correspond to r, y, x in [**21**] respectively.) We note two consequences of the calculation in Proposition 7.13.

Corollary 7.14. *Let* $\gamma \in \Gamma(J)_{\mathbb{R}}$ *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be its image in* $\mathrm{SL}(2, \mathbb{R})$. *The factor of automorphy of the* γ*-action on* $\mathcal{L}$ *with respect to the* I*-trivialization* $\mathcal{L} \simeq I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{D}}$ *is* $c\tau + d$.

Proof. In view of (2.3), this follows by looking at the coefficients of f_1 in the equations in the proof of Proposition 7.13. □

This gives a computational explanation of the $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\mathcal{L} \simeq \pi^* \mathcal{L}_J$ in Lemma 5.9. We also provide a computational proof of Lemma 7.2.

Corollary 7.15 (cf. Lemma 7.2). *Let* $\gamma \in \Gamma(J)_{\mathbb{R}}$ *and* $\omega_J = e((l_{J,\Gamma}, Z))$ *be as in §7.1. Then* $\gamma^* \omega_J = j_\gamma(\tau, z) \omega_J$ *for a function* $j_\gamma(\tau, z)$ *of* (τ, z) *which does not depend on the* w*-component.*

Proof. Since $l_{J,\Gamma} = \beta_0 l_J$, if we express $Z = (\tau, z, w)$, we have

$$\omega_J = e((l_{J,\Gamma}, Z)) = e((\beta_0 l_J, w v_J)) = e(\beta_0 w).$$

Therefore, if we denote by $\gamma^* w$ the w-component of $\gamma(Z)$, we have

$$(\gamma^*\omega_J)/\omega_J = e(\beta_0(\gamma^* w - w)). \tag{7.19}$$

It remains to observe from Proposition 7.13 that $\gamma^* w - w$ depends only on (τ, z). □

The function $j_\gamma(\tau, z)$ is the inverse of the factor of automorphy of the γ-action (= pullback by γ^{-1}) on the conormal bundle Θ_J of Δ_J with respect to the ω_J-trivialization. Thus $j_\gamma(\tau, z)$ is the multiplier in the slash operator by γ on Θ_J with respect to the ω_J-trivialization. By (7.19), $j_\gamma(\tau, z)$ is explicitly written as follows.

$$j_\gamma(\tau, z) = \begin{cases} e(\beta_0\alpha) & \gamma = (\alpha, 0, 0) \\ 1 & \gamma = (0, v_1, 0) \\ e(-\beta_0(v_2, z) - 2^{-1}\beta_0(v_2, v_2)\tau) & \gamma = (0, 0, v_2) \\ e\left(\frac{\beta_0 c(z,z)}{2(c\tau+d)}\right) & \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{cases} \tag{7.20}$$

If we divide $\Gamma(J)_{\mathbb{R}}$ by $U(J)_{\mathbb{R}}$, these coincide with the multipliers in the slash operator in [**44**] p.248 with $k = 0$ and the quadratic space $V(J)_{\mathbb{Q}}(-\beta_0)$. (We identify the half-integral matrix F in [**44**] with the even lattice with Gram matrix $2F$, and this lattice tensored with $\mathbb{Q}$ corresponds to our $V(J)_{\mathbb{Q}}(-\beta_0)$.)

7.4.2. Translation to classical Jacobi forms. Now, using the coordinates prepared in §7.4.1, we describe Jacobi forms with $\lambda = 0$ in a more classical manner. We identify $\Delta_J \simeq \mathbb{H} \times V(J)$ by (7.16) and accordingly use the coordinates (τ, z) on Δ_J. We put $q_J = e(\tau) = e((v_J, Z))$ and $q_{J,\Gamma} = e((v_{J,\Gamma}, Z))$ (as in (7.9)) for $Z = (\tau, z) \in \Delta_J$. Since $v_{J,\Gamma} = \beta_0^{-1} v_J$, then $q_{J,\Gamma} = e(\beta_0^{-1}\tau) = (q_J)^{\beta_0^{-1}}$. We also write $\beta_2 = \beta_0^{-1}\beta_1$.

Let $\phi \in J_{k,m}(\Gamma(J)_{\mathbb{Z}})$ be a Jacobi form of weight $(0, k)$ and index m in the sense of Definition 7.10. Via the (I, ω_J)-trivialization and the basis e_1 of I,

$$\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m} \simeq (I_{\mathbb{C}}^{\vee})^{\otimes k} \otimes \mathcal{O}_{\Delta_J} \simeq \mathcal{O}_{\Delta_J},$$

we regard ϕ as a scalar-valued function on Δ_J. Let $V(J)(\beta_0 m)$ be the scaling of the quadratic space $V(J)$ by $\beta_0 m$.

Lemma 7.16. *We identify $V(J) = V(J)(\beta_0 m)$ as a $\mathbb{C}$-linear space naturally and regard ϕ as a function on $\Delta_J \simeq \mathbb{H} \times V(J)(\beta_0 m)$. Then ϕ has a Fourier expansion of the form*

$$\phi(\tau, z) = \sum_{n \in \beta_2\mathbb{Z}} \sum_{l \in K_J(\beta_0 m)^{\vee}} a(n, l)\, q^l\, q_J^n, \qquad \tau \in \mathbb{H},\ z \in V(J)(\beta_0 m).$$

Here $q^l = e((l,z))$ with (l,z) being the pairing in $V(J)(\beta_0 m)$, and K_I is some full lattice in $V(J)_{\mathbb{Q}}$ such that $K_I(\beta_0)$ is an even lattice. The holomorphicity condition at the I-cusp is $2n \geq |(l,l)|$.

Proof. Recall from (7.9) that ϕ as a function on $\mathbb{H} \times V(J)$ has a Fourier expansion of the form

$$\phi(\tau, z) = \sum_{n \in \beta_1 \mathbb{Z}} \sum_{l \in K'} a(n,l)\, q^l\, q_{J,\Gamma}^n, \qquad \tau \in \mathbb{H},\ z \in V(J),$$

where K' is some full lattice in $V(J)_{\mathbb{Q}}$ and $q^l = e((l,z))$. (The vectors l in (7.9) are $l \otimes e_1$ here.) The I-cusp condition is $2nm \geq |(l,l)|$. We substitute $q_{J,\Gamma} = (q_J)^{\beta_0^{-1}}$ and rewrite $\beta_0^{-1} n$ as n. Then this expression is rewritten as

$$\phi(\tau, z) = \sum_{n \in \beta_2 \mathbb{Z}} \sum_{l \in K'} a(n,l)\, q^l\, q_J^n, \qquad \tau \in \mathbb{H},\ z \in V(J),$$

with the I-cusp condition being $2n\beta_0 m \geq |(l,l)|$. By enlarging K', we may assume that $K' = K_I^\vee$ for a lattice $K_I \subset V(J)_{\mathbb{Q}}$ such that $K_I(\beta_0)$ is even.

Next we identify $V(J) = V(J)(\beta_0 m)$ as a $\mathbb{C}$-linear space, which multiplies the quadratic form by $\beta_0 m$. This identification maps the lattice $K_I^\vee \subset V(J)$ to the lattice $\beta_0 m K_I(\beta_0 m)^\vee \subset V(J)(\beta_0 m)$. Then, by multiplying the index lattice $K_I^\vee$ by $(\beta_0 m)^{-1}$ and identifying it with $K_I(\beta_0 m)^\vee$ by this scaling, the Fourier expansion of ϕ as a function on $\mathbb{H} \times V(J)(m\beta_0)$ is written as

$$\phi(\tau, z) = \sum_{n \in \beta_2 \mathbb{Z}} \sum_{l \in K_I(\beta_0 m)^\vee} a(n,l)\, q^l\, q_J^n, \qquad \tau \in \mathbb{H},\ z \in V(J)(\beta_0 m),$$

where (l,z) in $q^l = e((l,z))$ is the pairing in $V(J)(\beta_0 m)$. The I-cusp condition is then rewritten as $2n \geq |(l,l)|$ for $l \in K_I(\beta_0 m)^\vee$. □

Here we passed from $q_{J,\Gamma}$ to q_J because the latter does not depend on Γ, and passed from $V(J)$ to $V(J)(\beta_0 m)$ in order to match our holomorphicity condition at the I-cusp to the holomorphicity condition at $i\infty$ of Skoruppa [**44**] p.249.

Next we shrink the integral Jacobi group $\overline{\Gamma(J)_{\mathbb{Z}}}$ to a subgroup of simpler form. We let $\Gamma_J \subset \mathrm{SL}(J)$ be the intersection of $\Gamma(J)_{\mathbb{Z}}$ with the lifted group $\mathrm{SL}(J_{\mathbb{Q}}) \subset \Gamma(J)_{\mathbb{Q}}$. (This is different from the notation in §6.3 in general.) Note that Γ_J does not depend on I (but on $J_{\mathbb{Q}}^\vee \subset L_{\mathbb{Q}}$). The splitting (7.17) defines an isomorphism

$$\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}} \simeq \mathrm{SL}(J_{\mathbb{Q}}) \ltimes (V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}),$$

where $\mathrm{SL}(J_{\mathbb{Q}})$ acts on $V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}$ by its natural action on $J_{\mathbb{Q}}$. We fix this splitting of $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. The inclusion $\Gamma(J)_{\mathbb{Z}} \subset \Gamma(J)_{\mathbb{Q}}$ defines a canonical injective map $\overline{\Gamma(J)_{\mathbb{Z}}} \to \Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$. Its image is not necessarily a semi-product. Elements in the intersection $\overline{\Gamma(J)_{\mathbb{Z}}} \cap (V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i)$ are images of elements of $\Gamma(J)_{\mathbb{Z}}$ of the form $E_{\alpha e_1 \wedge e_2} \circ E_{v \otimes e_i}$, $v \in V(J)_{\mathbb{Q}}$, but $\alpha e_1 \wedge e_2 \in U(J)_{\mathbb{Q}}$ is not necessarily contained in $U(J)_{\mathbb{Z}}$ in general. We remedy these two subtle problems by passing to a subgroup of $\overline{\Gamma(J)_{\mathbb{Z}}}$ as follows.

Lemma 7.17. *There exists a full lattice K_I' in $V(J)_{\mathbb{Q}}$ such that*

$$\Gamma_J \ltimes (K_I' \otimes_{\mathbb{Z}} J) \ \subset \ \overline{\Gamma(J)_{\mathbb{Z}}} \tag{7.21}$$

as subgroups of $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$, and for each $i = 1, 2$, the subgroup $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_i$ of this semi-product is contained in the image of $W(J)_{\mathbb{Z}} \cap (V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i)$ in $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$, where $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ is the component of $W(J)_{\mathbb{Q}}$ in (7.18).

Proof. The intersection of $W(J)_{\mathbb{Z}}$ with the component $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ in (7.18) is a full lattice in $V(J)_{\mathbb{Q}} \otimes \mathbb{Q}e_i$ and hence can be written as $K_i \otimes_{\mathbb{Z}} \mathbb{Z}e_i$ for some full lattice K_i in $V(J)_{\mathbb{Q}}$. We put $K_I' = K_1 \cap K_2$. Then the second property holds by construction. Since $J = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2$, it follows that

$$K_I' \otimes_{\mathbb{Z}} J \ \subset \ \overline{\Gamma(J)_{\mathbb{Z}}} \cap (V(J)_{\mathbb{Q}} \otimes J_{\mathbb{Q}}).$$

Since we also have $\Gamma_J \subset \overline{\Gamma(J)_{\mathbb{Z}}} \cap \mathrm{SL}(J_{\mathbb{Q}})$ by construction, the inclusion (7.21) is verified. □

The second property in Lemma 7.17 means that $E_{v\otimes e_1}, E_{v\otimes e_2} \in W(J)_{\mathbb{Z}}$ for $v \in K_I'$, and their images in $\Gamma(J)_{\mathbb{Q}}/U(J)_{\mathbb{Q}}$ form the subgroups $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_1$, $K_I' \otimes_{\mathbb{Z}} \mathbb{Z}e_2$ in (7.21) respectively. Their factors of automorphy on Θ_J are given by the second and the third line in (7.20) respectively. This is why we require the second property in Lemma 7.17.

We can now state the translation of Jacobi forms in a precise form. For an even negative-definite lattice K', let $J_{k,K'}(\Gamma_J)$ be the space of Jacobi forms of weight k and index lattice $K'(-1)$ for the group $\Gamma_J < \mathrm{SL}(J) \simeq \mathrm{SL}(2,\mathbb{Z})$ in the sense of Skoruppa [**44**] p.249. (In the notation of [**44**], $K'(-1)$ is the positive-definite even lattice with Gram matrix $2F$, and corresponds to the $\mathbb{Z}^n$ in the Heisenberg group in [**44**] p.248. The dual lattice of $K'(-1)$ corresponds to the index $\mathbb{Z}^n$ in the Fourier expansion in [**44**] p.249.)

Proposition 7.18. *There exists a full lattice K in $V(J)_{\mathbb{Q}}$ such that $K(\beta_0)$ is an even lattice and we have an embedding*

$$J_{k,m}(\Gamma(J)_{\mathbb{Z}}) \hookrightarrow J_{k,K(\beta_0 m)}(\Gamma_J)$$

for every $m > 0$ and $k \in \mathbb{Z}$.

Proof. The correspondence is summarized as follows:

(1) Start from a section ϕ of $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$ over Δ_J.

(2) Choose a rank 1 primitive sublattice $I \subset J$ and identify ϕ with a holomorphic function on Δ_J by the (I, ω_J)-trivialization of $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$.
(3) Identify $\Delta_J \simeq \mathbb{H} \times V(J)(\beta_0 m)$ by the coordinates in §7.4.1 and the scaling $V(J) \simeq V(J)(\beta_0 m)$.
(4) In this way ϕ is identified with a holomorphic function on $\mathbb{H} \times V(J)(\beta_0 m)$.

We shall show that this defines a well-defined map from $J_{k,m}(\Gamma(J)_{\mathbb{Z}})$ to $J_{k,K(\beta_0 m)}(\Gamma_J)$ for a suitable lattice $K \subset V(J)_{\mathbb{Q}}$.

We replace K_I in Lemma 7.16 and K'_I in Lemma 7.17 by their intersection $K_I \cap K'_I$ and rewrite it as K_I. Then Lemma 7.16 says that our Jacobi form ϕ viewed as a function on $\mathbb{H} \times V(J)(\beta_0 m)$ by the above procedure has the same shape of Fourier expansion as that of Jacobi forms of weight k and index lattice $K_I(\beta_0 m)$ at $i\infty$ in the sense of [**44**] p.249. Our I-cusp condition $2n \geq |(l, l)|$ agrees with the holomorphicity condition at $i\infty$ in [**44**]. By Corollary 7.14 and (7.20), we see that the factor of automorphy for the action of $\Gamma_J \ltimes (K_I \otimes J)$ on $\mathcal{L}^{\otimes k} \otimes \Theta_J^{\otimes m}$ with respect to the (I, ω_J)-trivialization agrees with the factor of automorphy for the slash operator $|_{k,V(J)(\beta_0 m)}$ in [**44**] p.248. Therefore the function ϕ satisfies the transformation rule of [**44**] p.249 (Definition (i)) for the group $\Gamma_J < \mathrm{SL}(J)$ with weight k and index lattice $K_I(\beta_0 m)$. In particular, the function ϕ is also holomorphic (in the sense of [**44**]) at the cusps equivalent to I under Γ_J.

It remains to cover all cusps. The coincidence of the automorphy factors on $\mathrm{SL}(J_{\mathbb{R}})$ implies that the function $\phi|_{k,V(J)(\beta_0 m)}\gamma$ for $\gamma \in \mathrm{SL}(J)$ is identified with the section $\gamma^*\phi$ via the (I, ω_J)-trivialization. Then we have

the section ϕ is holomorphic at the γI-cusp in our sense
$\Leftrightarrow$ the section $\gamma^*\phi$ is holomorphic at the I-cusp in our sense
$\Leftrightarrow$ the function $\phi|_{k,V(J)(\beta_0 m)}\gamma$ is holomorphic at $i\infty$ in the sense of [**44**].

The first equivalence follows from Lemma 7.11, and the second equivalence follows by applying the argument so far to the Jacobi form $\gamma^*\phi$ for $\gamma^{-1}\Gamma(J)_{\mathbb{Z}}\gamma \subset \Gamma(J)_{\mathbb{Q}}$ (with $J = \gamma(J)$ and $U(J)_{\mathbb{Z}}$ unchanged). Here the index lattice for $\gamma^*\phi$ is determined from the Fourier expansion of $\gamma^*\phi$ at the I-cusp with the group $\gamma^{-1}\Gamma(J)_{\mathbb{Z}}\gamma$, by the procedure in Lemma 7.16. We denote it by $K_{\gamma I}(\beta_0 m)$, with $K_{\gamma I}$ a full lattice in $V(J)_{\mathbb{Q}}$. This may be in general different from K_I.

Then we take representatives $I_1 = I, I_2, \cdots, I_N$ of Γ_J-equivalence classes of rank 1 primitive sublattices of J and put

$$K = \bigcap_i K_{I_i} \quad \subset V(J)_{\mathbb{Q}}.$$

As a function on $\mathbb{H}\times V(J)(\beta_0 m)$, ϕ satisfies the transformation rule of Jacobi forms of weight k and index lattice $K(\beta_0 m)$ for $\Gamma_J < \mathrm{SL}(J)$, and is holomorphic at the cusps $I_1 = i\infty, I_2, \cdots, I_N$ of $\mathbb{H}_J \simeq \mathbb{H}$ in the sense of [**44**]. If $\gamma(I_i)$, $\gamma \in \Gamma_J$, is an arbitrary cusp of $\mathbb{H}_J$, the holomorphicity of $\phi = \phi|_{k,V(J)(\beta_0 m)}\gamma$ at I_i implies that of ϕ at $\gamma(I_i)$. Thus ϕ is holomorphic at all cusps, namely $\phi \in J_{k,K(\beta_0 m)}(\Gamma_J)$. □

Proposition 7.18 implies the following.

Proposition 7.19. *We have* $J_{k,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ *when* $k < n/2 - 1$.

Proof. This holds because $J_{k,K'}(\Gamma_J) = 0$ when $k < \mathrm{rk}(K')/2 = n/2 - 1$ (see [**44**] p.251). □

CHAPTER 8

Filtrations associated to 1-dimensional cusps

Let L, Γ, J be as in §7. In this chapter we introduce filtrations on the automorphic vector bundles canonically associated to the J-cusp, and study its basic properties. These filtrations will play a fundamental role in the study of the Fourier-Jacobi expansion. Our geometric approach will be effective here. In §8.1 we define the filtration on the second Hodge bundle $\mathcal{E}$. This induces filtrations on general automorphic vector bundles $\mathcal{E}_{\lambda,k}$ (§8.2). In §8.3 we study these filtrations from the viewpoint of representations of a parabolic subgroup. In §8.4, as the first application of our filtration, we prove that vector-valued Jacobi forms decompose, in a certain sense, into scalar-valued Jacobi forms of various weights. The second application will be given in §9.

8.1. J-filtration on $\mathcal{E}$

In this section we define a filtration on $\mathcal{E}$ canonically associated to J. For $[\omega] \in \mathcal{D}$ we consider the filtration

$$0 \subset \omega^{\perp} \cap J_{\mathbb{C}} \subset \omega^{\perp} \cap J_{\mathbb{C}}^{\perp} \subset \omega^{\perp} \tag{8.1}$$

on $\omega^{\perp} = \omega^{\perp} \cap L_{\mathbb{C}}$.

Lemma 8.1. *Let $p\colon \omega^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ be the projection. Then $p(\omega^{\perp} \cap J_{\mathbb{C}})$ has dimension 1 and $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}) = p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$ in $\omega^{\perp}/\mathbb{C}\omega$.*

Proof. Since $(\omega, J) \not\equiv 0$, we have $\dim(\omega^{\perp} \cap J_{\mathbb{C}}) = 1$. The fact that $\mathbb{C}\omega \not\subset J_{\mathbb{C}}$ then implies that $p(\omega^{\perp} \cap J_{\mathbb{C}})$ has dimension 1. Next we prove the second assertion. It is clear that $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}) \subset p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$. Since $p(\omega^{\perp} \cap J_{\mathbb{C}})^{\perp}$ is of codimension 1 in $\omega^{\perp}/\mathbb{C}\omega$ by the first assertion, it is sufficient to show that $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})$ is of codimension 1 too. Since $\mathbb{C}\omega \not\subset J_{\mathbb{C}}$, we have $(\omega, J^{\perp}) \not\equiv 0$. This implies that $\omega^{\perp} \cap J_{\mathbb{C}}^{\perp}$ is of codimension 1 in $J_{\mathbb{C}}^{\perp}$, and so of codimension 2 in $\omega^{\perp}$. The fact that $\mathbb{C}\omega \not\subset J_{\mathbb{C}}^{\perp}$ implies that the projection $\omega^{\perp} \cap J_{\mathbb{C}}^{\perp} \to \omega^{\perp}/\mathbb{C}\omega$ is injective. Hence $p(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})$ is of codimension 1 in $\omega^{\perp}/\mathbb{C}\omega$. □

Let $\mathcal{E}_J$ be the sub line bundle of $\mathcal{E}$ whose fiber over $[\omega] \in \mathcal{D}$ is the image of $\omega^{\perp} \cap J_{\mathbb{C}}$ in $\omega^{\perp}/\mathbb{C}\omega$. This is an isotropic sub line bundle of $\mathcal{E}$. Taking the image of (8.1) in $\omega^{\perp}/\mathbb{C}\omega$ and varying $[\omega] \in \mathcal{D}$, we obtain the filtration

$$0 \subset \mathcal{E}_J \subset \mathcal{E}_J^{\perp} \subset \mathcal{E} \tag{8.2}$$

on $\mathcal{E}$. We call it the *J-filtration* on $\mathcal{E}$. By construction, this is $\Gamma(J)_{\mathbb{R}}$-invariant.

We calculate the graded quotients of the J-filtration. Let $\pi\colon \mathcal{D} \to \mathbb{H}_J$ be the projection to the J-cusp and $\mathcal{L}_J$ be the Hodge bundle on $\mathbb{H}_J$. We write $V(J) = (J^{\perp}/J)_{\mathbb{C}}$ as before.

Proposition 8.2. *We have $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphisms*

$$\mathcal{E}_J \simeq \pi^*\mathcal{L}_J, \qquad \mathcal{E}_J^{\perp}/\mathcal{E}_J \simeq V(J) \otimes \mathcal{O}_{\mathcal{D}}, \qquad \mathcal{E}/\mathcal{E}_J^{\perp} \simeq \pi^*\mathcal{L}_J^{-1}. \tag{8.3}$$

Proof. We begin with $\mathcal{E}_J$. Let $[\omega] \in \mathcal{D}$. The fiber of $\mathcal{E}_J$ over $[\omega]$ is the line $\omega^{\perp} \cap J_{\mathbb{C}} \subset J_{\mathbb{C}}$, while that of $\pi^*\mathcal{L}_J$ is the image of $\mathbb{C}\omega$ in $(L/J^{\perp})_{\mathbb{C}}$. In order to compare these two lines, we consider the canonical isomorphisms

$$(L/J^{\perp})_{\mathbb{C}} \to J_{\mathbb{C}}^{\vee} \leftarrow J_{\mathbb{C}}. \tag{8.4}$$

Here the first map is induced by the pairing on L, and the second map is induced by the canonical symplectic form $J \times J \to \wedge^2 J \simeq \mathbb{Z}$ on J. The second map sends a line in $J_{\mathbb{C}}$ to its annihilator in $J_{\mathbb{C}}^{\vee}$. In (8.4), the above two lines are both sent to the line $(\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}}$ in $J_{\mathbb{C}}^{\vee}$ (the pairing of $J_{\mathbb{C}}$ with $\mathbb{C}\omega$). This gives the canonical isomorphism

$$(\pi^*\mathcal{L}_J)_{[\omega]} = \mathrm{Im}(\mathbb{C}\omega \to (L/J^{\perp})_{\mathbb{C}}) \;\; \to \;\; \omega^{\perp} \cap J_{\mathbb{C}} = (\mathcal{E}_J)_{[\omega]}.$$

Varying $[\omega]$, we obtain a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\pi^*\mathcal{L}_J \simeq \mathcal{E}_J$.

Consequently, we obtain the description of the last graded quotient

$$\mathcal{E}/\mathcal{E}_J^{\perp} \;\simeq\; \mathcal{E}_J^{\vee} \;\simeq\; \pi^*\mathcal{L}_J^{-1},$$

where the first map is induced by the quadratic form on $\mathcal{E}$.

Finally, we consider the middle graded quotient $\mathcal{E}_J^{\perp}/\mathcal{E}_J$. The fiber of this vector bundle over $[\omega] \in \mathcal{D}$ is $(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})/(\omega^{\perp} \cap J_{\mathbb{C}})$. We have a natural map

$$(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})/(\omega^{\perp} \cap J_{\mathbb{C}}) \to J_{\mathbb{C}}^{\perp}/J_{\mathbb{C}} = V(J). \tag{8.5}$$

This is clearly injective. Since the source and the target have the same dimension, this map is an isomorphism. Varying $[\omega]$, we obtain a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{D}}$. □

Next we choose a rank 1 primitive sublattice I of J and describe the J-filtration under the I-trivialization.

Proposition 8.3. *The I-trivialization $\mathcal{E} \simeq V(I)\otimes\mathcal{O}_{\mathcal{D}}$ sends the J-filtration* (8.2) *on $\mathcal{E}$ to the filtration*

$$(0 \subset J/I \subset J^{\perp}/I \subset I^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$$

on $V(I) \otimes \mathcal{O}_{\mathcal{D}}$.

Proof. Since the I-trivialization $V(I)\otimes\mathcal{O}_{\mathcal{D}} \to \mathcal{E}$ preserves the quadratic forms, it suffices to check that this sends $(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ to $\mathcal{E}_J$. Recall that the I-trivialization at $[\omega] \in \mathcal{D}$ is the composition map

$$I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}} \to \omega^{\perp} \cap I_{\mathbb{C}}^{\perp} \to \omega^{\perp}/\mathbb{C}\omega. \tag{8.6}$$

The inverse of the first map sends the line $\omega^{\perp} \cap J_{\mathbb{C}}$ in $\omega^{\perp} \cap I_{\mathbb{C}}^{\perp}$ to the line $J_{\mathbb{C}}/I_{\mathbb{C}}$ in $I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}}$, and the second map sends $\omega^{\perp} \cap J_{\mathbb{C}}$ to $(\mathcal{E}_J)_{[\omega]}$ by definition. Therefore (8.6) sends $J_{\mathbb{C}}/I_{\mathbb{C}}$ to $(\mathcal{E}_J)_{[\omega]}$. This proves our assertion. □

The J-filtration descends to a filtration on the descent of $\mathcal{E}$ to $\mathcal{X}(J) = \mathcal{D}/U(J)_{\mathbb{Z}}$. We consider the canonical extension over the partial toroidal compactification $\overline{\mathcal{X}(J)}$.

Proposition 8.4. *The J-filtration on $\mathcal{E}$ over $\mathcal{X}(J)$ extends to a filtration on the canonical extension of $\mathcal{E}$ over $\overline{\mathcal{X}(J)}$ by $\overline{\Gamma(J)}_{\mathbb{R}}$-invariant sub vector bundles. The isomorphisms* (8.3) *for the graded quotients on $\mathcal{X}(J)$ extend to isomorphisms between the canonical extensions of both sides over $\overline{\mathcal{X}(J)}$.*

Proof. We choose a rank 1 primitive sublattice I of J. Recall that the canonical extension of $\mathcal{E}$ is defined via the I-trivialization $\mathcal{E} \to V(I)\otimes\mathcal{O}_{\mathcal{X}(J)}$. By Proposition 8.3, the I-trivialization sends the sub vector bundles $\mathcal{E}_J$, $\mathcal{E}_J^{\perp}$ of $\mathcal{E}$ to the sub vector bundles $(J/I)_{\mathbb{C}}\otimes\mathcal{O}_{\mathcal{X}(J)}$, $(J^{\perp}/I)_{\mathbb{C}}\otimes\mathcal{O}_{\mathcal{X}(J)}$ of $V(I)\otimes\mathcal{O}_{\mathcal{X}(J)}$ respectively. The latter clearly extend to the sub vector bundles $(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$, $(J^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ of $V(I) \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$ respectively. This means that $\mathcal{E}_J$, $\mathcal{E}_J^{\perp}$ extend to sub vector bundles of the canonical extension of $\mathcal{E}$. They are still $\overline{\Gamma(J)}_{\mathbb{R}}$-invariant by continuity.

We prove that the isomorphisms (8.3) extend over $\overline{\mathcal{X}(J)}$. We begin with $\mathcal{E}_J \simeq \pi^*\mathcal{L}_J$. For each $[\omega] \in \mathcal{D}$ we have the following commutative diagram of isomorphisms between 1-dimensional linear spaces:

$$\begin{array}{ccc} \omega^{\perp} \cap J_{\mathbb{C}} & \xrightarrow{p_1} & (\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}} \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_4} \\ J_{\mathbb{C}}/I_{\mathbb{C}} & \xrightarrow[p_2]{} & I_{\mathbb{C}}^{\vee}. \end{array}$$

Here p_1 is restriction of the second isomorphism $J_{\mathbb{C}} \to J_{\mathbb{C}}^{\vee}$ in (8.4), p_2 is the map induced from this $J_{\mathbb{C}} \to J_{\mathbb{C}}^{\vee}$, p_3 is the natural projection, and p_4 is the restriction of the natural map $J_{\mathbb{C}}^{\vee} \to I_{\mathbb{C}}^{\vee}$ to the line $(\mathbb{C}\omega, \cdot)|_{J_{\mathbb{C}}}$ of $J_{\mathbb{C}}^{\vee}$. Recall from the proof of Proposition 8.2 that p_1 is identified with the isomorphism $\mathcal{E}_J \to \pi^*\mathcal{L}_J$ at $[\omega]$ after the canonical isomorphism $J_{\mathbb{C}}^{\vee} \simeq (L/J^{\perp})_{\mathbb{C}}$. Varying $[\omega]$, we obtain the following commutative diagram of isomorphisms between line bundles on $\mathcal{X}(J)$:

$$\begin{array}{ccc} \mathcal{E}_J & \xrightarrow{p_1} & \pi^*\mathcal{L}_J \\ {\scriptstyle p_3}\downarrow & & \downarrow{\scriptstyle p_4} \\ (J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)} & \xrightarrow[p_2]{} & I_{\mathbb{C}}^{\vee} \otimes \mathcal{O}_{\mathcal{X}(J)}. \end{array}$$

Here p_1 is the isomorphism we want to extend, p_2 is the constant homomorphism, p_3 is the I-trivialization of $\mathcal{E}_J$, and p_4 is the pullback of the I-trivialization of $\mathcal{L}_J$ (cf. Remark 5.11). By construction, the canonical extension of $\mathcal{E}_J$ is given via p_3. Similarly, by the proof of Proposition 5.10, the canonical extension of $\pi^*\mathcal{L}_J$ is given via p_4. Since p_2 is constant, it extends over $\overline{\mathcal{X}(J)}$. Then this commutative diagram shows that p_1 extends to an isomorphism between the canonical extensions of $\mathcal{E}_J$ and $\pi^*\mathcal{L}_J$.

Next we consider $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$. We observe that for each $[\omega] \in \mathcal{D}$, the natural composition

$$(\omega^{\perp} \cap J_{\mathbb{C}}^{\perp})/(\omega^{\perp} \cap J_{\mathbb{C}}) \to (J_{\mathbb{C}}^{\perp}/I_{\mathbb{C}})/(J_{\mathbb{C}}/I_{\mathbb{C}}) \to J_{\mathbb{C}}^{\perp}/J_{\mathbb{C}},$$

where the first isomorphism comes from $\omega^{\perp} \cap I_{\mathbb{C}}^{\perp} \to I_{\mathbb{C}}^{\perp}/I_{\mathbb{C}}$, coincides with the isomorphism (8.5) defining $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ at $[\omega]$. Therefore the isomorphism $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ in (8.3) factorizes as

$$\mathcal{E}_J^{\perp}/\mathcal{E}_J \to (J^{\perp}/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)}/(J/I)_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{X}(J)} \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)},$$

where the first isomorphism is induced by the I-trivialization and hence gives the canonical extension of $\mathcal{E}_J^{\perp}/\mathcal{E}_J$, and the second isomorphism is the constant homomorphism. The constancy of the second isomorphism ensures that it extends over $\overline{\mathcal{X}(J)}$. This shows that the isomorphism $\mathcal{E}_J^{\perp}/\mathcal{E}_J \to V(J) \otimes \mathcal{O}_{\mathcal{X}(J)}$ in (8.3) extends to an isomorphism between the canonical extensions.

Finally, the extendability of $\mathcal{E}/\mathcal{E}_J^{\perp} \simeq \pi^*\mathcal{L}_J^{-1}$ follows from the extendability of $\mathcal{E}_J \simeq \pi^*\mathcal{L}_J$ and the fact that the quadratic form on $\mathcal{E}$ extends over the canonical extension (by construction). $\square$

8.2. J-filtration on $\mathcal{E}_{\lambda,k}$

In this section we use the J-filtration on $\mathcal{E}$ to define a filtration on a general automorphic vector bundle $\mathcal{E}_{\lambda,k}$.

We begin with a recollection from linear algebra. Let V be a $\mathbb{C}$-linear space of finite dimension endowed with a decreasing filtration of length 3:

$$0 \subset F^1V \subset F^0V \subset F^{-1}V = V.$$

We denote by $\mathrm{Gr}^r V = F^rV/F^{r+1}V$ the r-th graded quotient. (By convention, $F^2V = 0$.) Let $d > 0$. On the tensor product $V^{\otimes d}$ we have a decreasing filtration of length $2d+1$ defined by

$$F^r V^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1} V \otimes F^{i_2} V \otimes \cdots \otimes F^{i_d} V, \qquad -d \leq r \leq d, \tag{8.7}$$

where $\vec{i} = (i_1, \cdots, i_d)$ run over all multi-indices such that $|\vec{i}| = i_1 + \cdots + i_d$ is equal to r. The graded quotient $\mathrm{Gr}^r V^{\otimes d} = F^r V^{\otimes d} / F^{r+1} V^{\otimes d}$ is canonically isomorphic to

$$\mathrm{Gr}^r V^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} \mathrm{Gr}^{i_1} V \otimes \mathrm{Gr}^{i_2} V \otimes \cdots \otimes \mathrm{Gr}^{i_d} V. \tag{8.8}$$

This construction of filtration is well-known in the case $d = 2$; the construction for general d is obtained inductively.

We apply this construction relatively to the J-filtration on the second Hodge bundle $\mathcal{E}$. We write $F^1\mathcal{E} = \mathcal{E}_J$, $F^0\mathcal{E} = \mathcal{E}_J^{\perp}$, $F^{-1}\mathcal{E} = \mathcal{E}$, and define a decreasing filtration

$$0 \subset F^d \mathcal{E}^{\otimes d} \subset F^{d-1}\mathcal{E}^{\otimes d} \subset \cdots \subset F^{-d}\mathcal{E}^{\otimes d} = \mathcal{E}^{\otimes d}$$

of length $2d+1$ on $\mathcal{E}^{\otimes d}$ by

$$F^r \mathcal{E}^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1}\mathcal{E} \otimes F^{i_2}\mathcal{E} \otimes \cdots \otimes F^{i_d}\mathcal{E}, \qquad -d \leq r \leq d.$$

This is a filtration by $\Gamma(J)_{\mathbb{R}}$-invariant sub vector bundles.

Lemma 8.5. *We have a $\Gamma(J)_{\mathbb{R}}$-equivariant isomorphism*

$$\mathrm{Gr}^r \mathcal{E}^{\otimes d} \simeq \pi^* \mathcal{L}_J^{\otimes r} \otimes \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})}, \tag{8.9}$$

where $b(\vec{i}) \geq 0$ is the number of components i_* of $\vec{i} = (i_1, \cdots, i_d)$ equal to 0.

Proof. By (8.8) we have

$$\mathrm{Gr}^r \mathcal{E}^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} \mathrm{Gr}^{i_1}\mathcal{E} \otimes \cdots \otimes \mathrm{Gr}^{i_d}\mathcal{E}. \tag{8.10}$$

By Proposition 8.2, each factor $\mathrm{Gr}^{i_*}\mathcal{E}$ is isomorphic to $\pi^*\mathcal{L}_J$, $V(J) \otimes \mathcal{O}_{\mathcal{D}}$, $\pi^*\mathcal{L}_J^{-1}$ according to $i_* = 1, 0, -1$ respectively. Let $a(\vec{i}), b(\vec{i}), c(\vec{i})$ be the number of components i_* of $\vec{i} = (i_1, \cdots, i_d)$ equal to $1, 0, -1$ respectively. Then (8.10) can be written more explicitly as

$$\mathrm{Gr}^r \mathcal{E}^{\otimes d} \simeq \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})} \otimes \pi^* \mathcal{L}_J^{\otimes a(\vec{i}) - c(\vec{i})}.$$

We have $a(\vec{i}) - c(\vec{i}) = |\vec{i}| = r$. □

Since $\mathrm{Gr}^{-i}\mathcal{E} \simeq (\mathrm{Gr}^{i}\mathcal{E})^{\vee}$, the expression (8.10) shows that we have the duality

$$\mathrm{Gr}^{-r}\mathcal{E}^{\otimes d} \simeq (\mathrm{Gr}^{r}\mathcal{E}^{\otimes d})^{\vee},$$

by sending an index $\vec{i} = (i_1, \cdots, i_d)$ to its dual index $(-i_1, \cdots, -i_d)$.

By Proposition 8.3, the I-trivialization $\mathcal{E}^{\otimes d} \simeq V(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$ sends the sub vector bundle $F^r\mathcal{E}^{\otimes d}$ of $\mathcal{E}^{\otimes d}$ to the sub vector bundle $F^rV(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$ of $V(I)^{\otimes d} \otimes \mathcal{O}_{\mathcal{D}}$, where $F^rV(I)^{\otimes d}$ is the filtration (8.7) applied to $V = V(I)$, $F^1V = (J/I)_{\mathbb{C}}$ and $F^0V = (J^{\perp}/I)_{\mathbb{C}}$. This implies that the filtration $F^{\bullet}\mathcal{E}^{\otimes d}$ on $\mathcal{E}^{\otimes d}$ over $\mathcal{X}(J)$ extends to a filtration on the canonical extension of $\mathcal{E}^{\otimes d}$ over $\overline{\mathcal{X}(J)}$ by sub vector bundles. (We use the same notation.)

Now we consider a general automorphic vector bundle $\mathcal{E}_{\lambda,k} = \mathcal{E}_{\lambda} \otimes \mathcal{L}^{\otimes k}$. Let $d = |\lambda|$. Recall from §3.2 that $\mathcal{E}_{\lambda} = c_{\lambda} \cdot \mathcal{E}^{[d]}$ is defined as an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant sub vector bundle of $\mathcal{E}^{\otimes d}$, where $c_{\lambda} = b_{\lambda}a_{\lambda}$ is the Young symmetrizer for λ. We define a decreasing filtration on $\mathcal{E}_{\lambda}$ by taking the intersection with $F^r\mathcal{E}^{\otimes d}$ inside $\mathcal{E}^{\otimes d}$:

$$F^r\mathcal{E}_{\lambda} = \mathcal{E}_{\lambda} \cap F^r\mathcal{E}^{\otimes d}, \qquad -d \leq r \leq d.$$

Then we take the twist by $\mathcal{L}^{\otimes k}$:

$$F^r\mathcal{E}_{\lambda,k} = F^r\mathcal{E}_{\lambda} \otimes \mathcal{L}^{\otimes k}.$$

This is a $\Gamma(J)_{\mathbb{R}}$-invariant filtration on $\mathcal{E}_{\lambda,k}$. We call it the *J-filtration* on $\mathcal{E}_{\lambda,k}$. This is a standard filtration on $\mathcal{E}_{\lambda,k}$ that can be induced from the J-filtration on $\mathcal{E}$. In Proposition 8.13, we will prove that the range of the level r reduces to $-\lambda_1 \leq r \leq \lambda_1$.

Remark 8.6. We also have the following natural expressions of $F^r\mathcal{E}_{\lambda}$:

$$F^r\mathcal{E}_{\lambda} = c_{\lambda}(\mathcal{E}^{[d]} \cap F^r\mathcal{E}^{\otimes d}) = \mathcal{E}^{[d]} \cap c_{\lambda}(F^r\mathcal{E}^{\otimes d}).$$

These equalities hold because we have $c_{\lambda}(F^r\mathcal{E}^{\otimes d}) \subset F^r\mathcal{E}^{\otimes d}$ by the $\mathfrak{S}_d$-invariance of $F^r\mathcal{E}^{\otimes d}$ and c_{λ} is an idempotent up to scalar multiplication.

Let

$$F^rV(I)_{\lambda} = V(I)_{\lambda} \cap F^rV(I)^{\otimes d}, \qquad -d \leq r \leq d, \tag{8.11}$$

be the similar filtration on $V(I)_{\lambda}$. The I-trivialization $\mathcal{E}_{\lambda} \simeq V(I)_{\lambda} \otimes \mathcal{O}_{\mathcal{D}}$ sends the J-filtration $F^{\bullet}\mathcal{E}_{\lambda}$ on $\mathcal{E}_{\lambda}$ to the filtration $F^{\bullet}V(I)_{\lambda} \otimes \mathcal{O}_{\mathcal{D}}$ on $V(I)_{\lambda} \otimes \mathcal{O}_{\mathcal{D}}$. This implies that the J-filtration on $\mathcal{E}_{\lambda,k}$, after descending to $\mathcal{X}(J)$, extends to a filtration on the canonical extension of $\mathcal{E}_{\lambda,k}$ over $\overline{\mathcal{X}(J)}$ by $\overline{\Gamma(J)}_{\mathbb{R}}$-invariant sub vector bundles.

Proposition 8.7. *At the boundary divisor Δ_J of $\overline{\mathcal{X}(J)}$, we have a $\overline{\Gamma(J)}_{\mathbb{R}}$-equivariant isomorphism*

$$\mathrm{Gr}^r(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \simeq (\pi_2^*\mathcal{L}_J^{\otimes r+k})^{\oplus \alpha(r)}, \tag{8.12}$$

where $\alpha(r) \geq 0$ is the rank of $\mathrm{Gr}^r\mathcal{E}_{\lambda}$ and π_2 is the projection $\Delta_J \to \mathbb{H}_J$.

PROOF. Since $\mathcal{L}|_{\Delta_J} \simeq \pi_2^*\mathcal{L}_J$ by Proposition 5.10, it suffices to prove this assertion in the case $k = 0$. By Lemma 8.5, we have a $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant embedding

$$\mathrm{Gr}^r\mathcal{E}_\lambda \hookrightarrow \mathrm{Gr}^r\mathcal{E}^{\otimes|\lambda|} \simeq (\pi^*\mathcal{L}_J^{\otimes r})^{\oplus b}$$

over $\mathcal{X}(J)$ for some $b > 0$. By Proposition 8.4, this embedding extends over $\overline{\mathcal{X}(J)}$. By restricting it to Δ_J, we obtain a $\overline{\Gamma(J)_{\mathbb{R}}}$-equivariant embedding

$$\mathrm{Gr}^r(\mathcal{E}_\lambda|_{\Delta_J}) \hookrightarrow (\pi_2^*\mathcal{L}_J^{\otimes r})^{\oplus b}.$$

The image of this embedding is a $\overline{\Gamma(J)_{\mathbb{R}}}$-invariant sub vector bundle of $(\pi_2^*\mathcal{L}_J^{\otimes r})^{\oplus b}$. Since the Heisenberg group $W(J)_{\mathbb{R}} \subset \Gamma(J)_{\mathbb{R}}$ acts on each fiber of $\pi_2 \colon \Delta_J \to \mathbb{H}_J$ transitively, this image can be written as $\pi_2^*\mathcal{F}$ for some $\mathrm{SL}(J_{\mathbb{R}})$-invariant sub vector bundle $\mathcal{F}$ of $(\mathcal{L}_J^{\otimes r})^{\oplus b}$. By the $\mathrm{SL}(J_{\mathbb{R}})$-invariance, $\mathcal{F}$ is isomorphic to a direct sum of copies of $\mathcal{L}_J^{\otimes r}$. □

Before finishing this section, we look at two typical examples.

EXAMPLE 8.8. Let $\lambda = (1^d)$ with $0 < d < n$, namely $V_\lambda = \wedge^d V$. We have $\wedge^i\mathcal{E}_J = 0$ if $i > 1$ and $(\wedge^i\mathcal{E}_J^\perp) \wedge (\wedge^j\mathcal{E}) = \wedge^{i+j}\mathcal{E}$ if $j > 0$. This shows that the J-filtration on $\wedge^d\mathcal{E}$ reduces to the following filtration of length 3:

$$0 \ \subset\ \mathcal{E}_J \wedge (\wedge^{d-1}\mathcal{E}_J^\perp) \ \subset\ \wedge^d\mathcal{E}_J^\perp + \mathcal{E}_J \wedge (\wedge^{d-1}\mathcal{E}) \ \subset\ \wedge^d\mathcal{E}.$$

These three subspaces have level 1, 0, −1 respectively. (Note that $\wedge^{d-1}\mathcal{E} = (\wedge^{d-2}\mathcal{E}_J^\perp) \wedge \mathcal{E}$ in the second term and $\wedge^d\mathcal{E} = (\wedge^{d-1}\mathcal{E}_J^\perp) \wedge \mathcal{E}$ in the last term.) The three graded quotients are respectively isomorphic to

$$\mathcal{E}_J \otimes \wedge^{d-1}(\mathcal{E}_J^\perp/\mathcal{E}_J) \simeq \wedge^{d-1}V(J) \otimes \pi^*\mathcal{L}_J,$$

$$\wedge^d(\mathcal{E}_J^\perp/\mathcal{E}_J) \oplus \wedge^{d-2}(\mathcal{E}_J^\perp/\mathcal{E}_J) \simeq (\wedge^d V(J) \oplus \wedge^{d-2}V(J)) \otimes \mathcal{O}_{\mathcal{D}},$$

$$(\mathcal{E}/\mathcal{E}_J^\perp) \otimes \wedge^{d-1}(\mathcal{E}_J^\perp/\mathcal{E}_J) \simeq \wedge^{d-1}V(J) \otimes \pi^*\mathcal{L}_J^{-1}.$$

Here $\wedge^{d-2}V(J) = 0$ when $d = 1$, and $\wedge^d V(J) = 0$ when $d = n - 1$.

EXAMPLE 8.9. The J-filtration on $\mathrm{Sym}^d\mathcal{E}$ has length $2d + 1$, with subspaces

$$F^r\mathrm{Sym}^d\mathcal{E} = \sum_{\substack{a+b+c=d \\ a-c=r}} \mathrm{Sym}^a\mathcal{E}_J \cdot \mathrm{Sym}^b\mathcal{E}_J^\perp \cdot \mathrm{Sym}^c\mathcal{E}, \qquad -d \leq r \leq d.$$

The graded quotient $\mathrm{Gr}^r\mathrm{Sym}^d\mathcal{E}$ is isomorphic to

$$\pi^*\mathcal{L}_J^{\otimes r} \otimes (\mathrm{Sym}^{d-|r|}V(J) \oplus \mathrm{Sym}^{d-|r|-2}V(J) \oplus \cdots \oplus \mathrm{Sym}^{0\, or\, 1}V(J)).$$

This shows that the J-filtration on the main irreducible component $\mathcal{E}_{(d)}$ of $\mathrm{Sym}^d\mathcal{E}$ has length $2d + 1$ with graded quotient

$$\mathrm{Gr}^r\mathcal{E}_{(d)} \simeq \pi^*\mathcal{L}_J^{\otimes r} \otimes \mathrm{Sym}^{d-|r|}V(J), \qquad -d \leq r \leq d. \tag{8.13}$$

8.3. J-filtration and representations

In this section we study the J-filtration, in its I-trivialized form, from the viewpoint of representations of a parabolic subgroup. As consequences, we determine the range of possible levels, and also relate the Siegel operator (§6) to the J-filtration.

We choose a rank 1 primitive sublattice $I \subset J$. Let $P(J/I)_{\mathbb{C}}$ be the stabilizer of the isotropic line $(J/I)_{\mathbb{C}} \subset V(I)$ in $\mathrm{O}(V(I))$. As in (6.4), $P(J/I)_{\mathbb{C}}$ sits in the exact sequence

$$0 \to U(J/I)_{\mathbb{C}} \to P(J/I)_{\mathbb{C}} \to \mathrm{GL}((J/I)_{\mathbb{C}}) \times \mathrm{O}(V(J)) \to 1, \tag{8.14}$$

where $U(J/I)_{\mathbb{C}} \simeq V(J) \otimes (J/I)_{\mathbb{C}}$ is the unipotent radical of $P(J/I)_{\mathbb{C}}$ consisting of the Eichler transvections of $V(I)$ with respect to $(J/I)_{\mathbb{C}}$. The filtration

$$(F^r V(I))_{-1 \le r \le 1} \;=\; (0 \subset (J/I)_{\mathbb{C}} \subset (J^{\perp}/I)_{\mathbb{C}} \subset V(I))$$

on $V(I)$ is $P(J/I)_{\mathbb{C}}$-invariant. The unipotent radical $U(J/I)_{\mathbb{C}}$ acts on the graded quotients trivially, so they are representations of

$$\mathrm{GL}((J/I)_{\mathbb{C}}) \times \mathrm{O}(V(J)) \;\simeq\; \mathbb{C}^* \times \mathrm{O}(n-2, \mathbb{C}).$$

Specifically,

- $\mathrm{Gr}^1 V(I) = (J/I)_{\mathbb{C}}$ is the weight 1 character of $\mathbb{C}^*$.
- $\mathrm{Gr}^0 V(I) = V(J)$ is the standard representation of $\mathrm{O}(V(J))$.
- $\mathrm{Gr}^{-1} V(I) = (J/I)_{\mathbb{C}}^{\vee}$ is the weight -1 character of $\mathbb{C}^*$.

Let $d > 0$. As in (8.7), let

$$F^r V(I)^{\otimes d} = \sum_{|\vec{i}|=r} F^{i_1} V(I) \otimes \cdots \otimes F^{i_d} V(I), \qquad -d \le r \le d,$$

be the induced filtration on $V(I)^{\otimes d}$. This is $P(J/I)_{\mathbb{C}}$-invariant. By (8.8), the unipotent radical $U(J/I)_{\mathbb{C}}$ acts on the graded quotients $\mathrm{Gr}^r V(I)^{\otimes d}$ trivially. Hence $\mathrm{Gr}^r V(I)^{\otimes d}$ is a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$. Specifically, by the same calculation as in Lemma 8.5, we have

$$\mathrm{Gr}^r V(I)^{\otimes d} \simeq \chi_r \boxtimes \bigoplus_{|\vec{i}|=r} V(J)^{\otimes b(\vec{i})}, \tag{8.15}$$

where χ_r is the weight r character of $\mathbb{C}^*$. If we take a lift of $\mathbb{C}^* \times \mathrm{O}(V(J))$ in (8.14), we have a decomposition

$$V(I)^{\otimes d} \simeq \bigoplus_{r=-d}^{d} \mathrm{Gr}^r V(I)^{\otimes d} \tag{8.16}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$ because $\mathbb{C}^* \times \mathrm{O}(V(J))$ is reductive. By (8.15), this is the weight decomposition with respect to $\mathbb{C}^*$.

Now let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(V(I)) \simeq \mathrm{O}(n, \mathbb{C})$. As in (8.11), let

$$F^r V(I)_\lambda = V(I)_\lambda \cap F^r V(I)^{\otimes |\lambda|}$$

be the filtration induced on $V(I)_\lambda$. This is a $P(J/I)_{\mathbb{C}}$-invariant filtration, and $U(J/I)_{\mathbb{C}}$ acts on the graded quotients trivially. By the above argument, if we take a lift of $\mathbb{C}^* \times \mathrm{O}(V(J))$ in (8.14), we have a decomposition

$$V(I)_\lambda \simeq \bigoplus_r \mathrm{Gr}^r V(I)_\lambda \tag{8.17}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$, and this agrees with the weight decomposition for $\mathbb{C}^*$ with $\mathrm{Gr}^r V(I)_\lambda$ being the weight r subspace.

Proposition 8.10. *Let* $\lambda \neq \det$. *We have*

$$F^{\lambda_1+1} V(I)_\lambda = 0, \qquad F^{-\lambda_1} V(I)_\lambda = V(I)_\lambda. \tag{8.18}$$

Thus the filtration $F^\bullet V(I)_\lambda$ *has length* $\leq 2\lambda_1 + 1$*, from level* $-\lambda_1$ *to* λ_1*. Moreover, we have*

$$F^{\lambda_1} V(I)_\lambda = V(I)_\lambda^{U(J/I)_{\mathbb{C}}}. \tag{8.19}$$

Proof. This is purely a representation-theoretic calculation. We write $V = V(I)$ and take a basis $e_1, \cdots, e_n$ of V such that $(J/I)_{\mathbb{C}} = \mathbb{C}e_1$, $(e_i, e_j) = 1$ if $i + j = n + 1$, and $(e_i, e_j) = 0$ otherwise. We also write $P = P(J/I)_{\mathbb{C}}$ and $U = U(J/I)_{\mathbb{C}}$. (The same notation as in the proof of Proposition 6.3.) We identify $V(J)$ with $V' = \langle e_2, \cdots, e_{n-1} \rangle$. This defines a lift $\mathbb{C}^* \times \mathrm{O}(V') \hookrightarrow P$. Then $\mathbb{C}^*$ acts on $\mathbb{C}e_1$ by weight 1, on V' by weight 0, and on $\mathbb{C}e_n$ by weight -1.

We first prove (8.18). Recall from (3.1) that

$$V_\lambda \subset \wedge^{{}^t\lambda_1} V \otimes \cdots \otimes \wedge^{{}^t\lambda_{\lambda_1}} V. \tag{8.20}$$

Since the weights of $\mathbb{C}^*$ on each space $\wedge^i V$ are only $-1, 0, 1$, the weights of $\mathbb{C}^*$ on the right hand side of (8.20) are contained in the range $[-\lambda_1, \lambda_1]$. Therefore the weights of $\mathbb{C}^*$ on V_λ are contained in $[-\lambda_1, \lambda_1]$. Since $\mathrm{Gr}^r V_\lambda$ is the weight r subspace for the action of $\mathbb{C}^*$, this shows that $\mathrm{Gr}^r V_\lambda \neq 0$ only when $-\lambda_1 \leq r \leq \lambda_1$. This implies (8.18).

Next we prove (8.19). In Proposition 6.3, we proved that $V_\lambda^U \simeq \chi_{\lambda_1} \boxtimes W$ as a representation of $\mathbb{C}^* \times \mathrm{O}(V')$ where W is a representation of $\mathrm{O}(V') \simeq \mathrm{O}(n-2, \mathbb{C})$. (We do not use precise information on W.) In particular, $\mathbb{C}^*$ acts on V_λ^U by weight λ_1. This means that $V_\lambda^U \subset F^{\lambda_1} V_\lambda$. On the other hand, since U acts trivially on

$$\mathrm{Gr}^{\lambda_1} V_\lambda \;=\; F^{\lambda_1} V_\lambda / F^{\lambda_1+1} V_\lambda \;=\; F^{\lambda_1} V_\lambda,$$

we also see that $F^{\lambda_1} V_\lambda \subset V_\lambda^U$. Therefore $F^{\lambda_1} V_\lambda = V_\lambda^U$. □

We have the following duality between the graded quotients.

Lemma 8.11. *We have* $\mathrm{Gr}^r V(I)_\lambda \simeq \mathrm{Gr}^{-r} V(I)_\lambda$ *as representations of* $\mathrm{O}(V(J))$.

Proof. We keep the notation as in the proof of Proposition 8.10 and take the $\mathbb{C}^* \times \mathrm{O}(V')$-decomposition (8.17) of V_λ. Let ι be the involution of V which exchanges e_1 and e_n and acts on $V' = \langle e_2, \cdots, e_{n-1}\rangle$ trivially. Thus ι and $\mathbb{C}^* = \mathrm{SO}(\langle e_1, e_n\rangle)$ generate $\mathrm{O}(\langle e_1, e_n\rangle)$. The involution ι normalizes $\mathbb{C}^* \times \mathrm{O}(V')$. Its adjoint action acts on $\mathbb{C}^*$ by $\alpha \mapsto \alpha^{-1}$, and acts on $\mathrm{O}(V')$ trivially. Therefore the action of ι on V_λ maps the weight r subspace $\mathrm{Gr}^r V_\lambda$ to the weight $-r$ subspace $\mathrm{Gr}^{-r} V_\lambda$, and this map is $\mathrm{O}(V')$-equivariant. □

It will be useful to know that the graded quotients in level $-\lambda_1$ and λ_1 are indeed nontrivial.

Lemma 8.12. *Let* $\lambda \neq \det$. *We have* $\mathrm{Gr}^{\lambda_1} V(I)_\lambda \neq 0$ *and* $\mathrm{Gr}^{-\lambda_1} V(I)_\lambda \neq 0$.

Proof. We keep the notation as in the proof of Proposition 8.10. Recall from (3.2) that V_λ contains the vector

$$(e_1 \wedge \cdots \wedge e_{{}^t\lambda_1}) \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_2}) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_{{}^t\lambda_{\lambda_1}}).$$

Since ${}^t\lambda_1 < n$ by $\lambda \neq \det$, this vector is contained in the weight λ_1 subspace for the $\mathbb{C}^*$-action. Therefore $\mathrm{Gr}^{\lambda_1} V_\lambda \neq 0$. The nontriviality of $\mathrm{Gr}^{-\lambda_1} V_\lambda$ then follows from Lemma 8.11. □

Since (8.17) is the weight decomposition for $\mathbb{C}^*$, we can write

$$\mathrm{Gr}^r V(I)_\lambda \simeq \chi_r \boxtimes V(J)_{\lambda'(r)}$$

as a representation of $\mathbb{C}^* \times \mathrm{O}(V(J))$, where $V(J)_{\lambda'(r)}$ is some (in general reducible) representation of $\mathrm{O}(V(J)) \simeq \mathrm{O}(n-2, \mathbb{C})$. The representation $V(J)_{\lambda'(r)}$ can be understood through the restriction rule of V_λ for $\mathrm{SO}(2,\mathbb{C}) \times \mathrm{O}(n-2,\mathbb{C}) \subset \mathrm{O}(n,\mathbb{C})$. See [**31**] and [**33**] for a description of this restriction rule in terms of the Littlewood-Richardson numbers.

By translating the conclusions of Proposition 8.10 and Lemmas 8.11 and 8.12 by the I-trivialization, we obtain the following consequence for the J-filtration on $\mathcal{E}_\lambda$.

Proposition 8.13. *Let* $\lambda \neq \det$. *The* J*-filtration* $F^\bullet\mathcal{E}_\lambda$ *on* $\mathcal{E}_\lambda$ *satisfies*

$$F^{\lambda_1+1}\mathcal{E}_\lambda = 0, \qquad F^{-\lambda_1}\mathcal{E}_\lambda = \mathcal{E}_\lambda$$

and

$$F^{\lambda_1}\mathcal{E}_\lambda = \mathrm{Gr}^{\lambda_1}\mathcal{E}_\lambda \neq 0, \qquad \mathrm{Gr}^{-\lambda_1}\mathcal{E}_\lambda \neq 0.$$

Thus $F^\bullet\mathcal{E}_\lambda$ *has length* $\leq 2\lambda_1 + 1$, *from level* $-\lambda_1$ *to* λ_1. *The graded quotients* $\mathrm{Gr}^r\mathcal{E}_\lambda$ *and* $\mathrm{Gr}^{-r}\mathcal{E}_\lambda$ *have the same rank. Moreover,* $F^{\lambda_1}\mathcal{E}_\lambda$ *coincides with the sub vector bundle* $\mathcal{E}_\lambda^J$ *of* $\mathcal{E}_\lambda$ *defined in §6.2.*

REMARK 8.14. (1) By this description of $\mathcal{E}^J_\lambda$, some of the results of §6.2 also follow from the results of §8.2.

(2) The isomorphism (8.12) can be written better as

$$\mathrm{Gr}^r(\mathcal{E}_{\lambda,k}|_{\Delta_J}) \simeq \pi_2^*\mathcal{L}_J^{\otimes r+k} \otimes V(J)_{\lambda'(r)}.$$

8.4. Decomposition of Jacobi forms

In this section we use the J-filtration on $\mathcal{E}_{\lambda,k}$ to show that vector-valued Jacobi forms decompose, in a sense, into some tuples of scalar-valued Jacobi forms.

PROPOSITION 8.15. *Let* $\lambda \neq \det$. *There exists an injective map*

$$J_{\lambda,k,m}(\Gamma(J)_\mathbb{Z}) \hookrightarrow \bigoplus_{r=-\lambda_1}^{\lambda_1} J_{k+r,m}(\Gamma(J)_\mathbb{Z})^{\oplus\alpha(r)}, \tag{8.21}$$

where $\alpha(r)$ *is the rank of* $\mathrm{Gr}^r\mathcal{E}_\lambda$.

PROOF. We use the notation in §7. Let $F^rJ_{\lambda,k,m}(\Gamma(J)_\mathbb{Z})$ be the subspace of $J_{\lambda,k,m}(\Gamma(J)_\mathbb{Z})$ consisting of Jacobi forms which take values in the sub vector bundle $F^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ of $\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$. This defines a filtration on $J_{\lambda,k,m}(\Gamma(J)_\mathbb{Z})$ from level $r=-\lambda_1$ to λ_1. By the exact sequence

$$0 \to F^{r+1}\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m} \to F^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m} \to \mathrm{Gr}^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m} \to 0$$

and Proposition 8.7, we obtain an embedding

$$\begin{aligned}\mathrm{Gr}^r(J_{\lambda,k,m}(\Gamma(J)_\mathbb{Z})) &\hookrightarrow H^0(\Delta_J, \mathrm{Gr}^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m})^{\overline{\Gamma(J)}_\mathbb{Z}} \\ &\simeq H^0(\Delta_J, (\pi_2^*\mathcal{L}_J^{\otimes r+k})^{\oplus\alpha(r)}\otimes\Theta_J^{\otimes m})^{\overline{\Gamma(J)}_\mathbb{Z}}.\end{aligned}$$

The image of this embedding is contained in $J_{r+k,m}(\Gamma(J)_\mathbb{Z})^{\oplus\alpha(r)}$, namely holomorphic at the cusps of $\mathbb{H}_J$. Indeed, if we take the (I,ω_J)-trivialization at $I\subset J$, the quotient homomorphism $F^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}\to\mathrm{Gr}^r\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$ is identified with the quotient homomorphism

$$F^rV(I)_\lambda\otimes(I_\mathbb{C}^\vee)^{\otimes k}\otimes\mathcal{O}_{\Delta_J}\to\mathrm{Gr}^rV(I)_\lambda\otimes(I_\mathbb{C}^\vee)^{\otimes k}\otimes\mathcal{O}_{\Delta_J}.$$

Since this is constant over Δ_J, its effect on the Fourier expansion of a Jacobi form is just reducing each Fourier coefficient from $F^rV(I)_\lambda\otimes(I_\mathbb{C}^\vee)^{\otimes k}$ to $\mathrm{Gr}^rV(I)_\lambda\otimes(I_\mathbb{C}^\vee)^{\otimes k}$, so the Fourier coefficients still satisfy the holomorphicity condition at the I-cusp.

Therefore we obtain a canonical embedding

$$\mathrm{Gr}^r(J_{\lambda,k,m}(\Gamma(J)_\mathbb{Z}))\hookrightarrow J_{r+k,m}(\Gamma(J)_\mathbb{Z})^{\oplus\alpha(r)}. \tag{8.22}$$

Finally, if we choose a splitting of the filtration $F^\bullet J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}})$, we obtain a (non-canonical) isomorphism

$$J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}}) \simeq \bigoplus_{r=-\lambda_1}^{\lambda_1} \mathrm{Gr}^r(J_{k,\lambda,m}(\Gamma(J)_{\mathbb{Z}})).$$

This defines an embedding as claimed. □

As the proof shows, the embedding (8.21) is not canonical: it requires a choice of a splitting of the filtration $F^\bullet J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$. But at least the last subspace is canonically determined:

Corollary 8.16. *Let* $\lambda \neq \det$. *We have a canonical embedding*

$$J_{k+\lambda_1,m}(\Gamma(J)_{\mathbb{Z}}) \otimes V(J)_{\lambda'} \hookrightarrow J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$$

where $\lambda' = (\lambda_2 \geq \cdots \geq \lambda_{n-1})$.

Proof. The last (= level λ_1) subspace $F^{\lambda_1} J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ is the space of Jacobi forms with values in $F^{\lambda_1}\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. By Proposition 8.13 and Theorem 6.1, this sub vector bundle is isomorphic to $\pi_2^*\mathcal{L}_J^{\otimes k+\lambda_1} \otimes V(J)_{\lambda'} \otimes \Theta_J^{\otimes m}$. □

Example 8.17. Let $n = 3$ and $\lambda = (d)$. In this case, in view of (8.13), the embedding (8.21) takes the form

$$J_{(d),k,m}(\Gamma(J)_{\mathbb{Z}}) \hookrightarrow \bigoplus_{r=-d}^{d} J_{k+r,m}(\Gamma(J)_{\mathbb{Z}}).$$

In the context of Siegel modular forms of genus 2, Ibukiyama-Kyomura [**28**] found an *isomorphism* of the same shape for a certain type of integral Jacobi groups. (In our notation, $L = 2U \oplus \langle -2 \rangle$, $K = \langle -2 \rangle$, $J \subset 2U$ the standard one, $U(J)_{\mathbb{Z}} = \wedge^2 J$, and $\overline{\Gamma(J)}_{\mathbb{Z}} = \Gamma_J \ltimes (K \otimes J)$.) The method of Ibukiyama and Kyomura is different, based on differential operators. It might be plausible that their decomposition essentially agrees with that of us.

Proposition 8.15 and Proposition 7.18 enable us to deduce some basic results for vector-valued Jacobi forms from those for scalar-valued Jacobi forms. We present two such consequences.

Corollary 8.18. *Let* $\lambda \neq \det$. *We have* $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ *when* $k + \lambda_1 < n/2 - 1$.

Proof. In this case, all weights $k + r$ in (8.21) satisfy $k + r \leq k + \lambda_1 < n/2 - 1$. Then we have $J_{k+r,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ by Proposition 7.19. □

Corollary 8.19. $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ *has finite dimension. Moreover, we have the following asymptotic estimates:*

$$\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = O(k) \qquad (k \to \infty),$$
$$\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) = O(m^{n-2}) \qquad (m \to \infty).$$

PROOF. By Proposition 8.15 and Proposition 7.18, we have

$$\begin{aligned} \dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}}) &\leq \sum_{r=-\lambda_1}^{\lambda_1} \alpha(r) \cdot \dim J_{k+r,m}(\Gamma(J)_{\mathbb{Z}}) \\ &\leq \sum_{r=-\lambda_1}^{\lambda_1} \alpha(r) \cdot \dim J_{k+r,K(\beta_0 m)}(\Gamma_J), \end{aligned}$$

where K, β_0, Γ_J do not depend on λ, k, m. By the dimension formula of Skoruppa ([**44**] Theorem 6), we see that each $J_{k+r,K(\beta_0 m)}(\Gamma_J)$ is finite-dimensional and

$$\dim J_{k+r,K(\beta_0 m)}(\Gamma_J) = O(k) \qquad (k \to \infty),$$
$$\dim J_{k+r,K(\beta_0 m)}(\Gamma_J) = O(\det K(\beta_0 m)) = O(m^{n-2}) \qquad (m \to \infty).$$

These imply the asymptotic estimates for $\dim J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$. □

REMARK 8.20. From Proposition 8.10, we have imposed the assumption $\lambda \neq \det$. This was necessary in our representation-theoretic calculation. Indeed, (8.19) and Lemma 8.12 do not hold for $\lambda = \det$. On the other hand, since $\Gamma(J)_{\mathbb{Z}} \subset \mathrm{SO}^+(L)$, Jacobi forms with $\lambda = \det$ are the same as those with $\lambda = 1$ (scalar-valued Jacobi forms) as far as $\Gamma(J)_{\mathbb{Z}}$ is concerned. The difference arises when we consider the action by the full stabilizer $\Gamma(J)_{\mathbb{Z}}^*$, which may contain an element of determinant -1.

CHAPTER 9

Vanishing theorem I

Let L be a lattice of signature $(2, n)$ with $n \geq 3$. We assume that L has Witt index 2, i.e., has a rank 2 isotropic sublattice. This is always satisfied when $n \geq 5$. Let Γ be a finite-index subgroup of $O^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition with ${}^t\lambda_1 + {}^t\lambda_2 \leq n$ which expresses an irreducible representation of $O(n, \mathbb{C})$. We assume $\lambda \neq 1, \det$. In this chapter, as an application of the J-filtration, we prove the following vanishing theorem.

THEOREM 9.1. *Let $\lambda \neq 1, \det$. If $k < \lambda_1 + n/2 - 1$, then $M_{\lambda,k}(\Gamma) = 0$. In particular, we have $M_{\lambda,k}(\Gamma) = 0$ whenever $k < n/2$.*

This generalizes the well-known vanishing theorem $M_k(\Gamma) = 0$ for $0 < k < n/2-1$ in the scalar-valued case. This classical fact can be deduced from the vanishing of scalar-valued Jacobi forms (Fourier-Jacobi coefficients) of weight $< n/2 - 1$. Our proof of Theorem 9.1 is a natural generalization of this approach. The outline is as follows.

The first step is to take the projection $\mathcal{E}_{\lambda,k} \to \mathrm{Gr}^{-\lambda_1}\mathcal{E}_{\lambda,k}$ to the first graded quotient of the J-filtration for each 1-dimensional cusp J. Then we apply the classical vanishing theorem of scalar-valued Jacobi forms (Proposition 7.19) to $\mathrm{Gr}^{-\lambda_1}\mathcal{E}_{\lambda,k}$. This tells us that when $k - \lambda_1 < n/2 - 1$, the Fourier coefficients of a modular form at a 0-dimensional cusp $I \subset J$ are contained in a proper subspace of $V(I)_{\lambda,k}$. Finally, running J over all 1-dimensional cusps containing I, we find that the Fourier coefficients are zero.

The second step of this argument (and hence the bound in Theorem 9.1) could be improved for some specific (Γ, L) if a stronger vanishing theorem of classical Jacobi forms is available (cf. Remark 9.4). Theorem 9.1 would be a prototype in this direction.

Let us look at Theorem 9.1 in the cases $n = 3, 4$ under the accidental isomorphisms.

EXAMPLE 9.2. Let $n = 3$. Recall from Example 3.4 that the orthogonal weight $(\lambda, k) = ((d), k)$ corresponds to the $GL(2, \mathbb{C})$-weight $(\rho_1, \rho_2) = (k + d, k - d)$ for Siegel modular forms of genus 2. In this case, the bound in Theorem 9.1 is $k < d + 1/2$, namely $k \leq d$. This is rewritten as $\rho_2 \leq 0$. This is the same bound as the vanishing theorem of Freitag [**15**] and Weissauer [**47**] for Siegel modular forms of genus 2.

In the case of Siegel modular forms of genus 2, the idea to use Jacobi forms to derive a vanishing theorem of vector-valued modular forms seems to go back to Ibukiyama. See [**26**] Section 6 (and also [**27**] p.54). Our proof of Theorem 9.1 can be regarded as a generalization of the argument of Ibukiyama.

Example 9.3. Let $n = 4$. Recall from Example 3.5 that the orthogonal weight $(\lambda, k) = ((d), k)$ corresponds to the weight $(r, \rho \boxtimes \rho)$ with $r = k - d$ and $\rho = \mathrm{Sym}^d$ for Hermitian modular forms of degree 2. In this case, the bound in Theorem 9.1 is $k < d + 1$, i.e., $k \leq d$. Thus Theorem 9.1 says that there is no nonzero Hermitian modular form of degree 2 and weight $(r, \rho \boxtimes \rho)$ with $\rho = \mathrm{Sym}^d \neq 1$ when $r \leq 0$. Furthermore, our second vanishing theorem (Theorem 11.1 (1)) says that there is no nonzero cusp form when $r \leq 1$.

The rest of this chapter is as follows. In §9.1 we prove Theorem 9.1. In §9.2 we give an application of Theorem 9.1 to the vanishing of holomorphic tensors of small degree on the modular variety $\mathcal{F}(\Gamma)$.

9.1. Proof of Theorem 9.1

In this section we prove Theorem 9.1. Let $\lambda \neq 1, \det$ and assume that $k - \lambda_1 < n/2 - 1$. For a rank 2 primitive isotropic sublattice J of L, we denote by $F_J\mathcal{E}_{\lambda,k} = F_J^{-\lambda_1+1}\mathcal{E}_{\lambda,k}$ the level $-\lambda_1 + 1$ (= the first) sub vector bundle of $\mathcal{E}_{\lambda,k}$ in the J-filtration. Here we add J in the notation in order to indicate the cusp.

Step 1. *Every Jacobi form in* $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ *takes values in the sub vector bundle* $F_J\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ *of* $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$.

Proof. Recall from (8.22) that we have an embedding

$$\mathrm{Gr}^{-\lambda_1}(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) \hookrightarrow J_{k-\lambda_1,m}(\Gamma(J)_{\mathbb{Z}})^{\oplus \alpha(-\lambda_1)}.$$

Since $k - \lambda_1 < n/2 - 1$, we have $J_{k-\lambda_1,m}(\Gamma(J)_{\mathbb{Z}}) = 0$ by Proposition 7.19. Therefore $\mathrm{Gr}^{-\lambda_1}(J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})) = 0$, which means that every Jacobi form in $J_{\lambda,k,m}(\Gamma(J)_{\mathbb{Z}})$ takes values in $F_J\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$. □

Now let $f \in M_{\lambda,k}(\Gamma)$. We want to prove that $f = 0$. We fix a rank 1 primitive isotropic sublattice I of L and let $f = \sum_l a(l)q^l$ be the Fourier expansion of f at the I-cusp, where $a(l) \in V(I)_{\lambda,k}$. For a rank 2 primitive isotropic sublattice J of L containing I, we denote by $F_JV(I)_\lambda = F_J^{-\lambda_1+1}V(I)_\lambda$ the level $-\lambda_1 + 1$ subspace in the J-filtration (8.11) on $V(I)_\lambda$ and write

$$F_JV(I)_{\lambda,k} = F_JV(I)_\lambda \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k} \subset V(I)_{\lambda,k}.$$

Step 2. *Every Fourier coefficient* $a(l)$ *is contained in the subspace* $F_JV(I)_{\lambda,k}$ *of* $V(I)_{\lambda,k}$.

Proof. Let σ_J be the isotropic ray in $U(I)_{\mathbb{R}}$ corresponding to J. If $l \in \sigma_J$, then $a(l)$ appears as a Fourier coefficient of the restriction $f|_{\Delta_J}$ of f to Δ_J. By Lemma 6.2 and Proposition 8.10, we see that $a(l)$ is contained in $F_J^{\lambda_1}V(I)_{\lambda,k} \subset F_J V(I)_{\lambda,k}$.

Next let $l \notin \sigma_J$. Then $a(l)$ appears as a Fourier coefficient of the m-th Fourier-Jacobi coefficient ϕ_m of f for some $m > 0$ along the J-cusp (see §7.1). By Proposition 7.12, ϕ_m is a Jacobi form of weight (λ, k) and index m. By Step 1, ϕ_m as a section of $\mathcal{E}_{\lambda,k} \otimes \Theta_J^{\otimes m}$ takes values in the sub vector bundle $F_J\mathcal{E}_{\lambda,k}\otimes\Theta_J^{\otimes m}$. Since the I-trivialization over $\overline{\mathcal{X}(J)}$ sends $F_J\mathcal{E}_{\lambda,k}$ to $F_J V(I)_{\lambda,k} \otimes \mathcal{O}_{\overline{\mathcal{X}(J)}}$, this implies that the Jacobi form ϕ_m, regarded as a $V(I)_{\lambda,k}$-valued function on Δ_J via the (I, ω_J)-trivialization, takes values in the subspace $F_J V(I)_{\lambda,k}$ of $V(I)_{\lambda,k}$. It follows that its Fourier coefficients $a(l)$ are contained in $F_J V(I)_{\lambda,k}$. □

Step 3. *Every Fourier coefficient $a(l)$ is zero.*

Proof. Let $W = \bigcap_{J \supset I} F_J V(I)_\lambda$. By applying Step 2 to all $J \supset I$, we find that $a(l)$ is contained in $W \otimes (I_{\mathbb{C}}^{\vee})^{\otimes k}$. We shall prove that $W = 0$. Since $(J/I)_{\mathbb{Q}}$ runs over all isotropic lines in $V(I)_{\mathbb{Q}}$ in the definition of W and

$$F_{\gamma J} V(I)_\lambda = \gamma(F_J V(I)_\lambda)$$

for $\gamma \in \mathrm{O}(V(I)_{\mathbb{Q}})$, we see that W is an $\mathrm{O}(V(I)_{\mathbb{Q}})$-invariant subspace of $V(I)_\lambda$. Since $\mathrm{O}(V(I)_{\mathbb{Q}})$ is Zariski dense in $\mathrm{O}(V(I))$, we find that W is $\mathrm{O}(V(I))$-invariant. But $V(I)_\lambda$ is irreducible as a representation of $\mathrm{O}(V(I))$, so we have either $W = 0$ or $W = V(I)_\lambda$. Since $F_J V(I)_\lambda \neq V(I)_\lambda$ by Lemma 8.12, we have $W \neq V(I)_\lambda$. Therefore $W = 0$. This finishes the proof of Theorem 9.1. □

Remark 9.4. At least when V_λ remains irreducible as a representation of $\mathrm{SO}(n, \mathbb{C})$, it is also possible to replace the argument in Step 3 by an argument using the symmetry of the Fourier coefficients in Proposition 3.6 and the Zariski density of $\overline{\Gamma(I)_{\mathbb{Z}}}$ as in the proof of Proposition 3.7. This approach allows improvement of Theorem 9.1 when a stronger vanishing theorem of scalar-valued Jacobi forms holds for $\Gamma(J)_{\mathbb{Z}}$.

9.2. Vanishing of holomorphic tensors

In this section, as an application of Theorem 9.1, we deduce vanishing of holomorphic tensors of small degree on the modular variety $\mathcal{F}(\Gamma) = \Gamma\backslash\mathcal{D}$. To be more precise, let X be the regular locus of $\mathcal{F}(\Gamma)$. Sections of $(\Omega_X^1)^{\otimes k}$ are called *holomorphic tensors* on X. Among them, those which extend holomorphically over a smooth projective compactification of X are a birational invariant of $\mathcal{F}(\Gamma)$.

Theorem 9.5. *When* $0 < k < n/2 - 1$*, we have* $H^0(X, (\Omega_X^1)^{\otimes k}) = 0$*. In particular,* $H^0(\tilde{X}, (\Omega_{\tilde{X}}^1)^{\otimes k}) = 0$ *for any smooth projective model* $\tilde{X}$ *of* $\mathcal{F}(\Gamma)$*.*

Proof. Let $\pi \colon \mathcal{D} \to \mathcal{F}(\Gamma)$ be the projection. We can pullback sections of $(\Omega_X^1)^{\otimes k}$ to Γ-invariant sections of $(\Omega_{\pi^{-1}(X)}^1)^{\otimes k}$. They extend holomorphically over $\mathcal{D}$ because the complement of $\pi^{-1}(X)$ in $\mathcal{D}$ is of codimension ≥ 2. Hence we have an embedding

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow H^0(\mathcal{D}, (\Omega_{\mathcal{D}}^1)^{\otimes k})^{\Gamma}. \tag{9.1}$$

Recall from (2.5) that $\Omega_{\mathcal{D}}^1 \simeq \mathcal{E} \otimes \mathcal{L}$. If we denote by $\mathrm{St}^{\otimes k} = \bigoplus_\alpha V_{\lambda(\alpha)}$ the irreducible decomposition of $\mathrm{St}^{\otimes k}$, we thus obtain an embedding

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \bigoplus_\alpha M_{\lambda(\alpha),k}(\Gamma). \tag{9.2}$$

When $\lambda(\alpha) \neq 1, \det$, we have $M_{\lambda(\alpha),k}(\Gamma) = 0$ for $k < n/2$ by Theorem 9.1. The determinant character does not appear in the irreducible decomposition of $\mathrm{St}^{\otimes k}$ if $k < n$ ([**39**] Theorem 8.21). Finally, when $\lambda(\alpha) = 1$, we have $M_k(\Gamma) = 0$ for $0 < k < n/2 - 1$ as it is classically known. Therefore $H^0(X, (\Omega_X^1)^{\otimes k}) = 0$ when $0 < k < n/2 - 1$. □

We can also classify possible types of holomorphic tensors on X in the next few degrees $n/2 - 1 \leq k \leq n/2$.

Proposition 9.6. *We write* $N(k) = k!/2^{k/2}(k/2)!$ *when k is even.*
(1) Let $k = [(n-1)/2]$*. Then we have an embedding*

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \begin{cases} 0 & n \equiv 0, 3 \mod 4, \\ M_k(\Gamma)^{\oplus N(k)} & n \equiv 1, 2 \mod 4. \end{cases}$$

(2) Let $k = n/2$ *with n even. Then we have an embedding*

$$H^0(X, (\Omega_X^1)^{\otimes k}) \hookrightarrow \begin{cases} M_{\wedge^k,k}(\Gamma) & n \equiv 2 \mod 4, \\ M_{\wedge^k,k}(\Gamma) \oplus M_k(\Gamma)^{\oplus N(k)} & n \equiv 0 \mod 4. \end{cases}$$

The component $M_{\wedge^k,k}(\Gamma)$ *in (2) gives the holomorphic differential forms of degree* $k = n/2$*. The component* $M_k(\Gamma)^{\oplus N(k)}$ *in both (1) and (2) corresponds to the trivial summands in* $\mathrm{St}^{\otimes k}$*. In both (1) and (2), the embedding is an isomorphism when* $\langle \Gamma, -\mathrm{id} \rangle$ *contains no reflection.*

Proof. We keep the same notation as in the proof of Theorem 9.5.

(1) When $\lambda(\alpha) \neq 1, \det$, we still have $M_{\lambda(\alpha),k}(\Gamma) = 0$ for $k < n/2$ by Theorem 9.1. The determinant character does not appear too. By [**39**] Exercise 12.2, $\mathrm{St}^{\otimes k}$ does not contain the trivial representation when k is odd, while it occurs with multiplicity $N(k)$ when k is even.

(2) When $\lambda(\alpha) \neq \wedge^d$ with $0 \leq d \leq n$, we have $\lambda_1 \geq 2$ and so $M_{\lambda(\alpha),n/2}(\Gamma) = 0$ by Theorem 9.1. By [**39**] Theorem 8.21, the representations $\wedge^d$ with $d > n/2$ or $d \not\equiv n/2 \mod 2$ do not appear in $\mathrm{St}^{\otimes n/2}$, and $\wedge^{n/2}$ occurs with multiplicity 1. The multiplicity of the trivial summand is as before. It remains to consider $\wedge^d$ with $0 < d < n/2$ and $d \equiv n/2 \mod 2$. We apply our second vanishing theorem (Theorem 11.1 (2)). This says that $M_{\wedge^d,n/2}(\Gamma) = 0$ when $n/2 \leq n - d - 2$, namely $d \leq n/2 - 2$.

Finally, when $\langle \Gamma, -\mathrm{id} \rangle$ contains no reflection, the projection $\mathcal{D} \to \mathcal{F}(\Gamma)$ is unramified in codimension 1 by [**22**]. Then (9.1) and (9.2) are isomorphisms, and so the above embeddings are isomorphisms. □

REMARK 9.7. (1) The weight $k = [(n-1)/2]$ in Proposition 9.6 (1) is the so-called *singular weight* when n is even, and the *critical weight* when n is odd, for scalar-valued modular forms. Since $M_k(\Gamma) \neq 0$ in general for these weights, the bound in Theorem 9.5 is optimal as a general bound.

(2) Theorem 9.5 and Proposition 9.6 imply in particular vanishing of holomorphic differential forms of degree $< n/2$ on X. Via the extension theorem of Pommerening [**41**], this can also be deduced from the vanishing of the corresponding Hodge components in the L^2-cohomology (cf. [**4**]).

CHAPTER 10

Square integrability

Let L be a lattice of signature $(2,n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. In this chapter we study convergence of the Petersson inner product

$$\int_{\mathcal{F}(\Gamma)} (f,g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$$

for $f, g \in M_{\lambda,k}(\Gamma)$, where $(\ ,\ )_{\lambda,k}$ is the Petersson metric on the vector bundle $\mathcal{E}_{\lambda,k}$ and $\mathrm{vol}_{\mathcal{D}}$ is the invariant volume form on $\mathcal{D}$.

For $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ let $\bar{\lambda} = (\lambda_1 - \lambda_n, \cdots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$ be the associated highest weight for $\mathrm{SO}(n, \mathbb{C})$ (see §3.6.1). We denote by $|\bar{\lambda}|$ the sum of all components of $\bar{\lambda}$. Our results are summarized as follows.

Theorem 10.1. *Let $f, g \in M_{\lambda,k}(\Gamma)$ with $\lambda \neq 1, \det$.*

(1) If f is a cusp form, then $\int_{\mathcal{F}(\Gamma)} (f,g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty$.

(2) When $k \geq n + |\bar{\lambda}| - 1$, f is a cusp form if and only if $\int_{\mathcal{F}(\Gamma)} (f,f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty$.

(3) When $k \leq n - |\bar{\lambda}| - 2$, we always have $\int_{\mathcal{F}(\Gamma)} (f,g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} < \infty$.

See Remark 10.13 for the scalar-valued case. The assertion (1) should be more or less standard. The assertions (2) and (3) give a characterization of square integrability except in the range

$$n - |\bar{\lambda}| - 1 \ \leq\ k \ \leq\ n + |\bar{\lambda}| - 2. \tag{10.1}$$

The assertion (3) is in fact an intermediate step in the proof of our second vanishing theorem (Theorem 11.1), where we eventually prove that $M_{\lambda,k}(\Gamma) = 0$ when $k \leq n - |\bar{\lambda}| - 2$.

This chapter starts with defining the Petersson metrics on the Hodge bundles explicitly (§10.1) and calculating them over the tube domain (§10.2). In §10.3 we give some asymptotic estimates needed in the proof of Theorem 10.1. In §10.4 we prove Theorem 10.1.

10.1. Petersson metrics

In this section we explicitly define the Petersson metrics on the Hodge bundles $\mathcal{L}$ and $\mathcal{E}$, and hence on the automorphic vector bundles $\mathcal{E}_{\lambda,k}$.

We begin with $\mathcal{L}$. By the definition of $\mathcal{D}$, the Hermitian form $(\cdot, \bar{\cdot})$ on $L_{\mathbb{C}}$ is positive on the lines parametrized by $\mathcal{D}$. Thus restriction of this Hermitian form defines a Hermitian metric on each fiber of $\mathcal{L}$, and hence an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{L}$. We call it the *Petersson metric* on $\mathcal{L}$ and denote it by $(\, , \,)_{\mathcal{L}}$.

Next we consider $\mathcal{E}$. We first define the real part of $\mathcal{E}$. We write $\underline{L_{\mathbb{R}}}$ for the product real vector bundle $L_{\mathbb{R}} \times \mathcal{D}$, which we regard as a sub real vector bundle of $L_{\mathbb{C}} \otimes \mathcal{O}_{\mathcal{D}}$ in the natural way. Then we define a sub real vector bundle of $\underline{L_{\mathbb{R}}}$ by

$$\mathcal{E}_{\mathbb{R}} := \mathcal{L}^{\perp} \cap \underline{L_{\mathbb{R}}} = (\mathcal{L} \oplus \bar{\mathcal{L}})^{\perp} \cap \underline{L_{\mathbb{R}}}.$$

This is a real vector bundle of rank n. By the second expression, the fiber of $\mathcal{E}_{\mathbb{R}}$ over $[\omega] \in \mathcal{D}$ is the negative-definite subspace

$$\langle \mathrm{Re}(\omega), \mathrm{Im}(\omega) \rangle^{\perp} \cap L_{\mathbb{R}} \tag{10.2}$$

of $L_{\mathbb{R}}$ (cf. §2.1). The $\mathrm{O}^+(L_{\mathbb{R}})$-action on $\underline{L_{\mathbb{R}}}$ preserves the sub vector bundle $\mathcal{E}_{\mathbb{R}}$. The natural homomorphism

$$\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathcal{L}^{\perp} \to \mathcal{E}$$

gives an $\mathrm{O}^+(L_{\mathbb{R}})$-equivariant C^{∞}-isomorphism between $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathcal{E}$. This defines a real structure of $\mathcal{E}$.

By the description (10.2) of the fibers, the real vector bundle $\mathcal{E}_{\mathbb{R}}$ is naturally endowed with an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant negative-definite quadratic form. We take the (-1)-scaling to turn it to positive-definite. This is a Riemannian metric on $\mathcal{E}_{\mathbb{R}}$. It extends to a Hermitian metric on $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ in the usual way. (Explicitly, the Hermitian pairing between two vectors v, w is the quadratic pairing between v and $\bar{w}$.) Via the C^{∞}-isomorphism $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}$, we obtain an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}$. We call it the *Petersson metric* on $\mathcal{E}$ and denote it by $(\, , \,)_{\mathcal{E}}$.

The Petersson metric on $\mathcal{E}$ induces an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}^{\otimes d}$, and hence by restriction an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}_{\lambda}$ with $|\lambda| = d$. Taking the tensor product with the Petersson metric on $\mathcal{L}^{\otimes k}$, we obtain an $\mathrm{O}^+(L_{\mathbb{R}})$-invariant Hermitian metric on $\mathcal{E}_{\lambda,k}$. We call it the *Petersson metric* on $\mathcal{E}_{\lambda,k}$ and denote it by $(\, , \,)_{\lambda,k}$.

Remark 10.2. When L is the primitive integral cohomology of a lattice-polarized $K3$ surface X with period $[\omega] \in \mathcal{D}$, we have the identifications $\mathcal{L}_{[\omega]} = H^{2,0}(X)$, $\mathcal{E}_{\mathbb{R},[\omega]} = H^{1,1}_{\mathrm{prim}}(X, \mathbb{R})$, and $\mathcal{E}_{\mathbb{R},[\omega]} \otimes_{\mathbb{R}} \mathbb{C} \to \mathcal{E}_{[\omega]}$ is identified with $H^{1,1}_{\mathrm{prim}}(X, \mathbb{C}) \to H^{2,0}(X)^{\perp}/H^{2,0}(X)$. On $H^{2,0}(X)$ and $H^{1,1}_{\mathrm{prim}}(X, \mathbb{C})$ we have the *Hodge metrics* defined by $\int_X \alpha \wedge \bar{\beta}$ and $-\int_X \alpha \wedge \bar{\beta}$ respectively (see [**46**] §6.3.2). Thus the Petersson metrics on $\mathcal{L}$ and $\mathcal{E}$ are essentially the Hodge metrics in this geometric setting.

Let I be a rank 1 primitive isotropic sublattice of L. For a vector v of $V(I)_{\mathbb{R}} = (I^{\perp}/I)_{\mathbb{R}}$, let s_v be the section of $\mathcal{E}$ which corresponds to the constant section v of $V(I) \otimes \mathcal{O}_{\mathcal{D}}$ by the I-trivialization $V(I) \otimes \mathcal{O}_{\mathcal{D}} \simeq \mathcal{E}$. We compute the Hermitian pairing between these distinguished sections. We choose and fix a lift $V(I)_{\mathbb{R}} \hookrightarrow I^{\perp}_{\mathbb{R}}$ of $V(I)_{\mathbb{R}}$ and regard vectors of $V(I)_{\mathbb{R}}$ as vectors of $I^{\perp}_{\mathbb{R}} \subset L_{\mathbb{R}}$ in this way.

Lemma 10.3. *Let $v_1, v_2 \in V(I)_{\mathbb{R}}$. The pairing of the sections s_{v_1}, s_{v_2} of $\mathcal{E}$ with respect to the Petersson metric $(\, , \,)_{\mathcal{E}}$ is given by*

$$(s_{v_1}([\omega]), s_{v_2}([\omega]))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(\omega)) \cdot (v_2, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))}$$

for $[\omega] \in \mathcal{D}$. In the right hand side, $(\, , \,)$ is the quadratic form on $L_{\mathbb{R}}$, and ω is normalized so as to have real pairing with $I_{\mathbb{R}}$. In particular, $(s_{v_1}, s_{v_2})_{\mathcal{E}}$ is $\mathbb{R}$-valued.

Proof. Let $[\omega] \in \mathcal{D}$. We choose a nonzero vector $l \in I$. We may normalize ω so that $(l, \omega) = 1$. For $v \in V(I)_{\mathbb{R}} \subset I^{\perp}_{\mathbb{R}}$ we write

$$\alpha(v) = \frac{(v, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))} = \frac{(v, \mathrm{Im}(\omega))}{(\mathrm{Re}(\omega), \mathrm{Re}(\omega))} \quad \in \mathbb{R}$$

and define a vector of $L_{\mathbb{C}}$ by

$$s'_v([\omega]) = v - (v, \omega)l + \sqrt{-1}\alpha(v)\omega. \tag{10.3}$$

Claim 10.4. s'_v is a section of $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and is the image of s_v under the C^{∞}-isomorphism $\mathcal{E} \to \mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

We prove Claim 10.4. The conditions to be checked are

$$(\mathrm{Re}(s'_v([\omega])), \omega) = 0, \quad (\mathrm{Im}(s'_v([\omega])), \omega) = 0, \quad s'_v([\omega]) \in s_v([\omega]) + \mathbb{C}\omega.$$

Since $s_v([\omega]) = v - (v, \omega)l + \mathbb{C}\omega$ by Lemma 2.6, the last condition follows from the definition of s'_v. We check the first equality. Since

$$\mathrm{Re}(s'_v([\omega])) = v - (v, \mathrm{Re}(\omega))l - \alpha(v) \cdot \mathrm{Im}(\omega),$$

we see that

$$\begin{aligned} (\mathrm{Re}(s'_v([\omega])), \omega) &= (v, \omega) - (v, \mathrm{Re}(\omega)) - \sqrt{-1}\alpha(v)(\mathrm{Im}(\omega), \mathrm{Im}(\omega)) \\ &= (v, \omega) - (v, \mathrm{Re}(\omega)) - \sqrt{-1}(v, \mathrm{Im}(\omega)) \\ &= 0. \end{aligned}$$

In the first equality we used $(\mathrm{Re}(\omega), \mathrm{Im}(\omega)) = 0$. The equality $(\mathrm{Im}(s'_v([\omega])), \omega) = 0$ can be verified similarly. This proves Claim 10.4.

We return to the proof of Lemma 10.3. We take two vectors $v_1, v_2 \in V(I)_{\mathbb{R}}$. By definition, $(s_{v_1}([\omega]), s_{v_2}(\omega]))_{\mathcal{E}}$ is the pairing of $s'_{v_1}([\omega])$ and $s'_{v_2}([\omega])$ with respect to the Hermitian form on $\mathcal{E}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. This in turn is the pairing of the vectors $s'_{v_1}([\omega])$ and $\overline{s'_{v_2}([\omega])}$ of $L_{\mathbb{C}}$ with respect to the (-1)-scaling of the quadratic form on $L_{\mathbb{C}}$. By the expression (10.3) of $s'_v([\omega])$, we can calculate

$$\begin{aligned} & -(s_{v_1}([\omega]), s_{v_2}([\omega]))_{\mathcal{E}} \\ = \; & (v_1 - (v_1, \omega)l + \sqrt{-1}\alpha(v_1)\omega, \; v_2 - (v_2, \bar{\omega})l - \sqrt{-1}\alpha(v_2)\bar{\omega}) \\ = \; & (v_1, v_2) + \alpha(v_1)\alpha(v_2)(\omega, \bar{\omega}) - 2\alpha(v_1)(\mathrm{Im}(\omega), v_2) - 2\alpha(v_2)(\mathrm{Im}(\omega), v_1). \end{aligned}$$

Since we have

$$\begin{aligned} & \alpha(v_1)\alpha(v_2)(\omega, \bar{\omega}) \; = \; 2\alpha(v_1)(\mathrm{Im}(\omega), v_2) \; = \; 2\alpha(v_2)(\mathrm{Im}(\omega), v_1) \\ = \; & \frac{2\,(v_1, \mathrm{Im}(\omega))\,(v_2, \mathrm{Im}(\omega))}{(\mathrm{Im}(\omega), \mathrm{Im}(\omega))}, \end{aligned}$$

this proves Lemma 10.3. □

REMARK 10.5. By the expression (10.3), the imaginary part of $s'_v([\omega])$ is nonzero for general $[\omega]$. This shows that the real structure on $\mathcal{E} \simeq V(I) \otimes \mathcal{O}_{\mathcal{D}}$ given by $\mathcal{E}_{\mathbb{R}}$ is different from that given by $V(I)_{\mathbb{R}}$. Nevertheless, the Petersson metric on the real part given by $V(I)_{\mathbb{R}}$ is $\mathbb{R}$-valued by Lemma 10.3.

Let $\mathrm{vol}_{\mathcal{D}}$ be the invariant volume form on $\mathcal{D}$. The Petersson metric $(\ ,\ )_{\det,n}$ of weight $(\lambda, k) = (\det, n)$ gives an invariant metric on the canonical bundle $K_{\mathcal{D}} \simeq \mathcal{L}^{\otimes n} \otimes \det$, where det stands for the determinant character (cf. Example 2.2). This can be used to express $\mathrm{vol}_{\mathcal{D}}$ as follows. If Ω is an arbitrary nonzero vector of $(K_{\mathcal{D}})_{[\omega]}$ over a point $[\omega]$ of $\mathcal{D}$, the volume form $\mathrm{vol}_{\mathcal{D}}$ at $[\omega]$ is written as

$$\mathrm{vol}_{\mathcal{D}}([\omega]) = \frac{\Omega \wedge \bar{\Omega}}{(\Omega, \Omega)_{\det,n}} \tag{10.4}$$

up to a constant independent of $[\omega]$. Indeed, the right hand side does not depend on the choice of Ω, and the differential form of degree (n, n) on $\mathcal{D}$ defined by the right hand side is clearly $\mathrm{O}^+(L_{\mathbb{R}})$-invariant, so it should coincide with $\mathrm{vol}_{\mathcal{D}}$ up to constant.

10.2. Petersson metrics on the tube domain

Let I be a rank 1 primitive isotropic sublattice of L. We calculate the Petersson metrics on $\mathcal{L}, \mathcal{E}$ over the tube domain $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$. We choose a rank 1 isotropic sublattice $I' \subset L$ with $(I, I') \neq 0$. Recall that the choice of I' determines a tube domain realization $\mathcal{D} \to \mathcal{D}_I$. We take a generator l of I and identify $U(I)_{\mathbb{Q}} \simeq V(I)_{\mathbb{Q}}$ accordingly.

LEMMA 10.6. *On the tube domain $\mathcal{D}_I$ we have*

$$(s_l(Z), s_l(Z))_{\mathcal{L}} = 2(\mathrm{Im}(Z), \mathrm{Im}(Z)), \tag{10.5}$$

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(Z)) \cdot (v_2, \mathrm{Im}(Z))}{(\mathrm{Im}(Z), \mathrm{Im}(Z))}, \tag{10.6}$$

for $Z \in \mathcal{D}_I$. Here s_l is the section of $\mathcal{L}$ corresponding to the dual vector of l, v_1, v_2 are vectors of $V(I)_{\mathbb{R}}$, and $(\ ,\ )$ in the right hand sides are the natural quadratic form on $V(I)_{\mathbb{R}} \simeq U(I)_{\mathbb{R}}$.

Proof. We begin with $(\ ,\ )_{\mathcal{L}}$. We can view the section s_l over $\mathcal{D}_I$ as a function $\mathcal{D}_I \to L_{\mathbb{C}}$ which lifts the inverse $\mathcal{D}_I \to \mathcal{D}$ of the tube domain realization and satisfies $(s_l, l) \equiv 1$. Let l' be the vector of $I'_{\mathbb{Q}}$ with $(l, l') = 1$, and we identify $V(I)_{\mathbb{Q}}$ with $(I_{\mathbb{Q}} \oplus I'_{\mathbb{Q}})^{\perp}$. Then we can explicitly write s_l as

$$s_l(Z) = l' + Z - 2^{-1}(Z, Z)l \quad \in L_{\mathbb{C}}$$

for $Z \in \mathcal{D}_I \subset V(I)$. Thus we have

$$\begin{aligned} (s_l(Z), s_l(Z))_{\mathcal{L}} &= (s_l(Z), \overline{s_l(Z)}) = (Z, \bar{Z}) - (Z, Z)/2 - \overline{(Z, Z)}/2 \\ &= 2(\mathrm{Im}(Z), \mathrm{Im}(Z)). \end{aligned}$$

Next we calculate $(\ ,\ )_{\mathcal{E}}$. By Lemma 10.3, we have

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = -(v_1, v_2) + \frac{2 \cdot (v_1, \mathrm{Im}(s_l(Z))) \cdot (v_2, \mathrm{Im}(s_l(Z)))}{(\mathrm{Im}(s_l(Z)), \mathrm{Im}(s_l(Z)))}.$$

Since

$$\mathrm{Im}(s_l(Z)) = \mathrm{Im}(Z) - 2^{-1}\mathrm{Im}((Z, Z))l,$$

we see that

$$(\mathrm{Im}(s_l(Z)), \mathrm{Im}(s_l(Z))) = (\mathrm{Im}(Z), \mathrm{Im}(Z)), \quad (v_i, \mathrm{Im}(s_l(Z))) = (v_i, \mathrm{Im}(Z)).$$

This proves (10.6). □

At each point $Z \in \mathcal{D}_I$, the Petersson metric on $\mathcal{E}$ can be understood as follows. We take an $\mathbb{R}$-basis $v_1, \cdots, v_n$ of $V(I)_{\mathbb{R}}$ such that $v_1 \in \mathbb{R}\mathrm{Im}(Z)$ and $(v_i, \mathrm{Im}(Z)) = 0$ for $i > 1$. Then, by (10.6), we have

$$(s_{v_i}(Z), s_{v_j}(Z))_{\mathcal{E}} = \begin{cases} (v_1, v_1) & i = j = 1 \\ -(v_i, v_j) & i, j > 1 \\ 0 & i = 1, j > 1 \end{cases}$$

The right hand side can be seen as the positive-definite modification of the hyperbolic quadratic form on $V(I)_{\mathbb{R}}$ given by taking the (-1)-scaling of the negative-definite subspace $\mathrm{Im}(Z)^{\perp}$. The Petersson metric on $\mathcal{E}_Z \simeq V(I)$ is the Hermitian extension of this modified real metric on $V(I)_{\mathbb{R}}$ to $V(I)$.

Finally, we recall the expression of $\mathrm{vol}_{\mathcal{D}}$ over $\mathcal{D}_I$. Let vol_I be a flat volume form on $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$. Then, as it is well-known, we have

$$\mathrm{vol}_{\mathcal{D}} = (\mathrm{Im}(Z), \mathrm{Im}(Z))^{-n}\mathrm{vol}_I. \tag{10.7}$$

This can be seen by substituting $\Omega = s_l^{\otimes n} \otimes v_0$ in (10.4) and using (10.5), where v_0 is a nonzero vector of det. The section $s_l^{\otimes n} \otimes v_0$ of $\mathcal{L}^{\otimes n} \otimes \det$ corresponds to a flat canonical form on $\mathcal{D}_I \subset U(I)_{\mathbb{C}}$ by its $U(I)_{\mathbb{C}}$-invariance.

10.3. Asymptotic estimates on the tube domain

In this section we prepare some estimates of the Petersson metrics on $\mathcal{E}_{\lambda,k}$ over the tube domain $\mathcal{D}_I$. This will be a main ingredient in the proof of Theorem 10.1. We keep the setting of §10.2.

We choose an $\mathbb{R}$-basis $\{v_i\}_i$ of the real part $(V(I)_{\mathbb{R}})_\lambda$ of $V(I)_\lambda$. Then $\{v_i\}_i$ is also a $\mathbb{C}$-basis of $V(I)_\lambda$. Let s_i' be the section of $\mathcal{E}_\lambda$ corresponding to v_i via the I-trivialization $\mathcal{E}_\lambda \simeq V(I)_\lambda \otimes \mathcal{O}_{\mathcal{D}}$ and let $s_i = s_i' \otimes s_l^{\otimes k}$. Then $\{s_i\}_i$ is a frame of $\mathcal{E}_{\lambda,k}$ corresponding to a basis of $V(I)_{\lambda,k}$ by the I-trivialization. Accordingly, we express a section f of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D} \simeq \mathcal{D}_I$ as $f = \sum_i f_i s_i$ with f_i a scalar-valued holomorphic function on $\mathcal{D}_I$.

Lemma 10.7. *There exist real homogeneous polynomials $\{P_{ij}\}_{i,j}$ on $U(I)_{\mathbb{R}}$ of degree $\leq 2|\lambda|$ determined by the basis $\{v_i\}_i$ of $(V(I)_{\mathbb{R}})_\lambda$ such that*

$$(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} = \sum_{i,j} f_i\, \bar{g}_j \cdot P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|}\ \mathrm{vol}_I \tag{10.8}$$

for all sections $f = \sum_i f_i s_i$, $g = \sum_i g_i s_i$ of $\mathcal{E}_{\lambda,k}$ over $\mathcal{D}_I$. The matrix $(P_{ij}(\mathrm{Im}(Z)))_{i,j}$ is symmetric and positive-definite for $Z \in \mathcal{D}_I$.

Proof. The section s_i' is an $\mathbb{R}$-linear combination of $|\lambda|$-fold tensor products of the distinguished sections s_v of $\mathcal{E}$ associated to $v \in V(I)_{\mathbb{R}}$. (Recall that $V_\lambda \subset V^{\otimes|\lambda|}$.) The equation (10.6) can be written as

$$(s_{v_1}(Z), s_{v_2}(Z))_{\mathcal{E}} = \frac{-(v_1, v_2)(\mathrm{Im}(Z), \mathrm{Im}(Z)) + 2(v_1, \mathrm{Im}(Z))(v_2, \mathrm{Im}(Z))}{(\mathrm{Im}(Z), \mathrm{Im}(Z))}.$$

The numerator is a real homogeneous polynomial of $\mathrm{Im}(Z)$ of degree ≤ 2. Therefore the Petersson paring between s_i' and s_j' can be written as

$$(s_i'(Z), s_j'(Z))_\lambda = P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{-|\lambda|} \tag{10.9}$$

for a real homogeneous polynomial P_{ij} of $\mathrm{Im}(Z)$ of degree $\leq 2|\lambda|$. Together with (10.5) and (10.7), we obtain

$$(s_i(Z), s_j(Z))_{\lambda,k}\mathrm{vol}_{\mathcal{D}} = P_{ij}(\mathrm{Im}(Z)) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|}\ \mathrm{vol}_I.$$

This proves (10.8). Since the matrix $((s_i'(Z), s_j'(Z))_\lambda)_{i,j}$ is real symmetric and positive-definite, so is $(P_{ij}(\mathrm{Im}(Z)))_{i,j}$ by (10.9). □

Let Γ be a finite-index subgroup of $\mathrm{O}^+(L)$ and let $\mathcal{X}(I) = \mathcal{D}_I/U(I)_{\mathbb{Z}}$. We take a regular $\Gamma(I)_{\mathbb{Z}}$-admissible cone decomposition Σ_I of $\mathcal{C}_I^+ \subset U(I)_{\mathbb{R}}$ in the sense of §3.5.1. Let $\mathcal{X}(I)^{\Sigma_I}$ be the associated partial toroidal compactification of $\mathcal{X}(I)$. Let σ be a cone in Σ_I of dimension c. By the regularity of

Σ_I, we can write $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_c$ such that $v_1, \cdots, v_c$ is a part of a $\mathbb{Z}$-basis of $U(I)_{\mathbb{Z}}$, say $v_1, \cdots, v_n$. Let $l_1, \cdots, l_n \in U(I)_{\mathbb{Z}}^{\vee}$ be the dual basis of $v_1, \cdots, v_n$. Then $z_i = (l_i, Z)$, $1 \leq i \leq n$, are flat coordinates on $U(I)_{\mathbb{C}}$. We have

$$\mathrm{vol}_I = dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$$

up to constant. We write $q_i = e(z_i)$ for $1 \leq i \leq c$. Let Δ_σ be the boundary stratum of $\mathcal{X}(I)^{\Sigma_I}$ corresponding to the cone σ, and $\Delta_i = \Delta_{v_i}$ be the boundary divisor corresponding to the ray $\mathbb{R}_{\geq 0}v_i$. Then $q_1, \cdots, q_c, z_{c+1}, \cdots, z_n$ give local coordinates around Δ_σ. The divisor Δ_i is defined by $q_i = 0$, and Δ_σ is defined by $q_1 = \cdots = q_c = 0$. We write $q_i = r_i e(\theta_i)$ with $r_i = |q_i|$ and $0 \leq \theta_i < 1$. Then

$$\begin{aligned} \mathrm{vol}_I &= \frac{dq_1}{q_1} \wedge \frac{d\bar{q}_1}{\bar{q}_1} \wedge \cdots \wedge \frac{dq_c}{q_c} \wedge \frac{d\bar{q}_c}{\bar{q}_c} \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n \\ &= (r_1 \cdots r_c)^{-1} dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_c \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n \qquad (10.10) \end{aligned}$$

up to constant.

We want to give an asymptotic estimate of the right hand side of (10.8) as $q_1, \cdots, q_c \to 0$. We take an arbitrary base point $Z_0 \in \mathcal{D}_I$ and consider a flow of points of the form

$$Z = Z(t_1, \cdots, t_c) = Z_0 + \sqrt{-1}(t_1 v_1 + \cdots + t_c v_c), \qquad t_1, \cdots, t_c \to \infty. \qquad (10.11)$$

This flow converges to a point of Δ_σ as $t_1, \cdots, t_c \to \infty$, and every point of Δ_σ can be obtained in this way. Let $v_0 = \mathrm{Im}(Z_0)$. This is a vector in the positive cone C_I.

LEMMA 10.8. *The following asymptotic estimates hold as $t_1, \cdots, t_c \to \infty$.*

$$P_{ij}(\mathrm{Im}(Z)) = O((t_1 + \cdots + t_c)^{2|\lambda|}), \qquad (10.12)$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = O((t_1 + \cdots + t_c)^2), \qquad (10.13)$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z))^{-1} = O((t_1 + \cdots + t_c)^{-1}). \qquad (10.14)$$

PROOF. We have $\mathrm{Im}(Z) = v_0 + \sum_i t_i v_i$. Since P_{ij} is a real homogeneous polynomial of degree $\leq 2|\lambda|$ on $U(I)_{\mathbb{R}}$, we see that $P_{ij}(v_0 + \sum_i t_i v_i)$ is a real inhomogeneous polynomial of $t_1, \cdots, t_c$ of degree $\leq 2|\lambda|$. This implies (10.12). Next we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = (v_0, v_0) + 2\sum_i (v_0, v_i) t_i + 2\sum_{i \neq j} (v_i, v_j) t_i t_j + \sum_i (v_i, v_i) t_i^2.$$

The estimate (10.13) is obvious from this expression. Since $v_0 \in C_I$ and $v_1, \cdots, v_c \in \overline{C_I}$, all coefficients in the right hand side are nonnegative; possibly except for (v_i, v_i) with $i \geq 1$, they are furthermore positive. Therefore we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) > 2\sum_i (v_0, v_i)t_i > C \cdot \sum_i t_i$$

for some constant $C > 0$. This implies (10.14). □

LEMMA 10.9. *In a small neighborhood of an arbitrary point of Δ_σ, we have*

$$P_{ij}(\mathrm{Im}(Z)) = O((-\log r_1 \cdots r_c)^{2|\lambda|}), \tag{10.15}$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) = O((-\log r_1 \cdots r_c)^2), \tag{10.16}$$

$$(\mathrm{Im}(Z), \mathrm{Im}(Z))^{-1} = O((-\log r_1 \cdots r_c)^{-1}), \tag{10.17}$$

as $q_1, \cdots, q_c \to 0$.

PROOF. We consider the flow (10.11) with Z_0 varying over the range $\mathrm{Re}(Z_0) \in U(I)_{\mathbb{R}}/U(I)_{\mathbb{Z}}$ and $v_0 = \mathrm{Im}(Z_0)$ in a small neighborhood of an arbitrary point of C_I. Since

$$r_i = |q_i| = \exp(-2\pi(l_i, \mathrm{Im}(Z))) = \exp(-2\pi(l_i, v_0) - 2\pi t_i),$$

we have

$$t_i = -(2\pi)^{-1}\log r_i - (l_i, v_0). \tag{10.18}$$

The constant term $-(l_i, v_0)$ depends on $v_0 = \mathrm{Im}(Z_0)$ continuously. Therefore our assertions follow by substituting $t_i \sim -(2\pi)^{-1}\log r_i$ in the estimates in Lemma 10.8 and using $\log r_1 + \cdots + \log r_c = \log r_1 \cdots r_c$. □

Summing up the calculations so far, we obtain the following asymptotic estimate of $(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$.

PROPOSITION 10.10. *Let $f = \sum_i f_i s_i$ and $g = \sum_i g_i s_i$ be as in Lemma 10.7. In a small neighborhood of an arbitrary point of Δ_σ, we have*

$$\begin{aligned}(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} &= \sum_{i,j} f_i\, \bar{g}_j \cdot O((-\log r_1 \cdots r_c)^{\alpha}) \cdot (r_1 \cdots r_c)^{-1} \\ &\quad \times\ dr_1 \wedge \cdots \wedge dr_c \wedge d\theta_1 \wedge \cdots \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n\end{aligned}$$

as $q_1, \cdots, q_c \to 0$, where

$$\alpha = \begin{cases} 2k - 2n & k \geq n + |\lambda| \\ k - n + |\lambda| & k < n + |\lambda|. \end{cases}$$

Proof. By substituting (10.15) and (10.10) in the right hand side of (10.8), we obtain

$$\begin{aligned} & (f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} \\ = \; & \sum_{i,j} f_i \, \bar{g}_j \cdot O((-\log r_1 \cdots r_c)^{2|\lambda|}) \cdot (\mathrm{Im}(Z), \mathrm{Im}(Z))^{k-n-|\lambda|} \\ & \times \; (r_1 \cdots r_c)^{-1} \cdot dr_1 \wedge \cdots \wedge dr_c \wedge d\theta_1 \wedge \cdots \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n. \end{aligned}$$

Then, according to whether the power degree $k - n - |\lambda|$ of $(\mathrm{Im}(Z), \mathrm{Im}(Z))$ is nonnegative or negative, we use (10.16) and (10.17) respectively. □

Before going to §10.4, we recall the following exercise in calculus.

Lemma 10.11. *Let $m \in \mathbb{Z}$. The integral*

$$\lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/2} \frac{1}{(\log r)^m \cdot r} dr$$

converges if $m \geq 2$, and diverges if $m \leq 1$.

Proof. This can be seen from

$$\left(\frac{1}{(\log r)^{m-1}} \right)' = \frac{1-m}{(\log r)^m \cdot r}$$

when $m \neq 1$, and $(\log(-\log r))' = ((\log r) \cdot r)^{-1}$ when $m = 1$. □

10.4. Proof of Theorem 10.1

Now we prove Theorem 10.1. We begin with some reductions. For the proof of Theorem 10.1, there is no loss of generality even if we replace the given group Γ by a subgroup of finite index. Thus we may assume that Γ is neat. In particular, Γ is contained in $\mathrm{SO}^+(L)$. By Proposition 3.12 (1), when ${}^t\lambda_1 > n/2$, we have $\mathcal{E}_\lambda \simeq \mathcal{E}_{\bar{\lambda}}$ as $\mathrm{SO}^+(L_{\mathbb{R}})$-equivariant vector bundles. This isomorphism preserves the Petersson metrics up to constant by their uniqueness as $\mathrm{SO}^+(L_{\mathbb{R}})$-invariant Hermitian metrics. Thus we have a natural isomorphism $M_{\lambda,k}(\Gamma) \simeq M_{\bar{\lambda},k}(\Gamma)$ which preserves the Petersson inner product up to constant. Since the highest weight for the partition $\bar{\lambda}$ is $\bar{\lambda}$ itself, the assertions of Theorem 10.1 for weight (λ, k) follow from those for weight $(\bar{\lambda}, k)$. Thus we may assume that ${}^t\lambda_1 \leq n/2$.

We take a smooth toroidal compactification $\mathcal{F}(\Gamma)^\Sigma$ of $\mathcal{F}(\Gamma)$ where the fans Σ_I are regular. We take a subdivision of Σ_I as follows.

Lemma 10.12. *There exists a $\Gamma(I)_{\mathbb{Z}}$-admissible and regular subdivision Σ_I' of Σ_I such that every cone in Σ_I' contains at most one isotropic ray.*

Proof. We take representatives $\tau_1, \cdots, \tau_N$ of $\overline{\Gamma(I)_{\mathbb{Z}}}$-equivalence classes of 2-dimensional cones spanned by two isotropic rays. For each τ_a, we choose a rational vector from the interior of τ_a. This vector has positive norm, and the ray it generates divides τ_a. Letting $\overline{\Gamma(I)_{\mathbb{Z}}}$ act, we obtain a division of every 2-dimensional cone τ spanned by two isotropic rays. This is well-defined because $\overline{\Gamma(I)_{\mathbb{Z}}}$ is torsion-free and so acts on the set of such cones freely. The collection of these divisions is $\overline{\Gamma(I)_{\mathbb{Z}}}$-invariant.

The division of τ uniquely induces a division of every cone σ having τ as a face, because σ is simplicial. Explicitly, if $\sigma = \mathbb{R}_{\geq 0}v_1 + \cdots + \mathbb{R}_{\geq 0}v_c$, $\tau = \mathbb{R}_{\geq 0}v_1 + \mathbb{R}_{\geq 0}v_2$ and $v_0 \in \tau$ is the division vector, we add the wall $\mathbb{R}_{\geq 0}v_0 + \mathbb{R}_{\geq 0}v_3 + \cdots + \mathbb{R}_{\geq 0}v_c$. The collection of these new walls defines a $\overline{\Gamma(I)_{\mathbb{Z}}}$-invariant subdivision of the fan Σ_I such that every cone contains at most one isotropic ray. Taking its regular subdivision ([**2**] p.186), we obtain a desired subdivision. □

Thus our reduced situation is: Γ is neat, ${}^t\lambda_1 \leq n/2$ so that $\bar{\lambda} = \lambda$, and every cone in Σ_I contains at most one isotropic ray. (The last property will be used only in the proof of the assertion (3).)

The integral $\int_{\mathcal{F}(\Gamma)}(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges if for every boundary point x of $\mathcal{F}(\Gamma)^{\Sigma}$ there exists a neighborhood $U = U_x$ of x such that $\int_U(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ converges. Thus, for the proof of (1) and (3) of Theorem 10.1, it suffices to verify the convergence of the integral over U. Conversely, when $f = g$, if $\int_U(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges around some boundary point x, then $\int_{\mathcal{F}(\Gamma)}(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges because $(f, f)_{\lambda,k}$ is nonnegative, real-valued. Therefore, for the proof of (2) of Theorem 10.1, it suffices to show that the integral $\int_U(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges at some U when f is not a cusp form.

Recall that we have étale maps $\mathcal{X}(I)^{\Sigma_I} \to \mathcal{F}(\Gamma)^{\Sigma}$ and $\overline{\mathcal{X}(J)} \to \mathcal{F}(\Gamma)^{\Sigma}$ which give local charts around the boundary points of $\mathcal{F}(\Gamma)^{\Sigma}$. Moreover, we have an étale gluing map $\overline{\mathcal{X}(J)} \to \mathcal{X}(I)^{\Sigma_I}$ for $I \subset J$. Thus the problem is reduced to estimating $\int_U(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ for a small neighborhood U of a boundary point of $\mathcal{X}(I)^{\Sigma_I}$ over a 0-dimensional cusp I. We are thus in the situation of §10.3. In what follows, we use the notation in §10.3.

(1) We first prove the assertion (1) of Theorem 10.1. By Proposition 10.10, the local integral $\int_U(f, g)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ can be bounded from above by

$$\lim_{\varepsilon_1,\cdots,\varepsilon_c\to 0}\int_{\varepsilon_1}^{a_1}\cdots\int_{\varepsilon_c}^{a_c}\int_0^1\cdots\int_0^1\int_{U'} \sum_{i,j} f_i\,\bar{g}_j\cdot O((-\log r_1\cdots r_c)^N)\cdot(r_1\cdots r_c)^{-1} \times dr_1\wedge\cdots\wedge dr_c\wedge d\theta_1\wedge\cdots\wedge d\theta_c\wedge dz_{c+1}\wedge\cdots\wedge d\bar{z}_n$$

for some integer $N > 0$, where $a_1, \cdots, a_c > 0$ are small constants and U' is a small open set in Δ_σ with coordinates $z_{c+1}, \cdots, z_n$. If f is a cusp form, its components f_i vanish at the boundary divisors $\Delta_1, \cdots, \Delta_c$ by Lemma 3.9. Therefore we have $f_i = q_1 \cdots q_c \cdot O(1)$. Similarly we have $g_j = O(1)$. We also have $-\log r_1 \cdots r_c \leq \prod_{l=1}^{c}(-\log r_l)$. Then the above integral can be bounded from above by

$$\lim_{\varepsilon_1, \cdots, \varepsilon_c \to 0} \int_{\varepsilon_1}^{a_1} \cdots \int_{\varepsilon_c}^{a_c} \prod_{l=1}^{c} O((-\log r_l)^N)\, dr_1 \wedge \cdots \wedge dr_c.$$

This integral converges because $\int_\varepsilon^a (\log r)^N dr$ converges in $\varepsilon \to 0$. Hence $\int_U (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ converges if f is a cusp form. This proves the assertion (1) of Theorem 10.1.

(3) Next we prove the assertion (3) of Theorem 10.1. Let $k \leq n - |\lambda| - 2$. When σ has no isotropic ray, f and g vanish at the boundary divisors $\Delta_1, \cdots, \Delta_c$ by Lemma 3.9. (Recall our assumption $\lambda \neq 1, \det$.) Therefore we can give a similar (actually stronger) estimate as in the case (1) above, which implies that $\int_U (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ converges. We consider the case when σ has an isotropic ray, say $\mathbb{R}_{\geq 0} v_1$. Since other rays $\mathbb{R}_{\geq 0} v_2, \cdots, \mathbb{R}_{\geq 0} v_c$ are not isotropic by our assumption, we see from Lemma 3.9 that f and g vanish at $\Delta_2, \cdots, \Delta_c$. Therefore we have $f = q_2 \cdots q_c \cdot O(1)$ and $g = q_2 \cdots q_c \cdot O(1)$. By substituting these estimates in the second case of Proposition 10.10, we see that

$$\begin{aligned}(f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}} &= (r_2 \cdots r_c) \cdot O(1) \cdot O((-\log r_1 \cdots r_c)^{k-n+|\lambda|}) \cdot r_1^{-1} \\ &\quad \times dr_1 \wedge \cdots \wedge dr_c \wedge d\theta_1 \wedge \cdots \wedge d\theta_c \wedge dz_{c+1} \wedge \cdots \wedge d\bar{z}_n.\end{aligned}$$

We have $(-\log r_1 \cdots r_c)^{-1} \leq (-\log r_1)^{-1}$. Therefore $\int_U (f, g)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ can be bounded from above by

$$\lim_{\varepsilon_1 \to 0} \int_{\varepsilon_1}^{a_1} O((-\log r_1)^{k-n+|\lambda|} \cdot r_1^{-1})\, dr_1.$$

Since $k - n + |\lambda| \leq -2$ by the assumption, this integral converges by Lemma 10.11.

(2) Finally, we prove the assertion (2) of Theorem 10.1. When L has Witt index ≤ 1, we have $S_{\lambda,k}(\Gamma) = M_{\lambda,k}(\Gamma)$ by Proposition 3.7. Thus we may assume that L has Witt index 2. Let $k \geq n + |\lambda| - 1$ and assume that f is not a cusp form. Then f does not vanish identically at a boundary divisor $\Delta = \Delta_\sigma$ corresponding to an isotropic ray $\sigma = \mathbb{R}_{\geq 0} v$ for some 0-dimensional cusp I. We shall show that $\int_U (f, f)_{\lambda,k} \mathrm{vol}_{\mathcal{D}}$ diverges for a general point x of Δ. Thus we consider the case $c = 1$. We rewrite $q_1 = r_1 e(\theta_1)$ as $q = re(\theta)$, and also denote $Z' = (z_2, \cdots, z_c)$ which give local charts on Δ.

We go back to the flow $Z = Z_0 + \sqrt{-1}tv$ in §10.3. Then $P_{ij}(\mathrm{Im}(Z))$ is a real polynomial of t whose coefficients depend continuously on $v_0 = \mathrm{Im}(Z_0)$. Therefore, by substituting (10.18), we see that in a neighborhood of x,

$$P_{ij}(\mathrm{Im(Z)}) = Q_{ij}(\log r)$$

for a real polynomial Q_{ij} of one variable whose coefficients depend continuously on Z'. Moreover, as in the proof of Lemma 10.8, we have

$$(\mathrm{Im}(Z), \mathrm{Im}(Z)) \ = \ (v_0, v_0) + 2(v_0, v)t \ \sim \ -C \log r$$

for some constant $C = C(Z') > 0$ depending continuously on Z'. Therefore, by the same calculation as in §10.3, we see that

$$(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} \geqslant \sum_{i,j} f_i \bar{f}_j \, Q_{ij}(\log r) \, (-\log r)^{k-n-|\lambda|} \, r^{-1} \, dr \wedge d\theta \wedge \cdots$$

as $r \to 0$.

We take the base change of the frame $(s_i)_i$ by a $\mathrm{GL}_N(\mathbb{C})$-valued holomorphic function $A = A(Z')$ of Z' around x so that $f_1 \to 1$ and $f_i \to 0$ for $i > 1$ as $r \to 0$. This is possible because $f \neq 0 \in V(I)_{\lambda,k}$ around x. Then the real symmetric matrix $Q = (Q_{ij})_{i,j}$ is replaced by the Hermitian matrix ${}^t\bar{A}QA$, which we denote by $Q' = (Q'_{ij})_{i,j}$. Each Q'_{ij} is a $\mathbb{C}$-polynomial of $\log r$ whose coefficients depend continuously on Z'. Since the Hermitian matrix Q' is positive-definite when r is small, we have in particular $Q'_{11} \neq 0$. Then

$$(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} \ \geqslant \ Q'_{11}(\log r) \, (-\log r)^{k-n-|\lambda|} \, r^{-1} \, dr \wedge d\theta \wedge \cdots$$

as $r \to 0$. Since Q'_{11} is a nonzero real polynomial and $k - n - |\lambda| \geq -1$ by our assumption, we obtain

$$(f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}} \ \geqslant \ (-\log r)^{-1} \, r^{-1} \, dr \wedge d\theta \wedge \cdots$$

as $r \to 0$. By Lemma 10.11, this implies that the integral $\int_U (f, f)_{\lambda,k}\mathrm{vol}_{\mathcal{D}}$ diverges. This completes the proof of Theorem 10.1. □

Remark 10.13. As the proof shows, the assertion (1) of Theorem 10.1 holds even when $\lambda = 1, \det$. Similarly, the assertion (2) holds also for $\lambda = 1, \det$ at least when L has Witt index 2. On the other hand, the proof of (3) makes use of Proposition 3.7, which requires $\lambda \neq 1, \det$.

CHAPTER 11

Vanishing theorem II

Let L be a lattice of signature $(2, n)$ with $n \geq 3$ and Γ be a finite-index subgroup of $\mathrm{O}^+(L)$. Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ be a partition expressing an irreducible representation of $\mathrm{O}(n, \mathbb{C})$. We assume $\lambda \neq 1, \det$. Therefore $\lambda_1 > 0$ and $\lambda_n = 0$. In this chapter we prove our second type of vanishing theorem. We define the *corank* of λ, denoted by $\mathrm{corank}(\lambda)$, as the maximal index $1 \leq i \leq [n/2]$ such that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_i \quad \text{and} \quad \lambda_n = \lambda_{n-1} = \cdots = \lambda_{n+1-i} = 0.$$

Let

$$\bar{\lambda} = (\bar{\lambda}_1, \cdots, \bar{\lambda}_{[n/2]}) = (\lambda_1 - \lambda_n, \lambda_2 - \lambda_{n-1}, \cdots, \lambda_{[n/2]} - \lambda_{n+1-[n/2]})$$

be the highest weight for $\mathrm{SO}(n, \mathbb{C})$ associated to λ. Then $\mathrm{corank}(\lambda)$ is the maximal index i such that $\bar{\lambda}_1 = \bar{\lambda}_2 = \cdots = \bar{\lambda}_i$. Let $|\bar{\lambda}| = \sum_i \bar{\lambda}_i$ be as in §10.

Our second vanishing theorem is the following.

Theorem 11.1. *Let* $\lambda \neq 1, \det$. *If* $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$*, there is no nonzero square integrable modular form of weight* (λ, k)*. In particular,*
(1) $S_{\lambda,k}(\Gamma) = 0$ *when* $k < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$.
(2) $M_{\lambda,k}(\Gamma) = 0$ *when* $k < n - |\bar{\lambda}| - 1$.

We compare Theorem 11.1 and Theorem 9.1. The bound $n/2 + \lambda_1 - 1$ in Theorem 9.1 is smaller than the main bound $n + \lambda_1 - \mathrm{corank}(\lambda) - 1$ in Theorem 11.1 because $\mathrm{corank}(\lambda) \leq [n/2]$. However, Theorem 11.1 is about square integrable modular forms, while Theorem 9.1 is about the whole $M_{\lambda,k}(\Gamma)$, so Theorem 11.1 does not supersede Theorem 9.1. The comparison of Theorem 11.1 (1) and Theorem 9.1 raises the question if we have convergent Eisenstein series in the range

$$n/2 + \lambda_1 - 1 \;\leq\; k \;<\; n + \lambda_1 - \mathrm{corank}(\lambda) - 1.$$

As for the comparison of Theorem 11.1 (2) and Theorem 9.1, it depends on λ which $n - |\bar{\lambda}| - 1$ or $n/2 - 1 + \lambda_1$ is larger. Roughly speaking, Theorem 11.1 (2) is stronger when $|\bar{\lambda}|$ is small, while Theorem 9.1 is stronger when λ_1 is large. Thus Theorem 11.1 and Theorem 9.1 are rather complementary.

The proof of Theorem 11.1 follows the same strategy as Weissauer's vanishing theorem for vector-valued Siegel modular forms ([**47**]). If we have a square integrable modular form $f \neq 0$, we can construct a unitary highest weight module for the Lie algebra of $\mathrm{SO}^+(L_{\mathbb{R}})$ by a standard procedure (cf. [**24**], [**47**] for the Siegel case). By computing its weight and comparing it with the classification of unitarizable highest weight modules ([**13**], [**12**], [**29**]), we obtain the bound $k \geq n + \lambda_1 - \mathrm{corank}(\lambda) - 1$. The more specific assertions (1), (2) in Theorem 11.1 are derived from combination with Theorem 10.1.

The rest of this chapter is devoted to the proof of Theorem 11.1. The construction of highest weight module occupies §11.1 and §11.2, and the concluding step is done in §11.3.

11.1. Lifting to the Lie group

In this section we work with $G = \mathrm{SO}^+(L_{\mathbb{R}})$. We lift a square integrable modular form on $\mathcal{D}$ to a square integrable function on G in a standard way. We choose a base point $[\omega_0] \in \mathcal{D}$. Let $K \simeq \mathrm{SO}(2,\mathbb{R}) \times \mathrm{SO}(n,\mathbb{R})$ be the stabilizer of $[\omega_0]$ in G. We denote by $\mathfrak{g}, \mathfrak{k}$ the Lie algebras of G, K respectively. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$ with respect to $\mathfrak{k}$, and $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ be the eigendecomposition for the adjoint action of $\mathfrak{so}(2,\mathbb{R}) \subset \mathfrak{k}$ on $\mathfrak{p}$. Then $\mathfrak{p}$ is identified with the real tangent space $T_{[\omega_0],\mathbb{R}}\mathcal{D}$ of $\mathcal{D}$ at $[\omega_0]$, and the decomposition $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ corresponds to the decomposition $T_{[\omega_0],\mathbb{R}}\mathcal{D} \otimes_{\mathbb{R}} \mathbb{C} = T^{1,0}_{[\omega_0]}\mathcal{D} \oplus T^{0,1}_{[\omega_0]}\mathcal{D}$. For each point $[\omega] = g([\omega_0])$ of $\mathcal{D}$, the g-action gives an isomorphism $\mathfrak{p}_- \to T^{0,1}_{[\omega]}\mathcal{D}$. This isomorphism is unique up to the adjoint action of K.

The Lie group $P_- = \exp(\mathfrak{p}_-)$ is abelian and is the unipotent radical of the stabilizer of $[\omega_0]$ in $\mathrm{SO}(L_{\mathbb{C}})$ (see, e.g., [**2**] pp.107–108). Therefore, in view of (1.3), P_- coincides with the group of Eichler transvections of $L_{\mathbb{C}}$ with respect to the isotropic line $\mathbb{C}\omega_0$. In particular, P_- acts trivially on $\mathbb{C}\omega_0 = \mathcal{L}_{[\omega_0]}$ and $\omega_0^{\perp}/\mathbb{C}\omega_0 = \mathcal{E}_{[\omega_0]}$. We will use this property in the proof of Claim 11.3 (3) below.

Now let λ be a partition for $\mathrm{O}(n,\mathbb{C})$ and $\bar{\lambda}$ be the associated highest weight for $\mathrm{SO}(n,\mathbb{C})$. To start with $\mathrm{O}(n,\mathbb{C})$ is somewhat roundabout here, but this is for consistency with the formulation of Theorem 11.1 and eventually with other chapters. We first consider the case when V_λ remains irreducible as a representation of $\mathrm{SO}(n,\mathbb{C})$ (cf. §3.6.1). Let $W_{\bar{\lambda},k}$ be the finite-dimensional irreducible $\mathbb{C}$-representation of $K \simeq \mathrm{SO}(n,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$ with highest weight $(\bar{\lambda}, k)$.

LEMMA 11.2. *Assume that either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. Let $f \neq 0$ be a square integrable modular form of weight (λ, k) for a finite-index subgroup Γ of $\mathrm{SO}^+(L)$. Then there exists a smooth function $\phi_f \neq 0$ on G with the following properties.*

(1) $\phi_f \in L^2(\Gamma \backslash G)$.

(2) $\mathfrak{p}_- \cdot \phi_f = 0$. (Here $\mathfrak{g}$ acts on ϕ_f as the derivative of the right G-translations.)

(3) The linear subspace of $L^2(\Gamma\backslash G)$ spanned by the right K-translations of ϕ_f is finite-dimensional and is isomorphic to $W_{\bar{\lambda},k}^{\vee}$ as a K-representation.

Proof. We choose a rank 1 primitive isotropic sublattice I of L and let $j(g, [\omega])$ be the factor of automorphy associated to the I-trivialization $\mathcal{E}_{\lambda,k} \simeq V(I)_{\lambda,k} \otimes \mathcal{O}_{\mathcal{D}}$. The homomorphism

$$K \to \mathrm{End}(V(I)_{\lambda,k}), \qquad k \mapsto j(k, [\omega_0]), \tag{11.1}$$

defines a representation of K on $V(I)_{\lambda,k} \simeq (\mathcal{E}_{\lambda,k})_{[\omega_0]}$. This is irreducible of highest weight $(\bar{\lambda}, k)$ by our assumption on λ. The Petersson metric on $(\mathcal{E}_{\lambda,k})_{[\omega_0]}$ is K-invariant. Via the I-trivialization at $[\omega_0]$, this defines a K-invariant Hermitian metric on $V(I)_{\lambda,k}$. The induced constant Hermitian metric on the product vector bundle $G \times V(I)_{\lambda,k}$ over G corresponds to the Petersson metric on $\mathcal{E}_{\lambda,k}$ through the isomorphism

$$\mathcal{E}_{\lambda,k} \simeq G \times_K (\mathcal{E}_{\lambda,k})_{[\omega_0]} \simeq G \times_K V(I)_{\lambda,k}. \tag{11.2}$$

Via the I-trivialization we regard the modular form f as a $V(I)_{\lambda,k}$-valued holomorphic function on $\mathcal{D}$. We define a $V(I)_{\lambda,k}$-valued smooth function $\tilde{f}$ on G by

$$\tilde{f}(g) = j(g, [\omega_0])^{-1} \cdot f(g([\omega_0])), \qquad g \in G.$$

This is the $V(I)_{\lambda,k}$-valued function on G that corresponds to the section f of $\mathcal{E}_{\lambda,k}$ via the G-equivariant isomorphism (11.2).

Claim 11.3. The $V(I)_{\lambda,k}$-valued function $\tilde{f}$ satisfies the following.

(1) $\tilde{f}(\gamma g) = \tilde{f}(g)$ for $\gamma \in \Gamma$.

(2) $\tilde{f}(gk) = k^{-1}(\tilde{f}(g))$ for $k \in K$, where k^{-1} acts on $V(I)_{\lambda,k}$ by (11.1).

(3) $\mathfrak{p}_- \cdot \tilde{f} = 0$.

(4) $\tilde{f}$ is square integrable over $\Gamma\backslash G$ with respect to the Haar measure on G and the Hermitian metric on $V(I)_{\lambda,k}$.

All these properties should be standard. We supply an argument for the sake of completeness (cf. [**24**] for the Siegel modular case). The property (1) follows from the Γ-invariance of f, and the property (2) is just the invariance of $\tilde{f}$ under the K-action on $G \times V(I)_{\lambda,k}$. Both (1) and (2) can also be checked directly by using the cocycle condition for $j(g, [\omega])$.

The property (4) holds because we have

$$\begin{aligned} \int_{\Gamma\backslash G} (\tilde{f}(g), \tilde{f}(g))\, d\mu_G &= \int_{\Gamma\backslash\mathcal{D}} \mathrm{vol}_{\mathcal{D}} \int_K (\tilde{f}(g), \tilde{f}(g))\, d\mu_K \\ &= \int_{\Gamma\backslash\mathcal{D}} (f, f)_{\lambda,k}\, \mathrm{vol}_{\mathcal{D}} \end{aligned}$$

up to constant, where $d\mu_G, d\mu_K$ are the Haar measures on G, K respectively, and $(\ ,\ )$ in the first line is the Hermitian metric on $V(I)_{\lambda,k}$.

Finally, we check the property (3). We have

$$X \cdot \tilde{f}(g) = (X \cdot j(g, [\omega_0])^{-1})\, f(g([\omega_0])) + j(g, [\omega_0])^{-1}\, (X \cdot f(g([\omega_0])))$$

for $X \in \mathfrak{p}_-$. Then $X \cdot f(g([\omega_0])) = 0$ by the holomorphicity of f. As for the first term, since P_- fixes $[\omega_0]$ and acts trivially on $(\mathcal{E}_{\lambda,k})_{[\omega_0]}$ as noticed before, we have

$$j(g \exp(tX), [\omega_0]) = j(g, \exp(tX)([\omega_0])) \circ j(\exp(tX), [\omega_0]) = j(g, [\omega_0]).$$

This shows that $X \cdot j(g, [\omega_0]) = 0$ and so

$$X \cdot j(g, [\omega_0])^{-1} = -j(g, \omega_0)^{-1} \circ (X \cdot j(g, [\omega_0])) \circ j(g, [\omega_0])^{-1} = 0.$$

Therefore $X \cdot \tilde{f} = 0$. Claim 11.3 is thus verified.

We go back to the proof of Lemma 11.2. The property (2) in Claim 11.3 means that $\tilde{f}$ as a vector of the K-representation

$$L^2(\Gamma\backslash G,\ V(I)_{\lambda,k}) \simeq L^2(\Gamma\backslash G) \otimes V(I)_{\lambda,k} \simeq L^2(\Gamma\backslash G) \otimes W_{\bar{\lambda},k}$$

is K-invariant. Therefore it corresponds to a nonzero K-homomorphism $\Phi_f : W_{\bar{\lambda},k}^{\vee} \to L^2(\Gamma\backslash G)$, which must be injective by the irreducibility of $W_{\bar{\lambda},k}^{\vee}$. The image of Φ_f consists of the scalar-valued functions $L \circ \tilde{f}$ for $L \in V(I)_{\lambda,k}^{\vee}$. By the irreducibility, the K-orbit of any such nonzero vector generates the image of Φ_f. Then we put $\phi_f = L \circ \tilde{f}$ for an arbitrary $L \neq 0$. The property (3) in Claim 11.3 implies the property (2) in Lemma 11.2. This finishes the proof of Lemma 11.2. □

11.2. Highest weight modules

In this section we construct from ϕ_f a unitary highest weight module of $\mathfrak{g}$. The result is summarized in Propositions 11.4 and 11.5.

First we recall the theory of highest weight modules following [**13**], [**12**], [**25**] and specialized to $G = \mathrm{SO}^+(L_{\mathbb{R}})$. Let $\mathfrak{k}_0 = \mathfrak{so}(2, \mathbb{R})$ and $\mathfrak{k}_1 = \mathfrak{so}(n, \mathbb{R})$. Then $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{k}_1$, $\mathfrak{k}_0$ is the center of $\mathfrak{k}$, and $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ is the semi-simple part of $\mathfrak{k}$. We take a maximal abelian subalgebra $\mathfrak{h}$ of $\mathfrak{k}$. Then $\mathfrak{h}_{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$. We have $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{h}_1$ with $\mathfrak{h}_1 = \mathfrak{h} \cap \mathfrak{k}_1$ being a maximal abelian subalgebra of $\mathfrak{k}_1$. We may take a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_{\mathbb{C}}$ constructed from the root data in $\mathfrak{h}_{\mathbb{C}}$ which is the direct sum of a Borel subalgebra of $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_-$ (rather than $\mathfrak{p}_+$).

Let $\tilde{\rho} \in \mathfrak{h}_{\mathbb{C}}^{\vee}$ be a weight which is dominant and integral with respect to $\mathfrak{k}_{\mathbb{C}}$ (rather than $\mathfrak{g}_{\mathbb{C}}$). According to the decomposition $\mathfrak{h} = \mathfrak{k}_0 \oplus \mathfrak{h}_1$, we can write

$$\tilde{\rho} = (\rho, \alpha), \qquad \rho \in (\mathfrak{h}_1)_{\mathbb{C}}^{\vee},\ \ \alpha \in (\mathfrak{k}_0)_{\mathbb{C}}^{\vee} \simeq \mathbb{C},$$

with ρ a dominant and integral weight for $(\mathfrak{k}_1)_{\mathbb{C}} = \mathfrak{so}(n, \mathbb{C})$. Here we identify $(\mathfrak{k}_0)^\vee_{\mathbb{C}} \simeq \mathbb{C}$ by the pairing with the unique maximal non-compact positive root. (In the notation of [**13**] §4, $\rho = (m_2, \cdots, m_n)$ and $\alpha = m_1$; in the notation of [**12**] §10 – §11, $\rho = (\lambda_2, \cdots, \lambda_n)$ and $\alpha = \lambda_1 + z$.) We denote by $\mathbb{C}_{\rho,\alpha}$ the 1-dimensional module of $\mathfrak{h}_{\mathbb{C}}$ of weight (ρ, α). We can regard $\mathbb{C}_{\rho,\alpha}$ as a module of $\mathfrak{b}$ naturally. We also denote by $W_{\rho,\alpha}$ the finite-dimensional irreducible module of $\mathfrak{k}_{\mathbb{C}}$ of highest weight (ρ, α). This is compatible with the notation in §11.1.

Let $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}})$ and $\mathfrak{U}(\mathfrak{b})$ be the universal enveloping algebras of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{b}$ respectively. Let

$$M(\rho, \alpha) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{b})} \mathbb{C}_{\rho,\alpha}$$

be the Verma module of $\mathfrak{g}_{\mathbb{C}}$ with highest weight (ρ, α). The module $M(\rho, \alpha)$ has a unique irreducible quotient $L(\rho, \alpha)$ (see [**25**] §1.3). This is called the *irreducible highest weight module* of $\mathfrak{g}_{\mathbb{C}}$ with highest weight (ρ, α). The module $L(\rho, \alpha)$ is also a unique irreducible quotient of the generalized (or parabolic) Verma module

$$N(\rho, \alpha) = \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-)} W_{\rho,\alpha},$$

because $N(\rho, \alpha)$ is also a quotient of $M(\rho, \alpha)$ (see [**25**] §9.4). The highest weight module $L(\rho, \alpha)$ is said to be *unitarizable* if it is isomorphic as a $\mathfrak{g}_{\mathbb{C}}$-module to the K-finite part of a unitary representation of G.

Now we go back to modular forms on $\mathcal{D}$.

Proposition 11.4. *Assume that either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. If we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$, then the irreducible highest weight module $L(\bar{\lambda}^\vee, -k)$ is unitarizable.*

Proof. Let V_f be the minimal Hilbert subspace of $L^2(\Gamma\backslash G)$ which contains the right G-translations of the function ϕ_f in Lemma 11.2. This is a sub unitary representation of $L^2(\Gamma\backslash G)$. The K-finite part $(V_f)_K$ of V_f is a $(\mathfrak{g}, K)$-module. Let V_0 be the subspace of $(V_f)_K$ generated by the right K-translations of ϕ_f. By Lemma 11.2 (3), V_0 is isomorphic to $W^\vee_{\bar{\lambda},k} = W_{\bar{\lambda}^\vee,-k}$ as a K-representation. By Lemma 11.2 (2), V_0 is annihilated by $\mathfrak{p}_-$. Indeed, for $X \in \mathfrak{p}_-$ and $k \in K$, we have

$$k^{-1} \cdot (X \cdot (k \cdot \phi_f)) = \mathrm{Ad}_{k^{-1}}(X) \cdot \phi_f = 0$$

because the adjoint action of K preserves $\mathfrak{p}_-$. Therefore the natural homomorphism $\mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathbb{C}} V_0 \twoheadrightarrow (V_f)_K$ descends to a surjective homomorphism

$$N(\bar{\lambda}^\vee, -k) \simeq \mathfrak{U}(\mathfrak{g}_{\mathbb{C}}) \otimes_{\mathfrak{U}(\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-)} V_0 \twoheadrightarrow (V_f)_K$$

from the generalized Verma module $N(\bar{\lambda}^\vee, -k)$. By the minimality of the quotient $L(\bar{\lambda}^\vee, -k)$, this in turn implies that there exists a surjective homomorphism

$$(V_f)_K \twoheadrightarrow L(\bar{\lambda}^\vee, -k).$$

Since $(V_f)_K$ is unitarizable, so is $L(\bar{\lambda}^\vee, -k)$. □

So far we have considered the case when V_λ remains irreducible as an $\mathrm{SO}(n,\mathbb{C})$-representation. It remains to consider the exceptional case $n = 2m$, ${}^t\lambda_1 = m$ where V_λ gets reducible. In that case, Proposition 11.4 is modified as follows. For a highest weight $\rho = (\rho_1, \cdots, \rho_m)$ for $\mathrm{SO}(2m,\mathbb{C})$, we write $\rho^\dagger = (\rho_1, \cdots, \rho_{m-1}, -\rho_m)$ as in §3.6.1.

Proposition 11.5. *Let $n = 2m$ be even and ${}^t\lambda_1 = m$. Suppose that we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$. Then either $L(\bar{\lambda}^\vee, -k)$ or $L((\bar{\lambda}^\dagger)^\vee, -k)$ is unitarizable.*

Proof. According to the decomposition of $\mathcal{E}_\lambda$ in Proposition 3.12 (2), we can write $f = (f_+, f_-)$ with f_+ of weight $(\bar{\lambda}, k)$ and f_- of weight $(\bar{\lambda}^\dagger, k)$ with respect to $\mathrm{SO}(n,\mathbb{R}) \times \mathrm{SO}(2,\mathbb{R})$. We have either $f_+ \neq 0$ or $f_- \neq 0$. Then we can do the same construction for the nonzero component $f_\pm$ as before, by using the component-wise I-trivialization (3.23). □

Finally, we recall the classification of unitarizable irreducible highest weight modules ([**12**], [**29**], [**13**]). For our purpose, we restrict ourselves to those weights (ρ, α) such that $\alpha \in \mathbb{Z}$ and ρ is a highest weight for $\mathrm{SO}(n,\mathbb{C})$ (rather than $\mathfrak{so}(n,\mathbb{C})$). In this situation, the version in [**13**] is convenient to use. For such a weight $\rho = (\rho_1, \cdots, \rho_{[n/2]})$, we denote by $\mathrm{corank}(\rho)$ the maximal index i such that $\rho_1 = \rho_2 = \cdots = \rho_{i-1} = |\rho_i|$.

Theorem 11.6 ([**13**], [**12**], [**29**]). *Let $\rho = (\rho_1, \cdots, \rho_{[n/2]})$ be a highest weight for $\mathrm{SO}(n,\mathbb{C})$. Assume that $\rho_1 \neq 0$, i.e., ρ nontrivial. Let $\alpha \in \mathbb{Z}$. Then the irreducible highest weight module $L(\rho, \alpha)$ is unitarizable if and only if $-\alpha \geq n + \rho_1 - \mathrm{corank}(\rho) - 1$.*

Here we follow [**13**] Theorem 4.2 and Theorem 4.3, with $\alpha = m_1$, $\rho = (m_2, \cdots, m_n)$ and $\mathrm{corank}(\rho) = i - 1$ in the notation there. A complete classification of unitary irreducible highest weight modules for general (ρ, α) (and also for other Lie groups) is given in [**12**] and [**29**]. For the proof of Theorem 11.1, we just use the "only if" part of Theorem 11.6.

Remark 11.7. In fact, the result of [**12**] tells us more than unitarizability. Let $\rho_1 > 0$. By the calculation of "the first reduction point" in [**12**] Lemma 10.3 and Lemma 11.3, we see that the generalized Verma module $N(\rho, \alpha)$ is already irreducible when $-\alpha > n + \rho_1 - \mathrm{corank}(\rho) - 1$. Thus $L(\rho, \alpha) = N(\rho, \alpha)$ in that case. Furthermore, according to [**12**] Theorem 2.4 (b), $L(\rho, \alpha)$ belongs to the holomorphic discrete series when $-\alpha > n+\rho_1-1$, and to the limit of holomorphic discrete series when $-\alpha = n + \rho_1 - 1$. Note that $\alpha = \lambda_1 + z$ in the notation of [**12**] §10 – §11, and this λ_1 corresponds to $-\rho_1 - n + 1$ in our notation, so z in [**12**] is $\alpha + n + \rho_1 - 1$ here.

11.3. Proof of Theorem 11.1

With the preliminaries in §11.1 and §11.2, we can now complete the proof of Theorem 11.1. Let $n \geq 3$ and $\lambda \neq 1, \det$. We first consider the case when either n is odd or $n = 2m$ is even with ${}^t\lambda_1 \neq m$. Suppose that we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$. Then the highest weight module $L(\bar{\lambda}^\vee, -k)$ is unitarizable by Proposition 11.4. By applying Theorem 11.6 to $(\rho, \alpha) = (\bar{\lambda}^\vee, -k)$, we see that (λ, k) must satisfy

$$k \geq n + (\bar{\lambda}^\vee)_1 - \mathrm{corank}(\bar{\lambda}^\vee) - 1.$$

Recall from §3.6.1 that $\bar{\lambda}^\vee = \bar{\lambda}$ in the case $n \not\equiv 2 \bmod 4$ and $\bar{\lambda}^\vee = \bar{\lambda}^\dagger$ in the case $n \equiv 2 \bmod 4$. Since $n \geq 3$, we have $\bar{\lambda}_1 = (\bar{\lambda}^\dagger)_1$ and so

$$(\bar{\lambda}^\vee)_1 = \bar{\lambda}_1 = \lambda_1 \tag{11.3}$$

in both cases. Since $\mathrm{corank}(\bar{\lambda}^\dagger) = \mathrm{corank}(\bar{\lambda})$, we also have

$$\mathrm{corank}(\bar{\lambda}^\vee) = \mathrm{corank}(\bar{\lambda}) = \mathrm{corank}(\lambda) \tag{11.4}$$

by the definition of $\mathrm{corank}(\lambda)$. (Note that all components of $\bar{\lambda}$ are nonnegative.) Hence (λ, k) satisfies the bound

$$k \geq n + \lambda_1 - \mathrm{corank}(\lambda) - 1. \tag{11.5}$$

This proves the main assertion of Theorem 11.1. The assertion (1) for $S_{\lambda,k}(\Gamma)$ is then a consequence of Theorem 10.1 (1). As for the assertion (2), we note that the inequality

$$n - |\bar{\lambda}| - 1 < n + \lambda_1 - \mathrm{corank}(\lambda) - 1$$

holds, because $\mathrm{corank}(\bar{\lambda}) \leq |\bar{\lambda}|$ and $\lambda_1 > 0$. Therefore, when $k < n - |\bar{\lambda}| - 1$, any modular form of weight (λ, k) is square integrable by Theorem 10.1 (3), but at the same time its weight violates the bound (11.5). This implies that $M_{\lambda,k}(\Gamma) = 0$ in this case.

Next we consider the exceptional case when $n = 2m$ is even and ${}^t\lambda_1 = m$. Note that $\bar{\lambda} = \lambda$ in this case. If we have a square integrable modular form $f \neq 0 \in M_{\lambda,k}(\Gamma)$, then either $L(\bar{\lambda}^\vee, -k)$ or $L((\bar{\lambda}^\dagger)^\vee, -k)$ is unitarizable by Proposition 11.5. Since $\lambda^\vee = \lambda$ or $\lambda^\dagger$, this means that either $L(\lambda, -k)$ or $L(\lambda^\dagger, -k)$ is unitarizable. By Theorem 11.6, we obtain the bound

$$k \geq \min(n + \lambda_1 - \mathrm{corank}(\lambda) - 1,\ n + (\lambda^\dagger)_1 - \mathrm{corank}(\lambda^\dagger) - 1).$$

Since $(\lambda^\dagger)_1 = \lambda_1$ and $\mathrm{corank}(\lambda^\dagger) = \mathrm{corank}(\lambda)$ as before, this reduces to the same bound as (11.5). The rest of the argument is similar to the non-exceptional case. This completes the proof of Theorem 11.1. □

Remark 11.8. Since Theorem 11.1 (2) is derived from Theorem 10.1 (3), this part could be improved if we could improve the characterization of square integrability in the remaining range (10.1).

Remark 11.9. Let $V_f \subset L^2(\Gamma\backslash G)$ be the unitary representation attached to a square integrable modular form $f \in M_{\lambda,k}(\Gamma)$, say in the non-exceptional case. Recall from the proof of Proposition 11.4 that

$$N(\bar{\lambda}^\vee, -k) \twoheadrightarrow (V_f)_K \twoheadrightarrow L(\bar{\lambda}^\vee, -k).$$

If we apply Remark 11.7 to $(\rho, \alpha) = (\bar{\lambda}^\vee, -k)$ and use (11.3) and (11.4), we find that

$$(V_f)_K \simeq L(\bar{\lambda}^\vee, -k) \simeq N(\bar{\lambda}^\vee, -k)$$

when $k \geq n + \lambda_1 - \operatorname{corank}(\lambda)$. The unitary representation V_f belongs to the holomorphic discrete series when $k \geq n+\lambda_1$, and to the limit of holomorphic discrete series when $k = n + \lambda_1 - 1$.

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