The Upper Bound for the Lebesgue Constant for Lagrange Interpolation in Equally Spaced Points of the Triangle

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An upper bound for the Lebesgue constant (the supremum norm) of the operator of interpolation of a function in equally spaced points of a triangle by a polynomial of total degree less than or equal to n is obtained. Earlier, the rate of increase of the Lebesgue constants with respect to n for an arbitrary d-dimensional simplex was established by the author. The explicit upper bound proved in this article refines this result for $d = 2$.

Keywords: multivariate polynomial interpolation, Lebesgue constant, the norm of the interpolation operator.

MSC: 65D05

1. Introduction

Let Δ be a non-degenerate simplex in $\mathbb{R}^d$ with vertices $\bar{a}_1, \dots, \bar{a}_{d+1}$. Any point $u \in \Delta$ can be written as

$$u = \sum_{r=1}^{d+1} \lambda_r \bar{a}_r, \quad \sum_{r=1}^{d+1} \lambda_r = 1, \quad \lambda_r \geq 0 \quad \text{for} \quad 1 \leq r \leq d+1.$$

The elements of the tuple $\lambda = (\lambda_1, \dots, \lambda_{d+1})$ are called the barycentric coordinates of the point u. Further we will identify u with $\lambda \overset{def}{=} (\lambda_1, \dots, \lambda_{d+1}) \in \Delta$. Let's denote

$$I = \left\{ i = (i_1, \dots, i_{d+1}) : \quad i_1, \dots, i_{d+1} \in \mathbb{Z}_+, \quad \sum_{r=1}^{d+1} i_r = n \right\},$$

where $\mathbb{Z}_+$ is the set of non-negative integers. Let $P_n^d = P_n^d[f] = P_n^d[f](u)$ be a polynomial of total degree less than or equal to n interpolating the values of the function $f \in C(\Delta)$ at equidistant nodes a_i of the simplex Δ , where

$$a_i = \left(\frac{i_1}{n}, \dots, \frac{i_{d+1}}{n} \right), \quad i = (i_1, \dots, i_{d+1}) \in I.$$

If we take the polynomials l_j of total degree less than or equal to n defined by the conditions

$$l_j(a_i) = \delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad i, j \in I,$$

then

$$P_n^d(u) = \sum_{i \in I} f(a_i) l_i(u).$$

The polynomials l_j are called Lagrange's fundamental interpolating polynomials.

Denote by $\mathcal{L}_n^d = \mathcal{L}_n^d(u) = \mathcal{L}_n^d(\lambda)$ the Lebesgue function of the Lagrange interpolation process at equally spaced nodes of the simplex, i.e., the norm of a functional in the space $C(\Delta)$ that assigns to each continuous function $f \in C(\Delta)$ the value of its interpolation polynomial at the point $u \in \Delta$. By Λ_n^d we denote the Lebesgue constant of this interpolation process, i.e., the norm of an operator from $C(\Delta)$ to $C(\Delta)$ which assigns to each continuous function its interpolation polynomial $P_n^d[f]$. Thus

$$\mathcal{L}_n^d(u) = \mathcal{L}_n^d(\lambda) = \sup_{\substack{f \in C(\Delta) \\ f \neq 0}} \frac{|P_n^d(u)|}{\|f\|_{C(\Delta)}}, \qquad \Lambda_n^d = \sup_{\substack{f \in C(\Delta) \\ f \neq 0}} \frac{\|P_n^d\|_{C(\Delta)}}{\|f\|_{C(\Delta)}}.$$

It is known that

$$\Lambda_n^d = \max_{u\in\Delta} \mathcal{L}_n^d(u) = \max_{\lambda\in\Delta} \mathcal{L}_n^d(\lambda), \tag{1}$$

$$\mathcal{L}_n^d(\lambda) = \sum_{i\in I} |l_i(\lambda)|. \tag{2}$$

For $d = 1$, in 1940 Turetskii [1] proved the following asymptotics of the Lebesgue constant:

$$\Lambda_n^1 = \frac{2^{n+1}}{e\ n\ln n}(1+\varepsilon_n), \quad \text{where} \quad \lim_{n\to\infty} \varepsilon_n = 0. \tag{3}$$

This result was also published in his book of 1968 [2]. In 1961, (3) was independently proved again by Schönhage [3] (more precisely, the formulation of the result in [3] is similar to the formula (3); however, a slightly stronger result is proved in [3], where the first terms of the asymptotic expansion of the Lebesgue constant Λ_n^1 for $n \to \infty$ were given). A review of other results can be found in the article by Trefethen and Weideman [4]. In 1992, Mills and Smith [5] obtained an asymptotic expansion for $\ln \Lambda_n^1$, which refines the result from [3]. In 2004, Eisinberg, Fedele, Franz [6] got a new asymptotic formula for Λ_n^1 and numerically demonstrated the advantage of the formula for $33 \le n \le 200$ in terms of the relative error.

In 1983, for arbitrary nonnegative integers d, n Bos [7] proved that

$$\Lambda_n^d \le \begin{pmatrix} 2n-1 \\ n \end{pmatrix} \asymp \frac{4^n}{\sqrt{n}}\ ; \quad \Lambda_n^d \to \begin{pmatrix} 2n-1 \\ n \end{pmatrix} \quad \text{if} \quad d\to\infty.$$

In 1988, Bloom [8] established that

$$\lim_{n\to\infty} \frac{\ln \Lambda_n^d}{n} = \ln 2,$$

and in 2005, the author [9] proved that

$$C_1 \frac{2^n}{n\ln n} \le \Lambda_n^d \le C_2(d)\frac{2^n}{n\ln n}\ , \tag{4}$$

where C_1 is some positive constant, and $C_2(d)$ may depend on d (but does not depend on n).

The purpose of this paper is to obtain an upper bound of Λ_n^d for $d = 2$ with preservation of the known rate of increase with respect to n, i.e. essentially to estimate the constant $C_2(d)$ in (4) for $d = 2$. Namely, for $n \ge 4$, we prove that

$$\Lambda_n^2 \le (7+\mu_n)\frac{2^{n+1}}{en(\ln n - \ln 2)}\left(1+\frac{15}{n-3}\right), \quad \text{where} \quad \mu_n \le \frac{4en(\ln n)^3}{2^n/3} + \frac{en^2\ln n}{2^n} \to 0 \quad \text{as} \quad n\to\infty.$$

The paper is organized as follows. In Sections $2-4$, some auxiliary results are given. In Sections $5-7$, the problem of estimating Λ_n^2 is reduced to the problem of estimating the function $\mathcal{L}_n^2(\lambda)$ for the case $0 \le \lambda_2, \lambda_3 \le 1/n$. In Section 8, the main theorem 2 is proved, which gives an upper bound for Λ_n^2. Everywhere we assume that $\sum\limits_{k=s}^{p} a_k = 0$ and $\prod\limits_{k=s}^{p} a_k = 1$ if $p < s$.

2. Lagrange's Fundamental Interpolating Polynomials

It is known that

$$l_i(\lambda) = \frac{\omega_i(\lambda)}{\omega_i(a_i)}\ , \quad i \in I,$$

where

$$\omega_i(\lambda) = \prod_{s_1=0}^{i_1-1}\left(\lambda_1 - \frac{s_1}{n}\right)\prod_{s_2=0}^{i_2-1}\left(\lambda_2 - \frac{s_2}{n}\right)\ldots\prod_{s_{d+1}=0}^{i_{d+1}-1}\left(\lambda_{d+1} - \frac{s_{d+1}}{n}\right) =$$

$$= \left(\frac{1}{n}\right)^n \frac{\Gamma(n\lambda_1+1)}{\Gamma(n\lambda_1-i_1+1)} \cdots \frac{\Gamma(n\lambda_{d+1}+1)}{\Gamma(n\lambda_{d+1}-i_{d+1}+1)}\,.$$

This explicit formula was given by Nicolaides [10]. Indeed, $\omega_i(a_j) = 0$ for $i \neq j$ since if $i, j \in I$, $i \neq j$, then there is $q \in \{1, \ldots, d+1\}$ such than $j_q < i_q$ and hence $\prod_{s_q=0}^{i_q-1}\left(\frac{j_q}{n} - \frac{s_q}{n}\right) = 0$. Since $\omega_i(a_i) = (1/n)^n\, \Gamma(i_1+1)\ldots\Gamma(i_{d+1}+1) = (1/n)^n\, i_1!i_2!\ldots i_{d+1}!$, then

$$l_i(\lambda) = \frac{\Gamma(n\lambda_1+1)\ldots\Gamma(n\lambda_{d+1}+1)}{\Gamma(n\lambda_1-i_1+1)\ldots\Gamma(n\lambda_{d+1}-i_{d+1}+1)} \cdot \frac{1}{i_1!i_2!\ldots i_{d+1}!}\,. \tag{5}$$

When obtaining estimates, we will exclude from consideration those arguments of Γ-functions that are poles, since these cases correspond to the situation when the corresponding summands $|l_i(\lambda)|$ from (2) are equal to zero.

3. Properties of the Lebesgue Function

Further, let $d = 2$, Δ is a triangle, $n \geq 4$, $\mathcal{L}_n = \mathcal{L}_n(\lambda) = \mathcal{L}_n^2(\lambda)$.

Lemma 1. *The Lebesgue function $\mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3)$ is a symmetric function of its arguments.*

Proof. Let's prove, for example, that $\mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3) = \mathcal{L}_n(\lambda_2, \lambda_1, \lambda_3)$. This follows from (2) and (5), because for any $i = (i_1, i_2, i_3) \in I$ there is $j = (j_1, j_2, j_3) = (i_2, i_1, i_3) \in I$ for which $l_i(\lambda_1, \lambda_2, \lambda_3) = l_j(\lambda_2, \lambda_1, \lambda_3)$. The other cases are proved in the same way. □

Lemma 2. $\Lambda_n = \max\limits_{\substack{\lambda_1 \geq \lambda_2 \geq 0 \\ \lambda_1 \geq \lambda_3 \geq 0 \\ 0 \leq \lambda_1+\lambda_2+\lambda_3 \leq 1}} \mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3).$

Proof. This equality follows from the fact that the function $\mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3)$ is symmetric. □

Note that if a point u with barycentric coordinates $(\lambda_1, \lambda_2, \lambda_3)$ belongs to Δ then there are numbers $r_1, r_2, r_3 \in \mathbb{Z}_+$, $r_1+r_2+r_3 = n-1$, $\alpha_1 \in [-1, 1]$, $\alpha_2, \alpha_3 \in [0, 1]$, $\alpha_1+\alpha_2+\alpha_3 = 1$, such that $n\lambda_s = r_s+\alpha_s$, $s = 1, 2, 3$ (see, for example, Fig. 1).

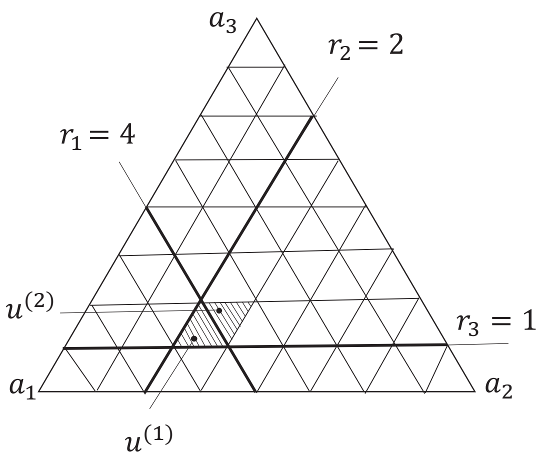

Fig. 1. Equally spaced nodes in the case $n = 8$. Here $r_1 = 4$, $r_2 = 2$, $r_3 = 1$, $0 < \alpha_1, \alpha_2, \alpha_3 < 1$ for $u^{(1)}$ or $r_1 = 4$, $r_2 = 2$, $r_3 = 1$, $-1 < \alpha_1 < 0$, $0 < \alpha_2, \alpha_3 < 1$ for $u^{(2)}$.

So, let $n\lambda_s = r_s + \alpha_s$, $r_1 + r_2 + r_3 = n - 1$, $-1 \leq \alpha_1 \leq 1$, $0 \leq \alpha_2, \alpha_3 \leq 1$. Denote $\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) = \mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3)$, and

$$a_{i_s}(\lambda_s) = \left|\frac{\Gamma(n\lambda_s+1)}{i_s!\Gamma(n\lambda_s-i_s+1)}\right| = \frac{\Gamma(r_s+\alpha_s+1)}{i_s!\,|\Gamma(r_s+\alpha_s-i_s+1)|}\,, \quad s = 1, 2, 3.$$

Also, let us define the sums $S_k = S_k(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3)$, $1 \le k \le 6$, as follows:

$$S_1 = \sum_{i_2=0}^{r_2} \sum_{i_3=0}^{r_3} |l_i(\lambda)|, \quad S_3 = \sum_{i_2=0}^{r_2} \sum_{i_3=0}^{r_1} |l_i(\lambda)|,$$

$$S_4 = \sum_{i_2=0}^{r_2-1} \sum_{i_3=r_3+1}^{n-r_1-1-i_2} |l_i(\lambda)|, \quad S_5 = \sum_{i_2=r_2+1}^{n-r_1-1} \sum_{i_3=0}^{n-i_2} |l_i(\lambda)|, \quad S_6 = \sum_{i_2=r_2+1}^{n-r_3-1} \sum_{i_1=0}^{n-r_3-1-i_2} |l_i(\lambda)|.$$

$$S_2 = S_{2,1} + S_{2,2} + S_{2,3},$$

where

$$S_{2,1} = \sum_{i_2=n-r_1}^{n-r_3} \sum_{i_1=n-r_3-i_2}^{n-i_2} |l_i(\lambda)|, \; S_{2,2} = \sum_{i_2=n-r_3+1}^{n} \sum_{i_1=0}^{n-i_2} |l_i(\lambda)|, \; S_{2,3} = \sum_{i_2=r_2+1}^{n-r_1-1} \sum_{i_1=n-r_3-i_2}^{r_1} |l_i(\lambda)|.$$

Then

$$|l_i(\lambda)| = \prod_{s=1}^{3} a_{i_s}(\lambda_s) = \prod_{s=1}^{3} \frac{\Gamma(r_s+\alpha_s+1)}{i_s! \, |\Gamma(r_s+\alpha_s-i_s+1)|} \,, \tag{6}$$

$$\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) = \mathcal{L}_n(\lambda_1, \lambda_2, \lambda_3) = \sum_{k=1}^{6} S_k. \tag{7}$$

Taking into account (1) and Lemma 2, we can state that in order to estimate Λ_n it is sufficient to estimate the maximum of the function $\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) = \mathcal{L}_n(\lambda_1, \lambda_2, \lambda)$ for $\lambda_2 \le \lambda_1$, $\lambda_3 \le \lambda_1$, so we further assume that

$$r_1 + \alpha_1 \ge r_2 + \alpha_2, \quad r_1 + \alpha_1 \ge r_3 + \alpha_3 \tag{8}$$

and we will estimate $\mathfrak{L}_n = \mathcal{L}_n$ under constraints (8). Denote

$$\delta_k = S_k(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) - S_k(r_1 - 1, r_2, r_3 + 1, \alpha_1, \alpha_2, \alpha_3), \quad 1 \le k \le 6. \tag{9}$$

The purpose of Section 5 is to estimate the values of δ_k under the constraints $r_1 - 1 + \alpha_1 \ge r_2 + \alpha_2$, $r_1 - 1 + \alpha_1 \ge r_3 + 1 + \alpha_3$, $-1 < \alpha_1 < 1$. Since $r_1, r_2, r_3 \in \mathbb{Z}_+$ then

$$r_1 \ge r_3 + 2 \ge 2.$$

4. Some Auxiliary Results

We will use the notation $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)}$.

Lemma 3. $$\frac{1}{b\begin{pmatrix} a-1 \\ b \end{pmatrix}} - \frac{1}{(b+1)\begin{pmatrix} a \\ b+1 \end{pmatrix}} = \frac{1}{b\begin{pmatrix} a \\ b \end{pmatrix}} .$$

Proof. The proof follows from the fact that

$$\frac{1}{(b+1)\begin{pmatrix} a \\ b+1 \end{pmatrix}} + \frac{1}{b\begin{pmatrix} a \\ b \end{pmatrix}} = \frac{\Gamma(b)\Gamma(a-b)}{\Gamma(a+1)}(b + a - b) = \frac{1}{b\begin{pmatrix} a-1 \\ b \end{pmatrix}} .$$

□

Lemma 4. *Let* $\mathcal{D}^* = \mathcal{D}^*(p,m,x,y) = \sum_{\kappa=0}^{p+1} \frac{\binom{x+1}{\kappa}}{y\binom{m-\kappa}{y}} - \sum_{\kappa=0}^{p} \frac{\binom{x}{\kappa}}{(y+1)\binom{m-\kappa}{y+1}}$. *Then*

$$\mathcal{D}^* = \sum_{\kappa=0}^{p} \frac{2\binom{x}{\kappa}}{y\binom{m-\kappa}{y}} + \frac{\binom{x}{p+1}}{y\binom{m-p-1}{y}} . \tag{10}$$

Proof. Let's make the following transformations:

$$\mathcal{D}^* = \sum_{\kappa=0}^{p+1} \frac{\binom{x}{\kappa}}{y\binom{m-\kappa}{y}} + \sum_{\kappa=1}^{p+1} \frac{\binom{x}{\kappa-1}}{y\binom{m-\kappa}{y}} - \sum_{\kappa=0}^{p} \frac{\binom{x}{\kappa}}{(y+1)\binom{m-\kappa}{y+1}} .$$

Let's change the variable $\kappa = \nu + 1$ in the second sum and use the lemma 3 for $a = m - \nu - 1$, $b = y$. Then

$$\mathcal{D}^* = \sum_{\kappa=0}^{p+1} \frac{\binom{x}{\kappa}}{y\binom{m-\kappa}{y}} + \left(\sum_{\nu=0}^{p} \frac{\binom{x}{\nu}}{y\binom{m-\nu-1}{y}} - \sum_{\kappa=0}^{p} \frac{\binom{x}{\kappa}}{(y+1)\binom{m-\kappa}{y+1}} \right)$$

$$= \sum_{\kappa=0}^{p+1} \frac{\binom{x}{\kappa}}{y\binom{m-\kappa}{y}} + \sum_{\kappa=0}^{p} \frac{\binom{x}{\kappa}}{y\binom{m-\kappa}{y}} ,$$

and (10) follows from this equality. ☐

Lemma 5. *Let*

$$\mathcal{D}^{**} = \mathcal{D}^{**}(m,q,r,x,y) = \sum_{\kappa=q}^{m-1-r} \frac{1}{(x+1)\binom{m-\kappa}{x+1} y\binom{\kappa}{y}} - \sum_{\kappa=q+1}^{m-r} \frac{1}{x\binom{m-\kappa}{x}(y+1)\binom{\kappa}{y+1}} .$$

Then

$$\mathcal{D}^{**} = \frac{1}{x\binom{r}{x} y\binom{m-r}{y}} - \frac{1}{x\binom{m-q}{x} y\binom{q}{y}} .$$

Proof. Using Lemma 3 with $a = m - \kappa$, $b = r + \alpha$ we have

$$\mathcal{D}^{**} = \sum_{\kappa=q}^{m-1-r} \frac{1}{x\binom{m-\kappa-1}{x} y\binom{\kappa}{y}} - \sum_{\kappa=q}^{m-1-r} \frac{1}{x\binom{m-\kappa}{x} y\binom{\kappa}{y}}$$

$$- \sum_{\kappa=q+1}^{m-r} \frac{1}{x\binom{m-\kappa}{x}(y+1)\binom{\kappa}{y+1}} .$$

In the first sum, we make the change of variable $\kappa = \nu - 1$. Add terms with the same κ from the second and third sums, leaving the term with $\kappa = q$ in the second sum and the term with $\kappa = m - r$ in the third sum unchanged. Then

$$\mathcal{D}^{**} = \sum_{\nu=q+1}^{m-r} \frac{1}{x\binom{m-\nu}{x} y\binom{\nu-1}{y}} - \sum_{\kappa=q+1}^{m-1-r} \frac{1}{x\binom{m-\kappa}{x} y\binom{\kappa-1}{y}}$$

$$- \frac{1}{x\binom{m-q}{x} y\binom{q}{y}} - \frac{1}{x\binom{r}{x}(y+1)\binom{m-r}{y+1}}$$

$$= \left(\frac{1}{x\binom{r}{x} y\binom{m-r-1}{y}} - \frac{1}{x\binom{r}{x}(y+1)\binom{m-r}{y+1}} \right) - \frac{1}{x\binom{m-q}{x} y\binom{q}{y}}$$

$$= \frac{1}{x\binom{r}{x} x\binom{m-r}{x}} - \frac{1}{x\binom{m-q}{x} y\binom{q+1}{y}}.$$

□

Lemma 6. *Let* $\mathcal{D}^{***} = \mathcal{D}^{***}(p, q, m, x, y) = \sum_{\kappa=q}^{p} \binom{x}{\kappa}\binom{y}{m-\kappa} - \sum_{\kappa=q-1}^{p-1} \binom{x-1}{\kappa}\binom{y+1}{m-\kappa}.$

Then

$$\mathcal{D}^{***} = \binom{x-1}{p}\binom{y}{m-p} - \binom{x-1}{q-1}\binom{y}{m-q+1}.$$

Proof. Notice that

$$D^{***} = \sum_{\kappa=q}^{p-1} \binom{x-1}{\kappa}\binom{y}{m-\kappa} + \sum_{\kappa=q}^{p-1} \binom{x-1}{\kappa-1}\binom{y}{m-\kappa} + \binom{x}{p}\binom{y}{m-p}$$

$$- \sum_{\kappa=q}^{p-1} \binom{x-1}{\kappa}\binom{y}{m-\kappa} - \sum_{\kappa=q}^{p-1} \binom{x-1}{\kappa}\binom{y}{m-\kappa-1} - \binom{x-1}{q-1}\binom{y+1}{m-q+1}.$$

The first sum in the first line and the first sum in the second line give 0, in the second sum of the first line we make the change of variable $\kappa = s + 1$. Then

$$\mathcal{D}^{***} = \sum_{s=q-1}^{p-2} \binom{x-1}{s}\binom{y}{m-s-1} - \sum_{\kappa=q}^{p-1} \binom{x-1}{\kappa}\binom{y}{m-\kappa-1} + \binom{x-1}{p}\binom{y}{m-p}$$

$$+ \binom{x-1}{p-1}\binom{y}{m-p} - \binom{x-1}{q-1}\binom{y}{m-q+1} - \binom{x-1}{q-1}\binom{y}{m-q}$$

$$= \sum_{s=q-1}^{p-1} \binom{x-1}{s}\binom{y}{m-s-1} - \sum_{\kappa=q-1}^{p-1} \binom{x-1}{\kappa}\binom{y}{m-\kappa-1} \binom{x-1}{p}\binom{y}{m-p}$$

$$- \binom{x-1}{q-1}\binom{y}{m-q+1} = \binom{x-1}{p}\binom{y}{m-p} - \binom{x-1}{q-1}\binom{y}{m-q+1}.$$

□

Lemma 7. *Let* $g(\alpha) = \dfrac{\Gamma(a+\alpha)}{\Gamma(b+\alpha)}$*. If* $a \geq b \geq \alpha$*, then the function* $g(\alpha)$ *is nondecreasing. If* $b \geq a \geq \alpha$*, then the function* $g(\alpha)$ *is nonincreasing.*

Proof. The assertion of the lemma follows from the fact that

$$g'(\alpha) = \frac{\Gamma(a+\alpha)\psi(a+\alpha)\Gamma(b+\alpha) - \Gamma(a+\alpha)\Gamma(b+\alpha)\psi(b+\alpha)}{(\Gamma(b+\alpha))^2}$$

$$= \frac{\Gamma(a+\alpha)\Gamma(b+\alpha)}{(\Gamma(b+\alpha))^2}\left(\psi(a+\alpha) - \psi(b+\alpha)\right),$$

where $\psi(t)$ is the digamma function. Since $\psi(t)$ increases on $(0, +\infty)$ then $g'(\alpha) \geq 0$ for $a \geq b$ and $g'(\alpha) \leq 0$ for $a \leq b$. □

5. Estimations of δ_s, $1 \leq \delta_s \leq 6$

Recall that we exclude from consideration those λ and i for which the argument of at least one of the Γ–functions in the denominators is its pole, since these cases correspond to the situation $l_i(\lambda) = 0$, i.e., the corresponding term in (2) is equal to zero. Also recall that we are considering the case $r_1 - 1 + \alpha_1 \geq r_2 + \alpha_2$, $r_1 - 1 + \alpha_1 \geq r_3 + 1 + \alpha_3$. In particular, $r_1 \geq r_3 + 2 \geq 2$. Everywhere below we assume that $i = (i_1, i_2, i_3) \in I$, i.e. $i_1 + i_2 + i_3 = n$.

Lemma 8. $\delta_1 \geq -2^{r_2+r_3+2} + 2^{r_3+1}$.

Proof. Due to Lemma 4 with $p = r_3$, $m = n - i_2$, $x = r_3 + \alpha_3$, $y = r_1 + \alpha_1$ we have

$$\delta_1 = \frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_3=0}^{r_3}\frac{\Gamma(r_1+\alpha_1+1)\Gamma(i_1-r_1-\alpha_1)}{i_1!}\,\frac{\Gamma(r_3+\alpha_3+1)}{i_3!\;\Gamma(r_3+\alpha_3+1-i_3)}$$

$$-\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_3=0}^{r_3+1}\frac{\Gamma(r_1+\alpha_1)\Gamma(i_1-r_1-\alpha_1+1)}{i_1!}\,\frac{\Gamma(r_3+\alpha_3+2)}{i_3!\;\Gamma(r_3+\alpha_3+2-i_3)}$$

$$= -\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\left(\sum_{i_3=0}^{r_3+1}\frac{\binom{r_3+1+\alpha_3}{i_3}}{(r_1+\alpha_1)\binom{i_1}{r_1+\alpha_1}} - \sum_{i_3=0}^{r_3}\frac{\binom{r_3+\alpha_3}{i_3}}{(r_1+\alpha_1+1)\binom{i_1}{r_1+\alpha_1+1}}\right)$$

$$= -\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\mathcal{D}^*(r_3, n-i_2, r_3+\alpha_3, r_1+\alpha_1)$$

$$= -\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\left(\sum_{i_3=0}^{r_3}\frac{2\binom{r_3+\alpha_3}{i_3}}{(r_1+\alpha_1)\binom{n-i_2-i_3}{r_1+\alpha_1}} + \frac{\binom{r_3+\alpha_3}{r_3+1}}{(r_1+\alpha_1)\binom{n-i_2-r_3-1}{r_1+\alpha_1}}\right).$$

Since $0 \leq i_2 \leq r_2$, $0 \leq i_3 \leq r_3$, $i_1 = n - i_2 - i_3 \geq n - r_2 - r_3 = r_1 + 1$, then all the terms in the obtained sums are non-negative and all the gamma functions in fractions $\dfrac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \dbinom{a}{b}$ are non-negative. Then

$$\delta_1 \geq -\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\left(\sum_{i_3=0}^{r_3}\frac{2\binom{r_3+1}{i_3}}{(r_1+\alpha_1)\binom{r_1+1}{r_1+\alpha_1}} + \frac{1}{(r_1+\alpha_1)\binom{r_1}{r_1+\alpha_1}}\right)$$

$$= -\frac{|\sin\pi\alpha_1|}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\left(\sum_{i_3=0}^{r_3}\frac{2\Gamma(r_1+\alpha_1)\Gamma(2-\alpha_1)}{(r_1+1)!}\binom{r_3+1}{i_3}+\frac{\Gamma(r_1+\alpha_1)\Gamma(1-\alpha_1)}{r_1!}\right)$$

$$\geq -\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_3=0}^{r_3}\frac{2(1-\alpha_1)}{(r_1+1)|\Gamma(\alpha_1)|}\binom{r_3+1}{i_3} - \sum_{i_2=0}^{r_2}\frac{a_{i_2}(\lambda_2)}{|\Gamma(\alpha_1)|} \geq -\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_3=0}^{r_3+1}\binom{r_3+1}{i_3}$$

$$= -\sum_{i_2=0}^{r_2}\frac{\Gamma(r_2+\alpha_2+1)}{i_2!\;\Gamma(r_2+\alpha_2+1-i_2)}2^{r_3+1} \geq -2^{r_3+1}\sum_{i_2=0}^{r_2}\frac{(r_2+1)!}{i_2!\;(r_2+1-i_2)!} \geq -2^{r_2+r_3+2}+2^{r_3+1}$$

(we used Lemma 7 in the last but one inequality). □

Lemma 9. $\delta_3 \geq -\frac{2^{r_2-1}}{r_1}$.

Proof. Due to Lemma 4 with $p - r_1 - 1$, $m - n - i_2$, $x = r_1 + \alpha_1 - 1$, $y = r_3 + \alpha_3 + 1$ we obtain

$$\delta_3 = \frac{\sin\pi\alpha_3}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_1=0}^{r_1}\frac{\Gamma(r_3+\alpha_3+1)\Gamma(i_3-r_3-\alpha_3)}{i_3!}\;\frac{\Gamma(r_1+\alpha_1+1)}{i_1!\;\Gamma(r_1+\alpha_1+1-i_1)}$$

$$-\frac{\sin\pi\alpha_3}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\sum_{i_1=0}^{r_1-1}\frac{\Gamma(r_3+\alpha_3+2)\Gamma(i_3-r_3-1-\alpha_3)}{i_3!}\;\frac{\Gamma(r_1+\alpha_1)}{i_1!\;\Gamma(r_1+\alpha_1-i_1)}$$

$$= \frac{\sin\pi\alpha_3}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\mathcal{D}^*(r_1-1, n-i_2, r_1+\alpha_1-1, r_3+\alpha_3+1)$$

$$= \frac{\sin\pi\alpha_3}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\left(\sum_{i_1=0}^{r_1-1}\frac{2\binom{r_1-1+\alpha_1}{i_1}}{(r_3+\alpha_3+1)\binom{n-i_2-i_1}{r_3+\alpha_3+1}}+\frac{\binom{r_1-1+\alpha_1}{r_1}}{(r_3+\alpha_3+1)\binom{n-i_2-r_1}{r_3+\alpha_3+1}}\right).$$

Let $\alpha_1 \geq 0$. Since $0 \leq i_1 \leq r_1 - 1$, $0 \leq i_2 \leq r_2$, $i_3 = n - i_1 - i_2 \geq n - r_1 + 1 - r_2 = r_3$, then then all terms in the bracketed sum are non-negative, because all the gamma functions in fractions $\frac{\Gamma(a+1)}{\Gamma(b+1)\Gamma(a-b+1)} = \binom{a}{b}$ are non-negative, i.e., $\delta_3 \geq 0$. If $\alpha_1 < 0$, then

$$\delta_3 \geq -\frac{\sin\pi\alpha_3}{\pi}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2)\frac{\Gamma(r_1+\alpha_1)}{\Gamma(r_1+1)|\Gamma(\alpha_1)|}\;\frac{\Gamma(r_3+\alpha_3+1)\Gamma(n-i_2-r_1-r_3-\alpha_3)}{\Gamma(n-i_2-r_1+1)}$$

$$\geq -\frac{1}{3\pi r_1}\sum_{i_2=0}^{r_2}\frac{\Gamma(r_2+\alpha_2+1)}{i_2!\Gamma(r_2+\alpha_2-i_2+1)}\;\frac{(r_3+1)!(r_2-i_2)!}{(r_2+r_3+1-i_2)!} \geq -\frac{1}{3\pi r_1}\sum_{i_2=0}^{r_2}\binom{r_2+1}{i_2} \geq -\frac{2^{r_2-1}}{r_1}.$$

□

Lemma 10. $\delta_4 \geq -\frac{2^{r_2-1}}{r_1}$.

Proof. Due to Lemma 5 with $m = n - i_2$, $q = r_3 + 1$, $r = r_1$, $x = r_1 + \alpha_1$, $y = r_3 + \alpha_3 + 1$ we obtain

$$\delta_4 = \frac{|\sin\pi\alpha_1|\sin\pi\alpha_3}{\pi^2}\sum_{i_2=0}^{r_2-1} a_{i_2}(\lambda_2)\left(\sum_{i_3=r_3+1}^{n-r_1-1-i_2}\frac{\Gamma(r_1+\alpha_1+1)\Gamma(i_1-r_1-\alpha_1)}{i_1!}\frac{\Gamma(r_3+\alpha_3+1)\Gamma(i_3-r_3-\alpha_3)}{i_3!}\right.$$

$$\left.-\sum_{i_3=r_3+2}^{n-r_1-i_2}\frac{\Gamma(r_1+\alpha_1)\Gamma(i_1-r_1+1-\alpha_1)}{i_1!}\;\frac{\Gamma(r_3+\alpha_3+2)\Gamma(i_3-r_3-1-\alpha_3)}{i_3!}\right)$$

$$= \frac{|\sin\pi\alpha_1|\sin\pi\alpha_3}{\pi^2}\sum_{i_2=0}^{r_2-1} a_{i_2}(\lambda_2)\mathcal{D}^{**}(n-i_2, r_3+1, r_1, r_1+\alpha_1, r_3+\alpha_3+1)$$

$$= \frac{|\sin\pi\alpha_1|\sin\pi\alpha_3}{\pi^2}\sum_{i_2=0}^{r_2-1} a_{i_2}(\lambda_2)\left(\frac{1}{(r_1+\alpha_1)\binom{r_1}{r_1+\alpha_1}(r_3+\alpha_3+1)\binom{n-i_2-r_1}{r_3+\alpha_3+1}}\right.$$

$$\left. - \frac{1}{(r_1+\alpha_1)\binom{n-i_2-r_3-1}{r_1+\alpha_1}(r_3+\alpha_3+1)\binom{r_3+1}{r_3+\alpha_3+1}}\right)$$

$$\geq -\frac{|\sin\pi\alpha_1|\sin\pi\alpha_3}{\pi^2}\sum_{i_2=0}^{r_2-1}\frac{a_{i_2}(\lambda_2)}{(r_1+\alpha_1)\binom{n-i_2-r_3-1}{r_1+\alpha_1}(r_3+\alpha_3+1)\binom{r_3+1}{r_3+\alpha_3+1}}$$

$$= -\frac{|\sin\pi\alpha_1|\sin\pi\alpha_3}{\pi^2}\sum_{i_2=0}^{r_2-1} a_{i_2}(\lambda_2)\ \frac{\Gamma(r_1+\alpha_1)\Gamma(n-i_2-r_3-r_1-\alpha_1)}{\Gamma(n-i_2-r_3)}\ \frac{\Gamma(r_3+\alpha_3+1)\Gamma(1-\alpha_3)}{\Gamma(r_3+2)}\ . \quad (11)$$

Note that

$$\frac{\sin\pi\alpha_3}{\pi}\ \frac{\Gamma(r_3+\alpha_3+1)\Gamma(1-\alpha_3)}{\Gamma(r_3+2)} = \frac{\Gamma(r_3+\alpha_3+1)}{\Gamma(r_3+2)\Gamma(\alpha_3)} \leq \Gamma(r_3+2)\Gamma(r_3+2)\Gamma(1) \leq 1. \quad (12)$$

Consider μ defined by

$$\mu \stackrel{def}{=} \frac{|\sin\pi\alpha_1|}{\pi}\ \frac{\Gamma(r_1+\alpha_1)\Gamma(n-i_2-r_3-r_1-\alpha_1)}{\Gamma(n-i_2-r_3)}$$

and make transformations

$$\mu = \frac{|\sin\pi\alpha_1|}{\pi}\ \frac{\Gamma(r_1+\alpha_1)\Gamma(r_2+1-i_2-\alpha_1)}{\Gamma(n-i_2-r_3)} = \frac{|\sin\pi\alpha_1|}{\pi}\ \frac{\Gamma(\alpha_1)\Gamma(1-\alpha_1)}{(n-i_2-r_3-1)!}\prod_{s=0}^{r_1-1}(s+\alpha_1)\prod_{\kappa=1}^{r_2-i_2}(\kappa-\alpha_1)$$

$$= \frac{(1+\alpha_1)|\alpha_1|(1-\alpha_1)}{(n-i_2-r_3-1)!}\prod_{s=2}^{r_1-1}(s+\alpha_1)\prod_{\kappa=2}^{r_2-i_2}(\kappa-\alpha_1) \leq \frac{1}{2(n-i_2-r_3-1)!}\prod_{s=2}^{r_1-1}(s+\alpha_1)\prod_{\kappa=2}^{r_2-i_2}(\kappa-\alpha_1).$$

If $i_2 = r_2 - 1$, then

$$\mu \leq \frac{(r_1+\alpha_1-1)\dots(2+\alpha_1)}{2(r_1+r_2-r_2+1)!} \leq \frac{r_1(r_1-1)\dots 3}{2(r_1+1)!} \leq \frac{1}{4r_1}\ . \quad (13)$$

If $i_2 \leq r_2 - 2$, then

$$\mu \leq \frac{3\ \dots\ (r_1+r_2-i_2-2)}{2(r_1+r_2-i_2)!} \leq \frac{1}{2r_1^2} \leq \frac{1}{4r_1}\ . \quad (14)$$

It follows from (11), (12), (13), (14) that

$$\delta_4 \geq -\frac{1}{4r_1}\sum_{i_2=0}^{r_2} a_{i_2}(\lambda_2) = -\frac{1}{4r_1}\sum_{i_2=0}^{r_2}\left|\frac{\Gamma(r_2+\alpha_2+1)}{i_2!\ \Gamma(r_2+\alpha_2+1-i_2)}\right| \geq -\frac{1}{4r_1}\sum_{i_2=0}^{r_2}\binom{r_2+1}{i_2} \geq -\frac{2^{r_2-1}}{r_1}.$$

□

Lemma 11. $\delta_6 \geq \sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1-1+\alpha_1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1}$.

Proof. Using Lemma 4 with $p = n - r_3 - 2 - i_2$, $m = n - i_2$, $x = r_1 + \alpha_1 - 1$, $y = r_3 + \alpha_3 + 1$ we obtain

$$\delta_6 = \sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2) \sum_{i_1=0}^{n-r_3-1-i_2} \frac{\Gamma(r_1+\alpha_1+1)}{i_1!\,|\Gamma(r_1+\alpha_1-i_1+1)|} \frac{\Gamma(r_3+\alpha_3+1)}{i_3!\,|\Gamma(r_3+\alpha_3-i_3+1)|}$$

$$- \sum_{i_2=r_2+1}^{n-r_3-2} a_{i_2}(\lambda_2) \sum_{i_1=0}^{n-r_3-2-i_2} \frac{\Gamma(r_1+\alpha_1)}{i_1!\,|\Gamma(r_1+\alpha_1-i_1)|} \frac{\Gamma(r_3+\alpha_3+2)}{i_3!\,|\Gamma(r_3+\alpha_3-i_3+2)|}$$

$$= \frac{\sin\pi\alpha_3}{\pi} \sum_{i_2=r_2+1}^{n-r_3-2} a_{i_2}(\lambda_2) \left(\sum_{i_1=0}^{n-r_3-1-i_2} \frac{\Gamma(r_1+\alpha_1+1)}{i_1!\Gamma(r_1+\alpha_1-i_1+1)} \frac{\Gamma(r_3+\alpha_3+1)\Gamma(i_3-r_3-\alpha_3)}{i_3!} \right.$$

$$\left. - \sum_{i_1=0}^{n-r_3-2-i_2} \frac{\Gamma(r_1+\alpha_1)}{i_1!\Gamma(r_1+\alpha_1-i_1)} \frac{\Gamma(r_3+\alpha_3+2)\Gamma(i_3-r_3-\alpha_3-1)}{i_3!} \right)$$

$$+ \frac{\sin\pi\alpha_3}{\pi} a_{n-r_3-1}(\lambda_2) \frac{\Gamma(r_1+\alpha_1+1)}{0!\ \Gamma(r_1+\alpha_1+1)} \frac{\Gamma(r_3+\alpha_3+1)\Gamma(1-\alpha_3)}{(r_3+1)!}$$

$$= \frac{\sin\pi\alpha_3}{\pi} \sum_{i_2=r_2+1}^{n-r_3-2} a_{i_2}(\lambda_2) \mathcal{D}^*(n-r_3-2-i_2, n-i_2, r_1+\alpha_1-1, r_3+\alpha_3+1)$$

$$+ \frac{\sin\pi\alpha_3}{\pi} \frac{a_{n-r_3-1}(\lambda_2)}{(r_3+1+\alpha_3)\binom{r_3+1}{r_3+1+\alpha_3}}$$

$$= \frac{\sin\pi\alpha_3}{\pi} \sum_{i_2=r_2+1}^{n-r_3-2} a_{i_2}(\lambda_2) \left(\sum_{i_1=0}^{n-r_3-i_2-2} \frac{2\binom{r_1-1+\alpha_1}{i_1}}{(r_3+1+\alpha_3)\binom{n-i_2-i_1}{r_3+1+\alpha_3}} \right.$$

$$\left. + \frac{\binom{r_1-1+\alpha_1}{n-r_3-i_2-1}}{(r_3+1+\alpha_3)\binom{r_3+1}{r_3+1+\alpha_3}} \right) + \frac{\sin\pi\alpha_3}{\pi} \frac{a_{n-r_3-1}(\lambda_2)}{(r_3+1+\alpha_3)\binom{r_3+1}{r_3+1+\alpha_3}}$$

$$= \frac{\sin\pi\alpha_3}{\pi} \sum_{i_2=r_2+1}^{n-r_3-2} a_{i_2}(\lambda_2) \sum_{i_1=0}^{n-r_3-i_2-2} \frac{2\binom{r_1-1+\alpha_1}{i_1}}{(r_3+1+\alpha_3)\binom{n-i_2-i_1}{r_3+1+\alpha_3}}$$

$$+ \sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2) \binom{r_1-1+\alpha_1}{n-r_3-i_2-1} \binom{r_3+\alpha_3}{r_3+1}.$$

Since all the terms in the first sum are non-negative, we obtain the assertion of the lemma. ☐

Lemma 12. $\delta_5 \geq -\sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \binom{r_3+1}{i_3} - \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \binom{r_3+\alpha_3}{n-r_1-i_2} \frac{\Gamma(r_1+\alpha_1)}{\Gamma(r_1+1)|\Gamma(\alpha_1)|}.$

Proof. Using Lemma 4 with $p = n - r_1 - i_2 - 1$, $m = n - i_2$, $x = r_3 + \alpha_3$, $y = r_1 + \alpha_1$ we obtain

$$\delta_5 = \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \frac{\Gamma(r_1+\alpha_1+1)}{i_1!\,|\Gamma(r_1+\alpha_1-i_1+1)|} \frac{\Gamma(r_3+\alpha_3+1)}{i_3!\,|\Gamma(r_3+\alpha_3-i_3+1)|}$$

$$
- \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-i_2} \frac{\Gamma(r_1+\alpha_1)}{i_1!\,|\Gamma(r_1+\alpha_1-i_1)|} \; \frac{\Gamma(r_3+\alpha_3+2)}{i_3!\,|\Gamma(r_3+\alpha_3-i_3+2)|}
$$

$$
= -\frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \Bigg(\sum_{i_3=0}^{n-r_1-i_2} \frac{\Gamma(r_1+\alpha_1)\Gamma(i_1-r_1+1-\alpha_1)}{i_1!} \; \frac{\Gamma(r_3+\alpha_3+2)}{i_3!\;\Gamma(r_3+\alpha_3-i_3+2)}
$$

$$
- \sum_{i_3=0}^{n-r_1-1-i_2} \frac{\Gamma(r_1+\alpha_1+1)\Gamma(i_1-r_1-\alpha_1)}{i_1!} \; \frac{\Gamma(r_3+\alpha_3+1)}{i_3!\;\Gamma(r_3+\alpha_3-i_3+1)} \Bigg)
$$

$$
- \frac{|\sin \pi\alpha_1|}{\pi} a_{n-r_1}(\lambda_2) \frac{\Gamma(r_1+\alpha_1)\Gamma(1-\alpha_1)}{r_1!} \; \frac{\Gamma(r_3+\alpha_3+2)}{0!\;\Gamma(r_3+\alpha_3+2)}
$$

$$
= -\frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \mathcal{D}^*(n-r_1-i_2-1, n-i_2, r_3+\alpha_3, r_1+\alpha_1)
$$

$$
- \frac{|\sin \pi\alpha_1|}{\pi} \; a_{n-r_1}(\lambda_2) \frac{\Gamma(r_1+\alpha_1)\Gamma(1-\alpha_1)}{r_1!}
$$

$$
= -\frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-i_2-1} \frac{2\begin{pmatrix} r_3+\alpha_3 \\ i_3 \end{pmatrix}}{(r_1+\alpha_1)\begin{pmatrix} n-i_2-i_3 \\ r_1+\alpha_1 \end{pmatrix}}
$$

$$
- \frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \frac{\begin{pmatrix} r_3+\alpha_3 \\ n-r_1-i_2 \end{pmatrix}}{(r_1+\alpha_1)\begin{pmatrix} r_1 \\ r_1+\alpha_1 \end{pmatrix}}.
$$

Now we can estimate $|\delta_5|$ as follows:

$$
|\delta_5| \le \frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \frac{2\begin{pmatrix} r_3+\alpha_3 \\ i_3 \end{pmatrix}}{(r_1+\alpha_1)\begin{pmatrix} r_1+1 \\ r_1+\alpha_1 \end{pmatrix}}
$$

$$
+ \frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \begin{pmatrix} r_3+\alpha_3 \\ n-r_1-i_2 \end{pmatrix} \frac{\Gamma(r_1+\alpha_1)\Gamma(1-\alpha_1)}{r_1!}
$$

$$
= \frac{|\sin \pi\alpha_1|}{\pi} \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} 2\begin{pmatrix} r_3+\alpha_3 \\ i_3 \end{pmatrix} \frac{\Gamma(r_1+\alpha_1)\Gamma(2-\alpha_1)}{\Gamma(r_1+2)}
$$

$$
+ \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \begin{pmatrix} r_3+\alpha_3 \\ n-r_1-i_2 \end{pmatrix} \frac{\Gamma(r_1+\alpha_1)}{\Gamma(r_1+1)|\Gamma(\alpha_1)|}
$$

$$
\le \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \begin{pmatrix} r_3+\alpha_3 \\ i_3 \end{pmatrix} \frac{2(1-\alpha_1)\Gamma(r_1+\alpha_1)}{(r_1+1)\Gamma(r_1+1)|\Gamma(\alpha_1)|}
$$

$$
+ \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \begin{pmatrix} r_3+\alpha_3 \\ n-r_1-i_2 \end{pmatrix} \frac{\Gamma(r_1+\alpha_1)}{\Gamma(r_1+1)|\Gamma(\alpha_1)|}
$$

$$
\le \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \begin{pmatrix} r_3+1 \\ i_3 \end{pmatrix} + \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \begin{pmatrix} r_3+\alpha_3 \\ n-r_1-i_2 \end{pmatrix} \frac{\Gamma(r_1+\alpha_1)}{\Gamma(r_1+1)|\Gamma(\alpha_1)|}.
$$

□

Lemma 13.

$$\delta_2 = -\sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1}+\sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}.$$

Proof. Denote

$$\delta_{2,k} = S_{2,k}(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) - S_{2,k}(r_1-1, r_2, r_3+1, \alpha_1, \alpha_2, \alpha_3), \quad 1 \le k \le 3.$$

Using Lemma 6 we obtain

$$\delta_{2,1} = \sum_{i_2=n-r_1}^{n-r_3} a_{i_2}(\lambda_2) \sum_{i_1=n-r_3-i_2}^{n-i_2} \binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$- \sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2) \sum_{i_1=n-r_3-1-i_2}^{n-i_2} \binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$= \sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2)\left(\sum_{i_1=n-r_3-i_2}^{n-i_2}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} - \sum_{i_1=n-r_3-1-i_2}^{n-i_2}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}\right)$$

$$+ a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} + a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$= \sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2)\left(D^{***}(r_1+\alpha_1, r_3+\alpha_3, n-i_2, n-r_3-i_2, n-i_2) - \binom{r_1+\alpha_1-1}{n-i_2}\right)$$

$$+ a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} + a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$= \sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2)\left(\binom{r_1+\alpha_1-1}{n-i_2} - \binom{r_1+\alpha_1-1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1} - \binom{r_1+\alpha_1-1}{n-i_2}\right)$$

$$+ a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} + a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$= -\sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1}$$

$$+ a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} + a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}.$$

Now we estumate $\delta_{2,2}$ and $\delta_{2,3}$:

$$\delta_{2,2} = \sum_{i_2=n-r_3+1}^{n} a_{i_2}(\lambda_2)\sum_{i_1=0}^{n-i_2}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3} - \sum_{i_2=n-r_3}^{n} a_{i_2}(\lambda_2)\sum_{i_1=0}^{n-i_2}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$= \sum_{i_2=n-r_3+1}^{n} a_{i_2}(\lambda_2)\left(\sum_{i_1=0}^{n-i_2}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3}{i_3} + \sum_{i_1=1}^{n-i_2}\binom{r_1+\alpha_1-1}{i_1-1}\binom{r_3+\alpha_3}{i_3}\right)$$

$$-\sum_{i_1=0}^{n-i_2}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3}{i_3}-\sum_{i_1=0}^{n-i_2-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3}{i_3-1}\Bigg)$$

$$-a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}=-a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3},$$

$$\delta_{2,3}=\sum_{i_2=r_2+1}^{n-r_1-1}a_{i_2}(\lambda_2)\sum_{i_1=n-r_3-i_2}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$-\sum_{i_2=r_2+1}^{n-r_1}a_{i_2}(\lambda_2)\sum_{i_1=n-r_3-i_2-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$=\sum_{i_2=r_2+1}^{n-r_1-1}a_{i_2}(\lambda_2)\Bigg(\sum_{i_1=n-r_3-i_2}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}-\sum_{i_1=n-r_3-i_2-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}\Bigg)$$

$$-a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$=\sum_{i_2=r_2+1}^{n-r_1-1}a_{i_2}(\lambda_2)D^{***}(r_1+\alpha_1,r_3+\alpha_3,r_1,n-r_3-i_2,n-i_2)$$

$$-a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$=\sum_{i_2=r_2+1}^{n-r_1-1}a_{i_2}(\lambda_2)\Bigg(\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}-\binom{r_1+\alpha_1-1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1}\Bigg)$$

$$-a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}.$$

Finally we have

$$\delta_2=\delta_{2,1}+\delta_{2,2}+\delta_{2,3}=-\sum_{i_2=n-r_1+1}^{n-r_3-1}a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1}$$

$$+a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}+a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{i_3}$$

$$-a_{n-r_3}(\lambda_2)\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$+\sum_{i_2=r_2+1}^{n-r_1-1}a_{i_2}(\lambda_2)\Bigg(\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}-\binom{r_1+\alpha_1-1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1}\Bigg)$$

$$-a_{n-r_1}(\lambda_2)\sum_{i_1=r_1-r_3-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{i_3}$$

$$= -\sum_{i_2=n-r_1+1}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1}$$

$$+ a_{n-r_1}(\lambda_2)\left(\sum_{i_1=r_1-r_3}^{r_1}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{r_1-i_1} - \sum_{i_1=r_1-r_3-1}^{r_1-1}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{r_1-i_1}\right)$$

$$+ a_{n-r_3}(\lambda_2)\left(\sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1}{i_1}\binom{r_3+\alpha_3}{r_3-i_1} - \sum_{i_1=0}^{r_3}\binom{r_1+\alpha_1-1}{i_1}\binom{r_3+\alpha_3+1}{r_3-i_1}\right)$$

$$+ \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2)\left(\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2} - \binom{r_1+\alpha_1-1}{n-r_3-i_2-1}\binom{r_3+\alpha_3}{r_3+1}\right)$$

$$= -\sum_{\substack{i_2=r_2+1\\ i_2\neq n-r_1}}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1}$$

$$+ a_{n-r_1}(\lambda_2)D^{***}(r_1+\alpha_1, r_3+\alpha_3, r_1, r_1-r_3, r_1)$$

$$+ a_{n-r_3}(\lambda_2)\left(\binom{r_3+\alpha_3}{r_3} - \binom{r_1+\alpha_1-1}{r_3} + D^{***}(r_1+\alpha_1, r_3+\alpha_3, r_3, 1, r_3)\right)$$

$$+ \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}$$

$$= -\sum_{\substack{i_2=r_2+1\\ i_2\neq n-r_1}}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1}$$

$$+ a_{n-r_1}(\lambda_2)\left(\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{0} - \binom{r_1+\alpha_1-1}{r_1-r_3-1}\binom{r_3+\alpha_3}{r_3+1}\right)$$

$$+ a_{n-r_3}(\lambda_2)\left(\binom{r_3+\alpha_3}{r_3} - \binom{r_1+\alpha_1-1}{r_3} + \binom{r_1+\alpha_1-1}{r_3} - \binom{r_3+\alpha_3}{r_3}\right)$$

$$+ \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}$$

$$= -\sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{n-r_3-1-i_2}\binom{r_3+\alpha_3}{r_3+1} + \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2)\binom{r_1+\alpha_1-1}{r_1}\binom{r_3+\alpha_3}{n-r_1-i_2}.$$

□

6. Reducing to the case $r_1 = n-1-r_2$, $r_3 = 0$

Lemma 14. *Let $r_s \in \mathbb{Z}_+$, $1 \le s \le 3$, $r_1+r_2+r_3 = n-1$, $-1 < \alpha_1 < 1$, $0 \le \alpha_2, \alpha_3 < 1$, $r_1 - 1 + \alpha_1 \ge r_2 + \alpha_2$, $r_1 - 1 + \alpha_1 \ge r_3 + 1 + \alpha_3$. Then*

$$\mathfrak{L}_n(r_1-1, r_2, r_3+1, \alpha_1, \alpha_2, \alpha_3) \le \mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) + 2^{r_2+r_3+2} + \frac{2^{r_2}}{r_1} - 1 + \begin{cases} 2^{r_3} & \text{if } \ r_2 \ge 1, \\ 2^{r_3+1}\ln n & \text{if } \ r_2 = 0. \end{cases} \tag{15}$$

Proof. Recall that $r_1 \geq r_3 + 2$, $n \geq 4$. Consider the difference

$$\omega(r_1, r_2, r_3) \overset{def}{=} \mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) - \mathfrak{L}_n(r_1 - 1, r_2, r_3 + 1, \alpha_1, \alpha_2, \alpha_3).$$

Due to (7), (9) and Lemmas 8, 9, 10, 11, 12, 13, we obtain

$$\omega(r_1, r_2, r_3) = \sum_{s=1}^{6} \delta_s \geq -2^{r_2+r_3+2} + 2^{r_3+1} - \frac{2^{r_2-1}}{r_1} - \frac{2^{r_2-1}}{r_1} + \sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2) \binom{r_1 - 1 + \alpha_1}{n - r_3 - i_2 - 1} \binom{r_3 + \alpha_3}{r_3 + 1}$$

$$- \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \binom{r_3+1}{i_3} - \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \binom{r_3 + \alpha_3}{n - r_1 - i_2} \frac{\Gamma(r_1 + \alpha_1)}{\Gamma(r_1 + 1)|\Gamma(\alpha_1)|}$$

$$- \sum_{i_2=r_2+1}^{n-r_3-1} a_{i_2}(\lambda_2) \binom{r_1 + \alpha_1 - 1}{n - r_3 - 1 - i_2} \binom{r_3 + \alpha_3}{r_3 + 1} + \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \binom{r_1 + \alpha_1 - 1}{r_1} \binom{r_3 + \alpha_3}{n - r_1 - i_2}$$

$$= -2^{r_2+r_3+2} + 2^{r_3+1} - \frac{2^{r_2}}{r_1} - \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \binom{r_3+1}{i_3}$$

$$- \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \binom{r_3 + \alpha_3}{n - r_1 - i_2} \frac{\Gamma(r_1 + \alpha_1)}{\Gamma(r_1 + 1)} \left(\frac{1}{|\Gamma(\alpha_1)|} - \frac{1}{\Gamma(\alpha_1)} \right).$$

If $0 < \alpha_1 < 1$, then $\dfrac{1}{|\Gamma(\alpha_1)|} - \dfrac{1}{\Gamma(\alpha_1)} = 0$. If $-1 < \alpha_1 < 0$, then $\dfrac{1}{|\Gamma(\alpha_1)|} - \dfrac{1}{\Gamma(\alpha_1)} \leq 1$. Thus

$$\omega(r_1, r_2, r_3) \geq -2^{r_2+r_3+2} + 2^{r_3+1} - \frac{2^{r_2}}{r_1} - \sum_{i_2=r_2+1}^{n-r_1-1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-1-i_2} \binom{r_3+1}{i_3} - \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \binom{r_3 + \alpha_3}{n - r_1 - i_2}$$

$$\geq -2^{r_2+r_3+2} + 2^{r_3+1} - \frac{2^{r_2}}{r_1} - \sum_{i_2=r_2+1}^{n-r_1} a_{i_2}(\lambda_2) \sum_{i_3=0}^{n-r_1-i_2} \binom{r_3+1}{i_3}$$

$$\geq -2^{r_2+r_3+2} + 2^{r_3+1} - \frac{2^{r_2}}{r_1} - (2^{r_3+1} - 1) \frac{\sin \pi\alpha_2}{\pi} \sum_{i_2=r_2+1}^{n-r_1} \frac{\Gamma(r_2 + \alpha_2 + 1)\Gamma(i_2 - r_2 - \alpha_2)}{i_2!}.$$

If $i_2 = r_2 + 1$, then

$$\frac{\sin \pi\alpha_2}{\pi} \frac{\Gamma(r_2 + \alpha_2 + 1)\Gamma(i_2 - r_2 - \alpha_2)}{i_2!} = \frac{\Gamma(r_2 + \alpha_2 + 1)}{(r_2 + 1)!\ \Gamma(\alpha_2)} \leq \frac{\Gamma(r_2 + 2)}{(r_2 + 1)!\ \Gamma(1)} \leq 1.$$

Let $r_2 \neq 0$. If $i_2 \geq r_2 + 2$, then taking into account the fact that $(1+\alpha_2)\alpha_2(1-\alpha_2) \leq 0.5$ for $0 \leq \alpha_2 \leq 1$, we obtain

$$\frac{\sin \pi\alpha_2}{\pi} \Gamma(r_2 + \alpha_2 + 1)\Gamma(i_2 - r_2 - \alpha_2) = \frac{\sin \pi\alpha_2}{\pi} \Gamma(\alpha_2)\Gamma(1 - \alpha_2) \prod_{s=0}^{r_2} (s + \alpha_2) \prod_{\kappa=1}^{i_2-r_2-1} (\kappa - \alpha_2)$$

$$= (1+\alpha_2)\alpha_2(1-\alpha_2) \prod_{s=2}^{r_2} (s+\alpha_2) \prod_{\kappa=2}^{i_2-r_2-1} (\kappa - \alpha_2) \leq \frac{2}{2} \prod_{s=2}^{r_2} (s+\alpha_2) \prod_{\kappa=3}^{i_2-r_2-1} (\kappa - \alpha_2) \leq \prod_{s=3}^{r_2+1} s \prod_{\kappa=3}^{i_2-r_2-1} \kappa \leq \prod_{\kappa=3}^{i_2-2} \kappa.$$

Then

$$\frac{\sin \pi\alpha_2}{\pi} \sum_{i_2=r_2+1}^{n-r_1} \frac{\Gamma(r_2 + \alpha_2 + 1)\Gamma(i_2 - r_2 - \alpha_2)}{i_2!} \leq \sum_{i_2=r_2+1}^{n-r_1} \frac{1}{i_2!} \prod_{\kappa=3}^{i_2-2} \kappa = \sum_{i_2=r_2+1}^{n-r_1} \frac{1}{2(i_2 - 1)i_2}$$

$$\leq 1+\frac{1}{2}\sum_{i_2=r_2+2}^{n-r_1}\frac{1}{(i_2-1)i_2}=1+\frac{1}{2(r_2+1)}-\frac{1}{2(n-r_1)}\leq 1+\frac{1}{2(r_2+1)}\leq\frac{3}{2}\,,$$

and

$$\omega(r_1,r_2,r_3)\geq -2^{r_2+r_3+2}+2^{r_3+1}-\frac{2^{r-2}}{r_1}-(2^{r_3+1}-1)\frac{3}{2}\geq -2^{r_2+r_3+2}-\frac{2^{r-2}}{r_1}-2^{r_3}+1.$$

If $r_2=0$, then

$$\frac{\sin\pi\alpha_2}{\pi}\sum_{i_2=r_2+1}^{n-r_1}\frac{\Gamma(r_2+\alpha_2+1)\Gamma(i_2-r_2-\alpha_2)}{i_2!}=\frac{\sin\pi\alpha_2}{\pi}\sum_{i_2=1}^{n-r_1}\frac{\Gamma(1+\alpha_2)\Gamma(i_2-\alpha_2)}{i_2!}$$

$$\leq 1+\frac{\sin\pi\alpha_2}{\pi}\sum_{i_2=2}^{n-r_1}\frac{\alpha_2\Gamma(\alpha_2)\Gamma(1-\alpha_2)(1-\alpha_2)\ldots(i_2-1-\alpha_2)}{i_2!}$$

$$\leq 1+\sum_{i_2=2}^{n-r_1}\frac{\alpha_2(1-\alpha_2)\ldots(i_2-1-\alpha_2)}{i_2!}\leq 1+\frac{1}{2}\sum_{i_2=2}^{n-r_1}\frac{1}{i_2}\leq 1+\frac{1}{4}\ln(n-r_1)\leq\ln n,$$

and

$$\omega(r_1,0,r_3)\geq -2^{r_2+r_3+2}+2^{r_3+1}-\frac{2^{r-2}}{r_1}-(2^{r_3+1}-1)\ln 2\geq -2^{r_2+r_3+2}-\frac{2^{r-2}}{r_1}-2^{r_3+1}\ln 2+1.$$

Finally we have

$$\omega(r_1,r_2,r_3)\geq -2^{r_2+r_3+2}-\frac{2^{r_2}}{r_1}+1-\left\{\begin{array}{ll}2^{r_3} & \text{if}\;\; r_2\geq 1,\\ 2^{r_3+1}\ln n & \text{if}\;\; r_2=0,\end{array}\right.$$

and it follows from this that (15) holds. □

Lemma 15. *Let $r_s\in\mathbb{Z}_+$, $1\leq s\leq 3$, $r_1+r_2+r_3=n-1$, $-1<\alpha_1<1$, $0\leq\alpha_2,\alpha_3<1$, $r_1+\alpha_1\geq r_2+\alpha_2$, $r_1+\alpha_1\geq r_3+\alpha_3$. Then*

$$\mathfrak{L}_n(r_1,r_2,r_3,\alpha_1,\alpha_2,\alpha_3)\leq\mathfrak{L}_n(r_1+r_3,r_2,0,\alpha_1,\alpha_2,\alpha_3)+2^{r_2+r_3+2}+2^{r_2}-r_3+\left\{\begin{array}{ll}2^{r_3} & \textit{if}\;\; r_2\geq 1,\\ 2^{r_3+1}\ln n & \textit{if}\;\; r_2=0.\end{array}\right.$$

Proof. Let $r_2\geq 1$. Applying r_3 times lemma 14, we obtain

$$\mathfrak{L}_n(r_1,r_2,r_3,\alpha_1,\alpha_2,\alpha_3)\leq\mathfrak{L}_n(r_1+r_3,r_2,0,\alpha_1,\alpha_2,\alpha_3)+\sum_{s=0}^{r_3-1}2^{r_2+s+2}+\sum_{s=0}^{r_3-1}\frac{2^{r_2}}{r_1}+\sum_{s=0}^{r_3-1}2^s-\sum_{s=0}^{r_3-1}1$$

$$\leq\mathfrak{L}_n(r_1+r_3,r_2,0,\alpha_1,\alpha_2,\alpha_3)+2^{r_2+r_3+2}+2^{r_2}+2^{r_3}-r_3.$$

The case $r_2=0$ is treated similarly. □

7. Reducing to the case $r_1=n-1$, $r_2=r_3=0$

Lemma 16. *Let $r_s\in\mathbb{Z}_+$, $1\leq s\leq 3$, $r_3=0$, $r_1+r_2=n-1$, $-1<\alpha_1<1$, $0\leq\alpha_2,\alpha_3<1$, $r_1+\alpha_1\geq r_2+\alpha_2$. Then*

$$\mathfrak{L}_n(r_1,r_2,0,\alpha_1,\alpha_2,\alpha_3)\leq\mathfrak{L}_n(r_1+r_2,0,0,\alpha_1,\alpha_2,\alpha_3)+2^{r_2+2}+1-r_2+2^{r_2+1}\ln n.$$

Proof. We treat this case similarly to Section 5, where the parameter r_2 plays the role of the parameter r_3.

□

Theorem 1. *Let* $r_s \in \mathbb{Z}_+$, $1 \le s \le 3$, $r_1+r_2+r_3 = n-1$, $-1 < \alpha_1 < 1$, $0 \le \alpha_2, \alpha_3 < 1$, $r_1+\alpha_1 \ge r_2+\alpha_2$, $r_1 + \alpha_1 \ge r_3 + \alpha_3$. *Then*

$$\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) \le \mathfrak{L}_n(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3) + 2^{2n/3}(10 + 2\ln n).$$

Proof. We apply Lemmas 15 and 16. If $r_2 \ne 0$, then

$$\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) - \mathfrak{L}_n(r_1+r_2+r_3, 0, 0, \alpha_1, \alpha_2, \alpha_3) \le 2^{r_2+r_3+2} + 2^{r_2} + 2^{r_3} + 2^{r_2+2} + 1 + 2^{r_2+1}\ln n - (r_2+r_3)$$

$$\le 4 \cdot 2^{r_2+r_3} + 2^{r_2}(1 + 4 + 2\ln n) + 2^{r_3} \le 4 \cdot 2^{r_2+r_3} + 2^{r_2+r_3}(6 + 2\ln n) \le 2^{2n/3}(10 + 2\ln n).$$

If $r_2 = 0$, then

$$\mathfrak{L}_n(r_1, r_2, r_3, \alpha_1, \alpha_2, \alpha_3) - \mathfrak{L}_n(r_1 + r_2 + r_3, 0, 0, \alpha_1, \alpha_2, \alpha_3) \le 2^{r_3+2} + 1 + 2^{r_3+1}\ln n \le 2^{2n/3}(10 + 2\ln n).$$

□

8. The upper bound of Λ_n

8.1. Statement of the main result. The purpose of this section is to prove the following main theorem of this article.

Theorem 2. *Let* $n \ge 4$. *Then*

$$\Lambda_n = \Lambda_{n,d} \le (7 + \mu_n)\frac{2^{n+1}}{en(\ln n - \ln 2)}\left(1 + \frac{15}{n-3}\right), \tag{16}$$

where

$$\mu_n \le \frac{3en(\ln n)^3}{2^{n/3}} + \frac{en^2 \ln n}{2^n} \to 0 \quad as \quad n \to \infty.$$

Let $k, m, p \in \mathbb{Z}_+$. If k, p are odd, we assume that

$$\sum_{s=k/2}^{n} = \sum_{s=(k+1)/2}^{n} \quad \text{and} \quad \sum_{s=m}^{p/2} = \sum_{s=m}^{(p-1)/2}.$$

Proof. Lemma 2 and Theorem 1 show that

$$\Lambda_n \le \max_{\substack{-1\le\alpha_1\le1 \\ 0\le\alpha_2,\alpha_3\le1 \\ 0\le\alpha_1+\alpha_2+\alpha_3\le1}} \mathfrak{L}_n(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3) + 2^{2n/3}(10 + 2\ln n). \tag{17}$$

To estimate $\mathfrak{L}_n(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3)$ let's estimate each term in (7) for $r_1 = n-1$, $r_2 = r_3 = 0$. Denote $\bar{\mathcal{S}}_k = S_k(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3)$, $1 \le k \le 6$. Then

$$\mathfrak{L}_n(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3) = \sum_{k=1}^{6} \bar{\mathcal{S}}_k. \tag{18}$$

Everywhere we assume that $i = (i_1, i_2, i_3) \in I$, $i_1, i_2, i_3 \in \mathbb{Z}_+$, $i_1 + i_2 + i_3 = n$, $\lambda_1 = (n-1+\alpha_1)/n$, $\lambda_2 = \alpha_2/n$, $\lambda_3 = \alpha_3/n$, $\lambda = (\lambda_1, \lambda_2, \lambda_3)$, $-1 < \alpha_1 < 1$, $0 < \alpha_2, \alpha_3 < 1$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Now we will deal with the sums $\bar{\mathcal{S}}_k$.

8.2. The upper bound of $\bar{\mathcal{S}}_6 = S_6(n-1, 0, 0, \alpha_1, \alpha_2, \alpha_3)$. Denote

$$\varepsilon(i_1) = \varepsilon = \begin{cases} 0, & \text{if } \frac{n-i_1}{2} \in \mathbb{N}, \\ 1, & \text{if } \frac{n-i_1}{2} \notin \mathbb{N}. \end{cases}$$

If $\frac{n-i_1}{2} \in \mathbb{Z}_+$ then the polynomial $l_i(\lambda)$ such that $i = \left(i_1, \frac{n-i_1}{2}, \frac{n-i_1}{2}\right)$ denote $\mathfrak{p}(i_1)$, i.e.,

$$\mathfrak{p}(i_1) = l_i(\lambda)\Big|_{i=\left(i_1, \frac{n-i_1}{2}, \frac{n-i_1}{2}\right)}.$$

Consider $\bar{\mathcal{S}}_6$ and represent it as a sum of terms

$$\bar{\mathcal{S}}_6 = \sum_{s=1}^{4} \sigma_s,$$

where

$$\sigma_1 = \sum_{i_1=0}^{n-4} \left(\sum_{i_2=1}^{\frac{n-i_1-1}{2}} |l_i(\lambda)| + \frac{1-\varepsilon(i_1)}{2}\mathfrak{p}(i_1) \right), \quad \sigma_2 = \sum_{i_1=0}^{n-4} \left(\sum_{i_3=1}^{\frac{n-i_1-1}{2}} |l_i(\lambda)| + \frac{1-\varepsilon(i_1)}{2}\mathfrak{p}(i_1) \right),$$

$$\sigma_3 = |l_i(\lambda)|\Big|_{i=(n-3,1,2)} + |l_i(\lambda)|\Big|_{i=(n-3,2,1)}, \quad \sigma_4 = |l_i(\lambda)|\Big|_{i=(n-2,1,1)}.$$

Before estimating the summands of σ_s, we prove several auxiliary lemmas. The following auxiliary lemma is similar to the lemma from [1], the proof is given here for the convenience of the reader.

Lemma 17. *Let*

$$\Phi(\alpha, m) = \frac{\sin \pi\alpha}{\pi} \, \frac{\Gamma(1+\alpha)\Gamma(m-\alpha)}{m!},$$

where $0 < \alpha < 1$, m is an integer, $m \geq 2$. Then

$$\Phi(\alpha, m) \leq \frac{\alpha(1-\alpha)2^{\alpha}}{m^{1+\alpha}}.$$

If $m = 1$, then $\Phi(\alpha, m) = \Phi(\alpha, 1) = \alpha$.

Proof [2]. For $m = 1, 2$ the inequality can be checked directly by substituting m. Let $m \geq 3$. First, we estimate the value

$$\varphi(\alpha, m) \overset{def}{=} \prod_{s=2}^{m-1} (s-\alpha).$$

Take the logarithm and make the following transformations:

$$\ln \varphi(\alpha, m) = \sum_{s=2}^{m-1} \ln(s-\alpha) = \sum_{s=2}^{m-1} \ln s - \sum_{s=2}^{m-1} (\ln s - \ln(s-\alpha)) = \ln(m-1)! - \sum_{s=2}^{m-1} \ln\left(1 + \frac{\alpha}{s-\alpha}\right)$$

$$\leq \ln(m-1)! - \sum_{s=2}^{m-1} \frac{1}{1+\frac{\alpha}{s-\alpha}} \, \frac{\alpha}{s-\alpha} = \ln(m-1)! - \alpha \sum_{s=2}^{m-1} \frac{1}{s} \leq \ln(m-1)! - \alpha \int_{s=2}^{m} \frac{ds}{s} = \ln(m-1)! - \alpha \ln m + \alpha \ln 2.$$

So, $\varphi(\alpha, m) \leq \dfrac{(m-1)!2^{\alpha}}{m^{\alpha}}$. Then

$$\Phi(\alpha, m) = \frac{\alpha\Gamma(m-\alpha)}{m! \, \Gamma(1-\alpha)} = \frac{\alpha}{m!} \prod_{s=1}^{m-1} (s-\alpha) = \frac{\alpha(1-\alpha)}{m!} \varphi(\alpha, m) \leq \frac{\alpha(1-\alpha)(m-1)!2^{\alpha}}{m! \, m^{\alpha}} = \frac{\alpha(1-\alpha)2^{\alpha}}{m^{1+\alpha}}.$$

□

Lemma 18. *Let $\alpha \geq 1$, $m > 1$, $\phi(\alpha, m) = \dfrac{\alpha}{m^{\alpha}}$. Then $\phi(\alpha, m) \leq \dfrac{1}{e \ln m}$.*

Proof. Consider the function $\phi(\alpha,m)$ as a function of α on $(0,+\infty)$. Since $\phi'_\alpha = m^{-\alpha}(1-\alpha\ln m)$, the point $\alpha = 1/\ln m$ is the maximum point of the function ϕ. Then

$$\phi(\alpha,m) \le \phi\left(\frac{1}{\ln m}\,, m\right) = \frac{1}{\ln m\; m^{1/\ln m}} = \frac{1}{e\ln m}\ .$$

□

Lemma 19. *Let $n \ge 4$. Then*

$$\sum_{s=1}^{n} \frac{2^s}{s} \le \frac{2^{n+1}}{n} + \frac{2^{n+1}}{(n-3)^2}\ . \tag{19}$$

Proof. For $n = \overline{4,8}$ the inequality (19) is verified by direct calculations. Let $n \ge 9$. We proceed by induction. Suppose the inequality (19) to be true for $n = \mathfrak{t}-1$, i.e.,

$$\sum_{s=1}^{\mathfrak{t}-1} \frac{2^s}{s} \le \frac{2^{\mathfrak{t}}}{\mathfrak{t}-1} + \frac{2^{\mathfrak{t}}}{(\mathfrak{t}-4)^2} \stackrel{def}{=} \varphi_1(\mathfrak{t}).$$

Now prove that (19) holds for $n = \mathfrak{t}$. This is equivalent to the inequality

$$\sum_{s=1}^{\mathfrak{t}-1} \frac{2^s}{s} \le \frac{2^{\mathfrak{t}}}{\mathfrak{t}} + \frac{2^{\mathfrak{t}+1}}{(\mathfrak{t}-3)^2} \stackrel{def}{=} \varphi_2(\mathfrak{t}).$$

Thus, it suffices for us to prove that $\varphi_1(\mathfrak{t}) \le \varphi_2(\mathfrak{t})$. Consider the difference

$$\varphi_2(\mathfrak{t}) - \varphi_1(\mathfrak{t}) = \frac{2^{\mathfrak{t}}(3\mathfrak{t}^3 - 40\mathfrak{t}^2 + 145\mathfrak{t} - 144)}{\mathfrak{t}(\mathfrak{t}-1)(\mathfrak{t}-3)^2(\mathfrak{t}-4)^2} = \frac{2^{\mathfrak{t}} g(\mathfrak{t})}{\mathfrak{t}(\mathfrak{t}-1)(\mathfrak{t}-3)^2(\mathfrak{t}-4)^2}\ ,$$

where $g(\mathfrak{t}) = 3\mathfrak{t}^3 - 40\mathfrak{t}^2 + 145\mathfrak{t} - 144$. Since $g'(\mathfrak{t}) > 0$ for $\mathfrak{t} \ge 7$ and $g(9) > 0$, then $g(\mathfrak{t}) > 0$ for all $\mathfrak{t} \ge 9$, which implies that $\varphi_1(\mathfrak{t}) \le \varphi_2(\mathfrak{t})$ for all $\mathfrak{t} \ge 9$. □

Denote

$$\nu_p(s) = \begin{cases} \alpha_p & \text{if} \quad s = 1, \\ \alpha_p(1-\alpha_p)2^{\alpha_p} & \text{if} \quad s \ge 2, \end{cases}$$

where $p = 2, 3$.

Lemma 20. *Let $i_2 \ge 1$, $i_3 \ge 1$, $i_2 + i_3 \ge 3$. Then*

$$|l_i(\lambda)| \le \frac{(1-\alpha_3)}{e(\ln n - \ln 2)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2} i_3(i_3-1)} \quad (i_2 \ge 1, i_3 \ge 2), \tag{20}$$

$$|l_i(\lambda)| \le \frac{(1-\alpha_2)}{e(\ln n - \ln 2)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{\nu_3(i_3)}{i_3^{1+\alpha_3} i_2(i_2-1)} \quad (i_2 \ge 2, i_3 \ge 1). \tag{21}$$

Proof. We will prove (20). Let $i_2, i_3 \ge 1$. Due to (6) and Lemmas 7, 17 we have

$$l_i(\lambda) = \prod_{s=1}^{3} a_{i_s}(\lambda_s) = \frac{\sin\pi\alpha_2 \sin\pi\alpha_3}{\pi^2} \frac{\Gamma(n+\alpha_1)}{i_1!\ \Gamma(n+\alpha_1-i_1)} \frac{\Gamma(1+\alpha_2)\Gamma(i_2-\alpha_2)}{i_2!} \frac{\Gamma(1+\alpha_3)\Gamma(i_3-\alpha_3)}{i_3!}$$

$$= \frac{\sin\pi\alpha_3}{\pi} \frac{\Gamma(n+1-\alpha_2-\alpha_3)}{i_1!\ \Gamma(n+1-i_1-\alpha_2-\alpha_3)} \frac{\sin\pi\alpha_2}{\pi} \frac{\Gamma(1+\alpha_2)\Gamma(i_2-\alpha_2)}{i_2!} \frac{\Gamma(1+\alpha_3)\Gamma(i_3-\alpha_3)}{i_3!}$$

$$\le \frac{\sin\pi\alpha_3}{\pi} \frac{\Gamma(n+1-\alpha_3)}{i_1!\ \Gamma(n+1-i_1-\alpha_3)} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2}} \frac{\Gamma(1+\alpha_3)\Gamma(i_3-\alpha_3)}{i_3!}$$

$$= \frac{\sin \pi\alpha_3}{\pi} \frac{\Gamma(1+\alpha_3)\Gamma(n+1-\alpha_3)}{(n+1)!} \frac{(n+1)!}{i_1!} \frac{nu_2(i_2)}{i_2^{1+\alpha_2}} \frac{\Gamma(i_3-\alpha_3)}{i_3!\ \Gamma(n+1-i_1-\alpha_3)}$$

$$\leq \frac{\alpha_3(1-\alpha_3)2^{\alpha_3}}{(n+1)^{1+\alpha_3}} \frac{(n+1)!}{i_1!} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2}} \frac{\Gamma(i_3-1)}{i_3!\ \Gamma(n+1-i_1-1)}.$$

By Lemma 18, we have

$$\frac{\alpha_3 2^{\alpha_3}}{(n+1)^{\alpha_3}} = \phi\left(\alpha_3, \frac{n+1}{2}\right) \leq \frac{1}{e(\ln(n+1)-\ln 2)} \leq \frac{1}{e(\ln n - \ln 2)}$$

which implies

$$|l_i(\lambda)| \leq \frac{1-\alpha_3}{e(\ln n - \ln 2)} \frac{\Gamma(n+1)}{i_1!} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2}} \cdot \frac{\Gamma(i_3-\alpha_3)}{i_3!\ \Gamma(n+1-i_1-\alpha_3)}. \tag{22}$$

If $i_3 \geq 2$, $i_2 \geq 1$, then due to (22) and Lemma 7, we obtain

$$l_i(\lambda) \leq \frac{1-\alpha_3}{e(\ln n - \ln 2)} \frac{\Gamma(n+1)}{i_1!} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2}} \frac{\Gamma(i_3-1)}{i_3!\ \Gamma(n-i_1)} = \frac{1-\alpha_3}{e(\ln n - \ln 2)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{\nu_2(i_2)}{i_2^{1+\alpha_2} i_3(i_3-1)}.$$

The case of (21) is similar. ☐

Lemma 21. *Let $(n-i_1)$ be even, $i_1 \leq n-4$. Then*

$$\mathfrak{p}(i_1) \leq \frac{1}{e(\ln n - 1)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{4\alpha_2(1-\alpha_2)}{(n-i_1)(n-i_1-1)}. \tag{23}$$

Proof. Consider $\mathfrak{p}(i_1)$ and apply Lemma 20. We also take into account that $i_1 \leq n-4$. Then

$$\mathfrak{p}(i_1) \leq \frac{1}{e(\ln n - 1)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \left. \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}(1-\alpha_3)}{i_2^{1+\alpha_2} i_3(i_3-1)} \right|_{i_2=i_3=\frac{n-i_1}{2}}$$

$$= \frac{1}{e(\ln n - 1)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{8\alpha_2(1-\alpha_2)(1-\alpha_3)4^{\alpha_2}}{(n-i_1)^{2+\alpha_2}(n-i_1-2)}$$

$$\leq \frac{1}{e(\ln n - 1)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{1}{(n-i_1)^2} \frac{8\alpha_2(1-\alpha_2)(1-\alpha_3)4^{\alpha_2}}{4^{\alpha_2}2}$$

$$\leq \frac{1}{e(\ln n - 1)} \frac{\Gamma(n+1)}{i_1!\ \Gamma(n-i_1)} \frac{4\alpha_2(1-\alpha_2)}{(n-i_1)(n-i_1-1)}.$$

☐

Lemma 22. *The following inequality holds:*

$$\sigma_s \leq \frac{C_0}{e(\ln n - 1)} \sum_{i_1=0}^{n-4} \binom{n}{n-i_1} \frac{1}{n-i_1-1}, \quad s = 1, 2, \tag{24}$$

where

$$C_0 = \frac{(\ln 2)^2 + 12\ln 2 + 28}{4\ln 2 + 12} < 2.5.$$

Proof. We will prove (24) for $s = 1$. Let $\tilde{\mathfrak{p}}(i_1)$ be a function such that

$$\mathfrak{p}(i_1) = \frac{1}{e(\ln n - 1)} \frac{n!}{i_1!\ (n-1-i_1)!} \tilde{\mathfrak{p}}(i_1).$$

Using Lemma 20 we have

$$\sigma_1 \le \frac{1}{e(\ln n - 1)} \sum_{i_1=0}^{n-4} \frac{n!}{i_1!\,(n-1-i_1)!} \left(\frac{\alpha_2}{(n-i_1-1)(n-i_1-2)} \right.$$

$$\left. + \sum_{i_2=2}^{(n-i_1-1)/2} \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{i_2^{1+\alpha_2} i_3(i_3-1)} + \frac{1-\varepsilon(i_1)}{2}\tilde{\mathfrak{p}}(i_1) \right). \tag{25}$$

Now we estimate the terms in brackets as follows:

$$h_1(i_1) \stackrel{def}{=} \frac{\alpha_2}{(n-i_1-1)(n-i_1-2)} = \frac{\alpha_2}{(n-i_1)} \left(\frac{1}{n-i_1-1} + \frac{2}{(n-i_1-1)(n-i_1-2)} \right)$$

$$\le \frac{\alpha_2}{(n-i_1)} \left(\frac{1}{n-i_1-1} + \frac{2}{2(n-i_1-1)} \right) = \frac{2\alpha_2}{(n-i_1)(n-i_1-1)}, \tag{26}$$

$$h_2(i_1) \stackrel{def}{=} \sum_{i_2=2}^{(n-i_1-1)/2} \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{i_2^{1+\alpha_2} i_3(i_3-1)} = \sum_{i_2=2}^{(n-i_1-1)/2} \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{i_2^{1+\alpha_2}(n-i_1-i_2)(n-i_1-i_2-1)}$$

$$= \sum_{i_2=2}^{(n-i_1-1)/2} \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{n-i_1} \left(\frac{1+(n-i_1-i_2)-(n-i_1-i_2)}{i_2^{\alpha_2}(n-i_1-i_2)(n-i_1-i_2-1)} + \frac{1}{i_2^{1+\alpha_2}(n-i_1-i_2-1)} \right)$$

$$\le \alpha_2(1-\alpha_2)2^{\alpha_2} \sum_{i_2=2}^{(n-i_1-1)/2} \left(\frac{1}{n-i_1} \left(\frac{1}{2^{\alpha_2}(n-i_1-i_2-1)} - \frac{1}{2^{\alpha_2}(n-i_1-i_2)} \right) \right.$$

$$\left. + \frac{1}{(n-i_1)(n-i_1-1)} \left(\frac{1}{i_2^{\alpha_2}(n-i_1-i_2-1)} + \frac{1}{i_2^{1+\alpha_2}} \right) \right)$$

$$\le \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{n-i_1} \int_2^{(n-i_1+\varepsilon)/2} \left(\frac{1}{2^{\alpha_2}(n-i_1-t-1)} - \frac{1}{2^{\alpha_2}(n-i_1-t)} \right) dt$$

$$+ \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{(n-i_1)(n-i_1-1)} \left(\int_2^{(n-i_1+\varepsilon)/2} \frac{dt}{2^{\alpha_2}(n-i_1-t-1)} + \int_1^{(n-i_1-2+\varepsilon)/2} \frac{dt}{t^{1+\alpha_2}} \right)$$

$$= \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{2^{\alpha_2}(n-i_1)} \left(-\ln\frac{n-i_1-2-\varepsilon}{2} + \ln(n-i_1-3) + \ln\frac{n-i_1-\varepsilon}{2} - \ln(n-i_1-2) \right)$$

$$+ \frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{2^{\alpha_2}(n-i_1)(n-i_1-1)} \left(-\ln\frac{n-i_1-2-\varepsilon}{2} + \ln(n-i_1-3) - \frac{2^{\alpha_2}\,2^{\alpha_2}}{\alpha_2(n-i_1-2+\varepsilon)^{\alpha_2}} + \frac{1}{\alpha_2} \right)$$

$$\le \frac{\alpha_2(1-\alpha_2)}{n-i_1} \left(-\ln(n-i_1-2-\varepsilon) + \ln(n-i_1-3) + \ln(n-i_1-\varepsilon) - \ln(n-i_1-2) \right)$$

$$+ \frac{\alpha_2(1-\alpha_2)}{(n-i_1)(n-i_1-1)} \left(\ln 2 + \frac{1}{\alpha_2} \right).$$

If $\varepsilon = 0$, then

$$-\ln(n-i_1-2-\varepsilon) + \ln(n-i_1-3) + \ln(n-i_1-\varepsilon) - \ln(n-i_1-2)$$

$$= -\ln(n-i_1-2)^2 + \ln(n-i_1-3) + \ln(n-i_1)$$

$$\le -\ln(n-i_1-1)(n-i_1-3) + \ln(n-i_1-3) + \ln(n-i_1) = -\ln(n-i_1-1) + \ln(n-i_1)$$

$$= \ln\left(1 + \frac{1}{n - i_1 - 1}\right) \le \frac{1}{n - i_1 - 1}.$$

If $\varepsilon = 1$, then

$$-\ln(n - i_1 - 2 - \varepsilon) + \ln(n - i_1 - 3) + \ln(n - i_1 - \varepsilon) - \ln(n - i_1 - 2)$$

$$= -\ln(n - i_1 - 3) + \ln(n - i_1 - 3) + \ln(n - i_1 - 1) - \ln(n - i_1 - 2) = \ln\left(1 + \frac{1}{n - i_1 - 2}\right)$$

$$\le \frac{1}{n - i_1 - 2} = \frac{1}{n - i_1 - 1}\left(1 + \frac{1}{n - i_1 - 2}\right) \le \frac{3}{2(n - i_1 - 1)}.$$

Thus

$$h_2(i_1) \le \frac{\alpha_2(1 - \alpha_2)}{(n - i_1)(n - i_1 - 1)}\left(1 + \frac{\varepsilon(i_1)}{2} + \ln 2 + \frac{1}{\alpha_2}\right). \tag{27}$$

Further, taking into account (23), we have

$$h_3(i_1) \overset{def}{=} \frac{1 - \varepsilon(i_1)}{2}\tilde{\mathfrak{p}}(i_1) \le \frac{1 - \varepsilon(i_1)}{2}\frac{4\alpha_2(1 - \alpha_2)}{(n - i_1)(n - i_1 - 1)}. \tag{28}$$

It follows from (25), (26), (27), (28) that

$$\sigma_1 \le \frac{}{e(\ln n - 1)}\sum_{i_1=0}^{n-4}\frac{n!}{i_1!\ (n - 1 - i_1)!}\frac{1}{(n - i_1)(n - i_1 - 1)}$$

$$\times\left(2\alpha_2 + \alpha_2(1 - \alpha_2)\left(1 + \frac{\varepsilon(i_1)}{2} + \ln 2 + \frac{1}{\alpha_2}\right) + 4\alpha_2(1 - \alpha_2)\frac{1 - \varepsilon(i_1)}{2}\right)$$

$$= \frac{1}{e(\ln n - 1)}\sum_{i_1=0}^{n-4}\binom{n}{n - i_1}\frac{1}{n - i_1 - 1}\left(2\alpha_2 + \alpha_2(1 - \alpha_2)\left(3 + \ln 2 - \frac{3\varepsilon(i_1)}{2} + \frac{1}{\alpha_2}\right)\right)$$

$$\le \frac{1}{e(\ln n - 1)}\sum_{i_1=0}^{n-4}\binom{n}{n - i_1}\frac{2\alpha_2 + (1 - \alpha_2)\left((3 + \ln 2)\alpha_2 + 1\right)}{n - i_1 - 1}. \tag{29}$$

Consider the function

$$\eta(\alpha_2) = 2\alpha_2 + (1 - \alpha_2)\left((3 + \ln 2)\alpha_2 + 1\right).$$

The point $\alpha_2 = \dfrac{4 + \ln 2}{6 + 2\ln 2}$ is the maximum point of this function, which implies that

$$\eta(\alpha_2) \le \eta\left(\frac{4 + \ln 2}{6 + 2\ln 2}\right) = \frac{(\ln 2)^2 + 12\ln 2 + 28}{4\ln 2 + 12} < 2.5. \tag{30}$$

From (29) and (30) we obtain (24) for $s = 1$. The case $s = 2$ is proved in the same way. □

Lemma 23. $\sigma_3 \le \dfrac{3}{e(\ln n - 1)}\dbinom{n}{n - 3}\dfrac{1}{2}.$

Proof. Consider σ_3 and make estimates using Lemma 20:

$$\sigma_3 \le \frac{1 - \alpha_3}{e(\ln n - 1)}\frac{\Gamma(n + 1)}{(n - 3)!\ \Gamma(3)}\frac{\alpha_2}{2 \cdot 1} + \frac{1 - \alpha_2}{e(\ln n - 1)}\frac{\Gamma(n + 1)}{(n - 3)!\ \Gamma(3)}\frac{\alpha_3}{2 \cdot 1}$$

$$= \frac{1}{e(\ln n - 1)}\frac{\Gamma(n + 1)}{(n - 3)!\ \Gamma(4)}\ \frac{3}{2}\ (\alpha_2(1 - \alpha_3) + \alpha_3(1 - \alpha_2)). \tag{31}$$

The maximum of the function $\zeta(\alpha_2, \alpha_3) = \alpha_2(1 - \alpha_3) + \alpha_3(1 - \alpha_2)$, $0 \le \alpha_2, \alpha_3 \le 1$, equals 1. Then from (31) we obtain the assertion of the lemma. □

Lemma 24. $\sigma_4 \le \dfrac{2}{e(\ln n-1)}\left(\begin{array}{c} n \\ n-2 \end{array}\right).$

Proof. Due to the estimate (22) found in the proof of the lemma 20 we obtain

$$\sigma_4 = |l_i(\lambda)|\Big|_{i=(n-2,1,1)} \le \frac{1-\alpha_3}{e(\ln n-1)}\frac{\Gamma(n+1)}{(n-2)!}\ \frac{\alpha_2}{1^{1+\alpha_2}}\cdot\frac{\Gamma(1-\alpha_3)}{1!\ \Gamma(n+1-(n-2)-\alpha_3)}$$

$$= \frac{\alpha_2(1-\alpha_3)}{e(\ln n-1)}\frac{n!}{(n-2)!}\ \frac{2}{2} \le \frac{2}{e(\ln n-1)}\left(\begin{array}{c} n \\ n-2 \end{array}\right).$$

□

To estimate the sum $\bar{\mathcal{S}}_6$ we prove the following lemma. A similar result was proved in [2]. The proposed proof is given for the convenience of the reader.

Lemma 25. *Let $n \ge 4$. Then*

$$\sum_{i_1=0}^{n-2}\frac{n!}{i_1!\ (n-i_1)!}\ \frac{1}{n-i_1-1} = \sum_{s=2}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)\frac{1}{s-1} \le \frac{2^{n+1}}{n}\left(1+\frac{15}{n-3}\right). \tag{32}$$

Proof [2]. First note that

$$\Upsilon(n) \overset{def}{=} \sum_{i_1=0}^{n-2}\frac{n!}{i_1!\ (n-i_1)!}\ \frac{1}{n-i_1-1} = \sum_{s=2}^{n}\left(\begin{array}{c} n \\ s \end{array}\right)\frac{1}{s-1} = \sum_{s=2}^{n}\int_0^1\left(\begin{array}{c} n \\ s \end{array}\right)t^{s-2}dt$$

$$= \int_0^1\frac{(1+t)^n-nt-1}{t^2}dt = \int_0^1\sum_{s=1}^{n-1}(n-s)(1+t)^{s-1}dt = \sum_{s=1}^{n-1}\frac{n-s}{s}(1+t)^s\Big|_0^1 = \sum_{s=1}^{n-1}\frac{n-s}{s}(2^s-1)$$

$$= \sum_{s=1}^{n}\frac{n-s}{s}2^s - \sum_{s=1}^{n}\frac{n-s}{s} = n\sum_{s=1}^{n}\frac{2^s}{s} - \sum_{s=1}^{n}2^s - n\sum_{s=1}^{n}\frac{1}{s} + n = n\sum_{s=1}^{n}\frac{2^s}{s} - \frac{2(2^n-1)}{2-1} - n\sum_{s=1}^{n}\frac{1}{s} + n.$$

This equality and (19) imply that

$$\Upsilon(n) \le n\left(\frac{2^{n+1}}{n}+\frac{2^{n+1}}{(n-3)^2}\right) - 2^{n+1} + 2 - n\sum_{s=2}^{n}\frac{1}{s} = \frac{2^{n+1}n}{(n-3)^2} + 2 - n\sum_{s=2}^{n}\frac{1}{s}$$

$$\le \frac{2^{n+1}}{n}\frac{n^2}{(n-3)^2} \le \frac{2^{n+1}}{n}\left(1+\frac{15}{n-3}\right).$$

□

Now we can estimate $\bar{\mathcal{S}}_6$.

Lemma 26. $\bar{\mathcal{S}}_6 \le \dfrac{5\cdot 2^{n+1}}{en(\ln n-1)}\left(1+\dfrac{15}{n-3}\right).$

Proof. Since $\bar{\mathcal{S}}_6 = \sigma_1+\ldots+\sigma_4$, then from Lemmas 22, 23, 24 we obtain

$$\bar{\mathcal{S}}_6 \le \frac{C_0+C_0}{e(\ln n-1)}\sum_{i_1=0}^{n-4}\left(\begin{array}{c} n \\ n-i_1 \end{array}\right)\frac{1}{n-i_1-1} + \frac{3}{e(\ln n-1)}\left(\begin{array}{c} n \\ n-3 \end{array}\right)\frac{1}{2} + \frac{2}{e(\ln n-1)}\left(\begin{array}{c} n \\ n-2 \end{array}\right)$$

$$\le \frac{5}{e(\ln n-1)}\sum_{i_1=0}^{n-2}\left(\begin{array}{c} n \\ n-i_1 \end{array}\right)\frac{1}{n-i_1-1}.$$

This estimate and (32) imply the assertion of the lemma. □

8.3. The upper bounds for other sums.

Lemma 27. *The following inequalities hold:*

$$\bar{\mathcal{S}}_1+\bar{\mathcal{S}}_2+\bar{\mathcal{S}}_5 \le \frac{2^{n+1}}{en(\ln n-\ln 2)}\left(1+\frac{15}{n-3}\right)\left(1+\frac{e(\ln n-1)n(n+1)}{2^{n+1}}\right),$$

$$\bar{\mathcal{S}}_3+\bar{\mathcal{S}}_4 \le \frac{2^{n+1}}{en(\ln n-\ln 2)}\left(1+\frac{15}{n-3}\right)\left(1+\frac{e(\ln n-1)n^2}{2^{n+1}}\right).$$

Proof. Recall that $r_3=0$. Then

$$\bar{\mathcal{S}}_1+\bar{\mathcal{S}}_5+\bar{\mathcal{S}}_2=\sum_{i_2=0}^{n}|l_i(\lambda)|\Big|_{i=(n-i_2,i_2,0)}.$$

Consider $l_i(\lambda)$ for $i_2=0,1$:

$$|l_i(\lambda)|\Big|_{i=(n,0,0)}=\frac{\Gamma(n+\alpha_1)}{n!\,\Gamma(\alpha_1)}\le\frac{\Gamma(n+1)}{n!\,\Gamma(\alpha_1)}\le 1, \tag{33}$$

$$|l_i(\lambda)|\Big|_{i=(n-1,1,0)}=\frac{\Gamma(n+\alpha_1)}{(n-1)!\,\Gamma(1+\alpha_1)}\frac{\Gamma(1+\alpha_2)}{\Gamma(\alpha_2)}\le\frac{\Gamma(n+1)}{(n-1)!\,\Gamma(1+1)}\alpha_2\le n \tag{34}$$

(we used Lemma 7 to obtain the inequality (34)).

Now we have to estimate the sum

$$\sigma_0\overset{def}{=}\sum_{i_2=2}^{n}|l_i(\lambda)|\Big|_{i=(n-i_2,i_2,0)}=\sum_{i_2=2}^{n}\frac{\sin\pi\alpha_2}{\pi}\frac{\Gamma(n+\alpha_1)}{(n-i_2)!\,\Gamma(i_2+\alpha_1)}\frac{\Gamma(1+\alpha_2)\Gamma(i_2-\alpha_2)}{i_2!}$$

$$=\sum_{i_2=2}^{n}\frac{\sin\pi\alpha_2}{\pi}\frac{\Gamma(n+1-\alpha_2-\alpha_3)}{(n-i_2)!\,\Gamma(i_2+1-\alpha_2-\alpha_3)}\frac{\Gamma(1+\alpha_2)\Gamma(i_2-\alpha_2)}{i_2!}.$$

Lemma 7 provides the estimate

$$\sigma_0\le\sum_{i_2=2}^{n}\frac{\sin\pi\alpha_2}{\pi}\frac{\Gamma(n+1-\alpha_2)\Gamma(1+\alpha_2)}{(n+1)!}\frac{(n+1)!\,\Gamma(i_2-\alpha_2)}{i_2!\,(n-i_2)!\,\Gamma(i_2+1-\alpha_2)}.$$

Using Lemmas 17, 18, 25 we obtain

$$\sigma_0\le\sum_{i_2=2}^{n}\frac{\alpha_2(1-\alpha_2)2^{\alpha_2}}{(n+1)^{1+\alpha_2}}\frac{(n+1)!\,\Gamma(i_2-\alpha_2)}{i_2!\,(n-i_2)!\,\Gamma(i_2+1-\alpha_2)}\le\sum_{i_2=2}^{n}\frac{1-\alpha_2}{e(\ln n-\ln 2)}\frac{n!}{i_2!\,(n-i_2)!}\,\frac{1}{i_2-\alpha_2}$$

$$\le\frac{1-\alpha_2}{e(\ln n-\ln 2)}\sum_{i_2=2}^{n}\binom{n}{i_2}\frac{1}{i_2-1}\le\frac{1}{e(\ln n-\ln 2)}\frac{2^{n+1}}{n}\left(1+\frac{15}{n-3}\right). \tag{35}$$

So, the first inequality of the lemma follows from (33), (34), (35). The second inequality is proved in a similar way. ☐

8.4. Completion of the proof of Theorem 2. It follows from (17), (18) and Lemmas 26, 27 that

$$\Lambda_n\le(7+\bar{\mu}_1+\bar{\mu}_2)\frac{2^{n+1}}{en(\ln n-\ln 2)}\left(1+\frac{15}{n-3}\right), \tag{36}$$

where

$$\bar{\mu}_1\le\frac{e(\ln n-1)n(n+1)}{2^{n+1}}+\frac{e(\ln n-1)n^2}{2^{n+1}}=\frac{en(2n\ln n+\ln n-2n-1)}{2^{n+1}}\le\frac{en^2\ln n}{2^n}, \tag{37}$$

$$\bar{\mu}_2 \le 2^{2n/3}(10 + 2\ln n)\left(\frac{2^{n+1}}{en(\ln n - \ln 2)}\left(1 + \frac{15}{n-3}\right)\right)^{-1}$$
$$\le \frac{en}{2^{n/3}}(5 + \ln n)(\ln n - \ln 2)\frac{n-3}{n+12} \le \frac{en(\ln n)^2}{2^{n/3}}\left(\frac{5 - \ln 2}{\ln n} + 1 - \frac{\ln 2}{(\ln n)^2}\right).$$

The point $t^* = \dfrac{10\ln 2}{5 - \ln 2}$ is the only maximum point of the function $\chi(t) = \dfrac{5 - \ln 2}{t} + 1 - \dfrac{5\ln 2}{t^2}$, $t > 0$, and $\chi(t^*) < 3$. Then

$$\bar{\mu}_2 < \frac{3en(\ln n)^2}{2^{n/3}}. \tag{38}$$

The estimates (36), (37), (38) imply (16). The proof of Theorem 2 is complete.

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