MAGIC PARTIALLY FILLED ARRAYS ON ABELIAN GROUPS

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Abstract. In this paper we introduce a special class of partially filled arrays. A magic partially filled array $\mathrm{MPF}_\Omega(m,n;s,k)$ on a subset Ω of an abelian group $(\Gamma,+)$ is a partially filled array of size $m\times n$ with entries in Ω such that (i) every $\omega\in\Omega$ appears once in the array; (ii) each row contains s filled cells and each column contains k filled cells; (iii) there exist (not necessarily distinct) elements $x,y\in\Gamma$ such that the sum of the elements in each row is x and the sum of the elements in each column is y. In particular, if $x=y=0_\Gamma$, we have a zero-sum magic partially filled array ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$. Examples of these objects are magic rectangles, Γ-magic rectangles, signed magic arrays, (integer or non integer) Heffter arrays. Here, we give necessary and sufficient conditions for the existence of a magic rectangle with empty cells, i.e., of an $\mathrm{MPF}_\Omega(m,n;s,k)$ where $\Omega=\{1,2,\ldots,nk\}\subset\mathbb{Z}$. We also construct zero-sum magic partially filled arrays when Ω is the abelian group Γ or the set of its nonzero elements.

1. Introduction

The aim of this paper is to introduce and study the following class of partially filled arrays (that is, matrices where some cells are allowed to be empty), whose elements belong to an abelian group.

Definition 1.1. A *magic partially filled array* $\mathrm{MPF}_\Omega(m,n;s,k)$ on a subset Ω of an abelian group $(\Gamma,+)$ is a partially filled array of size $m\times n$ with entries in Ω such that

(a) every $\omega\in\Omega$ appears once in the array;
(b) each row contains s filled cells and each column contains k filled cells;
(c) there exist (not necessarily distinct) elements $x,y\in\Gamma$ such that the sum of the elements in each row is x and the sum of the elements in each column is y.

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Throughout this paper, we will always assume $|\Omega| > 1$. So, necessary conditions for the existence of an $\mathrm{MPF}_\Omega(m,n;s,k)$ are $2 \le s \le n$, $2 \le k \le m$ and $|\Omega| = ms = nk$.

We came along with this definition considering two recent generalizations of magic rectangles. We recall that a *magic rectangle* $\mathrm{MR}(m,n)$ is an $m \times n$ array whose entries are the integers $1, 2, \ldots, mn$, each appearing once in such a way that the sum of the elements in each row is a constant x and the sum of the elements in each column is a constant y. These are well known objects: as shown in [23, 24], an $\mathrm{MR}(m,n)$ exists if and only if $m,n > 1$, $mn > 4$ and $m \equiv n \pmod 2$. In our terms, a magic rectangle is a tight $\mathrm{MPF}_\Omega(m,n;n,m)$ where $\Omega = \{1,2,\ldots,mn\} \subset (\mathbb{Z},+)$. In [25], Khodkar and Leach considered magic rectangles with some empty cells. They gave partial results about the existence of an $\mathrm{MR}(m,n;s,k)$, i.e., in our terminology, of an $\mathrm{MPF}_\Omega(m,n;s,k)$ where $\Omega = \{1,2,\ldots,nk\} \subset \mathbb{Z}$. On the other hand, starting from the concept of a magic square with elements on an abelian group [31], Cichacz studied in [7] the existence of a magic $m \times n$ rectangle with elements in an abelian group Γ of order mn (i.e., an $\mathrm{MPF}_\Gamma(m,n;n,m)$).

In [18, 19], Froncek introduced the notion of magic rectangle set $\mathrm{MRS}(m,n;c)$. Similarly, also Cichacz was interested in magic rectangle sets on abelian groups.

Definition 1.2. [7] A Γ-*magic rectangle set* $\mathrm{MRS}_\Gamma(m,n;c)$ on an abelian group $(\Gamma,+)$ of order mnc is a set of c arrays of size $m \times n$, whose entries are elements of Γ, each appearing once, with all row sums in each rectangle equal to a constant $x \in \Gamma$ and all column sums in each rectangle equal to a constant $y \in \Gamma$.

Even if Froncek provided in [20] necessary and sufficient conditions for the existence of magic rectangle sets, the construction of an $\mathrm{MRS}_\Gamma(m,n;c)$ is, in general, still an open problem, see [8, 9]. In particular, the following conjecture has been proposed by Cichacz and Hinc, where $\mathcal{G}$ denotes the set of all finite abelian groups that either have odd order or contain more than one involution (i.e., an element of order two).

Conjecture 1.3. [8] *Let $m,n > 1$ and $c \ge 1$. An $\mathrm{MRS}_\Gamma(m,n;c)$ exists if and only if m and n are both even or $\Gamma \in \mathcal{G}$ and $\{m,n\} \neq \{2\ell+1, 2\}$.*

In the same spirit, we introduce the following definition.

Definition 1.4. A *magic partially filled array set* $\mathrm{MPFS}_\Omega(m,n;s,k;c)$ on a subset Ω of an abelian group $(\Gamma,+)$ is a set of c partially filled arrays of size $m \times n$ with entries in Ω such that

- (a) every $\omega \in \Omega$ appears once and in a unique array;
- (b) for every array, each row contains s filled cells and each column contains k filled cells;
- (c) there exist (not necessarily distinct) elements $x, y \in \Gamma$ such that, for every array, the sum of the elements in each row is x and the sum of the elements in each column is y.

One of our main goals is the construction of magic rectangle sets with empty cells, denoted by $\mathrm{MRS}(m,n;s,k;c)$, which are nothing but $\mathrm{MPFS}_\Omega(m,n;s,k;c)$, where $\Omega = \{1,\ldots,nkc\} \subset \mathbb{Z}$. Note that the aforementioned (tight) magic rectangle sets $\mathrm{MRS}(m,n;c)$ studied by Froncek correspond to the case $\mathrm{MPFS}_\Omega(m,n;n,m;c)$, where $\Omega = \{1,\ldots,mnc\} \subset \mathbb{Z}$. We are also interested in constructing magic partially filled arrays (sets), where the elements of each row and of each columns sum to zero.

Definition 1.5. Given a subset Ω of an abelian group $(\Gamma,+)$, we say that an $\mathrm{MPF}_\Omega(m,n;s,k)$ is a *zero-sum magic partially filled array* (and we write ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$) if the elements in each row and in each column sum to $0 \in \Gamma$. Similarly, we speak about a *zero-sum magic partially filled array set* (writing ${}^0\mathrm{MPFS}_\Omega(m,n;s,k;c)$) if, for every array, the elements in each row and in each column sum to $0 \in \Gamma$.

Examples of ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$ are the signed magic arrays, denoted by $\mathrm{SMA}(m,n;s,k)$ in [26]: they correspond to the case $\Omega = \{0, \pm 1, \pm 2, \ldots, \pm(nk-1)/2\} \subset \mathbb{Z}$ if nk is odd or $\Omega = \{\pm 1, \pm 2, \ldots, \pm nk/2\} \subset \mathbb{Z}$ if nk is even. Also the Heffter arrays, introduced by Archdeacon in [2], can be viewed as zero-sum magic partially filled arrays.

Definition 1.6. A *Heffter array* $\mathrm{H}(m,n;s,k)$ is an $m \times n$ partially filled array with elements in the cyclic group $(\mathbb{Z}_{2nk+1}, +)$ such that

(a) for every $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, either x or $-x$ appears in the array;
(b) each row contains s filled cells and each column contains k filled cells;
(c) the elements in every row and column sum to 0 in $\mathbb{Z}_{2nk+1}$.

In [5] it was proved that a square Heffter array $\mathrm{H}(n,n;k,k)$ exists for all $n \geq k \geq 3$, while in [3] the authors proved the existence of a $\mathrm{H}(m,n;n,m)$ for all $m,n \geq 3$. The first results about non-square Heffter arrays with empty cells have been obtained in [27, 29]. They confirm the following.

Conjecture 1.7. [2, Conjecture 6.3] *Given four integers m,n,s,k such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$, there exists a Heffter array $\mathrm{H}(m,n;s,k)$.*

A Heffter array $\mathrm{H}(m,n;s,k)$ is a ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$ where Ω is a subset of size nk of $\mathbb{Z}_{2nk+1} \setminus \{0\}$ such that $\Omega \cap -\Omega = \emptyset$. In other words, $\Omega \cup -\Omega$ is a partition of $\mathbb{Z}_{2nk+1} \setminus \{0\}$. In [13] the authors proposed a generalization of the notion of Heffter array studying *relative* Heffter arrays: a relative Heffter array $\mathrm{H}_t(m,n;s,k)$ is a ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$ where $\Omega \cup -\Omega$ is a partition of $\mathbb{Z}_{2nk+t} \setminus J$, with J being the subgroup of $\mathbb{Z}_{2nk+t}$ of size t. We refer to [4, 6, 12, 15, 16, 17, 29] for results about the existence and the properties of these objects. Further generalizations of Heffter arrays are described in [10, 11, 14, 28].

Magic and zero-sum magic partially filled arrays are worth to be studied not only because they generalize several combinatorial objects, as we previously explained, but also because of their connection with magic labelings (see [22, Sections 5.1 and 5.7]). We briefly describe how these labelings can be obtained.

A bipartite biregular graph $G(V,E)$ is a graph whose vertex set can be written as disjoint union $V = V_1 \cup V_2$ with $|V_1| = m$, $|V_2| = n$, and where each vertex of V_1 is connected with exactly s vertices of V_2, and each vertex of V_2 is connected with exactly k vertices of V_1. Now, let M be an $\mathrm{MPF}_\Omega(m,n;s,k)$ (or, a ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$). We associate to M a bipartite biregular graph $\Phi_M = G(V,E)$ by taking a set V_1 of m points, a set V_2 of n points, and drawing an edge $e_{i,j}$ between the i-th vertex of V_1 and the j-vertex of V_2 if the cell (i,j) of M is not empty. Define the labeling $f_M : E \to \Omega \subseteq \Gamma$, where $f_M(e_{i,j})$ is the entry of the corresponding cell (i,j) of M.

According to [30], a graph is *magic* if there is a labeling of its edges with distinct positive integers such that for each vertex v the sum of the labels of all edges incident with v is the same for all v. A such labeling is said to be a *magic labeling*. A magic labeling is called *supermagic* if the set of edge labels consists of consecutive positive integers. It is clear that a square $\mathrm{MR}(n,n;k,k)$, say M, produces a supermagic labeling of the graph Φ_M, since M has the same row and column sums.

Now, take a finite abelian group Γ of order $\ell > 1$, and write Γ^* for $\Gamma \setminus \{0_\Gamma\}$. A Γ-*supermagic labeling* of a graph $G(V,E)$ with $|E| = \ell$ is a bijection from E to Γ such that the sum of labels of all incident edges of every vertex $v \in V$ is equal to the same element $x \in \Gamma$, see [21]. If M is a ${}^0\mathrm{MPF}_\Gamma(m,n;s,k)$, then the function f_M is a Γ-supermagic labeling of Φ_M, where the constant x is 0_Γ.

A graph G is said to be *zero-sum* Γ-*magic* if there exists a labeling of the edges of G with elements of Γ^* such that, for each vertex v, the sum of the labels of the edges incident with v is equal to 0_Γ, see [1]. If M is a ${}^0\mathrm{MPF}_{\Gamma^*}(m,n;s,k)$, then Φ_M is a zero-sum Γ-magic graph. Note that in this case the labeling f_M is a bijection.

When the zero-sum magic array ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$ is actually a Heffter array, there are further applications to cyclic cycle decompositions (see [2]); when it is a tight Γ-magic rectangle, there are applications to cryptography, scheduling and statistical design of experiments (see [8] and the references therein).

Finally, we briefly describe our main achievements. The first one, proved in Section 3, extends Froncek's result about tight magic rectangle sets, providing necessary and sufficient conditions for the existence of an $\mathrm{MRS}(m,n;s,k;c)$.

Theorem 1.8. *Let m,n,s,k,c be five positive integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. An $\mathrm{MRS}(m,n;s,k;c)$ exists if and only if either nkc is odd, or s and k are both even and $sk > 4$.*

Next, keeping in mind the connection with Γ-supermagic labelings and zero-sum Γ-magic graphs, we will focus our attention on zero-sum magic partially filled arrays (sets) where Ω is a finite abelian group Γ, or the set Γ^* of its nonzero elements. In particular, we will prove the following result.

Theorem 1.9. *Let m,n,s,k,c be five positive integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. Set $d = \gcd(s,k)$.*

1. *A ${}^0\mathrm{MPFS}_{\mathbb{Z}_{nkc}}(m,n;s,k;c)$ exists if and only if nkc is odd.*
2. *If $d \equiv 0 \pmod 4$, then there exists a ${}^0\mathrm{MPFS}_\Gamma(m,n;s,k;c)$ for every abelian group $\Gamma \in \mathcal{G}$ of order nkc.*
3. *If nk is odd and $\gcd(n,d-1) = 1$, then there exists a ${}^0\mathrm{MPF}_{\mathbb{Z}_d \oplus \mathbb{Z}_{nk/d}}(m,n;s,k)$.*
4. *A ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{2nc+1}}(2,n;n,2;c)$ exists if and only if $n \geq 3$.*
5. *A ${}^0\mathrm{MPF}_{\mathbb{Z}^*_{mn+1}}(m,n;n,m)$ exists if and only if $mn > 5$ is even.*
6. *Let Γ be an abelian group of order $2n+1$. There exists a tight ${}^0\mathrm{MPF}_{\Gamma^*}(2,n;n,2)$ if and only if $\Gamma \notin \{\mathbb{Z}_5, \mathbb{Z}_3 \oplus \mathbb{Z}_3\}$.*

2. Notation, examples and preliminary results

Given two integers $a \leq b$, we denote by $[a,b]$ the set consisting of the integers $a, a+1, \ldots, b$. If $a > b$, then $[a,b]$ is empty. We denote by (i,j) the cell in the i-th row and j-th column of a partially filled array A, while $\mathcal{E}(A)$ denotes the *list* of the entries of the filled cells of A. We also write $\mathcal{E}(i,j)$ to indicate the entry of the cell (i,j) of A. Given a sequence $S = (B_1, B_2, \ldots, B_r)$ of partially filled arrays, we set $\mathcal{E}(S) = \cup_i \mathcal{E}(B_i)$.

We recall that a finite nontrivial abelian group can be written as a direct sum

$$\mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \ldots \oplus \mathbb{Z}_{n_\ell}$$

of cyclic groups $\mathbb{Z}_{n_i}$ of order $n_i > 1$. In particular, we can take these integers n_i in such a way that n_i divides n_{i+1} for all $i \in [1, \ell-1]$. The elements of a cyclic group $(\mathbb{Z}_n, +)$ of order n will be denoted by $[x]_n$. In other words, $[x]_n$ is the image of $x \in \mathbb{Z}$ by the canonical projection $\pi : \mathbb{Z} \to \mathbb{Z}_n$. More in general, given a direct sum $\Gamma_1 \oplus \Gamma_2 \oplus \ldots \oplus \Gamma_\ell$ of abelian groups, its elements will be denoted by $(x_1, x_2, \ldots, x_\ell)$, where $x_i \in \Gamma_i$ for all $i = 1, 2, \ldots, \ell$.

In the following, it will be convenient to denote an $\mathrm{MPF}_\Omega(n, n; k, k)$, an $\mathrm{MPFS}_\Omega(n, n; k, k; c)$, a ${}^0\mathrm{MPF}_\Omega(n, n; k, k)$ and a ${}^0\mathrm{MPFS}_\Omega(n, n; k, k; c)$, respectively, by $\mathrm{MPF}_\Omega(n; k)$, $\mathrm{MPFS}_\Omega(n; k; c)$, ${}^0\mathrm{MPF}_\Omega(n; k)$ and ${}^0\mathrm{MPFS}_\Omega(n; k; c)$. Furthermore, we denote a tight ${}^0\mathrm{MPF}_\Omega(m, n; n, m)$ by ${}^0\mathrm{MPF}_\Omega(m, n)$, and a tight ${}^0\mathrm{MPFS}_\Omega(m, n; n, m; c)$ by ${}^0\mathrm{MPFS}_\Omega(m, n; c)$.

Here, some examples. The arrays

$[0]_4$	$[1]_4$
$[3]_4$	$[2]_4$

and

$([0]_2, [0]_2)$	$([0]_2, [1]_2)$
$([1]_2, [0]_2)$	$([1]_2, [1]_2)$

are two magic rectangles: on the left-hand side we have an $\mathrm{MPF}_{\mathbb{Z}_4}(2; 2)$; on the right-hand side, we have an $\mathrm{MPF}_{\mathbb{Z}_2 \oplus \mathbb{Z}_2}(2; 2)$. The arrays

$([0]_2, [0]_6)$	$([1]_2, [0]_6)$	$([0]_2, [5]_6)$	$([1]_2, [1]_6)$
$([1]_2, [2]_6)$	$([1]_2, [5]_6)$	$([0]_2, [3]_6)$	$([0]_2, [2]_6)$
$([1]_2, [4]_6)$	$([0]_2, [1]_6)$	$([0]_2, [4]_6)$	$([1]_2, [3]_6)$

and

$([0]_2, [0]_6)$	$([0]_2, [2]_6)$	$([0]_2, [4]_6)$	
	$([1]_2, [0]_6)$	$([1]_2, [3]_6)$	$([0]_2, [3]_6)$
$([0]_2, [5]_6)$		$([1]_2, [5]_6)$	$([1]_2, [2]_6)$
$([0]_2, [1]_6)$	$([1]_2, [4]_6)$		$([1]_2, [1]_6)$

are, respectively, a ${}^0\mathrm{MPF}_\Gamma(3, 4)$ and a ${}^0\mathrm{MPF}_\Gamma(4; 3)$, where $\Gamma = \mathbb{Z}_2 \oplus \mathbb{Z}_6$.

Example 2.1. For any odd $n \geq 3$, the sum of the integers $1, 2, \ldots, n$ is a multiple of n, being equal to $n \cdot \frac{n+1}{2}$. Then, the array $A = (a_{i,j})$, defined by $a_{i,j} = ([i]_n, [j]_n)$ for all $i, j \in [1, n]$, is a ${}^0\mathrm{MPF}_{\mathbb{Z}_n \oplus \mathbb{Z}_n}(n, n)$.

We recall that, given an abelian group Γ, we set $\Gamma^* = \Gamma \setminus \{0_\Gamma\}$. Furthermore, $\mathcal{G}$ denotes the set of all finite abelian groups that either have odd order or contain more than one involution. Since the sum of all the elements of Γ is equal to the sum of its involutions, it is easy to see that

$$\sum_{g \in \Gamma} g = \sum_{\substack{x \in \Gamma, \\ 2x = 0}} x = \begin{cases} \iota & \text{if } \Gamma \text{ has a unique involution } \iota, \\ 0 & \text{otherwise.} \end{cases}$$

It follows that if a ${}^0\mathrm{MPFS}_\Omega(m, n; s, k; c)$ exists for some $\Omega \in \{\Gamma, \Gamma^*\}$, then $\Gamma \in \mathcal{G}$; in particular, Γ cannot be a cyclic group of even order. Moreover, if there exists a ${}^0\mathrm{MPFS}_\Gamma(m, n; s, k; c)$, then either $|\Gamma| = nkc$ is odd or $nkc \equiv 0 \pmod 4$.

The arrays

$[1]_{13}$	$[2]_{13}$	$[10]_{13}$	
	$[3]_{13}$	$[4]_{13}$	$[6]_{13}$
$[5]_{13}$		$[12]_{13}$	$[9]_{13}$
$[7]_{13}$	$[8]_{13}$		$[11]_{13}$

and

$[1]_{13}$		$[3]_{13}$		$[9]_{13}$	
	$[2]_{13}$		$[5]_{13}$		$[6]_{13}$
$[12]_{13}$		$[10]_{13}$		$[4]_{13}$	
	$[11]_{13}$		$[8]_{13}$		$[7]_{13}$

are, respectively, a ${}^0\mathrm{MPF}_{\mathbb{Z}_{13}^*}(4; 3)$ and a ${}^0\mathrm{MPF}_{\mathbb{Z}_{13}^*}(4, 6; 3, 2)$.

Let $A = (a_{i,j})$ be a partially filled square array of size n. We say that the element $a_{i,j}$ belongs to the diagonal D_r if $j - i \equiv r \pmod{n}$. We say that A is ℓ-diagonal if the nonempty cells of A are exactly those of ℓ consecutive diagonals. In particular, if A is an $\mathrm{MPF}_\Omega(n;k)$, then we say that A is diagonal if it is k-diagonal. We also say that an $\mathrm{MPFS}_\Omega(n;k;c)$ is diagonal if every member of this set is a k-diagonal partially filled array (similarly for a zero-sum magic array). For instance, this is a diagonal ${}^0\mathrm{MPF}_{\mathbb{Z}_{21}^*}(5;4)$:

$[1]_{21}$	$[19]_{21}$		$[6]_{21}$	$[16]_{21}$
$[20]_{21}$	$[5]_{21}$	$[15]_{21}$		$[2]_{21}$
$[18]_{21}$	$[4]_{21}$	$[9]_{21}$	$[11]_{21}$	
	$[14]_{21}$	$[8]_{21}$	$[13]_{21}$	$[7]_{21}$
$[3]_{21}$		$[10]_{21}$	$[12]_{21}$	$[17]_{21}$

The following theorem shows how it is possible to construct rectangular magic partially filled arrays starting from diagonal square ones. This result was actually proven in [29] for Heffter arrays, but the proof can be easily adapted to the more general context of magic partially filled array sets.

Theorem 2.2. *Let m,n,s,k,c be five positive integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. Let Ω be a subset of size nkc of an abelian group Γ. Set $d = \gcd(s,k)$. If there exists a diagonal $\mathrm{MPFS}_\Omega\left(\frac{nk}{d};d;c\right)$, then there exists an $\mathrm{MPFS}_\Omega(m,n;s,k;c)$. In particular, if there exists a diagonal ${}^0\mathrm{MPFS}_\Omega\left(\frac{nk}{d};d;c\right)$, then there exists a ${}^0\mathrm{MPFS}_\Omega(m,n;s,k;c)$.*

Sets of zero-sum magic partially filled arrays can be constructed using a sort of Kronecker product. For $i = 1,2$, let A_i be a partially filled array of size $a_i \times b_i$ whose elements belong to an abelian group Γ_i. We define the "Kronecker product" between A_1 and A_2 as the partially filled array $A_1 \otimes A_2$ of size $(a_1a_2) \times (b_1b_2)$ obtained as follows. Set $A_1 = (u_{i,j})$ and $A_2 = (v_{x,y})$, and take ℓ, h in such a way that $\ell \in [1, a_1a_2]$ and $h \in [1, b_1b_2]$. Write $\ell = q_1a_2 + r_1$ and $h = q_2b_2 + r_2$ with $1 \leq r_1 \leq a_2$ and $1 \leq r_2 \leq b_2$. Then, the cell (ℓ, h) of $A_1 \otimes A_2$ is nonempty if and only if the cell (q_1+1, q_2+1) of A_1 and the cell (r_1, r_2) of A_2 are nonempty. In such case, $\mathcal{E}(\ell,h) = (u_{q_1+1,q_2+1}, v_{r_1,r_2}) \in \Gamma_1 \oplus \Gamma_2$.

For instance, take the following partially filled arrays:

$A_1 =$

0		1
2	3	

, $A_2 =$

4	5		
		6	7
8			9

.

Then, the product $A_1 \otimes A_2$ is

$(0,4)$	$(0,5)$							$(1,4)$	$(1,5)$		
		$(0,6)$	$(0,7)$							$(1,6)$	$(1,7)$
$(0,8)$			$(0,9)$					$(1,8)$			$(1,9)$
$(2,4)$	$(2,5)$			$(3,4)$	$(3,5)$						
		$(2,6)$	$(2,7)$			$(3,6)$	$(3,7)$				
$(2,8)$			$(2,9)$				$(3,9)$				

.

Lemma 2.3. *If there exist a ${}^0\mathrm{MPFS}_{\Omega_1}(m_1,n_1;s_1,k_1;c_1)$ and a ${}^0\mathrm{MPFS}_{\Omega_2}(m_2,n_2;s_2,k_2;c_2)$, where $\Omega_1 \subseteq \Gamma_1$ and $\Omega_2 \subseteq \Gamma_2$, then there exists a ${}^0\mathrm{MPFS}_{\Omega_1\times\Omega_2}(m_1m_2,n_1n_2;s_1s_2,k_1k_2;c_1c_2)$, where $\Omega_1 \times \Omega_2 \subseteq \Gamma_1 \oplus \Gamma_2$.*

Proof. Let A_i be an array of the set ${}^0\mathrm{MPFS}_{\Omega_1}(m_1, n_1; s_1, k_1; c_1)$ and B_j be an array of the set ${}^0\mathrm{MPFS}_{\Omega_2}(m_2, n_2; s_2, k_2; c_2)$. Then $C = A_i \otimes B_j$ is a partially filled array of size $(m_1m_2) \times (n_1n_2)$ such that: every row of C contains s_1s_2 elements and each column of C contains k_1k_2 elements; the elements of each row of C sum to $(s_2 \cdot 0_{\Gamma_1}, s_1 \cdot 0_{\Gamma_2}) = 0_\Gamma$; the elements of each column of C sum to $(k_2 \cdot 0_{\Gamma_1}, k_1 \cdot 0_{\Gamma_2}) = 0_\Gamma$; $\mathcal{E}(C) = \{(x, y) \mid x \in \mathcal{E}(A_i), y \in \mathcal{E}(B_j)\}$. We conclude that the set $\{A_i \otimes B_j \mid i \in [1, c_1], j \in [1, c_2]\}$ is a ${}^0\mathrm{MPFS}_{\Omega_1\times\Omega_2}(m_1m_2, n_1n_2; s_1s_2, k_1k_2; c_1c_2)$. □

3. Magic rectangle sets with empty cells

In this section we solve the existence problem of a magic rectangle set with empty cells. Our starting point is Froncek's result about (tight) magic rectangle sets. As we recalled in Section 1, Froncek studied these objects, proving the following.

Theorem 3.1. [20, Theorem 3.2] *Let m, n, c be positive integers such that $m, n \geq 2$. A magic rectangle set $\mathrm{MRS}(m, n; n, m; c)$ exists if and only if either mnc is odd, or m, n are both even and $mn > 4$ (c arbitrary).*

Remark 3.2. Suppose that an $\mathrm{MRS}(m, n; s, k; c)$ exists. Then, for every array of this set, the elements of each row and each column sum, respectively, to $\frac{s(nkc+1)}{2}$ and to $\frac{k(nkc+1)}{2}$. So, if nkc is even, then s and k must be both even.

First, we consider the case when k (or s) is equal to 2.

Proposition 3.3. *Let m, n, s, c be four positive integers such that $2 \leq s \leq n$ and $ms = 2n$. An $\mathrm{MRS}(m, n; s, 2; c)$ exists if and only if $s \geq 4$ is even.*

Proof. Set $N = 2nc + 1$. Suppose that an $\mathrm{MRS}(m, n; s, 2; c)$ exists. By Remark 3.2, s must be even. Furthermore, it is easy to see that an $\mathrm{MRS}(n, n; 2, 2; c)$ does not exist. We conclude that s must be an even integer greater than 2.

Now, suppose that $s \geq 4$ is even and write $s = 2\bar{s}$. Hence, we have $n = m\bar{s}$. Our construction of an $\mathrm{MRS}(m, m\bar{s}; 2\bar{s}, 2; c)$ depends on the parity of $\bar{s}$, and uses some basic blocks. Given an integer $x \geq 0$, we construct two 2-diagonal $m \times m$ partially filled arrays U_x, V_x:

$U_x =$

$x+1$	$N-(x+2)$			
	$x+2$	$N-(x+3)$		
		$\ddots$	$\ddots$	
			$x+(m-1)$	$N-(x+m)$
$N-(x+1)$				$x+m$

,

$V_x =$

$x+m$	$N-(x+m-1)$			
	$x+(m-1)$	$N-(x+m-2)$		
		$\ddots$	$\ddots$	
			$x+2$	$N-(x+1)$
$N-(x+m)$				$x+1$

.

In both cases, the elements of each column sum to N; the row sums of U_x are $(N-1, \ldots, N-1, N+m-1)$, while the row sums of V_x are $(N+1, \ldots, N+1, N-m+1)$. Furthermore, $\mathcal{E}(U_x) = \mathcal{E}(V_x) = [x+1, x+m] \cup [N-(x+m), N-(x+1)]$. Now, let A_x be the partially filled array obtained by the juxtaposition of U_x and V_{x+m}. Then, A_x is an $\mathrm{MPF}_\Omega(m, 2m; 4, 2)$, where $\Omega = \mathcal{E}(A_x) = [x+1, x+2m] \cup [N-(x+2m), N-(x+1)]$, the elements of every column sum to N, and the elements of every row sum to $2N$.

Assume $\bar{s} = 2t$ with $t \geq 1$. For every $\ell \geq 0$, let R_ℓ be the $m \times n$ partially filled array obtained by the juxtaposition of $A_{2m(t\ell)}, A_{2m(t\ell+1)}, \ldots, A_{2m(t\ell+t-1)}$. By construction, every column of R_ℓ contains 2 filled cells, and every row contains $4t = s$ filled cells. The elements of each column sum to N, while the elements of each row sum to $2Nt = N\bar{s}$. Furthermore,

$$\mathcal{E}(R_\ell) = [m\bar{s}\ell + 1, m\bar{s}(\ell+1)] \cup [N - m\bar{s}(\ell+1), N - (m\bar{s}\ell + 1)].$$

Our $\mathrm{MRS}(m, n; s, 2; c)$ consists of $R_0, R_1, \ldots, R_{c-1}$. In fact,

$$\bigcup_{\ell=0}^{c-1} \mathcal{E}(R_\ell) = [1, m\bar{s}c] \cup [N - m\bar{s}c, N-1] = [1, nc] \cup [nc+1, 2nc] = [1, 2nc].$$

Next, assume $\bar{s} = 2t+1$ with $t \geq 1$. Given an integer $x \geq 0$, we construct a 2-diagonal $m \times m$ partially filled array W_x as follows:

$W_x =$

$x+(2m-1)$	$N-(x+2m-3)$			
	$x+(2m-3)$	$N-(x+2m-5)$		
		$\ddots$	$\ddots$	
			$x+3$	$N-(x+1)$
$N-(x+2m-1)$				$x+1$

The elements of each column of W_x sum to N, while the row sums are $(N+2, \ldots, N+2, N+2-2m)$. Furthermore, $\mathcal{E}(W_x) = \{x+1, x+3, \ldots, x+(2m-1)\} \cup \{N-(x+2m-1), N-(x+2m-3), \ldots, N-(x+1)\}$.

Let $Y_x, \widetilde{Y}_x$ be the partially filled arrays obtained by the juxtaposition of U_x, U_{x+m}, W_{x+2m} and of $W_{x+1}, U_{x+2m}, U_{x+3m}$, respectively. Then, Y_x and $\widetilde{Y}_x$ are two $\mathrm{MPF}_\Omega(m, 3m; 6, 2)$, where

$$\begin{aligned}
\Omega = \mathcal{E}(Y_x) &= [x+1, x+2m] \cup [N-(x+2m), N-(x+1)] \\
&\cup \{x+2m+1, x+2m+3, \ldots, x+4m-1\} \\
&\cup \{N-(x+4m-1), N-(x+4m-3), \ldots, N-(x+2m+1)\}, \\
\Omega = \mathcal{E}(\widetilde{Y}_x) &= \{x+2, x+4, \ldots, x+2m\} \cup \{N-(x+2m), N-(x+2m-2), \ldots, \\
&N-(x+2)\} \cup [x+2m+1, x+4m] \cup [N-(x+4m), N-(x+2m+1)].
\end{aligned}$$

In both cases, the elements of each row and each column sum, respectively, to $3N$ and to N.

For every $\ell \geq 0$, let $R_{2\ell}, R_{2\ell+1}$ be the $m \times n$ partially filled arrays obtained by the juxtaposition, respectively, of

$$A_{2m(2t+1)\ell}, A_{2m((2t+1)\ell+1)}, \ldots, A_{2m((2t+1)\ell+t-2)}, Y_{2m((2t+1)\ell+t-1)},$$

and of

$$\widetilde{Y}_{2m((2t+1)\ell+t)}, A_{2m((2t+1)\ell+t+2)}, A_{2m((2t+1)\ell+t+3)}, \ldots, A_{2m((2t+1)\ell+2t)}.$$

In both cases, every column contains 2 filled cells and every row contains $4(t-1)+6 = s$ filled cells; the elements of each column sum to N, while the elements of each row sum to $2N(t-1) + 3N = N\bar{s}$. Furthermore,

$$\begin{array}{rcl} \mathcal{E}(R_{2\ell}) & = & [ms\ell+1, ms\ell+2mt] \cup [N-(ms\ell+2mt), N-(ms\ell+1)] \\ & & \cup\{ms\ell+2mt+1, ms\ell+2mt+3, \ldots, ms\ell+2mt+2m-1\} \\ & & \cup\{N-(ms\ell+2mt+2m-1), N-(ms\ell+2mt+2m-3), \ldots, \\ & & N-(ms\ell+2mt+1)\}, \\ \mathcal{E}(R_{2\ell+1}) & = & \{ms\ell+2mt+2, ms\ell+2mt+4, \ldots, ms\ell+2mt+2m\} \\ & & \cup\{N-(ms\ell+2mt+2m), N-(ms\ell+2mt+2m-2), \ldots, \\ & & N-(ms\ell+2mt+2)\} \cup [ms\ell+2m(t+1)+1, ms(\ell+1)] \\ & & \cup[N-ms(\ell+1), N-(ms\ell+2m(t+1)+1)]. \end{array}$$

Note that

$$\mathcal{E}(R_{2\ell}) \cup \mathcal{E}(R_{2\ell+1}) = [ms\ell+1, ms(\ell+1)] \cup [N-ms(\ell+1), N-(ms\ell+1)].$$

Our $\mathrm{MRS}(m,n;s,2;c)$ consists of $R_0, R_1, \ldots, R_{c-1}$. In fact, if c is even, then

$$\bigcup_{\ell=0}^{(c-2)/2} (\mathcal{E}(R_{2\ell}) \cup \mathcal{E}(R_{2\ell+1})) = [1, m\bar{s}c] \cup [N-m\bar{s}c, N-1] = [1, 2nc].$$

If c is odd, then $\mathcal{E}(R_{c-1}) = [m\bar{s}(c-1)+1, m\bar{s}(c+1)]$ and

$$\begin{array}{rcl} \mathcal{E}(R_{c-1}) \cup \bigcup_{\ell=0}^{(c-3)/2} (\mathcal{E}(R_{2\ell}) \cup \mathcal{E}(R_{2\ell+1})) & = & [m\bar{s}(c-1)+1, m\bar{s}(c+1)] \cup [1, m\bar{s}(c-1)] \\ & & \cup[m\bar{s}(c+1)+1, 2m\bar{s}c] \\ & = & [1, 2nc]. \end{array}$$

This concludes our proof. □

As recalled in Section 1, a signed magic array $\mathrm{SMA}(m,n;s,k)$ is a ${}^0\mathrm{MPF}_\Omega(m,n;s,k)$, where $\Omega = \left[-\frac{nk-1}{2}, +\frac{nk-1}{2}\right]$ if nk is odd or $\Omega = \left[-\frac{nk}{2}, -1\right] \cup \left[1, \frac{nk}{2}\right]$ if nk is even.

Proposition 3.4. *Let m,n,s,k,c be five positive integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. There exists an* $\mathrm{MRS}(m,n;s,k;c)$ *if and only if either nkc is odd, or s and k are both even.*

Proof. Set $d = \gcd(s,k)$ and suppose that the product nkc is odd. Assume $d \neq 1$, hence $d \geq 3$. By [25, Corollary 7], there exists a diagonal MRS $\left(\frac{nk}{d}; d; c\right)$: hence, the existence of an $\mathrm{MRS}(m,n;s,k;c)$ follows from Theorem 2.2. Assume $d = 1$. Since s and k are coprime, we can write $m = kr$ and $n = sr$, where $r \geq 1$ is an odd integer. Let $A_0, \ldots, A_{rc-1}$ be the arrays of a magic rectangle set $\mathrm{MRS}(k,s;rc)$, whose existence is guaranteed by Theorem 3.1. Write $A_\ell = \left(a_{i,j}^{(\ell)}\right)$, with $i \in [1,k]$, $j \in [1,s]$ and $\ell \in [0, rc-1]$. We construct c partially filled arrays of size $m \times n$ by taking empty arrays R_t, $t \in [0, c-1]$, of size $(kr) \times (sr)$ and filling them in such a way that the entry of the cell $(ku+i, su+j)$, $u \in [0, r-1]$, of R_t is $a_{i,j}^{(rt+u)}$. Hence, the arrays $R_0, \ldots, R_{c-1}$ so constructed are the members of an $\mathrm{MRS}(m,n;s,k;c)$. In fact, its entries are the integers of $[1, ksrc] = [1, nkc]$, each row of R_t contains s filled cells and each column contains k filled cells; the elements of every row sum to $s\frac{ksrc+1}{2} = s\frac{nkc+1}{2}$, while the elements of every column sum to $k\frac{nkc+1}{2}$.

Now, suppose that s and k are both even. By [29, Proposition 5.7] there exists a shiftable $\mathrm{SMA}(m,n;s,k)$, say A, that is, a signed magic array where every row and every column contains an equal number of positive and negative entries. For every $\ell \in [1,c]$, let R_ℓ be the array obtained from A by replacing every positive entry x of A with $x + \frac{nk}{2}(2c-\ell)$ and replacing every negative entry y with $y + \frac{nk}{2}\ell + 1$. So, the elements of each row of R_ℓ sum to $\frac{s}{2}(\frac{nk}{2}(2c-\ell) + \frac{nk}{2}\ell + 1) = \frac{s}{2}(nkc+1)$ and the elements of each column sum to $\frac{k}{2}(nkc+1)$. Furthermore, $\mathcal{E}(R_\ell) = \left[\frac{nk}{2}(\ell-1)+1, \frac{nk}{2}\ell\right] \cup \left[\frac{nk}{2}(2c-\ell)+1, \frac{nk}{2}(2c+1-\ell)\right]$. It follows that

$$\bigcup_{\ell=1}^{c} \mathcal{E}(R_\ell) = \left[1, \frac{nk}{2}c\right] \cup \left[\frac{nk}{2}c+1, \frac{nk}{2}2c\right] = [1, nkc],$$

and so $R_1, \ldots, R_c$ are the members of an $\mathrm{MRS}(m,n;s,k;c)$.

This proves the existence of an $\mathrm{MRS}(m,n;s,k;c)$ whenever nkc is odd or n and k are both even. Vice-versa, if an $\mathrm{MRS}(m,n;s,k;c)$ exists, either nkc is odd, or s and k are both even, by Remark 3.2. □

Proof of Theorem 1.8. If $s,k \geq 3$, the result follows from Proposition 3.4. Also, since the transpose of an $\mathrm{MPF}_\Omega(m,n;s,k)$ is an $\mathrm{MPF}_\Omega(n,m;k,s)$, we may assume $k=2$, and hence the result follows from Proposition 3.3. □

Corollary 3.5. *Let m,n,s,k be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. If nk is odd, then there exists an $\mathrm{SMA}(m,n;s,k)$.*

Proof. Let A be an $\mathrm{MR}(m,n;s,k)$, whose existence follows from Theorem 1.8 for $c=1$. Set $w = \frac{nk+1}{2}$, so that the elements of each row of A sum to sw, while the elements of each column sum to kw. Replacing each entry $x \in [1,nk]$ of A with $x-w$, we obtain an $\mathrm{SMA}(m,n;s,k)$, say B. In fact, $\mathcal{E}(B) = [1-w, nk-w] = \left[-\frac{nk-1}{2}, +\frac{nk-1}{2}\right]$. Moreover, the elements of each row of B sum to $sw - sw = 0$; similarly, the elements of each column sum to 0. □

4. Some constructions for the case $\Omega = \Gamma$

First, we consider the case when Γ is a cyclic group. Thanks to the results of Section 3 we obtain the following.

Corollary 4.1. *Let m,n,s,k,c be five positive integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. A ${}^0\mathrm{MPFS}_{\mathbb{Z}_{nkc}}(m,n;s,k;c)$ exists if and only if nkc is odd.*

Proof. If a ${}^0\mathrm{MPFS}_{\mathbb{Z}_{nkc}}(m,n;s,k;c)$ exists, then nkc must be odd. In this case, by Theorem 1.8 there exists an $\mathrm{MRS}(m,n;s,k;c)$, whose members are, say, $A_1, \ldots, A_c$. Write $w = \frac{nkc+1}{2}$ and replace each entry $x \in [1,nkc]$ of A_ℓ with $[x-w]_{nkc} \in \mathbb{Z}_{nkc}$, obtaining a partially filled array B_ℓ. Then, $\{B_1, \ldots, B_c\}$ is a ${}^0\mathrm{MPFS}_{\mathbb{Z}_{nkc}}(m,n;s,k;c)$. In fact, the elements of each row of B_ℓ sum to $[sw]_{nkc} - s[w]_{nkc} = [0]_{nkc}$; similarly, the elements of each column sum to $[0]_{nkc}$. Furthermore,

$$\begin{aligned}\bigcup_{\ell=1}^{c} \mathcal{E}(B_\ell) &= \{[1-w]_{nkc}, [2-w]_{nkc}, \ldots, [nkc-w]_{nkc}\} \\ &= \left\{\left[-\tfrac{nk-1}{2}\right]_{nkc}, \left[-\tfrac{nk-1}{2}+1\right]_{nkc}, \ldots, \left[\tfrac{nk-1}{2}\right]_{nkc}\right\} = \mathbb{Z}_{nkc}.\end{aligned}$$

□

Example 4.2. We start by taking the arrays A_0, A_1, A_2 of an $\mathrm{MRS}(3,5;3)$:

$A_0 =$

1	35	9	31	39
24	28	20	27	16
44	6	40	11	14

, $A_1 =$

2	36	7	32	38
45	4	41	12	13
22	29	21	25	18

,

$A_2 =$

3	34	8	33	37
43	5	42	10	15
23	30	19	26	17

.

Following the proofs of Proposition 3.4 and Corollary 4.1, we get the ${}^0\mathrm{MPF}_{\mathbb{Z}_{45}}(9,15;5,3)$ of Figure 1.

23	12	31	8	16										
1	5	42	4	38										
21	28	17	33	36										
					24	13	29	9	15					
					22	26	18	34	35					
					44	6	43	2	40					
										25	11	30	10	14
										20	27	19	32	37
										0	7	41	3	39

.

FIGURE 1. A ${}^0\mathrm{MPF}_{\mathbb{Z}_{45}}(9,15;5,3)$, where each entry x must be read as $[x]_{45}$.

Now, we completely solve the case when $k \equiv 0 \pmod 4$ (there is no need to assume that Γ is noncyclic).

Proposition 4.3. *Suppose $n \geq k \geq 4$ with $k \equiv 0 \pmod 4$. Then, there exists a diagonal ${}^0\mathrm{MPFS}_\Gamma(n;k;c)$ for every $\Gamma \in \mathcal{G}$ of order nkc.*

Proof. Since $k \equiv 0 \pmod 4$ and $\Gamma \in \mathcal{G}$, we can write $\Gamma = \mathbb{Z}_{2a} \oplus \mathbb{Z}_{2b} \oplus \Gamma'$, where $a, b \geq 1$ and $|\Gamma'| \geq 1$. Let $\Lambda = [1,a] \times [1,b] \times \Gamma' \subset \mathbb{Z} \oplus \mathbb{Z} \oplus \Gamma'$. For any $(x,y,g) \in \Lambda$, consider the following 3×2 partially filled array with elements in Γ:

$B(x,y,g) =$

$([2x]_{2a}, [2y]_{2b}, g)$	$-([2x+1]_{2a}, [2y]_{2b}, g)$
$-([2x]_{2a}, [2y+1]_{2b}, g)$	$([2x+1]_{2a}, [2y+1]_{2b}, g)$

.

Note that the elements in the nonempty rows sum to $([-1]_{2a}, [0]_{2b}, 0_{\Gamma'})$ and $([1]_{2a}, [0]_{2b}, 0_{\Gamma'})$, while the elements of the columns sum to $([0]_{2a}, [-1]_{2b}, 0_{\Gamma'})$ and $([0]_{2a}, [1]_{2b}, 0_{\Gamma'})$. We use this 3×2 block for constructing partially filled arrays whose rows and columns sum to 0_Γ. Define $\mathcal{B} = \{B(x,y,g) \mid (x,y,g) \in \Lambda\}$: this is a set of cardinality $ab|\Gamma'| = \frac{|\Gamma|}{4} = \frac{nkc}{4}$. So, write $\mathcal{B} = \left(X_1, X_2, \ldots, X_{\frac{k}{4}nc}\right)$. Taking an empty $n \times n$ array A_1, arrange the first n blocks of $\mathcal{B}$ in such a way that the element of the cell $(1,1)$ of X_j fills the cell (j,j) of A_1 (we work modulo n on row/column indices). In this way, we fill the diagonals $D_{n-2}, D_{n-1}, D_0, D_1$. In particular, every row has 4 filled cells and every column has 4 filled cells. Looking at the rows, the elements belonging to the diagonals D_0, D_1 sum to $([-1]_{2a}, [0]_{2b}, 0_{\Gamma'})$, while the elements belonging to the diagonals D_{n-2}, D_{n-1} sum to $([1]_{2a}, [0]_{2b}, 0_{\Gamma'})$. Looking at the columns, the elements belonging to the diagonals D_0, D_{n-2} sum to $([0]_{2a}, [-1]_{2b}, 0_{\Gamma'})$, while the elements belonging to the diagonals D_1, D_{n-1} sum to $([0]_{2a}, [1]_{2b}, 0_{\Gamma'})$. Then A_1 has row/column sums equal to 0_Γ.

Applying this process $\frac{k}{4}$ times (working with $X_{n+1}, X_{n+2}, \ldots, X_{2n}$ on the diagonals D_2, D_3, D_4, D_5, and so on), we obtain a partially filled array A_1, whose rows and columns have exactly k filled cells. Finally, we repeat this entire process $c-1$ times, obtaining a set $A_1, \ldots, A_c$ of partially filled arrays. To prove that this set is a diagonal ${}^0\mathrm{MPFS}_\Gamma(n;k;c)$ it suffices to check that $\bigcup_{\ell=1}^{c} \mathcal{E}(A_\ell) = \mathcal{E}(\mathcal{B})$ is equal to Γ.

Considering the four entries of each X_i, we can write $\mathcal{E}(\mathcal{B}) = S_1 \cup S_2 \cup S_3 \cup S_4$, where

$$\begin{array}{rcl} S_1 & = & \{([2x]_{2a}, [2y]_{2b}, g) \mid (x,y,g) \in \Lambda\},\\ S_2 & = & \{-([2x+1]_{2a}, [2y]_{2b}, g) \mid (x,y,g) \in \Lambda\},\\ S_3 & = & \{-([2x]_{2a}, [2y+1]_{2b}, g) \mid (x,y,g) \in \Lambda\},\\ S_4 & = & \{([2x+1]_{2a}, [2y+1]_{2b}, g) \mid (x,y,g) \in \Lambda\}. \end{array}$$

Clearly, the sets S_1, S_2, S_3, S_4 are pairwise disjoint. Fixed $x, y \in \mathbb{Z}$, we have $-[2x+1]_{2a} = [2x'+1]_{2a}$ and $-[2y]_{2b} = [2y']_{2b}$ for some $x' \in [1,a]$ and some $y' \in [1,b]$. It follows that

$$\begin{array}{rcl} S_2 & = & \{([2x+1]_{2a}, [2y]_{2b}, g) \mid (x,y,g) \in \Lambda\},\\ S_3 & = & \{([2x]_{2a}, [2y+1]_{2b}, g) \mid (x,y,g) \in \Lambda\}. \end{array}$$

Then $\mathcal{E}(\mathcal{B}) = \mathbb{Z}_{2a} \oplus \mathbb{Z}_{2b} \oplus \Gamma' = \Gamma$. □

Following the proof of the previous proposition, we construct a ${}^0\mathrm{MPFS}_{\mathbb{Z}_6\oplus\mathbb{Z}_2\oplus\mathbb{Z}_4}(6;4;2)$, where each entry xyz must be read as $([x]_6, [y]_2, [z]_4)$:

000	500			410	310
311	001	503			413
010	110	002	502		
	013	111	003	501	
		012	112	200	300
303			011	113	201

,

202	302			212	512
513	203	301			211
412	312	400	100		
	411	313	401	103	
		210	510	402	102
101			213	511	403

.

Now, we provide a construction when k is odd.

Proposition 4.4. *Suppose that $n \geq k \geq 3$ are odd integers such that $\gcd(n, k-1) = 1$. Then there exists a diagonal ${}^0\mathrm{MPF}_{\mathbb{Z}_k\oplus\mathbb{Z}_n}(n;k)$.*

Proof. Write $n = 2a+1$ and $k = 2b+1$. Take an empty $n \times n$ array and fill the diagonals $D_{n-b}, \ldots, D_{n-1}, D_0, D_1, \ldots, D_b$ as follows. Note that we give the elements for each diagonal, starting with those belonging to the first row. For $\ell \in [1,b]$,

$$\begin{array}{rl} D_\ell: & ([2\ell]_k, [1]_n),\ ([2\ell]_k, [2]_n),\ \ldots, ([2\ell]_k, [n]_n);\\ D_{n-\ell}: & ([2\ell-1]_k, [1]_n),\ ([2\ell-1]_k, [2]_n),\ \ldots, ([2\ell-1]_k, [n]_n);\\ D_0: & ([0]_k, [1-k]_n),\ ([0]_k, [2(1-k)]_n)\ \ldots, ([0]_k, [n(1-k)]_n). \end{array}$$

Call A the partially filled array so obtained. By construction, we fill k cells in each row and each column of A. Also, we have

$$\bigcup_{\ell=1}^{b} (\mathcal{E}(D_\ell) \cup \mathcal{E}(D_{n-\ell})) = \{(x,y) \mid x \in \mathbb{Z}_k^*,\ y \in \mathbb{Z}_n\}$$

and $\mathcal{E}(D_0) = \{([0]_k, y) \mid y \in \mathbb{Z}_n\}$, whence $\mathcal{E}(A) = \mathbb{Z}_k \oplus \mathbb{Z}_n$. For every $i \in [1,n]$, the i-th row of A contains the element $([0]_k, [i(1-k)]_n)$ and the elements $([x]_k, [i]_n)$ with $x \in [1, k-1]$. The sum of these elements is

$$([0]_k, [i(1-k)]_n) + \sum_{x=1}^{k-1} ([x]_k, [i]_n) = ([0]_k, [i(1-k) + (k-1)i]_n) = ([0]_k, [0]_n).$$

The i-th column of A contains the element $([0]_k, [i(1-k)]_n)$ and the elements $([2\ell-1]_k, [\ell+i]_n)$, $([2\ell]_k, [n-\ell+i]_n)$ with $\ell \in [1, b]$. The sum of these elements is

$$([0]_k, [i(1-k)]_n) + \sum_{\ell=1}^{b} (([2\ell-1]_k, [\ell+i]_n) + ([2\ell]_k, [i-\ell]_n)) =$$
$$([0]_k, [i(1-k)]_n) + ([0]_k, [2ib]_n) = ([0]_k, [i(1-k) + i(k-1)]_n) = ([0]_k, [0]_n).$$

This proves that A is a ${}^0\mathrm{MPF}_{\mathbb{Z}_k \oplus \mathbb{Z}_n}(n; k)$. □

Here, a diagonal ${}^0\mathrm{MPF}_{\mathbb{Z}_5 \oplus \mathbb{Z}_7}(7; 5)$ obtained following the proof of the previous proposition:

$([0]_5,[3]_7)$	$([2]_5,[1]_7)$	$([4]_5,[1]_7)$			$([3]_5,[1]_7)$	$([1]_5,[1]_7)$
$([1]_5,[2]_7)$	$([0]_5,[6]_7)$	$([2]_5,[2]_7)$	$([4]_5,[2]_7)$			$([3]_5,[2]_7)$
$([3]_5,[3]_7)$	$([1]_5,[3]_7)$	$([0]_5,[2]_7)$	$([2]_5,[3]_7)$	$([4]_5,[3]_7)$		
	$([3]_5,[4]_7)$	$([1]_5,[4]_7)$	$([0]_5,[5]_7)$	$([2]_5,[4]_7)$	$([4]_5,[4]_7)$	
		$([3]_5,[5]_7)$	$([1]_5,[5]_7)$	$([0]_5,[1]_7)$	$([2]_5,[5]_7)$	$([4]_5,[5]_7)$
$([4]_5,[6]_7)$			$([3]_5,[6]_7)$	$([1]_5,[6]_7)$	$([0]_5,[4]_7)$	$([2]_5,[6]_7)$
$([2]_5,[0]_7)$	$([4]_5,[0]_7)$			$([3]_5,[0]_7)$	$([1]_5,[0]_7)$	$([0]_5,[0]_7)$

.

5. Some constructions for the case $\Omega = \Gamma^*$

Also in this case, we firstly consider the cyclic case. Thanks to Theorem 1.8 we obtain the following result.

Lemma 5.1. *Let m, n, s, k, c be five positive integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. Suppose that s and k are both even and such that $(s,k) \neq (2,2)$. Then, there exists a ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{nkc+1}}(m, n; s, k; c)$.*

Proof. By Theorem 1.8, there exists an $\mathrm{MRS}(m, n; s, k; c)$, say $\mathcal{B} = \{A_1, A_2, \ldots, A_c\}$. Call $\widetilde{\mathcal{B}} = \{B_1, B_2, \ldots, B_c\}$ the set obtained by replacing each entry $x \in \mathbb{Z}$ of A_i with $[x] \in \mathbb{Z}_{nkc+1}$. Then, $\mathcal{E}(\widetilde{\mathcal{B}}) = \{[x]_{nkc+1} \mid x \in [1, nkc]\} = \mathbb{Z}^*_{nkc+1}$. Since s is even and the elements of each row of A_i sum to $\frac{s}{2}(nkc+1)$, the elements of each row of B_i sum to $[0]_{nkc+1}$. Similarly for the columns. We conclude that $\widetilde{\mathcal{B}}$ is a ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{nkc+1}}(m, n; s, k; c)$. □

We also make use of the known results about signed magic arrays. We recall that in [26] it was proved that an $\mathrm{SMA}(2, n; n, 2)$ exists if and only if $n \equiv 0, 3 \pmod 4$, while an $\mathrm{SMA}(m, n; n, m)$ exists for all $m, n > 2$.

Theorem 5.2. [29] *Let m, n, s, k be four integers such that $3 \leq s \leq n$, $3 \leq k \leq m$ and $ms = nk$. There exists an $\mathrm{SMA}(m, n; s, k)$ whenever $\gcd(s, k) \geq 2$, or $s \equiv 0 \pmod 4$, or $k \equiv 0 \pmod 4$. Furthermore, there exists a diagonal $\mathrm{SMA}(n, n; k, k)$ for any $n \geq k \geq 3$.*

Replacing the entry $x \in \mathbb{Z}$ of an $\mathrm{SMA}(m, n; s, k)$, with the element $[x]_{nk+1}$ of $\mathbb{Z}_{nk+1}$, we get the following.

Lemma 5.3. *Suppose there exists an $\mathrm{SMA}(m, n; s, k)$, where nk is even. Then there exists a ${}^0\mathrm{MPF}_{\mathbb{Z}^*_{nk+1}}(m, n; s, k)$.*

From Theorem 5.2 and Lemma 5.3 we obtain the following result.

Corollary 5.4. *Let m, n, s, k be four integers such that $2 \leq s \leq n$, $2 \leq k \leq m$ and $ms = nk$. If nk is even, then there exists a ${}^0\mathrm{MPF}_{\mathbb{Z}^*_{nk+1}}(m, n; s, k)$ in each of the following cases:*

1. $s = n \geq 3$ *and* $k = m \geq 3$, *or* $m = n$ *and* $s = k \geq 3$;
2. $k = m = 2$ *and* $s = n \equiv 0, 3 \pmod 4$, *or* $s = n = 2$ *and* $k = m \equiv 0, 3 \pmod 4$;
3. $s \equiv 0 \pmod 4$ *or* $k \equiv 0 \pmod 4$;
4. $\gcd(s, k) \geq 2$.

Proposition 5.5. *A tight ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{2nc+1}}(2, n; c)$ exists if and only if $n \geq 3$.*

Proof. As already observed, there is no ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{2nc+1}}(2, n; c)$ when $n = 1, 2$. If $n \geq 4$ is even, we apply Lemma 5.1 taking $m = k = 2$ and $n = s$. So, we may assume that $n \geq 3$ is odd.

We first consider the case $c \geq 3$. Let H be a Heffter array $\mathrm{H}(c, n; n, c)$ (see Definition 1.6), whose existence was proved in [3]. So, there exists a subset Ω of $\mathbb{Z}_{2nc+1}$ such that $\Omega \cup -\Omega$ is a partition of $\mathbb{Z}^*_{2nc+1}$. Let $R_1, R_2, \ldots, R_c$ be the rows of H: by definition, the elements of each R_i sum to $[0]_{2nc+1}$. For every $i \in [1, c]$ we construct the $2 \times n$ array A_i, by taking

$$A_i = \begin{array}{|c|}\hline R_i \\ \hline -R_i \\ \hline \end{array}.$$

By construction, the elements of each column of A_i sum to $[0]_{2nc+1}$. Also, taking $\mathcal{B} = \{A_1, A_2, \ldots, A_c\}$, we have $\mathcal{E}(\mathcal{B}) = \Omega \cup -\Omega = \mathbb{Z}^*_{2nc+1}$, showing that $\mathcal{B}$ is a ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{2nc+1}}(2, n; c)$.

Now, we deal with the case $c = 1$. If $n \equiv 3 \pmod 4$, then the result follows from Corollary 5.4. So, suppose $n \equiv 1 \pmod 4$, and write $n = 4\ell + 5$ with $\ell \geq 0$. Take the following subsets of $[1, n]$:

$$\Lambda_+ = \{\ell + 1\} \cup [\ell + 3, 3\ell + 5] \quad \text{and} \quad \Lambda_- = [1, \ell] \cup \{\ell + 2\} \cup [3\ell + 6, 4\ell + 5].$$

Then $\Lambda_+ \cup \Lambda_- = [1, n]$ and

$$\begin{array}{rcl} \sum\limits_{\lambda \in \Lambda_+} \lambda & = & (\ell + 1) + \binom{3\ell+6}{2} - \binom{\ell+3}{2} = 4\ell^2 + 15\ell + 13, \\ \sum\limits_{\lambda \in \Lambda_-} \lambda & = & \binom{\ell+1}{2} + (\ell + 2) + \binom{4\ell+6}{2} - \binom{3\ell+6}{2} = 4\ell^2 + 7\ell + 2. \end{array}$$

Now, let $(i_1, \ldots, i_{2\ell+4})$ be any ordering of Λ_+ and $(j_1, \ldots, j_{2\ell+1})$ be any ordering of Λ_-. Take the following $2 \times n$ array with elements in $\mathbb{Z}^*_{2n+1}$:

$$A = \begin{array}{|c|c|c|c|c|c|}\hline [+i_1]_{2n+1} & \cdots & [+i_{2\ell+4}]_{2n+1} & [-j_1]_{2n+1} & \cdots & [-j_{2\ell+1}]_{2n+1} \\ \hline [-i_1]_{2n+1} & \cdots & [-i_{2\ell+4}]_{2n+1} & [+j_1]_{2n+1} & \cdots & [+j_{2\ell+1}]_{2n+1} \\ \hline \end{array}.$$

Clearly, each column sums to $[0]_{2n+1}$. Furthermore, the elements of the first row of A sum to $[4\ell^2 + 15\ell + 13]_{2n+1} - [4\ell^2 + 7\ell + 2]_{2n+1} = [8\ell + 11]_{2n+1} = [0]_{2n+1}$. This shows that A is a ${}^0\mathrm{MPF}_{\mathbb{Z}^*_{2n+1}}(2, n)$, i.e., a ${}^0\mathrm{MPFS}_{\mathbb{Z}^*_{2n+1}}(2, n; 1)$.

Finally, we consider the case $c = 2$. Set $N = 4n + 1$ and define the following blocks with elements in $\mathbb{Z}_N$, where x is a positive integer:

$$\begin{array}{rcl} U_3 & = & \begin{array}{|c|c|c|}\hline [1]_N & [3]_N & -[4]_N \\ \hline -[1]_N & -[3]_N & [4]_N \\ \hline \end{array}, \\ V_3 & = & \begin{array}{|c|c|c|}\hline [2]_N & [5]_N & -[7]_N \\ \hline -[2]_N & -[5]_N & [7]_N \\ \hline \end{array}, \end{array}$$

U_9 =

$[1]_N$	$[2]_N$	$[3]_N$	$[4]_N$	$-[5]_N$	$[12]_N$	$[13]_N$	$-[14]_N$	$-[16]_N$
$-[1]_N$	$-[2]_N$	$-[3]_N$	$-[4]_N$	$[5]_N$	$-[12]_N$	$-[13]_N$	$[14]_N$	$[16]_N$

,

V_9 =

$[6]_N$	$[7]_N$	$[8]_N$	$[9]_N$	$[10]_N$	$[11]_N$	$-[15]_N$	$-[17]_N$	$-[19]_N$
$-[6]_N$	$-[7]_N$	$-[8]_N$	$-[9]_N$	$-[10]_N$	$-[11]_N$	$[15]_N$	$[17]_N$	$[19]_N$

,

W_x =

$[x]_N$	$-[x+2]_N$	$-[x+3]_N$	$[x+5]_N$
$-[x]_N$	$[x+2]_N$	$[x+3]_N$	$-[x+5]_N$

.

Note that all these blocks have rows and columns that sum to $[0]_N$.

If $n = 5$, we take

A_1 =

$[1]_{21}$	$[2]_{21}$	$[13]_{21}$	$[16]_{21}$	$[10]_{21}$
$[20]_{21}$	$[19]_{21}$	$[8]_{21}$	$[5]_{21}$	$[11]_{21}$

,

A_2 =

$[3]_{21}$	$[17]_{21}$	$[6]_{21}$	$[7]_{21}$	$[9]_{21}$
$[18]_{21}$	$[4]_{21}$	$[15]_{21}$	$[14]_{21}$	$[12]_{21}$

.

If $n \equiv 3 \pmod 4$, write $n = 4\ell + 3$ and define A_1 as the $2 \times n$ array obtained by the juxtaposition of U_3 and the blocks W_{6+4j} with $j \in [0, \ell - 1]$ (so, A_1 coincides with U_3 when $n = 3$). Also, let A_2 be the $2 \times n$ array defined by the juxtaposition of V_3 and the blocks W_{n+3+4j} with $j \in [0, \ell-1]$. We have $\mathcal{E}(A_1) = \{\pm[z]_N \mid z \in \Psi_1\}$ and $\mathcal{E}(A_2) = \{\pm[z]_N \mid z \in \Psi_2\}$, where

$$\begin{aligned} \Psi_1 &= \{1, 3, 4\} \cup \{6\} \cup [8, n+2] \cup \{n+4\}, \\ \Psi_2 &= \{2, 5, 7\} \cup \{n+3\} \cup [n+5, 2n-1] \cup \{2n+1\}. \end{aligned}$$

If $n \equiv 1 \pmod 4$, with $n \geq 9$, write $n = 4\ell + 9$ and define A_1 as the $2 \times n$ array obtained by the juxtaposition of U_9 and the blocks W_{18+4j} with $j \in [0, \ell - 1]$. Also, let A_2 be the $2 \times n$ array defined by the juxtaposition of V_9 and the blocks W_{n+9+4j} with $j \in [0, \ell - 1]$. We have $\mathcal{E}(A_1) = \{\pm[z]_N \mid z \in \Psi_1\}$ and $\mathcal{E}(A_2) = \{\pm[z]_N \mid z \in \Psi_2\}$, where

$$\begin{aligned} \Psi_1 &= \{1, 2, 3, 4, 5, 12, 13, 14, 16\} \cup \{18\} \cup [20, n+8] \cup \{n+10\}, \\ \Psi_2 &= \{6, 7, 8, 9, 10, 11, 15, 17, 19\} \cup \{n+9\} \cup [n+11, 2n-1] \cup \{2n+1\}. \end{aligned}$$

In all the previous cases, we get $\mathcal{E}(A_1) \cup \mathcal{E}(A_2) = \{\pm[z]_{4n+1} \mid z \in [1, 2n]\} = \mathbb{Z}_{4n+1}^*$, and hence the set $\{A_1, A_2\}$ is a ${}^0\mathrm{MPFS}_{\mathbb{Z}_{4n+1}^*}(2, n; 2)$. □

For instance, we can construct a ${}^0\mathrm{MPF}_{\mathbb{Z}_{19}^*}(2, 9)$ following the proof of the previous proposition. In this case, $\Lambda_+ = \{2, 4, 5, 6, 7, 8\}$ and $\Lambda_- = \{1, 3, 9\}$:

$[2]_{19}$	$[4]_{19}$	$[5]_{19}$	$[6]_{19}$	$[7]_{19}$	$[8]_{19}$	$[18]_{19}$	$[16]_{19}$	$[10]_{19}$
$[17]_{19}$	$[15]_{19}$	$[14]_{19}$	$[13]_{19}$	$[12]_{19}$	$[11]_{19}$	$[1]_{19}$	$[3]_{19}$	$[9]_{19}$

.

Corollary 5.6. *Let $m, n \geq 2$. There exists a ${}^0\mathrm{MPF}_{\mathbb{Z}_{mn+1}^*}(m, n)$ if and only if $mn + 1$ is odd and greater than* 5.

We now consider the general case of any finite abelian group, not necessarily cyclic.

Lemma 5.7. *Let Γ_1, Γ_2 be two finite abelian groups with $|\Gamma_1| = 2a + 1$ and $|\Gamma_2| = 2b + 1$. Suppose that $2a + 1$ divides $2b + 1$ and that there exists a ${}^0\mathrm{MPF}_{\Gamma_2^*}(2, b)$. Then, there exists a ${}^0\mathrm{MPF}_{\Gamma^*}(2, a + b + 2ab)$, where $\Gamma = \Gamma_1 \oplus \Gamma_2$.*

Proof. Note that Γ_1, Γ_2 belong to $\mathcal{G}$, since they both have odd order. Let $(g_1, \ldots, g_{2a})$ be any ordering of the elements of Γ_1^* such that $g_{a+\ell} = -g_\ell$ for all $\ell \in [1, a]$, and let $(h_1, \ldots, h_{2b+1})$

be any ordering of the elements of Γ_2. Let $V=(v_{i,j})$ be a ${}^0\mathrm{MPF}_{\Gamma_2^*}(2,b)$. We construct a $2\times(a+b+2ab)$ array C as follows.

First, take the $2\times b$ array $T=(t_{i,j})$, where $t_{i,j}=(0_{\Gamma_1},v_{i,j})$ for any $i=1,2$ and any $j\in[1,b]$. Since the elements of each row and each column of V sum to 0_{Γ_2}, the elements of each row and each column of T sum to $(0_{\Gamma_1},0_{\Gamma_2})$. Now, for every $\ell\in[1,a]$, we construct a $2\times(2b+1)$ array $U_\ell=\left(u_{i,r}^{(\ell)}\right)$ setting $u_{1,r}^{(\ell)}=(g_\ell,h_r)$ and $u_{2,r}^{(\ell)}=(-g_\ell,-h_r)$ for every $r\in[1,2b+1]$. Note that the elements of each column of U_ℓ sum to $(0_{\Gamma_1},0_{\Gamma_2})$, while the elements of each row sum to $\pm\left(|\Gamma_2|g_\ell,\sum\limits_{r=1}^{2b+1}h_r\right)=(\pm|\Gamma_2|g_\ell,0_{\Gamma_2})$. Since the order of Γ_1 divides the order of Γ_2, applying Lagrange's theorem, we obtain $|\Gamma_2|g_\ell=0_{\Gamma_1}$, whence $(\pm|\Gamma_2|g_\ell,0_{\Gamma_2})=(0_{\Gamma_1},0_{\Gamma_2})$. Finally, we construct the array C:

$$C=\boxed{T}\boxed{U_1}\boxed{\cdots}\boxed{U_a}.$$

By the previous observations, the elements of each row and each column of C sum to 0_Γ. Furthermore, $\mathcal{E}(T)=\{(0,h)\mid h\in\Gamma_2^*\}$ and $\mathcal{E}(U_\ell)=\{\pm(g_\ell,h)\mid h\in\Gamma_2\}$. Hence, $\mathcal{E}(C)=\mathcal{E}(T)\cup\left(\bigcup\limits_{\ell=1}^{a}\mathcal{E}(U_\ell)\right)=\Gamma^*$. This proves that C is a ${}^0\mathrm{MPF}_{\Gamma^*}(2,a+b+2ab)$. □

Corollary 5.8. *Let Γ be an abelian group of order $2n+1\geq 3$. There exists a ${}^0\mathrm{MPF}_{\Gamma^*}(2,n)$ if and only if $\Gamma\notin\{\mathbb{Z}_3,\mathbb{Z}_5,\mathbb{Z}_3\oplus\mathbb{Z}_3\}$.*

Proof. First, decompose Γ as the direct sum $\mathbb{Z}_{2n_1+1}\oplus\ldots\oplus\mathbb{Z}_{2n_r+1}$ of cyclic groups of order $2n_i+1$ such that $2n_i+1$ divides $2n_{i+1}+1$ for all $i\in[1,r-1]$. Suppose $2n_r+1\geq 7$. In this case, there exists a ${}^0\mathrm{MPF}_{\mathbb{Z}^*_{2n_r+1}}(2,n_r)$ by Proposition 5.5. If $r=1$, our proof is complete. If $r\geq 2$, applying Lemma 5.7, there also exists a ${}^0\mathrm{MPF}_{\Gamma_2^*}\left(2,\frac{(2n_r+1)(2n_{r-1}+1)-1}{2}\right)$, where $\Gamma_2=\mathbb{Z}_{2n_{r-1}+1}\oplus\mathbb{Z}_{2n_r+1}$. Again, if $r=2$, our proof is complete; otherwise, we apply repeatedly Lemma 5.7, obtaining at the end a ${}^0\mathrm{MPF}_{\Gamma^*}(2,n)$.

So, we are left to consider the cases when Γ is an elementary abelian group of exponent $2n_r+1\in\{3,5\}$. We have already seen that there is no ${}^0\mathrm{MPF}_{\mathbb{Z}_3^*}(2,1)$ and no ${}^0\mathrm{MPF}_{\mathbb{Z}_5^*}(2,2)$. Also, it is not difficult to prove that there is no ${}^0\mathrm{MPF}_{\Gamma^*}(2,4)$ when $\Gamma=\mathbb{Z}_3\oplus\mathbb{Z}_3$. Assume $2n_r+1=3$ and $r\geq 3$. We first take the following ${}^0\mathrm{MPF}_{(\mathbb{Z}_3\oplus\mathbb{Z}_3\oplus\mathbb{Z}_3)^*}(2,13)$, where the entry xyz must be read as $([x]_3,[y]_3,[z]_3)$:

001	010	011	021	100	101	102	110	111	221	210	212	122
002	020	022	012	200	202	201	220	222	112	120	121	211

Then, we proceed as before applying repeatedly Lemma 5.7 until we obtain a ${}^0\mathrm{MPF}_{\Gamma^*}(2,n)$. Assume $2n_r+1=5$ and $r\geq 2$. In this case, it suffices to apply the previous argument starting with the following ${}^0\mathrm{MPF}_{(\mathbb{Z}_5\oplus\mathbb{Z}_5)^*}(2,12)$, where the entry xy must be read as $([x]_5,[y]_5)$:

01	02	10	11	12	13	41	20	21	33	32	24
04	03	40	44	43	42	14	30	34	22	23	31

□

6. Conclusions

In this paper we introduced the concepts of magic and zero-sum magic partially filled array whose elements belong to a subset Ω of an abelian group Γ. We think these arrays are worth to be studied not only because they generalize well known objects such as magic rectangles,

Γ-magic rectangles, signed magic arrays, integer/non integer/relative Heffter arrays, but also because of their connection with Γ-supermagic labelings and with zero-sum Γ-magic graphs.

One of our main achievements is the complete solution for the existence of magic rectangle sets with empty cells $\mathrm{MRS}(m,n;s,k;c)$ (Theorem 1.8), which extends Froncek's result about tight magic rectangle sets. We have also investigated two significant cases: when Ω is the full group Γ and when Ω is the set Γ^* of nonzero elements of Γ. The main results about these two cases are described in Theorem 1.9, that we can now prove.

Proof of Theorem 1.9. Items (1), (5) and (6) follow from Corollaries 4.1, 5.6 and 5.8; item (4) follows from Proposition 5.5. To prove (2) and (3), set $d=\gcd(s,k)$. If $d\equiv 0 \pmod 4$, there exists a diagonal ${}^0\mathrm{MPFS}_\Gamma\left(\frac{nk}{d};d;c\right)$ for every $\Gamma\in\mathcal{G}$ of order nkc, by Proposition 4.3. If nk is odd and $\gcd(n,d-1)=1$, then $d\geq 3$ is odd: so, there exists a diagonal ${}^0\mathrm{MPF}_{\mathbb{Z}_d\oplus\mathbb{Z}_{nk/d}}\left(\frac{nk}{d};d\right)$ by Proposition 4.4. In both cases, we apply Theorem 2.2. □

References

[1] S. Akbari, F. Rahmati, S. Zare, Zero-sum magic labelings and null sets of regular graphs, *Electron. J. Combin.* **21** (2014), #P2.17.

[2] D.S. Archdeacon, Heffter arrays and biembedding graphs on surfaces, *Electron. J. Combin.* **22** (2015), #P1.74.

[3] D.S. Archdeacon, T. Boothby, J.H. Dinitz, Tight Heffter arrays exist for all possible values, *J. Combin. Des.* **25** (2017), 5–35.

[4] K. Burrage, D.M. Donovan, N.J. Cavenagh, E.Ş. Yazıcı, Globally simple Heffter arrays $H(n;k)$ when $k\equiv 0,3 \pmod 4$, *Discrete Math.* 343 (2020), #111787.

[5] N.J. Cavenagh, J.H. Dinitz, D.M. Donovan, E.Ş. Yazıcı, The existence of square non-integer Heffter arrays, *Ars Math. Contemp.* **17** (2019), 369–395.

[6] N.J. Cavenagh, D.M. Donovan, E.Ş. Yazıcı, Biembeddings of cycle systems using integer Heffter arrays, *J. Combin. Des*, 28 (2020), 900–922.

[7] S. Cichacz, A Γ-magic rectangle set and group distance magic labeling, IWOCA 2014, in: *Lecture Notes in Comput. Sci.* **8986**, Springer, 2015, 122–127.

[8] S. Cichacz, T. Hinc, A magic rectangle set on Abelian groups and its application, *Discrete Appl. Math.* **288** (2021), 201–210.

[9] S. Cichacz, T. Hinc, A note on magic rectangle set $\mathrm{MRS}_\Gamma(2k+1,4;4l+2)$, *J. Combin. Des.* **29** (2021), 502–507.

[10] S. Costa, S. Della Fiore, Existence of λ-fold non-zero sum Heffter arrays through local considerations, preprint available at `https://arxiv.org/abs/2209.02309`.

[11] S. Costa, S. Della Fiore, A. Pasotti, Non-zero sum Heffter arrays and their applications, *Discrete Math.* **345** (2022), #112952.

[12] S. Costa, F. Morini, A. Pasotti, M.A. Pellegrini, Globally simple Heffter arrays and orthogonal cyclic cycle decompositions, *Australas. J. Combin.* 72 (2018), 549–593.

[13] S. Costa, F. Morini, A. Pasotti, M.A. Pellegrini, A generalization of Heffter arrays, *J. Combin. Des.* **28** (2020), 171–206.

[14] S. Costa, A. Pasotti, On λ-fold relative Heffter arrays and biembedding multigraphs on surfaces, *European J. Combin.*, **97** (2021), #103370.

[15] S. Costa, A. Pasotti, M.A. Pellegrini, Relative Heffter arrays and biembeddings, *Ars Math. Contemp.* **18** (2020), 241–271.

[16] J.H. Dinitz, A.R.W. Mattern, Biembedding Steiner triple systems and n-cycle systems on orientable surfaces, *Australas. J. Combin.* 67 (2017), 327–344.

[17] J.H. Dinitz, A. Pasotti, A survey of Heffter arrays, in preparation.

[18] D. Froncek, Handicap distance antimagic graphs and incomplete tournaments, *AKCE Int. J. Graphs Comb.* **10** (2013), 119–127.

[19] D. Froncek, Handicap incomplete tournaments and ordered distance antimagic graphs, *Congr. Numer.* **217** (2013), 93–99.

[20] D. Froncek, Magic rectangle sets of odd order, *Australas. J. Combin.* **67** (2017), 345–351.

[21] D. Froncek, J. McKeown, J. McKeown, M. McKeown, Z_{2nm}-supermagic labeling of $C_n \Box C_m$, *Indones. J. Comb.* **2** (2018), 57–71.
[22] J.A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combin.* **5** (1998), #DS6, Twenty-fourth edition.
[23] T. Harmuth, Über magische Quadrate und ähniche Zahlenfiguren, *Arch. Math. Phys.* **66** (1881), 286–313.
[24] T. Harmuth, Über magische Rechtecke mit ungeraden Seitenzahlen, *Arch. Math. Phys.* **66** (1881), 413–447.
[25] A. Khodkar, D. Leach, Magic rectangles with empty cells, *Util. Math.* **116** (2020), 45–56.
[26] A. Khodkar, C. Schulz, N. Wagner, Existence of some signed magic arrays, *Discrete Math.* **340** (2017), 906–926.
[27] F. Morini, M.A. Pellegrini, On the existence of integer relative Heffter arrays, *Discrete Math.* **343** (2020), #112088.
[28] F. Morini, M.A. Pellegrini, Magic rectangles, signed magic arrays and integer λ-fold relative Heffter arrays, *Australas. J. Combin.* **80** (2021), 249–280.
[29] F. Morini, M.A. Pellegrini, Rectangular Heffter arrays: a reduction theorem, *Discrete Math.* **345** (2022), #113073.
[30] B.M. Stewart, Supermagic complete graphs. *Canadian J. Math.* **19** (1967), 427–438.
[31] Y. Wen, H. Sun, Note on magic squares and magic cubes on abelian groups, *J. Math. Res. Exposition* **17** (1997), 176–178.