WEAK AND STRONG VERSIONS OF THE KOLMOGOROV 4/5-LAW FOR STOCHASTIC BURGERS EQUATION

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ABSTRACT. For solutions of the space-periodic stochastic 1d Burgers equation we establish two versions of the Kolmogorov 4/5-law which provides an asymptotic expansion for the third moment of increments of turbulent velocity fields. We also prove for this equation an analogy of the Landau objection to possible universality of Kolmogorov's theory of turbulence, and show that the third moment is the only one which admits a universal asymptotic expansion.

1. The 4/5 law

The Kolmogorov theory of turbulence, known as the K41 theory (see in [16, 13]), examines homogeneous turbulence, corresponding to velocity fields u(t,x) which are random fields, stationary in time t, homogeneous and isotropic in the space-variable x. Using dimension analysis and arguing on physical level of rigour, Kolmogorov made a number of remarkable predictions concerning small-scale properties of velocity fields with large Reynolds numbers R, corresponding to increments u(t,x+r) - u(t,x). Here the vectors r are such that their lengths |r| belong to the inertial range, which is an interval in \mathbb{R}_+ , formed by real numbers which are "small but not too small" in term of R and the rate of dissipation of energy $\varepsilon = \nu \mathbf{E} |\nabla u(t,x)|^2$, where $\nu > 0$ is the kinematic viscosity of the fluid. One of these predictions is the 4/5-law, stating that for large R and for r from the inertial range

(1.1)
$$\mathbf{E}\left[\left(u(t,x+r)-u(t,x)\right)\cdot\frac{r}{|r|}\right]^{3}=-\frac{4}{5}\varepsilon|r|.$$

Later the law was intensively discussed by physicists and was re-proved, using physical arguments, always related to the original Kolmogorov's arguing (see [13, Sec. 6.2] and [11, Sec. 2.2.2]). Rigorous verification of the 4/5-law and other laws of the K41 theory remains an outstanding open mathematical problem. Recently a progress was achieved in [5]. There – as it is often the case since 1960's – 3d turbulent flows are modelled by solutions of the stochastic 3d NSE with small viscosity $\nu > 0$ on the torus \mathbb{T}^3 . It is known that the latter equation has stationary solutions. Taking such a solutions $u^{\nu}(t,x)$ and assuming that it meets the assumption $\nu \mathbf{E} \|u(t)\|_{L_2}^2 = o(1)$ as $\nu \to 0$ they prove that (1.1) holds after averaging in r over a sphere of radius |r|. The relation is established for all |r| from an interval in \mathbb{R}_+ whose left end goes to zero with ν , but whose relation with the inertial range is not clear.

In order to understand better turbulence and the laws of the K41 theory, starting 1940's physicists use stochastic 1d models, where the most popular one is given by the stochastic space-periodic Burgers equation, see [12, 3, 4]. The equation describes fictitious 1d "burgers fluid" and turbulence in it, called by U. Frisch burgulence. Our goal in this work is to rigorously derive for this equation two relations, which may be regarded as weak and strong forms of the 4/5-law (1.1).

In the next Section 2 we discuss the stochastic space-periodic Burgers equation, following the book [7]. There we develop the notation and state properties of the equation's solutions,

used in the rest of the paper. Then in Section 3 we prove a weak form of the 4/5-law for the Burgers equation. There we show that for any solution u(t,x) of the equation, its cubed increments $(u(t,x+l)-u(t,x))^3$ with l from the inertial range for burgulence, averaged in x, in ensemble and locally averaged in time, behave as -Const l, uniformly in small non-negative viscosities. See relations (3.4), (3.5).

In Section 4 we prove a strong form of the 4/5-law for the Burgers equation. Namely, there in Theorem 4.1 we show that if $u^{\nu st}(t,x)$ is a stationary in time solution of the equation with a positive viscosity ν^{-1} and $\varepsilon^B = \nu \mathbf{E} \int |u_x(t,x)|^2 dx$ is its rate of dissipation of energy, then, for any t,

(1.2)
$$\mathbf{E} \int ((u(t,x+l) - u(t,x))^3 dx = -12\varepsilon^B l + o(l) \quad \text{as } \nu \to 0.$$

The relation is proved to hold for l from a strongly inertial range, which is "just a bit smaller" than the inertial range for burgulence. The latter is the segment $[c_1\nu,c]$, where the constants c_1,c_2 depend on the force, applied to the "burgers fluid" (see a discussion after Theorem 3.1), and the strongly inertial range is defined in (4.8). In Corollary 4.2 we show that (1.2) as well holds for any solution of Burgers equation, asymptotically as $t \to \infty$. We also prove in Theorem 4.1 that a stationary solution of the inviscid Burgers equation satisfies (1.2) with removed limit " $\nu \to 0$ " and o(l) replaced by $O(l^3)$. Relation (1.2) is the form in which the "4/5-law for burgulence" appears in works of physicists, justified by heuristic arguments. E.g. see [17, 9].

In Section 5 we discuss the criticism of possible universality of the K41 theory, made by Landau. It implies (on the physical level of rigour) that in the frame of K41 the only universal relation for moments of increments u(t, x+r) - u(t,x) is the 4/5-law (1.1) for cubic moments. We state a reformulation for burgulence of the Landau claim and rigorously prove it.

Notation. For a metric space M we denote by $\mathcal{P}(M)$ the set of probability Borel measures on M, for a random variable ξ , valued in M, denote by $\mathcal{D}(\xi) \in \mathcal{P}(M)$ its distribution, and for a function f and a measure μ on M denote $\langle f, \mu \rangle = \int_M f \, d\mu$. The arrow \rightarrow stands for weak convergence of measures.

2. Stochastic Burgers equation

2.1. The setting and well posedness of the equation. The stochastic Burgers equation under periodic boundary conditions has the following form:

(2.1)
$$u_t(t,x) + u(t,x)u_x(t,x) - \nu u_{xx}(t,x) = \partial_t \xi(t,x), \qquad t \ge 0, \ x \in S^1 := \mathbb{R}/\mathbb{Z},$$
 where the viscosity coefficient satisfies $\nu \in (0,1]$, and

(2.2)
$$\xi(t,x) \coloneqq \sum_{s \in \mathbb{Z}^*} b_s \beta_s(t) e_s(x).$$

Here $\{b_s\}$ are real numbers, $\{\beta_s\}$ are standard independent Wiener processes defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and $\{e_s(x), s \in \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}\}$ is an usual trigonometric basic in the space of 1-periodic functions with zero mean-value. Namely, for a $k \geq 1$, $e_k(x) = \sqrt{2} \cos(2\pi kx)$ and $e_{-k}(x) = \sqrt{2} \sin(2\pi kx)$. We suppose that the process ξ is non-zero and is sufficiently smooth in x:

(2.3)
$$B_0 > 0$$
, $B_M < \infty$ for some $M \ge 5$, where $B_m := \sum |2\pi s|^{2m} b_s^2 \le \infty$.
Since always $\nu \le 1$, then below $\nu > 0$ stands for $0 < \nu \le 1$.

¹All such solutions have the same distribution.

As the space-mean value of $\xi(t,x)$ is zero, then $\int_{S^1} u(t,x) dx$ is an integral of motion for the equation, and we always assume that it vanishes:

$$\int u(t,x)\,dx \equiv 0.$$

For short we will write the solutions $u^{\nu}(t,x)$ of (2.1) as $u^{\nu}(t)$ or just u(t). If u_0 is a random field, independent from ξ (e.g. u_0 is non-random), then by $u^{\nu}(t,x;u_0) = u^{\nu}(t;u_0)$ we will denote a solution of (2.1), equal u_0 at t=0. By H we denote the space of L_2 -functions on S^1 with zero mean, given the L_2 -norm $\|\cdot\|$ (so $\{e_s\}$ is its Hilbert basis). For $m \in \mathbb{N} \cup \{0\}$ we denote by H^m the Sobolev space

$$H^m = \{ v \in H : \partial^m v \in H \},\$$

given the homogeneous norm $||u||_m = ||\partial^m u||$ (so $||u||_0 = ||u||$). By $|\cdot|_p$, $1 \le p \le \infty$, we denote the L_p -norm on S^1 , and abbreviate $L_p(S^1)$ to L_p .

It is well known that under assumption (2.3) eq. (2.1) is well posed in spaces H^m , $1 \le m \le M$, and defines in spaces H^m with $1 \le m \le M - 1$ Markov processes with continuous trajectories. E.g. see [8, 7] and references in [7]. Moreover, if $u_0 \in H^1$ is non-random, then

$$(2.4) \mathbf{E}\exp(\sigma \|u^{\nu}(t;u_0)\|^2) \le C$$

for some $\sigma, C > 0$ (depending on u_0, ν and the random force). If $u_0 \in H^m$, $1 \le m \le M$, then

(2.5)
$$\mathbf{E} \|u^{\nu}(t; u_0)\|_{m}^{2} \le C_m \quad \forall \ t \ge 0,$$

where again C_m depends on u_0, ν and ξ . A solution $u^{\nu} = u^{\nu}(t; u_0)$ may be constructed by Galerkin's method. Then u^{ν} is obtained as a limit of Galerkin's approximations $u^{(N)}(t) \in \text{span}\{e_s, |s| \leq N\} \subset H$, which also satisfy estimates (2.4), (2.5). If $u_0 \in H^m$ with $1 \leq m < M$, then a.s.

$$(2.6) u^{(N)}(t) \to u^{\nu}(t) \text{ in } H^{m-1} \text{ as } N \to \infty, \text{ uniformly in } t \in [0, T],$$

for any finite T > 0. See in [7] Theorems 1.4.2 and 1.4.4.

2.2. Main estimates for solutions. A remarkable property of solutions u^{ν} of eq. (2.1) is given by Oleinik's estimates: If $u_0 \in H^1$ is a r.v., independent from ξ , then for each $0 < q < \infty$ and $0 < \theta \le 1$ there exists $C = C(q, B_4) > 0$ such that for every $\nu > 0$ and $t \ge \theta$ the solution $u^{\nu}(t) = u^{\nu}(t; u_0)$ satisfies

(2.7)
$$\mathbf{E}|u_x^{\nu+}(t)|_{\infty}^q \le C\theta^{-q},$$

(2.8)
$$\mathbf{E}(|u^{\nu}(t)|_{\infty}^{q} + |u_{x}^{\nu}(t)|_{1}^{q}) \le C\theta^{-q}$$

(in (2.7) $u_x^{\nu+}(t) = u_x^{\nu+}(t,x)$ is a positive part of the function $u_x^{\nu}(t,x)$). We stress that C does not depend on ν and u_0 . These relations imply crucial lower and upper estimates on Sobolev norms of solutions. To state them we need a definition: for a random process $f^{\omega}(t)$ and $\sigma \geq 0, T > 0$ we denote

(2.9)
$$\langle\!\langle f \rangle\!\rangle \coloneqq \langle\!\langle f \rangle\!\rangle_T^{T+\sigma} = \frac{1}{\sigma} \int_T^{T+\sigma} \mathbf{E} f(s) \, ds.$$

The following result is proved in [7, Section 2.3]:

Theorem 2.1. For each $\theta > 0$, any $\mathbb{N} \ni m \leq M$ and for every random variable $u_0 \in H^1$, independent from ξ , solution $u^{\nu}(t) = u^{\nu}(t; u_0)$ satisfies

1)
$$\mathbf{E} \| u^{\nu}(t) \|_{m}^{2} \leq C'_{m} \nu^{-(2m-1)}$$
 for $t \geq \theta$, where C'_{m} depends on θ and $B_{\max(4,m)}$.

2) There exists $T_*(\theta) > 0$ such that

(2.10)
$$C_m^{-1} \nu^{-(2m-1)} \le \langle \langle \mathbf{E} \| u^{\nu} \|_m^2 \rangle \rangle \le C_m \nu^{-(2m-1)}$$

for some $C_m \ge 1$. Here the averaging $\langle \langle \cdot \rangle \rangle = \langle \langle \cdot \rangle \rangle_T^{T+\sigma}$ corresponds to arbitrary constants $\sigma \ge \theta$ and $T \ge T_*$. The constants C_m depend on θ , T_* and ξ , but not on $\sigma \ge \theta$, $T \ge T_*$ and $\nu \in (0,1]$.

We will write $A \sim B$ if the two quantities satisfy $C_1B \leq A \leq C_2B$ for some $C_1, C_2 > 0$ which do not depend on ν and σ, T as in the theorem. Then (2.10) may be written as

$$\langle \langle \mathbf{E} \| u^{\nu} \|_m^2 \rangle \rangle \sim \nu^{-(2m-1)}$$
.

For m=0 relations for the norm $\|u^{\nu}(t;u_0)\|_0$ are different. Namely from (2.8) it follows that $\mathbf{E}\|u^{\nu}(t)\|^2 \leq C_0'$ for $t\geq \theta$, and it is shown in [7] that for the brackets as in the theorem's assumptions, $\langle\!\langle \mathbf{E}\|u^{\nu}\|^2\rangle\!\rangle \sim 1$. The quantity $\frac{1}{2}\mathbf{E}\|u^{\nu}(t)\|^2$ is the (averaged) energy of the "1d flow" $u^{\nu}(t,x)$. Formally applying Ito's formula to $\frac{1}{2}\|u^{\nu}(t)\|^2$ and taking the expectation we get the balance of energy relation

(2.11)
$$\frac{1}{2}\mathbf{E}\|u^{\nu}(t)\|^{2} - \frac{1}{2}\mathbf{E}\|u^{\nu}(0)\|^{2} = -\mathbf{E}\int_{0}^{t}\nu\|u^{\nu}(s)\|_{1}^{2}ds + \frac{1}{2}B_{0}t.$$

See [7, Chapter 1.4] for its rigorous derivation if u_0 is sufficiently smooth.

2.3. **The mixing.** Equation (2.1) is mixing. It means that there exists a unique measure $\mu_{\nu} \in \mathcal{P}(H^{M+1})$ such that for any r.v. $u_0 \in H^1$, independent from ξ ,

(2.12)
$$\mathcal{D}u^{\nu}(t;u_0) \rightharpoonup \mu_{\nu} \text{ in } \mathcal{P}(H^{M-1}) \text{ as } t \to \infty,$$

see [7, Chapter 3.3]. The rate of convergence in (2.12) may depend on ν (but it becomes ν -independent if we regard $\mathcal{D}u^{\nu}(t)$ and μ_{ν} as measures in L_p , see [7, Chapter 4.2]). If u_0 is a r.v. such that $\mathcal{D}u_0 = \mu_{\nu}$, then $\mathcal{D}u^{\nu}(t; u_0) \equiv \mu_{\nu}$. Such solutions u^{ν} are called *stationary*.

2.4. **Inviscid limit.** When $\nu \to 0$, a solution $u^{\nu}(t; u_0)$ converges to a limit, known as an entropy solution of eq. $(2.1)|_{\nu=0}$. This result may be established in a number of different ways. In the form, given in the theorem below, it is proved in [7, Chapter 8], following Kruzkov's approach [14], based on a version of Oleinik's estimates (2.7), (2.8).

Theorem 2.2. Let T > 0, $u_0 \in H^2$ be a non-random function and $u^{\nu} = u^{\nu}(t; u_0)$. Then there exists a random field $u^0(t, x) := u^{0\omega}(t, x; u_0)$ such that almost surely, for any $1 \le p < \infty$,

- 1) $u^0 \in L_{\infty}([0,T] \times S^1) \cap C([0,T]; L_p);$
- 2) $u^{\nu}(t) \rightarrow u^{0}(t)$ in L_{p} , uniformly in $t \in [0,T]$;
- 3) for any $t \ge \theta > 0$ and $0 < q < \infty$, $\mathbf{E}|u^0(t)|_p^q \le C(q, B_4)\theta^{-q}$;
- 4) $u^0(0,x) = u_0$ and $u^0(t,x)$ satisfies eq. (2.1) in the sense of generalised functions.

Eq. $(2.1)|_{\nu=0}$ with a prescribed initial data has many generalised solution, but its entropy solution, defined by the limiting construction above, exists and is unique. Entropy solutions extend to a mixing process in L_1 :

Theorem 2.3. Solutions $u^0(t; u_0)$ extend by continuity in u_0 to a Markov process in L_1 such that for every $u_0 \in L_1$, $u^0(t; u_0)$ a.s. is continuous in t and meets the estimate in item 3) of Theorem 2.2. This process is mixing: there is a measure $\mu_0 \in \mathcal{P}(L_1)$, satisfying $\mu_0(\cap_{q<\infty}L_q)=1$, such that for every r.v. $u_0 \in L_1$, independent from ξ ,

(2.13)
$$\mathcal{D}u^{0}(t;u_{0}) \rightharpoonup \mu_{0} \quad in \quad \mathcal{P}(L_{p}) \quad as \ t \to \infty,$$

for any $p < \infty$. If $\mathcal{D}u_0 = \mu_0$, then solution $u^0(t; u_0)$ is stationary: $\mathcal{D}u^0(t; u_0) \equiv \mu_0$.

See [7, Chapter 8.5] and see [10, 4] for another approach to stationary solutions of the inviscid Burgers equation (2.1).

3. Moments of increments $u^{\nu}(t, x+r) - u^{\nu}(t, x)$ and a weak form of the 4/5-law for Burgers equation.

Let us adopt some more notation. For a function v(x) on S^1 and |l| < 1 we denote $v^l = v(x+l)$ and set $\delta^l v(x) = v^l(x) - v(x)$. The absolute moments of increments in x of a solution $u^{\nu}(t,x)$ for (2.1) with respect to the brackets $\langle \cdot \rangle$ and averaging in x are

$$\langle\!\langle \int |\delta^l u^{\nu}(t,x)|^p dx \rangle\!\rangle = \langle\!\langle |\delta^l u^{\nu}(t)|_p^p \rangle\!\rangle =: S_{p,l} = S_{p,l}(u^{\nu}) < \infty, \quad p > 0.$$

Obviously if $u^{\nu}(t)$ is a stationary solution, then $S_{p,l}(u^{\nu}) = \mathbf{E} \int |\delta^l u^{\nu}(t,x)|^p dx$. The function $(p,l) \mapsto S_{p,l}(u^{\nu})$ is called the *structure function* of a solution u^{ν} . Since the function $v(x) \mapsto |\delta^l v|_p^p$ is continuous on $L_{\max(1,p)}$, then in view of item 3) of Theorem 2.2 the structure function $S_{p,l}(u^0)$ for entropy solutions $u^0(t,x)$ also is well defined and finite.

Careful analysis of solutions u^{ν} with $\nu \geq 0$ and of estimates (2.7), (2.8), (2.10) implies that for any random initial data $u_0 \in H^1$ (independent from ξ) the structure function $S_{p,l}(u^{\nu})$ obeys the following law:

Theorem 3.1. There exist constants $c_*, c > 0$ and $c_1 \ge 1$, and for each p > 0 there exists $C_p \ge 1$, all depending only on the random force in eq. (2.1) and on θ, T_* as in Theorem 2.1, such that for each $\nu \in [0, c_*]$ and p > 0 we have:

1) if
$$l \in [c_1 \nu, c]$$
, then

(3.1)
$$C_p^{-1}l^{\min(1,p)} \le S_{p,l}(u^{\nu}) \le C_p l^{\min(1,p)};$$

2) if $l \in [0, c_1 \nu)$, then

$$C_p^{-1} l^p \nu^{1-\max(1,p)} \le S_{p,l}(u^{\nu}) \le C_p l^p \nu^{1-\max(1,p)}$$

(for $\nu = 0$ this assertion is empty).

We see from this result that statistically the increments $|\delta^l u^{\nu}(t)|$, $\nu > 0$, with $l \in [0, c_1 \nu]$ behave "linearly in l", while for $l \in [c_1 \nu, c]$ their behaviour is non-linear. So the interval $[0, c_1 \nu]$ is the dissipation range for burgulence, described by the Burgers equation (2.1), while $[c_1 \nu, c]$ is the inertial range. The frontier $c_1 \nu$ between the two interval is the dissipation or inner scale of the flow. The constants c_1 and c, depending on the random force, may change from one group of results to another.

On the physical level of rigour the first assertion of the theorem was proved in [1]. Rigorously for $\nu > 0$ it was established in [6], using some ides from [1]. For a complete proof of assertions 1) and 2) see [7, Chapter 7.2].

Now we start to discuss moments (not absolute ones) of increments $\delta^l u^{\nu}$ of solutions $u^{\nu}(t;u_0)$ as above:

(3.2)
$$S_{p,l}^{s}(u^{\nu}) = \langle \langle s_{p,l}(u^{\nu}(t)) \rangle \rangle, \quad s_{p,l}(v) = \int (\delta^{l}v^{\nu}(x))^{p} dx, \quad \nu > 0, \quad 0 \le l < 1,$$

where $p \in \mathbb{N}$ (the upper index s stands for "skew"). If p is an even number then $S_{p,l}^s = S_{p,l}$, but for an odd p the two moments are different. As $S_{1,l}^s = 0$, then the first non-trivial skew moment is the third one. Let us examine it. Since any real number x may be written as $x = 2x^+ - |x|$, then

$$S_{3,l}^s = -S_{3,l} + 2\langle\langle \int \left(\delta^l u^{\nu}\right)^+\right)^3 dx\rangle\rangle.$$

But for any x, $\left(\delta^l u^{\nu}(x)\right)^+ \leq \int_x^{x+l} (u_x^{\nu})^+ (t,y) dy \leq l |(u_x^{\nu})^+|_{\infty}$. So in view of (2.7) the second term in the r.h.s. of (3.3) is bounded by Cl^3 , uniformly in $\nu > 0$. From here, (3.3) and (3.1) with p = 3 we get that for a suitable c' > 0 and all $\nu \in (0, c_*]$,

$$(3.4) -C_1'l \le S_{3,l}^s(u^{\nu}) \le -C_2'l \quad \text{if} \quad l \in [c_1\nu, c'],$$

for some $C_1' \ge C_2' > 0$, independent from $\nu > 0$ and solution u^{ν} .

Since in view of (2.8) with p=4 the family of functions $\int (\delta^l u^{\nu}(t,x))^3 dx$, $\nu > 0$, is uniformly integrable in variables $(t,\omega) \in [\sigma,\sigma+T] \times \Omega$, then passing to the limit in (3.4) as $\nu \to 0$ using item 2) of Theorem 2.2 with p=3 we find that entropy solutions $u^0=u^0(t;u_0),\ u_0\in H^2$, satisfy

$$-C_1'l \le S_{3,l}^s(u^0) \le -C_2'l \quad \text{if} \quad l \in [0, c'].$$

Relation (3.4) + (3.5), valid for all small $\nu \ge 0$, is a weak form of the 4/5-law (1.1) for Burgers equation. Literally the same argument shows that the two relations hold for all moments $S_{p,l}^s$ with odd $p \ge 3$ (the constants C_1', C_2' should be modified).

In the next section we will see that some ideas, originated in K41, allow to strengthen the asymptotic behaviour (3.4) for $S_{3,l}^s$ (as $\nu \to 0$) with l in the inertial range to a real asymptotic, if l belongs not to the whole inertial range, but to some large part of the latter.

4. Strong form of the 4/5-law for Burgers equation

Assuming that in (1.1) the velocity field u is homogeneous and isotropic in x (not necessarily stationary in t), for any $p \in \mathbb{N}$ define the p-th moment $S_p^{\parallel}(t,r)$ of a longitudinal increment $(u(t,x+r)-u(t,x))\cdot (r/|r|)$ as $\mathbf{E}\big[\big(u(t,x+r)-u(t,x)\big)\cdot (r/|r|)\big]^p$ (so the l.h.s. of (1.1) is $S_3^{\parallel}(t,r)$). Following Kolmogorov, proofs of the 4/5-law in physical works, e.g. in [13, 11], as well as in the rigorous paper [5], crucially use the Karman–Howard–Monin formula (which is rather a class of formulas, see for them the references above and relation (5.5.5) in [2]). The formula relates time-derivative of the second moment $S_2^{\parallel}(t,r)$ with derivatives in r of the third moment $S_3^{\parallel}(t,r)$. Variations of the formula (e.g. see in [11]) instead of the second moments S_2^{\parallel} analyse the correlations $\mathbf{E}u(t,x)\cdot u(t,x+r)$, closely related to S_2^{\parallel} . Thus motivated let us examine time-derivatives of correlations of a solution $u^{\nu}(t,x)=u^{\nu}(t,x;u_0)$ of Burgers equation with $\nu>0$ and a non-random initial data $u_0\in H^{M-1}$, i.e. of

$$\int u^{\nu}(t,x)u^{\nu l}(t,x) \, dx =: f^{l}(u^{\nu}(t)).$$

Abbreviating $u^{\nu}(t)$ to u(t), formally applying the Ito formula to $f^{l}(u(t))$ and taking the expectation we get:

(4.1)
$$\frac{d}{dt}\mathbf{E}f^{l}(u(t)) = \mathbf{E}\left(-df^{l}(u)(uu_{x}) + \nu df^{l}(u)(u_{xx}) + \frac{1}{2}\sum b_{s}^{2}d^{2}f^{l}(u)(e_{s}, e_{s})\right)$$
$$=: \mathbf{E}\left(-I_{1}(t) + I_{2}(t) + I_{3}(t)\right).$$

Since $df^l(u)v = \int (uv^l + u^lv)dx = \int (uv^l + uv^{-l})dx$ and

$$(\partial/\partial l)u^l(x) = u_x^l(x),$$

²There, to adjust the formula to periodic boundary conditions, it is integrated in dr with suitable densities. It turns out that thus obtained "weak KHM formula" is sufficient for a "conditional" derivation of relation (1.1).

we get that

$$I_{1}(t) = \int (uu^{l}u_{x}^{l} + uu^{-l}u_{x}^{-l})dx = \frac{1}{2}\frac{\partial}{\partial l}\int (u(u^{l})^{2} - u(u^{-l})^{2})dx = \frac{1}{2}\frac{\partial}{\partial l}\int (u(u^{l})^{2} - u^{l}u^{2})dx.$$

Recalling that the functional $s_{3,l}$ was defined in (3.2), we have $s_{3,l}(v(x)) = \int (\delta^l v(x))^3 dx = 3 \int (v^l v^2 - (v^l)^2 v) dx$. So

$$I_1(t) = -\frac{1}{6} \frac{\partial}{\partial l} s_{3,l}(u(t)).$$

Similar, using (4.2) we get

$$I_2(t) = \nu \int \left(uu_{xx}^l + uu_{xx}^{-l}\right) dx = \nu \frac{\partial^2}{\partial l^2} \int \left(uu^l + uu^{-l}\right) dx = 2\nu \frac{\partial^2}{\partial l^2} f^l(u(t)).$$

Since $d^2 f^l(u)(v,v) = 2f^l(v)$, then relation (4.1) may be re-written as

(4.3)
$$\frac{d}{dt}\mathbf{E}f^{l}(u(t)) = \frac{1}{6}\mathbf{E}\frac{\partial}{\partial l}s_{3,l}(u(t)) + 2\nu\mathbf{E}\frac{\partial^{2}}{\partial l^{2}}f^{l}(u(t)) + \tilde{B}_{0}(l),$$

where $\tilde{B}_0(l) := \sum_s b_s^2 f^l(e_s) = \sum_s b_s^2 \cos(2\pi s l)$.

We have obtained (4.1) by a formal application of Ito's formula to the infinite-dimensional stochastic process $u^{\nu}(t;u_0) \in H^{M-1}$ with $u_0 \in H^{M-1}$. But Galerkin's approximations $u^{(N)}(t)$ to solutions $u^{\nu}(t)$ satisfy finite-dimensional stochastic systems. Estimates (2.4), (2.5) also hold for them and imply the validity of Ito's formula for $u^{(N)}$'s. The latter has the form (4.1) with Ito's term $\mathbf{E}I_3(t)$ modified to $\mathbf{E}\frac{1}{2}\sum_{|s|\leq N}b_s^2d^2f^l(u)(e_s,e_s)$. Then passing to a limit as $N\to\infty$ using (2.6) and the uniform in N estimates we justify the validity of (4.1) for $u(t)=u^{\nu}(t;u_0)$. (Doing that we write the Ito equation in the integrated in time form.) Cf. [7], where the energy balance (2.11) is established in a similar way. Since solutions $u^{\nu}(t)\in H^{M-1}$ meet estimates (2.4), (2.5) with $m=M-1\geq 4$, then the given above formal transformation from (4.1) to (4.3) also is rigorous.

Relation (4.3) is a version of the Karman–Howarth–Monin formula for the stochastic Burgers equation.

Now let $\mu_{\nu} \in \mathcal{P}(H^{M+1})$ be the stationary measure for eq. (2.1) (see Section 2.3), and let $u^{\nu st}(t)$ be a corresponding stationary solution. Then $u^{\nu st}(t) = u^{\nu}(t; u_0)$, where $\mathcal{D}u_0 = \mu_{\nu}$. Using estimate (2.8), where $u^{\nu} = u^{\nu st}$, we see that all terms in the integrated in time relation (4.3) with $u(t) = u^{\nu}(t; u_0)$, are integrable in $\mu_{\nu}(du_0)$. Performing this integration we get that equality (4.3) stays true for $u = u^{\nu st}(t)$. Then the l.h.s. of (4.3) vanishes, so the relation takes form

$$(\partial/\partial l)\mathbf{E}(s_{3,l}(u^{\nu st}(t))) = -12\nu(\partial^2/\partial l^2)\mathbf{E}(f^l(u^{\nu st}(t))) - 6\tilde{B}_0(l).$$

Since $s_{3,0}(u(x)) \equiv 0$ and since by $(4.2) (\partial/\partial l) f^l(u)|_{l=0} = \int u(x) u_x(x) dx = 0$, then integrating this equality in dl we find that

(4.4)
$$\mathbf{E}(s_{3,l}(u^{\nu st}(t))) = -12\nu(\partial/\partial l)\mathbf{E}(f^l(u^{\nu st}(t))) - 6\int_0^l \tilde{B}_0(r)dr.$$

Next, convergence (2.12), estimate (2.8), item 1) of Theorem 2.1 and Fatou's lemma imply that for all $\nu > 0$,

(4.5)
$$\mathbf{E} \| u^{\nu s t}(t) \|_1^2 \le C \nu^{-1},$$

for all $\nu > 0$ and a suitable C. Consider the first term in the r.h.s of (4.4). Dropping the factor -12ν and using (4.2) we write its modulus as

$$|\mathbf{E} \int u u_x^l dx| = |\mathbf{E} \int (u(t,x) - u(t,x+l)) u_x(x+l) dx|$$

$$\leq \left[\mathbf{E} \int (u(t,x) - u(t,x+l))^2 dx \right]^{1/2} \left[\mathbf{E} \int u_x(t,x)^2 dx \right]^{1/2}$$

(we used that $\int u(x+l)u_x(x+l)dx = 0$). Since u is a stationary solution, then the first factor in the r.h.s. equals $S_{2,l}^{1/2}(u^{\nu st})$. So in view of Theorem 3.1 and estimate (4.5) the first term in the r.h.s. of (4.4) is $O(\sqrt{l}\sqrt{\nu})$.

In view of (2.3), \tilde{B}_0 is an even C^2 -function. Since $\tilde{B}_0(0) = B_0$, then $\int_0^l \tilde{B}_0(r) dr = B_0 l + O(l^3)$. Using in (4.4) the estimates for the terms in its r.h.s. which we have just obtained we find that

(4.6)
$$\mathbf{E}(s_{3,l}(u^{\nu st}(t))) = -6B_0l + O(l^3) + O(\sqrt{l}\sqrt{\nu}).$$

By relation (8.5.4) in [7], $\mu_{\nu} \to \mu_{0}$ in $\mathcal{P}(L_{3})$ as $\nu \to 0$. Next, by convergence (2.12) and estimate (2.8) (with q=4), $\langle |u|_{3}^{4}, \mu_{\nu} \rangle \leq C$ uniformly in $\nu > 0$ (e.g. see [7, Corollary 11.1.7]). Since functional $s_{3,l}$ is continuous on L_{3} and $s_{3,l}(u) \leq C|u|_{3}^{3}$, then we derive from here that

$$\lim_{\nu \to 0} \langle s_{3,l}, \mu_{\nu} \rangle = \langle s_{3,l}, \mu_{0} \rangle.$$

Let $u^{0\,st}(t)$ be a stationary entropy solutions of eq. (2.1)| $_{\nu=0}$, $\mathcal{D}(u^{0\,st}(t)) \equiv \mu_0$ (see Theorem 2.3). Then $\langle s_{3,l}, \mu_0 \rangle = \mathbf{E} s_{3,l}(u^{0\,st}(t))$. So passing in (4.6) to the limit as $\nu \to 0$, we get that

(4.7)
$$\mathbf{E}(s_{3,l}(u^{0st}(t))) = -6B_0l + O(l^3).$$

If in (4.6) l belongs to the inertial range $[c_1\nu, c]$, then the norm of third term in the r.h.s. of (4.6) is bounded by $Cc_1^{-1/2}l$. Assuming that c_1 is sufficiently big, we obtain from (4.6) another proof of the weak law (3.4) for stationary solutions $u^{\nu st}(t)$ (since $S_{3,l}^s(u^{\nu st}) = \mathbf{E}s_{3,l}(u^{\nu st}(t))$). Now let l belongs to a "strongly inertial range", i.e.

$$(4.8) l \in [L(\nu), c],$$

for any fixed function $L(\nu)$ such that

$$L(\nu) \to 0$$
 and $L(\nu)/\nu \to \infty$ as $\nu \to 0$.

Then $\sqrt{l}\sqrt{\nu} = o(l)$ as $\nu \to 0$ and we arrive at the main result of this work:

Theorem 4.1. Let $u^{\nu st}(t)$, $\nu > 0$, be a stationary solution of eq. (2.1) and l satisfies (4.8). Then

(4.9)
$$S_{3,l}^{s}(u^{\nu st}(t)) = \mathbf{E}(s_{3,l}(u^{\nu st}(t))) = -6B_0l + o(l) \quad as \ \nu \to 0,$$

where o(l) depends only on the function $L(\nu)$ and the random force ξ . While the stationary entropy solution $u^{0\,st}(t)$ satisfies (4.7).

Due to the balance relation (2.11), for stationary solution $u^{\nu st}$ with $\nu > 0$ the rate of dissipation of energy is given by

$$\varepsilon^B = \frac{1}{2}B_0.$$

³Equivalently, $\langle s_{3l}, \mu_0 \rangle = -6B_0l + O(l^3)$ as $l \to 0$.

So (4.9) may be written as

$$(4.11) S_{3,l}^s(u^{\nu st}(t)) = -12\varepsilon^B l + o(l) \quad \text{as} \quad \nu \to 0.$$

Combining the theorem's result with (2.12) and (2.13) we get

Corollary 4.2. 1) Let $\nu > 0$, and $u_0 \in H^1$ be a r.v., independent from ξ . Then for any l as in (4.8) we have

$$\lim_{t \to \infty} \mathbf{E}(s_{3,l}(u^{\nu}(t;u_0))) = -6B_0l + o(l) \text{ as } \nu \to 0.$$

2) If $\nu = 0$ and $u_0 \in L_1$ is a r.v., independent from ξ , then

$$\lim_{t \to \infty} \mathbf{E}(s_{3,l}(u^0(t; u_0))) = -6B_0l + O(l^3) \text{ as } l \to 0.$$

The proof of Theorem 4.1 crucially uses that the moment which we analyse is cubic. Indeed, for the proof the Ito term I_3 should be a constant, for that the functional f^l should be quadratic, and then the term I_1 in (4.11) is of the third order in u. But this does not imply that moments $S_{p,l}^s$ with integers $p \neq 3$ do not admit asymptotic expansions in l. In the next section we show that an asymptotic expansion of $S_{p,l}^s$ with an integer $p \geq 2$, $p \neq 3$, is not possible if in addition we require that its leading term "is universal".

Remark 4.3. While suitable analogies of Theorems 2.1, 2.2, 3.1 and of the weak 4/5-law (3.5) hold for solutions of the free Burgers equation $(2.1)_{\xi=0}$ with a non-zero smooth initial data (see [7, Section 10.11]), we see no way to establish for the latter equation a reasonable analogy of Theorem 4.1.

5. On the Landau objection to universality in K41 and burgulence

The celebrated 2/3-law of K41 states that for turbulent velocity fields u(t,x) as those, treated by the theory, the second moments $S_2^{\parallel}(r)^4$ of longitudinal increments of u behave as $(\varepsilon|r|)^{2/3}$. Originally Kolmogorov insisted on the universality of the law and claimed that $S_2^{\parallel}(r) = C^K(\varepsilon|r|)^{2/3} + o((\varepsilon|r|)^{2/3})$, where C^K is an absolute constant. But this universality was put in doubt by Landau who suggested a physical argument, implying that a relation for a moment of velocity increment may be universal only if the value of the moment, suggested by the relation, is linear in the rate of energy dissipation ε (like relation (1.1) for the third moment). See in [16] a footnote at page 126 and see [13, Section 6.4]. The goal of this section is to show that for burgulence, indeed, the only universal relation for the moments $S_{p,l}^s$ is relation (4.11) for the cubic one (which is linear in ε^B).

Namely, for a stationary solution $u^{\nu st}(t,x)$ of stochastic Burgers equation (2.1) and an integer $p \ge 2$ consider the following relation for the p-th moment $S_{p,l}^s$ of increments of $u^{\nu st}$:

$$(5.1) S_{p,l}^s(u^{\nu st}(t)) = C_*(\varepsilon^B l)^q + o(\varepsilon^B l)^q \text{ as } \nu \to 0,$$

where l is any number from the inertial range $[c_1\nu, c]$ and q > 0. We address the following question: for which p and q relation (5.1) holds with a *universal* constant C_* , independent from the random force ξ ?

Theorem 5.1. If relation (5.1) holds for any random force ξ , satisfying (2.3), with a C_* , independent from ξ , then

$$p = 3, q = 1, C_* = -12.$$

⁴The moments do not depend on t by the assumed stationarity of u.

Proof. Let us abbreviate $u^{\nu st}(t)$ to u(t). We take some real number $\mu > 1$ and set $\tilde{\xi}(\tau) := \mu^{-\frac{1}{2}} \xi(\mu \tau)$. This also is a standard Wiener process. Denote $w(\tau, x) := \mu u(\mu \tau, x)$. Then w is a stationary solution of equation

$$(5.2) w_{\tau}(\tau, x) + w(\tau, x)w_{x}(\tau, x) - \nu^{\mu}w_{xx}(\tau, x) = \mu^{\frac{3}{2}}\partial_{\tau}\tilde{\xi}(\tau, x), \nu^{\mu} = \nu\mu.$$

Consider the inertial range $J^1 = [c_1\nu, c]$ for eq. (2.1) and inertial range $J^\mu = [c_1^\mu\nu, c^\mu]$ for eq. (5.2). For small ν their intersection $J = J^1 \cap J^\mu$ is not empty. For $l \in J$ relation (5.1) holds for u which solves eq. (2.1) and for w, solving eq. (5.2). Since $S_{p,l}(w) = \mu^p S_{p,l}(u)$ and as $\varepsilon_w^B = \mu^3 \varepsilon_u^B$ in view of (4.10), then from here

$$\mu^p C_* \left(\varepsilon_u^B l\right)^q + o(\varepsilon_u^B l)^q = C_* \left(\mu^3 \varepsilon_u^B l\right)^q + o(\varepsilon_u^B l)^q$$

for $l \in J$ and all small ν . As $\mu > 1$, then by this equality q = p/3. On the other hand, it follows from Theorem 3.1 if p is even and from (3.4) and a discussion after (3.5) if p is odd that $S_{p,l}(u) \sim -l$ for any integer $p \geq 2$. Thus in (5.1) q = 1, and so p = 3q = 3. Then by Theorem 4.1 $C_* = -12$ and the theorem is proved.

Remark 5.2. 1) The result of Theorem 5.1 remains true with the same proof if relation (5.1) is claimed to hold not for all l from the inertial range, but only for l from a strongly inertial range as in (4.8). In this form asymptotic (5.1) with p = 3 and q = 1 indeed is valid by Theorem 4.1.

2) We do not know if for some integer $p \ge 2$, different from 3, asymptotical expansion for $S_{p,l}(u^{\nu st}(t))$ of the form (5.1), valid for all l from the inertial range (or from a strongly inertial range) may hold with a constant C_* which depends on the random force ξ .

ACKNOWLEDGEMENT

The authors are thankful to A. Boritchev for discussion. PG was supported by Natural Science Foundation of Jilin Province (Grant No. YDZJ202201ZYTS306) and the Fundamental Research Funds for the Central Universities. Both authors were supported by the Ministry of Science and Higher Education of the Russian Federation (megagrant No. 075-15-2022-1115).

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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⁵This is in line with the relation $|u(t, x + r) - u(t, x)| \approx (\varepsilon |r|)^{1/3}$ which appears in the theory of turbulence due to a basic dimension argument, without any relation to the equations, describing the fluid. See [16, (32,1)].

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