

COMPLETE POSITIVITY OF COMULTIPLICATION AND PRIMARY CRITERIA FOR UNITARY CATEGORIFICATION

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ABSTRACT. In this paper, we investigate quantum Fourier analysis on subfactors and unitary fusion categories. We prove the complete positivity of the comultiplication for subfactors and derive a primary n -criterion of unitary categorification of multifusion rings. It is stronger than the Schur product criterion when $n \geq 3$. The primary criterion could be transformed into various criteria which are easier to check in practice even for noncommutative, high-rank, high-multiplicity, multifusion rings. More importantly, the primary criterion could be localized on a sparse set, so that it works for multifusion rings with sparse known data. We give numerous examples to illustrate the efficiency and the power of these criteria.

Key words. Quantum Fourier analysis, Complete positivity of comultiplication, subfactors, unitary categorification, fusion rings.

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1. INTRODUCTION

Jones discovered a quantum knot invariant, the Jones polynomial [19], inspired by the classification of Jones index of subfactors [18]. Witten extended the Jones polynomial as a three manifold invariant through certain partition function in topological quantum field theory in [43]. Atiyah [1] introduced a mathematical axiomatization for topological quantum field theories, which computes the topological invariants for quantum field theories. One can construct 2+1 topological quantum field theory from modular tensor categories [39] and from spherical categories [41]. In addition, the partition function has reflection positivity, if the tensor categories are unitary which naturally come from subfactors [34].

We study the analytic aspects of reflection positivity and the Fourier duality of subfactors in quantum Fourier analysis [13, 15, 5, 16, 17, 29, 10], which has led to various applications in subfactor theory [26, 23, 25, 5, 2]. The positivity of convolution for subfactors, called the quantum Schur product theorem (Theorem 4.1 in [23]), turned out to be a surprisingly efficient analytic obstruction of unitary categorification of fusion rings, called the Schur product criterion (Proposition 8.3 and Corollary 8.5 in [26]). This adds an extra dimension to algebraic obstructions for categorification that have been found in [6, 35, 36, 27].

In this paper, we show that the complete positivity of comultiplication is more fundamental than the positivity of convolution, which encodes stronger analytic obstructions of unitary categorification. In a companion paper [10], we prove various convolution inequalities based on the positivity of comultiplication, which provide analytic obstructions as well. Suppose \mathcal{R} is a fusion ring [28] with fusion coefficients $N_{i,j}^k$, $1 \leq i, j, k \leq m$, and fusion matrices $M_i = N_{i,\cdot}$. We prove in Theorem 3.12 that if \mathcal{R} has a unitary categorification, then

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i}{\|M_i\|} \right)^{\otimes n} \geq 0. \quad (1.1)$$

We call this inequality the **primary n -criterion**. The primary n -criterion is stronger than the Schur product criterion when $n \geq 3$, more friendly to noncommutative fusion rings, and easier to check by computer in practice.

More importantly, for any subset $S \subseteq \{1, 2, \dots, m\}$, denote M_i^S to be the submatrix of M_i , we can derive the following localized categorification criteria in Theorem 4.1 from Inequality (1.1),

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} \geq 0. \quad (1.2)$$

By Frobenius reciprocity, $N_{i,j}^k = N_{j,k^*}^{i^*}$, which only depends on the local fusion rules on S . We call Inequality (1.2) the **S -localized n -criterion**. The size of the checking matrices in the

primary criteria is large when the rank of the fusion ring is large. It would take a long time for computers to check the positivity. Fortunately, the submatrices inherit the positivity of matrices. Based on this fact, the localized criteria make it more efficient to check the positivity when the size of the local set S is fixed. Moreover, the localized criteria are still valid when only sparse data of the fusion rules are known.

Such local obstructions are rare. The triple/quadruple point obstructions [9, 21, 31, 40, 12, 37] are local obstructions to eliminate certain bipartite graphs with a triple/quadruple point from principal graph of subfactors, which are crucial in the small-index classification of subfactors [14]. Technically, a subfactor could produce a “fusion ring” with 2-color. Inspired by this, we generalize our results to multifusion rings in §6, including a general version of the localized categorification criterion in Theorem 6.2. Applying this criterion, we eliminate a large family of bipartite graphs of the following form from principal graphs of subfactor with certain dimension bounds (See Theorem 6.5 and Remark 6.2).

x_1 — x_2 $\begin{matrix} \curvearrowright \\ \vdots \\ \curvearrowleft \end{matrix}$ x_3 \cdots

ℓ arcs

(1.3)

It is the first local obstruction for principal graphs with arbitrary high multiplicities.

When n grows, the computational complexity of checking positivity in Inequality (1.1) grows exponentially $O(m^{3n})$. We applied the Hadamard product to reduce the computation complexity to $O(m^3 + m^2 \log n)$. For any subset $S \subseteq \{1, 2, \dots, m\}$ and unitary matrices $U_j \in M_{|S|}(\mathbb{C})$, $j = 1, \dots, n$, we can derive **reduced twisted n -criterion** in Theorem 5.1 from Inequality (1.2),

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{U_j M_i^S U_j^*}{\|M_i\|} \right)_{*_{j=1}^n} \geq 0, \tag{1.4}$$

where $*$ is the Hadamard product and $*_{j=1}^n$ means the Hadamard product of the n matrices. This twisted version is better in practice whenever there are enough zero entries of fusion matrices. We could choose proper unitary matrices to make it computable even some entries of fusion matrices are unknown (See Theorem 5.2 for application). Applying these criteria to fusion rings, we eliminate numerous fusing rings with sparse data from unitary categorification. We present the corresponding results as follows.

Fusion rings always pass primary 1, 2-criteria. By checking the dataset of [42] with computer assistance, among the 28451 fusion rings, 19738 ones do not pass primary 3-criterion (about 68.37%). In particular, the primary 3-criterion is more efficient for simple fusion rings. Among 13893 simple fusion rings in [42], exactly 11729 ones can be excluded from unitary categorification by primary 3-criterion (about 84.4%). One of the three simple noncommutative fusion rings in the dataset of [42] can be excluded by primary 3-criterion. We expect primary criteria to be powerful for fusion rings of high ranks.

Localized criteria allow us to eliminate fusion rings and bipartite graphs in a single time (See Subsection 4.3 and Theorem 6.5). \mathcal{K}_7 is excluded by only two points local set in §4.2. Other

9 fusion rings are also excluded with a small perturbation of the Frobenius-Perron dimensions in §4.2.1. By applying the reduced criteria to a simple integral fusion ring of rank 8, we see that localized n -criteria are locally inequivalent for different n (See §5.2). A family of infinitely many simple fusion rings with unbounded dimension are excluded by reduced twisted criterion in Theorem 5.3.

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Data availability: The datasets generated during and/or analysed during the current study are available from the corresponding author on reasonable request.

2. PRELIMINARIES

In this section, we will recall the notion of fusion ring, fusion bi-algebra, subfactor and the connections between them, see [26] for further details.

Let \mathbb{N} be the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $\mathbb{R}_{\geq 0}$ be the set of non-negative real numbers. Let \mathbb{C} be the complex field.

A based ring \mathcal{R} is a free \mathbb{Z} -module with basis $\{x_j\}_{j \in I}$ such that

- (1) $x_i x_j = \sum_{k \in I} N_{ij}^k x_k$, where $N_{ij}^k \in \mathbb{N}$ and the sum here is a finite sum.
- (2) The identity 1 is an \mathbb{N} -linear combination of the basis elements.
- (3) There exists an involution $*$ on I such that the induced map

$$\sum_{j \in I} n_j x_j \mapsto \sum_{j \in I} n_j x_{j^*}$$

is an anti-involution of the ring \mathcal{R} .

- (4) The linear functional $\tau : \mathcal{R} \rightarrow \mathbb{Z}$ given by $\tau(x_j) = 1$ if x_j occurs in the decomposition of 1 and 0 otherwise satisfies that

$$\tau(x_i x_j) = \delta_{i, j^*}, \quad i, j \in I.$$

A unital based ring is a based ring such that the identity 1 is a basis element. A multifusion ring is a based ring of finite rank. A fusion ring is a unital based ring of finite rank.

Furthermore, \mathcal{R} is called a fusion algebra, if the condition $N_{ij}^k \in \mathbb{N}$ is released as $N_{ij}^k \geq 0$ [26, Definition 2.1]. We obtain a finite dimensional unital C^* -algebra $\mathcal{B} := \mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$ equipped with a faithful trace τ . In this paper, we identify \mathcal{R} as \mathcal{B} if there is no confusion.

We obtain another abelian C^* -algebra \mathcal{A} with the basis $\{x_j\}_{j \in I}$ of a fusion ring \mathcal{R} , a multiplication \diamond and an involution $\#$,

$$\begin{aligned} x_i \diamond x_j &= \delta_{ij} d(x_i)^{-1} x_i, \\ x_i^\# &= x_i, \end{aligned} \tag{2.1}$$

where $d(x_j)$ is the operator norm of x_j in \mathcal{B} , also called the Perron-Frobenius dimension of x_j . Then d extends to a faithful trace on \mathcal{A} . The Fourier transform $\mathfrak{F}: \mathcal{A} \rightarrow \mathcal{B}$ is defined as

$$\mathfrak{F}(x_i) = x_i, \quad \forall i \in I.$$

Then \mathfrak{F} is a unitary from $L^2(\mathcal{A}, d)$ to $L^2(\mathcal{B}, \tau)$. The quintuple $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ is called the fusion bialgebra arising from \mathcal{R} . Given a fusion bialgebra $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$, $(\mathcal{A}, \mathcal{B}, \lambda_1 d, \lambda_2 \tau, \lambda_1^{\frac{1}{2}} \lambda_2^{-\frac{1}{2}} \mathfrak{F})$ is also a fusion bialgebra, for any gauge parameters $\lambda_1, \lambda_2 > 0$. The multiplication \diamond on \mathcal{A} induces a convolution on \mathcal{B} :

$$x_i * x_j := \mathfrak{F}(x_i \diamond x_j) = \delta_{ij} d(x_i)^{-1} x_i. \tag{2.2}$$

The study of the positivity of the convolution led to the Schur product criterion [26, Proposition 8.3] for unitary categorification of fusion rings.

Jones introduced planar algebras [20] to study the standard invariants of subfactors. Suppose \mathcal{P} is a subfactor planar algebra with Jones index δ^2 , and $\mathcal{P}_{2,+}$ is abelian, then the quintuple $(\mathcal{P}_{2,+}, \mathcal{P}_{2,-}, tr_{2,+}, tr_{2,-}, \mathfrak{F}_s)$ is a fusion bialgebra. Here $tr_{2,\pm}$ are the Markov traces on $\mathcal{P}_{2,\pm}$:

$$tr_{2,\pm}(x) = \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{x} \\ \text{---} \\ \text{---} \end{array} \tag{2.3}$$

Moreover, the multiplication and the convolution on $\mathcal{P}_{2,\pm}$ are defined as

$$xy = \begin{array}{c} \text{---} \\ \boxed{y} \\ \text{---} \\ \boxed{x} \\ \text{---} \\ \text{---} \end{array}, \quad x * y = \begin{array}{c} \text{---} \\ \boxed{x} \quad \boxed{y} \\ \text{---} \end{array}.$$

The string Fourier transform $\mathfrak{F}_s: \mathcal{P}_{2,\pm} \rightarrow \mathcal{P}_{2,\mp}$ is

$$\mathfrak{F}_s(x) = \begin{array}{c} \text{---} \\ \text{---} \\ \boxed{x} \\ \text{---} \\ \text{---} \end{array} \tag{2.4}$$

We refer the readers to [20, 3, 4, 27, 23, 15, 13] and references therein for further study on $\mathcal{P}_{2,\pm}$ of subfactor planar algebras.

A fusion bialgebra $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ is called subfactorizable if it comes from the quintuple $(\mathcal{P}_{2,+}, \mathcal{P}_{2,-}, tr_{2,+}, tr_{2,-}, \mathfrak{F}_s)$ of a subfactor planar algebra up to gauge parameters (λ_1, λ_2) .

If a fusion ring \mathcal{R} admits a unitary categorification \mathcal{C} , then the quantum double construction of \mathcal{C} produces a subfactor planar algebra which subfactorizes \mathcal{R} . For readers' convenience, we briefly recall this process in §8.3 in the Appendix.

The convolution is positive on $\mathcal{P}_{2,\pm}$ for a subfactor planar algebra, called the quantum Schur product theorem [23, Theorem 4.1]. However, the convolution may not be positive on \mathcal{B} of a fusion bialgebra. Due to the subfactorization process, the failure of the quantum Schur product theorem turned out to be a surprisingly efficient obstruction of unitary categorification of fusion rings, called the Schur product criterion [26, Proposition 8.3]. This criterion could be reformulated for commutative fusion rings as follows [26, Corollary 8.5].

Corollary 2.1. *Suppose \mathcal{R} is a commutative fusion ring with a basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Let $(\lambda_{i,j})_{1 \leq i, j \leq m}$ be the character table such that $\lambda_{i,1} = \|M_i\|$. If \mathcal{R} admits a unitary categorification, then for any m -tuple $\vec{j} = (j_1, \dots, j_m)$ and $n \geq 1$,*

$$\nu_{\vec{j}} := \sum_{i=1}^m \frac{\prod_{k=1}^n \lambda_{i,j_k}}{\lambda_{i,1}^{n-2}} \geq 0. \quad (2.5)$$

For each n , Equation (2.5) is equivalent to that the convolution of $n - 1$ positive operators is positive. Therefore, the case $n = 3$ implies the case $n \geq 3$. For noncommutative fusion rings, the corresponding statement in terms of irreducible representations of \mathcal{R} is given in [26, Proposition 8.3], which is inconvenient to check (by computer) in practice.

It is worth mentioning that $\{\nu_{\vec{j}}\}$ are the eigenvalues of the primary matrix for commutative fusion rings. Etingof, Nikshych and Ostrik considered these eigenvalues as invariants of fusion rings and discussed their integrality property in [8]. It will be interesting to study the integrality property of these eigenvalues of the primary matrix for fusion rings in general. In particular, if Kaplansky's 6th conjecture holds for spherical categories, then $\{\delta^{2(n-2)} \nu_{\vec{j}}\}$ are algebraic integers for Grothendieck rings of spherical categories.

3. PRIMARY CRITERIA

In this section, we prove a family of criteria for unitary categorification of fusion rings in Theorem 3.12, which are stronger than the Schur product criterion. We also provide concrete examples and computation results to illustrate the efficiency of the criteria.

3.1. Complete positivity and Fourier Multiple. Suppose \mathcal{A} and \mathcal{B} are C^* -algebras and \mathcal{A} is finite dimensional. Let τ be a faithful trace on \mathcal{A} and Ω be the vacuum vector in the GNS construction. Let \mathcal{A}' be the commutant algebra of \mathcal{A} on $L^2(\mathcal{A})$. Let J be the modular conjugation, $J(a\Omega) = a^*\Omega$. Then conjugation of J is a map from \mathcal{A} to \mathcal{A}' and $J\mathcal{A}J = \mathcal{A}'$ in Tomita-Takesaki theory

Proposition 3.1. *Let $\{a_j\}_{j=1}^m$ be an orthonormal basis of \mathcal{A} with respect to τ . Then*

$$\sum_{j=1}^m J a_j J \otimes a_j \geq 0$$

in $\mathcal{A}' \otimes \mathcal{A}$.

Proof. Define a linear map $T: \mathcal{A}' \otimes \mathcal{A} \rightarrow \mathbb{C}$,

$$J(x)J \otimes y \mapsto \tau(x^*y).$$

Then T is a positive linear functional. Indeed,

$$\begin{aligned} T \left(\sum_{j=1}^n J(x_j)J \otimes y_j \right) \left(\sum_{j=1}^n J(x_j)J \otimes y_j \right)^* &= \sum_{i,j=1}^n T(J(x_i x_j^*)J \otimes x_i x_j^*) \\ &= \sum_{i,j=1}^n T(x_i^* x_i x_j^* x_j) \\ &\geq 0. \end{aligned}$$

It is clear that

$$\begin{aligned} \left\langle J(x)J \otimes y, \sum_{j=1}^m J a_j J \otimes a_j \right\rangle &= \sum_{j=1}^m \langle J(x)J, J(a_j)J \rangle \langle y, a_j \rangle \\ &= \sum_{j=1}^m \langle a_j, x \rangle \langle y, a_j \rangle \\ &= \langle y, x \rangle. \end{aligned}$$

By Riesz representation theorem and the positivity of T , we have $\sum_{j=1}^m J a_j J \otimes a_j \geq 0$. \square

Definition 3.1. For a linear map $\Phi: \mathcal{A} \rightarrow \mathcal{B}$, we define its Fourier multiplier $\hat{\Phi}$ in $\mathcal{A}' \otimes \mathcal{B}$ as

$$\hat{\Phi} = \sum_j J(a_j)J \otimes \Phi(a_j),$$

where $\{a_j\}$ is an orthonormal basis of \mathcal{A} w.r.t. τ .

Note that $\hat{\Phi}$ is independent of the choice of the basis.

Proposition 3.2. We define

$$a * \hat{\Phi} := (\Omega^* \otimes I_{\mathcal{B}})((a \otimes I_{\mathcal{B}})\hat{\Phi})(\Omega \otimes I_{\mathcal{B}}),$$

where $I_{\mathcal{B}}$ is the identity of \mathcal{B} . Then $a * \hat{\Phi} = \Phi(a)$.

Proof. We have

$$\begin{aligned}
a * \hat{\Phi} &= \sum_j (\Omega^* \otimes I_{\mathcal{B}})((a \otimes I_{\mathcal{B}})(J(a_j)J \otimes \Phi(a_j)))(\Omega \otimes I_{\mathcal{B}}), \\
&= \sum_j \Omega^* a J(a_j) J \Omega \otimes \Phi(a_j) \\
&= \sum_j \Omega^* a a_j^* \Omega \otimes \Phi(a_j) \\
&= \sum_j \tau(a a_j^*) \Phi(a_j) \\
&= \Phi(a).
\end{aligned}$$

□

Proposition 3.3. *For a linear map $\Phi : \mathcal{A} \rightarrow \mathcal{B}$, we have that Φ is completely positive if and only if $\hat{\Phi} \geq 0$.*

Proof. If $\hat{\Phi} \geq 0$, then $\hat{\Phi}^{1/2}$ is in $\mathcal{A}' \otimes \mathcal{B}$. We have that

$$\begin{aligned}
\Phi(a) &= (\Omega^* \otimes I_{\mathcal{B}})((a \otimes I_{\mathcal{B}})\hat{\Phi})(\Omega \otimes I_{\mathcal{B}}) \\
&= (\Omega^* \otimes I_{\mathcal{B}})(\hat{\Phi}^{1/2}(a \otimes I_{\mathcal{B}})\hat{\Phi}^{1/2})(\Omega \otimes I_{\mathcal{B}}).
\end{aligned}$$

So Φ is completely positive. On the other hand,

$$\hat{\Phi} = I \otimes \Phi \left(\sum_{j=1}^m J a_j J \otimes a_j \right).$$

By Proposition 3.1, $\hat{\Phi} \geq 0$. □

3.2. Comultiplication and Primary Matrix. Suppose \mathcal{R} is a fusion ring with basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be the fusion bialgebra arising from \mathcal{R} .

Definition 3.2. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. We define the **comultiplication** $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ as a linear map such that*

$$\Delta(x_j) = \frac{1}{d_j} x_j \otimes x_j,$$

where d_j is the quantum dimension of x_j . Moreover, we define the **higher comultiplication** $\Delta^{(n)} : \mathcal{B} \rightarrow \mathcal{B}^{\otimes(n+1)}$, $n \geq 2$, as a linear map such that

$$\Delta^{(n)}(x_j) = \frac{1}{d_j^n} x_j^{\otimes(n+1)}.$$

The convolution on \mathcal{B} in (2.2) induces a linear map $\mathcal{B} \otimes \mathcal{B} \rightarrow \mathcal{B}$: $\sum a_i \otimes b_i \mapsto \sum a_i * b_i$. It is clear that

$$\langle x_i * x_j, x_k \rangle = \langle x_i \otimes x_j, \Delta(x_k) \rangle. \quad (3.1)$$

So the comultiplication is the dual operator of this linear map.

Proposition 3.4. *If the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is positive then the convolution is also positive, i.e., $x * y \geq 0$, for $x, y \geq 0$, $x, y \in \mathcal{B}$.*

Proof. For any $x, y, z \geq 0$, $x, y, z \in \mathcal{B}$, we have

$$\langle x * y, z \rangle = \langle x \otimes y, \Delta(z) \rangle \geq 0. \quad (3.2)$$

So $x * y \geq 0$. □

Definition 3.3. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. For any $n \geq 1$, we define*

$$T_n(\mathcal{B}) = \sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i}{\|M_i\|} \right)^{\otimes n}, \quad (3.3)$$

where $\|M_i\|$ is the operator norm of M_i acting on \mathbb{C}^m . We call it the **primary n -matrix** of \mathcal{B} . For simplicity, we use T_n instead $T_n(\mathcal{B})$ if there is no confusion.

Proposition 3.5. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. Then the primary 1-matrix is positive.*

Proof. Let $e = \sum_{j=1}^m d_j x_j$. Then $e = e^*$ and $e^2 = \text{FPdim}_{\mathbb{C}}(\mathcal{R})e$. So e is a multiple of a projection. Thus $e \geq 0$. The representation of e on the basis $\{x_j\}$ is

$$\sum_{j=1}^m \|M_j\| M_j.$$

So it is positive. □

Proposition 3.6. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. Then the primary 2-matrix is positive.*

Proof. By Proposition 3.1, we have

$$\sum_{j=1}^m J x_j J \otimes x_j \geq 0.$$

The representation of this operator on the basis $\{J(x_i)J \otimes x_j\}$ is

$$\sum_{j=1}^m M_j \otimes M_j.$$

So it is positive. □

Proposition 3.7. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. Then the comultiplication $\Delta: \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$ is completely positive if and only if the primary 3-matrix is positive. If the comultiplication Δ is completely positive, then the primary n -matrix is positive for all n .*

Proof. For the comultiplication $\Delta : \mathcal{R} \rightarrow \mathcal{R} \otimes \mathcal{R}$, its Fourier multiplier is an operator in $\mathcal{R}' \otimes \mathcal{R} \otimes \mathcal{R}$. We have

$$\hat{\Delta} = \sum_{j=1}^m J(x_j)J \otimes \Delta(x_j) = \sum_{j=1}^m d_j^{-1} J(x_j)J \otimes x_j \otimes x_j.$$

The representation of $\hat{\Delta}$ on the basis $\{J(x_i)J\Omega \otimes x_j\Omega \otimes x_k\Omega\}$ is

$$\sum_{j=1}^m \|M_j\|^{-1} M_j \otimes M_j \otimes M_j.$$

By Proposition 3.3, if the primary 3-matrix is positive, then Δ is completely positive. Then $\Delta^{(n)}$ is a positive map and $\Delta^{(n)}(e)$ is a positive operator. Therefore, the primary n -matrix as a representation of $\Delta^{n-1}(e)$ on the basis is positive. \square

Proposition 3.8. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. If the primary n -matrix is positive, then the primary $(n-1)$ -matrix is also positive.*

Proof. Recall that the representation of e on the basis $\{x_j\}$ is $\sum_{j=1}^m \|M_j\| M_j$, denoted by M . Then $M \geq 0$ and $M_i M = d_i M$. Let Tr_1 be the partial trace from $M_m(\mathbb{C})^{\otimes n}$ onto $M_m(\mathbb{C})^{\otimes(n-1)}$. Then

$$\text{Tr}_1[(M \otimes I)T_n(M \otimes I)] = \text{Tr}(M^2)T_{n-1} \geq 0.$$

So the primary $(n-1)$ -matrix is positive. \square

In summary, we have

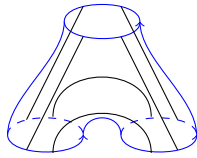
Theorem 3.9. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. The following statements are equivalent:*

- (1) Δ is completely positive;
- (2) the primary 3-matrix is positive;
- (3) the primary n -matrix is positive for some $n \geq 3$;
- (4) the primary n -matrix is positive for any $n \in \mathbb{N}$.

3.3. Main Results. Suppose \mathcal{P} is a subfactor planar algebra with index δ^2 . We define the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$ as a linear map such that

$$\langle \Delta(z), x \otimes y \rangle = \langle z, x * y \rangle, \quad \forall x, y, z \in \mathcal{P}_{2,-}. \quad (3.4)$$

Switching the input discs and the output disc of the convolution tangle, we obtain the following surface tangle representing the comultiplication, see [24] for the theory of surface tangles and surface algebras:



Theorem 3.10. *Suppose \mathcal{P} is a subfactor planar algebra. Then the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$ is completely positive.*

Proof. From Equation (3.4), we have

$$tr_{2,-} \left(\begin{array}{c} \begin{array}{|c|c|} \hline x & y \\ \hline \end{array} \\ \Delta(z) \end{array} \right) = tr_{2,-} \left(\begin{array}{c} \begin{array}{|c|c|} \hline x & y \\ \hline \end{array} \\ z \end{array} \right) = tr_{2,-} \left(\begin{array}{c} \begin{array}{|c|c|} \hline x & y \\ \hline \end{array} \\ z \end{array} \right)$$

Let

$$\Psi: \mathcal{P}_{4,\pm} \rightarrow \mathcal{P}_{2,\pm}^{\otimes 2}$$

be the trace-preserving conditional expectation. Then one could obtain that Δ has graphical representation as follows:

$$\Delta(z) = \Psi \left(\begin{array}{c} \text{---} \cup \text{---} \\ \square \\ \text{---} \cap \text{---} \end{array} \right). \quad (3.5)$$

This indicates that Δ is a composition (up to a scalar) of a $*$ -isomorphism and a conditional expectation. So Δ is completely positive. \square

Remark 3.1. *The positivity of the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$ indicates the positivity of the convolution: $x * y \geq 0$, for $x, y \geq 0$, $x, y \in \mathcal{P}_{2,-}$, the Schur product theorem [23, Theorem 4.1].*

Proposition 3.11. *Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be a fusion bialgebra. If it is subfactorizable, then the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is completely positive and the primary n -matrix in (3.3) is positive for all n .*

Proof. Suppose the fusion bialgebra arises from a subfactor planar algebra \mathcal{P} . Then the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is consistent with the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$. So by Theorem 3.10, it is completely positive. By Theorem 3.9, the primary n -matrix is positive for all n . \square

A fusion ring \mathcal{R} admits a unitary categorification means it is the Grothendieck ring of a unitary fusion category. For the definition of fusion category, we refer the readers to [7] and [6, Section 1.12]. The Grothendieck ring of a fusion category is a fusion ring. A unitary fusion category is a fusion category with a unitary structure (See e.g. [6, Remark 9.4.7]). If \mathcal{R} is the Grothendieck ring of a unitary fusion category, then the canonical fusion bialgebra associated to the fusion ring is subfactorizable by quantum double construction [26, Proposition 7.4].

Let $(\mathcal{P}_{2,+}, \mathcal{P}_{2,-}, \text{Tr}_{2,+}, tr_{2,-}, \mathfrak{F})$ be the canonical fusion bialgebra associated to the canonical Frobenius algebra γ of $\mathcal{C} \otimes \overline{\mathcal{C}}$ in the quantum double construction, see Appendix 8.2. Here $\text{Tr}_{2,+}$

is the unnormalized Markov trace on $\mathcal{P}_{2,+}$ and $\mathfrak{F} = \delta\mathfrak{F}_s$. From Equations (3.4) and (8.1), we have that the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$ satisfying

$$\begin{aligned} \langle \Delta(\mathfrak{F}(\beta_j)), \mathfrak{F}(\beta_k) \otimes \mathfrak{F}(\beta_i) \rangle &= \langle \mathfrak{F}(\beta_j), \mathfrak{F}(\beta_k) * \mathfrak{F}(\beta_i) \rangle \\ &= \frac{\delta_{k,i}}{d_i} \langle \mathfrak{F}(\beta_j), \mathfrak{F}(\beta_k) \rangle \\ &= \frac{\delta_{k,i}\delta_{jk}}{d_i}. \end{aligned}$$

We see that

$$\Delta(\mathfrak{F}(\beta_j)) = \frac{1}{d_j} \mathfrak{F}(\beta_j) \otimes \mathfrak{F}(\beta_j). \quad (3.6)$$

Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be the canonical fusion bialgebra associated to the Grothendieck ring \mathcal{R} of a unitary fusion category \mathcal{C} . Then it is isomorphic to $(\mathcal{P}_{2,+}, \mathcal{P}_{2,-}, \text{Tr}_{2,+}, \text{tr}_{2,-}, \mathfrak{F})$ by mapping x_j to β_j [26, Proposition 7.4]. So the comultiplication $\Delta: \mathcal{P}_{2,-} \rightarrow \mathcal{P}_{2,-} \otimes \mathcal{P}_{2,-}$ and the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ are consistent.

Theorem 3.12 (Primary criteria). *Suppose \mathcal{R} is a fusion ring with basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_j is the fusion matrix of x_j . If \mathcal{R} admits a unitary categorification, then for any $n \geq 1$, we have*

$$\sum_{j=1}^m \|M_j\|^2 \left(\frac{M_j}{\|M_j\|} \right)^{\otimes n} \geq 0. \quad (3.7)$$

Proof. Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be the fusion bialgebra associated to \mathcal{R} . Then it is subfactorizable. So the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is completely positive by Proposition 3.11. Further by Theorem 3.9, the primary n -matrix is positive for all n . \square

Definition 3.4. *For any $n \geq 1$, we call Inequality (3.7) as **primary n -criterion** of unitary categorification of fusion rings.*

Remark 3.2. *Though primary n -criterion are equivalent when $n \geq 3$, we will show that they are not locally equivalent in §5.*

Proposition 3.13. *Suppose \mathcal{R} is a fusion ring. If \mathcal{R} passes the primary 3-criterion then \mathcal{R} passes the Schur product criterion.*

Proof. Let $(\mathcal{A}, \mathcal{B}, d, \tau, \mathfrak{F})$ be the fusion bialgebra arising from the fusion ring \mathcal{R} . If \mathcal{R} passes the primary 3-criterion, then the primary 3-matrix is positive. Theorem 3.9 indicates that the comultiplication $\Delta: \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}$ is completely positive. By Proposition 3.4, we have $x * y \geq 0$ for any $x, y \geq 0$, $x, y \in \mathcal{B}$. So \mathcal{R} passes the Schur product criterion. \square

In [42], many fusion rings of low rank were produced by computers. One of the three simple non-commutative fusion rings in the dataset of [42] can be excluded from unitary categorification by primary 3-criterion.

Proposition 3.14. *Let \mathcal{R}_6 be the following non-commutative simple fusion ring of rank 6, type $[[1, 1], [7 + 2\sqrt{13}, 3], [11 + 3\sqrt{13}, 2]]$ with fusion matrices:*

$$\begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 4 & 2 & 2 & 2 & 2 & 0 & 2 & 2 & 1 & 4 & 2 & 0 & 2 & 1 & 2 & 2 & 4 & 0 & 2 & 4 & 2 & 5 & 4 & 0 & 2 & 2 & 4 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 2 & 4 & 0 & 1 & 3 & 1 & 3 & 3 & 1 & 2 & 3 & 3 & 3 & 1 & 0 & 4 & 1 & 3 & 5 & 4 & 0 & 2 & 3 & 3 & 4 & 5 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 2 & 4 & 2 & 1 & 2 & 3 & 3 & 1 & 3 & 0 & 1 & 1 & 3 & 3 & 3 & 0 & 2 & 3 & 3 & 5 & 4 & 0 & 4 & 3 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 2 & 4 & 5 & 4 & 0 & 2 & 3 & 3 & 5 & 4 & 0 & 4 & 3 & 1 & 5 & 4 & 1 & 5 & 5 & 5 & 5 & 7 & 0 & 4 & 4 & 4 & 7 & 7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 4 & 2 & 4 & 5 & 0 & 4 & 1 & 3 & 4 & 5 & 0 & 2 & 3 & 3 & 4 & 5 & 0 & 4 & 4 & 4 & 7 & 7 & 1 & 5 & 5 & 5 & 7 & 5 \end{matrix}$$

Then \mathcal{R}_6 admits no unitary categorification.

Proof. The primary 3-matrix T_3 of \mathcal{R}_6 has a negative eigenvalue $\simeq -1.176375$ (See Appendix 8.1.1 for SageMath code). So \mathcal{R}_6 admits no unitary categorification by Theorem 3.12. \square

Moreover, we complete the full computation on their dataset in [42]:

- Among the 28451 fusion rings, exactly 19738 does not pass primary 3-criterion (about 69.37%);
- Among the 14558 non-simple ones, exactly 8009 does not pass primary 3-criterion (about 55.01%);
- Among the 13893 simple ones, exactly 11729 ones does not pass primary 3-criterion (about 84.4%).

It turns out that primary 3-criterion is more efficient for simple fusion rings.

Remark 3.3. *The primary criteria in Inequality (3.7) are very easy to check, and friendly to computers for all fusion rings including non-commutative cases. It is hard to check non-commutative fusion rings by the Schur product criterion (See Proposition 8.3 in [26]) since we have to know all the irreducible unital $*$ -representations of the non-commutative fusion rings. So checking noncommutative fusion rings is an advantage of the primary criteria.*

Remark 3.4. *We need to compute the full character table of fusion matrices to apply the Schur product criterion. But the full fusion rules are not always known, so the Schur product criterion may not be valid all the time. In §4 and §5, we introduce localized versions of primary criteria to deal with this problem, which could check fusion rings with sparse data.*

4. LOCALIZED CRITERIA

In this section, we localize the primary criteria via the basic fact that submatrices inherit the positivity of matrices. In this way, we only need to check the positivity of a small submatrix instead of a full large matrix. This would improve the efficiency of calculations. Moreover, this localized method makes it applicable to fusion rings with incomplete data, which is superior to the Schur product criterion.

4.1. Results and Definitions.

Theorem 4.1 (Localized criteria). *Suppose \mathcal{R} is a fusion ring with basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset. If \mathcal{R} admits a unitary*

categorification, then

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} \geq 0, \quad \forall n \geq 1, \quad (4.1)$$

where M_i^S is the sub-matrix of M_i with rows and columns in S .

Proof. Let $P_S: \mathbb{C}^m \rightarrow \mathbb{C}^{|S|}$ be the projection. By Inequality (3.7), we have

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} = P_S^{\otimes n} \sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i}{\|M_i\|} \right)^{\otimes n} P_S^{*\otimes n} \geq 0.$$

This completes the proof of the theorem. \square

Definition 4.1. For any $n \geq 1$ and $S \subseteq \{1, 2, \dots, m\}$, we call Inequality (4.1) **S -localized n -criterion** of unitary categorification of fusion rings.

Definition 4.2. For any $n \geq 1$ and $S \subseteq \{1, 2, \dots, m\}$, we call

$$T_n^S(\mathcal{R}) = \sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} \quad (4.2)$$

primary (S, n) -**matrix** of \mathcal{R} . For simplicity, we use T_n^S instead of $T_n^S(\mathcal{R})$ if there is no confusion.

4.2. \mathcal{K}_7 Revisited and Perturbations. We denote by \mathcal{K}_7 the simple integral fusion ring of rank 7, FPdim 210, type $[[1, 1], [5, 3], [6, 1], [7, 2]]$, with fusion matrices (See page 54 in [26]):

$$\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}, \begin{array}{cccccccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}, \begin{array}{cccccccc} 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}$$

With the character table of these fusion matrices, \mathcal{K}_7 is excluded from unitary categorification by the Schur product criterion in [26].

In this subsection, we exclude \mathcal{K}_7 again by the localized criterion with a local set with only two points. In this way, the computations are much simpler. Let $S = \{x_6, x_7\}$. Then

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = M_3^S = M_4^S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, M_5^S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, M_6^S = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, M_7^S = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

The primary $(S, 3)$ -matrix T_3^S of \mathcal{K}_7 is

$$\frac{1}{210} \begin{pmatrix} 1426 & 536 & 536 & 286 & 536 & 286 & 286 & 1001 \\ 536 & 1156 & 286 & 446 & 286 & 446 & 1001 & 526 \\ 536 & 286 & 1156 & 446 & 286 & 1001 & 446 & 526 \\ 286 & 446 & 446 & 976 & 1001 & 526 & 526 & 476 \\ 536 & 286 & 286 & 1001 & 1156 & 446 & 446 & 526 \\ 286 & 446 & 1001 & 526 & 446 & 976 & 526 & 476 \\ 286 & 1001 & 446 & 526 & 446 & 526 & 976 & 476 \\ 1001 & 526 & 526 & 476 & 526 & 476 & 476 & 886 \end{pmatrix}.$$

It is not positive since it has a negative eigenvalue $\simeq -0.62949$ (see Appendix 8.1.2). So \mathcal{K}_7 does not admit a unitary categorification by Theorem 4.1.

4.2.1. *Perturbation of Frobenius-Perron dimensions.* Localized criteria allow us to do some perturbations of the Frobenius-Perron dimensions. We obtain a bound of FPdim, which provides a sufficient condition for a fusion ring's exclusion from unitary categorification. This helps us to exclude another nine non-integral fusion rings from unitary categorification.

Suppose that \mathcal{R} is a fusion ring with basis $\{I = x_1, \dots, x_7\}$ and M_i are fusion matrices such that

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = M_3^S = M_4^S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, M_5^S = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, M_6^S = \begin{pmatrix} 0 & 3 \\ 3 & 1 \end{pmatrix}, M_7^S = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix},$$

where $S = \{x_6, x_7\}$. We assume that d_j is the Frobenius-Perron dimension and they are variables here. Let

$$T_3^S = \sum_{i=1}^7 \frac{1}{d_i} (M_i^S)^{\otimes 3}$$

where $d_i \geq 1$ and $d_1 = 1$. The exclusion of fusion ring \mathcal{K}_7 can be seen as the application of the following proposition to $(d_1, \dots, d_7) = (1, 5, 5, 5, 6, 7, 7)$.

Proposition 4.2. *The matrix T_3^S has a negative eigenvalue if*

$$d_6 < \max_{i=1,2} Q_{i,+}(d_2, d_3, d_4, d_5, d_7),$$

where $Q_{i,\pm} = \frac{-B_i \pm \sqrt{\Delta_i}}{2A_i}$, $\Delta_i = B_i^2 - 4A_i C_i$, and

$$A_1 = d_5^2 d_7^2 + 25d_5^2 d_7 + 12d_5 d_7^2 + 125d_5^2 + 120d_5 d_7 + 27d_7^2,$$

$$B_1 = -9(d_5 d_7 - 15d_5 - 12d_7) d_5 d_7,$$

$$C_1 = -729d_5^2 d_7^2,$$

$$\begin{aligned} A_2 = & d_2 d_3 d_4 d_5^3 d_7^3 + 67d_2 d_3 d_4 d_5^3 d_7^2 + 32d_2 d_3 d_4 d_5^2 d_7^3 + 7d_2 d_3 d_5^3 d_7^3 + 7d_2 d_4 d_5^3 d_7^3 + 7d_3 d_4 d_5^3 d_7^3 \\ & + 1025d_2 d_3 d_4 d_5^3 d_7 + 962d_2 d_3 d_4 d_5^2 d_7^2 + 172d_2 d_3 d_5^3 d_7^2 + 172d_2 d_4 d_5^3 d_7^2 + 172d_3 d_4 d_5^3 d_7^2 \\ & + 195d_2 d_3 d_4 d_5 d_7^3 + 54d_2 d_3 d_5^2 d_7^3 + 54d_2 d_4 d_5^2 d_7^3 + 54d_3 d_4 d_5^2 d_7^3 + 3375d_2 d_3 d_4 d_5^3 + 4600d_2 d_3 d_4 d_5^2 d_7 \\ & + 665d_2 d_3 d_5^3 d_7 + 665d_2 d_4 d_5^3 d_7 + 665d_3 d_4 d_5^3 d_7 + 1833d_2 d_3 d_4 d_5 d_7^2 + 432d_2 d_3 d_5^2 d_7^2 + 432d_2 d_4 d_5^2 d_7^2 \\ & + 432d_3 d_4 d_5^2 d_7^2 + 216d_2 d_3 d_4 d_7^3 + 63d_2 d_3 d_5 d_7^3 + 63d_2 d_4 d_5 d_7^3 + 63d_3 d_4 d_5 d_7^3, \end{aligned}$$

$$\begin{aligned} B_2 = & 18(d_2 d_3 d_4 d_5^2 d_7^2 + 12d_2 d_3 d_4 d_5^2 d_7 - 7d_2 d_3 d_4 d_5 d_7^2 - 5d_2 d_3 d_5^2 d_7^2 - 5d_2 d_4 d_5^2 d_7^2 - 5d_3 d_4 d_5^2 d_7^2 \\ & - 405d_2 d_3 d_4 d_5^2 - 471d_2 d_3 d_4 d_5 d_7 - 123d_2 d_3 d_5^2 d_7 - 123d_2 d_4 d_5^2 d_7 - 123d_3 d_4 d_5^2 d_7 - 96d_2 d_3 d_4 d_7^2 \\ & - 33d_2 d_3 d_5 d_7^2 - 33d_2 d_4 d_5 d_7^2 - 33d_3 d_4 d_5 d_7^2) d_5 d_7, \end{aligned}$$

$$C_2 = -729(d_2 d_3 d_4 d_5 d_7 + 27d_2 d_3 d_4 d_5 + 8d_2 d_3 d_4 d_7 + d_2 d_3 d_5 d_7 + d_2 d_4 d_5 d_7 + d_3 d_4 d_5 d_7) d_5^2 d_7^2.$$

Proof. If a corner of T_3^S has a negative eigenvalue then so is T_3^S . Let N be the 7×7 top left corner of T_3^S . By Lemma 4.3 and the fact that $d_i \geq 1$ for all i , the sign of $\det(N)$ is the sign of $A_2d_6^2 + B_2d_6 + C_2$, so if $Q_{k,-} < d_6 < Q_{k,+}$, with $k = 2$, then N has a negative eigenvalue. But Lemma 4.3 states also that for $k = 1$. Finally, observe that $Q_{k,-} < 0$ for $k = 1, 2$. The result follows. \square

Lemma 4.3. *Let N be the 7×7 top left corner of T_3^S . Then*

$$\det(N) = \frac{(A_1d_6^2 + B_1d_6 + C_1)^2(A_2d_6^2 + B_2d_6 + C_2)}{d_2d_3d_4d_5^7d_6^6d_7^7},$$

with the notations of Proposition 4.2. Moreover, N has an eigenvalue which is negative for $Q_{1,-} < d_6 < Q_{1,+}$.

Proof. See the SageMath computation in Appendix 8.1.4. \square

Remark 4.1. *Observe that $Q_{1,+} < \frac{9}{2}(1 + \sqrt{37}) = 31.87243\dots$ and $Q_{2,+} < 9(\sqrt{10} - 1) = 19.4604989\dots$*

Remark 4.2. *Note that $\dim = [1, 5, 5, 5, 6, 7, 7]$ is covered by $Q_{2,+}$ only, because $Q_{2,+}(5, 5, 5, 6, 7) = 8.882676\dots > 7$, whereas $Q_{1,+}(5, 5, 5, 6, 7) = 6.537671\dots < 7$.*

Remark 4.3. *The converse of Proposition 4.2 is false, because for $(d_2, d_3, d_4, d_5, d_7) = (5, 5, 5, 6, 7)$, the matrix T_3^S has a negative eigenvalue if $d_6 < \alpha = 9.5934\dots$, see Appendix 8.1.5.*

Remark 4.4. *To show how an ultimate update of Proposition 4.2 should be complex to state, note that α (in Remark 4.3) is a root of an irreducible polynomial of degree 4 in $\mathbb{Z}[X]$. It can be written as follows (see Appendix 8.1.6):*

$$\alpha = \frac{B}{a} + \frac{1}{2}\sqrt{-A - i/j/A - k/b/B + l/b} + \frac{m}{n}$$

where

$$A = \left(\frac{f + c\sqrt{eI}}{d} \right)^{1/3}, \quad B = \sqrt{bA + g/A + h},$$

and

$a = 421113102$, $b = 44334061169015601$, $c = 250462485504$, $d = 345734333761887148583413$,
 $e = 5213988190704773354123819324759655$, $f = 64964979666121194605838007808$,
 $g = 149108026701745210176$, $h = 3010843992410706004$, $i = 16567558522416134464$,
 $j = 4926006796557289$, $k = 1151178558485955841275891856$, $l = 6021687984821412008$,
 $m = 241694327$, $n = 210556551$.

In fact, $\alpha = Q(5, 5, 5, 6, 7)$ where Q is an algebraic function requiring about one million characters (or about 500 pages) to be written (at least before any simplification). A laptop needed about 24 hours to compute it. Of course, we can compute α directly, see Appendix. In addition, it seems that $Q < \beta = \frac{9}{2}(1 + \sqrt{37})$. So if $d_6 \geq \beta$ then T_3^S should not have a negative eigenvalue.

A natural question is that whether we could exclude fusion rings by the restriction condition on PFdim in Proposition 4.2. The answer is yes. With the help of computers, we find another 9 simple (non-integral) fusion rings of rank 7 and multiplicity ≤ 6 , having the same local data for $S = \{x_6, x_7\}$, as above, and whose exclusion from unitary categorification is covered by Proposition 4.2.

| | | | | | | |
|---------------|-------------------|-------------------|-------------------|-------------------|-----------------|-----------------|
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 1 1 0 0 1 1 | 0 1 0 0 0 1 1 1 | 0 0 0 1 1 1 1 1 | 0 0 0 1 1 1 1 1 | 0 0 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 1 0 0 1 1 1 1 | 1 0 1 1 0 1 1 1 | 0 0 1 0 1 1 1 1 | 0 1 0 1 0 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 0 0 1 1 1 1 1 | 0 0 1 0 1 1 1 1 | 1 1 0 1 0 1 1 1 | 0 1 1 0 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 0 1 1 1 1 1 1 | 0 1 0 1 1 1 1 1 | 0 1 1 0 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 1 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
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| 0 0 1 0 0 0 0 | 0 0 1 1 1 1 1 1 | 1 1 1 0 0 0 1 1 1 | 0 1 0 0 1 1 1 1 | 0 1 0 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 0 1 1 1 1 1 1 | 0 1 0 0 0 1 1 1 1 | 1 1 0 1 0 1 1 1 | 0 1 1 0 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 1 1 1 1 1 1 1 | 0 1 0 1 1 1 1 1 | 0 1 1 0 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 2 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 1 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 2 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 1 1 1 2 3 3 3 | 0 1 1 2 1 1 1 1 | 0 1 2 1 2 2 2 2 | 0 2 1 2 2 1 1 1 | 0 3 1 2 1 1 1 1 | 0 3 1 2 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 1 1 2 1 1 1 1 | 1 1 0 1 0 1 1 1 | 0 2 1 0 1 1 1 1 | 0 1 0 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 2 1 2 1 2 2 2 | 0 2 1 0 1 1 1 1 | 1 1 0 2 1 2 2 2 | 0 2 1 1 1 1 1 1 | 0 2 1 2 1 1 1 1 | 0 2 1 2 1 1 1 1 |
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| 0 0 0 0 0 1 0 | 0 3 1 2 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 2 1 2 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 3 1 2 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 2 1 2 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 4 2 4 3 4 4 4 | 0 2 1 3 1 1 1 1 | 0 4 3 3 3 3 3 3 | 0 3 1 3 1 1 1 1 | 0 4 1 3 1 1 1 1 | 0 4 1 3 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 2 1 3 1 1 1 1 | 1 1 0 1 0 1 1 1 | 0 3 1 1 1 1 1 1 | 0 1 0 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 4 3 3 3 3 3 3 | 0 3 1 1 1 1 1 1 | 1 3 1 4 2 3 3 3 | 0 3 1 2 1 1 1 1 | 0 3 1 3 1 1 1 1 | 0 3 1 3 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 3 1 3 1 1 1 1 | 0 1 0 1 1 1 1 1 1 | 0 3 1 2 1 1 1 1 | 1 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 4 1 3 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 3 1 3 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 4 1 3 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 3 1 3 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 1 1 0 0 0 1 1 | 0 1 3 1 1 1 1 1 | 0 0 1 0 1 1 1 1 | 0 0 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 1 3 1 1 1 1 1 | 1 3 0 3 4 5 5 | 0 1 3 1 1 1 1 1 | 0 1 4 1 1 1 1 1 | 0 1 5 1 1 1 1 1 | 0 1 5 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 0 1 0 1 1 1 1 | 0 1 3 1 1 1 1 1 | 1 0 1 1 0 1 1 1 | 0 1 1 0 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 0 1 1 1 1 1 1 | 0 1 4 1 1 1 1 1 | 0 1 1 0 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 1 1 1 1 1 1 1 | 0 1 5 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 1 1 1 1 1 1 1 | 0 1 5 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 0 1 1 0 1 1 1 | 0 1 3 2 1 1 1 1 | 0 1 2 0 1 1 1 1 | 0 0 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 1 3 2 1 1 1 1 | 1 3 3 3 4 5 5 | 0 2 3 3 2 2 2 2 | 0 1 4 2 1 1 1 1 | 0 1 5 2 1 1 1 1 | 0 1 5 2 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 1 2 0 1 1 1 1 | 0 2 3 3 2 2 2 2 | 1 0 3 0 1 2 2 2 | 0 1 2 1 1 1 1 1 | 0 1 2 2 1 1 1 1 | 0 1 2 2 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 0 1 1 1 1 1 1 | 0 1 4 2 1 1 1 1 | 0 1 2 1 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 1 1 1 1 1 1 1 | 0 1 5 2 1 1 1 1 | 0 1 2 2 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 1 1 1 1 1 1 1 | 0 1 5 2 1 1 1 1 | 0 1 2 2 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 5 5 3 3 4 4 4 | 0 5 3 4 3 3 3 3 | 0 3 4 0 2 2 2 2 | 0 3 3 2 1 1 1 1 | 0 4 3 2 1 1 1 1 | 0 4 3 2 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 5 3 4 3 3 3 3 | 1 3 1 5 3 4 4 | 0 4 5 0 0 0 0 0 | 0 3 3 0 1 1 1 1 | 0 3 4 0 1 1 1 1 | 0 3 4 0 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 3 4 0 2 2 2 2 | 0 4 5 0 0 0 0 0 | 1 0 0 2 2 3 3 3 | 0 2 0 2 1 1 1 1 | 0 2 0 3 1 1 1 1 | 0 2 0 3 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 3 3 2 1 1 1 1 | 0 3 3 0 1 1 1 1 | 0 2 0 2 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 4 3 2 1 1 1 1 | 0 3 4 0 1 1 1 1 | 0 2 0 3 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 4 3 2 1 1 1 1 | 0 3 4 0 1 1 1 1 | 0 2 0 3 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 1 1 0 0 0 1 1 | 0 1 4 1 1 1 1 1 | 0 0 1 0 1 1 1 1 | 0 0 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 1 4 1 1 1 1 1 | 1 4 4 4 5 6 6 | 0 1 4 1 1 1 1 1 | 0 1 5 1 1 1 1 1 | 0 1 6 1 1 1 1 1 | 0 1 6 1 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 0 1 0 1 1 1 1 | 0 1 4 1 1 1 1 1 | 1 0 1 1 0 1 1 1 | 0 1 1 0 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 0 1 1 1 1 1 1 | 0 1 5 1 1 1 1 1 | 0 1 1 0 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 1 1 1 1 1 1 1 | 0 1 6 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 1 1 1 1 1 1 1 | 0 1 6 1 1 1 1 1 | 0 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |
| 1 0 0 0 0 0 0 | 0 1 0 0 0 0 0 | 0 0 1 0 0 0 0 | 0 0 0 1 0 0 0 | 0 0 0 0 1 0 0 | 0 0 0 0 0 1 0 | 0 0 0 0 0 0 1 |
| 0 1 0 0 0 0 0 | 1 1 5 4 3 4 4 4 | 0 5 2 5 1 1 1 1 | 0 4 5 2 3 3 3 3 | 0 3 1 3 1 1 1 1 | 0 4 1 3 1 1 1 1 | 0 4 1 3 1 1 1 1 |
| 0 0 1 0 0 0 0 | 0 5 2 5 1 1 1 1 | 1 2 1 2 3 4 4 | 0 5 2 3 2 2 2 2 | 0 1 3 2 1 1 1 1 | 0 1 4 2 1 1 1 1 | 0 1 4 2 1 1 1 1 |
| 0 0 0 1 0 0 0 | 0 4 5 2 3 3 3 3 | 0 5 2 3 2 2 2 2 | 1 2 3 6 2 3 3 3 | 0 3 2 2 1 1 1 1 | 0 3 2 3 1 1 1 1 | 0 3 2 3 1 1 1 1 |
| 0 0 0 0 1 0 0 | 0 3 1 3 1 1 1 1 | 0 1 3 2 1 1 1 1 | 0 3 2 2 1 1 1 1 | 1 1 1 1 1 1 1 1 | 0 1 1 1 1 1 2 1 | 0 1 1 1 1 1 2 |
| 0 0 0 0 0 1 0 | 0 4 1 3 1 1 1 1 | 0 1 4 2 1 1 1 1 | 0 3 2 3 1 1 1 1 | 0 1 1 1 1 1 2 1 | 1 1 1 1 2 0 3 | 0 1 1 1 1 3 1 |
| 0 0 0 0 0 0 1 | 0 4 1 3 1 1 1 1 | 0 1 4 2 1 1 1 1 | 0 3 2 3 1 1 1 1 | 0 1 1 1 1 1 2 | 0 1 1 1 1 3 1 | 1 1 1 1 2 1 2 |


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100000000 010000000 001000000 000100000 000010000 000001000 000000100 000000010 000000001
010000000 001110000 100000011 001101111 001010111 000111111 010111111 010111111 010111111
001000000 100000011 010110000 010101111 010011111 000111111 000111111 001111111 001111111
000100000 001101111 010101111 111031222 000302222 011122222 011222222 011222222 011222222
000010000, 001011111, 010011111, 000302222, 111031222, 011212222, 011222222, 011222222 011222222
000001000 000111111 000111111 011122222 011212222 111222222 01122232 01122223 01122223
000000100 010111111 001111111 011222222 011222222 011222232 111222305 01122251 01122251
000000001 010111111 001111111 011222222 011222222 011222223 011222251 111222314
    
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100000000 010000000 001000000 000100000 000010000 000001000 000000100 000000010 000000001
010000000 001110000 100000011 001101111 001010111 000111111 010111111 010111111 010111111
001000000 100000011 010110000 010101111 010011111 000111111 000111111 001111111 001111111
000100000 001101111 010101111 111121222 000212222 011122222 011222222 011222222 011222222
000010000, 001011111, 010011111, 000212222, 111121222, 011212222, 011222222, 011222222 011222222
000001000 000111111 000111111 011122222 011212222 111222222 01122232 01122223 01122223
000000100 010111111 001111111 011222222 011222222 011222232 111222305 01122251 01122251
000000001 010111111 001111111 011222222 011222222 011222223 011222251 111222314
    
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100000000 010000000 001000000 000100000 000010000 000001000 000000100 000000010 000000001
010000000 001110000 100000011 001101111 001010111 000111111 010111111 010111111 010111111
001000000 100000011 010110000 010101111 010011111 000111111 000111111 001111111 001111111
000100000 001101111 010101111 011202222 100221222 011122222 011222222 011222222 011222222
000010000, 001101111, 010101111, 100221222, 011202222, 011212222, 011222222, 011222222 011222222
000001000 000111111 000111111 011122222 011212222 111222222 01122232 01122223 01122223
000000100 010111111 001111111 011222222 011222222 011222232 111222305 01122251 01122251
000000001 010111111 001111111 011222222 011222222 011222223 011222251 111222314
    
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They are all excluded by the Schur product criterion in [26]. Here, we exclude them again by localized criterion in a single time. Let $S = \{x_7, x_8\}$. Then they have the same local fusion matrices:

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = M_3^S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, M_4^S = M_5^S = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}.$$

$$M_6^S = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, M_7^S = \begin{pmatrix} 0 & 5 \\ 5 & 1 \end{pmatrix}, M_8^S = \begin{pmatrix} 5 & 1 \\ 1 & 4 \end{pmatrix}.$$

So the primary $(S, 3)$ -matrix T_3^S of them are same as follows:

$$\frac{1}{132} \begin{pmatrix} 2095 & 755 & 755 & 463 & 755 & 463 & 463 & 1746 \\ 755 & 1820 & 463 & 700 & 463 & 700 & 1746 & 727 \\ 755 & 463 & 1820 & 700 & 463 & 1746 & 700 & 727 \\ 463 & 700 & 700 & 1600 & 1746 & 727 & 727 & 711 \\ 755 & 463 & 463 & 1746 & 1820 & 700 & 700 & 727 \\ 463 & 700 & 1746 & 727 & 700 & 1600 & 727 & 711 \\ 463 & 1746 & 700 & 727 & 700 & 727 & 1600 & 711 \\ 1746 & 727 & 727 & 711 & 727 & 711 & 711 & 1435 \end{pmatrix}.$$

It is not positive since it has a negative eigenvalue $\simeq -2.948812$ (see Appendix 8.1.3). Hence these five fusion rings admit no unitary categorification by Theorem 4.1.

Remark 4.6. *This example of five fusion rings indicates that one could exclude more than one fusion rings by checking the positivity of only one matrix. This could not be done by the Schur product criterion. So localized criteria could improve the efficiency to check fusion rings.*

5. REDUCED TWISTED CRITERIA

In this section, we twist the fusion (sub)matrices and take Hadamard product of fusion (sub)matrices to obtain much more powerful criteria. Recall that the computational complexity of checking fusion rings of rank m by primary n -criterion grows exponentially $O(m^{3n})$. Note that the order of the matrix remains the same under the Hadamard product. So we apply the Hadamard product to primary criteria and reduce the computation complexity to $O(m^3 + m^2 \log n)$. Moreover, the twisting of the fusion (sub)matrices leads to corresponding criteria, which allow us to exclude fusion rings with unknown data.

5.1. Results and Examples.

Theorem 5.1 (Reduced twisted criteria). *Suppose \mathcal{R} is a fusion ring with basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset and $U_j \in M_{|S|}(\mathbb{C})$ are unitary matrices. If \mathcal{R} admits a unitary categorification, then for any $n \geq 1$, we have*

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{*n} \geq 0, \quad \text{Reduced} \quad (5.1)$$

$$\sum_{i=1}^m \|M_i\|^2 U_j^{\otimes_{j=1}^n} \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} (U_j^*)^{\otimes_{j=1}^n} \geq 0, \quad \text{Twisted} \quad (5.2)$$

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{U_j M_i^S U_j^*}{\|M_i\|} \right)^{*_{j=1}^n} \geq 0, \quad \text{Reduced twisted} \quad (5.3)$$

where $*$ is the Hadamard product of matrices.

Proof. Define

$$S^{(n)} := \{ \underbrace{(i, i, \dots, i)}_n : i \in S \}, \quad S^{(1)} = S.$$

Then

$$(U_j M_i^S U_j^*)^{*_{j=1}^n} = P_{S^{(n)}} (U_j M_i^S U_j^*)^{\otimes_{j=1}^n} P_{S^{(n)}}.$$

Theorem 4.1 implies

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{U_j M_i^S U_j^*}{\|M_i\|} \right)^{*_{j=1}^n} = P_{S^{(n)}} U_j^{\otimes_{j=1}^n} \sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{\otimes n} U_j^{* \otimes_{j=1}^n} P_{S^{(n)}} \Big|_{S^{(n)} \times S^{(n)}} \geq 0.$$

This completes the proof of the theorem. \square

Remark 5.1. *If there are enough zero entries of the fusion matrices, then we can eliminate the unknown entries of the fusion matrices by choosing proper unitary matrices.*

Remark 5.2. *In Theorem 5.1, when the fusion matrices M_j is twisted by unitary matrices U_j , we have*

$$\sum_{i=1}^m \|M_i\|^2 \left(\frac{(U_j M_i U_j^*)^S}{\|M_i\|} \right)^{\otimes_{j=1}^n} \geq 0.$$

This gives a different twisted positivity for a fusion ring.

Now we apply the criteria to a family of fusion rings.

Theorem 5.2. *Let \mathcal{R} be a fusion ring with basis $\{x_1 = 1, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Suppose that*

$$\begin{aligned} x_2 x_2^* &= I + s x_2 + \ell x_3 \\ x_3 x_3^* &= I + t x_2 + k x_3 + \sum_{i=4}^m \lambda_i x_i \end{aligned}$$

for $s, t, k, \lambda_i \geq 0, \ell \geq 1, i \geq 4$. Then we have the following statements:

(1) *If x_2 is self-dual and there exist $n \geq 2, a, b \geq 1, a + b = n$, such that*

$$\left(\frac{s^a t^b}{d_2^{n-2}} + \frac{\ell^a k^b}{d_3^{n-2}} + 1 \right) \left(\frac{s^b t^a}{d_2^{n-2}} + \frac{\ell^b k^a}{d_3^{n-2}} + 1 \right) - \left(\frac{\ell^n}{d_2^{n-2}} + \frac{t^n}{d_3^{n-2}} \right)^2 < 0, \quad (5.4)$$

then \mathcal{R} admits no unitary categorification.

(2) *If x_2 is not self-dual and there exist $n \geq 2, a, b \geq 1, a + b = n$, such that*

$$\frac{2s^n}{d_2^{n-2}} + 1 - \frac{s^a \tau(x_2^3)^b + s^b \tau(x_2^3)^a}{d_2^{n-2}} < 0, \quad (5.5)$$

then \mathcal{R} admits no unitary categorification, where τ is the faithful tracial state on \mathcal{R} .

Proof. (1) If x_2 is self-dual, then x_3 is also self-dual. Take the local data $S = \{2, 3\}$. Then

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = \begin{pmatrix} s & \ell \\ \ell & t \end{pmatrix}, M_3^S = \begin{pmatrix} \ell & t \\ t & k \end{pmatrix}, M_i^S = \begin{pmatrix} 0 & * \\ * & \lambda_i \end{pmatrix}, \quad i \geq 4.$$

Let $U_1 = \dots = U_a = I, U_{a+1} = \dots = U_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. By the assumption that $a, b \geq 1$ and $\{x_i : 4 \leq i \leq m\} = \{x_i^* : 4 \leq i \leq m\}$, we have

$$\sum_{i=4}^m \frac{1}{d_i^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

for some $\lambda > 0$. Applying Theorem 5.1 to \mathcal{R} , we have

$$\begin{aligned} & \det \left(\sum_{i=1}^m \frac{1}{d_i^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} \right) \\ & \leq \det \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{d_2^{n-2}} \begin{pmatrix} s^a t^b & \ell^n \\ \ell^n & s^b t^a \end{pmatrix} + \frac{1}{d_3^{n-2}} \begin{pmatrix} \ell^a k^b & t^n \\ t^n & \ell^b k^a \end{pmatrix} + \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix} \right] \\ & \leq \left(\frac{s^a t^b}{d_2^{n-2}} + \frac{\ell^a k^b}{d_3^{n-2}} + 1 \right) \left(\frac{s^b t^a}{d_2^{n-2}} + \frac{\ell^b k^a}{d_3^{n-2}} + 1 \right) - \left(\frac{\ell^n}{d_2^{n-2}} + \frac{t^n}{d_3^{n-2}} \right)^2 \\ & < 0. \quad (\text{From inequality (5.4)}) \end{aligned}$$

Hence \mathcal{R} admits no unitary categorification.

(2) If x_2 is not self-dual, then $x_3 = x_2^*$ and $s = k = t = \ell$. Take the local data $S = \{2, 3\}$. Then

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = \begin{pmatrix} s & \tau(x_2^3) \\ s & s \end{pmatrix}, M_3^S = \begin{pmatrix} s & s \\ \tau(x_2^3) & s \end{pmatrix}, M_i^S = \begin{pmatrix} 0 & * \\ * & \lambda_i \end{pmatrix}, \quad i \geq 4.$$

With the same process, we could obtain the conclusion. \square

Remark 5.3. *In the proof of Theorem 5.2, the unknown data λ_i is removed by the following equation:*

$$(U_j M_i^S U_j^*)^{*j=1} = \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}.$$

So the reduced twist criteria allow us to check fusion rings with unknown data, which is superior to the Schur product criterion.

In the following, we give an example of a family of fusion rings excluded by reduced twisted criteria.

Theorem 5.3. *Suppose $\mathcal{R}_{4,k}$ is a ring generated by the following 4 matrices:*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & k & 0 & 1 \\ 0 & 0 & k & 1 \\ 0 & 1 & 1 & k \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & k & 1 \\ 1 & k & 0 & 0 \\ 0 & 1 & 0 & k \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & k \\ 0 & 1 & 0 & k \\ 1 & k & k & 1 \end{pmatrix}.$$

Then $\mathcal{R}_{4,k}$ is a simple fusion ring admitting no unitary categorification for any $k \geq 3$.

Proof. By computations, we have the following fusion rules:

$$\begin{aligned} x_2^2 &= 1 + kx_2 + x_4, \\ x_3^2 &= 1 + kx_2, \\ x_4^2 &= 1 + k(x_2 + x_3) + x_4, \\ x_2x_3 &= kx_3 + x_4, \\ x_3x_4 &= x_2 + kx_4, \\ x_2x_4 &= x_2 + x_3 + kx_4. \end{aligned}$$

Take local data $S = \{2, 3\}$. Then $\mathcal{R}_{4,k}$ satisfies the condition in Theorem 5.2 by switching x_2 and x_3 . By the Collatz-Wielandt formula for non-negative matrix, we have $d_3 = \|x_3\| \leq k + 1 \leq \|x_2\| = d_2$. Then

$$\sqrt{k^2 + k + 1} \leq \|x_3\| \leq k + 1.$$

Inequality (5.4) ($n = 3$) is equivalent to

$$f(\|x_3\|) = \|x_3\|^3 - k^3\|x_3\|^2 + (k^4 - 1)\|x_3\| + k^3 < 0.$$

Note that $f'(x) < 0$ for any $x \in [\sqrt{k^2 + k + 1}, k + 1]$ when $k \geq 3$. It is not difficult to check that $f(\sqrt{k^2 + k + 1}) < 0$ when $k \geq 5$. So $f(x) < 0$ for any $x \in [\sqrt{k^2 + k + 1}, k + 1]$ when $k \geq 5$. Thus, $f(\|x_3\|) < 0$ and $\mathcal{R}_{4,k}$ admits no unitary categorification when $k \geq 5$. On the other hand, $\mathcal{R}_{4,3}$ and $\mathcal{R}_{4,4}$ could be excluded by direct computations. \square

Remark 5.4. *The reduced twist criteria make it applicable to exclude a family of infinitely many simple fusion rings. This is not easy to complete by the Schur product criterion since the character table of $\mathcal{R}_{4,k}$ are very complex and it is hard to compute.*

5.2. Inequivalence of Localized Criteria. Let \mathcal{R} be any one of the five fusion rings of rank 8 in subsection 4.3. Recall that $S = \{x_7, x_8\}$. Applying Theorem 5.1 for $n = 3$, $U_1 = U_2 = I$, $U_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we obtain

$$\sum_{i=1}^8 \frac{1}{\|M_i\|} (U_j M_i^S U_j^*)^{*_{j=1}^3} = \frac{1}{330} \begin{pmatrix} 4286 & 4101 \\ 4101 & 3736 \end{pmatrix}.$$

It is not positive since its determinant $\simeq -7.4$. Hence we see that these five fusion rings admit no unitary categorifications by reduced twisted criteria. For $n = 4$, $U_1 = U_2 = I$, $U_3 = U_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have that

$$\sum_{i=1}^8 \frac{1}{\|M_i\|^2} (U_j M_i^S U_j^*)^{*_{j=1}^4} \simeq \begin{pmatrix} 4.85 & 4.88 \\ 4.88 & 4.85 \end{pmatrix}.$$

It is not positive either.

Now let us consider the limit of $\sum_{i=1}^n \frac{1}{\|M_i\|^2} (U_j M_i^S U_j^*)^{*_{j=1}^n}$ when n goes to infinity.

Proposition 5.4. *Suppose \mathcal{R} is a fusion ring with basis $\{x_1 = I, x_2, \dots, x_m\}$ and M_i is the fusion matrix of x_i . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset. Define $L_i^S = (a_{kj})_{k,j=1}^m \in M_m(\mathbb{C})$ such that*

$$a_{kj} = \begin{cases} 1 & \text{if } j \text{ or } k \in S \text{ and } N_{i,j}^k = \|M_i\|, \\ 0 & \text{otherwise.} \end{cases}$$

If \mathcal{R} admits a unitary categorification, then

$$\sum_{i=1}^m \|M_i\|^2 L_i^S \geq 0. \quad (5.6)$$

Proof. Note that $N_{i,j}^k \leq \|M_i\|$. Let $U_i = I$. Theorem 5.1 implies that

$$\sum_{i=1}^m \|M_i\|^2 L_i^S = \lim_{n \rightarrow \infty} \sum_{i=1}^m \|M_i\|^2 \left(\frac{M_i^S}{\|M_i\|} \right)^{*n} \geq 0.$$

This completes the proof of the proposition. \square

Note that for any unitary matrices U_j , we have

$$\lim_{n \rightarrow \infty} \sum_{i=1}^8 \frac{1}{\|M_i\|^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} = I.$$

We see that it is positive when n is sufficiently large. This implies localized criteria are not all equivalent.

6. MULTIFUSION RINGS AND PRINCIPAL GRAPHS

The Grothendieck ring of a multifusion category is a multifusion ring. We could ask whether a multifusion ring is a Grothendieck ring of a multifusion category. By similar arguments, we see that the criteria obtained in this paper are true for unitary multifusion categorification of a multifusion ring. The principal graph of a subfactor provides partial data of the multifusion ring of its bimodule category, see §8.2 in the Appendix. We apply the localized criteria to provide obstruction of bipartite graphs from being principle graphs for subfactors of finite depth.

6.1. Results and Examples. By a similar argument in the proof of Theorem 3.12, we have the following result.

Theorem 6.1. *Suppose \mathcal{R} is a multifusion ring with basis $\{x_1, x_2, \dots, x_m\}$ and M_j is the fusion matrix of x_j . If \mathcal{R} admits a unitary multifusion categorification, then for any $n \geq 1$, we have*

$$\sum_{j=1}^m \|M_j\|^2 \left(\frac{M_j}{\|M_j\|} \right)^{\otimes n} \geq 0. \quad (6.1)$$

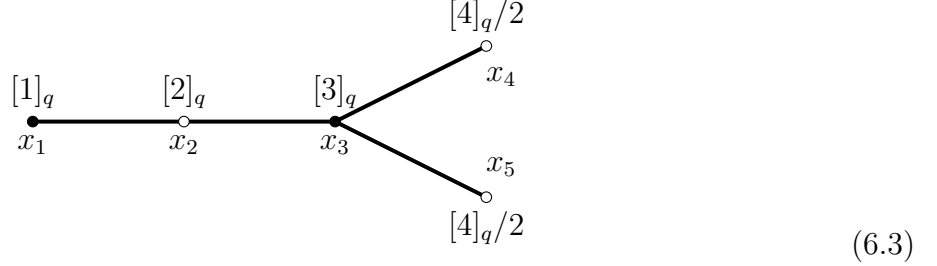
All localized criteria for unitary categorification of fusion rings obtained before in this paper are true for unitary multifusion categorification of a multifusion ring. Hence the localized positivity are obstructions for constructing principal graphs of subfactors of finite depth. Applying the idea of localization from Theorem 4.1, we have

Theorem 6.2. *Suppose \mathcal{R} is a multifusion ring with basis $\{x_1, x_2, \dots, x_m\}$ and M_j is the fusion matrix of x_j . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset. If \mathcal{R} admits a unitary multifusion categorification, then for any $n \geq 1$, we have*

$$\sum_j d_j^2 \left(\frac{M_j^S}{d_j} \right)^{\otimes n} \geq 0, \quad (6.2)$$

where d_j is the quantum dimension of x_j for $j = 1, \dots$

Example 6.1. Let D_5 be the following bipartite graph: $q = e^{\pi i/8}$,



We take the local set $S = \{x_1, x_4\}$. Then

$$T_1^S = \sum_{j=1}^5 d_j M_j^S = \begin{pmatrix} 1 & [4]_q/2 \\ [4]_q/2 & 1 \end{pmatrix}$$

is not positive. So D_5 is not a principal graph of a subfactor.

Applying the idea of Theorem 5.1, we further have

Theorem 6.3. Suppose \mathcal{R} is a multifusion ring with basis $\{x_1, x_2, \dots, x_m\}$ and M_j is the fusion matrix of x_j . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset, and $U_j \in M_{|S|}(\mathbb{C})$ are unitary matrices. If \mathcal{R} admits a unitary multifusion categorification, then for any $n \geq 1$, we have

$$\sum_i d_i^2 \left(\frac{M_i^S}{d_i} \right)^{*n} \geq 0, \quad \text{Reduced} \quad (6.4)$$

$$\sum_i d_i^2 U_j^{\otimes_{j=1}^n} \left(\frac{M_i^S}{d_i} \right)^{\otimes n} (U_j^*)^{\otimes_{j=1}^n} \geq 0, \quad \text{Twisted} \quad (6.5)$$

$$\sum_i d_i^2 \left(\frac{U_j M_i^S U_j^*}{d_i} \right)^{*_{j=1}^n} \geq 0, \quad \text{Reduced twisted} \quad (6.6)$$

where $*$ is the Hadamard product of matrices.

Note that multifusion categories are multicolored. For graded multifusion rings, we have

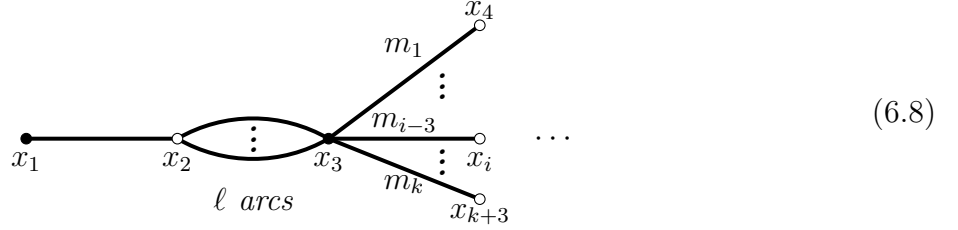
Theorem 6.4. Suppose \mathcal{R} is a \mathbb{Z}_k -graded multifusion ring with basis $\{x_1, x_2, \dots, x_m\}$ and M_j is the fusion matrix of x_j . Let $S \subseteq \{1, 2, \dots, m\}$ be a subset. If \mathcal{R} admits a unitary multifusion categorification, then

$$\sum_{i=1}^m (-1)^{|i|} d_i^2 \left(\frac{M_i^S}{d_i} \right)^{\otimes n} \geq 0, \quad (6.7)$$

where $|i|$ is the grading of x_i and d_i is the quantum dimension of x_i for $i = 1, \dots$.

6.2. Bipartite Graphs with Local data Excluded. In this subsection, we will present a family of bipartite graphs excluded from principal graphs of subfactors by the reduced twisted criteria.

Theorem 6.5. *Let \mathcal{G} be a bipartite graph of finite depth with the following local form $\ell \geq 1$:*



Let $M = \sum_{j=1}^k m_j^2 + \ell^2 - 1$ and d_2 is the quantum dimension of x_2 . If there exist $n \geq 2$, $a, b \geq 1$, $a + b = n$, such that

$$\left(\frac{\ell^{2a-2} M^b}{(d_2^2 - 1)^{n-2}} + 1 \right) \left(\frac{\ell^{2b-2} M^a}{(d_2^2 - 1)^{n-2}} + 1 \right) - \frac{\ell^{2n}}{d_2^{2n-4}} < 0, \quad (6.9)$$

then \mathcal{G} can not be a principal graph of a subfactor.

Proof. Suppose that \mathcal{G} is a principal graph of a subfactor. Let \mathcal{R} be the multifusion ring arising from \mathcal{G} , generated by $\{x_1, x_2, \dots\}$ with fusion matrices M_i , where x_i is the direct sum of two irreducible bimodules. Let d_i be the quantum dimension of x_i . Note that x_2, x_3 are self-dual. We have

$$\begin{aligned} x_2^2 &= 1 + \ell x_3 \\ x_3^2 &= 1 + \frac{M}{\ell} x_3 + \sum_{i \geq k+4} \lambda_i x_i. \end{aligned}$$

Then $d_3 = (d_2^2 - 1)/\ell$. Take local data $S = \{2, 3\}$, we have

$$M_1^S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, M_2^S = \begin{pmatrix} 0 & \ell \\ \ell & 0 \end{pmatrix}, M_3^S = \begin{pmatrix} \ell & 0 \\ 0 & M \end{pmatrix}, M_i^S = \begin{pmatrix} 0 & * \\ * & \lambda_i \end{pmatrix}, \quad i \geq 4.$$

Let $U_1 = \dots = U_a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $U_{a+1} = \dots = U_n = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $\{x_i : i \geq 4\} = \{x_i^* : i \geq 4\}$, we have

$$\sum_{i \geq 4} \frac{1}{d_i^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$$

for some $\lambda > 0$. So from Inequality (6.9),

$$\det \sum_{i=1}^m \frac{1}{d_i^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} \leq \det \sum_{i=1}^3 \frac{1}{d_i^{n-2}} (U_j M_i^S U_j^*)^{*_{j=1}^n} < 0.$$

By Theorem 6.3, \mathcal{R} admits no unitary multifusion categorification. This leads to a contradiction. \square

Remark 6.1. *The key point in the proof of Theorem 6.5 is that we use the reduced twist criterion to remove the unknown data λ_i . So it is practicable to check the positivity of the matrix even some entries are known. This is a big advantage of the reduced twist criterion.*

Remark 6.2. *Suppose the bipartite graph \mathcal{G} in Theorem 6.5 is a principal graph of a subfactor with finite index δ^2 , we have $M = \dim_{\mathbb{C}}(\mathcal{P}_{3,\pm}) - \ell^4 - \ell^2 - 2$ and $\dim_{\mathbb{C}}(\mathcal{P}_{2,\pm}) = \ell^2 + 1$. So Theorem 6.5 indicates that there does not exist a principal graph of a subfactor of index δ^2 such that*

- (1) *It has the following local graph*

$$\begin{array}{c} \bullet \text{---} \bullet \text{---} \bullet \text{---} \dots \\ x_1 \quad x_2 \quad x_3 \\ \text{\scriptsize } \ell \text{ arcs} \end{array} \quad (6.10)$$

- (2) *There exist $n \geq 2$, $a, b \geq 1$, $a + b = n$, such that*

$$\left(\frac{\ell^{2a-2} M^b}{(\delta^2 - 1)^{n-2}} + 1 \right) \left(\frac{\ell^{2b-2} M^a}{(\delta^2 - 1)^{n-2}} + 1 \right) - \frac{\ell^{2n}}{\delta^{2n-4}} < 0. \quad (6.11)$$

In particular, let $a = b = 2$. Suppose that $\ell^2 \geq \delta$. Then Inequality (6.11) is equivalent to

$$\dim_{\mathbb{C}} \mathcal{P}_{3,\pm} < \frac{\delta^2 - 1}{\ell} \sqrt{\frac{\ell^4}{\delta^2} - 1} + \ell^4 + \ell^2 + 2. \quad (6.12)$$

So Theorem 6.5 allows us to eliminate a large family of bipartite graphs from principal graphs of subfactors with certain dimension bounds.

Next, we present a special case of Theorem 6.5 when $a = b = 2$ and $n = 4$.

Corollary 6.6. *Suppose \mathcal{G} is the bipartite graph defined in Theorem 6.5. If*

$$\frac{\sqrt{2}M}{\ell} < d_2 - \frac{1}{d_2} < d_2 < \frac{\ell^2}{\sqrt{2}}, \quad (6.13)$$

then \mathcal{G} can not be a principal graph of a subfactor.

Proof. When $a = b = 2$, inequality (6.9) is equivalent to

$$\frac{\ell^2 M^2}{(d_2^2 - 1)^2} + 1 < \frac{\ell^4}{d_2^2}.$$

It is not difficult to check that inequality (6.13) indicates

$$\frac{\ell^2 M^2}{(d_2^2 - 1)^2} < \frac{1}{2} \cdot \frac{\ell^4}{d_2^2}, \quad \text{and} \quad 1 < \frac{1}{2} \cdot \frac{\ell^4}{d_2^2}.$$

Adding this two inequalities, we know that inequality (6.9) holds. So from Theorem 6.5, \mathcal{G} can not be a principal graph of a subfactor. \square

Remark 6.3. *Suppose that $M \leq A\ell^2$, where A is a positive constant. If*

$$\sqrt{2}A\ell < d_2 < \frac{\ell^2}{\sqrt{2}},$$

then inequality (6.13) holds. So \mathcal{G} can not be a principal graph of a subfactor.

7. QUESTIONS

Unitary multifusion categories are unitary multitensor categories [38] whose ranks could be infinite and the Grothendieck ring of a unitary multitensor categories is a based ring. A natural question is whether a based ring is a Grothendieck ring of a unitary multitensor category. Inspired by localized criteria for multifusion rings in Section 6, we propose the following question.

Question 1. *Suppose \mathcal{R} is a based ring with basis $\{x_1, x_2, \dots\}$. Let S be a finite subset of \mathbb{N} and M_j^S is the fusion matrix of x_j restricted on S . If \mathcal{R} admits a unitary multitensor categorification, does the following inequality hold for all $n \in \mathbb{N}$:*

$$\sum_j d_j^2 \left(\frac{M_j^S}{d_j} \right)^{\otimes n} \geq 0? \quad (7.1)$$

Note that these unitary categorification criteria are also criteria of subfactorization of fusion bialgebras (See Definition 7.3 in [26]). There are subfactorizable fusion rings that admit no unitary categorification. For example, the subfactor planar algebra $\mathbb{Z}_n * TL(\delta)$, such that $\delta^2 - 1 = \sqrt{\delta^2 - 1} + n$, is a subfactorization of the following near group fusion ring,

$$\mathcal{R}_n := \{b, g^k, k \in \mathbb{Z}_n : g^n = 1, gb = bg = b, b^2 = b + \sum_{k=1}^n g^k\}.$$

We see that \mathcal{R}_n passes primary criteria because it is subfactorizable. However, \mathcal{R}_n admits no unitary categorification when $n \geq 2$, due to Theorem 1.1 in [11].

Bisch-Haagerup constructed the following sequence of fusion rings in 1994,

$$BH_r := \langle \rho, \tau : \rho^2 = \rho + 1, \tau^2 = 1, (\rho\tau)^r = (\tau\rho)^r \rangle.$$

and they asked that whether they have unitary categorifications. It had been widely believed that the whole sequence of fusion rings have unitary categorification. A surprising answer was given in [22] that BH_r admits a unitary categorification if and only if $r = 0, 1, 2, 3$. (This is a slightly weaker version of the original question and answer on possible compositions of A_3 and A_4 subfactors.) It is natural to ask the following question.

Question 2. *For each $r \geq 4$, does BH_r pass the primary 3-criterion? If so, then is BH_r subfactorizable?*

8. APPENDIX

8.1. SageMath computations.

8.1.1. *Computation for Proposition 3.14.*

```

sage: MM=[
....: [[1,0,0,0,0,0],[0,1,0,0,0,0],[0,0,1,0,0,0],[0,0,0,1,0,0],[0,0,0,0,1,0],[0,0,0,0,0,1]],
....: [[0,1,0,0,0,0],[1,4,2,2,2,2],[0,2,2,1,2,4],[0,2,1,2,4,2],[0,2,2,4,5,4],[0,2,4,2,4,5]],
....: [[0,0,1,0,0,0],[0,2,2,1,4,2],[0,1,3,1,3,3],[1,2,3,3,1,3],[0,2,3,3,5,4],[0,4,1,3,4,5]],
....: [[0,0,0,1,0,0],[0,2,1,2,2,4],[1,2,3,3,3,1],[0,1,1,3,3,3],[0,4,3,1,5,4],[0,2,3,3,4,5]],
....: [[0,0,0,0,1,0],[0,2,4,2,5,4],[0,4,1,3,5,4],[0,2,3,3,5,4],[1,5,5,5,5,7],[0,4,4,4,7,7]],
....: [[0,0,0,0,0,1],[0,2,2,4,4,5],[0,2,3,3,4,5],[0,4,3,1,4,5],[0,4,4,4,7,7],[1,5,5,5,7,5]]
....: ]
sage: L=[matrix(m) for m in MM]
sage: dim=[m.norm() for m in L]
sage: M=sum((L[i].tensor_product(L[i])).tensor_product(L[i])/(dim[i]) for i in range(6))
sage: E=M.eigenvalues()
sage: E[24]
-1.176375045085803

```

8.1.2. *Computation for Subsection 4.2.*

```

sage: MM=[[ [1,0],[0,1]],[ [1,1],[1,1]],[ [1,1],[1,1]],[ [2,1],[1,2]],[ [0,3],[3,1]],[ [3,1],[1,2] ] ]
sage: L=[matrix(m) for m in MM]
sage: dim=[1,5,5,5,6,7,7]
sage: M=sum((L[i].tensor_product(L[i])).tensor_product(L[i])/(dim[i]) for i in range(7))
sage: E=M.eigenvalues()
sage: E[4]
-0.6294949095094869

```

8.1.3. *Computation for Subsection 4.3.*

```

sage: MM=[[ [1,0],[0,1]],[ [1,1],[1,1]],[ [1,1],[1,1]],[ [2,2],[2,2]],[ [2,2],[2,2]],[ [3,2],[2,3]],[ [0,5],[5,1]],[ [5,1],[1,4] ] ]
sage: L=[matrix(m) for m in MM]
sage: dim=[1,5,5,10,10,11,12,12]
sage: M=sum((L[i].tensor_product(L[i])).tensor_product(L[i])/(dim[i]) for i in range(8))
sage: E=M.eigenvalues()
sage: E[4]
-2.948812175750019

```

8.1.4. *Computation for Lemma 4.3.*

```

sage: MM=[[ [1,0],[0,1]],[ [1,1],[1,1]],[ [1,1],[1,1]],[ [2,1],[1,2]],[ [0,3],[3,1]],[ [3,1],[1,2] ] ]
sage: L=[matrix(m) for m in MM]
sage: var('x2,x3,x4,x5,x6,x7')
sage: dim=[1,x2,x3,x4,x5,x6,x7]
sage: M=sum((L[i].tensor_product(L[i])).tensor_product(L[i])/(dim[i]) for i in range(7))
sage: N=M[:7,:7]
sage: N.determinant().factor()
(x2*x3*x4*x5^3*x6^2*x7^3 + 67*x2*x3*x4*x5^3*x6^2*x7^2 + 18*x2*x3*x4*x5^3*x6*x7^3
+ 32*x2*x3*x4*x5^2*x6^2*x7^3 + 7*x2*x3*x5^3*x6^2*x7^3 + 7*x2*x4*x5^3*x6^2*x7^3
+ 7*x3*x4*x5^3*x6^2*x7^3 + 1025*x2*x3*x4*x5^3*x6^2*x7 + 216*x2*x3*x4*x5^3*x6*x7^2
+ 962*x2*x3*x4*x5^2*x6^2*x7^2 + 172*x2*x3*x5^3*x6^2*x7^2 + 172*x2*x4*x5^3*x6^2*x7^2

```

```

+ 172*x3*x4*x5^3*x6^2*x7^2 - 729*x2*x3*x4*x5^3*x7^3 - 126*x2*x3*x4*x5^2*x6*x7^3
- 90*x2*x3*x5^3*x6*x7^3 - 90*x2*x4*x5^3*x6*x7^3 - 90*x3*x4*x5^3*x6*x7^3
+ 195*x2*x3*x4*x5*x6^2*x7^3 + 54*x2*x3*x5^2*x6^2*x7^3 + 54*x2*x4*x5^2*x6^2*x7^3
+ 54*x3*x4*x5^2*x6^2*x7^3 + 3375*x2*x3*x4*x5^3*x6^2 - 7290*x2*x3*x4*x5^3*x6*x7
+ 4600*x2*x3*x4*x5^2*x6^2*x7 + 665*x2*x3*x5^3*x6^2*x7 + 665*x2*x4*x5^3*x6^2*x7
+ 665*x3*x4*x5^3*x6^2*x7 - 19683*x2*x3*x4*x5^3*x7^2 - 8478*x2*x3*x4*x5^2*x6*x7^2
- 2214*x2*x3*x5^3*x6*x7^2 - 2214*x2*x4*x5^3*x6*x7^2 - 2214*x3*x4*x5^3*x6*x7^2
+ 1833*x2*x3*x4*x5*x6^2*x7^2 + 432*x2*x3*x5^2*x6^2*x7^2 + 432*x2*x4*x5^2*x6^2*x7^2
+ 432*x3*x4*x5^2*x6^2*x7^2 - 5832*x2*x3*x4*x5^2*x7^3 - 729*x2*x3*x5^3*x7^3
- 729*x2*x4*x5^3*x7^3 - 729*x3*x4*x5^3*x7^3 - 1728*x2*x3*x4*x5*x6*x7^3
- 594*x2*x3*x5^2*x6*x7^3 - 594*x2*x4*x5^2*x6*x7^3 - 594*x3*x4*x5^2*x6*x7^3
+ 216*x2*x3*x4*x6^2*x7^3 + 63*x2*x3*x5*x6^2*x7^3 + 63*x2*x4*x5*x6^2*x7^3
+ 63*x3*x4*x5*x6^2*x7^3)*(x5^2*x6^2*x7^2 + 25*x5^2*x6^2*x7 - 9*x5^2*x6*x7^2
+ 12*x5*x6^2*x7^2 + 125*x5^2*x6^2 + 135*x5^2*x6*x7 + 120*x5*x6^2*x7 - 729*x5^2*x7^2
+ 108*x5*x6*x7^2 + 27*x6^2*x7^2)^2/(x2*x3*x4*x5^7*x6^6*x7^7)
sage: E=N.eigenvalues()
sage: E[4].factor()
1/2*(2*x5*x6*x7 + 25*x5*x6 - 9*x5*x7 + 12*x6*x7 - sqrt(125*x5^2*x6^2 + 9*(333*x5^2 - 72*x5*x6
+ 4*x6^2)*x7^2 - 30*(33*x5^2*x6 - 4*x5*x6^2)*x7))/(x5*x6*x7)
sage: S=solve((2*x5*x6*x7 + 25*x5*x6 - 9*x5*x7 + 12*x6*x7)^2 - 125*x5^2*x6^2 + 9*(333*x5^2
- 72*x5*x6) + 4*x6^2)*x7^2 - 30*(33*x5^2*x6 - 4*x5*x6^2)*x7,x6)
sage: SS=solve(x5^2*x6^2*x7^2 + 25*x5^2*x6^2*x7 - 9*x5^2*x6*x7^2 + 12*x5*x6^2*x7^2
+ 125*x5^2*x6^2 + 135*x5^2*x6*x7 + 120*x5*x6^2*x7 - 729*x5^2*x7^2 + 108*x5*x6*x7^2
+ 27*x6^2*x7^2,x6)
sage: S==SS
True

```

8.1.5. Computation for Remark 4.3.

```

sage: ddim=[1,5,5,5,6,x6,7]
sage: MM=sum((L[i].tensor_product(L[i])).tensor_product(L[i])/(ddim[i]) for i in range(7))
sage: MM.determinant().factor()
1/22136835840*(210556551*x6^4 - 966777308*x6^3 - 19784561832*x6^2 + 60951456720*x6
+ 306237561840)*(2495*x6^2 + 5544*x6 - 142884)^2/x6^8
sage: S=solve(210556551*x6^4 - 966777308*x6^3 - 19784561832*x6^2 + 60951456720*x6
+ 306237561840,x6)
sage: [s.rhs().n(digits=40) for s in S]
[-8.335767036491464773340488283566158298175 - 2.116481932246399247255558980887840928053e-42*I,
-2.911937593466257812583289613470318862572 + 2.116481932246399247255558980887840928053e-42*I,
6.245789619885724844645019429070095813233 - 3.429094729194210149876597305845817158163e-42*I,
9.593447799759457999876934309097061722308 + 3.429094729194210149876597305845817158163e-42*I]

```

The complex parts above are zero (they appear just due to numerical approximations).

8.1.6. Computation for Remark 4.4.

```

sage: R = ZZ['x6']
sage: x6 = R.gen()
sage: P=210556551*x6^4 - 966777308*x6^3 - 19784561832*x6^2 + 60951456720*x6 + 306237561840
sage: P.is_irreducible()
True
sage: a=421113102

```

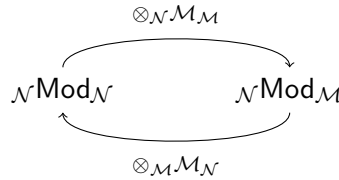
```

.....: b=44334061169015601
.....: c=250462485504
.....: d=345734333761887148583413
.....: e=5213988190704773354123819324759655
.....: f=64964979666121194605838007808
.....: g=149108026701745210176
.....: h=3010843992410706004
.....: i=16567558522416134464
.....: j=4926006796557289
.....: k=1151178558485955841275891856
.....: l=6021687984821412008
.....: m=241694327
.....: n=210556551
.....: A=(c/d*I*sqrt(e) + f/d)^(1/3)
.....: B=sqrt(b*A + g/A + h)
.....: alpha = 1/a*B+ 1/2*sqrt(-A - i/j/A - k/b/B+ l/b) + m/n
sage: str(S[3].rhs().expand())==str(alpha)
True

```

8.2. Principal Graphs. For the definition of fusion category and multifusion category, we refer the readers to [7] and [6, Section 1.12]. The Grothendieck ring of a (multi)fusion category is a (multi)fusion ring. A unitary (multi)fusion category is a (multi)fusion category with a unitary structure, see e.g. [6, Remark 9.4.7].

Let $\mathcal{N} \subseteq \mathcal{M}$ be a finite index subfactor. Its subfactor planar algebra captures the bimodule category of the subfactor, $\{\mathcal{N}\text{Mod}_{\mathcal{N}}, \mathcal{N}\text{Mod}_{\mathcal{M}}, \mathcal{M}\text{Mod}_{\mathcal{M}}, \mathcal{M}\text{Mod}_{\mathcal{N}}\}$. This bimodule category is a unitary multitensor category with the 0-morphisms being factors \mathcal{N} and \mathcal{M} ; the 1-morphisms being bimodules over factors; and the 2-morphisms being the bimodule homomorphisms. The subcategories $\mathcal{N}\text{Mod}_{\mathcal{N}}$ and $\mathcal{M}\text{Mod}_{\mathcal{M}}$ are unitary tensor categories. If they have finitely many simple objects, then the subfactor is called finite depth. Take the generating bimodule $\tau = \mathcal{N}\mathcal{M}_{\mathcal{M}}$. Its conjugate $\bar{\tau}$ is the bimodule ${}_{\mathcal{M}}\mathcal{M}_{\mathcal{N}}$. Moreover, $\gamma = \tau\bar{\tau} = \mathcal{N}\mathcal{M}_{\mathcal{N}}$. It defines a Frobenius algebra (γ, m, ι) in $\mathcal{N}\text{Mod}_{\mathcal{N}}$, where $m \in \text{hom}(\gamma^2, \gamma)$ is the multiplication on \mathcal{M} and ι is the inclusion from \mathcal{N} to \mathcal{M} . Moreover, $\tilde{\gamma} = \bar{\tau}\tau$ induces a Frobenius algebra in $\mathcal{M}\text{Mod}_{\mathcal{M}}$. Conversely, given a Frobenius algebra in a unitary tensor category, one can construct a subfactor planar algebra as above, see e.g. [32].



The principal graph of the subfactor $\mathcal{N} \subseteq \mathcal{M}$ is a bipartite graph. Its vertices are equivalence classes of irreducible bimodules over $(\mathcal{N}, \mathcal{N})$ and $(\mathcal{N}, \mathcal{M})$. The number of edges connecting two vertices, an $(\mathcal{N}, \mathcal{N})$ bimodule Y and an $(\mathcal{N}, \mathcal{M})$ bimodule Z , is the multiplicity of the equivalence class of Z as a sub bimodule of $Y \otimes X$, where $X = \mathcal{N}\mathcal{M}_{\mathcal{M}}$. The dimension vector of the bipartite graph is a function from the vertices of the graph to \mathbb{R}^+ . Its value at a vertex is defined to be the dimension of the corresponding bimodule. The dual principal graph is defined similarly for $(\mathcal{M}, \mathcal{M})$ and $(\mathcal{M}, \mathcal{N})$ bimodules. The adjacent matrix of the principal graph is a submatrix of the fusion matrix of X .

8.3. Quantum Double Construction. Given a unitary fusion category, one can obtain a subfactor planar algebra from the quantum double construction. Here, we recall the construction with notations in [30]. Let \mathcal{C} be a unitary fusion category and $\text{Irr} = \{X_1 = 1, \dots, X_m\}$ the set of all simple objects. For each object X_j , we

denote by $d_j := \text{FPdim}(X_j)$ the Frobenius-Perron dimension of X_j . Denote by $\text{FPdim}(\mathcal{C})$ the dimension of \mathcal{C} , which is $\sum_{j=1}^m \text{FPdim}(X_j)^2$. Let \mathcal{C}^{op} be the opposite category of \mathcal{C} . Then

$$\gamma = \bigoplus_{X \in \text{Irr}} X \boxtimes X^{op}$$

is a Frobenius $*$ -algebra in $\mathcal{C} \boxtimes \mathcal{C}^{op}$, where the multiplication \mathbf{m} is given by

$$\mathbf{m} = \text{FPdim}(\mathcal{C})^{1/4} \bigoplus_{j,k,t=1}^m (d_j d_k d_t)^{1/2} \sum_{\alpha \in \text{ONB}(X_j, X_k; X_t)} \alpha \boxtimes \bar{\alpha}$$

and the canonical unit is $\iota : 1 \boxtimes 1^{op} \rightarrow \gamma$; $\text{ONB}(X_j, X_k; X_t)$ is an orthonormal basis of $\text{hom}_{\mathcal{C}}(X_j \otimes X_k, X_t)$. One can construct a subfactor planar algebra \mathcal{P} with a simple object τ associated to a single string such that $\gamma = \tau \bar{\tau}$. The single quon space $\mathcal{P}_{2,+} := \text{hom}_{\mathcal{P}}(\tau \bar{\tau}, \tau \bar{\tau}) \cong \text{hom}_{\mathcal{C} \boxtimes \mathcal{C}^{op}}(\gamma, \gamma)$ has an orthonormal basis $\{\beta_j\}_{j=1}^m$, where $\beta_j = d_j^{-1} \mathbb{1}_{X_j \boxtimes X_j^{op}}$, and $\mathbb{1}_{X_j \boxtimes X_j^{op}}$ is the identity map on $X_j \boxtimes X_j^{op}$. The morphism β_j is represented as a square-like diagram in the planar algebra

$$\beta_j = \begin{array}{c} \diagup \quad \diagdown \\ | \\ \diagdown \quad \diagup \\ j \end{array} .$$

The multiplication of the morphisms is a vertical composition and the convolution is a horizontal composition. Moreover, (Propositions 4.1, 4.2 in [30])

$$\begin{aligned} \beta_j \beta_k &= \delta_{j,k} d_j^{-1} \beta_j , \\ \beta_j * \beta_k &= \delta^{-1} \sum_{s=1}^m N_{j,k}^s \beta_s , \end{aligned}$$

where $\delta_{j,k}$ is the Kronecker delta function, $\delta^2 = \sum_{j=1}^m d_j^2$ is the Jones index of the subfactor planar algebra, identical to the Frobenius-Perron dimension of \mathcal{C} . Moreover, $\mathcal{P}_{2,-} := \text{hom}_{\mathcal{P}}(\bar{\tau} \tau, \bar{\tau} \tau) \cong \text{hom}_{Z(\mathcal{C})}(\tilde{\gamma}, \tilde{\gamma})$, where $Z(\mathcal{C})$ is the Drinfeld center of \mathcal{C} and $\tilde{\gamma} = \bar{\tau} \tau$ induces a Frobenius algebra in $Z(\mathcal{C})$. Furthermore, $Z(\mathcal{C})$ and $\mathcal{C} \boxtimes \mathcal{C}^{op}$ are Morita equivalent w. r. t. the Frobenius algebras, see e.g. [33].

The Fourier transform $\mathfrak{F}_s : \mathcal{P}_{2,\pm} \rightarrow \mathcal{P}_{2,\mp}$ is a clockwise 90° rotation, which intertwines the multiplication and the convolution. Pictorially,

$$\mathfrak{F}_s(\beta_j) = \begin{array}{c} \diagdown \quad \diagup \\ | \\ \diagup \quad \diagdown \\ j \end{array} ,$$

and

$$\mathfrak{F}_s(\beta_j) * \mathfrak{F}_s(\beta_k) = \delta_{j,k} d_j^{-1} \mathfrak{F}_s(\beta_j) , \quad (8.1)$$

$$\mathfrak{F}_s^{-1}(\beta_j) \mathfrak{F}_s^{-1}(\beta_k) = \delta^{-1} \sum_{t=1}^m N_{j,k}^t \mathfrak{F}_s^{-1}(\beta_t) . \quad (8.2)$$

By Equation (8.2), $\mathcal{P}_{2,-}$ is isomorphic to the C^* algebra $\mathcal{R} \otimes_{\mathbb{Z}} \mathbb{C}$.

One can construct a subfactor planar algebra from a unitary multifusion categories following the quantum double construction in a similar way.

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