# $C^*$ -IRREDUCIBILITY OF COMMENSURATED SUBGROUPS

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ABSTRACT. Given a commensurated subgroup  $\Lambda$  of a group  $\Gamma$ , we completely characterize when the inclusion  $\Lambda \leq \Gamma$  is  $C^*$ -irreducible and provide new examples of such inclusions. In particular, we obtain that  $\mathrm{PSL}(n,\mathbb{Z}) \leq \mathrm{PGL}(n,\mathbb{Q})$  is  $C^*$ -irreducible for any  $n \in \mathbb{N}$ , and that the inclusion of a  $C^*$ -simple group into its abstract commensurator is  $C^*$ -irreducible.

The main ingredient that we use is the fact that the action of a commensurated subgroup  $\Lambda \leq \Gamma$  on its Furstenberg boundary  $\partial_F \Lambda$  can be extended in a unique way to an action of  $\Gamma$  on  $\partial_F \Lambda$ . Finally, we also investigate the counterpart of this extension result for the universal minimal proximal space of a group.

### 1. Introduction

A group  $\Gamma$  is said to be  $C^*$ -simple if its reduced  $C^*$ -algebra  $C^*_r(\Gamma)$  is simple. After the breakthrough characterizations of  $C^*$ -simplicity in [KK17] and [BKKO17], several directions of research applying the new methods in different settings arose.

One of the recent interesting directions is investigating when inclusions of groups  $\Lambda \leq \Gamma$  are  $C^*$ -irreducible, in the sense that every intermediate  $C^*$ -algebra B in  $C^*_r(\Lambda) \subset B \subset C^*_r(\Gamma)$  is simple. In [Rør21], Rørdam started a systematic study of this property and provided a dynamical criterion for an inclusion of groups to be  $C^*$ -irreducible. Together with results in [Amr21], [Urs22] and [BO23], this has provided a complete characterization of  $C^*$ -irreducibility of an inclusion in the case that  $\Lambda$  is a normal subgroup of  $\Gamma$ .

Recall that a subgroup  $\Lambda$  of a group  $\Gamma$  is said to be *commensurated* if, for any  $g \in \Gamma$ ,  $\Lambda \cap g\Lambda g^{-1}$  has finite index in  $\Lambda$ . This is a much more flexible generalization of normal subgroups and finite-index subgroups. For example, for every  $n \geq 2$ ,  $\mathrm{PSL}(n,\mathbb{Z})$  is an infinite-index commensurated subgroup of the simple group  $\mathrm{PSL}(n,\mathbb{Q})$ .

In this work, we generalize the above characterization of  $C^*$ -irreducibility to commensurated subgroups (see Theorem 3.5). The main ingredient in our proof is the fact that the action of  $\Lambda$  on its Furstenberg boundary  $\partial_F \Lambda$  can be uniquely extended to an action of  $\Gamma$  on  $\partial_F \Lambda$  if  $\Lambda$  is a commensurated subgroup in  $\Gamma$  (see Theorem 3.1).

As one of the applications, we show that, if  $\Gamma$  is a  $C^*$ -simple group, then the inclusion of  $\Gamma$  in its abstract commensurator  $Comm(\Gamma)$  is  $C^*$ -irreducible (see Corollary 3.14). To our best knowledge, this is also the first observation of the fact that, if  $\Gamma$  is a  $C^*$ -simple group, then  $Comm(\Gamma)$  is  $C^*$ -simple as well.

Given a subgroup  $\Lambda$  of a group  $\Gamma$ , Ursu introduced in [Urs22] a universal  $\Lambda$ strongly proximal  $\Gamma$ -boundary  $B(\Gamma, \Lambda)$  and showed that, if  $\Lambda \leq \Gamma$ , then  $B(\Gamma, \Lambda) =$ 

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 $\partial_F \Lambda$ . In Section 4, we generalize this fact to commensurated subgroups and also observe that, in general,  $B(\Gamma, \Lambda)$  is not extremally disconnected.

Finally, we also show that, given a commensurated subgroup  $\Lambda$  of a group  $\Gamma$ , the action of  $\Lambda$  on its universal minimal proximal space  $\partial_p \Lambda$  can also be extended in a unique way to an action of  $\Gamma$  on  $\partial_p \Lambda$  (see Theorem 5.1), and use this fact for concluding that, for a certain locally finite commensurated subgroup G of Thompson's group V, the resulting action of V on  $\partial_p G$  is free (see Example 5.4).

### 2. Preliminaries

Given a compact Hausdorff space X, we denote by  $\operatorname{Prob}(X)$  the space of regular probability measures on X. An action of a group  $\Gamma$  on X by homeomorphisms is said to be minimal if X does not contain any non-trivial closed invariant subset, and to be topologically free if, for any  $g \in \Gamma \setminus \{e\}$ , the set  $\{x \in X : gx = x\}$  has empty interior (if  $\Gamma$  is countable, then  $\Gamma \curvearrowright X$  is topologically free if and only if the set of points in X which are not fixed by any non-trivial element of  $\Gamma$  is dense in X). The action is said to be proximal if, given  $x, y \in X$ , there is a net  $(g_i) \subset \Gamma$  such that the nets  $(g_ix)$  and  $(g_iy)$  converge and  $\lim g_ix = \lim g_iy$ . We say that the action is strongly proximal if the induced action  $\Gamma \curvearrowright \operatorname{Prob}(X)$  is proximal. The action is called a boundary action (or X is a  $\Gamma$ -boundary) if it is both minimal and strongly proximal. We denote by  $\partial_F\Gamma$  the Furstenberg boundary of  $\Gamma$ , i.e., the universal  $\Gamma$ -boundary (see [Gla76, Section III.1]). The group  $\Gamma$  is  $C^*$ -simple if and only if  $\Gamma \curvearrowright \partial_F\Gamma$  is free ([BKKO17, Theorem 3.1]).

Given  $\Gamma$ -boundaries X and Y, if there exists  $\varphi \colon X \to Y$  a homeomorphism which is  $\Gamma$ -equivariant ( $\Gamma$ -isomorphism), then it follows from [Gla76, Lemma II.4.1] that  $\varphi$  is the unique  $\Gamma$ -isomorphism between X and Y.

Let  $\Lambda \leq \Gamma$  be a finite-index subgroup. Then any strongly proximal  $\Gamma$ -action is also  $\Lambda$ -strongly proximal ([Gla76, Lemma II.3.1]) and any  $\Gamma$ -boundary is also a  $\Lambda$ -boundary ([Gla76, Lemma II.3.2]). Furthermore, by [Gla76, Theorem II.4.4], which is stated for the universal minimal proximal space but whose proof also works for the Furstenberg boundary, the action  $\Lambda \curvearrowright \partial_F \Lambda$  can be extended to  $\Gamma \curvearrowright \partial_F \Lambda$  and  $\partial_F \Lambda$  is  $\Gamma$ -isomorphic to  $\partial_F \Gamma$ . In particular,  $\partial_F \Lambda$  and  $\partial_F \Gamma$  are also  $\Lambda$ -isomorphic.

Given a group isomorphism  $\psi \colon \Gamma_1 \to \Gamma_2$ , by universality there is a unique homeomorphism  $\tilde{\psi} \colon \partial_F \Gamma_1 \to \partial_F \Gamma_2$  such that  $\tilde{\psi}(gx) = \psi(g)\tilde{\psi}(x)$  for any  $g \in \Gamma_1$  and  $x \in \partial_F \Gamma_1$ .

Given a group  $\Gamma$ , let  $\operatorname{Sub}(\Gamma)$  be the space of subgroups of  $\Gamma$  endowed with the pointwise convergence topology and with the  $\Gamma$ -action given by conjugation. Given a subgroup  $\Lambda \leq \Gamma$ , a  $\Lambda$ -uniformly recurrent subgroup (URS) is a non-empty closed  $\Lambda$ -invariant minimal set  $\mathcal{U} \subset \operatorname{Sub}(\Gamma)$ . Moreover, we say that  $\mathcal{U}$  is amenable if one (equivalently all) of its elements is amenable. By [Ken20, Theorem 4.1], a group  $\Gamma$  is  $C^*$ -simple if and only if it does not admit any non-trivial amenable  $\Gamma$ -uniformly recurrent subgroup.

An inclusion of groups  $\Lambda \leq \Gamma$  is said to be  $C^*$ -irreducible if every intermediate  $C^*$ -algebra of  $C_r^*(\Lambda) \subset C_r^*(\Gamma)$  is simple.

Given  $\Lambda \leq \Gamma$  and  $g \in \Gamma$ , let  $g^{\Lambda} := \{hgh^{-1} : h \in \Lambda\}$ . We say that  $\Gamma$  is icc relatively to  $\Lambda$  if, for any  $g \in \Gamma \setminus \{e\}$ ,  $|g^{\Lambda}| < \infty$ . The group  $\Gamma$  is said to be icc if it is icc relatively to itself.

### 3. C\*-irreducibility of commensurated subgroups

Let  $\Gamma$  be a group. Two subgroups  $\Lambda_1, \Lambda_2 \leq \Gamma$  are said to be *commensurable* if  $[\Lambda_1 : \Lambda_1 \cap \Lambda_2] < \infty$  and  $[\Lambda_2 : \Lambda_1 \cap \Lambda_2] < \infty$ . Notice that this is an equivalence relation.

A subgroup  $\Lambda \leq \Gamma$  is said to be *commensurated* if, for any  $g \in \Gamma$ ,  $\Lambda$  is commensurable with  $g\Lambda g^{-1}$ . Equivalently, for any  $g \in \Gamma$ ,  $[\Lambda : \Lambda \cap g\Lambda g^{-1}] < \infty$ . In this case, we write  $\Lambda \leq_c \Gamma$ . In the literature, this notion is also referred to by saying that  $\Lambda$  is an almost normal subgroup of  $\Gamma$  or that  $(\Gamma, \Lambda)$  is a Hecke pair.

The following result generalizes [Gla76, Theorem II.4.4] and [Oza14, Lemma 20].

**Theorem 3.1.** Let  $\Lambda \leq_c \Gamma$ . Then  $\Lambda \curvearrowright \partial_F \Lambda$  extends in a unique way to an action of  $\Gamma$  on  $\partial_F \Lambda$ .

Proof. Given  $g \in \Gamma$ , let  $\varphi_g \colon \partial_F \Lambda \to \partial_F (\Lambda \cap g \Lambda g^{-1})$  be the  $(\Lambda \cap g \Lambda g^{-1})$ -isomorphism. Also let  $\psi_g \colon \partial_F (\Lambda \cap g^{-1} \Lambda g) \to \partial_F (\Lambda \cap g \Lambda g^{-1})$  be the homeomorphism such that for all  $h \in \Lambda \cap g^{-1} \Lambda g$  and  $x \in \partial_F (\Lambda \cap g^{-1} \Lambda g)$  we have  $\psi_g(hx) = ghg^{-1}\psi_g(x)$ . Let  $T_g := (\varphi_g)^{-1}\psi_g \varphi_{g^{-1}} \colon \partial_F \Lambda \to \partial_F \Lambda$ . We claim that  $g \mapsto T_g$  is a  $\Gamma$ -action which extends  $\Lambda \cap \partial_F \Lambda$ .

Given  $h \in \Lambda \cap g^{-1}\Lambda g$  and  $x \in \partial_F \Lambda$ , one can readily check that  $T_g(hx) = ghg^{-1}T_g(x)$ .

Given  $g,h \in \Gamma$ , we have that  $[\Lambda : \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)] < \infty$ . Furthermore, given  $k \in \Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh)$  and  $x \in \partial_F \Lambda$ , we have  $T_{gh}(kx) = (gh)k(gh)^{-1}T_{gh}(x)$ . On the other hand,  $T_gT_h(kx) = (gh)k(gh)^{-1}T_gT_h(x)$ . In particular,  $(T_gT_h)^{-1}T_{gh}$  is a  $(\Lambda \cap h^{-1}\Lambda h \cap (gh)^{-1}\Lambda(gh))$ -automorphism, hence  $T_{gh} = T_gT_h$ .

Finally, given  $g \in \Lambda$ , we have that  $x \mapsto g^{-1}T_g(x)$  is a  $(\Lambda \cap g^{-1}\Lambda g)$ -automorphism, so that  $g^{-1}T_g = \mathrm{Id}_{\partial_F\Lambda}$ .

**Remark 3.2.** The existence part of Theorem 3.1 was shown by Dai and Glasner in [DG19, Theorem 6.1] using a different method and assuming that  $\Gamma$  is countable.

Given a subset S of a group  $\Gamma$ , let  $C_{\Gamma}(S)$  be the *centralizer* of S in  $\Gamma$ . In the next result, we follow the argument of [BKKO17, Lemma 5.3].

**Lemma 3.3.** Let  $\Lambda \leq_c \Gamma$  and consider  $\Gamma \curvearrowright \partial_F \Lambda$ . Given  $s \in \Gamma$ , if  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ , then  $Fix(s) = \partial_F \Lambda$ . Conversely, if  $\Lambda \curvearrowright \partial_F \Lambda$  is free and  $Fix(s) \neq \emptyset$ , then  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ .

*Proof.* If  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ , then, given  $h \in \Lambda \cap s^{-1}\Lambda s$  and  $x \in \partial_F \Lambda$ , we have s(hx) = hs(x). Since  $[\Lambda : \Lambda \cap s^{-1}\Lambda s] < \infty$ , we conclude that s acts trivially on  $\partial_F \Lambda$ .

Suppose now that  $\Lambda \curvearrowright \partial_F \Lambda$  is free and  $\operatorname{Fix}(s) \neq \emptyset$ . Given  $t \in A := \{t \in \Lambda \cap s^{-1}\Lambda s : t\operatorname{Fix}(s) \cap \operatorname{Fix}(s) \neq \emptyset\}$ , we have that the action of  $sts^{-1}$  and t coincide on  $\operatorname{Fix}(s) \cap t^{-1}\operatorname{Fix}(s)$ . Since  $sts^{-1}$ ,  $t \in \Lambda$  and  $\Lambda \curvearrowright \partial_F \Lambda$  is free, we obtain that  $t = sts^{-1}$ . Since, by [BKKO17, Lemma 5.1], A generates  $\Lambda \cap s^{-1}\Lambda s$ , we conclude that  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ .

The proof of the following result is an adaptation of the argument in [Ken20, Remark 4.2] and its hypothesis is the same as in [Rør21, Theorem 5.3.(ii)].

**Proposition 3.4.** Let  $\Lambda \leq \Gamma$ . Suppose that there exists a  $\Gamma$ -boundary X such that, for any  $\mu \in \operatorname{Prob}(X)$ , there exists a net  $(g_i) \subset \Lambda$  such that  $g_i\mu$  converges to  $\delta_x$ ,

for some  $x \in X$ , on which  $\Gamma$  acts freely. Then  $\Gamma$  does not admit any non-trivial amenable  $\Lambda$ -URS.

Proof. Suppose  $\mathcal{U}$  is a non-trivial amenable  $\Lambda$ -URS, and take  $K \in \mathcal{U}$ . Since K is amenable, there exists  $\mu \in \operatorname{Prob}(X)$  fixed by K. Let  $(g_i) \subset \Lambda$  be a net such that  $g_i \mu \to \delta_x$ , for some  $x \in X$ , on which  $\Gamma$  acts freely. By taking a subnet, we may assume that  $g_i K g_i^{-1} \to L \in \operatorname{Sub}(\Gamma)$ . Take  $g \in L \setminus \{e\}$  and  $(k_i) \subset K$  such that  $g_i k_i g_i^{-1} = g$  for i sufficiently big. Then

$$\delta_x = \lim g_i \mu = \lim g_i k_i \mu = \lim g_i k_i g_i^{-1} g_i \mu = g \delta_x,$$

contradicting the fact that  $\Gamma$  acts freely on x.

The following result generalizes [Urs22, Theorems 1.3 and 1.9] and [BO23, Theorem 6.4], as well as the claim about finite-index subgroups in [R $\phi$ r21, Theorem 5.3].

**Theorem 3.5.** Let  $\Lambda \leq_c \Gamma$ . The following conditions are equivalent:

- (1)  $\Lambda \leq \Gamma$  is  $C^*$ -irreducible;
- (2)  $\Lambda$  is  $C^*$ -simple and  $\Gamma$  is icc relatively to  $\Lambda$ ;
- (3)  $\Lambda$  is  $C^*$ -simple and, for any  $s \in \Gamma \setminus \{e\}$ , we have that  $s \notin C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ ;
- (4)  $\Gamma \curvearrowright \partial_F \Lambda$  is free;
- (5) There is no non-trivial amenable  $\Lambda$ -URS of  $\Gamma$ ;
- (6)  $\Lambda$  is  $C^*$ -simple and  $\Gamma \curvearrowright \partial_F \Lambda$  is faithful.

*Proof.* (1)  $\Longrightarrow$  (2) follows from [Rør21, Remark 3.8 and Proposition 5.1].

(2)  $\Longrightarrow$  (3). Suppose that there is  $s \in \Gamma \setminus \{e\}$  such that  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ . Take  $g_1, \ldots, g_n \in \Lambda$  left coset representatives for  $\frac{\Lambda}{\Lambda \cap s^{-1}\Lambda s}$ . Then

$$s^{\Lambda} = \{g_i k s k^{-1} g_i^{-1} : 1 \le i \le n, k \in \Lambda \cap s^{-1} \Lambda s\} = \{g_i s g_i^{-1} : 1 \le i \le n\}$$

is finite.

- $(3) \Longrightarrow (4)$  follows from Lemma 3.3.
- $(4) \Longrightarrow (1)$  follows from [Rør21, Theorem 5.3].
- (5)  $\Longrightarrow$  (2). If  $\Lambda$  is not  $C^*$ -simple, then it contains a non-trivial amenable  $\Lambda$ -uniformly recurrent subgroup. If  $\Gamma$  is not icc relatively to  $\Lambda$ , there exists  $s \in \Gamma \setminus \{e\}$  such that  $s^{\Lambda}$  is finite. Hence the  $\Lambda$ -orbit of  $\langle s \rangle$  is a finite non-trivial amenable  $\Lambda$ -uniformly recurrent subgroup.
  - $(4) \Longrightarrow (5)$  follows from Proposition 3.4.
  - $(3) \iff (6)$  follows from Lemma 3.3.

**Remark 3.6.** In [Rør21, Theorem 5.3], Rørdam showed that an inclusion  $\Lambda \leq \Gamma$  satisfying the hypothesis of Proposition 3.4 is  $C^*$ -irreducible, and asked whether the converse holds. We do not know whether the converse of Proposition 3.4 holds and whether the absence of non-trivial amenable  $\Lambda$ -URS of  $\Gamma$  is equivalent to  $\Lambda \leq \Gamma$  being  $C^*$ -irreducible in general.

Corollary 3.7. Given  $n \in \mathbb{N}$ , the inclusion

$$PSL(n, \mathbb{Z}) \leq PGL(n, \mathbb{Q})$$

is  $C^*$ -irreducible.

*Proof.* It was shown in [BCdlH94] that  $PSL(n, \mathbb{Z})$  is  $C^*$ -simple.

Let  $U(n, \mathbb{Z})$  be the group of units of the ring  $M_n(\mathbb{Z})$ . By [Kri90, Corollary V.5.3],  $U(n, \mathbb{Z}) \leq_c \mathrm{GL}(n, \mathbb{Q})$ . Since  $[U(n, \mathbb{Z}) : \mathrm{SL}(n, \mathbb{Z})] = 2$ , we conclude that  $\mathrm{SL}(n, \mathbb{Z}) \leq_c \mathrm{SL}(n, \mathbb{Z}) \leq_c \mathrm{SL}(n, \mathbb{Z})$ 

 $GL(n, \mathbb{Q})$  as well. Since taking quotients preserves being commensurated, it follows that  $PSL(n, \mathbb{Z}) \leq_c PGL(n, \mathbb{Q})$ .

Let  $(e_{ij})_{1 \leq i,j \leq n} \in M_n(\mathbb{Z})$  be the matrix units and fix  $[a] \in \operatorname{PGL}(n,\mathbb{Q}) \setminus \{[\operatorname{Id}]\}$ . By taking conjugates of [a] by elements of the form  $[\operatorname{Id}+m \cdot e_{ij}] \in \operatorname{PSL}(n,\mathbb{Z}), m \in \mathbb{Z}, 1 \leq i \neq j \leq n$ , it is easy to see that  $[a]^{\operatorname{PSL}(n,\mathbb{Z})}$  is infinite, so that  $\operatorname{PGL}(n,\mathbb{Q})$  is icc relatively to  $\operatorname{PSL}(n,\mathbb{Z})$ .

The conclusion then follows from Theorem 3.5.

**Remark 3.8.** Let us sketch a different proof of Corollary 3.7 which gives the stronger statement that  $PSL(n, \mathbb{Z}) \leq PGL(n, \mathbb{R})$  is  $C^*$ -irreducible, where  $PGL(n, \mathbb{R})$  is seen as a discrete group.

Clearly, it suffices to show that, for any countable group  $\Gamma$  such that  $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma \leq \mathrm{PGL}(n,\mathbb{R})$ , the inclusion  $\mathrm{PSL}(n,\mathbb{Z}) \leq \Gamma$  is  $C^*$ -irreducible. By the argument in [Bry17, Example 3.4.3], the action of  $\mathrm{PGL}(n,\mathbb{R})$  on the projective space  $P^{n-1}(\mathbb{R})$  is topologically free. Since  $\mathrm{PSL}(n,\mathbb{Z}) \curvearrowright \mathrm{P}^{n-1}(\mathbb{R})$  is a boundary action, the result follows from [Rør21, Theorem 5.3].

Corollary 3.9. Let  $\Lambda$  be a finite-index subgroup of a group  $\Gamma$ . If  $\Gamma$  is  $C^*$ -simple, then  $\Lambda \leq \Gamma$  is  $C^*$ -irreducible. Conversely, if  $\Lambda$  is  $C^*$ -simple, then  $\Gamma$  is icc if and only if  $\Lambda \leq \Gamma$  is  $C^*$ -irreducible.

*Proof.* If  $\Gamma$  is  $C^*$ -simple, then  $\Gamma \curvearrowright \partial_F \Gamma$  is free. Since  $\partial_F \Gamma$  is  $\Gamma$ -isomorphic to  $\partial_F \Lambda$ , it follows that  $\Lambda < \Gamma$  is  $C^*$ -irreducible.

If  $\Gamma$  is icc, then, since  $[\Gamma : \Lambda] < \infty$ , it is also icc relatively to  $\Lambda$ , hence  $\Lambda \leq \Gamma$  is  $C^*$ -irreducible by Theorem 3.5. The last implication is immediate.

**Example 3.10.** The inclusion given by the Sanov subgroup  $\mathbb{F}_2 \leq \mathrm{PSL}(2,\mathbb{Z})$  is finite-index, hence it is  $C^*$ -irreducible by Corollary 3.9.

Free groups. Fix  $m, n \in \mathbb{N}$  such that  $2 \leq m < n$  and consider the free groups  $\mathbb{F}_m = \langle a_1, \dots, a_m \rangle \leq \langle a_1, \dots, a_n \rangle = \mathbb{F}_n$ . In [Rør21, Example 5.4], Rørdam observed that  $\mathbb{F}_m \leq \mathbb{F}_n$  is  $C^*$ -irreducible. Notice that  $\mathbb{F}_m$  is far from being commensurated in  $\mathbb{F}_n$ . In fact, given  $g \in \mathbb{F}_n \setminus \mathbb{F}_m$ , we have that  $\mathbb{F}_m \cap g\mathbb{F}_m g^{-1} = \{e\}$  (i.e.,  $\mathbb{F}_m$  is malnormal in  $\mathbb{F}_n$ ). In particular, this example is not covered by Theorems 3.1 and 3.5. Nonetheless, there does exist an extension to  $\mathbb{F}_n$  of the action  $\mathbb{F}_m \cap \partial_F \mathbb{F}_m$ , but it is far from being unique, since the generators  $a_{m+1}, \dots, a_n$  can be mapped into any homeomorphisms on  $\partial_F \mathbb{F}_m$ .

Furthermore, we claim that  $\mathbb{F}_m \leq \mathbb{F}_n$  satisfies condition (5) in Theorem 3.5. We will prove this by using Proposition 3.4.

Let

$$\partial \mathbb{F}_n := \{(x_i) \in \prod_{\mathbb{N}} \{a_1, a_1^{-1}, \dots, a_n, a_n^{-1}\} : \forall i \in \mathbb{N}, x_{i+1} \neq x_i^{-1}\}$$

be the Gromov boundary of  $\mathbb{F}_n$ , and consider the action of  $\mathbb{F}_n$  on  $\partial \mathbb{F}_n$  by left multiplication. Fix  $\mu \in \text{Prob}(\partial \mathbb{F}_n)$  and we will show that there is  $w \in \partial \mathbb{F}_n$  on which  $\mathbb{F}_n$  acts freely and such that  $\delta_w \in \overline{\mathbb{F}_m \mu}$ .

Let  $z_+ := (a_1)_{i \in \mathbb{N}} \in \partial \mathbb{F}_n$  and  $z_- := (a_1^{-1})_{i \in \mathbb{N}} \in \partial \mathbb{F}_n$ . Notice that, for all  $y \in \partial \mathbb{F}_n \setminus \{z_-\}$ , we have that, as  $k \to +\infty$ ,  $a_1^k y \to z_+$ . Furthermore,  $a_1$  fixes  $z_-$ .

It follows from the dominated convergence theorem that

$$a_1^k \mu \to \mu(\{z_-\})\delta_{z_-} + (1 - \mu(\{z_-\})\delta_{z_+}),$$

as  $k \to +\infty$ . In particular,  $\nu := \mu(\{z_-\})\delta_{z_-} + (1 - \mu(\{z_-\})\delta_{z_+} \in \overline{\mathbb{F}_n\mu}$ .

Let  $w:=a_1a_2^1a_1a_2^2a_1a_2^3\dots a_1a_2^la_1a_2^{l+1}\dots\in\partial\mathbb{F}_n$ . Since w is not eventually periodic, we have that  $\mathbb{F}_n$  acts freely on w. Given  $k\in\mathbb{N}$ , let  $g_k:=w_1\dots w_k\underline{a_2}\in\mathbb{F}_m$ . We have that  $g_kz_\pm=w_1\dots w_ka_2z_\pm\to w$ , as  $k\to+\infty$ . Therefore,  $\delta_w\in\overline{\mathbb{F}_m\nu}\subset\overline{\mathbb{F}_m\mu}$ , thus showing the claim.

Abstract commensurator. Let  $\Gamma$  be a group and  $\Omega$  be the set of isomorphisms between finite-index subgroups of  $\Gamma$ . Given  $\alpha, \beta \in \Omega$ , we say that  $\alpha \sim \beta$  if there exists a finite-index subgroup  $H \leq \operatorname{dom}(\alpha) \cap \operatorname{dom}(\beta)$  such that  $\alpha|_H = \beta|_H$ . Recall that the *abstract commensurator* of  $\Gamma$ , denoted by  $\operatorname{Comm}(\Gamma)$ , is the group whose underlying set is  $\Omega/\sim$ , with product given by composition (defined up to finite-index subgroup).

Let  $\Lambda$  be a commensurated subgroup of  $\Gamma$ . Given  $g \in \Gamma$ , let

$$\beta_g \colon \Lambda \cap g^{-1} \Lambda g \to \Lambda \cap g \Lambda g^{-1}$$

$$h \mapsto g h g^{-1}$$

and  $j_{\Lambda}^{\Gamma} \colon \Gamma \to \operatorname{Comm}(\Lambda)$  be the homomorphism given by  $j_{\Lambda}^{\Gamma}(g) := [\beta_g]$ . In order to ease the notation, we will sometimes denote  $j_{\Lambda}^{\Gamma}$  simply by j, and it will always be clear from the context what are the involved groups. Let us now collect a few elementary facts about j.

**Lemma 3.11.** Let  $\Gamma$  be a group. Then  $j_{\Gamma}^{\Gamma}(\Gamma) \leq_c \operatorname{Comm}(\Gamma)$ .

*Proof.* Fix  $[\alpha] \in \text{Comm}(\Gamma)$ . Given  $g \in \text{dom}(\alpha)$ , we have that  $[\alpha]j(g)[\alpha]^{-1} = j(\alpha(g))$ . In particular,  $j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1} \supset j(\text{Im}(\alpha))$ . Since  $[\Gamma : \text{Im}(\alpha)] < \infty$ , we conclude that  $[j(\Gamma) : j(\Gamma) \cap [\alpha]j(\Gamma)[\alpha]^{-1}] < \infty$ .

**Lemma 3.12.** Let  $\Lambda \leq_c \Gamma$ . Then  $\ker j_{\Lambda}^{\Gamma} = \{g \in \Gamma : |g^{\Lambda}| < \infty\}$ .

*Proof.* Given  $g \in \ker j$ , there exists a finite-index subgroup  $H \leq \Lambda \cap g^{-1}\Lambda g$  such that, for all  $h \in H$ ,  $ghg^{-1} = h$ , which implies that  $|g^{\Lambda}| < \infty$ . Conversely, if  $|g^{\Lambda}| < \infty$ , then  $H := \{k \in \Lambda : kg = gk\}$  is a finite-index subgroup of  $\Lambda$  and  $g \in \ker j$ .

As a consequence of Lemma 3.12, if  $\Gamma$  is an icc group, then  $j \colon \Gamma \to \operatorname{Comm}(\Gamma)$  is injective ([Kid11, Lemma 3.8.(i)]). The next result is known ([Kid11, Lemma 3.8.(iii)]). For the convenience of the reader, we provide the proof here.

**Lemma 3.13.** If  $\Gamma$  is an icc group, then  $Comm(\Gamma)$  is icc relatively to  $\Gamma$ .

*Proof.* Given  $[\alpha] \in \text{Comm}(\Gamma)$  and  $g \in \text{dom}(\alpha)$ , we have

$$j(g)[\alpha]j(g^{-1}) = j(g\alpha(g^{-1}))[\alpha].$$

If  $[\alpha] \neq e$ , then  $H := \{g \in \text{dom}(\alpha) : g = \alpha(g)\}$  has infinite-index in  $\text{dom}(\alpha)$ . Given  $g_1, g_2 \in \text{dom}(\alpha)$  such that  $g_1H \neq g_2H$ , one can readily check that  $g_1\alpha(g_1)^{-1} \neq g_2\alpha(g_2)^{-1}$ . From this, it follows immediately that  $[\alpha]^{\Gamma}$  is infinite.  $\square$ 

In [BO23, Corollary 6.6], Bédos and Omland showed that if  $\Gamma$  is a  $C^*$ -simple group, then  $\Gamma \leq \operatorname{Aut}(\Gamma)$  is  $C^*$ -irreducible. The same conclusion holds when we consider the abstract commensurator:

**Corollary 3.14.** Given a  $C^*$ -simple group  $\Gamma$ , we have that  $\Gamma \leq \operatorname{Comm}(\Gamma)$  is  $C^*$ -irreducible.

*Proof.* Recall that any  $C^*$ -simple group is icc (this follows, e.g., from Theorem 3.5). The result is then a consequence of Theorem 3.5 and Lemma 3.13.

**Remark 3.15.** Corollary 3.14 generalizes the fact proven in [LBMB18, Corollary 4.4] that, if Thompson's group F is  $C^*$ -simple, then Comm(F) is  $C^*$ -simple.

**Remark 3.16.** Let  $\mathbb{F}_n$  be a non-abelian free group of finite rank. Then Corollary 3.14 implies that  $\text{Comm}(\mathbb{F}_n)$  is  $C^*$ -simple. In particular, it does not admit any non-trivial amenable normal subgroup. It is an open problem whether  $\text{Comm}(\mathbb{F}_n)$  is a simple group ([CM18, Problem 7.2]).

#### 4. Relative boundaries

Given groups  $\Lambda \leq \Gamma$ , Ursu introduced in [Urs22, Proposition 4.1] a  $\Lambda$ -strongly proximal  $\Gamma$ -boundary  $B(\Gamma, \Lambda)$  which is universal with these properties.

Consider  $\Gamma := \mathrm{PSL}(2,\mathbb{Z})$  and the boundary action  $\Gamma \curvearrowright \mathbb{R} \cup \{\infty\}$ . The stabilizer  $\Gamma_{\infty}$  of  $\infty$  is isomorphic to  $\mathbb{Z}$  and consists of the translations  $g_n(x) := x + n, n \in \mathbb{Z}$ ,  $x \in \mathbb{R}$ .

**Proposition 4.1.** The action of  $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$  on  $B(\Gamma,\Gamma_{\infty})$  is topologically free but non-free. In particular,  $B(\Gamma,\Gamma_{\infty})$  is not extremally disconnected.

Proof. For any  $x \in \mathbb{R} \cup \{\infty\}$ , we have  $g_n(x) \to \infty$  as  $n \to +\infty$ . As a consequence of the dominated convergence theorem, it follows easily that  $\Gamma_\infty \curvearrowright \mathbb{R} \cup \{\infty\}$  is strongly proximal. Hence, there is a  $\Gamma$ -equivariant map  $B(\Gamma, \Gamma_\infty) \to \mathbb{R} \cup \{\infty\}$ . Since  $\Gamma_\infty \curvearrowright B(\Gamma, \Gamma_\infty)$  is strongly proximal, it follows from amenability of  $\Gamma_\infty$  that  $\Gamma_\infty$  fixes some point in  $B(\Gamma, \Gamma_\infty)$ . In particular,  $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$  is not free. On the other hand, since  $\Gamma \curvearrowright \mathbb{R} \cup \{\infty\}$  is topologically free, it follows from [BKKO17, Lemma 3.2] that  $\Gamma \curvearrowright B(\Gamma, \Gamma_\infty)$  is topologically free. As a consequence of [Fro71, Theorem 3.1],  $B(\Gamma, \Gamma_\infty)$  is not extremally disconnected.

Remark 4.2. Let  $\Gamma$  be a group. One of the key properties in the applications of  $\partial_F \Gamma$  to  $C^*$ -simplicity of  $\Gamma$  is the fact that  $C(\partial_F \Gamma)$  is injective, shown in [KK17, Theorem 3.12]. Proposition 4.1 implies that  $C(B(\Gamma, \Lambda))$  is not injective, in general. We believe that this is an evidence that  $B(\Gamma, \Lambda)$  is not likely to play the same role of the Furstenberg boundary in  $C^*$ -algebraic applications.

Our next aim is to show that, given  $\Lambda \leq_c \Gamma$ , it holds that  $B(\Gamma, \Lambda) = \partial_F \Lambda$ . We start with a result which we believe has its own interest.

**Theorem 4.3.** Let  $\Lambda \leq_c \Gamma$  and  $\Gamma \curvearrowright X$  a minimal action on a compact space such that  $\Lambda \curvearrowright X$  is proximal. Then  $\Lambda \curvearrowright X$  is minimal as well.

*Proof.* Let  $M \subset X$  be a closed non-empty  $\Lambda$ -invariant set. For any  $g \in \Gamma$ , we have that gM is  $g\Lambda g^{-1}$ -invariant.

Fix  $g_1, \ldots, g_n \in \Gamma$ . We have that  $H := \Lambda \cap g_1 \Lambda g_1^{-1} \cap \cdots \cap g_n \Lambda g_n^{-1}$  has finite index in  $\Lambda$ . In particular,  $H \cap X$  is proximal and admits a unique minimal component K. Since each  $g_i M$  is  $g_i \Lambda g_i^{-1}$ -invariant, we conclude that  $K \subset \bigcap_{i=1}^n g_i M$ .

By compactness of X, we obtain that  $L := \bigcap_{g \in \Gamma} gM \neq \emptyset$ . Since L is  $\Gamma$ -invariant, we have  $X = L \subset M$ .

The following is an immediate consequence of the previous theorem:

Corollary 4.4. Let  $\Lambda \leq_c \Gamma$ . If X is a  $\Gamma$ -boundary which is also  $\Lambda$ -strongly proximal, then X is a  $\Lambda$ -boundary.

By arguing as in [Urs22, Corollary 4.3], we conclude the following:

Corollary 4.5. If  $\Lambda \leq_c \Gamma$ , then  $B(\Gamma, \Lambda) = \partial_F \Lambda$ .

### 5. Commensurated subgroups and proximal actions

Given a group  $\Gamma$ , there exists a universal minimal proximal  $\Gamma$ -space  $\partial_p\Gamma$  ([Gla76, Theorem II.4.2]). It was shown in [FTVF19, Proposition 2.12] and [GTWZ21, Theorem 1.5] that a countable group  $\Gamma$  is icc if and only if  $\Gamma \curvearrowright \partial_p\Gamma$  is faithful, if and only if  $\Gamma \curvearrowright \partial_p\Gamma$  is free.

One can easily check that the statements of Theorem 3.1 and Lemma 3.3 hold with  $\partial_p \Lambda$  instead of  $\partial_F \Lambda$ , with the exact same proofs (in particular, [BKKO17, Lemma 5.1], which is needed in the proof of Lemma 3.3, uses only proximality). Thus, we obtain:

**Theorem 5.1.** Let  $\Lambda \leq_c \Gamma$ . Then  $\Lambda \curvearrowright \partial_p \Lambda$  extends in a unique way to an action of  $\Gamma$  on  $\partial_p \Lambda$ . Furthermore, given  $s \in \Gamma$ , if  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ , then  $Fix(s) = \partial_p \Lambda$ . Conversely, if  $\Lambda \curvearrowright \partial_p \Lambda$  is free and  $Fix(s) \neq \emptyset$ , then  $s \in C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ .

As a consequence, we obtain the following:

**Theorem 5.2.** Let  $\Lambda \leq_c \Gamma$  and suppose that  $\Lambda \curvearrowright \partial_p \Lambda$  is free. The following conditions are equivalent:

- (1)  $\Gamma$  is icc relatively with  $\Lambda$ ;
- (2) For any  $s \in \Gamma \setminus \{e\}$ , we have that  $s \notin C_{\Gamma}(\Lambda \cap s^{-1}\Lambda s)$ ;
- (3)  $\Gamma \curvearrowright \partial_p \Lambda$  is free;
- (4)  $\Gamma \curvearrowright \partial_p \Lambda$  is faithful.

*Proof.* The implications  $(1) \Longrightarrow (2) \Longrightarrow (3) \Longrightarrow (4)$  are proven as in Theorem 3.5.

 $(4) \Longrightarrow (1)$ . Suppose that there is  $g \in \Gamma \setminus \{e\}$  such that  $|g^{\Lambda}| < \infty$ . Then  $H := \{h \in \Lambda : gh = hg\}$  is a finite-index subgroup of  $\Lambda$ , hence  $H \cap \partial_p \Lambda$  is also minimal and proximal. Since the homeomorphism on  $\partial_p \Lambda$  given by g is H-equivariant, we conclude that g acts trivially on  $\partial_p \Lambda$ .

Remark 5.3. Given a group  $\Gamma$ , let  $L(\Gamma)$  be its group von Neumann algebra. Given  $\Lambda \leq \Gamma$ , it follows from [Rør21, Proposition 5.1] and [BO23, Corollary 4.3] that  $\Gamma$  is icc relatively to  $\Lambda$  if and only if any intermediate von Neumann algebra of  $L(\Lambda) \subset L(\Gamma)$  is a factor, if and only if any intermediate  $C^*$ -algebra of  $C^*_r(\Lambda) \subset C^*_r(\Gamma)$  is prime.

Let us now apply Theorem 5.2 to a certain locally finite commensurated subgroup of Thompson's group V.

**Example 5.4.** Let  $X := \{0,1\}$  and, given  $n \geq 0$ , let  $X^n$  be the set of words in X of length n. Given  $w \in X^n$ , let  $\mathcal{C}(w) := \{(s_n) \in X^{\mathbb{N}} : s_{[1,n]} = w\}$ . Recall that Thompson's group V is the group of homeomorphisms on  $X^{\mathbb{N}}$  consisting of elements g for which there exist two partitions  $\{\mathcal{C}(w_1), \ldots, \mathcal{C}(w_m)\}$  and  $\{\mathcal{C}(z_1), \ldots, \mathcal{C}(z_m)\}$  of  $\{0,1\}^{\mathbb{N}}$  such that  $g(w_is) = z_is$  for every  $1 \leq i \leq m$  and  $s \in X^{\mathbb{N}}$ .

Let us define inductively groups  $G_n$  acting by permutations on  $X^n$ .

Let  $G_1 := \mathbb{Z}_2$  acting non-trivially on X and, for  $n \in \mathbb{N}$ ,

$$G_{n+1} := \left(\bigoplus_{w \in X^n} \mathbb{Z}_2\right) \rtimes G_n,$$

where the action of  $G_{n+1}$  on  $X^{n+1}$  is defined as follows: given  $v \in X^n$ ,  $x \in X$ ,  $\sigma \in G_n$  and  $f \in \bigoplus_{X^n} \mathbb{Z}_2$ ,

$$(f,\sigma)(vx) := \sigma(v)f_{\sigma(v)}(x).$$

Let  $G := \lim_{n \in \mathbb{N}} G_n$ . Then G acts faithfully on  $X^{\mathbb{N}}$  and, as observed in [LB17, Proposition 7.11],  $G \leq_c V$ .

We claim that V is icc relatively with G. Given  $u \in X^n$ , let the rigid stabilizer of u, denoted by  $\mathrm{rist}_G(u)$ , be the subgroup of G consisting of the elements which, for every  $v \in X^n \setminus \{u\}$ , act as the identity on C(v). Given  $g \in G$ , there is  $\tilde{g} \in \mathrm{rist}_G(u)$  such that  $\tilde{g}(us) = ug(s)$  for any  $s \in X^{\mathbb{N}}$ . Clearly, the map  $g \mapsto \tilde{g}$  is an isomorphism from G to  $\mathrm{rist}_G(u)$ . Fix  $h \in V \setminus \{e\}$  and take  $w \in X^n$  and  $z \in X^m$  such that  $w \neq z$ ,  $n \geq m$  and h(ws) = zs for any  $s \in X^{\mathbb{N}}$ . Furthermore, take  $v \in X^{n-m}$  such that  $zv \neq w$ . Given  $s \in X^{\mathbb{N}}$ , we have that

(1) 
$$\{\tilde{g}h\tilde{g}^{-1}(wvs): \tilde{g} \in \operatorname{rist}_G(zv)\} = \{zvg(s): g \in G\}.$$

Since  $G \cap X^{\mathbb{N}}$  is faithful, it follows from (1) that  $|h^G| = \infty$ , thus proving the claim. From [GTWZ21, Theorem 1.5], we obtain that  $G \cap \partial_p G$  is free and from Theorem 5.2, we conclude that  $V \cap \partial_p G$  is free.

**Remark 5.5.** In [LBMB18, Theorem 1.5], Le Boudec and Matte Bon showed that Thompson's group V is  $C^*$ -simple, hence  $V \curvearrowright \partial_F V$  is free. However, their proof is done by showing that V does not admit non-trivial amenable URS, not by exhibitting a concrete topologically free V-boundary. It seems as an interesting problem to determine whether  $V \curvearrowright \partial_P G$  is strongly proximal, thus providing an alternative proof of  $C^*$ -simplicity of V.

Remark 5.6. In [BKKO17, Theorem 1.4], it was shown that the class of  $C^*$ -simple groups is closed by taking normal subgroups. Obviously, this class is not closed by taking commensurated subgroups, since any finite subgroup is commensurated. Moreover, Example 5.4 shows that, given  $\Lambda \leq_c \Gamma$  such that  $\Gamma$  is icc relatively to  $\Lambda$ ,  $C^*$ -simplicity of  $\Gamma$  does not pass to  $\Lambda$  in general.

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