# Submitted to *Mathematics of Operations Research* manuscript (Please, provide the manuccript number!)

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# Analysis of the primal-dual central path for nonlinear semidefinite optimization without the nondegeneracy condition

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We study properties of the central path underlying a nonlinear semidefinite optimization problem, called NSDP for short. The latest radical work on this topic was contributed by Yamashita and Yabe (2012): they proved that the Jacobian of a certain equation-system derived from the Karush-Kuhn-Tucker (KKT) conditions of the NSDP is nonsingular at a KKT point under the second-order sufficient condition (SOSC), the strict complementarity condition (SC), and the nondegeneracy condition (NC). This yields uniqueness and existence of the central path through the implicit function theorem. In this paper, we consider the following three assumptions on a KKT point: the enhanced SOSC, the SC, and the Mangasarian-Fromovitz constraint qualification. Under the absence of the NC, the Lagrange multiplier set is not necessarily a singleton and the nonsingularity of the above-mentioned Jacobian is no longer valid. Nonetheless, we establish that the central path exists uniquely, and moreover prove that the dual component of the path converges to the so-called analytic center of the Lagrange multiplier set. As another notable result, we clarify a region around the central path where Newton's equations relevant to primal-dual interior point methods are uniquely solvable.

Key words: nonlinear semidefinite optimization, primal-dual interior-point method, central path, nondegeneracy condition

**1. Introduction** We consider the following nonlinear semidefinite optimization problem:

Minimize 
$$f(x)$$
  
subject to  $G(x) \in \mathbb{S}_{+}^{m}$ , (1.1)  
 $h(x) = 0$ ,

where  $f : \mathbb{R}^n \to \mathbb{R}$ ,  $G : \mathbb{R}^n \to \mathbb{S}^m$ , and  $h : \mathbb{R}^n \to \mathbb{R}^s$  are twice continuously differentiable functions. Moreover,  $\mathbb{S}^m$  denotes the set of real  $m \times m$  symmetric matrices and  $\mathbb{S}^m_{++}$  (resp.  $\mathbb{S}^m_+$ ) stands for the set of  $m \times m$  real symmetric positive definite (resp. semidefinite) matrices. Throughout the paper, we often refer to problem (1.1) as NSDP. NSDP (1.1) contains a wide class of optimization problems. Indeed, when all the functions are affine with respect to x, it reduces to a linear semidefinite optimization problem (Vandenberghe and Boyd [56], Wolkowicz et al. [57]). When the function G is of the diagonal matrix form, it is regarded as a conventional nonlinear optimization problem (Mangasarian [37], Luenberger and Ye [35]). Moreover, it contains nonlinear second-order cone optimization problems (Kato and Fukushima [26], Bonnans and Ramírez [6]) by restricting the form of G appropriately.

The recent advance of researches on the NSDP is remarkable. Abundant practical applications of the NSDP can be found in a wide variety of fields, for example, structural optimization (Kočvara and Stingl

[28], Thore et al. [54], Takezawa et al. [52], Thore [53]), control (Scherer [46], Kočvara et al. [27], Hoi et al. [18], Leibfritz and Volkwein [33]), statistics (Qi and Sun [45]), finance (Konno et al. [30], Leibfritz and Maruhn [31]), positive semidefinite factorization (Vandaele et al. [55]), and so on. Elegant theoretical results on optimality conditions for the NSDP have been also developed. For example, the Karush-Kuhn-Tucker (KKT) conditions and the second-order conditions for the NSDP were studied in detail by Shapiro [47] and Forsgren [10]. Further examples are: the strong second-order conditions by Sun [49], sequential optimality conditions by Andreani et al. [2], the local duality by Qi [44], and the optimality conditions via squared slack variables by Lourenço et al. [34]. Along with such theoretical results, various algorithms have been proposed for solving the NSDP, for example, augmented Lagrangian methods (Kočvara and Stingl [28], Sun et al. [51, 50], Andreani et al. [2, 1], Fukuda and Lourenço [12], Huang et al. [20], Wu et al. [59]), sequential linear semidefinite optimization methods (Kanzow et al. [24]), sequential quadratic semidefinite optimization methods (Correa and Ramirez C [8], Freund et al. [11], Zhao and Chen [67, 68], Yamakawa and Okuno [60]), sequential quadratically constrained quadratic semidefinite optimization methods (Auslender [4]), exact penalty methods (Auslender [5]), interior point-type methods (Arahata et al. [3], Jarre [21], Kato et al. [25], Leibfritz and Mostafa [32], Okuno and Fukushima [43, 42], Okuno [41], Yamashita and Yabe [63], Yamashita et al. [64, 65], Yamakawa and Yamashita [62, 61]), homotopy methods (Yang and Yu [66]), and so forth.

In this paper, we study properties of the central path for the NSDP. The central path is a path formed by stationary points of the log-barrier penalized problem, and is a key concept of interior-point methods, abbreviated as IPMs, in solving a wide class of optimization problems including the NSDP. Many IPMs share the strategy of approaching a KKT point by following the central path approximately. Since the geometry of the central path is related to the performance of IPMs, it has been well studied under various settings. For example, Megiddo [38] presented an early work in this line for linear optimization or linear programming. Kojima et al. [29] and Monteiro and Tsuchiya [39] studied the central path for monotone complementarity problems under the absence of strict complementarity condition. Monteiro and Zou [40] worked with the existence of the central path for convex optimization problems. Wright and Orban [58] considered non-linear optimization problems and analyzed the properties of the central path under the absence of linear independence constraint qualification.

We briefly review the history of the central path of semidefinite optimization problems (SDPs). Concerning linear SDPs, Luo et al. [36] showed that the (primal-dual) central path converges to the analytic center under the presence of the strict complementarity condition. Sturm and Zhang [48] further proved that the derivative of the central path is convergent. Halická et al. [16] proved that the central path is convergent regardless of the strict complementarity, by means of the curve selection lemma from algebraic geometry, although it can fail to converge to the analytic center in the absence of the strict complementarity. Halická [15] established that the central path is analytic including the boundary. See also other works by Goldfarb and Scheinberg [13], Halická et al. [17], Kakihara et al. [22, 23], da Cruz Neto et al. [9], and so forth. More generally, Graña Drummond and Peterzil [14] worked with the existence and convergence of the central path of convex smooth SDP by assuming that the functions organizing the problem are analytic. While there are many such studies concerning linear and convex SDPs, those for the general NSDP (1.1) are very scarce.

The latest radical work for NSDP(1.1) along this research-topic was presented by Yamashita and Yabe [63]. The authors analyzed the local convergence property of the primal-dual IPM, called PDIPM for short, that was proposed in another article of theirs (Yamashita et al. [64]). This PDIPM is explained briefly as follows: in the algorithm, the barrier KKT (BKKT) conditions are derived by perturbing the KKT conditions, and the degree of perturbation is controlled by the so-called barrier parameter. See Section 2.3 for the precise definition of the BKKT conditions. The PDIPM approaches a KKT point by generating a sequence of approximate BKKT points while driving the barrier parameter to zero. To compute a BKKT point, the Newton method combined with scaling techniques is applied to an equation-system equivalent to the BKKT conditions. In [63], Yamashita and Yabe proved that the Jacobian of this equation-system is nonsingular at a KKT point under the following three conditions: the strict complementarity condition (SC), the second-order sufficient condition (SOSC), and the nondegeneracy condition (NC). Along with the classical implicit

function theorem, this fact yields that there exists a unique smooth path, i.e., a central path, passing through the focused KKT point, and this path is formed by BKKT points.

**Contribution** The main contribution of this paper is summarized as follows:

1. We prove that there exists a smooth central path under the SC, the enhanced SOSC, and the Mangasarian-Fromovitz constraint qualification (MFCQ) at a KKT point of the NSDP. We also prove that the central path converges to the KKT point and the analytic center of the corresponding Lagrange multiplier set. Since the NC is not assumed therein, the Lagrange multiplier set is compact and convex, but not necessarily a singleton, although the KKT point is a strict local optimum due to the enhanced SOSC. In such a situation, it is difficult (or impossible) to prove existence of the central path straightforwardly by means of the implicit function theorem.

2. Under the same conditions as above, we give a region around the central path where the Newton equation is solvable uniquely when applying the PDIPM.

Many of the analyses in literature on SDPs exploit the fact that the functions are analytic and thus so is the underlying central path. However, this methodology is no longer available in our setting since the functions of the NSDP are not assumed to be analytic. The manner of our analysis conducted in this paper is motivated from Wright and Orban [58] for nonlinear optimization, but ours is more complicated because the SOSC of the NSDP involves difficulty arising from the so-called sigma term. Furthermore, we deal with the nonlinear equality constraints together, whereas [58] does not.

**Notations and terminologies** Throughout the paper, we use the following notations as necessary: for a set *S*, we denote by int *S*, cl *S*, and bd *S* the topological interior, closure, and boundary of *S*, respectively. We denote the identity matrix in  $\mathbb{R}^{m \times m}$  by *I*, and for  $A \in \mathbb{R}^{m \times m}$ , we define  $\text{Sym}(A) := (A + A^{\top})/2$  and  $||A||_F := \sqrt{\text{trace}(A^{\top}A)}$ . For  $B \in \mathbb{R}^{m \times n}$ , we denote the kernel and image spaces of *B* by Ker *B* and Im *B*, respectively, that is, Ker  $B := \{x \in \mathbb{R}^n \mid Bx = 0\}$  and Im  $B := \{By \mid y \in \mathbb{R}^n\}$ . For  $X, Y \in \mathbb{S}^m$ , we define the inner product  $X \bullet Y$  by  $X \bullet Y := \text{trace}(XY)$ . We also define the linear operator  $\mathcal{L}_X : \mathbb{S}^m \to \mathbb{S}^m$  by

$$\mathcal{L}_X(Y) := XY + YX.$$

Denote the smallest eigenvalue of  $X \in \mathbb{S}^m$  by  $\lambda_{\min}(X)$ . For  $X \in \mathbb{S}^m_+$  and r > 0, we denote by  $X^{\frac{1}{r}}$  the unique solution  $U \in \mathbb{S}^m_+$  of  $U^r = X$ . For a function  $g : \mathbb{R}^n \to \mathbb{R}$ , we denote by  $\nabla g(x)$  or  $\nabla_x g(x)$  the gradient of g, namely,  $\nabla g(x) := (\frac{\partial g(x)}{\partial x_1}, \dots, \frac{\partial g(x)}{\partial x_n})^{\top} \in \mathbb{R}^n$  and, also denote by  $\nabla^2_{xx}g(x)$  the hessian of g, namely,  $\nabla^2_{xx}g(x) = (\frac{\partial^2 g(x)}{\partial x_i \partial x_j})_{1 \le i, j \le n} \in \mathbb{R}^{n \times n}$ . For  $\{A_k\}$  in a normed vector space with norm  $\|\cdot\|$  and  $\{b_k\} \subseteq \mathbb{R}$ , we write  $A_k = O(b_k)$  if there exists some M > 0 such that  $\|A_k\| \le M|b_k|$  for all k sufficiently large, and write  $A_k = o(b_k)$  if there exists some nonnegative sequence  $\{\alpha_k\} \subseteq \mathbb{R}$  such that  $\lim_{k \to \infty} \alpha_k = 0$  and  $\|A_k\| \le \alpha_k |b_k|$  for all k sufficiently large. We also say  $A_k = \Theta(b_k)$  if there exist  $M_1, M_2 > 0$  such that  $M_1|b_k| \le \|A_k\| \le M_2|b_k|$  for all k sufficiently large.

We also denote  $\mathbb{R}_{++} := \{a \in \mathbb{R} \mid a > 0\}, W := \mathbb{R}^n \times \mathbb{S}^m \times \mathbb{R}^s$ ,

$$\mathcal{W}_{++} := \{ (x, Y, z) \in \mathcal{W} \mid G(x) \in \mathbb{S}_{++}^m, Y \in \mathbb{S}_{++}^m \}, \ \mathcal{W}_+ := \{ (x, Y, z) \in \mathcal{W} \mid G(x) \in \mathbb{S}_+^m, Y \in \mathbb{S}_+^m \}.$$

For  $w := (x, Y, z) \in W$ , we define  $||w|| := \sqrt{||x||_2^2 + ||Y||_F^2 + ||z||_2^2}$ , where  $||\cdot||_2$  denotes the Euclidean norm. Lastly, relevant to the function *G* in NSDP (1.1), we define the following notations. For i = 1, 2, ..., n, we

Lastly, relevant to the function *G* in NSDP (1.1), we define the following notations. For i = 1, 2, ..., n, we write

$$\mathcal{G}_i(x) := \frac{\partial G(x)}{\partial x_i}.$$

For any  $x, d \in \mathbb{R}^n$  and  $Y \in \mathbb{S}^m$ , we write

$$\Delta G(x;d) := \sum_{i=1}^{n} d_i \mathcal{G}_i(x) \in \mathbb{S}^m, \ \mathcal{J}G(x)^* Y := \left[\mathcal{G}_1(x) \bullet Y, \mathcal{G}_2(x) \bullet Y, \dots, \mathcal{G}_n(x) \bullet Y\right]^\top \in \mathbb{R}^n.$$

Some more notations and symbols will be introduced for the main analysis. See the paragraph *Additional notations and symbols used hereafter* at the end of subsection 3.1.

Organization of the paper The rest of the paper is organized as follows. In section 2, we review some important concepts related to the NSDP such as the KKT conditions. In section 3, the main analysis is presented. In section 4, we conclude this paper with some remarks.

## 2. Preliminaries

#### **2.1. KKT conditions for NSDP** We introduce the KKT conditions for NSDP (1.1).

DEFINITION 1. We say that the Karush-Kuhn-Tucker (KKT) conditions for NSDP (1.1) hold at  $x \in \mathbb{R}^n$ if there exist a Lagrange multiplier matrix  $Y \in \mathbb{S}^m$  and vector  $z \in \mathbb{R}^s$  such that

$$\nabla_{x}L(w) = \nabla f(x) - \mathcal{J}G(x)^{*}Y + \nabla h(x)z = 0, \qquad (2.1)$$

$$G(x) \bullet Y = 0, \ G(x) \in \mathbb{S}_{+}^{m}, \ Y \in \mathbb{S}_{+}^{m},$$
 (2.2)

$$h(x) = 0, \tag{2.3}$$

where  $w := (x, Y, z) \in \mathcal{W}$  and  $L : \mathcal{W} \to \mathbb{R}$  denotes the Lagrange function for the NSDP, that is,

$$L(w) := f(x) - G(x) \bullet Y + h(x)^{\top} z$$

for any  $w \in W$ . Particularly, we call a triplet w = (x, Y, z) satisfying the KKT conditions a KKT triplet of NSDP (1.1), and also call x a KKT point of the NSDP. Moreover, given a KKT point x, we denote by  $\Lambda(x)$ the set of Lagrange multiplier pairs (Y, z) satisfying the KKT conditions at x, namely,

$$\Lambda(x) := \{(Y, z) \text{ satisfying } (2.1) - (2.3)\}.$$

As can be checked easily,  $\Lambda(x)$  is convex. Below, we define the Mangasarian-Fromovitz constraint qualification (MFCQ), under which the KKT conditions are ensured to be necessary optimality conditions for the NSDP.

DEFINITION 2. ([7, Definition 2.8.6]) Let  $x \in \mathbb{R}^n$  be a feasible point of NSDP(1.1). We say that the Mangasarian-Fromovitz constraint qualification (MFCQ) holds at x if  $\nabla h(x)$  is of full column rank and there exists a vector  $d \in \mathbb{R}^n$  such that  $G(x) + \Delta G(x; d) \in \mathbb{S}_{++}^m$  and  $\nabla h(x)^\top d = 0$ .

REMARK 1. The MFCQ is equivalent to the following Robinson's constraint qualification at a feasible point  $x \in \mathbb{R}^n$  [7, Corollary 2.101]:

$$\begin{bmatrix} O\\ 0 \end{bmatrix} \in \operatorname{int}\left(\left\{ \begin{bmatrix} G(x) + \Delta G(x; d) \\ \nabla h(x)^{\top} d \end{bmatrix} \middle| d \in \mathbb{R}^n \right\} - \begin{bmatrix} \mathbb{S}_{++}^m \\ \{0\} \end{bmatrix} \right).$$

REMARK 2. Let  $x \in \mathbb{R}^n$  be a local optimum of NSDP (1.1). Under the MFCQ, the KKT conditions hold at x, thus  $\Lambda(x) \neq \emptyset$ . In particular, the MFCQ implies that  $\Lambda(x)$  is compact. Conversely, when f is convex, h is affine, and G is matrix-convex in the sense of Bonnans and Shapiro [7, Section 5.3.2], a KKT point is a global optimum of (1.1).

There are several equivalent reformulations for the semidefinite complementarity condition (2.2), among which the simplest one is

$$G(x)Y = O, \ G(x) \in \mathbb{S}_+^m, \ Y \in \mathbb{S}_+^m, \tag{2.4}$$

and two other formulations are

$$Sym(G(x)Y) = O, \ G(x) \in \mathbb{S}_{+}^{m}, \ Y \in \mathbb{S}_{+}^{m},$$

$$G(x)^{\frac{1}{2}} VG(x)^{\frac{1}{2}} = O, \ G(x) \in \mathbb{S}^{m}, \ Y \in \mathbb{S}^{m}$$
(2.5)
(2.5)

$$G(x)^{\frac{1}{2}}YG(x)^{\frac{1}{2}} = O, \ G(x) \in \mathbb{S}_{+}^{m}, \ Y \in \mathbb{S}_{+}^{m}.$$
(2.6)

Based on the above two formulations, primal-dual interior point methods (PDIPMs) have been developed for solving NSDPs so far. For example, see Yamashita et al. [64] and Yamashita and Yabe [63] for PDIPM with (2.5) and also see Okuno [41] for that with (2.6).

Other fundamental properties of the complementarity condition Let  $x^* \in \mathbb{R}^n$  be a KKT point for the NSDP. With an appropriate orthogonal matrix  $P_* \in \mathbb{R}^{m \times m}$ , the matrix  $G(x^*)$  and an arbitrary dual matrix  $Y \in \mathbb{S}^m_+$  such that  $G_*Y = O$  holds can be factorized as

$$G(x^*) = P_* \begin{bmatrix} O & O \\ O & G_*^{\text{FF}} \end{bmatrix} P_*^{\mathsf{T}}, \ Y = P_* \begin{bmatrix} Y^{\text{EE}} & O \\ O & O \end{bmatrix} P_*^{\mathsf{T}},$$
(2.7)

where  $G_*^{\text{FF}} \in \mathbb{S}_{++}^{r_*}$  is a *diagonal* matrix with  $r_* := \operatorname{rank} G(x^*)$ , the positive real eigenvalues of  $G(x^*)$  are aligned on the diagonal line, and  $Y^{\text{EE}} \in \mathbb{S}_+^{m-r_*}$ .  $Y^{\text{EE}} \in \mathbb{S}_{++}^{m-r_*}$  does not necessarily hold. Without loss of generality, we may assume that the eigenvalues are placed in the ascending order on the diagonal. Needless to say,  $P_*$  is a matrix whose columns are eigenvectors of  $G(x^*)$ . Partition the matrix  $P_*$  as

$$P_* = [E_*, F_*],$$

where  $E_* \in \mathbb{R}^{m \times (m-r_*)}$  and  $F_* \in \mathbb{R}^{m \times r_*}$ . Note that each column of  $E_*$  represents an eigenvector of  $G_* := G(x^*)$  which corresponds to the zero-eigenvalue of  $G(x^*)$ , while that of  $F_*$  does to a positive eigenvalue of  $G_*$ . In terms of  $E_*$  and  $F_*$ , the two equations in (2.7) are transformed as

$$\begin{bmatrix} E_*^{\mathsf{T}}G_*E_* \ E_*^{\mathsf{T}}G_*F_* \\ F_*^{\mathsf{T}}G_*E_* \ F_*^{\mathsf{T}}G_*F_* \end{bmatrix} = \begin{bmatrix} O & O \\ O & G_*^{\mathsf{FF}} \end{bmatrix}, \quad \begin{bmatrix} E_*^{\mathsf{T}}YE_* \ E_*^{\mathsf{T}}YF_* \\ F_*^{\mathsf{T}}YE_* \ F_*^{\mathsf{T}}YF_* \end{bmatrix} = \begin{bmatrix} Y^{\mathsf{EE}} & O \\ O & O \end{bmatrix}.$$
(2.8)

We will often make use of formulation (2.8). For later use, we define the following notations: for the above  $P_* = [E_*, F_*]$  and given  $x, d \in \mathbb{R}^n$  and  $Y_* \in \mathbb{S}^m$ , we write

$$\begin{bmatrix} Y_{*}^{\text{EE}} & Y_{*}^{\text{EF}} \\ Y_{*}^{\text{FE}} & Y_{*}^{\text{FF}} \end{bmatrix} := \begin{bmatrix} E_{*}^{\text{T}} Y_{*} E_{*} & E_{*}^{\text{T}} Y_{*} F_{*} \\ F_{*}^{\text{T}} Y_{*} E_{*} & F_{*}^{\text{T}} Y_{*} F_{*} \end{bmatrix}, \begin{bmatrix} G^{\text{EE}} & G^{\text{EF}} \\ G^{\text{FE}} & G^{\text{FF}} \end{bmatrix} := \begin{bmatrix} E_{*}^{\text{T}} G(x) E_{*} & E_{*}^{\text{T}} G(x) F_{*} \\ F_{*}^{\text{T}} G(x) E_{*} & F_{*}^{\text{T}} G(x) F_{*} \end{bmatrix},$$
(2.9)

$$\begin{bmatrix} \Delta G^{\text{EE}}(x;d) \ \Delta G^{\text{EF}}(x;d) \\ \Delta G^{\text{FE}}(x;d) \ \Delta G^{\text{FF}}(x;d) \end{bmatrix} := \begin{bmatrix} E_*^{\top} \Delta G(x;d) E_* \ E_*^{\top} \Delta G(x;d) F_* \\ F_*^{\top} \Delta G(x;d) E_* \ F_*^{\top} \Delta G(x;d) F_* \end{bmatrix}.$$
(2.10)

**2.2. Second-order optimality conditions and relevant properties** In this subsection, we review the second-order necessary/sufficient conditions for the NSDP. Subsequently, we will describe the relevant properties briefly. For more detailed explanations, we refer readers to, e.g., [63, 47] or [7].

DEFINITION 3. Let  $x^*$  be a KKT point for the NSDP and consider the corresponding Lagrange multiplier set  $\Lambda(x^*)$ . Then, the nondegeneracy condition, strict complementarity conditions, and second-order condition are defined as follows:

**Nondegeneracy condition** Let  $r_* := \operatorname{rank} G(x^*)$  and let  $\{e_1, e_2, \dots, e_{m-r_*}\}$  be an orthonormal basis of the null space of  $G(x^*)$ . Moreover, denote

$$v_{ij} := (e_i^\top \mathcal{G}_1(x^*)e_j, \cdots, e_i^\top \mathcal{G}_n(x^*)e_j)^\top \in \mathbb{R}^n \ (1 \le i \le j \le m - r_*).$$

We say that the nondegeneracy condition holds at  $x^*$  if the vectors  $v_{ij} \in \mathbb{R}^n$   $(1 \le i \le j \le m - r_*)$  and  $\nabla h_i(x^*)$   $(i = 1, 2, ..., \ell)$  are linearly independent.

**Strict complementarity condition** Let  $Y \in \mathbb{S}_{+}^{m}$  be a Lagrange multiplier matrix at  $x^{*}$ , which means that  $G(x^{*})$  and Y satisfies the complementarity condition (2.2). We say that the strict complementarity condition holds at  $(x^{*}, Y)$  if  $G(x^{*}) + Y \in \mathbb{S}_{++}^{m}$ , which is equivalent to rank  $G(x^{*}) + \operatorname{rank} Y = m$  under (2.2).

Second-order conditions We say that the second-order necessary (resp., sufficient) condition holds at  $x^*$  if

$$\sup_{(Y,z)\in\Lambda(x^*)} d^{\top} \left( \nabla^2_{xx} L(x^*, Y, z) + \Omega(x^*, Y) \right) d \ge (\text{resp.}, >)0, \quad \forall d \in C(x^*) \setminus \{0\},$$
(2.11)

where  $C(x^*)$  is the critical cone at  $x^*$  and specifically represented as

$$C(x^*) = \left\{ d \in \mathbb{R}^n \,|\, \nabla f(x^*)^\top d = 0, \, \nabla h(x^*)^\top d = 0, \, \Delta G(x^*; d) \in T_{\mathbb{S}^m_+}(G(x^*)) \right\}.$$
(2.12)

Here,  $T_{\mathbb{S}^m_+}(G(x^*))$  denotes the tangent cone of  $\mathbb{S}^m_+$  at  $G(x^*)$  and is represented specifically as

$$T_{\mathbb{S}_{+}^{m}}(G(x^{*})) = \left\{ X \in \mathbb{S}^{m} \mid E_{*}^{\top} X E_{*}(=X^{\text{EE}}) \in S_{+}^{r_{*}} \right\}$$

Moreover, for any  $x \in \mathbb{R}^n$  and  $Y \in \mathbb{S}^m$ ,  $\Omega(x, Y)$  denotes the matrix in  $S^n$  whose (i, j)-th entry is given as

$$(\Omega(x,Y))_{i,j} := 2Y \bullet \mathcal{G}_i(x) \mathcal{G}(x)^{\dagger} \mathcal{G}_j(x)$$

for i, j = 1, 2, ..., n, where  $G(x)^{\dagger}$  denotes the Moore-Penrose inverse matrix of G(x).

REMARK 3. The nondegeneracy condition at  $x^*$  is a constraint qualification for the NSDP and yields the MFCQ. It reduces to the linear independence constraint qualification (LICQ) when nonlinear optimization is considered. As with the LICQ, the Lagrange multiplier set  $\Lambda(x^*)$  is a singleton under the nondegeneracy condition.

The term  $d^{\top}\Omega(x^*, Y)d$  in (2.11) is called the *sigma term* for the semi-definite constraint  $G(x) \in \mathbb{S}_+^m$ . We refer readers to [7] for a precise description of its background and properties. In the following lemma, the sigma term is expressed more specifically, thereby being ensured to be nonnegative.

LEMMA 1. For  $Y \in \mathbb{S}^m_+$  such that  $G_*Y = O$  and a direction  $d \in \mathbb{R}^n$ , it holds that

$$d^{\mathsf{T}}\Omega(x^*, Y)d = 2\mathrm{Tr}\left(Y^{\mathrm{EE}}\Delta G^{\mathrm{FE}}(x^*; d)(G_*^{\mathrm{FF}})^{-1}\Delta G^{\mathrm{EF}}(x^*; d)\right)$$
$$= 2\left\|(Y^{\mathrm{EE}})^{\frac{1}{2}}\Delta G^{\mathrm{FE}}(x^*; d)(G_*^{\mathrm{FF}})^{-\frac{1}{2}}\right\|_{\mathrm{F}}^2,$$

where  $Y^{\text{EE}}$  and  $G_*^{\text{FF}}$  are defined in (2.8), and moreover  $\Delta G^{\text{FE}}$  and  $\Delta G^{\text{EF}}$  in (2.10).

Proof. By straightforward calculation. See Appendix A.1 for details.

When we consider the standard nonlinear optimization where the nonnegative cone is set in the NSDP in place of the semidefinite cone, the sigma term always vanishes because  $\Delta G^{\text{FE}}(x^*;d) = O$  holds for any *d* in the above lemma, and thus it never appears in the second-order conditions. In contrast, in the NSDP, the sigma term reflects curvature of  $\mathbb{S}^m_+$  and is nonnegative for any  $d \neq 0$  and  $Y \in \mathbb{S}^m_+$  as shown in Lemma 1. With the help of this term, the second-order condition is more likely to hold even when  $\nabla^2_{xx}L$  is not positive semidefinite over the critical cone. However, this term makes the analysis for the NSDP more complicated than in nonlinear optimization.

Lastly, we mention useful facts associated with the second-order conditions in the following two necessary and sufficient optimality conditions.

Second-order necessary optimality for the NSDP [7, Theorem 3.45,5.88] Let  $x^* \in \mathbb{R}^n$  be a local optimum of NSDP (1.1) and suppose that the MFCQ holds there. Then, the second-order necessary condition holds at  $x^*$ .

Second-order sufficient optimality for the NSDP [7, Theorem 5.89] Suppose that  $x^*$  is a KKT point of NSDP (1.1) and, furthermore, the second-order sufficient condition holds. Then,  $x^*$  is a strict local optimum of NSDP (1.1). In particular, the quadratic growth condition holds, that is, there exists some q > 0 and vicinity  $\mathcal{N}(x^*)$  of  $x^*$  such that  $f(x) - f(x^*) \ge q ||x - x^*||^2$  for all  $x \in \mathcal{N}(x^*) \cap \mathcal{F}$ .

**2.3. BKKT conditions and central path** In this section, we introduce the barrier KKT (BKKT) conditions for the NSDP. The BKKT conditions are composed of (2.1), (2.3), and the following perturbed conditions for (2.4): for  $\mu > 0$ ,

$$G(x)Y = \mu I, \ G(x) \in \mathbb{S}_{++}^m, \ Y \in \mathbb{S}_{++}^m.$$
(2.13)

The parameter  $\mu$  is often referred to as barrier parameter, and *x* and (*x*, *Y*, *z*) satisfying the BKKT conditions are called a BKKT point and BKKT triplet, respectively. It is worth mentioning that condition (2.13) is equivalent to the condition obtained by replacing *O* with  $\mu I$  in (2.5) or (2.6). As  $\mu$  gets closer to 0, BKKT points are expected to approach the set of KKT points for the NSDP. A basic algorithmic policy of primal-dual interior point methods is to track BKKT triplets while driving  $\mu$  to 0, so as to reach a KKT triplet. In this paper, we will refer to a path formed by BKKT triplets as a *central path*.

## 3. Main analysis

**3.1.** Assumptions and outline of analysis Throughout Section 3,  $x^*$  denotes a KKT point of the NSDP, and is assumed to satisfy the following:

Assumption 1. The KKT point  $x^*$  satisfies the following three conditions:

- 1. There exists a Lagrange multiplier matrix  $Y \in \mathbb{S}_+^m$  satisfying the strict complementarity condition.
- 2. The enhanced second-order sufficient condition (ESOSC) holds: for all  $(Y, z) \in \Lambda(x^*)$ , it holds that

 $d^{\top} \left( \nabla_{xx}^2 L(x^*,Y,z) + \Omega(x^*,Y) \right) d > 0, \ \forall d \in C(x^*) \setminus \{0\}.$ 

3. The MFCQ holds at  $x^*$ .

The above ESOSC is indeed stronger than the second-order sufficient condition (SOSC) defined in (2.11), because, with arbitrarily chosen  $(\overline{Y}, \overline{z}) \in \Lambda(x^*)$ , we have  $\sup_{(Y,z)\in\Lambda(x^*)} d^{\top} (\nabla_{xx}^2 L(x^*, Y, z) + \Omega(x^*, Y)) d \ge d^{\top} (\nabla_{xx}^2 L(x^*, \overline{Y}, \overline{z}) + \Omega(x^*, \overline{Y})) d > 0$  for any  $d \in C(x^*) \setminus \{0\}$ , where the last inequality is due to the ESOSC. Under the ESOSC,  $x^*$  is a strict local optimum of the NSDP since the SOSC follows from the ESOSC as shown above. See also *Second-order sufficient optimality for the NSDP* at the end of subsection 2.2. The ESOSC holds, for example, when f is strongly convex and G and h are affine. It can be seen as a straightforward generalization of the strong second-order condition (SSOSC) considered by Wright and Orban [58] for nonlinear optimization. Though one may think it natural to refer to the condition as SSOSC, we call it ESOSC so as to distinguish it from the SSOSC for the NSDP studied by Sun [49]. Under the presence of the MFCQ, we ensure compactness and convexity of the Lagrange multiplier set  $\Lambda(x^*)$ , but  $\Lambda(x^*)$  is not necessarily a singleton(cf. Remark 3). Note that  $\nabla_{xx}^2 L$  is continuous, and so is  $\Omega(x^*, Y)$  with respect to  $Y \in \mathbb{S}_+^m$  such that  $G(x^*)Y = O$  from Lemma 1. This fact, the compactness of  $\Lambda(x^*)$ , and the ESOSC guarantee that there exists some  $\kappa > 0$  such that

$$\inf_{(Y,z)\in\Lambda(x^*)} d^{\mathsf{T}} \left( \nabla^2_{xx} L(x^*, Y, z) + \Omega(x^*, Y) \right) d \ge \kappa ||d||^2, \quad \forall d \in C(x^*) \setminus \{0\}.$$

$$(3.1)$$

*Goal and outline of the analysis:* The goal of the whole analysis we will conduct is to prove that under the above assumptions, there exists a unique and smooth central path converging to the KKT triplet

$$w^a := (x^*, Y_a, z^a),$$
 (3.2)

where  $(Y_a, z^a) \in \Lambda(x^*)$  is called an analytic center at  $x^*$ , defined formally in the next subsection. In order to achieve this goal, we will prove the following claims in order:

**Claim (i)** There exists a sequence of BKKT triplets  $\{w^k = (x^k, Y_k, z^k)\}$  converging to the KKT triplet  $w^a$  (cf. Theorem 1 in subsection 3.3).

**Claim (ii)** Any sequence of BKKT points  $\{x^k\}$  approaches the KKT point  $x^*$  asymptotically along a certain nonzero direction  $\xi^* \in \mathbb{R}^n$  in the sense that  $\lim_{k\to\infty} \frac{x^*-x^k}{\|x^*-x^k\|} = \frac{\xi^*}{\|\xi^*\|}$  (cf. Theorem 2 and Corollary 1 in subsection 3.4). This  $\xi^*$  is a unique *x*-component solution of a certain linear equation system related to the BKKT conditions.

**Claim (iii)** For any sufficiently small barrier parameter  $\mu$ , a corresponding BKKT point  $x(\mu)$  exists uniquely in the open ball  $\{x \in \mathbb{R}^n \mid ||x - x^* - \mu\xi^*|| < \rho\mu||\xi^*||\}$ , where  $\rho > 0$  is a certain small constant. Moreover, the Hessian of a certain barrier function is nonsingular at  $x(\mu)$ . (cf. Theorem 3 in subsection 3.6).

With the help of the above claims and the classical implicit function theorem, we will prove our main claim of the goal (cf. Theorem 4 in subsection 3.7 and Theorem 5 in subsection 3.8). Mind that henceforth, several proofs are deferred to the Appendix for the sake of readability.

Additional notations and symbols used hereafter In the remaining of Section 3 and the Appendix, we will use the symbols and the notations defined in (2.7)-(2.10) in addition to those introduced at the end of Section 1. In particular,  $P_* = [E_*, F_*]$  is an *arbitrarily* chosen orthogonal matrix defined for  $G(x^*)$  so that (2.7) holds. For the sake of simplicity, we often write

$$G_* := G(x^*), \ G_k := G(x^k).$$

Besides, we will make use of  $G_*^{ind}$  and  $G_k^{ind}$  ( $ind \in \{EE, FE, EF, FF\}$ ) defined by replacing G and Y in (2.9) with  $G_*$  and  $G_k$ , respectively. Furthermore,  $Y_k^{ind}$  and  $Y_a^{ind}$  ( $ind \in \{EE, FE, EF, FF\}$ ) are defined in the same way using  $Y_k$  and  $Y_a$ .

**3.2.** Existence of analytic center for NSDP The analytic center for the NSDP at  $x^*$  is formally defined as follows:

DEFINITION 4 (ANALYTIC CENTER FOR NSDP (1.1)). We say that  $(Y_a, z^a) \in \Lambda(x^*)$  is an analytic center of NSDP (1.1) at  $x^*$  if it is an optimum of

$$\min -\log \det Y^{\text{EE}} \text{ s.t. } (Y, z) \in \Lambda(x^*).$$
(3.3)

Here, we define  $\log 0 := -\infty$  by convention.

In the next proposition, we ensure existence and uniqueness of the analytic center at  $x^*$ . In other words, the KKT triplet  $w^a$  defined in (3.2) is well-defined.

PROPOSITION 1. Suppose that Assumption 1 holds. Then, an analytic center of NSDP (1.1) at  $x^*$  exists uniquely. In particular,  $(Y_a, z^a) \in \mathbb{S}^m \times \mathbb{R}^s$  is the analytic center at  $x^*$  if and only if  $(Y_a, z^a) \in \Lambda(x^*)$  and there exists some vector  $v \in \mathbb{R}^n$  such that

$$\Delta G^{\text{EE}}(x^*; v) = (Y_a^{\text{EE}})^{-1}, \ \nabla h(x^*)^{\mathsf{T}} v = 0.$$
(3.4)

Proof. See Appendix A.2.

**3.3.** Proof of Claim (i): convergence of BKKT triplets to KKT triplet with analytic center In this subsection, we will prove that there exists a sequence of BKKT points which converges to the KKT point  $x^*$ . Moreover, we will show that the corresponding dual sequence converges to the analytic center  $(Y_a, z^a)$ .

Let us define the following log-barrier function for the NSDP: for each  $\mu > 0$ 

$$\psi_{\mu}(x) := f(x) - \mu \log \det G(x).$$
 (3.5)

The following proposition states that there exists a sequence of local optima of barrier penalized NSDPs converging to  $x^*$ . Such local optima are BKKT points of the NSDP locally around  $x^*$ .

PROPOSITION 2. Let Assumption 1 hold and  $\{\mu_k\} \subseteq \mathbb{R}_{++}$  be an arbitrary decreasing sequence converging to 0. Then, there exists a sequence  $\{x^k\} \subseteq \mathbb{R}^n$  such that  $\lim_{k\to\infty} x^k = x^*$  and, for any  $k \ge \overline{K}$  with  $\overline{K}$  sufficiently large,  $x^k$  is a local optimum of

$$\min \psi_{\mu_k}(x) \text{ s.t. } h(x) = 0, \ G(x) \in \mathbb{S}_{++}^m.$$
(3.6)

Proof. The proof is analogous to those of classical results as to penalty methods [35], although it is different in dealing with the log determinant function and the semidefinite constraint. Nonetheless, the precise proof is given in Appendix A for completeness.  $\Box$ 

In Proposition 2, as  $\nabla h(x^*)$  is of full column rank, so is  $\nabla h(x^k)$  for any  $k \ge \overline{K}$  with  $\overline{K}$  large enough, and thus the KKT conditions for (3.6) holds at  $x^k$ . From the KKT conditions together with  $\nabla \psi_{\mu}(x) = \nabla f(x) - \mu \mathcal{J}G(x)^*G(x)^{-1}$ , there exists  $z^k \in \mathbb{R}^s$  such that

$$\nabla f(x^{k}) - \mu_{k} \mathcal{J} G(x^{k})^{*} G_{k}^{-1} + \nabla h(x^{k}) z^{k} = 0, \ h(x^{k}) = 0, \ G_{k} \in \mathbb{S}_{++}^{m},$$

which together with  $Y_k := \mu_k G_k^{-1} \in \mathbb{S}_{++}^m$  implies that  $x^k$  and  $w^k := (x^k, Y_k, z^k) \in \mathcal{W}_{++}$  are BKKT point and BKKT triplet for each  $k \ge \overline{K}$ , respectively.

In summary, as a consequence of Proposition 2, given a decreasing sequence  $\{\mu_k\} \subseteq \mathbb{R}_{++}$  converging to 0, there exists an integer  $\overline{K} > 0$  and  $\{w^k\} \subseteq W_{++}$  with  $w^k = (x^k, Y_k, z^k)$  such that

$$\lim_{k \to \infty} x^k = x^*, \ Y_k = \mu_k G_k^{-1} \in \mathbb{S}_{++}^m$$
(3.7)

and, for each  $k \ge \overline{K}$ ,

 $\nabla h(x^k)$ : full column rank,  $w^k$ : BKKT triplet with barrier parameter  $\mu_k$ .

In what follows, for the sake of brevity, we assume  $\overline{K} = 0$ . Moreover, we suppose that  $x^k \neq x^*$  for all k without loss of generality and define

$$d^k := x^k - x^*. (3.8)$$

Hereafter, we focus on those sequences  $\{w^k\}$  and  $\{d^k\}$ .

REMARK 4. In a quite similar manner to the proof of Yamashita et al. [64, Theorem 1], we ensure that, under the MFCQ at  $x^*$ , the sequence  $\{(Y_k, z^k)\}$  is bounded, and its accumulation point together with  $x^*$  fulfills the KKT conditions of NSDP (1.1).

In fact, the whole sequence  $\{(Y_k, z^k)\}$  converges to the analytic center  $(Y_a, z^a)$ . To prove this claim, we first present the following proposition, which claims that the convergence speeds of  $\{\mu_k\}$  and  $\{||d^k||\}$  towards zero are equivalent.

**PROPOSITION 3.** Suppose that Assumption 1 holds. Then, we have

$$\mu_k = \Theta(||d^k||).$$

Proof. See Appendix A.4.

Using this proposition, the convergence to the analytic center can be established.

THEOREM 1. Suppose that Assumption 1 holds. Then, the whole sequence  $\{(Y_k, z^k)\}$  converges to the analytic center  $(Y_a, z^a)$  of NSDP (1.1) at  $x^*$ , that is,  $\lim_{k\to\infty} w^k = w^a$ .

Proof. Note that  $\lim_{k\to\infty} d^k = 0$  from (3.7) and (3.8). Also, note that  $\{(Y_k, z^k)\}$  is bounded as was explained in Remark 4, and let  $(Y_*, z^*)$  be an arbitrary accumulation point of  $\{(Y_k, z^k)\}$ . For each  $k \ge 0$ , define  $\tilde{d}^k := \frac{d^k}{\|d^k\|}$ . Since  $\{\tilde{d}^k\}$  is bounded, it has at least one accumulation point, say  $\tilde{d}^*$ . Choose an arbitrary subsequence  $\{\tilde{d}^k\}_{k\in\mathcal{K}}$  which converges to  $\tilde{d}^*$ . From Proposition 3,  $\{\frac{\mu_k}{\|d^k\|}\}_{k\in\mathcal{K}}$  is bounded and any accumulation point, say  $\bar{\alpha} \in \mathbb{R}$ , is positive. Without loss of generality, we assume that  $\{\frac{\mu_k}{\|d^k\|}\}_{k\in\mathcal{K}}$  and  $\{(Y_k, z^k)\}_{k\in\mathcal{K}}$  to  $\bar{\alpha} > 0$  and  $(Y_*, z^*)$ , respectively, by taking a subsequence further if necessary.

Recalling that  $P_* = [E_*, F_*]$  is orthogonal, we have

$$\frac{\mu_{k}I_{m}}{\|d^{k}\|} = \frac{P_{*}^{\top}G_{k}Y_{k}P_{*}}{\|d^{k}\|} = \frac{P_{*}^{\top}(G(x^{*}) + \Delta G(x^{*}; d^{k}) + O(\|d^{k}\|^{2}))P_{*}P_{*}^{\top}Y_{k}P_{*}}{\|d^{k}\|} = \begin{bmatrix} O & O \\ \frac{1}{\|d^{k}\|}G_{*}^{\text{FF}}Y_{k}^{\text{EF}} \frac{1}{\|d^{k}\|}G_{*}^{\text{FF}}Y_{k}^{\text{FF}} \end{bmatrix} + P_{*}^{\top}\Delta G(x^{*}; \widetilde{d}^{k})P_{*}(P_{*}^{\top}Y_{k}P_{*}) + O(\|d^{k}\|).$$
(3.9)

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Taking into account that  $\lim_{k \in \mathcal{K} \to \infty} P_*^{\mathsf{T}} Y_k P_* = \begin{bmatrix} Y_*^{\mathsf{EE}} & O \\ O & O \end{bmatrix}$  with  $Y_*^{\mathsf{EE}} = E_*^{\mathsf{T}} Y_* E_*$  and driving  $k \in \mathcal{K} \to \infty$  in the (1, 1)-block component of (3.9), we obtain  $\Delta G^{\mathsf{EE}}(x^*; \tilde{d}^*) Y_*^{\mathsf{EE}} = \bar{\alpha} I_{r_*}$ , implying

$$(Y_*^{\rm EE})^{-1} = \Delta G^{\rm EE}(x^*; \bar{\alpha}^{-1} \widetilde{d}^*).$$
(3.10)

Moreover, for each  $k \in \mathcal{K}$ , it holds that

$$0 = \frac{h(x^k)}{||d^k||} = \frac{h(x^*) + \nabla h(x^*)^{\top} d^k + O(||d^k||^2)}{||d^k||} = \nabla h(x^*)^{\top} \widetilde{d^k} + O(||d^k||),$$

which along with driving  $k \in \mathcal{K} \to \infty$  and multiplying  $\overline{a}^{-1}$  implies  $\nabla h(x^*)^{\top}(\overline{a}^{-1}\widetilde{d}^*) = 0$ . Comparing this fact and (3.10) to condition (3.4) with  $v := \overline{a}\widetilde{d}^*$ , we ensure that  $(Y_*, z^*)$  is an analytic center of the NSDP at  $x^*$ , leading to  $(Y_a, z^a) = (Y_*, z^*)$  due to the uniqueness of analytic center by Proposition 1. Finally, recalling that  $(Y_*, z^*)$  is an arbitrary accumulation point of  $\{(Y_k, z^k)\}$ , we conclude that the whole sequence  $\{(Y_k, z^k)\}$  converges to  $(Y_a, z^a)$ . The proof is complete.

Before moving on to the next subsection, we show that  $||Y_k^{\text{EF}}||_F$  and  $||Y_k^{\text{FF}}||_F$  are bounded by  $O(\mu_k)$ .

PROPOSITION 4. Suppose that Assumption 1 holds. Then, we have

$$||Y_k^{\rm EF}||_{\rm F} = {\rm O}(\mu_k), ||Y_k^{\rm FF}||_{\rm F} = {\rm O}(\mu_k).$$

Proof. Note that  $\{(Y_k, z^k)\}$  is convergent by Theorem 1 and thus  $Y_k = O(1)$  and  $z^k = O(1)$ . Moreover,  $\Delta G(x^*; d^k) = \sum_{i=1}^n d_i^k \frac{\partial G(x^*)}{\partial x_i} = O(||d^k||)$ . Applying Taylor's expansion to  $G_k$  around  $x^*$  and using  $G_k Y_k = \mu_k I$  and  $G_* Y_* = G_* Y_a = O$  give

$$\mu_k I = (G_* + \Delta G(x^*; d^k) + O(||d^k||^2)) Y_k = G_*(Y_k - Y_a) + G_*Y_a + \Delta G(x^*; d^k)Y_k + O(||d^k||^2) = G_*(Y_k - Y_a) + O(||d^k||) = P_*^{\top} \begin{bmatrix} O & O \\ G_*^{\text{FF}}Y_k^{\text{FE}} & G_*^{\text{FF}}Y_k^{\text{FF}} \end{bmatrix} P_* + O(||d^k||),$$

where the third equality holds from  $G_*Y_a = O$ ,  $\Delta G(x^*; d^k)Y_k = O(||d^k||)$ , and  $O(||d^k||^2) = O(||d^k||)$ . Recall that  $P_*$  is an orthogonal matrix. Divide both the sides of the above by  $\mu_k$  and drive  $k \to \infty$ . From Proposition 3, we obtain

$$\frac{\|G_*^{\text{FF}}Y_k^{\text{FE}}\|_{\text{F}}}{\mu_k} = O(1), \ \frac{\|G_*^{\text{FF}}Y_k^{\text{FF}}\|_{\text{F}}}{\mu_k} = O(1).$$

which together with  $G_*^{\text{FF}} \in \mathbb{S}_{++}^{m-r_*}$  implies the desired assertions.

**3.4.** Proof of Claim (ii): convergence of BKKT points along a specific direction Let  $\{w^k = (x^k, Y_k, z^k)\} \subseteq W_{++}$  be a sequence of BKKT triplets as described right after Proposition 2. From Theorem 1,  $\{w^k\}$  converges to the KKT triplet  $w^* = (x^*, Y_a, z^a)$  with the analytic center  $(Y_a, z^a)$ . In this subsection, we study how  $d^k/\mu_k$  behaves asymptotically, wherein  $d^k$  is defined in (3.8).

We begin by considering the following equation-system that comes from the BKKT conditions of the symmetric form:

$$\nabla_x L(w) = 0, \ G(x)Y + YG(x) = 2\mu I, \ h(x) = 0, \tag{3.11}$$

where  $w = (x, Y, z) \in W_{++}$ . Suppose at this moment<sup>1</sup> that there exists a smooth function  $w(\cdot) : (0, \bar{\mu}] \to W_{++}$ with some  $\bar{\mu} > 0$  such that  $w(\mu)$  is a BKKT triplet for each  $\mu \in (0, \bar{\mu}]$  and we stand at  $w = w(\mu)$ . Differentiating equations (3.11) with respect to  $\mu$  results in

$$\nabla_{xx}^2 L(w)\dot{x} - \mathcal{J}G(x)^*\dot{Y} + \nabla h(x)\dot{z} = 0, \qquad (3.12)$$

$$\mathcal{L}_{G(x)}\dot{Y} + \mathcal{L}_Y \Delta G(x; \dot{x}) = 2I, \qquad (3.13)$$

$$\nabla h(x)^{\mathsf{T}} \dot{x} = 0. \tag{3.14}$$

<sup>1</sup> In Theorem 4, this assumption will be verified.

As for the definition of  $\mathcal{L}_{(\cdot)}(\cdot)$ , refer to the section of notations. For later use, in terms of the matrix function

$$\mathcal{A}(w) := \begin{bmatrix} \nabla_{xx}^2 \mathcal{L}(w) & -\mathcal{J}G(x)^* \ \nabla h(x) \\ \mathcal{L}_Y \mathcal{G}_1(x) \cdots \mathcal{L}_Y \mathcal{G}_n(x) & \mathcal{L}_{G(x)} & 0 \\ \nabla h(x)^\top & 0 & 0 \end{bmatrix},$$
(3.15)

we express the above equation-system (3.12)-(3.14) as

$$\mathcal{A}(w)\dot{w} = \begin{bmatrix} 0\\2I\\0 \end{bmatrix}.$$
 (3.16)

**REMARK 5.** The Newton equation to the BKKT system (3.11) is expressed as

$$\mathcal{A}(w)\Delta w = \begin{bmatrix} -\nabla_x L(w) \\ 2\mu I - \mathcal{L}_{G(x)} Y \\ -h(x) \end{bmatrix}.$$
(3.17)

This is often solved in the primal-dual interior point method for the NSDP [64, 65, 62, 61].

Now, relevant to equation (3.16), we consider the following equations defined at the KKT triplet  $w^a = (x^*, Y_a, z^a)$ :

$$U_{x^*}^{\top} \left( \nabla_{xx}^2 L(w^a) \Delta x - \mathcal{J}G(x^*)^* \Delta Y \right) = 0,$$
(3.18)

$$\mathcal{L}_{G_*}\Delta Y + \mathcal{L}_{Y_a}\Delta G(x^*;\Delta x) = 2I, \qquad (3.19)$$

$$\nabla h(x^*)^{\mathsf{T}} \Delta x = 0, \tag{3.20}$$

where  $U_{x^*}$  denotes an arbitrary matrix whose columns form an orthonormal basis of the subspace

$$\mathcal{U}_* := \{ d \in \mathbb{R}^n \, | \, \Delta G^{\text{EE}}(x^*; d) = O, \nabla h(x^*)^\top d = 0 \}$$
(3.21)

and we can write  $U_{x^*} \in \mathbb{R}^{n \times p_*}$  by letting  $p_*$  be the dimension of  $\mathcal{U}_*$ . Notice that the above equations (3.18)-(3.20) are derived by changing the variables in (3.12)-(3.14), pre-multiplying (3.12) by the matrix  $U_{x^*}^{\top}$ , and using the relation  $\nabla h(x^*)^{\top} U_{x^*} = 0$ . The following proposition holds as to the solution set of equations (3.18)-(3.20):

**PROPOSITION 5.** Suppose that Assumption 1 holds. Let

$$S := \{ (\Delta x, \Delta Y) \in \mathbb{R}^n \times \mathbb{S}^m : solution to (3.18) - (3.20) \}.$$

If  $S \neq \emptyset$ , then the following properties hold:

- 1.  $\Delta x$ -component in *S* is unique, written as  $\xi^* \in \mathbb{R}^n$ ;
- 2.  $\Delta Y^{\text{FF}} = (G_*^{\text{FF}})^{-1}, \ \Delta G^{\text{EE}}(x^*;\xi^*) = (Y_a^{\text{EE}})^{-1}, \ \Delta Y^{\text{EF}} = -Y_a^{\text{EE}} \Delta G^{\text{EF}}(x^*;\xi^*)(G_*^{\text{FF}})^{-1}.$

Proof. See Appendix A.5.

The following theorem shows that the limit of  $d^k/\mu_k$  is actually equal to the direction  $\xi^*$ , which is defined in the above proposition.

THEOREM 2. Suppose that Assumption 1 holds. Let  $d^k$  be the vector defined in (3.8) and  $\xi^*$  be the one defined in Proposition 5. Then, we have

$$\lim_{k\to\infty}\frac{d^k}{\mu_k}=\xi^*.$$

In particular,  $\xi^* \neq 0$  and  $\nabla h(x^*)^{\mathsf{T}} \xi^* = 0$ .

Proof. First, recall  $\lim_{k\to\infty} d^k = 0$ . From Proposition 3,  $\{d^k/\mu_k\}$  is bounded. Let  $\tilde{\xi} \in \mathbb{R}^n$  be an arbitrary accumulation point of  $\{d^k/\mu_k\}$ . In order to prove the assertion, it suffices to show that  $\tilde{\xi}$  is a  $\Delta x$ -component of the solution set of equations (3.18)-(3.20) because of item 1 of Proposition 5. Altering  $Y_k$  as

$$\widehat{Y}_k := P_* \begin{bmatrix} Y_a^{\text{EE}} & Y_k^{\text{EF}} \\ Y_k^{\text{FE}} & Y_k^{\text{FF}} \end{bmatrix} P_*^{\mathsf{T}}$$

for each k, we obtain

$$\begin{split} \|\widehat{Y}_{k} - Y_{a}\|_{\mathrm{F}} &= \left\| \begin{bmatrix} Y_{a}^{\mathrm{EE}} - Y_{a}^{\mathrm{EE}} & Y_{k}^{\mathrm{EF}} - Y_{a}^{\mathrm{EF}} \\ Y_{k}^{\mathrm{FE}} - Y_{a}^{\mathrm{FE}} & Y_{k}^{\mathrm{FF}} - Y_{a}^{\mathrm{FF}} \end{bmatrix} \right\|_{\mathrm{F}} \\ &= \left\| \begin{bmatrix} O & Y_{k}^{\mathrm{EF}} \\ Y_{k}^{\mathrm{FE}} & Y_{k}^{\mathrm{FF}} \end{bmatrix} \right\|_{\mathrm{F}} \\ &= O(||Y_{k}^{\mathrm{EF}}||_{\mathrm{F}} + ||Y_{k}^{\mathrm{FF}}||_{\mathrm{F}}) \\ &= O(\mu_{k}), \end{split}$$

where the last equality follows from Proposition 4, and thus  $\left\{\frac{1}{\mu_k}(\widehat{Y}_k - Y_a)\right\}$  is bounded and has at least one accumulation point, say  $\Delta Y_*$ . Without loss of generality, we assume that

$$\lim_{k \to \infty} \frac{1}{\mu_k} (\widehat{Y}_k - Y_a) = \Delta Y_*.$$
(3.22)

Note that

$$\nabla_{x}L(w^{k}) = \nabla_{x}L(x^{k}, Y_{a}, z^{a}) - \mathcal{J}G(x^{k})^{*}(Y_{k} - Y_{a}) + \nabla h(x^{k})(z^{k} - z^{a}) 
= \left(\nabla_{x}L(w^{a}) + \nabla_{xx}^{2}L(w^{a})d^{k} + O(||d^{k}||^{2})\right) - \mathcal{J}G(x^{k})^{*}(Y_{k} - Y_{a}) + \nabla h(x^{k})(z^{k} - z^{a}) 
= \nabla_{xx}^{2}L(w^{a})d^{k} - \mathcal{J}G(x^{k})^{*}(Y_{k} - Y_{a}) + \nabla h(x^{k})(z^{k} - z^{a}) + O(||d^{k}||^{2}) 
= \nabla_{xx}^{2}L(w^{a})d^{k} - \mathcal{J}G(x^{*})^{*}(Y_{k} - Y_{a}) + \nabla h(x^{*})(z^{k} - z^{a}) + O(||d^{k}||^{2}),$$
(3.23)

where the second equality follows from applying Taylor's expansion to  $\nabla_x L(x^k, Y_a, z^a)$  around  $x^*$  with respect to x and the third one from  $\nabla_x L(w^a) = 0$ . Moreover, the last one holds because  $o(||d^k||) + O(||d^k||^2) = o(||d^k||)$  and

$$-\mathcal{J}G(x^{k})^{*}(Y_{k}-Y_{a}) + \nabla h(x^{k})(z^{k}-z^{a}) = -\mathcal{J}G(x^{*})^{*}(Y_{k}-Y_{a}) + \nabla h(x^{*})(z^{k}-z^{a}) + o(||d^{k}||)$$

follows from Taylor's expansion of  $\mathcal{J}G(x^k)$  and  $\nabla h(x^k)$  around  $x^*$  again and  $\lim_{k\to\infty}(Y_k - Y_a, z^k - z^a) = (O, 0)$ . Then, it holds that

$$0 = \frac{1}{\mu_k} U_{x^*}^{\top} \nabla_x L(w^k)$$
  
=  $\frac{1}{\mu_k} U_{x^*}^{\top} \left( \nabla_{xx}^2 L(w^a) d^k - \mathcal{J}G(x^*)^* (Y_k - Y_a) \right) + \frac{1}{\mu_k} o(||d^k||)$   
=  $U_{x^*}^{\top} \left( \nabla_{xx}^2 L(w^a) \frac{d^k}{\mu_k} - \mathcal{J}G(x^*)^* \frac{(\widehat{Y}_k - Y_a)}{\mu_k} \right) + \frac{1}{\mu_k} o(||d^k||),$ 

where the first equality follows from  $\nabla_x L(w^k) = 0$ , the second one does from (3.23) and  $U_{x^*}^\top \nabla h(x^*) = 0$ , and the third one does from  $U_{x^*}^\top \mathcal{J}G(x^*)^* Y_k = U_{x^*}^\top \mathcal{J}G(x^*)^* \widehat{Y}_k$ . Driving  $k \to \infty$  above and using Proposition 3 and (3.22) imply

$$U_{x^*}^{\top} \left( \nabla_{xx}^2 L(w^a) \tilde{\xi} - \mathcal{J} G(x^*)^* \Delta Y_* \right) = 0$$

which is nothing but (3.18) with  $(\Delta x, \Delta Y) = (\tilde{\xi}, \Delta Y_*)$ .

Next, by  $G_k Y_k = \mu_k I$  from the BKKT conditions and also by noting  $G_* \widehat{Y}_k = G_* Y_k$  together with  $G_* Y_a = O$ , there holds that

$$I = \frac{1}{\mu_k} (G_k Y_k - G_* Y_a)$$
  
=  $\frac{1}{\mu_k} ((G_* + \Delta G(x^*; d^k) + O(||d^k||^2)) Y_k - G_* Y_a)$   
=  $\frac{1}{\mu_k} (G_*(\widehat{Y}_k - Y_a) + \Delta G(x^*; d^k) Y_k) + O(||d^k||),$ 

wherein by driving  $k \to \infty$ , symmetrizing, and using  $\lim_{k\to\infty} Y_k = Y_a$ , we gain (3.19) with  $(\Delta x, \Delta Y) = (\tilde{\xi}, \Delta Y_*)$ . Finally, we can prove (3.20) with  $(\Delta x, \Delta Y) = (\tilde{\xi}, \Delta Y_*)$  by driving k to  $\infty$  in the relation

$$0 = \frac{1}{\mu_k} (h(x^k) - h(x^*)) = \nabla h(x^*)^{\mathsf{T}} \frac{d^k}{\mu_k} + \mathcal{O}(||d^k||).$$
(3.24)

Consequently,  $(\tilde{\xi}, \Delta Y_*)$  solves (3.18)-(3.20). Hence,  $\tilde{\xi} = \xi^*$ , namely,  $\lim_{k \to \infty} d^k / \mu_k = \xi^*$  is ensured by using item 1 of Proposition 5. The remaining assertions  $\xi^* \neq 0$  and  $\nabla h(x^*)^{\mathsf{T}}\xi^* = 0$  follow immediately since  $||d^k|| = \Theta(\mu_k)$  from Proposition 3 and (3.24) holds. The proof is complete.

It is worth noting that we have multiple choices for  $\{(d^k, \mu_k)\}$ , while  $\xi^*$  is the constant vector that is uniquely determined as a  $\Delta x$ -component of the solution set to the equation-system (3.18)-(3.20). Nevertheless, according to Theorem 2, any  $\{d^k/\mu_k\}$  converges to  $\xi^*$ .

Theorem 2 yields the following corollary, a clear picture about how  $x^k$  approaches  $x^*$ .

COROLLARY 1. Under Assumption 1, we obtain  $\lim_{k\to\infty} \frac{x^k - x^*}{\|x^k - x^*\|} = \frac{\xi^*}{\|\xi^*\|}$ . This indicates that  $x^k$  approaches  $x^*$  along the direction  $-\xi^*$  asymptotically.

Proof. From Theorem 2, we have

$$\lim_{k \to \infty} \frac{x^k - x^*}{\|x^k - x^*\|} = \lim_{k \to \infty} \frac{x^k - x^*}{\mu_k} \frac{\mu_k}{\|x^k - x^*\|} = \frac{\xi^*}{\|\xi^*\|}$$

The proof is complete.

For  $\rho \in (0, 1)$  and  $\mu \ge 0$ , define

$$\mathcal{P}_{\rho}(\mu) := \{ x \in \mathbb{R}^n \mid ||x^* + \mu \xi^* - x|| < \rho \mu ||\xi^*|| \}.$$

**3.5.** Some properties on  $\mathcal{P}_{\rho}(\mu)$  In this subsection, we will present some propositions about properties relevant to  $\mathcal{P}_{\rho}(\mu)$ . The following proposition concerns the existence of a BKKT point in  $\mathcal{P}_{\rho}(\mu)$ .

**PROPOSITION 6.** *Choose*  $\rho \in (0, 1)$  *arbitrarily.* 

1. There exists some  $\bar{\mu}_{\rho} > 0$  such that, for any  $0 < \mu \leq \bar{\mu}_{\rho}$ , a BKKT point  $x_{\mu}$  with barrier parameter  $\mu$  exists in  $\mathcal{P}_{\rho}(\mu)$ , but never on the boundary  $\mathrm{bd}\mathcal{P}_{\rho}(\mu) := \{x \in \mathbb{R}^n \mid ||x^* + \mu\xi^* - x|| = \rho\mu||\xi^*||\}.$ 

2. Let  $\{\mu_k\} \subseteq \mathbb{R}_{++}$  and  $\{x(\mu_k)\}$  be arbitrary sequences of barrier parameters converging to 0 and corresponding BKKT points converging to the KKT point  $x^*$ , respectively. Then,  $x(\mu_k) \in \mathcal{P}_{\rho}(\mu_k)$  for any k large enough.

Proof. See Appendix A.6.

The next proposition shows existence and properties of  $\mu G(x)^{-1}$  for  $x \in \mathcal{P}_{\rho}(\mu)$ .

PROPOSITION 7. Choose  $\bar{\rho}_1 \in (0, 1]$  and  $\bar{\mu}_1 > 0$  sufficiently small. Then, the following properties hold: 1.  $G(x) \in \mathbb{S}^m_{++}$  holds for any  $\mu \in (0, \bar{\mu}_1]$  and  $x \in cl \mathcal{P}_{\bar{\rho}_1}(\mu)$ , and  $\left\{ \mu G(x)^{-1} \in \mathbb{S}^m_{++} \mid x \in cl \mathcal{P}_{\bar{\rho}_1}(\mu), \mu \in (0, \bar{\mu}_1] \right\}$  is bounded.

2. For any  $(\mu, \rho) \in (0, \bar{\mu}_1] \times (0, \bar{\rho}_1]$  and  $x \in cl \mathcal{P}_{\rho}(\mu)$ , there exists some  $K_1 > 0$  such that  $\|\mu G(x)^{-1} - Y_a\|_F \leq K_1(\rho + \mu)$ 

Proof. See Appendix A.7.

As  $\nabla h(x^*)$  is of full column rank since the MFCQ holds at  $x^*$  and  $\nabla h$  is continuous, there exists some closed ball  $\mathcal{B} \subseteq \mathbb{R}^n$  centered at  $x^*$  such that

$$x^* \in \mathcal{B} \subseteq \{ x \in \mathbb{R}^n \mid \nabla h(x) \text{ is of full-column rank } \}.$$
(3.25)

With sufficiently small  $\mu$  and  $\rho$ , cl $\mathcal{P}_{\rho}(\mu) \subseteq \mathcal{B}$  holds, which together with the first-half assertion of item 1 of Proposition 7 yields that, by re-taking smaller  $\bar{\mu}_1$  and  $\bar{\rho}_1$  if necessary,

$$\operatorname{cl}\mathcal{P}_{\rho}(\mu) \subseteq \{ x \in \mathbb{R}^n \mid G(x) \in \mathbb{S}_{++}^m \} \cap \mathcal{B}, \ \forall (\mu, \rho) \in (0, \bar{\mu}_1] \times (0, \bar{\rho}_1].$$
(3.26)

Next, we consider the following conditions:

$$x \in \mathrm{cl}\mathcal{P}_{\rho}(\mu),\tag{3.27}$$

$$\|Y - \mu G(x)^{-1}\|_{\rm F} \le \gamma_1 \mu, \tag{3.28}$$

$$\|z + (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla f(x) - \mathcal{J}G(x)^{*}Y)\| \le \gamma_{2}\mu.$$
(3.29)

As discussed in subsection 3.8, the set of (x, Y, z) which satisfies the above conditions is a neighborhood of the central path leading to the KKT triplet  $w^a$ . The following two propositions give crucial properties on this set. They will play important roles in proving Proposition 10 and Theorem 3 in subsection 3.6.

**PROPOSITION 8.** Let  $\gamma_1, \gamma_2 > 0$  and choose  $(\bar{\mu}_2, \bar{\rho}_2) \in (0, \bar{\mu}_1] \times (0, \bar{\rho}_1]$  sufficiently small. There exists some  $K_2 > 0$  such that

$$||x - x^*|| \le K_2 \mu, \quad \max(||Y - Y_a||_F, ||z - z^a||, ||\nabla_x L(w)||) \le K_2(\rho + \mu)$$
(3.30)

for all  $(\mu, \rho) \in (0, \bar{\mu}_2] \times (0, \bar{\rho}_2]$  and  $(x, Y, z) \in \mathbb{R}^n \times \mathbb{S}^m \times \mathbb{R}^s$  satisfying (3.27), (3.28), and (3.29).

Proof. See Appendix A.8.

PROPOSITION 9. Suppose that Assumption 1 holds. Let  $\gamma_1, \gamma_2 > 0$ . Choose  $(\bar{\mu}_3, \bar{\rho}_3) \in (0, \bar{\mu}_2] \times (0, \bar{\rho}_2]$  sufficiently small. For any  $(\mu, \rho) \in (0, \bar{\mu}_3] \times (0, \bar{\rho}_3]$  and  $w = (x, Y, z) \in \mathbb{R}^n \times \mathbb{S}^m \times \mathbb{R}^s$  satisfying (3.27), (3.28), and (3.29), we have

$$d^{\mathsf{T}}\nabla_{xx}^{2}L(w)d + \Delta G(x;d) \bullet \mathcal{L}_{G(x)}^{-1}\mathcal{L}_{Y}(\Delta G(x;d)) \ge \frac{\kappa}{2} ||d||^{2}, \quad \forall d \in \mathbb{R}^{n} \setminus \{0\} : \nabla h(x)^{\mathsf{T}}d = 0, \tag{3.31}$$

where  $\kappa > 0$  is the constant defined in (3.1).

Proof. See Appendix A.9.

**3.6.** Proof of Claim (iii): uniqueness of BKKT point for each barrier parameter In this section, in order to derive the smoothness of the central path by means of the classical implicit function theorem, we first transform NSDP (1.1) into an equivalent problem without equality constraints locally. Let  $\mathcal{M} :=$  $\{x \in \mathbb{R}^n \mid h(x) = 0\}$ . Under the presence of the full column rank of  $\nabla h(x^*)$  and twice continuous differentiability of *h*, there exists an open set  $U \subseteq \mathbb{R}^{n-s}$  together with a  $C^2$ -diffeomorphism  $\Phi : U \to \Phi(U) \subseteq \mathbb{R}^n$  such that  $x^* \in \Phi(U) \subseteq \mathcal{M}$ .<sup>2</sup> Then, we can take an open ball *V* such that  $\operatorname{cl} V \subsetneq U$ ,  $x^* \in \Phi(V)$ , and  $\nabla h(\Phi(v))$  is of full column rank for all  $v \in \operatorname{cl} V$ . Thus, there exists  $(\nabla h(x)^T \nabla h(x))^{-1}$  with  $x = \Phi(v)$  for all  $v \in \operatorname{cl} V$ .

<sup>&</sup>lt;sup>2</sup> More strictly speaking, there exists a  $C^2$  mapping  $\overline{\Phi} : U \to \mathbb{R}^n$  such that  $\overline{\Phi}(U)$  and U are diffeomorphic and furthermore  $h((\overline{\Phi}(v)^{\mathsf{T}}, v^{\mathsf{T}})^{\mathsf{T}}) = 0$  ( $v \in U$ ) holds by re-ordering the variables in x if necessary. Then, we let  $\Phi(v) := (\overline{\Phi}(v)^{\mathsf{T}}, v^{\mathsf{T}})^{\mathsf{T}}$ .

Let us give some relevant properties of  $\Phi$  for later use. Since V is bounded and  $\Phi$  is smooth on the open set  $U(\supseteq \operatorname{cl} V)$ , there exists a Lipshitz constant  $M_1 > 0$  such that

$$\|\Phi(u) - \Phi(v)\| \le M_1 \|u - v\|, \quad \forall u, v \in \text{cl } V.$$
(3.32)

Moreover, by noting that  $\Phi$  is a diffeomorphism, there exists  $M_2 > 0$  such that

$$\|\Phi^{-1}(x) - \Phi^{-1}(y)\| \le M_2 \|x - y\|, \quad \forall x, y \in \Phi(\operatorname{cl} V).$$

Since  $h(\Phi(v)) = 0$  ( $v \in V$ ), by differentiation with respect to v, we have

$$\nabla \Phi(v) \nabla_x h_i(\Phi(v)) = 0, \qquad (3.33)$$

$$\sum_{j=1}^{n} \frac{\partial h_i(\Phi(v))}{\partial x_j} \nabla^2 \Phi_j(v) + \nabla \Phi(v) \nabla_{xx}^2 h_i(\Phi(v)) \nabla \Phi(v)^{\mathsf{T}} = O$$
(3.34)

for each i = 1, 2, ..., s, where  $\Phi_i(v)$  stands for the *i*-th element of  $\Phi(v) \in \mathbb{R}^n$ . Note that  $\nabla \Phi(v)^{\top}$  is of full column rank for any  $v \in U$  because  $\Phi$  is a diffeomorphism on U and cl V is bounded by definition. From this fact, there exist some  $M_3, M_4 > 0$  such that

$$M_{3} \le \|\nabla \Phi(v)^{\top} y\| \le M_{4}, \ \forall (v, y) \in cl \ V \times \mathbb{R}^{n-s} : \|y\| = 1.$$
(3.35)

Since cl V is bounded and  $\nabla^2 \Phi$  is continuous on cl V, there exists some  $M_5 > 0$  such that

$$\max_{i=1,2,...,s} \|\nabla^2 \Phi_i(v)\|_{\mathsf{F}} \le M_5, \ \forall v \in \mathrm{cl} \, V.$$
(3.36)

Let dist $(x, \mathcal{M}) := \min_{y \in \mathcal{M}} ||x - y||$  for  $x \in \mathbb{R}^n$ . The following lemma holds.

LEMMA 2. It holds that

dist 
$$(\check{x}(\mu), \mathcal{M}) = O(\mu^2),$$
 (3.37)

where  $\check{x}(\mu) := x^* + \mu \xi^* \quad (\mu \ge 0).$ 

Proof. See Appendix A.10.

In terms of  $\Phi$ , NSDP(1.1) is reformulated as the following problem without equality constraints locally around  $v_* := \Phi^{-1}(x^*) \in V$ :

$$\min_{v \in V} f(\Phi(v)) \text{ s.t. } G(\Phi(v)) \in \mathbb{S}_+^m$$

Accordingly, we obtain the following barrier penalized problem for each  $\mu > 0$ :

$$\min_{v \in V} \Psi_{\mu}(v) := \psi_{\mu}(\Phi(v)) \quad \text{s.t.} \quad G(\Phi(v)) \in \mathbb{S}_{++}^{m}, \tag{3.38}$$

where  $\psi_{\mu}(x) = f(x) - \mu \log \det G(x)$  as defined in (3.5). The gradient and Hessian of  $\Psi_{\mu}$  are expressed as

$$\nabla \Psi_{\mu}(v) = \nabla \Phi(v) \nabla_{x} \psi_{\mu}(\Phi(v)), \qquad (3.39)$$

$$\nabla^2 \Psi_{\mu}(\nu) = \sum_{j=1}^n \frac{\partial \psi_{\mu}(\Phi(\nu))}{\partial x_j} \nabla^2 \Phi_j(\nu) + \nabla \Phi(\nu) \nabla_{xx}^2 \psi_{\mu}(\Phi(\nu)) \nabla \Phi(\nu)^{\mathsf{T}}, \qquad (3.40)$$

respectively. From (3.33) and the fact that  $\nabla h_1(x), \nabla h_2(x), \dots, \nabla h_s(x)$  are linearly independent at  $x = \Phi(v)$ and the dimension of Ker  $\nabla \Phi(v)$  is  $s, \nabla h_1(x), \dots, \nabla h_s(x)$  form a basis of Ker  $\nabla \Phi(v)$ . From this fact and equation (3.39), we see that  $\nabla \Psi_{\mu}(v) = 0$  if and only if  $\nabla_x \psi_{\mu}(\Phi(v)) = \nabla f(\Phi(v)) - \mu \mathcal{J}G(\Phi(v))G(\Phi(v))^{-1} \in$ Im $\nabla h(x)$ . Namely,

$$\nabla \Psi_{\mu}(v) = 0 \Leftrightarrow \Phi(v)$$
 is a BKKT point with barrier parameter  $\mu$ . (3.41)

PROPOSITION 10. Suppose that Assumption 1 holds. Choose  $(\bar{\rho}_4, \bar{\mu}_4) \in (0, \bar{\rho}_3] \times (0, \bar{\mu}_3]$  sufficiently small, where  $\bar{\rho}_3$  and  $\bar{\mu}_3$  are the constants defined in Proposition 9. Then, for any  $(\rho, \mu) \in (0, \bar{\rho}_4] \times (0, \bar{\mu}_4]$  and  $y \in \mathbb{R}^{n-s}$ with ||y|| = 1, we have

$$y^{\mathsf{T}}\nabla^{2}\Psi_{\mu}(v)y \ge \frac{\kappa M_{3}^{2}}{4}, \ \forall v \in V \cap \Phi^{-1}(\mathrm{cl}\mathcal{P}_{\rho}(\mu)),$$
(3.42)

where  $\kappa$  and  $M_3$  are defined in (3.1) and (3.35), respectively.

Proof. See Appendix A.11.

THEOREM 3. Suppose that Assumption 1 holds. Choose  $\rho \in (0, \bar{\rho}_4]$  arbitrarily, where  $\bar{\rho}_4$  is the constant defined in Proposition 10. Then, there exists some  $\bar{\mu}_{\rho} \in (0, \bar{\mu}_4]$  such that, for any  $\mu \in (0, \bar{\mu}_{\rho}]$  a unique BKKT point  $x(\mu)$  with barrier parameter  $\mu$  exists in  $\Phi(V) \cap \mathcal{P}_{\rho}(\mu)$ . In particular,  $\Phi^{-1}(x(\mu))$  is a unique solution of the equation  $\nabla \Psi_{\mu}(v) = 0$  in the open set  $V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$ . Moreover,  $\nabla^2 \Psi_{\mu}(\Phi^{-1}(x(\mu)))$  is positive definite, thus  $\Phi^{-1}(x(\mu))$  is a strict local optimum of problem (3.38).

Proof. For a fixed  $\rho \in (0, \bar{\rho}_4]$ , according to item 1 of Proposition 6,  $x(\mu) \in \mathcal{P}_{\rho}(\mu)$  holds for any sufficiently small  $\mu > 0$ . Since  $h(x(\mu)) = 0$  holds as  $x(\mu)$  is a BKKT point, we have  $x(\mu) \in \Phi(V)$ . Then, noting  $\lim_{\mu \to 0} x(\mu) = x^* \in \Phi(V)$ , we ensure  $x(\mu) \in \Phi(V) \cap \mathcal{P}_{\rho}(\mu)$  for any  $\mu$  small enough.

Next, by deriving a contradiction, we prove that such  $x(\mu)$  is unique in  $\Phi(V) \cap \mathcal{P}_{\rho}(\mu)$  for any sufficiently small  $\mu$ . Assume to the contrary that there exists an infinite sequence  $\{\mu_{\ell}\} \subseteq (0, \bar{\mu}_4]$  which converges to 0 and moreover accompanies two sequences  $\{x^{\ell}\}, \{y^{\ell}\} \subseteq \Phi(V) \cap \mathcal{P}_{\rho}(\mu_{\ell})$  such that for each  $\ell$ ,  $x^{\ell} \neq y^{\ell}$ , but  $x^{\ell}$  and  $y^{\ell}$  are both BKKT points with barrier parameter  $\mu_{\ell}$ . Hence,

$$v^{\ell} := \Phi^{-1}(x^{\ell}), \ \theta^{\ell} := \Phi^{-1}(y^{\ell})$$

exist in V, and in addition (3.41) yields

$$\nabla \Psi_{\mu_{\ell}}(v^{\ell}) = \nabla \Psi_{\mu_{\ell}}(\theta^{\ell}) = 0, \qquad (3.43)$$

where  $\Psi_{\mu}$  is defined in (3.38). Moreover, we have

$$x^{\ell}, y^{\ell} \in \Phi(V) \cap \mathcal{P}_{\rho}(\mu_{\ell})$$

for sufficiently large  $\ell$ . Let  $\check{x}_{\mathcal{M}}(\mu_{\ell}) \in \arg \min_{v \in \mathcal{M}} ||\check{x}(\mu_{\ell}) - y||$ , where  $\check{x}(\cdot)$  is defined in Lemma 2, and let

$$v_{\mu_{\ell}} := \Phi^{-1}(\check{x}_{\mathcal{M}}(\mu_{\ell})), \ V_{\ell} := \left\{ v \in V \ \Big| \ \|v - v_{\mu_{\ell}}\| < \frac{\rho\mu_{\ell}}{2M_1} \right\}$$
(3.44)

for each  $\ell$ . Then, by (3.37) in Lemma 2,

dist 
$$(\check{x}(\mu_{\ell}), \mathcal{M}) = ||\check{x}_{\mathcal{M}}(\mu_{\ell}) - \check{x}(\mu_{\ell})|| = O(\mu_{\ell}^2).$$
 (3.45)

For any  $v \in V_{\ell}$ ,

$$\begin{split} \|\Phi(v) - \check{x}(\mu_{\ell})\| &\leq \|\Phi(v) - \check{x}_{\mathcal{M}}(\mu_{\ell})\| + \|\check{x}_{\mathcal{M}}(\mu_{\ell}) - \check{x}(\mu_{\ell})\| \\ &\leq \|\Phi(v) - \Phi(v_{\mu_{\ell}})\| + O(\mu_{\ell}^{2}) \\ &\leq M_{1} \|v - v_{\mu_{\ell}}\| + O(\mu_{\ell}^{2}) \\ &\leq \frac{\rho\mu_{\ell}}{2} + O(\mu_{\ell}^{2}), \end{split}$$

where the second inequality follows from (3.44) and (3.45), the third from (3.32), and the fourth from (3.44) and  $v \in V_{\ell}$ . Hence, it holds that  $\|\Phi(v) - \check{x}(\mu)\| \le \rho \mu_{\ell}$ ,  $\forall v \in V_{\ell}$  for sufficiently large  $\ell$ , which yields that

$$\Phi(V_{\ell}) \subseteq \mathcal{P}_{\rho}(\mu_{\ell}). \tag{3.46}$$

Furthermore, since both  $x^{\ell}$  and  $y^{\ell}$  converge to  $x^*$  as  $\ell$  tends to  $\infty$  because of  $x^{\ell}, y^{\ell} \in \mathcal{P}_{\rho}(\mu_{\ell})$ , Theorem 2 implies

$$\|\check{x}(\mu_{\ell}) - x^{\ell}\| = \|x^* + \mu_{\ell}\xi^* - x^{\ell}\| = \mu_{\ell} \left\|\xi^* - \frac{x^{\ell} - x^*}{\mu_{\ell}}\right\| = o(\mu_{\ell}),$$
(3.47)

and also  $\|\check{x}(\mu_{\ell}) - y^{\ell}\| = o(\mu_{\ell})$  in a similar way. These relations along with the triangle inequality and (3.45) yield

$$\max\left(\|\check{x}_{\mathcal{M}}(\mu_{\ell})-x^{\ell}\|,\|\check{x}_{\mathcal{M}}(\mu_{\ell})-y^{\ell}\|\right)=o(\mu_{\ell}),$$

which implies that for sufficiently large  $\ell \ge 0$ , max  $(||\check{x}_{\mathcal{M}}(\mu_{\ell}) - x^{\ell}||, ||\check{x}_{\mathcal{M}}(\mu_{\ell}) - y^{\ell}||) \le \frac{\mu_{\ell}\rho}{4M_1M_2}$ , thus

$$\|v_{\mu_{\ell}} - v^{\ell}\| = \|\Phi^{-1}(\check{x}_{\mathcal{M}}(\mu_{\ell})) - \Phi^{-1}(x^{\ell})\| \le M_2 \|\check{x}_{\mathcal{M}}(\mu_{\ell}) - x^{\ell}\| \le \frac{\rho\mu_{\ell}}{4M_1}.$$

Therefore, we gain  $v^{\ell} \in V_{\ell}$  for  $\ell$  large enough. In a similar way, we can show  $\theta^{\ell} \in V_{\ell}$ . In short, from the above arguments we obtain that

$$\{v^{\ell}, \theta^{\ell}\} \subseteq V_{\ell} \tag{3.48}$$

for sufficiently large  $\ell$ .

From (3.46) together with  $V_{\ell} \subseteq V$ ,  $V_{\ell} \subseteq V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu_{\ell}))$  holds. Then, according to Proposition 10, when  $\rho \leq \bar{\rho}_4$  and  $\ell$  is so large that  $\mu_{\ell} \leq \bar{\mu}_4$ ,  $\nabla^2 \Psi_{\mu_{\ell}}(v)$  is positive definite for all  $v \in V_{\ell} \subseteq V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu_{\ell}))$ . Thus, problem (3.38) with *V* replaced by  $V_{\ell}$  can be viewed as a strongly convex problem for any large  $\ell$ . Therefore, a point  $v \in V_{\ell}$  which fulfills  $\nabla \Psi_{\mu_{\ell}}(v) = 0$  must be unique, which together with (3.43) and (3.48) implies  $\theta^{\ell} = v^{\ell}$ . This gives  $x^{\ell} = y^{\ell}$ , a contradiction. With this, we ensure that by setting  $\mu$  to be small enough, a BKKT point  $x(\mu)$  exists uniquely in  $\Phi(V) \cap \mathcal{P}_{\rho}(\mu)$ . Moreover, we also see  $\Phi(x(\mu))$  is a unique local optimum of (3.38) in  $V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$ . The positive definiteness of  $\nabla^2 \Psi_{\mu}(\Phi^{-1}(x(\mu)))$  is clear from Proposition 10 along with  $\Phi^{-1}(x(\mu)) \in V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$ . We thus obtain the desired assertion.

**3.7.** Main claim I: existence and uniqueness of central path By Theorem 3, we have ensured the uniqueness and existence of BKKT points around the KKT point  $x^*$ . In the following theorem, we prove that these BKKT points together with the corresponding Lagrange multiplier vectors and matrices form a smooth central path leading to the KKT triplet  $w^a = (x^*, Y_a, z^a)$ , and moreover show such a path is uniquely determined.

**THEOREM 4.** Suppose that Assumption 1 holds. For a sufficiently small  $\bar{\mu} > 0$ , there exists a unique central path  $w: (0, \bar{\mu}) \rightarrow W_{++}$  such that

1. it is smooth and, for each  $\mu \in (0, \overline{\mu})$ ,  $w(\mu)$  is a BKKT triplet of the NSDP with barrier parameter  $\mu$ . Moreover,

2.  $\lim_{\mu \to 0} w(\mu) = w^a$ .

Proof. We first show that there exists a unique smooth path  $v(\cdot)$  such that  $\nabla \Psi_{\mu}(v(\mu)) = 0$  and  $v(\mu) \in V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$  for each  $\mu \in (0,\bar{\mu})$  with some  $\bar{\mu}$ . Choose  $\rho \in (0,\bar{\rho}_4]$  arbitrarily and consider  $\bar{\mu}_{\rho} > 0$  defined as in Theorem 3. From Theorem 3, for each  $\mu \in (0,\bar{\mu}_{\rho}]$ , there exists a unique  $\bar{\nu}_{\mu} \in V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$  such that  $\nabla \Psi_{\mu}(\bar{\nu}_{\mu}) = 0$ . Moreover,  $\nabla^2 \Psi_{\mu}(\bar{\nu}_{\mu})$  is positive definite, thus nonsingular. By applying the implicit function theorem to the equation  $\nabla \Psi_{\mu}(v) = 0$ , there exists some  $l_{\mu} \in (0,\mu)$  and  $u_{\mu} \in (\mu, \min(2\mu, \bar{\mu}_{\rho}))$  together with a smooth path  $v_{\mu} : (l_{\mu}, u_{\mu}) \to V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$  satisfying  $v_{\mu}(\mu) = \bar{\nu}_{\mu}$  and  $\nabla \Psi_{t}(v_{\mu}(t)) = 0$  for each  $t \in (l_{\mu}, u_{\mu})$ . In fact,  $\Phi(v_{\mu}(t)) \in \mathcal{P}_{\rho}(t)$  holds for each  $t \in (l_{\mu}, u_{\mu})$  by re-taking smaller  $\bar{\mu}_{\rho}$  if necessary. For the proof, see the footnote<sup>3</sup>. Thus, due to Theorem 3 again, for each  $t \in (l_{\mu}, u_{\mu}), v_{\mu}(t)$  is the unique solution to  $\nabla \Psi_{t}(v) = 0$  in the

<sup>&</sup>lt;sup>3</sup> Suppose to the contrary that there exists a sequence  $\{\mu_\ell\}$  converging to 0 along with  $\{t_\ell\}$  such that  $t_\ell \in (l_{\mu_\ell}, u_\ell)$  and  $\Phi(v_{\mu_\ell}(t_\ell)) \in \mathcal{P}_\rho(\mu_\ell) \setminus \mathcal{P}_\rho(t_\ell)$ . By noting  $l_{\mu_\ell} < t_\ell < u_{\mu_\ell} \le 2\mu_\ell$ , (3.41), and the fact of  $\Phi(v_{\mu_\ell}(t_\ell)) \in \mathcal{P}_\rho(\mu_\ell)$ , it follows that  $\lim_{\ell \to \infty} t_\ell = 0$ ,  $\Phi(v_{\mu_\ell}(t_\ell))$  is a BKKT point with barrier parameter  $t_\ell$  and  $\lim_{\ell \to \infty} \Phi(v_{\mu_\ell}(t_\ell)) = x^*$ . However, according to item 2 of Proposition 6,  $\Phi(v_{\mu_\ell}(t_\ell)) \in \mathcal{P}_\rho(t_\ell)$  must hold for any  $\ell$  large enough, a contradiction to the assumption of  $\Phi(v_{\mu_\ell}(t_\ell)) \in \mathcal{P}_\rho(\mu_\ell) \setminus \mathcal{P}_\rho(t_\ell)$ .

open set  $V \cap \Phi^{-1}(\mathcal{P}_{\rho}(t))$  and  $\nabla^2 \Psi_t(v_{\mu}(t))$  is positive definite. Taking this fact into account and connecting the paths  $v_{\mu}(\cdot)$  constructed in the above way for each  $\mu \in (0, \bar{\mu})$  with  $\bar{\mu} := \bar{\mu}_{\rho}$ , we can ensure there exists a unique smooth path  $v : (0, \bar{\mu}) \to V$  such that  $\nabla \Psi_{\mu}(v(\mu)) = 0$  and  $v(\mu) \in V \cap \Phi^{-1}(\mathcal{P}_{\rho}(\mu))$  for each  $\mu \in (0, \bar{\mu})$ .

By letting  $x(\mu) := \Phi(v(\mu))$  for each  $\mu \in (0, \bar{\mu})$ ,  $x(\cdot)$  is smooth on  $(0, \bar{\mu})$  as  $v(\cdot)$  is smooth and  $\Phi : V \to \mathcal{M}$  is a diffeomorphism. Furthermore,  $x(\mu)$  is a BKKT point with barrier parameter  $\mu$  because of  $\nabla \Psi_{\mu}(\Phi(v(\mu))) = 0$  and (3.41). Since  $x(\mu) = \Phi(v(\mu)) \in \mathcal{P}_{\rho}(\mu)$ , it is clear from the definition of  $\mathcal{P}_{\rho}(\mu)$  that

$$\lim_{\mu \to 0} x(\mu) = x^*. \tag{3.49}$$

From (3.25) and (3.26),  $G(x(\mu)) \in \mathbb{S}_{++}^m$  and  $\nabla h(x(\mu))$  is of full column rank, and therefore  $G(x(\mu))^{-1}$  and  $(\nabla h(x(\mu))^\top \nabla h(x(\mu)))^{-1}$  exist. Since  $x(\mu)$  is a BKKT point as shown above, there exists  $z_{\mu} \in \mathbb{R}^s$  such that

$$\nabla_x L(x(\mu), \mu G(x(\mu))^{-1}, z_{\mu}) = \nabla f(x(\mu)) - \mu \mathcal{J}G(x(\mu))^* G(x(\mu))^{-1} + \nabla h(x(\mu)) z_{\mu} = 0.$$

Therefore, by premultiplying this equation with  $(\nabla h(x(\mu))^\top \nabla h(x(\mu)))^{-1} \nabla h(x(\mu))^\top$ , we obtain  $z_{\mu} = -(\nabla h(x(\mu))^\top \nabla h(x(\mu)))^{-1} \nabla h(x(\mu))^\top (f(x(\mu)) - \mathcal{J}G(x(\mu))^* Y(\mu))$  for each  $\mu \in (0, \bar{\mu})$ . Thus, we can define  $w(\cdot) : (0, \bar{\mu}) \to W_{++}$  by  $Y(\mu) := \mu G(x(\mu))^{-1}$  and  $z(\mu) := z_{\mu}$ . Since  $x(\cdot)$  is smooth, so is  $w(\cdot)$ , and  $w(\mu)$  is a BKKT triplet with (3.49). Recall that Theorem 1 implies any sequence of BKKT triplets  $\{w^k = (x^k, Y_k, z^k)\}$  such that  $\lim_{k\to\infty} x^k = x^*$  converges to  $w^a$ . Therefore,  $\lim_{\mu\to 0} w(\mu) = w^a$  follows. Finally, since  $x(\cdot)$  is uniquely determined as shown above and  $Y(\cdot)$  and  $z(\cdot)$  are uniquely constructed from  $x(\cdot)$ , we can conclude the uniqueness of  $w(\cdot)$ . The proof is complete.

**3.8.** Main claim II: unique solvability of the Newton equation in the primal-dual interior point **method** As remarked in remark 5,  $\mathcal{A}(w)$  is the coefficient matrix of the Newton equation to the BKKT system (3.11). In the following theorem,  $\mathcal{A}(w)$  is shown to be nonsingular near the central path.

THEOREM 5. Let the same assumptions as in Proposition 9 hold. For any  $w = (x, Y, z) \in W$  satisfying (3.27), (3.28), and (3.29), the matrix  $\mathcal{A}(w)$  defined in (3.15) is nonsingular.

Proof. We have only to show that  $\mathcal{A}(w)dw = 0$  when  $x \in \mathrm{cl}\mathcal{P}_{\rho}(\mu)$  for  $dw := (dx, dY, dz)^{\top} \in \mathcal{W}$  implies dw = 0. From  $\mathcal{A}(w)dw = 0$ , it holds that

$$\nabla_{xx}^2 L(w)dx - \mathcal{J}G(x)^*dY + \nabla h(x)dz = 0, \qquad (3.50)$$

$$\mathcal{L}_{G(x)}dY + \mathcal{L}_Y \Delta G(x; dx) = 0, \qquad (3.51)$$

$$\nabla h(x)^{\mathsf{T}} dx = 0. \tag{3.52}$$

Note that  $(G(x), Y) \in \mathbb{S}_{++}^m \times \mathbb{S}_{++}^m$  and  $\nabla h(x)$  is of full column rank due to  $x \in cl \mathcal{P}_{\rho}(\mu)$ , (3.25), and (3.26). Pre-multiplying (3.50) with  $dx^{\top}$  and substituting (3.52) and  $dY = -\mathcal{L}_{G(x)}^{-1}\mathcal{L}_Y \Delta G(x; dx)$  from (3.51) into it, we have  $dx^{\top} \nabla_{xx}^2 L(w) dx + \Delta G(x; dx) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_Y (\Delta G(x; dx)) = 0$ . Then, because of Proposition 9, dx = 0 must hold, which together with (3.51) implies dY = O. Moreover, (3.50) and the full column rank of  $\nabla h(x)$  give dz = 0. Hence, we obtain dw = 0 and thus the second assertion is obtained.

The set of  $w = (x, Y, z) \in W_{++}$  which fulfills (3.27)-(3.29), write  $N \subseteq W_{++}$ , contains any BKKT triplets with sufficiently small barrier parameters. Indeed, from item 2 of Proposition 6, (3.27) holds at any BKKT points  $x(\mu)$  with sufficiently small barrier parameter  $\mu$ , and  $(\nabla h(x(\mu))\nabla h(x(\mu))^{\top})^{-1}$  exists from (3.25) and (3.26). Moreover, the expressions on the left-hand sides of (3.28) and (3.29) are both equal to 0 because  $\nabla_x L(w) = 0$  and  $G(x)Y = \mu I$  hold on the central path, and hence (3.28) and (3.29) hold true. Therefore, Ncontains the central path. Thus, Theorem 5 indicates that the Newton equation (3.17) is uniquely solvable when w is close to the central path. This fact would be useful particularly when applying the Newton method in the primal-dual interior point method. **4.** Concluding remarks and future work In this paper, we have studied properties of a central path for nonlinear semidefinite optimization problems (NSDPs). Specifically, we have proven that, under the strict complementarity condition, strong second-order sufficient condition, and Mangasarian-Fromovitz constraint qualification, there exists a smooth central path which converges to a KKT triplet with an analytic center. In particular, given a KKT triplet, a central path leading to that KKT triplet is uniquely determined. Unlike the past results concerning the central path for the NSDP, the nondegeneracy condition is not assumed. The author believes that the results obtained in this paper will play a substantial role for further development of the primal-dual interior point method for the NSDP.

There exist two directions for future works. The first one is concerned with limiting behavior of the tangential direction  $\dot{x}(\mu)$  in the x-space as  $\mu \rightarrow 0$ . We have the following conjecture:

$$\lim_{\mu \to 0} \dot{x}(\mu) = \xi^*$$

where  $\xi^*$  was defined in Proposition 5. For nonlinear optimization, the corresponding result was proven by Wright and Orban [58, Theorem 12]. The second direction of future works is to mitigate the strict complementarity (SC) condition from our assumptions. The SC actually plays a key role in our analysis, particularly when establishing Proposition 3, a base for proving the subsequent theorems. Indeed, the following example, which is obtained from [58, Section 4] with slight modification, shows that Proposition 3 does not hold when the SC condition fails:

$$\min_{x_1, x_2, x_3} \frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \text{ s.t.} \begin{bmatrix} x_1 - 1 \ x_3 \\ x_3 \ x_2 \\ x_1 - 1 \ x_3 \\ x_3 \ x_2 \end{bmatrix} \in \mathbb{S}_+^4.$$

Its optimum  $x^*$  is only  $(1,0,0)^{\mathsf{T}}$ , and the set of corresponding dual matrices is

$$\left\{ \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \\ & \lambda_2 & 0 \\ & 0 & 0 \end{bmatrix} \middle| \lambda_1 + \lambda_2 = 1, \lambda_1 \ge 0, \lambda_2 \ge 0 \right\}.$$

It is easy to find that neither the SC nor the nondegeneracy condition holds, while both the MFCQ and ESOSC hold at  $x^*$ . For barrier parameter  $\mu > 0$ , the BKKT point is  $x(\mu) = (\frac{1+\sqrt{8\mu+1}}{2}, \sqrt{2\mu}, 0)^{\top}$ , which converges to the optimum  $x^*$  as  $\mu \to 0$ . However,  $||x(\mu) - x^*|| = \sqrt{\frac{8\mu+1-\sqrt{8\mu+1}}{2}} \le 2\sqrt{\mu} \neq \Theta(\mu)$ , and therefore Proposition 3 cannot hold without the SC.

Acknowledgments: The author thanks Professor Yoshiko Ikebe for much advice. He is also grateful to anonymous referees for many valuable comments and suggestions.

**A. Omitted Proofs** In this appendix, we give the proofs which are not shown in the main part of this paper.

**A.1. Proof of Lemma 1** The second equality follows from the direct calculation along with the fact of  $\Delta G^{\text{EF}} = (\Delta G^{\text{FE}})^{\top}$ , and the first one is derived from the following transformation:

$$d^{\mathsf{T}}\Omega(x^*, Y)d = 2\mathrm{Tr}\left(\sum_{j=1}^n \sum_{i=1}^n d_i d_j Y \mathcal{G}_i^* \mathcal{G}_i^* \mathcal{G}_j^*\right)$$
$$= 2\mathrm{Tr}\left((P_*^{\mathsf{T}} Y P_*) \left(\sum_{i=1}^n d_i P_*^{\mathsf{T}} \mathcal{G}_i^* P_*\right) \left(P_*^{\mathsf{T}} \mathcal{G}_*^* P_*\right) \left(\sum_{j=1}^n d_j P_*^{\mathsf{T}} \mathcal{G}_j^* P_*\right)\right)$$

$$2\mathrm{Tr}\left(\begin{bmatrix}Y^{\mathrm{EE}} & O\\ O & O\end{bmatrix}\begin{bmatrix}\Delta G^{\mathrm{EE}}(x^*;d) \ \Delta G^{\mathrm{FF}}(x^*;d)\\ \Delta G^{\mathrm{FE}}(x^*;d) \ \Delta G^{\mathrm{FF}}(x^*;d)\end{bmatrix}\begin{bmatrix}O & O\\ O & (G_*^{\mathrm{FF}})^{-1}\end{bmatrix}\begin{bmatrix}\Delta G^{\mathrm{EE}}(x^*;d) \ \Delta G^{\mathrm{FF}}(x^*;d)\\ \Delta G^{\mathrm{FE}}(x^*;d) \ \Delta G^{\mathrm{FF}}(x^*;d)\end{bmatrix}\right)$$
  
=2Tr( $Y^{\mathrm{EE}}\Delta G^{\mathrm{FE}}(x^*;d)(G_*^{\mathrm{FF}})^{-1}\Delta G^{\mathrm{EF}}(x^*;d))$ .

The proof is complete.

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A.2. Proof of Proposition 1 We first show the first assertion: the unique existence of the analytic center at  $x^*$ . Note that because of relation (2.7) for  $(Y, z) \in \Lambda(x^*)$ , (3.3) is equivalent to the following problem with respect to only *Y*:

$$\min_{\substack{Y \in \mathbb{S}^m \\ \text{s.t. } \nabla f(x^*) - \mathcal{J}G^{\text{EE}}(x^*)^* Y^{\text{EE}} \in \text{Im}\nabla h(x^*), \\ Y^{\text{EF}} = Y^{\text{FE}} = O, \quad Y^{\text{FF}} = O, \\ Y^{\text{EE}} \in \mathbb{S}_+^{m-r_*},$$
 (A.1)

where  $\mathcal{J}G^{\text{EE}}(x^*)^*Z := [(E_*^\top \mathcal{G}_i(x^*)E_*) \bullet Z]_{i=1}^n \in \mathbb{R}^n \text{ for } Z \in \mathbb{S}^{m-r_*}.$ 

We establish existence of optima of (3.3). By the strict complementarity condition as for the NSDP, there exists  $(Y, z) \in \Lambda(x^*)$  such that  $Y + G_* \in \mathbb{S}_{++}^m$ , which implies  $Y^{\text{EE}} \in \mathbb{S}_{++}^{m-r*}$ . This means that a finite objective value of (A.1) is attained at such a matrix *Y*. Moreover, as  $\Lambda(x^*)$  is convex and bounded from the MFCQ at  $x^*$  for the NSDP, so is the feasible region of (A.1). By combining these facts, (A.1) is ensured to have an optimum, say  $Y_a \in \mathbb{S}_{+}^m$ . From the full column rankness of  $\nabla h(x^*)$ , we see that the linear equation  $\nabla f(x^*) - \mathcal{J}G^{\text{EE}}(x^*)^*Y_a^{\text{EE}} + \nabla h(x^*)z = 0$  has a unique solution  $z \in \mathbb{R}^s$ , written  $z^a$ . This  $(Y_a, z^a)$  is nothing but an optimum of (3.3).

Next, consider the following problem:

$$\min_{Z} -\log \det Z 
\text{s.t. } \nabla f(x^*) - \mathcal{J}G^{\text{EE}}(x^*)^*Z \in \text{Im}\nabla h(x^*), 
Z \in \mathbb{S}_+^{m-r_*}.$$
(A.2)

For a feasible point *Y* of (A.1),  $Y^{\text{EE}}$  is clearly feasible to (A.2), and hence so is  $Y_a^{\text{EE}}$  to (A.2). Furthermore, we can ensure that  $Y_a^{\text{EE}}$  is optimal to (A.2). Indeed, if not, there exists *Z* such that *Z* is feasible to (A.2) and  $-\log \det Z < -\log \det Y_a^{\text{EE}}$ . Since  $Y := E_*ZE_*^{\top} \in \mathbb{S}_+^m$  is feasible to (A.1) and  $\log \det Y^{\text{EE}} = \log \det Z$ , we gain  $-\log \det Y^{\text{EE}} < -\log \det Y_a^{\text{EE}}$ , a contradiction to the optimality of  $Y_a^{\text{EE}}$  for (A.1). Lastly, since (A.2) is a strictly convex problem, we see that  $Y_a^{\text{EE}}$  is a *unique* optimum of (A.2).

In turn, we establish the uniqueness of  $(Y_a, z^a) \in \mathbb{S}^m_+ \times \mathbb{R}^\ell$  as optimum of (3.3). To derive a contradiction, assume that there exist two distinct optima  $(Y_a, z^a)$  and  $(\widetilde{Y}_a, \widetilde{z^a})$  at  $x^*$ , which yields that  $Y_a$  and  $\widetilde{Y}_a$  are both optima of (A.1) by the preceding argument. Thus, so are  $Y_a^{\text{EE}}$  and  $\widetilde{Y}_a^{\text{EE}}$  to (A.2), in particular  $Y_a^{\text{EE}} = \widetilde{Y}_a^{\text{EE}}$ , according to the preceding argument again. Hence, we have

$$P_*^{\mathsf{T}}(Y_{\mathsf{a}} - \widetilde{Y}_{\mathsf{a}})P_* = \begin{bmatrix} Y_{\mathsf{a}}^{\mathsf{EE}} - \widetilde{Y}_{\mathsf{a}}^{\mathsf{EE}} & O \\ O & O \end{bmatrix} = O.$$

Since  $P_*$  is nonsingular, we obtain  $Y_a = \widetilde{Y}_a$ , which together with the full column rankness of  $\nabla h(x^*)$  implies  $z^a = \widetilde{z^a}$ . Hence we ensure  $(Y_a, z^a) = (\widetilde{Y}_a, \widetilde{z^a})$ , which is a contradiction. Consequently, (3.3) has a unique optimum, and thus we obtain the first claim.

There remains to verify the second claim as for (3.4). If  $(Y_a, z^a)$  is the analytic center at  $x^*$ ,  $(Y_a, z^a) \in \Lambda(x^*)$  holds by definition, and from the above proof,  $Y_a^{\text{EE}}$  is the unique optimum of (A.2). Hence, by the KKT conditions of (A.2), there exists  $v \in \mathbb{R}^s$  such that (3.4) holds. Conversely, if such v exists and  $(Y_a, z^a) \in \Lambda(x^*)$ ,  $Y_a^{\text{EE}}$  solves (A.2), and hence  $E_*Y_a^{\text{EE}}E_*^{\top} = Y_a$  does (A.1). This means that  $(Y_a, z^a)$  is the analytic center. The whole proof is complete.

A.3. Proof of Proposition 2 Since  $x^*$  is a strict local optimum because of the ESOSC, we can take a compact set  $B \subseteq \mathbb{R}^n$  with nonempty interior such that  $x^* \in \text{int } B$  and it is a unique optimum of the problem

min 
$$f(x)$$
 s.t.  $h(x) = 0$ ,  $G(x) \in \mathbb{S}_{+}^{m}$ ,  $x \in B$ . (A.3)

Consider the sequence of the relevant barrier problems parameterized with  $\mu_k$  as in the following:

min 
$$f(x) - \mu_k \log \det G(x)$$
 s.t.  $h(x) = 0, \ G(x) \in \mathbb{S}^m_{++}, \ x \in B,$  (A.4)

and let  $x^k$  be an optimum of problem (A.4) for each k.

We will prove the theorem by showing that the above-defined sequence  $\{x^k\}$  is nothing but the desired one. To this end, it suffices to prove that  $\{x^k\}$  converges to  $x^*$ . Indeed, because  $x^* \in \text{int } B$ , the constraint  $x \in B$  for problem (A.4) is inactive at  $x^k$  for sufficiently large k, and thus  $x^k$  eventually becomes a local optimum of (3.6).

We write  $f_k := f(x^k)$  for each k and  $f_* := f(x^*)$  for the sake of simplicity. Recall  $G_k = G(x^k)$  and  $G_* = G(x^*)$ . We first consider the case (i) where  $G_* \in \mathbb{S}^m_+ \setminus \mathbb{S}^m_{++}$ , i.e.,  $G_*$  is on the boundary of  $\mathbb{S}^m_+$  and thus det  $G_* = 0$ . The proof for the other case (ii) where  $G_* \in \mathbb{S}^m_{++}$  will be given later. Letting  $\varphi_k := f_k - \mu_k \log \det G_k$  for each k, the first goal is to prove

$$\lim_{k \to \infty} \varphi_k = f_*. \tag{A.5}$$

Without loss of generality, by re-taking a smaller *B* with int  $B \ni x^*$  if necessary, we can suppose that det G(x) < 1 for all  $x \in B$  because of det  $G_* = 0$ , yielding

$$-\log \det G(x) > 0, \quad \forall x \in B, \tag{A.6}$$

which together with the feasibility of  $x^k$  for (A.3) implies

$$-\mu_k \log \det G_k > 0 > f_* - f_k.$$
(A.7)

Using the two inequalities in (A.7) yields

$$f_{*} < f_{k} < f_{k} - \mu_{k} \log \det G_{k} (= \varphi_{k}) \leq f_{k-1} - \mu_{k} \log \det G_{k-1} \leq f_{k-1} - \mu_{k-1} \log \det G_{k-1} (= \varphi_{k-1}),$$
(A.8)

where the third inequality follows from the optimality of  $x^k$  for problem (A.4) and the fourth one is due to  $\mu_k \le \mu_{k-1}$  and  $-\log \det G_{k-1} > 0$  from (A.6) and  $x^{k-1} \in B$ . From the above inequalities, we find that  $\{\varphi_k\}$  is a nonincreasing sequence such that it is bounded by  $f_*$  from below. Therefore, we ensure the existence of  $\lim_{k\to\infty} \varphi_k$  and moreover obtain

$$f_* \le \lim_{k \to \infty} \varphi_k. \tag{A.9}$$

To verify (A.5), there remains to prove the converse inequality. Related to  $\{x^k\}$ , under the MFCQ at  $x^*$ , we can construct another sequence  $\{x^{\ell(k)}\}$  feasible to problem (A.3) such that it converges to  $x^*$  and also satisfies det  $G_{\ell(k)} = \mu_k$  for each  $k \ge K$  with sufficiently large K > 0.<sup>4</sup> We then obtain  $\lim_{k\to\infty} \varphi_k \le f_*$  since  $a \log a \to 0$  as  $a \to 0+$  and  $\varphi_k \le f_{\ell(k)} - \mu_k \log \det G_{\ell(k)}$  holds by the definition of  $x^k$ . Together with (A.9), it derives the target equation (A.5).

<sup>&</sup>lt;sup>4</sup> This fact is verified as follows: From the MFCQ at  $x^*$ , there exists  $d \in \mathbb{R}^n$  such that  $G_* + \Delta G(x;d) \in \mathbb{S}_{++}^m$  and  $\nabla h(x^*)^\top d = 0$ . By the full column rankness of  $\nabla h(x^*)$ , we can ensure existence of a smooth curve  $x(\cdot) : [0,\bar{t}] \to \mathbb{R}^s$  with some  $\bar{t} > 0$  such that  $x(0) = x^*$ ,  $\dot{x}(0) = d$ , and x(t) is feasible to (A.4),  $\forall t \in (0,\bar{t}]$ . Particularly,  $G(x(t)) \in \mathbb{S}_{++}^m$  holds for all  $t \in (0,\bar{t}]$ . Therefore, as det G(x(t)) is continuous w.r.t.  $t \ge 0$  and takes 0 at t = 0 by the assumption  $G_* \in \mathbb{S}_+^m \setminus \mathbb{S}_{++}^m$ , we conclude that for any sufficiently small  $\alpha > 0$ , det  $G(x(t)) = \alpha$  is attained by some  $t \in (0,\bar{t}]$ . The proof is complete.

The convergence of  $\{x^k\}$  to  $x^*$  is not difficult to derive from (A.5). Letting  $\bar{x}$  be an arbitrary accumulation point of  $\{x^k\}$  and taking into consideration  $-\mu_k \log \det G_k > 0$  in  $\varphi_k$ , we get  $\limsup_{k\to\infty} \varphi_k \ge f(\bar{x})$ , which combined with (A.5) implies  $f_* \ge f(\bar{x})$ . By the feasibility of  $\bar{x}$  and the unique optimality of  $x^*$  for (A.3), we gain  $x^* = \bar{x}$ . Finally, since  $\bar{x}$  was an arbitrary accumulation point of  $\{x^k\}$ , we conclude that  $\lim_{k\to\infty} x^k = x^*$ .

We next consider case (ii) where  $G_* \in \mathbb{S}_{++}^m$ . Note that  $\log \det G_*$  is finite in this case. Without loss of generality, we may assume that  $\det G(x) > 0$  for all  $x \in B$ , by taking a smaller  $B(\ni x^*)$  if necessary. Let  $\bar{x}$  be an arbitrary accumulation point of  $\{x^k\}$  and note that  $\log \det G(\bar{x})$  is also finite since  $\det G(\bar{x}) > 0$  by virtue of  $\bar{x} \in B$ . By the optimality of  $x^k$  and feasibility of  $x^*$  to (A.4), it follows that

$$f_k - \mu_k \log \det G_k \le f_* - \mu_k \log \det G_*,$$

where driving  $k \to \infty$  and taking a subsequence if necessary imply  $f(\bar{x}) \le f_*$ . Then, in virtue of feasibility of  $\bar{x}$  and unique optimality of  $x^*$  for (A.3), we have  $x^* = \bar{x}$ . We hence conclude that  $\lim_{k\to\infty} x^k = x^*$  as for case(ii). Consequently, the desired result is obtained and the proof is complete.

**A.4. Proof of Proposition 3** We use the notations described before Proposition 3. In particular, recall (2.9) and (2.10).

*Proof of Proposition 3:* To begin with, for each  $k \ge 0$ , let

$$\widetilde{d}^k := \frac{d^k}{\|d^k\|}.$$

Since  $\{\widetilde{d}^k\}$  is bounded, it has at least one accumulation point, say  $\widetilde{d}^*$ . Choose an arbitrary subsequence  $\{\widetilde{d}^k\}_{k\in\mathcal{K}}$  which converges to  $\widetilde{d}^*$ . From remark 4,  $\{w^k\}_{k\in\mathcal{K}}$  has an accumulation point, say  $w^* := (x^*, Y_*, z^*)$ . Without loss of generality, we assume  $\lim_{\mathcal{K}\ni k\to\infty} w^k = w^*$ .

We prove the assertion by two steps. As the first step, we prove

$$\liminf_{k \to \infty} \frac{\mu_k}{\|d^k\|} > 0. \tag{A.10}$$

In order to derive a contradiction, suppose to the contrary that there exists a subsequence of  $\left\{\frac{\mu_k}{\|d^k\|}\right\}_{k \in \mathcal{K}}$  such that it converges to 0. We may assume  $\lim_{k \in \mathcal{K} \to \infty} \frac{\mu_k}{\|d^k\|} = 0$  by retaking  $\mathcal{K}$  if necessary. Since  $\lim_{k \in \mathcal{K} \to \infty} \widetilde{d^k} = \widetilde{d^*}, \widetilde{d^*}$  satisfies

$$\left(E_*^{\mathsf{T}}\Delta G(x^*; \widetilde{d}^*)E_* = \right)\Delta G^{\mathrm{EE}}(x^*; \widetilde{d}^*) \in \mathbb{S}_+^{r_*}, \ \nabla h(x^*)^{\mathsf{T}}\widetilde{d}^* = 0,$$
(A.11)

where these relations are derived from dividing the following equations by  $||d^k||$  and passing to the limit:

$$S_{++}^{r} \ni E_{*}^{\top}G_{k}E_{*} = E_{*}^{\top}(G_{k} - G_{*})E_{*} = \Delta G^{\text{EE}}(x^{*};d^{k}) + O(||d^{k}||^{2}),$$
  
$$0 = h(x^{k}) = h(x^{*}) + \nabla h(x^{*})^{\top}d^{k} + O(||d^{k}||^{2}).$$

As  $w^k = (x^k, Y_k, z^k)$  satisfies the BKKT conditions and  $P_* = [E_*, F_*]$  is an orthogonal matrix, we obtain, for each  $k \in \mathcal{K}$ ,

$$\begin{aligned} \frac{\mu_k I_{r_*}}{||d^k||} &= \frac{E_*^\top G_k Y_k E_*}{||d^k||} \\ &= \frac{E_*^\top \left(G_* + \Delta G(x^*; d^k) + \mathcal{O}(||d^k||^2)\right) \left[E_* \ F_*\right] \begin{bmatrix} E_*^\top \\ F_*^\top \end{bmatrix} Y_k E_*}{||d^k||}. \end{aligned}$$

which together with driving  $k \in \mathcal{K} \to \infty$  yields

$$\Delta G^{\rm EE}(x^*; \widetilde{d}^*) Y^{\rm EE}_* = O, \tag{A.12}$$

where we have used the relations  $G_*^{\text{EE}} = O$  and  $Y_*^{\text{FE}} = O$  from (2.8).

As  $w^* = (x^*, Y_*, z^*)$  and  $(x^*, Y_a, z^a)$  satisfy the KKT conditions, it follows that

$$\nabla f(x^*) = \mathcal{J}G(x^*)^* Y_* - \nabla h(x^*) z^*, \tag{A.13}$$

$$=\mathcal{J}G(x^*)^*Y_a - \nabla h(x^*)z^a. \tag{A.14}$$

Pre-multiplying both (A.13) and (A.14) by  $(\tilde{d}^*)^{\top}$  and noting (A.12) lead to

$$\nabla f(x^*)^{\mathsf{T}} \widetilde{d}^* = \operatorname{Tr}\left(\Delta G^{\operatorname{EE}}(x^*; \widetilde{d}^*) Y_{a}^{\operatorname{EE}}\right) = \operatorname{Tr}\left(\Delta G^{\operatorname{EE}}(x^*; \widetilde{d}^*) Y_{*}^{\operatorname{EE}}\right) = 0.$$
(A.15)

From (A.15) and (A.11), we ensure

$$\widetilde{d}^* \in C(x^*),\tag{A.16}$$

where  $C(x^*)$  is defined in (2.12). As  $Y_a^{\text{EE}}$  is positive definite by definition and  $\Delta G^{\text{EE}}(x^*; \widetilde{d}^*) \in \mathbb{S}_+^m$  follows from (A.11) again,  $\text{Tr}\left(\Delta G^{\text{EE}}(x^*; \widetilde{d}^*) Y_a^{\text{EE}}\right) = 0$  in (A.15) yields

$$\Delta G^{\rm EE}(x^*; \vec{d}^*) = 0. \tag{A.17}$$

Next, we transform  $(\widetilde{d}^*)^{\mathsf{T}} \mathcal{J} G(x^*)^* (Y_k - \overline{Y})$  as

$$\begin{split} (\widetilde{d}^{*})^{\mathsf{T}} \mathcal{J}G(x^{*})^{*}(Y_{k} - Y_{*}) &= \Delta G(x^{*}; \widetilde{d}^{*}) \bullet (Y_{k} - Y_{*}) \\ &= \mathrm{Tr}\left( \begin{bmatrix} E_{*}^{\mathsf{T}} \\ F_{*}^{\mathsf{T}} \end{bmatrix} \Delta G(x^{*}; \widetilde{d}^{*}) \begin{bmatrix} E_{*} F_{*} \end{bmatrix} \begin{bmatrix} F_{*}^{\mathsf{T}} \\ F_{*}^{\mathsf{T}} \end{bmatrix} (Y_{k} - Y_{*}) \begin{bmatrix} E_{*} F_{*} \end{bmatrix} \right) \\ &= \begin{bmatrix} \Delta G^{\mathrm{EE}}(x^{*}; \widetilde{d}^{*}) \Delta G^{\mathrm{EF}}(x^{*}; \widetilde{d}^{*}) \\ \Delta G^{\mathrm{FE}}(x^{*}; \widetilde{d}^{*}) \Delta G^{\mathrm{FF}}(x^{*}; \widetilde{d}^{*}) \end{bmatrix} \bullet \begin{bmatrix} Y_{k}^{\mathrm{EE}} - Y_{k}^{\mathrm{FF}} & Y_{k}^{\mathrm{EF}} \\ Y_{k}^{\mathrm{FE}} & Y_{k}^{\mathrm{FF}} \end{bmatrix} \\ &= \begin{bmatrix} O & \Delta G^{\mathrm{EF}}(x^{*}; \widetilde{d}^{*}) \Delta G^{\mathrm{FF}}(x^{*}; \widetilde{d}^{*}) \end{bmatrix} \bullet \begin{bmatrix} Y_{k}^{\mathrm{EE}} - Y_{k}^{\mathrm{FF}} & Y_{k}^{\mathrm{FF}} \\ Y_{k}^{\mathrm{FE}} & Y_{k}^{\mathrm{FF}} \end{bmatrix} \\ &= 2\mathrm{Tr}\left(\Delta G^{\mathrm{EF}}(x^{*}; \widetilde{d}^{*}) Y_{k}^{\mathrm{FE}}\right) + \mathrm{Tr}\left(\Delta G^{\mathrm{FF}}(x^{*}; \widetilde{d}^{*}) Y_{k}^{\mathrm{FF}}\right), \end{split}$$
(A.18)

where the second equality follows from the fact that  $[E_*, F_*](=P_*)$  is an orthogonal matrix and the fourth one is due to (A.17).

Since  $w^k$  and  $w^*$  satisfy the BKKT and KKT conditions, respectively, we have  $\nabla_x L(w^*) = \nabla_x L(w^k) = 0$  for each  $k \in \mathcal{K}$ , yielding

$$0 = (\widetilde{d}^{*})^{\top} \frac{(\nabla_{x}L(w^{k}) - \nabla_{x}L(w^{*}))}{||d^{k}||} = (\widetilde{d}^{*})^{\top} \frac{\nabla_{xx}^{2}L(w^{*})(d^{k}) - \mathcal{J}G(x^{*})(Y_{k} - Y_{*}) + \nabla h(x^{*})^{\top}(z^{k} - z^{*}) + O(||d^{k}||^{2})}{||d^{k}||} = \frac{(\widetilde{d}^{*})^{\top} \nabla_{xx}^{2}L(w^{*})(d^{k}) - \Delta G(x^{*};\widetilde{d}^{*}) \bullet (Y_{k} - Y_{*}) + O(||d^{k}||^{2})}{||d^{k}||} = (\widetilde{d}^{*})^{\top} \nabla_{xx}^{2}L(w^{*}) \frac{d^{k}}{||d^{k}||} - \frac{2\mathrm{Tr}\left(\Delta G^{\mathrm{EF}}(x^{*};\widetilde{d}^{*})Y_{k}^{\mathrm{FE}}\right) + \mathrm{Tr}\left(\Delta G^{\mathrm{FF}}(x^{*};\widetilde{d}^{*})Y_{k}^{\mathrm{FF}}\right)}{||d^{k}||} + O(||d^{k}||),$$
(A.19)

where the last equality follows from (A.18). Notice that the off-diagonal elements of  $P_*^{\mathsf{T}}G_kY_kP_*(=\mu_kI)$  are zeros for all *k*. Hence, for each  $k \in \mathcal{K} \geq 0$ , we have

$$O = F_*^{\top} G_k Y_k E_*$$
  
=  $F_*^{\top} G_k \left[ E_*, F_* \right] \begin{bmatrix} E_*^{\top} \\ F_*^{\top} \end{bmatrix} Y_k E_*$   
=  $F_*^{\top} G_k E_* Y_k^{\text{EE}} + F_*^{\top} G_k F_* Y_k^{\text{FE}}$ 

for each  $k \in \mathcal{K}$ . Substituting Taylor's expansion  $G_k = G_* + \Delta G(x^*; d^k) + O(||d^k||^2)$  into the last equation yields

$$(G_*^{\text{FE}} + \Delta G^{\text{FE}}(x^*; d^k))Y_k^{\text{EE}} + \left(G_*^{\text{FF}} + \Delta G^{\text{FF}}(x^*; d^k)\right)Y_k^{\text{FE}} = O(||d^k||^2)$$

where  $||Y_k||_F = O(1)$  was used for the last equality. Noting  $G_*^{FE} = O$  and dividing both the sides of the above by  $||d^k||$  give

$$\frac{\Delta G^{\text{FE}}(x^*; d^k)}{\|d^k\|} Y_k^{\text{EE}} + \left( G_*^{\text{FF}} \frac{Y_k^{\text{FE}}}{\|d^k\|} + \frac{\Delta G^{\text{FF}}(x^*; d^k)}{\|d^k\|} Y_k^{\text{FE}} \right) = \mathcal{O}(\|d^k\|)$$
(A.20)

for  $k \in \mathcal{K}$ . Note that  $\lim_{k\to\infty} Y_k^{\text{FE}} = O$  holds, which implies  $\lim_{k\to\infty} \frac{\Delta G^{\text{FF}}(x^*;d^k)}{\|d^k\|} Y_k^{\text{FE}} = O$ . Moreover, together with letting  $k \in \mathcal{K} \to \infty$ , equation (A.20) implies

$$\lim_{k \to \infty} \frac{Y_k^{\text{FE}}}{\|d^k\|} = -(G_*^{\text{FF}})^{-1} \Delta G^{\text{FE}}(x^*; \widetilde{d}^*) Y_*^{\text{EE}}.$$
(A.21)

On the other hand, the (2, 2)-block matrix of  $P_*^{\mathsf{T}}G_kY_kP_*/||d^k||(=\mu_kI/||d^k||)$  is calculated as

$$\frac{1}{||d^{k}||} \left( F_{*}^{\top} \left( G_{*} + \Delta G(x^{k}; d^{k}) \right) E_{*} Y_{k}^{\text{EF}} + F_{*}^{\top} \left( G_{*} + \Delta G_{k}(x^{k}; d^{k}) \right) F_{*} Y_{k}^{\text{FF}} \right) + \frac{\mathcal{O}(||d^{k}||^{2})}{||d^{k}||} \\
= \frac{1}{||d^{k}||} \left( F_{*}^{\top} \Delta G(x^{k}; d^{k}) E_{*} Y_{k}^{\text{EF}} + \left( G_{*}^{\text{FF}} + F_{*}^{\top} \Delta G_{k}(x^{k}; d^{k}) F_{*} \right) Y_{k}^{\text{FF}} \right) + \mathcal{O}(||d^{k}||), \tag{A.22}$$

where we have used

$$G_*^{\text{FE}} = O, \ G_*^{\text{FF}} = O$$
 (A.23)

from (2.8). In particular, from  $\lim_{k\to\infty} \Delta G(x^k; d^k) / ||d^k|| = \Delta G(x^*; \tilde{d}^*)$  and  $\lim_{k\to\infty} (Y_k^{\text{EF}}, Y_k^{\text{FF}}) = (O, O)$ , we see

$$\lim_{k\to\infty}\frac{F_*^{\mathsf{T}}\Delta G(x^k;d^k)E_*Y_k^{\mathsf{EF}}+F_*^{\mathsf{T}}\Delta G(x^k;d^k)F_*Y_k^{\mathsf{FF}}}{\|d^k\|}=O.$$

Since the limit of (A.22) is zero by the assumption  $\lim_{k\to\infty}\mu_k/||d^k|| = 0$  again and recalling that (A.22) is the (2,2)-block of  $\mu_k I/||d^k||$ , the above equation yields  $\lim_{k\to\infty} G_*^{\text{FF}}Y_k^{\text{FF}}/||d^k|| = O$ , which combined with the nonsingularity of  $G_*^{\text{FF}}$  induces

$$\lim_{k \to \infty} \frac{Y_k^{\text{FF}}}{\|d^k\|} = O. \tag{A.24}$$

By taking (A.21) and (A.24) into consideration and driving  $k \to \infty$  in the equation in (A.19), it holds that

$$(\widetilde{d}^*)^{\top} \nabla^2_{xx} L(w^*) \widetilde{d}^* = -2 \operatorname{Tr} \left( Y^{\text{EE}}_* \Delta G^{\text{EF}}(x^*; \widetilde{d}^*) (G^{\text{FF}}_*)^{-1} \Delta G^{\text{FE}}(x^*; \widetilde{d}^*) \right)$$

Combined with Lemma 1, this equation further implies

$$(\widetilde{d}^*)^{\mathsf{T}} \nabla^2_{xx} L(w^*) \widetilde{d}^* + (\widetilde{d}^*)^{\mathsf{T}} \Omega(x^*, Y_*) \widetilde{d}^* = 0.$$

However, in view of  $\tilde{d}^* \in C(x^*)$  by (A.16) and  $\tilde{d}^* \neq 0$  and by noting  $(Y_*, z^*) \in \Lambda(x^*)$ , the above equation contradicts the ESOSC. Therefore, we conclude (A.10).

In turn, we show  $\frac{\mu_k}{\|d^k\|} = O(1)$  as the second step. As  $w^k$  satisfies the BKKT conditions, we obtain, for each k,

$$\frac{\mu_k I_{r_*}}{\|d^k\|} = \frac{E_*^\top G_k Y_k E_*}{\|d^k\|} = \frac{E_*^\top (G_* + \mathcal{J}G(x^*)(d^k) + \mathcal{O}(\|d^k\|^2)) \left[E_* F_*\right] \left[\frac{E_*^\top}{F_*^\top}\right] Y_k E_*}{\|d^k\|},$$

which together with (A.23) yields

$$\lim_{k\to\infty}\frac{\mu_k I_{r_*}}{\|d^k\|} = \Delta G^{\text{EE}}(x^*; \widetilde{d}^*) Y^{\text{EE}}_*.$$

This means that the sequence  $\left\{\frac{\mu_k}{\|d^k\|}\right\}$  is bounded, and we thus obtain the desired consequence. By combining (A.10) and this fact, the proof is complete.

**A.5. Proof of Proposition 5** To start with, decompose a vector  $\Delta x \in \mathbb{R}^n$  into orthogonal component vectors as follows:

$$\Delta x = U_{x^*} \eta^1 + V_* \eta^2, \tag{A.25}$$

where  $(\eta^1, \eta^2) \in \mathbb{R}^{p_*} \times \mathbb{R}^{n-p_*}$  and  $V_* \in \mathbb{R}^{n \times (n-p_*)}$  is a matrix whose columns form an orthonormal basis of the orthogonal complement subspace of  $\mathcal{U}_*$ , where  $\mathcal{U}_*$  is defined in (3.21).

Let  $U_{x^*}^i$  be the *i*-th column of  $U_{x^*}$  for each  $i = 1, 2, ..., p_*$ . From (3.19), we have

$$\operatorname{Sym}\left(\begin{bmatrix} Y_{a}^{\operatorname{EE}} \Delta G^{\operatorname{EE}}(x^{*}; \Delta x) & Y_{a}^{\operatorname{EE}} \Delta G^{\operatorname{EF}}(x^{*}; \Delta x) \\ G_{*}^{\operatorname{FF}} \Delta Y^{\operatorname{FE}} & G_{*}^{\operatorname{FF}} \Delta Y^{\operatorname{FF}} \end{bmatrix}\right) = I,$$

of which the block components together with  $G_*^{\text{FF}} \in \mathbb{S}_{++}^{m-r_*}$  and  $Y_a^{\text{EE}} \in \mathbb{S}_{++}^{r_*}$  yield

$$\Delta Y^{\text{FF}} = (G_*^{\text{FF}})^{-1}, \tag{A.26}$$

$$\Delta G^{\rm EE}(x^*; \Delta x) = (Y^{\rm EE}_{a})^{-1}, \tag{A.27}$$

$$Y_{a}^{\text{EE}}\Delta G^{\text{EF}}(x^{*};\Delta x) + \Delta Y^{\text{EF}}G_{*}^{\text{FF}} = O.$$
(A.28)

By (3.18), we obtain

$$U_{x^*}^{\mathsf{T}} \nabla_{xx}^2 L(w^{\mathsf{a}}) \Delta x - \left( \begin{bmatrix} O & \Delta G^{\mathsf{EF}}(x^*; U_{x^*}^i) \\ \Delta G^{\mathsf{FE}}(x^*; U_{x^*}^i) & \Delta G^{\mathsf{FF}}(x^*; U_{x^*}^i) \end{bmatrix} \bullet P_*^{\mathsf{T}} \Delta Y P_* \right)_{i=1}^{p_*} = 0$$

leading to

$$U_{x^*}^{\mathsf{T}} \nabla_{xx}^2 L(w^a) \Delta x - \left( \Delta G^{\mathsf{FE}}(x^*; U_{x^*}^i) \bullet \Delta Y^{\mathsf{EF}} + \Delta G^{\mathsf{EF}}(x^*; U_{x^*}^i) \bullet \Delta Y^{\mathsf{FE}} + \Delta G^{\mathsf{FF}}(x^*; U_{x^*}^i) \bullet \Delta Y^{\mathsf{FF}} \right)_{i=1}^{p_*} = 0,$$

which is moreover rephrased as

$$U_{x^*}^{\top} \nabla_{xx}^2 L(w^{\rm a}) \Delta x - \left(2 \Delta G^{\rm FE}(x^*; U_{x^*}^i) \bullet \Delta Y^{\rm EF} + \Delta G^{\rm FF}(x^*; U_{x^*}^i) \bullet \Delta Y^{\rm FF}\right)_{i=1}^{p_*} = 0.$$

Combined with (A.26) and (A.28), this equation yields

$$U_{x^*}^{\mathsf{T}} \nabla_{xx}^2 L(w^{\mathsf{a}}) \Delta x + 2 \left( \operatorname{Tr} \left( \Delta G^{\operatorname{FE}}(x^*; U_{x^*}^i) Y_{\mathsf{a}}^{\operatorname{EE}} \Delta G^{\operatorname{EF}}(x^*; \Delta x) (G_*^{\operatorname{FF}})^{-1} \right) \right)_{i=1}^{p_*} = \left( \Delta G^{\operatorname{FF}}(x^*; U_{x^*}^i) \bullet (G_*^{\operatorname{FF}})^{-1} \right)_{i=1}^{p_*}$$

Decomposing  $\Delta x$  as in (A.25), we obtain from the above equation that

$$U_{x^*}^{\mathsf{T}} \nabla_{xx}^2 L(w^a) U_{x^*} \eta^1 + 2 \left( \sum_{j=1}^{p_*} \eta_j^1 \operatorname{Tr} \left( \Delta G^{\operatorname{FE}}(x^*; U_{x^*}^i) Y_a^{\operatorname{EE}} \Delta G^{\operatorname{EF}}(x^*; U_{x^*}^j) (G_*^{\operatorname{FF}})^{-1} \right) \right)_{i=1}^{p_*} \\ = - U_{x^*}^{\mathsf{T}} \nabla_{xx}^2 L(w^a) V_* \eta^2 - 2 \left( \operatorname{Tr} \left( \Delta G^{\operatorname{FE}}(x^*; U_{x^*}^i) Y_a^{\operatorname{EE}} \Delta G^{\operatorname{EF}}(x^*; V_* \eta^2) (G_*^{\operatorname{FF}})^{-1} \right) \right)_{i=1}^{p_*} \\ + \left( -\Delta G^{\operatorname{EF}}(x^*; U_{x^*}^i) \bullet (Y_a^{\operatorname{EE}})^{-1} + \left( \Delta G^{\operatorname{EF}}(x^*; U_{x^*}^i) - \Delta G^{\operatorname{FF}}(x^*; U_{x^*}^i) \right) \bullet (G_*^{\operatorname{FF}})^{-1} \right)_{i=1}^{p_*}, \tag{A.29}$$

where  $\eta^1 := (\eta_1^1, \eta_2^1, \dots, \eta_{p_*}^1)^\top$ . Next, we prove that  $\eta^1$  and  $\eta^2$  are uniquely determined. To this end, note that  $V_*\eta^2 \in \mathcal{U}_*^{\perp}$  by definition, and  $\Delta G^{\text{EE}}(x^*; V_*\eta^2) = (Y_a^{\text{EE}})^{-1}$  follows from (A.27) and  $\Delta G^{\text{EE}}(x^*; U_{x^*}\eta^1) = O$ . From this,  $V_*\eta^2$  turns out to be unique,<sup>5</sup> which together with the full column rank of  $V_*$  yields the uniqueness of  $\eta^2$ . In view of this fact and (A.29),  $\eta_1$  is also uniquely determined, because the matrix

$$U_{x^*}^{\top} \nabla_{xx}^2 L(w^{\mathrm{a}}) U_{x^*} + 2 \left( \mathrm{Tr} \left( \Delta G^{\mathrm{FE}}(x^*; U_{x^*}^i) Y_{\mathrm{a}}^{\mathrm{EE}} \Delta G^{\mathrm{EF}}(x^*; U_{x^*}^j) (G_*^{\mathrm{FF}})^{-1} \right) \right)_{1 \le i \le j \le p}$$

is actually positive definite by virtue of the ESOSC.

As a result,  $\Delta x = U_{x^*}\eta^1 + V_*\eta^2$  is the unique  $\Delta x$ -component of solutions to equations (3.18)-(3.20). Therefore, we ensure Item 1. Item 2 follows immediately from (A.26)-(A.28) with  $\Delta x = \xi^*$ . 

<sup>&</sup>lt;sup>5</sup> More precisely speaking, to derive the uniqueness of  $V_*\eta^2$ , we have made use of the following fundamental result from linear algebra: given  $A \in \mathbb{R}^{q_1 \times q_2}$  and  $b \in \mathbb{R}^{q_1}$ , assume that the linear equation  $A\theta = b$  has a nonempty solution set. Pick a solution *u* arbitrarily and decompose it as  $u = u^1 + u^2$  with  $u^1 \in \text{Ker } A$  and  $u^2 \in (\text{Ker } A)^{\perp}$ , where Ker A denotes the kernel or null space of the matrix A. Then,  $u^2$  is uniquely determined regardless of choice for u, whereas  $u^1$  is free in KerA. In the proof, there exist correspondences between Au = b and  $\Delta G^{\text{EE}}(x^*; U_{x^*}\eta^1 + V_*\eta^2) = (Y_a^{\text{EE}})^{-1}$ ,  $u^1$  and  $U_{x^*}\eta^1$ ,  $u^2$  and  $V_*\eta^2$ , and Ker A and  $\mathcal{U}_*$ , respectively.

A.6. Proof of Proposition 6 We show item 1. We first consider the first-half claim. For contradiction, assume that there exists an infinite sequence  $\{\mu_k\} \subseteq \mathbb{R}_{++}$  converging to 0 such that  $\mathcal{P}_{\rho}(\mu_k)$  does not contain a BKKT point with barrier parameter  $\mu_k$  for each *k*. According to Proposition 2,  $\{\mu_k\}$  accompanies a sequence of BKKT points  $\{x(\mu_k)\}$  which converges to the KKT point *x*<sup>\*</sup>. By the above assumption,  $x(\mu_k) \notin \mathcal{P}_{\rho}(\mu_k)$  for each *k*, implying  $||x(\mu_k) - x^* - \mu_k \xi^*|| / \mu_k \ge \rho ||\xi^*|| > 0$ . However, Theorem 2 implies

$$||x(\mu_k) - x^* - \mu_k \xi^*|| = o(\mu_k).$$
(A.30)

This is a contradiction. Hence, the first-claim claim is obtained. The second-half one can be also established by deriving a contradiction. Suppose to the contrary that there exist BKKT points  $\{x(\mu_k)\}$  with  $x(\mu_k) \in$ bd $\mathcal{P}_{\rho}(\mu_k)$ . By the definition of bd $\mathcal{P}_{\rho}(\mu_k)$ , we see  $\lim_{k\to\infty} x(\mu_k) = x^*$ , thus (A.30) holds again. However, this contradicts  $(x(\mu_k) - x^*)/\mu_k = \rho$ ,  $\forall k$  from  $x(\mu_k) \in$  bd $\mathcal{P}_{\rho}(\mu_k)$ . Item 2 also follows readily since the same relation as (A.30) is obtained from Theorem 2 again.

A.7. Proof of Proposition 7 We prove the first assertion in item 1. Write  $\widetilde{X} := P_*^T X P_*$  for any  $X \in \mathbb{S}^m$  by convension. In particular, we set  $G(\cdot)$  and  $\Delta G(x^*; \cdot)$  to X. Denote

$$R(x,\mu) := \widetilde{G}(x) - \widetilde{G}(x^*) - \mu \widetilde{\Delta G}(x^*;\xi^*).$$

We next consider to bound the magnitude of  $R(x,\mu)$  when  $(x,\mu)$  is varied. Recall  $G_*^{\text{EE}} = O$ ,  $G_*^{\text{EF}} = G_*^{\text{FE}} = O$ , and  $G_*^{\text{FF}} \in \mathbb{S}_{++}^{r_*}$ . It follows that

$$\frac{1}{\mu}\widetilde{G}(x) = \frac{1}{\mu}\widetilde{G}(x^*) + \widetilde{\Delta G}(x^*;\xi^*) + \frac{1}{\mu}R(x,\mu) 
= \begin{bmatrix} (Y_a^{\text{EE}})^{-1} + \frac{1}{\mu}R_1(x,\mu) & \Delta G^{\text{EF}}(x^*;\xi^*) + \frac{1}{\mu}R_2(x,\mu) \\ \Delta G^{\text{FE}}(x^*;\xi^*) + \frac{1}{\mu}R_2(x,\mu)^{\text{T}} & \Delta G^{\text{FF}}(x^*;\xi^*) + \frac{1}{\mu}G_*^{\text{FF}} + \frac{1}{\mu}R_3(x,\mu) \end{bmatrix},$$
(A.31)

where  $R_i(x,\mu)$  (*i* = 1, 2, 3) represent block submatrices of  $R(x,\mu)$  with appropriate sizes and the second equality follows from  $\Delta G^{\text{EE}}(x^*;\xi^*) = (Y_a^{\text{EE}})^{-1}$  by item-2 of Proposition 5. Taylor's expansion of  $\widetilde{G}$  at  $x^*$  gives

$$R(x,\mu) = \Delta G (x^*, x - x^* - \mu \xi^*) + O(||x - x^*||^2)$$
  
=  $O(\mu \rho ||\xi^*|| + ||x - x^*||^2)$   
=  $O(\mu \rho + \mu^2)$  (A.32)

for  $x \in cl \mathcal{P}_{\rho}(\mu)$ , where the last equality follows since  $||x - x^*|| \le \mu(\rho + 1)||\xi^*||$  by  $x \in cl \mathcal{P}_{\rho}(\mu)$ . By (A.32), the fact that  $(Y_a^{\text{EE}})^{-1} \in \mathbb{S}_{++}^{m-r_*}$ , and taking  $\mu_1 > 0$  and  $\rho_1 > 0$  so small,  $\frac{1}{\mu} ||R(x,\mu)||$  can be so small that the (1, 1)-block matrix of  $\frac{1}{\mu} \widetilde{G}(x)$  is symmetric positive definite for any  $(\rho, \mu) \in (0, \rho_1] \times (0, \mu_1]$ , that is to say,

$$Q(x,\mu) := (Y_{a}^{\text{EE}})^{-1} + \frac{1}{\mu} R_{1}(x,\mu) \in \mathbb{S}_{++}^{m-r_{*}}, \ \forall (\rho,\mu) \in (0,\rho_{1}] \times (0,\mu_{1}], \ x \in \text{cl}\mathcal{P}_{\rho}(\mu),$$
(A.33)

which along with (A.32) implies

$$Q(x,\mu)^{-1} = Y_{a}^{EE} \left(I + O(\rho + \mu)\right)^{-1}.$$
(A.34)

Meanwhile, the Schur complement of  $\frac{1}{u}\widetilde{G}(x)$  is expressed as

$$S_{c}(x,\mu) := \Delta G^{\text{FF}}(x^{*};\xi^{*}) + \frac{1}{\mu}G_{*}^{\text{FF}} + \frac{1}{\mu}R_{3} - \left(\Delta G^{\text{FE}}(x^{*};\xi^{*}) + \frac{1}{\mu}R_{2}^{\top}\right)Q^{-1}\left(\Delta G^{\text{EF}}(x^{*};\xi^{*}) + \frac{1}{\mu}R_{2}\right),$$

where we have dropped the arguments  $(x, \mu)$  from the functions  $R_1$ ,  $R_2$ ,  $R_3$ , and Q for simplicity. From (A.32), by re-taking  $(\mu_1, \rho_1)$  sufficiently small if necessary, we find that the above  $S_c$  is symmetric positive

definite for any  $\mu \in (0, \mu_1]$  because  $\frac{1}{\mu}G_*^{\text{FF}} \in \mathbb{S}_{++}^{r_*}$  is eventually dominant therein as  $\mu > 0$  gets smaller and  $x \in cl \mathcal{P}_{\rho}(\mu)$  holds by assumption. Hence,

$$S_{c}(x,\mu)^{-1} = O(\mu) \quad (x \in cl \mathcal{P}_{\rho}(\mu)).$$
 (A.35)

Moreover, in view of (A.31), from (A.33) and  $S_c \in \mathbb{S}_{++}^{r_*}$  shown above, we conclude  $\frac{1}{\mu}\widetilde{G}(x) \in \mathbb{S}_{++}^m$  for any  $(\rho,\mu) \in (0,\rho_1] \times (0,\mu_1]$  and  $x \in cl \mathcal{P}_{\rho}(\mu)$ , implying  $G(x) \in \mathbb{S}_{++}^m$ . Setting  $(\bar{\mu}_1,\bar{\rho}_1) := (\mu_1,\rho_1)$ , we ensure the first assertion.

We next prove the second assertion in item 1. Taking the inverse of  $\mu^{-1}\widetilde{G}(x)$  by applying the formula of the inverse of a partitioned matrix (e.g., Horn and Johnson [19, Section 0.7.3]) to (A.31), we obtain

$$\mu \widetilde{G}(x)^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^{\mathsf{T}} & M_{22} \end{bmatrix},\tag{A.36}$$

where each block component is defined as

$$\begin{split} M_{11} &:= Q(x,\mu)^{-1} + Q(x,\mu)^{-1} \left( \Delta G^{\text{EF}}(x^*;\xi^*) + \frac{1}{\mu} R_2(x,\mu) \right) S_c(x,\mu)^{-1} \left( \Delta G^{\text{EF}}(x^*;\xi^*) + \frac{1}{\mu} R_2(x,\mu) \right)^\top Q(x,\mu)^{-1}, \\ M_{12} &:= -Q(x,\mu)^{-1} \left( \Delta G^{\text{EF}}(x^*;\xi^*) + \frac{1}{\mu} R_2(x,\mu) \right) S_c(x,\mu)^{-1}, \\ M_{22} &:= S_c(x,\mu)^{-1}. \end{split}$$

Moreover, we have  $\frac{1}{\mu}R_i(x,\mu) = O(\rho + \mu)$  (*i* = 1, 2, 3) from (A.32). These facts together with (A.34), (A.35), and (A.36) yield

$$M_{11} = Y_{a}^{\text{EE}} \left( I + \mathcal{O} \left( \rho + \mu \right) \right)^{-1} + \mathcal{O}(\mu), M_{12} = \mathcal{O}(\mu), M_{22} = \mathcal{O}(\mu),$$
(A.37)

which together with  $\mu \overline{G}(x)^{-1} = \mu P_*^{\mathsf{T}} G(x)^{-1} P_*$  implies that  $\{ \mu G(x)^{-1} \mid x \in \operatorname{cl} \mathcal{P}_{\overline{\rho}_1}(\mu), \mu \in (0, \overline{\mu}_1] \}$  is bounded. In turn, we prove item 2. First, in view of (A.37), we obtain

$$M_{11} - Y_{a}^{EE} = Y_{a}^{EE} (I + O(\rho + \mu))^{-1} (I - (I + O(\rho + \mu))) + O(\mu)$$
  
= O(\(\rho + \mu\)). (A.38)

We drive  $(x, \mu) \rightarrow (x^*, 0)$  along with satisfying  $x \in cl \mathcal{P}_{\rho}(\mu)$ . From (A.37) and (A.38), it follows that

$$\mu \widetilde{G}(x)^{-1} - \widetilde{Y}_{a} = \begin{bmatrix} M_{11} - Y_{a}^{\text{EE}} & M_{12} \\ M_{12}^{\text{T}} & M_{22} \end{bmatrix} = \begin{bmatrix} O(\rho + \mu) & O(\mu) \\ O(\mu) & O(\mu) \end{bmatrix},$$

where  $\widetilde{Y}_a = P_*^{\top} Y_a P_*$ . Thus, we have  $\|\mu G(x)^{-1} - Y_a\|_F = \|\mu \widetilde{G}(x)^{-1} - \widetilde{Y}_a\|_F = O(\rho + \mu)$ . This means that there exists some  $K_1 > 0$  such that  $\|\mu G(x)^{-1} - Y_a\|_F \le K_1(\rho + \mu)$  as claimed.

**A.8. Proof of Proposition 8** First, from  $x \in cl \mathcal{P}_{\rho}(\mu)$ , it follows that  $||x - x^*|| \le ||x - x^* - \mu \xi^*|| + \mu ||\xi^*|| \le (\rho + 1)||\xi^*||\mu$ . Second, it follows that  $||Y - Y_a||_F \le ||Y - \mu G(x)^{-1}||_F + ||Y_a - \mu G(x)^{-1}||_F \le (\gamma_1 + K_1)\mu + K_1\rho$  from (3.28) and item 2 of Proposition 7. Moreover, since these inequalities yield

$$\begin{aligned} &\|z^{a} + (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla f(x) - \mathcal{J}G(x)^{*}Y) \| \\ &= \| (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla f(x) - \mathcal{J}G(x)^{*}Y + \nabla h(x)z^{a}) \| \\ &= \| (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla_{x}L(w^{a}) + O(\|Y - Y_{a}\|_{F} + \|x - x^{*}\|)) \| \\ &= O(\rho + \mu), \end{aligned}$$

where the second and third equalities are derived from applying Taylor's expansion to  $\nabla_x L(x, Y, z^a)$  at  $w^a$  and the facts that  $\nabla_x L(w^a) = 0$  and  $\|(\nabla h(x)^\top \nabla h(x))^{-1} \nabla h(x)^\top\|_F = O(1)$  for  $x \in \mathcal{B}$ , where  $\mathcal{B}$  is the ball defined in (3.25), we obtain

$$\begin{aligned} ||z - z^{a}|| &\leq ||z + (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla f(x) - \mathcal{J}G(x)^{*}Y)|| + \\ &||z^{a} + (\nabla h(x)^{\top} \nabla h(x))^{-1} \nabla h(x)^{\top} (\nabla f(x) - \mathcal{J}G(x)^{*}Y)|| \\ &= O(\rho + \mu), \end{aligned}$$

where we have used the assumption  $||z + (\nabla h(x)^\top \nabla h(x))^{-1} \nabla h(x)^\top (\nabla f(x) - \mathcal{J}G(x)^*Y)|| \le \gamma_2 \mu$ . Finally, using the above facts together with  $\nabla_x L(w^a) = 0$ , we have

$$\nabla_x L(w) = \nabla_x L(w^{a}) + O(||x - x^*|| + ||Y - Y_{a}||_{F} + ||z - z^{a}||) = O(\rho + \mu)$$

where the first equality follows from Taylor's expansion of  $\nabla_x L$  at  $w^a$ . As a consequence, by taking  $K_2 > 0$  sufficiently small, we ensure the desired inequalities.

**A.9. Proof of Proposition 9** To start with, choose  $\rho \leq \bar{\rho}_2$  and consider an arbitrary sequence  $\{w^{\ell} = (x^{\ell}, Y_{\ell}, z^{\ell})\}$  and  $\{\mu_{\ell}\} \subseteq (0, \bar{\mu}_2]$  such that  $\mu_{\ell} \to 0$  as  $\ell \to \infty$  and (3.27), (3.28), and (3.29) are fulfilled for each  $\ell$ . Write  $G_{\ell} := G(x^{\ell})$  for each  $\ell$ . From Proposition 8, we see that  $\{w^{\ell}\}$  is bounded and  $\lim_{\ell \to \infty} x^{\ell} = x^*$ . Note that  $G_{\ell}^{-1}$  exists by virtue of  $x^{\ell} \in cl \mathcal{P}_{\rho}(\mu_{\ell})$  and (3.26) along with  $\bar{\mu}_2 \leq \bar{\mu}_1$  and  $\bar{\rho}_2 \leq \bar{\rho}_1$ , and also note that  $\{\mu_{\ell}G_{\ell}^{-1}\}$  are bounded from item 1 of Proposition 7. Moreover, (3.28) implies that  $\{Y_{\ell}\}$  and  $\{\mu_{\ell}G_{\ell}^{-1}\}$  accumulate at identical points in  $\mathbb{S}_{+}^m$ . Denote an arbitrary accumulation point of  $\{(Y_{\ell}, z^{\ell})\}$  by  $(Y_*, z^*)$ . From the above argument and the fact that  $\|G_{\ell}Y_{\ell} - \mu_{\ell}I\|_{\mathsf{F}} = \|G_{\ell}(Y_{\ell} - \mu_{\ell}G_{\ell}^{-1})\|_{\mathsf{F}} \leq \|G_{\ell}\|_{\mathsf{F}} \|Y_{\ell} - \mu_{\ell}G_{\ell}^{-1}\|_{\mathsf{F}} \leq \gamma_1\mu_{\ell}\|G_{\ell}\|$ , we obtain

$$G_*Y_* = O, \ Y_* \in \mathbb{S}^m_+.$$
 (A.39)

Moreover, from Proposition 8, for any  $\ell$ , we have max  $(||Y_{\ell} - Y_{a}||_{F}, ||z^{\ell} - z^{a}||) \le K_{2}(\rho + \mu_{\ell})$ , where  $K_{2} > 0$  is the constant defined in Proposition 8. Then by driving  $\ell \to \infty$ , we obtain

$$\max\left(\|Y_* - Y_a\|_{\mathsf{F}}, \|z^* - z^a\|\right) \le K_2 \rho. \tag{A.40}$$

For  $X \in S^m$ , define  $\lambda_{\min}(X)$  as the least eigenvalue of X. From  $G_* + Y_a \in \mathbb{S}^m_{++}$  and (A.40), it follows that  $\lambda_{\min}(G_* + Y_*) \ge \lambda_{\min}(G_* + Y_a) + \lambda_{\min}(Y_* - Y_a) \ge \lambda_{\min}(G_* + Y_a) - ||Y_* - Y_a||_F \ge \lambda_{\min}(G_* + Y_a) - K_2\rho$ , which together with  $\lambda_{\min}(G_* + Y_a) > 0$  from  $G_* + Y_a \in \mathbb{S}^m_{++}$  implies

$$G_* + Y_* \in \mathbb{S}^m_{++}$$
 when  $0 < \rho \le \frac{\lambda_{\min}(G_* + Y_a)}{2K_2}$ . (A.41)

Next, let  $\widetilde{K} := \sum_{i=1}^{s} \|\nabla^2 h_i(x^*)\|_F + \frac{n(n+1)}{2} \max_{1 \le i, j \le n} \left\| \frac{\partial^2 G(x^*)}{\partial x_i \partial x_j} \right\|_F + n \|(G_*^{FF})^{-1}\|_F \max_{1 \le i \le n} \|E_* \mathcal{G}_i(x^*) F_*\|_F^2 > 0$  and recall that  $C(x^*)$  is a critical cone defined in (2.12). For any  $d \in C(x^*)$ , we have

$$\begin{aligned} d^{\top} \left( \nabla_{xx}^{2} L(x^{*}, Y_{*}, z^{*}) + \Omega(x^{*}, Y_{*}) \right) d \\ = d^{\top} \left( \nabla_{xx}^{2} L(w^{a}) + \left( \frac{\partial^{2} G(x^{*})}{\partial x_{i} \partial x_{j}} \bullet (Y_{*} - Y_{a}) \right)_{1 \le i, j \le n} + \sum_{i=1}^{s} \nabla^{2} h_{i}(x^{*})(z_{i}^{*} - z_{i}^{a}) \right) d \\ &+ d^{\top} \Omega(x^{*}, Y_{a}) d + 2 \mathrm{Tr} \left( \left( Y_{*}^{\mathrm{EE}} - Y_{a}^{\mathrm{EE}} \right) \Delta G^{\mathrm{FE}}(x^{*}; d) (G_{*}^{\mathrm{FF}})^{-1} \Delta G^{\mathrm{EF}}(x^{*}; d) \right) \\ \ge d^{\top} \left( \nabla_{xx}^{2} L(w^{a}) + \Omega(x^{*}, Y_{a}) \right) d - ||z^{*} - z^{a}|||d||^{2} \sum_{i=1}^{s} ||\nabla^{2} h_{i}(x^{*})||_{\mathrm{F}} - \frac{n(n+1)}{2} ||Y_{*} - Y_{a}||_{\mathrm{F}} ||d||^{2} \max_{1 \le i, j \le n} \left\| \frac{\partial^{2} G(x^{*})}{\partial x_{i} \partial x_{j}} \right\|_{\mathrm{F}} \\ &- n ||d||^{2} ||Y_{*}^{\mathrm{EE}} - Y_{a}^{\mathrm{EE}}||_{\mathrm{F}} ||(G_{*}^{\mathrm{FF}})^{-1}||_{\mathrm{F}} \max_{1 \le i \le n} ||E_{*} \mathcal{G}_{i}(x^{*})F_{*}||_{\mathrm{F}}^{2} \\ \ge d^{\top} \left( \nabla_{xx}^{2} L(w^{a}) + \Omega(x^{*}, Y_{a}) \right) d - \rho \widetilde{K} K_{2} ||d||^{2} \\ \ge (\kappa - \rho \widetilde{K} K_{2}) ||d||^{2}, \end{aligned}$$

where the first equality follows from Lemma 1 and (A.39), and the third inequality follows from (A.40) and  $||Y_*^{\text{EE}} - Y_a^{\text{EE}}||_F \le ||Y_* - Y_a||_F$  and the last inequality from ESOSC (3.1) with  $(Y, z) = (Y_a, z^a)$ . Thus, the last inequality implies

$$d^{\top} \left( \nabla_{xx}^{2} L(x^{*}, Y_{*}, z^{*}) + \Omega(x^{*}, Y_{*}) \right) d \ge \frac{\kappa ||d||^{2}}{2} \text{ when } 0 < \rho \le \frac{\kappa}{2\widetilde{K}K_{2}}.$$
 (A.42)

Hereafter, we set  $0 < \rho \le \min\left(\frac{\lambda_{\min}(G_*+Y_a)}{2K_2}, \frac{\kappa}{2\bar{\kappa}K_2}, \bar{\rho}_2\right)$  so that (A.41) and (A.42) hold. Notice that this choice of  $\rho$  is independent from the sequence  $\{w^\ell\}$ .

To prove the desired claim, we derive a contradiction by assuming to the contrary, that is, there exists infinite sequences<sup>6</sup>

$$\{\mu_\ell\} \subseteq \mathbb{R}_{++}, \ \{w^\ell := (x^\ell, Y_\ell, z^\ell)\} \subseteq \mathcal{W}_{++}, \ \{v^\ell\} \subseteq \mathbb{R}^n$$

such that  $\lim_{\ell \to \infty} \mu_{\ell} = 0$  and, for each  $\ell$ , it holds that  $\|v^{\ell}\| = 1$ ,  $\nabla h(x^{\ell})^{\top} v^{\ell} = 0$ , (3.27), (3.28), and (3.29) are fulfilled with  $(\mu, w) := (\mu_{\ell}, w^{\ell})$ , and

$$\mathcal{H}_{\ell} := (v^{\ell})^{\mathsf{T}} \nabla_{xx}^{2} L(w^{\ell}) v^{\ell} + \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell}} \left( \Delta G(x^{\ell}; v^{\ell}) \right) < \frac{\kappa}{2}.$$
(A.43)

By calculation, we have

$$\mathcal{H}_{\ell} = (v^{\ell})^{\top} \nabla^{2}_{xx} L(w^{\ell}) v^{\ell} + \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{\mu_{\ell} G_{\ell}^{-1}} \left( \Delta G(x^{\ell}; v^{\ell}) \right) + \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell} - \mu_{\ell} G_{\ell}^{-1}} \left( \Delta G(x^{\ell}; v^{\ell}) \right)$$

$$= (v^{\ell})^{\top} \nabla^{2}_{xx} L(w^{\ell}) v^{\ell} + \mu_{\ell} \Delta G(x^{\ell}; v^{\ell}) \bullet G_{\ell}^{-1} \Delta G(x^{\ell}; v^{\ell}) G_{\ell}^{-1} + \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell} - \mu_{\ell} G_{\ell}^{-1}} \left( \Delta G(x^{\ell}; v^{\ell}) \right)$$

$$= (v^{\ell})^{\top} \nabla^{2}_{xx} L(w^{\ell}) v^{\ell} + \mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G(x^{\ell}; v^{\ell}) G_{\ell}^{-\frac{1}{2}} \right\|_{\mathrm{F}}^{2} + \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell} - \mu_{\ell} G_{\ell}^{-1}} \left( \Delta G(x^{\ell}; v^{\ell}) \right).$$

$$(A.44)$$

As will be verified later on, we actually have the following relationships:

$$\lim_{\ell \to \infty} \Delta G(x^{\ell}; v^{\ell}) \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell} - \mu_{\ell} G_{\ell}^{-1}} \left( \Delta G(x^{\ell}; v^{\ell}) \right) = 0, \tag{A.45}$$

$$\liminf_{\ell \to \infty} \left( (v^{\ell})^{\mathsf{T}} \nabla_{xx}^{2} L(w^{\ell}) v^{\ell} + \mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G(x^{\ell}; v^{\ell}) G_{\ell}^{-\frac{1}{2}} \right\|_{\mathrm{F}}^{2} \right) \geq \frac{\kappa}{2}.$$
(A.46)

From these results and (A.44),  $\liminf_{\ell \to \infty} \mathcal{H}_{\ell} \geq \frac{\kappa}{2}$  holds. However, this contradicts hypothesis (A.43). Therefore, we have reached the first assertion.

**Proofs of** (A.45) **and** (A.46) For making the above proof complete, it remains to prove (A.45) and (A.46). We also suppose the same assumptions as those made for contradiction at the beginning of the above proof. In particular, we will use the same notations and symbols, such as  $\{w^{\ell}\}, \{v^{\ell}\}, \{v^{\ell}\}, \{w^{\ell}\}, \{w^{\ell}\},$ 

Before starting the proofs of (A.45) and (A.46), we shall give some preliminary results. First, note that  $\{w^{\ell}\}$  is bounded as described at the beginning of this section. Let  $w^*$  denote an accumulation point of  $\{w^{\ell}\}$ . Then, notice that the *x*-component of  $w^*$  is the KKT point  $x^*$  and denote the (Y, z)-component of  $w^*$  by  $(Y_*, z^*)$ . Moreover, let  $v^*$  be an accumulation point of  $\{v^{\ell}\}$ . Choose an orthogonal matrix  $P_{\ell}$  for each  $\ell$  so that  $G_{\ell}$  is diagonalized with  $P_{\ell}$  and the eigenvalues of the resultant diagonal matrix is aligned in the ascending order. By re-choosing  $P_*$  and taking a subsequence of  $\{(x^{\ell}, v^{\ell}, P_{\ell})\}$  if necessary, we can suppose, w.l.o.g<sup>7</sup>,

$$\lim_{\ell \to \infty} (w^{\ell}, v^{\ell}, P_{\ell}) = (w^*, v^*, P_*).$$
(A.47)

Note that, as  $\nabla h(x^{\ell})^{\top} v^{\ell} = 0$ ,  $||v^{\ell}|| = 1$  for each  $\ell$ , it follows that

$$\|v^*\| = 1, \ \nabla h(x^*)^\top v^* = 0. \tag{A.48}$$

Next, so as to match  $P_* = [E_*, F_*]$ , we partition  $P_\ell$  as

$$P_{\ell} = [E_{\ell}, F_{\ell}],$$

<sup>&</sup>lt;sup>6</sup> To abuse notation, we use  $w^{\ell}$  and  $\mu_{\ell}$  again to denote a sequence.

<sup>&</sup>lt;sup>7</sup> Recall that  $P_*$  was an arbitrary orthogonal matrix such that (2.7) holds. Even if  $P_*$  is reset as the limit of  $\{P_\ell\}$  here, it satisfies (2.7) again, and thus never affects the theoretical results established so far.

which along with (A.47) implies  $\lim_{\ell\to\infty} (E_{\ell}, F_{\ell}) = (E_*, F_*)$ . Let the resultant diagonal matrix obtained from  $G_{\ell} \in \mathbb{S}^m_{++}$  using  $P_{\ell}$  be  $D_{G_{\ell}}$ , and also let  $D^0_{G_{\ell}}$  and  $D^{++}_{G_{\ell}}$  be the block diagonal matrices of  $D_{G_{\ell}}$  that converge to the  $(m - r_*) \times (m - r_*)$  zero matrix and the positive diagonal matrix  $G^{\text{FF}}_*$ , respectively. Moreover, we often write simply

$$\widetilde{G}_{\ell} := P_{\ell} G_{\ell} P_{\ell}^{\mathsf{T}}$$

for each  $\ell$ . In summary, it holds that

$$\widetilde{G}_{\ell} = P_{\ell}G_{\ell}P_{\ell}^{\mathsf{T}} = \begin{bmatrix} D_{G_{\ell}}^{0} & O \\ O & D_{G_{\ell}}^{++} \end{bmatrix}, \quad \lim_{\ell \to \infty} (D_{G_{\ell}}^{0}, D_{G_{\ell}}^{++}) = (O, G_{*}^{\mathrm{FF}}).$$

Accordingly, we denote

$$\widetilde{Y}_{\ell} := P_{\ell} Y_{\ell} P_{\ell}^{\top}, \ \widetilde{\Delta G}_{\ell} := P_{\ell} \Delta G_{\ell} P_{\ell}^{\top}$$

with  $\Delta G_{\ell} := \Delta G(x^{\ell}; v^{\ell})$ . Furthermore, so as to match the partition pattern of  $\begin{bmatrix} 0 & 0 \\ 0 & G_{*}^{\text{FF}} \end{bmatrix}$ , partition a given matrix  $Z \in \mathbb{S}^{m}$  as

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^{\top} & Z_{22} \end{bmatrix}, \ Z_{11} \in \mathbb{S}^{m-r_*}, \ Z_{12} \in \mathbb{R}^{(m-r_*) \times r_*}, \ Z_{22} \in \mathbb{S}^{r_*}$$

Now, we start proving (A.45) and (A.46).

**Proof of** (A.45) First, recall that  $\mathcal{L}_X Y = XY + YX$  for  $X, Y \in \mathbb{S}^m$ . If  $X \in \mathbb{S}^{m}_{++}$ , the linear operator  $\mathcal{L}_X$  is invertible, namely,  $\mathcal{L}_X^{-1}$  exists. Next, note that, given  $W \in \mathbb{S}^m$ , a solution  $Z \in \mathbb{S}^m$  to  $\mathcal{L}_{\widetilde{G}_\ell} Z = W$  satisfies

$$Z_{11} = \mathcal{L}_{D_{G_{\ell}}^{0}}^{-1} W_{11}, \ Z_{22} = \mathcal{L}_{D_{G_{\ell}}^{++}}^{-1} W_{22}, \tag{A.49}$$

$$Z_{12}(i,j) = \frac{1}{D^0_{G_\ell}(i,i) + D^{++}_{G_\ell}(j,j)} W_{12}(i,j) \quad (1 \le i \le m - r_*, 1 \le j \le r_*),$$
(A.50)

which are verified by representing  $\mathcal{L}_{\widetilde{G}_{\ell}}Z = W$  as

$$\begin{bmatrix} D^0_{G_\ell} Z_{11} + Z_{11} D^0_{G_\ell} - W_{11} & D^0_{G_\ell} Z_{12} + Z_{12} D^{++}_{G_\ell} - W_{12} \\ D^{++}_{G_\ell} Z_{12}^\top + Z^\top_{12} D^0_{G_\ell} - W^\top_{12} & D^{++}_{G_\ell} Z_{22} + Z_{22} D^{++}_{G_\ell} - W_{22} \end{bmatrix} = O.$$

We have that  $||Y_{\ell} - \mu_{\ell}G_{\ell}^{-1}||_{F} \le \gamma_{1}\mu_{\ell}$  from (3.28) and  $\{P_{\ell}\}$  is bounded since  $P_{\ell}$  is an orthogonal matrix. These facts yield

$$\widetilde{Y}_{\ell} - \mu_{\ell} \widetilde{G}_{\ell}^{-1} = \mathcal{O}(\mu_{\ell}).$$
(A.51)

In view of (A.49) and (A.50) with  $W := \mathcal{L}_{\widetilde{Y}_{\ell}-\mu_{\ell}\widetilde{G}_{\ell}^{-1}} \Delta \widetilde{G}_{\ell}$ , the solution Z satisfies

$$Z_{11} = O(\|\widetilde{\Delta G}_{\ell}\|_{\rm F}), \ Z_{22} = O(\mu_{\ell}\|\widetilde{\Delta G}_{\ell}\|_{\rm F}), \ Z_{12}(i,j) = O(\mu_{\ell}\|\widetilde{\Delta G}_{\ell}\|_{\rm F}) \quad (1 \le i \le m - r_*, 1 \le j \le r_*),$$
(A.52)

where the first equation in (A.52) is derived from the fact

$$Z = \mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1} \mathcal{L}_{\frac{1}{\mu_{\ell}}(\widetilde{Y}_{\ell} - \mu_{\ell} \widetilde{G}_{\ell}^{-1})} \widetilde{\Delta G}_{\ell} = \mathcal{O}(\|\widetilde{\Delta G}_{\ell}\|_{F}),$$

which is ensured by (A.51) and the boundedness of  $\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}$ . (For the proof of the boundedness of  $\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}$ , see the footnote<sup>8</sup>.) Moreover, the second and third equations in (A.52) are implied by (A.50) and the right equation in (A.49). Using (A.52) again and noting that  $\{\Delta \widetilde{G}_{\ell}\}$  is bounded, we obtain

$$\Delta G_{\ell} \bullet \mathcal{L}_{G_{\ell}}^{-1} \mathcal{L}_{Y_{\ell} - \mu_{\ell} G_{\ell}^{-1}} \Delta G_{\ell} = \widetilde{\Delta G}_{\ell} \bullet \mathcal{L}_{\widetilde{G}_{\ell}}^{-1} \mathcal{L}_{\widetilde{Y}_{\ell} - \mu_{\ell} \widetilde{G}_{\ell}^{-1}} \widetilde{\Delta G}_{\ell}$$

<sup>8</sup> Note that, for any  $X \in \mathbb{S}^m$  having *m* eigenvalues  $\alpha_1 \leq \alpha_2 \leq \cdots, \leq \alpha_m$ , the linear operator  $\mathcal{L}_X$  is symmetric and has m(m + 1)/2 eigenvalues  $\alpha_1, \alpha_2, \ldots, \alpha_m, \{(\alpha_i + \alpha_j)/2\}_{i\neq j}$ . Letting  $(0 <)\lambda_1^{(\ell)} \leq \lambda_2^{(\ell)} \leq \cdots \lambda_m^{(\ell)}$  be the eigenvalues of  $\widetilde{G}_{\ell} \in \mathbb{S}_{+}^m$  for each  $\ell$ , the eigenvalues of the symmetric linear operator  $\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}$  are  $\mu_{\ell}/\lambda_1^{(\ell)}, \mu_{\ell}/\lambda_2^{(\ell)}, \cdots \mu_{\ell}/\lambda_m^{(\ell)}$ , and  $\{(\mu_{\ell}/\lambda_i^{(\ell)} + \mu_{\ell}/\lambda_j^{(\ell)})/2\}_{i\neq j}$ . Since  $\{\mu G(x)^{-1} \mid x \in \mathrm{cl} \mathcal{P}_{\bar{p}_1}(\mu), \mu \in (0, \bar{\mu}_1]\}$  is bounded from item 1 of Proposition 7, so are  $\{\mu_{\ell} G_{\ell}^{-1}\}$  and  $\{\mu_{\ell} \widetilde{G}_{\ell}^{-1}\}$ . Hence,  $\{\mu_{\ell}/\lambda_i^{(\ell)}\}$  is also bounded for each *i*, which together with  $\|\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}\|_{2} := \max_{Z \in \mathbb{S}^m; \|Z\|_{F} = 1} \|\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}Z\|_{F} \leq \mu_{\ell}/\lambda_1^{(\ell)}$  yields the boundedness of  $\{\mu_{\ell} \mathcal{L}_{\widetilde{G}_{\ell}}^{-1}\}$ .

$$= \widetilde{\Delta G}_{\ell} \bullet Z$$
  
= Tr  $\left( (\widetilde{\Delta G}_{\ell})_{11} Z_{11} + 2 (\widetilde{\Delta G}_{\ell})_{12} Z_{12}^{\top} + (\widetilde{\Delta G}_{\ell})_{22} Z_{22} \right)$   
=  $O \left( \| (\widetilde{\Delta G}_{\ell})_{11} \|_{F} + \mu_{\ell} \right).$  (A.53)

In order to prove the desired equation (A.45), it suffice to verify

$$\Delta G^{\text{EE}}(x^*;v^*) = O, \tag{A.54}$$

because (A.45) is verified by using (A.53) together with the fact that  $\lim_{\ell \to \infty} (\Delta G_{\ell})_{11} = \Delta G^{\text{EE}}(x^*; v^*) = O$  following from (A.54). To this end, we evaluate  $\mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G_{\ell} G_{\ell}^{-\frac{1}{2}} \right\|_{\text{F}}^{2}$  in (A.44) as follows:

$$\begin{split} \mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G_{\ell} G_{\ell}^{-\frac{1}{2}} \right\|_{F}^{2} &= \mu_{\ell} \operatorname{Tr} \left( G_{\ell}^{-1} \Delta G_{\ell} G_{\ell}^{-1} \Delta G_{\ell} \right) \\ &= \mu_{\ell} \operatorname{Tr} \left( \left\{ \begin{bmatrix} (D_{G_{\ell}}^{0})^{-1} & O \\ O & (D_{G_{\ell}}^{++})^{-1} \end{bmatrix} \begin{bmatrix} (\widetilde{\Delta G}_{\ell})_{11} & (\widetilde{\Delta G}_{\ell})_{12} \\ (\widetilde{\Delta G}_{\ell})_{21} & (\widetilde{\Delta G}_{\ell})_{22} \end{bmatrix} \right\}^{2} \right) \\ &= \mu_{\ell} \operatorname{Tr} \left( \begin{bmatrix} (D_{G_{\ell}}^{0})^{-1} (\widetilde{\Delta G}_{\ell})_{11} & (D_{G_{\ell}}^{0})^{-1} (\widetilde{\Delta G}_{\ell})_{12} \\ (D_{G_{\ell}}^{++})^{-1} (\widetilde{\Delta G}_{\ell})_{21} & (D_{G_{\ell}}^{++})^{-1} (\widetilde{\Delta G}_{\ell})_{22} \end{bmatrix}^{2} \right) \\ &= \mu_{\ell} \operatorname{Tr} \left( (D_{G_{\ell}}^{0})^{-1} (\widetilde{\Delta G}_{\ell})_{11} (D_{G_{\ell}}^{0})^{-1} (\widetilde{\Delta G}_{\ell})_{11} \right) + 2\mu_{\ell} \operatorname{Tr} \left( (D_{G_{\ell}}^{0})^{-1} (\widetilde{\Delta G}_{\ell})_{12} \right) \\ &+ \mu_{\ell} \operatorname{Tr} \left( (D_{G_{\ell}}^{++})^{-1} (\widetilde{\Delta G}_{\ell})_{22} (D_{G_{\ell}}^{++})^{-1} (\widetilde{\Delta G}_{\ell})_{22} \right), \end{split}$$

and herein we obtain

$$\lim_{\ell \to \infty} (c_\ell) = 0 \tag{A.55}$$

because  $\lim_{\ell \to \infty} \mu_{\ell} = 0$  and the matrices in the trace-part of  $(c_{\ell})$  are convergent. It holds that  $\lim_{\ell \to \infty} \mu_{\ell} G_{\ell}^{-1} = Y_*$  since  $\mu_{\ell} G_{\ell}^{-1}$  and  $Y_{\ell}$  accumulate at identical points due to (3.28), thus which together with (A.41) yields

$$\lim_{\ell \to \infty} \mu_{\ell} (D^0_{G_{\ell}})^{-1} = E^{\top}_* Y_* E_* = (\widetilde{Y}_*)_{11} \in \mathbb{S}^m_{++}, \tag{A.56}$$

which yields

$$\lim_{\ell \to \infty} (b_{\ell}) = 2 \operatorname{Tr} \left( (G_*^{\mathrm{FF}})^{-1} \Delta G^{\mathrm{FE}}(x^*; v^*) (\widetilde{Y}_*)_{11} \Delta G^{\mathrm{EF}}(x^*; v^*) \right) = (v^*)^{\mathsf{T}} \Omega(x^*, Y_*) v^*, \tag{A.57}$$

where the last equality follows from Lemma 1. Therefore, in view of (A.43), the last equality in (A.44), and (A.53), by noting that  $(v^{\ell})^{\mathsf{T}} \nabla_{xx}^2 L(w^{\ell}) v^{\ell}$  is bounded, we find that  $\mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G_{\ell} G_{\ell}^{-\frac{1}{2}} \right\|_{\mathsf{F}}^2$  is bounded, which together with (A.55) and (A.57) implies that  $\{(a_{\ell})\}$  is also bounded. From this fact together with (A.56), we ensure  $\Delta G^{\mathsf{EE}}(x^*; v^*) = \lim_{\ell \to \infty} (\Delta G_{\ell})_{11} = O$  thus (A.54). The proof of (A.45) is complete.

**Proof of** (A.46) From (A.54) and (A.48),  $v^* \in C(x^*)$  holds, where  $C(x^*)$  is the critical cone and expressed as in (2.12). Let

$$\Xi_{\ell} := (v^{\ell})^{\top} \nabla_{xx}^{2} L(w^{\ell}) v^{\ell} + \mu_{\ell} \left\| G_{\ell}^{-\frac{1}{2}} \Delta G(x^{\ell}; v^{\ell}) G_{\ell}^{-\frac{1}{2}} \right\|_{\mathrm{F}}^{2}$$

for each  $\ell$ . Since  $(a_{\ell}) \ge 0$  for each  $\ell$  and  $\{(a_{\ell})\}$  is bounded as shown in the proof of (A.45), any accumulation point of  $\{(a_{\ell})\}$  is nonnegative. Combining this fact with (A.55), (A.57), and (A.42) with  $d := v^*$  yields

$$\liminf_{\ell \to \infty} \Xi_{\ell} = (v^{*})^{\top} \left( \nabla_{xx}^{2} L(w^{*}) + \Omega(x^{*}, Y_{*}) \right) v^{*} + \liminf_{\ell \to \infty} (a_{\ell}) \ge \frac{\kappa}{2} ||v^{*}||^{2} = \frac{\kappa}{2}$$

which implies (A.46). The proof is complete.

#### **A.10. Proof of Lemma 2** First, we show that

$$\operatorname{dist}\left(\check{x}(\mu), \mathcal{M}\right) = \mathcal{O}(\|h(\check{x}(\mu))\|) \tag{A.58}$$

for  $\check{x}(\mu) \in \mathcal{B}$ . Note that (3.25) holds as for  $\mathcal{B}$ . To prove (A.58), assume to the contrary: there exists a sequence  $\{\mu_{\ell}\}$  converging to 0 such that

dist 
$$(\check{x}(\mu_{\ell}), \mathcal{M}) \neq 0, \ \forall \ell, \ \lim_{\ell \to \infty} \frac{\|h(\check{x}(\mu_{\ell}))\|}{\operatorname{dist}(\check{x}(\mu_{\ell}), \mathcal{M})} = 0.$$
 (A.59)

For each *l*, let  $y^{\ell} \in \arg\min_{y \in \mathcal{M} \cap \mathcal{B}} ||\check{x}(\mu_{\ell}) - y||$ . Then, we have that, for any  $\ell$  large enough,

$$\operatorname{dist}(\check{x}(\mu_{\ell}),\mathcal{M}) = \|\check{x}(\mu_{\ell}) - y^{\ell}\|$$
(A.60)

since  $\lim_{\ell\to\infty} \check{x}(\mu_{\ell}) = x^* \in \operatorname{int} \mathcal{B}$ ,  $\lim_{\ell\to\infty} \check{x}(\mu_{\ell}) - y^{\ell} = 0$ , and thus  $y^{\ell} \in \operatorname{int} \mathcal{B}$ . In what follows, we consider only  $\ell$  large enough and assume  $y^{\ell} \in \operatorname{int} \mathcal{B}$ . Since  $y^{\ell}$  solves  $\min_{y \in \mathcal{M}} \frac{1}{2} ||\check{x}(\mu) - y||^2$  and the LICQ holds at  $y^{\ell}$  from (3.25), the KKT conditions holds, namely there exists  $\eta^{\ell}$  such that

$$\check{x}(\mu_{\ell}) - y^{\ell} = \nabla h(y^{\ell})\eta^{\ell}.$$

From (A.59) and (A.60),  $\check{x}(\mu_{\ell}) - y^{\ell} = \nabla h(y^{\ell})\eta^{\ell} \neq 0$ . Letting  $\bar{v}$  be an accumulation point of  $\{(\check{x}(\mu_{\ell}) - y^{\ell})/||\check{x}(\mu_{\ell}) - y^{\ell}||\}$ , we may assume that

$$\frac{(\check{x}(\mu_{\ell}) - y^{\ell})}{\|\check{x}(\mu_{\ell}) - y^{\ell}\|} = \frac{\nabla h(y^{\ell})\eta^{\ell}}{\|\nabla h(y^{\ell})\eta^{\ell}\|} \to \bar{v} \quad (l \to \infty)$$
(A.61)

without loss of generality. It follows that  $\|\bar{v}\| = 1$ . As

$$\frac{\|h(\check{x}(\mu_{\ell}))\|}{\operatorname{dist}(\check{x}(\mu_{\ell}),\mathcal{M})} = \frac{\|h(\check{x}(\mu_{\ell})) - h(y^{\ell})\|}{\|\check{x}(\mu_{\ell}) - y^{\ell}\|} = \frac{\|\nabla h(y^{\ell})^{\top}(\check{x}(\mu_{\ell}) - y^{\ell}) + O(\|\check{x}(\mu_{\ell}) - y^{\ell}\|^{2})\|}{\|\check{x}(\mu_{\ell}) - y^{\ell}\|},$$

driving  $\ell \to \infty$  herein yields

$$\nabla h(x^*)^{\mathsf{T}} \bar{v} = 0. \tag{A.62}$$

Since  $\nabla h(x^*)$  is of full column rank,  $\nabla h$  is continuous, and  $\lim_{\ell \to \infty} y^{\ell} = x^*$ , (A.62) actually implies that there exists  $\{v^{\ell}\}$  such that it converges to  $\bar{v}$  and, for each  $\ell$ ,

$$\nabla h(y^{\ell})^{\mathsf{T}} v^{\ell} = 0. \tag{A.63}$$

Meanwhile, with (A.61) and  $\lim_{l\to\infty} v^{\ell} = \bar{v}$ , we gain

$$\lim_{\ell \to \infty} \frac{(v^{\ell})^{\top} \nabla h(y^{\ell}) \eta^{\ell}}{\|\nabla h(y^{\ell}) \eta^{\ell}\|} = \|\bar{v}\|^2,$$

which combined with (A.63) gives  $\|\bar{v}\|^2 = 0$ . However, this contradicts  $\|\bar{v}\| = 1$ . We thus obtain (A.58).

We next prove the desired relation (3.37). Note that  $h(\check{x}(\mu)) = h(x^* + \mu\xi^*) = O(\mu^2)$  holds by Taylor's expansion together with  $h(x^*) = 0 = \nabla h(x^*)^{\mathsf{T}}\xi^*$ , where the last equality follows from Theorem 2. With this fact along with (A.58), (3.37) is ensured. The proof is complete.

**A.11. Proof of Proposition 10** In order to prove Proposition 10, we start by showing the following lemma:

LEMMA A.1. For  $v \in V$  and  $y \in \mathbb{R}^{n-s}$ , let  $(x, Y, d) := (\Phi(v), \mu G(\Phi(v))^{-1}, \nabla \Phi(v)^{\top} y)$  and suppose  $G(x) \in \mathbb{S}_{++}^{m}$ . Moreover, let  $z \in \mathbb{R}^{s}$  and  $\mu > 0$ . Then, we have

$$y^{\mathsf{T}}\nabla^{2}\Psi_{\mu}(v)y = d^{\mathsf{T}}\nabla_{xx}^{2}L(x,Y,z)d + \sum_{j=1}^{n} \left(\frac{\partial L(x,Y,z)}{\partial x_{j}}y^{\mathsf{T}}\nabla^{2}\Phi_{j}(v)y\right) + \Delta G(x;d) \bullet \mathcal{L}_{G(x)}^{-1}\mathcal{L}_{Y}\left(\Delta G(x;d)\right).$$
(A.64)

Proof. By calculation, as regards the function  $\psi_{\mu}$  defined in (3.5), we have

$$\nabla \psi_{\mu}(x) = \nabla f(x) - \mu \mathcal{J}G(x)^{*}G(x)^{-1} = \nabla_{x}L(x,\mu G(x)^{-1},z) - \nabla h(x)z, \qquad (A.65)$$

$$d^{\top} \nabla^{2} \psi_{\mu}(x)d = d^{\top} \nabla^{2} f(x)d - \mu \left(\sum_{i=1}^{n} \sum_{j=1}^{n} d_{i}d_{j}\frac{\partial^{2}G(x)}{\partial x_{i}\partial x_{j}}\right) \bullet G(x)^{-1} + \mu ||G(x)^{-\frac{1}{2}} \Delta G(x;d)G(x)^{-\frac{1}{2}}||_{\mathrm{F}}^{2}$$

$$= d^{\top} \nabla^{2}_{xx}L(x,\mu G(x)^{-1},z)d - d^{\top} \left(\sum_{i=1}^{s} z_{i} \nabla^{2} h_{i}(x)\right)d + \mu ||G(x)^{-\frac{1}{2}} \Delta G(x;d)G(x)^{-\frac{1}{2}}||_{\mathrm{F}}^{2}. \qquad (A.66)$$

Moreover, it follows from (3.34) that

$$\sum_{j=1}^{n} \left( \sum_{i=1}^{s} z_i \frac{\partial h_i(\Phi(v))}{\partial x_j} \right) \nabla^2 \Phi_j(v) + \nabla \Phi(v) \left( \sum_{i=1}^{s} z_i \nabla_{xx}^2 h_i(\Phi(v)) \right) \nabla \Phi(v)^{\top} = O.$$
(A.67)

Then, with  $(x, Y, d) = (\Phi(v), \mu G(\Phi(v))^{-1}, \nabla \Phi(v)^{\top} y)$ , equation (3.40) yields

$$y^{\mathsf{T}} \nabla^{2} \Psi_{\mu}(v) y = d^{\mathsf{T}} \nabla^{2}_{xx} L(x, Y, z) d + \sum_{j=1}^{n} \frac{\partial \psi_{\mu}(\Phi(v))}{\partial x_{j}} y^{\mathsf{T}} \nabla^{2} \Phi_{j}(v) y - d^{\mathsf{T}} \left( \sum_{i=1}^{s} z_{i} \nabla^{2} h_{i}(x) \right) d + \mu \|G(x)^{-\frac{1}{2}} \Delta G(x; d) G(x)^{-\frac{1}{2}} \|_{\mathrm{F}}^{2}$$
$$= d^{\mathsf{T}} \nabla^{2}_{xx} L(x, Y, z) d + \sum_{j=1}^{n} \left( \frac{\partial \psi_{\mu}(x)}{\partial x_{j}} + \sum_{i=1}^{s} z_{i} \frac{\partial h_{i}(x)}{\partial x_{j}} \right) y^{\mathsf{T}} \nabla^{2} \Phi_{j}(v) y + \mu \|G(x)^{-\frac{1}{2}} \Delta G(x; d) G(x)^{-\frac{1}{2}} \|_{\mathrm{F}}^{2}$$
$$= d^{\mathsf{T}} \nabla^{2}_{xx} L(x, Y, z) d + \sum_{j=1}^{n} \left( \frac{\partial L(x, Y, z)}{\partial x_{j}} y^{\mathsf{T}} \nabla^{2} \Phi_{j}(v) y \right) + \Delta G(x; d) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_{Y} (\Delta G(x; d))$$

where the first equality follows from (3.34) and (A.66), the second from (A.67), the third from (A.65) and  $\mu \|G(x)^{-\frac{1}{2}} \Delta G(x;d) G(x)^{-\frac{1}{2}}\|_{\mathrm{F}}^{2} = \mu \Delta G(x;d) \bullet G(x)^{-1} \Delta G(x;d) G(x)^{-1} = \Delta G(x;d) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_{\mu G(x)^{-1}} (\Delta G(x;d)) = \Delta G(x;d) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_{F} (\Delta G(x;d)).$ 

**Proof of Proposition 10** First, let  $K_2$ ,  $M_3$ ,  $M_4 > 0$  be the constants defined in Proposition 8 and (3.35), and  $(\bar{\rho}_4, \bar{\mu}_4) \in (0, \bar{\rho}_3] \times (0, \bar{\mu}_3]$  be such that  $\max(\bar{\rho}_4, \bar{\mu}_4) \leq \kappa M_3^2/(8nK_2M_4^2M_5)$ . Choose  $(\rho, \mu) \in (0, \bar{\rho}_4] \times (0, \bar{\mu}_4]$ and  $v \in V \cap \Phi^{-1}(\operatorname{cl}\mathcal{P}_{\rho}(\mu))$  arbitrarily. Let  $x := \Phi(v) \in \operatorname{cl}\mathcal{P}_{\rho}(\mu)$  and recall  $\bar{\mu}_4 \leq \bar{\mu}_1$  and  $\bar{\rho}_4 \leq \bar{\rho}_1$ . Then, from (3.26), it holds that  $G(x) \in \mathbb{S}_{++}^m$  and  $(\nabla h(x)\nabla h(x)^{\top})^{-1}$  exists. Thus, we can define  $Y := \mu G(x)^{-1}$  and z := $-(\nabla h(x)\nabla h(x)^{\top})^{-1}\nabla h(x)^{\top}(\nabla f(x) - \mathcal{J}G(x)^*Y)$ , and these (x, Y, z) fulfills the conditions (3.27), (3.28), and (3.29). Letting  $d := \nabla \Phi(v)^{\top} y$ , we obtain  $\nabla h(x)^{\top} d = \sum_{i=1}^s \nabla h_i(x)^{\top} \nabla \Phi(v)^{\top} y = 0$  from (3.33). It holds that

$$y^{\top} \nabla^{2} \Psi_{\mu}(v) y = d^{\top} \nabla_{xx}^{2} L(x, Y, z) d + \Delta G(x; d) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_{Y} \Delta G(x; d) + \sum_{j=1}^{n} \left( \frac{\partial L(x, Y, z)}{\partial x_{j}} d^{\top} \nabla^{2} \Phi_{j}(v) d \right)$$
  

$$\geq d^{\top} \nabla_{xx}^{2} L(x, Y, z) d + \Delta G(x; d) \bullet \mathcal{L}_{G(x)}^{-1} \mathcal{L}_{Y} \Delta G(x; d) - \sum_{j=1}^{n} ||\nabla_{x} L(x, Y, z)|| ||\nabla^{2} \Phi_{j}(v)||_{\mathrm{F}} ||d||^{2}$$
  

$$\geq \frac{\kappa}{2} M_{3}^{2} - n K_{2} M_{4}^{2} M_{5}(\mu + \rho) \geq \frac{\kappa M_{3}^{2}}{4},$$

where the first equality follows from Lemma A.1, the third inequality from Proposition 8, Proposition 9, and (3.35) with  $d = \nabla \Phi(v)^{\top} y$ , and the last inequality from the above-mentioned definitions of  $\bar{\mu}_4$  and  $\bar{\rho}_4$ . The proof is complete.

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