

A recognition principle for iterated suspensions as coalgebras over the little cubes operad

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Abstract

Our main result is a recognition principle for iterated suspensions as coalgebras over the little cubes operads. Given a topological operad, we construct a comonad in pointed topological spaces endowed with the wedge product. We then prove an approximation theorem that shows that the comonad associated to the little n -cubes operad is weakly equivalent to the comonad $\Sigma^n \Omega^n$ arising from the suspension-loop space adjunction. Finally, our recognition theorem states that every little n -cubes coalgebra is homotopy equivalent to an n -fold suspension. These results are the Eckmann–Hilton dual of May’s foundational results on iterated loop spaces.

1 Introduction

Since the invention of operads by May, they have played an important role in many parts of mathematics and physics. The first application and the original motivation for their invention was for the study of iterated loop spaces (see [16] and [5]). Operads provide a way of, and a coherent framework for, studying objects equipped with many "multiplications", i.e. operations with multiple inputs and one output, satisfying certain homotopical coherences. An important class of such objects are n -fold loop spaces, which are algebras over the little n -cubes operad. May showed in his recognition principle a homotopical converse, namely that every little n -cubes algebra is weakly equivalent to an n -fold loop space; and further proved an approximation theorem which asserts that the monad associated to the little n -cubes operad is weakly equivalent to the monad $\Omega^n \Sigma^n$. This approximation theorem reduced the study of operations on the homology of iterated loop spaces to the combinatorics of the little cubes operads, a perspective which unravelled their complete algebraic structure (see [7]).

The goal of this paper is to prove the Eckmann–Hilton dual results of May’s work on iterated loop spaces. First of all, we construct a comonad in the category of pointed spaces associated to an operad. Next, we show that n -fold suspensions are coalgebras over the little n -cubes operad \mathcal{C}_n . More precisely we prove the following theorem.

Theorem A. *The n -fold reduced suspension of a pointed space X is a \mathcal{C}_n -coalgebra. More precisely, there is a natural and explicit operad map*

$$\nabla : \mathcal{C}_n \rightarrow \text{CoEnd}_{\Sigma^n X},$$

where $\text{CoEnd}_{\Sigma^n X}$ is the coendomorphism operad of $\Sigma^n X$. The map ∇ encodes the homotopy coassociativity and homotopy cocommutativity of the classical pinch map $\Sigma^n X \rightarrow \Sigma^n X \vee \Sigma^n X$. In particular, the pinch map is an operation associated to an element of $\mathcal{C}_n(2)$. Furthermore, for any based map $X \rightarrow Y$, the induced map $\Sigma^n X \rightarrow \Sigma^n Y$ extends to a morphism of \mathcal{C}_n -coalgebras.

All details will be explained later on. Bearing the above result in mind, it is natural to wonder if the Eckmann–Hilton dual of May’s celebrated recognition of iterated loop spaces is true in this new setting. This is indeed the case, as the following result shows.

Theorem B. *Let X be a \mathcal{C}_n -coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X , together with a weak equivalence of \mathcal{C}_n -coalgebras*

$$\Sigma^n \Gamma^n(X) \xrightarrow{\cong} X,$$

which is a retract in the category of pointed spaces. Therefore, every \mathcal{C}_n -coalgebra has the homotopy type of an n -fold reduced suspension.

Together, our theorems [A](#) and [B](#) provide the following intrinsic characterization of n -fold reduced suspensions as \mathcal{C}_n -coalgebras.

Corollary. *Every n -fold suspension is a \mathcal{C}_n -coalgebra, and if a pointed space is a \mathcal{C}_n -coalgebra then it is homotopy equivalent to an n -fold suspension.*

It is worth noting that this result already exists at the level of $\Sigma^n \Omega^n$ coalgebras, see [Theorem 4.9](#). Another celebrated result in [\[16\]](#) is the *approximation theorem*. It constitutes an essential step for proving the recognition principle for n -fold loop spaces, and it is also the key for unlocking certain computations on the homology of iterated loop spaces. Roughly speaking, the approximation theorem for loop spaces asserts that the free \mathcal{C}_n -algebra on a pointed space X is weakly equivalent to $\Omega^n \Sigma^n X$. We also prove the Eckmann–Hilton dual of this result. It reads as follows.

Theorem C. *For every $n \geq 1$, there is a natural morphism of comonads*

$$\alpha_n : \Sigma^n \Omega^n \longrightarrow C_n.$$

Furthermore, for every pointed space X , there is an explicit natural homotopy retract of pointed spaces

$$\Sigma^n \Omega^n X \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} C_n(X) \quad \begin{array}{c} \curvearrowright \\ \end{array}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

The comonad C_n in the statement above is constructed in a natural way from the little n -cubes operad. Essentially, it is an Eckmann–Hilton dualization of May’s monad associated to \mathcal{C}_n . To our knowledge, this comonad has not been studied elsewhere, and it seems to be an exciting new object that might shed light on further understanding n -fold reduced suspensions, as well as on other objects that support a coaction of the little n -cubes operad.

Let us place our work in historical context. It has been known for a long time that any $(n - 1)$ -connected CW complex of dimension less than or equal to $(2n - 1)$ has the homotopy type of a $(1$ -fold) suspension. In [\[3\]](#), [\[19\]](#), [\[10\]](#) and finally [\[14\]](#), this result was successively improved on. In modern language, these authors showed that an $(n - 1)$ -connected co- H -space equipped with an A_k comultiplication which is of dimension less than or equal to $k(n - 1) + 3$ is a suspension. The case of $k = \infty$ in [\[14\]](#) can be thought of as the E_1 -version of [Theorem B](#), although our proof strategy is very different. From a different angle, the case of iterated suspensions considered as coalgebras over (a homotopical version of) the $\Sigma^n \Omega^n$ -comonad was recently treated in [\[4\]](#), where the authors obtained a recognition principle for $(n + 1)$ -connected, n -fold (simplicial) suspensions. This last result differs from our [Theorem B](#) in several key respects. Firstly; our notions of coalgebra differ as they pass to a derived functor in the homotopy category of pointed spaces, while we consider only $\Sigma^n \Omega^n$ -coalgebras in the classical sense of coalgebras over comonads. Secondly; our result has the sharpest possible connectivity requirement. The most striking difference with all previous scholarship is that we make heavy use of the little n -cubes operad and the comonad C_n ; whereas these objects do not seem to have appeared in previous literature on the homotopy theory of iterated suspensions (with the exception of [\[11\]](#) in a very different context). In particular, there is no approximation theorem in [\[4\]](#).

One of the main contributions of this article is therefore providing a link between iterated suspensions and the combinatorics of the little cubes operads. Potentially, this connection can be exploited further.

To conclude, a few remarks are in order. The first remark is that to prove our theorems [B](#) and [C](#), we do not follow an Eckmann–Hilton dual approach to May’s proof in the case of iterated loop spaces. While we believe it may be possible to pursue this approach, we have found a framework

and proof which depends on explicit homotopies and hence avoids the use of quasi-fibrations and the construction of auxiliary spaces. In this sense, our approach is technically simpler. The approximation of suspensions is an independent result that we believe might have potential side applications. Finally, most of the results of this paper could have been stated using little n -disks instead of little n -cubes. However, using cubes significantly simplify many of the explicit formulae that appear when proving our results, and therefore we choose to present things this way.

1.1 Notation and conventions

All topological spaces are compactly generated and Hausdorff. We denote by I the unit interval in \mathbb{R} and by J its interior:

$$J = (0, 1) \subseteq [0, 1] = I.$$

The symmetric group on n letters is denoted S_n .

For $X = (X, *)$ a pointed space, it will be convenient to identify the r -fold wedge $X^{\vee r}$ as a subspace of the cartesian product $X^{\times r}$. To do so, consider


$$X^{\vee r} = \bigcup_{i=1}^r \{*\} \times \cdots \times \underbrace{X}_i \times \cdots \times \{*\} \subset X^{\times r}.$$

A point x in the i -th factor of the wedge $X^{\vee r}$ is therefore identified with the point $(*, \dots, *, x, *, \dots, *)$ having x at its i -th component and the base point at all others. We further use the convention that both $X^{\vee 0}$ and $X^{\times 0}$ are equal to the base point. Given pointed maps $\varphi_1, \dots, \varphi_r : X \rightarrow Y$, we denote by $(\varphi_1, \dots, \varphi_r)$ the induced map $X \rightarrow Y^{\times r}$ to the product. Here, we implicitly used the diagonal map $d : X \rightarrow X^{\times r}$ given by $d(x) = (x, \dots, x)$. To simplify the notation we will omit the diagonal from the notation when this is clear from the context. If the image of this map lands in the wedge subspace $Y^{\vee r}$, we denote the corresponding restriction by $\{\varphi_1, \dots, \varphi_r\}$. Thus, the curly brackets notation emphasizes that the map lands in the wedge rather than the product. We reserve the notation $\varphi_1 \vee \cdots \vee \varphi_r$ for the induced map $X^{\vee r} \rightarrow Y^{\vee r}$ given by

$$(\varphi_1 \vee \cdots \vee \varphi_r)(*, \dots, *, x_i, *, \dots, *) = (*, \dots, *, \varphi_i(x_i), *, \dots, *).$$

We frequently use the identification $\Sigma^n X = S^n \wedge X$ for the n -fold reduced suspension of a pointed space X . Thus, points in $\Sigma^n X$ will be denoted $[t, x]$, where $t \in S^n$ and $x \in X$. Since points in the suspensions are equivalence classes, we use the square brackets notation. From now on, we implicitly assume all suspensions are reduced.

We assume the reader is familiar with operad theory, especially in topological spaces, and we refer to [9]. We use the following conventions. An operad \mathcal{P} in a symmetric monoidal category $\mathcal{M} = (\mathcal{M}, \otimes, \mathbb{1})$ is *unitary* if $\mathcal{P}(0) = \mathbb{1}$, and *non-unitary* if $\mathcal{P}(0)$ is not defined (i.e., the underlying symmetric sequence of \mathcal{P} starts in arity 1). We borrow this nomenclature from [9, Section 2.2]. We will make heavy use of the operad of little n -cubes \mathcal{C}_n , considered as a unitary operad where $\mathcal{C}_n(0) = *$ is a single point.

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2 Coalgebras over topological operads

Given a unitary topological operad \mathcal{P} , we construct an explicit comonad $C_{\mathcal{P}}$ in *pointed spaces*. In Section 2.1, we carefully construct this comonad and study some of its basic properties. The

comonad $C_{\mathcal{P}}$ gives rise to the category of coalgebras over \mathcal{P} , also called \mathcal{P} -coalgebras. There is a second way of defining \mathcal{P} -coalgebras by using the coendomorphism operad that does not require the explicit construction of the comonad $C_{\mathcal{P}}$. This alternative construction has the advantage that it can be defined for all operads even when they are not necessarily unitary. The disadvantage is that it is not clear how to get an explicit comonad out of this definition. We explain this alternative construction and show that in the case of unitary operads it gives an equivalent notion of \mathcal{P} -coalgebras in Section 2.2. We specialize to the case in which \mathcal{P} is the operad \mathcal{C}_n of little n -cubes in Section 2.3, producing the central comonad of this paper. Finally, we prove Theorem A in Section 2.4 - that the n -fold reduced suspension of a pointed space is naturally a \mathcal{C}_n coalgebra. Therefore, the n -fold reduced suspensions are the paradigmatic examples of \mathcal{C}_n -coalgebras.

Remark 2.1. In our constructions of coalgebras, we are mixing pointed and unpointed spaces. All our operads live in the category of unpointed spaces while the coalgebras over the operads and associated comonads live in the category of pointed spaces.

2.1 Construction of topological comonads

In this section, we construct the mentioned comonad $C_{\mathcal{P}}$ in pointed spaces out of a unitary operad \mathcal{P} in unpointed spaces.

Let us first establish some preliminary notation. Denote

$$\text{Top} = (\text{Top}, \times, \{*\}) \quad \text{and} \quad \text{Top}_* = (\text{Top}_*, \vee, \{*\})$$

the symmetric monoidal categories of spaces endowed with the cartesian product \times , and pointed spaces endowed with the wedge product \vee , respectively. Let \mathcal{P} be a unitary operad in Top with composition map γ and denote the unitary operation by $*$ $\in \mathcal{P}(0)$. Define the *restriction operators*, for all $n \geq 1$ and $1 \leq i \leq n$, by inserting the unique point $*$ $\in \mathcal{P}(0)$ at the i -th component:

$$\begin{aligned} \mathcal{P}(n) &\xrightarrow{d_i} \mathcal{P}(n-1) \\ \theta &\longmapsto \gamma(\theta; \text{id}, \dots, *, \dots, \text{id}). \end{aligned}$$

Let $X \in \text{Top}_*$. The *wedge collapse maps*, defined for all $n \geq 1$ and $1 \leq i \leq n$, are given by collapsing the i -th factor in the wedge as follows:

$$\begin{aligned} X^{\vee n} &\xrightarrow{\pi_i} X^{\vee(n-1)} \\ (x_1, \dots, x_n) &\longmapsto (x_1, \dots, \hat{x}_i, \dots, x_n). \end{aligned}$$

Here, the r -fold wedge is seen inside the r -fold cartesian product, and the notation \hat{x}_i means that we are sending the i -th component to the basepoint.

Notation 2.2. If \mathcal{P} is a unitary operad and X is a pointed space, we denote

$$\text{Tot}(\mathcal{P}, X) := \prod_{n \geq 0} \text{Map}_{S_n}(\mathcal{P}(n), X^{\vee n}).$$

Each space $\text{Map}_{S_n}(\mathcal{P}(n), X^{\vee n})$ consists of the equivariant maps from the arity n component of \mathcal{P} equipped with its usual S_n -action to the n -fold wedge of X with itself endowed with the S_n -action that permutes the coordinates of its points by $\sigma \cdot (x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$. We frequently disregard the 0-th component in the infinite product above, since the mapping space $\text{Map}(\mathcal{P}(0), X^{\vee 0})$ is just a point. It can therefore be ignored in all computations that follow. Thus, the point $(f_0, f_1, f_2, \dots) \in \text{Tot}(\mathcal{P}, X)$ will be denoted (f_1, f_2, \dots) . The topology on the space $\text{Tot}(\mathcal{P}, X)$ is the usual product topology.

We are ready to define the underlying endofunctor of our comonad $C_{\mathcal{P}}$.

Definition 2.3. Let \mathcal{P} be a unitary operad in Top . Define the endofunctor in pointed spaces

$$\begin{aligned} C_{\mathcal{P}} : \text{Top}_* &\longrightarrow \text{Top}_* \\ X &\longmapsto C_{\mathcal{P}}(X), \end{aligned}$$

where

$$C_{\mathcal{P}}(X) = \{ \alpha = (f_1, f_2, \dots) \in \text{Tot}(\mathcal{P}, X) \mid \pi_i f_n = f_{n-1} d_i \text{ for all } n \geq 2 \text{ and } 1 \leq i \leq n \}$$

is the subspace of $\text{Tot}(\mathcal{P}, X)$ formed by those sequences (f_1, f_2, \dots) that commute with the restriction operators and wedge collapse maps. That is, for all $n \geq 2$ and $1 \leq i \leq n$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{P}(n) & \xrightarrow{f_n} & X^{\vee n} \\ d_i \downarrow & & \pi_i \downarrow \\ \mathcal{P}(n-1) & \xrightarrow{f_{n-1}} & X^{\vee(n-1)} \end{array}$$

The base point of $C_{\mathcal{P}}(X)$ is the sequence $\alpha = (f_1, f_2, \dots)$ where each f_r has image the base point of $X^{\vee r}$. Since the base point of the wedge $X^{\vee r}$ is fixed by the S_r -action, the base point is well-defined. If $f: X \rightarrow Y$ is a pointed map, then $C_{\mathcal{P}}(f): C_{\mathcal{P}}(X) \rightarrow C_{\mathcal{P}}(Y)$ is defined by

$$C_{\mathcal{P}}(f)(\alpha) = (f \circ f_1, (f \vee f) \circ f_2, \dots, (f \vee \dots \vee f) \circ f_n, \dots).$$

The n th term in the sequence above is given by

$$(f \vee \dots \vee f) \circ f_n: \mathcal{P}(n) \xrightarrow{f_n} X^{\vee n} \xrightarrow{f \vee \dots \vee f} Y^{\vee n}.$$

Remarks 2.4.

1. The idea of defining $C_{\mathcal{P}}$ above as a subspace of $\text{Tot}(\mathcal{P}, X)$ arises from an Eckmann–Hilton dualization of May’s definition of the monad associated to an operad [16]. Recall that the monad M_n in pointed spaces defined in *loc. cit.* by using the little n -cubes operad is given by

$$M_n(X) = \left(\prod_{r \geq 0} \mathcal{C}_n(r) \times X^{\times r} \right) / \sim,$$

where \sim is the equivalence relation that glues level r to level $r + 1$ by combining the restriction operators with the insertion of the base point, $(d_i(c), y) \sim (c, s_i(y))$, and imposing the compatibility with the group action, $(c \cdot \sigma, y) \sim (c, \sigma \cdot y)$.¹

2. The compatibility condition of a sequence $\alpha \in \text{Tot}(\mathcal{P}, X)$ with the restriction operators and wedge collapse maps,

$$\pi_i f_n = f_{n-1} d_i, \quad \text{for all } n \geq 1 \text{ and } 1 \leq i \leq n \tag{1}$$

is the precise condition needed to incorporate a counit to the coalgebras in pointed spaces that result from the comonad $C_{\mathcal{P}}$. See Remark 2.17 for further details.

3. The comonad $C_{\mathcal{P}}$ can be constructed in more general symmetric monoidal categories. For the applications that we give in this paper, we are only interested in the category of topological spaces.

Our next goal is to endow the endofunctor $C_{\mathcal{P}}$ with a comonad structure. Before doing so, we make two elementary observations that will simplify some of our proofs later on. We will use the following notation: if h_1, \dots, h_r is a family of maps such that the composition

$$h_1 \circ \dots \circ h_{i-1} \circ h_{i+1} \circ \dots \circ h_r$$

makes sense, then we denote the expression above by

$$h_1 \cdots \widehat{h_i} \cdots h_r.$$

That is, the hat $\widehat{(-)}$ on top of the i -th map indicates that this component is removed from the composition. The first observation is the following.

¹Here, $(c, y) \in \mathcal{C}_n(r) \times X^{\times(r-1)}$, $s_i(y)$ is the point of $X^{\times r}$ where we insert the base point at the i -th component, and $\sigma \in S_r$.

Lemma 2.5. A sequence $(f_1, f_2, \dots) \in C_{\mathcal{P}}(X)$ is determined by its first component $f_1 : \mathcal{P}(1) \rightarrow X$. That is, we can recursively write, for all $r \geq 2$,

$$f_r = \left\{ f_1 \widehat{d}_1 d_2 \cdots d_r, f_1 d_1 \widehat{d}_2 d_3 \cdots d_r, \dots, f_1 d_1 d_2 \cdots d_{r-1} \widehat{d}_r \right\},$$

where the d_i 's are the maps that insert $*$ in $\mathcal{P}(0)$ into the i th entry.

Recall that the term on the right hand side above follows the notation from Section 1.1.

Proof. Let $\alpha = (f_1, f_2, \dots) \in C_{\mathcal{P}}(X)$. Before we give a general proof of the lemma we first work out the $r = 2$ case since this makes the general argument clearer. Let

$$f_2 : \mathcal{P}(2) \rightarrow X \vee X$$

be the second component of α . Denote by $q_i : X \vee X \rightarrow X$ the projection onto the i -th factor of the wedge, for $i = 1, 2$. There are identifications $q_i = \pi_{3-i}$, where $\pi_1, \pi_2 : X \vee X \rightarrow X$ are the corresponding wedge collapse maps. Then,

$$f_2 = \{q_1 f_2, q_2 f_2\} = \{\pi_2 f_2, \pi_1 f_2\} = \{f_1 d_2, f_1 d_1\} = \left\{ f_1 \widehat{d}_1 d_2, f_1 d_1 \widehat{d}_2 \right\}.$$

In the third equality above, we used the Equation (1) for $n = 2$. The proof for general f_r follows a slight generalization of the case just proven, where we recursively use the identities of Equation (1) for all n between 2 and r . Thus, let

$$f_r : \mathcal{P}(r) \rightarrow X^{\vee r}$$

be the r th component of α . Denote by $q_i : X^{\vee r} \rightarrow X$ the projection onto the i -th factor of the wedge, for $i = 1, \dots, r$. There are identifications

$$q_i = \pi_1 \pi_2 \cdots \widehat{\pi}_i \cdots \pi_r, \quad \text{for all } i = 1, \dots, r.$$

Recall the hat $\widehat{\pi}_i$ indicates that we omit the i -th term. There is a slight but harmless abuse of notation above, since the π_j 's that appear in the expression of q_i have different domains. Then,

$$\begin{aligned} f_r &= \{q_1 f_r, q_2 f_r, \dots, q_r f_r\} \\ &= \{\widehat{\pi}_1 \pi_2 \pi_3 \cdots \pi_r f_r, \pi_1 \widehat{\pi}_2 \pi_3 \pi_4 \cdots \pi_r f_r, \dots, \pi_1 \pi_2 \cdots \pi_{r-1} \widehat{\pi}_r f_r\} \\ &= \{\widehat{\pi}_1 \pi_2 \pi_3 \cdots (\pi_r f_r), \pi_1 \widehat{\pi}_2 \pi_3 \pi_4 \cdots (\pi_r f_r), \dots, \pi_1 \pi_2 \cdots (\pi_{r-1} f_r)\} \\ &= \{\widehat{\pi}_1 \pi_2 \pi_3 \cdots (f_{r-1} d_r), \pi_1 \widehat{\pi}_2 \pi_3 \pi_4 \cdots (f_{r-1} d_r), \dots, \pi_1 \pi_2 \cdots (f_{r-1} d_{r-1})\} \\ &= \dots \\ &= \{\widehat{\pi}_1 \pi_2 (f_2 d_3 \cdots d_r), \pi_1 \widehat{\pi}_2 (f_2 d_3 \cdots d_r), \pi_1 f_2 (d_2 \widehat{d}_3 d_4 \cdots d_r), \pi_1 f_2 (d_2 \cdots d_{r-1} \widehat{d}_r)\} \\ &= \left\{ f_1 \widehat{d}_1 d_2 \cdots d_r, f_1 d_1 \widehat{d}_2 d_3 \cdots d_r, \dots, f_1 d_1 d_2 \cdots d_{r-1} \widehat{d}_r \right\}. \end{aligned}$$

This completes the proof. □

The result above tells us that any sequence $\alpha = (f_1, f_2, \dots) \in C_{\mathcal{P}}(X)$ can be written as

$$\alpha = (f_1, f_2, f_3, \dots) = (f_1, \{f_1 d_2, f_1 d_1\}, \{f_1 d_2 d_3, f_1 d_1 d_3, f_1 d_1 d_2\}, \dots).$$

However, it does not assert that any map $\mathcal{P}(1) \rightarrow X$ can be extended to a sequence in $C_{\mathcal{P}}(X)$ whose first component is the given map. In fact, that is usually not the case. Below, we give a characterization when \mathcal{P} is a unitary operad in topological spaces.

Let us point out the second observation. We need the following notation. If X is a pointed space, and $f : \mathcal{P}(1) \rightarrow X$ is any map, define for all $r \geq 2$ and $1 \leq i \leq r$ the collection of maps

$$f_r^i := f(d_1 \cdots \widehat{d}_i \cdots d_r) : \mathcal{P}(r) \rightarrow X.$$

The map

$$f_r := \{f_r^1, \dots, f_r^r\} : \mathcal{P}(r) \rightarrow X^{\vee r}$$

is then defined by first applying the diagonal map $\mathcal{P}(r) \rightarrow \mathcal{P}(r)^{\times r}$ and then the product of the f_r^i . The map above usually lands in the product but it restricts to the wedge if, and only if, the map belongs to the underlying space of the comonad.

Proposition 2.6. *Let X be a pointed space. Then the space $C_{\mathcal{P}}(X)$ is homeomorphic to the subspace of $\text{Map}(\mathcal{P}(1), X)$ given by all those maps $f_1 : \mathcal{P}(1) \rightarrow X$ such that for any fixed $r \geq 2$, the maps f_r^i are all the base point except for at most a single index i . In particular, the image of the map*

$$f_r := (f_r^1, \dots, f_r^r) : \mathcal{P}(r) \rightarrow X^{\times r}$$

is contained in the subspace $X^{\vee r} \subseteq X^{\times r}$. Furthermore, each

$$f_r : \mathcal{P}(r) \rightarrow X^{\vee r}$$

is S_r -equivariant. Under this identification, the value $C_{\mathcal{P}}(\phi)$ on a pointed map $\phi : X \rightarrow Y$ is the postcomposition with ϕ :

$$\begin{aligned} C_{\mathcal{P}}(X) &\xrightarrow{C_{\mathcal{P}}(\phi)} C_{\mathcal{P}}(Y) \\ f &\longmapsto C_{\mathcal{P}}(\phi)(g) = \phi \circ f. \end{aligned}$$

Proof. The fact that for a fixed $r \geq 2$, the map f_r^i is the base point for all indexes i except for at most one, implies that the map

$$f_r = (f_r^1, \dots, f_r^r) : \mathcal{P}(r) \rightarrow X^{\times r}$$

has its image in the wedge. Thus, it is correct to write $f_r = \{f_r^1, \dots, f_r^r\}$.

\Rightarrow Let $(f_1, f_2, \dots) \in C_{\mathcal{P}}(X)$, then we want to show that f_r^i is the base point for all i except for at most one. It is a straightforward consequence of Lemma 2.5 that the component f_1 of the sequence gives rise to the family of maps $\{f_r^i\}$ of the statement, with $f_r = \{f_r^1, \dots, f_r^r\}$. So, the implication follows.

\Leftarrow Let $f_1 : \mathcal{P}(1) \rightarrow X$ be a map giving rise to the family of maps $\{f_r^i\}$ and f_r satisfying the hypotheses of the statement. Then we want to show that this indeed belongs to $C_{\mathcal{P}}(X)$. Form the sequence

$$(f_1, f_2, \dots) \in \text{Tot}(\mathcal{P}, X).$$

It suffices to check that for every $r \geq 2$ and $1 \leq i \leq r$, the identity $f_{r-1}d_i = \pi_i f_r$ holds. To do so, we will make use of the following fact and notation for maps induced onto a wedge of pointed spaces: given pointed spaces W, Y, Z and maps $\varphi_1, \dots, \varphi_n : Y \rightarrow Z$ such that $\{\varphi_1, \dots, \varphi_r\} : Y \rightarrow Z^{\vee n}$ is well-defined, then for any map $g : W \rightarrow Y$, we have

$$\{\varphi_1, \dots, \varphi_r\} \circ g = \{\varphi_1 \circ g, \dots, \varphi_r \circ g\} : W \rightarrow Z^{\vee n}.$$

Thus, fix some $r \geq 2$ and $1 \leq i \leq r$. On the one hand,

$$\pi_i f_r = \pi_i \{f_1 \widehat{d}_1 \cdots d_r, \dots, f_1 d_1 \cdots \widehat{d}_r\} = \{f_1 \widehat{d}_1 \cdots d_r, \dots, \cancel{f_1 d_1 \cdots \widehat{d}_i \cdots d_r}, \dots, f_1 d_1 \cdots \widehat{d}_r\}. \quad (2)$$

Above, the strike-through indicates that the i -th component is not part of the sequence. On the other hand,

$$f_{r-1}d_i = \{f_1 \widehat{d}_1 \cdots d_{r-1}, \dots, f_1 d_1 \cdots \widehat{d}_{r-1}\} \circ d_i = \{f_1 \widehat{d}_1 \cdots d_{r-1} \circ d_i, \dots, f_1 d_1 \cdots \widehat{d}_{r-1} \circ d_i\}. \quad (3)$$

It suffices to check that, for any j with $1 \leq j \leq r-1$, the j -th component of the sequence (2) is equal to the j -th component of the sequence (3). This is a straightforward check, taking into account whether $j \leq i$ or $j \geq i$, and using the simplicial identities satisfied by the d_k 's - namely, that $d_i d_j = d_{j-1} d_i$ for $i < j$. \square

Proposition 2.6 above is very useful, as we will see in Section 3. Remark that this result identifies the space $C_{\mathcal{P}}(X)$ as the subspace of $\text{Map}(\mathcal{P}(1), X)$ formed by those maps satisfying an extra property. Bear in mind that, under this identification, the evaluation of $C_{\mathcal{P}}$ on a morphism $\phi : X \rightarrow Y$ corresponds to the postcomposition with ϕ .

Before going on, we introduce some notation that will be useful later.

Notation 2.7. We will occasionally use the following notation for the composition of the restriction operators:

$$D_i = d_1 \cdots \widehat{d_i} \cdots d_r : \mathcal{P}(r) \rightarrow \mathcal{P}(1).$$

These choices will simplify the formulae in what follows, making our results more readable. Remark also that, for any operation $\theta \in \mathcal{P}(r)$, the resulting operation $D_i(\theta) \in \mathcal{P}(1)$ is exactly

$$D_i(\theta) = \gamma(\theta; *, \dots, *, \underbrace{\text{id}_{\mathcal{P}}}_i, *, \dots, *),$$

where γ is the composition map of \mathcal{P} , the element $\text{id}_{\mathcal{P}} \in \mathcal{P}(1)$ is the operadic unit, and $*$ $\in \mathcal{P}(0)$ is the unitary operation. In other words, $D_i(\theta)$ retains the unary operation determined by the i -th input of θ . For example, if $\mathcal{P} = \mathcal{C}_n$ is the little n -cubes operad and $\theta = (c_1, \dots, c_r) \in \mathcal{C}_n(r)$ is a configuration of r little n -cubes, then $D_i(\theta) = c_i$ is the i -th little n -cube of the configuration, seen as an element of $\mathcal{C}_n(1)$.

Let us finally equip the endofunctor $C_{\mathcal{P}}$ with natural transformations $\varepsilon : C_{\mathcal{P}} \rightarrow \text{id}_{\text{Top}_*}$ and $\Delta : C_{\mathcal{P}} \rightarrow C_{\mathcal{P}} \circ C_{\mathcal{P}}$ that makes it a comonad. From now on, to lighten notation, we denote $C = C_{\mathcal{P}}$, assuming that the operad \mathcal{P} is understood.

Definition 2.8. Let $C = C_{\mathcal{P}} : \text{Top}_* \rightarrow \text{Top}_*$ be the endofunctor of Definition 2.3. Define the natural transformations

$$\varepsilon : C \rightarrow \text{id}_{\text{Top}_*} \quad \text{and} \quad \Delta : C \rightarrow C \circ C$$

level-wise as follows.

- The counit structure map is defined by

$$\begin{aligned} \varepsilon_X : C(X) &\longrightarrow X \\ \alpha = (f_1, f_2, \dots) &\longmapsto \varepsilon_X(\alpha) := f_1(\text{id}_{\mathcal{P}}). \end{aligned}$$

Here, $\text{id}_{\mathcal{P}} \in \mathcal{P}(1)$ is the operadic unit.

- We next define the coproduct structure map

$$\Delta_X : C(X) \rightarrow C(C(X)).$$

To do so, let $\alpha = (f_1, f_2, \dots) \in C(X)$. Then $\Delta_X(\alpha) = (\bar{f}_1, \bar{f}_2, \dots)$ is an element of the space $C(Z)$, with $Z = C(X)$. Thus, it is formed by a sequence of maps

$$\bar{f}_r : \mathcal{P}(r) \rightarrow C(X)^{\vee r}$$

satisfying the compatibility conditions

$$\pi_i \bar{f}_r = \bar{f}_{r-1} d_i, \quad \text{for } r \geq 2 \text{ and } 1 \leq i \leq r.$$

Because of Lemma 2.5 we only need to define the arity one component $\bar{f}_1 : \mathcal{P}(1) \rightarrow C(X)$, and extend it as a sequence by the formula

$$\bar{f}_r = \{\bar{f}_1 D_1, \dots, \bar{f}_1 D_r\},$$

where $D_i = d_1 \cdots \widehat{d_i} \cdots d_r$.

For the definition above to be complete and correct, we require two steps:

Step 1. Define $\bar{f}_1 : \mathcal{P}(1) \rightarrow C(X)$.

Step 2. Check that $\bar{f}_1 D_i = *$ is the base point for all indexes i , except for at most a single one.

Where Step 2 follows from Proposition 2.6.

Step 1 Denote by γ the operadic composition map of \mathcal{P} . Define $\bar{f}_1 : \mathcal{P}(1) \rightarrow C(X)$ by

$$\bar{f}_1(\mu) = (g_1^\mu, g_2^\mu, \dots) \quad \text{for all } \mu \in \mathcal{P}(1),$$

where the maps $g_r^\mu : \mathcal{P}(r) \rightarrow X^{\vee r}$ in the sequence are as follows. The first one is:

$$g_1^\mu : \mathcal{P}(1) \rightarrow X, \quad g_1^\mu(\theta) := f_1(\gamma(\mu; \theta)),$$

for $\theta \in \mathcal{P}(1)$. That is, $g_1^\mu = f_1(\gamma(\mu; -))$. The rest of the maps g_r^μ are recursively defined by the formula

$$g_r^\mu : \mathcal{P}(r) \rightarrow X^{\vee r}$$

$$g_r^\mu(\theta) = \{g_1^\mu D_1(\theta), \dots, g_1^\mu D_r(\theta)\} = \{g_1^\mu(\gamma(\theta; \text{id}_{\mathcal{P}}, *, \dots, *)), \dots, g_1^\mu(\gamma(\theta; *, \dots, *, \text{id}_{\mathcal{P}}))\}$$

For $\theta \in \mathcal{P}(r)$. We will check below that the image of g_r^μ is indeed contained in the wedge $X^{\vee r}$. The family of maps g_r^μ can be explicitly described. Let us first describe $g_2^\mu : \mathcal{P}(2) \rightarrow X \vee X$. Using, in the order given, the recursive definition of g_2^μ , the definitions of D_i and of g_1^μ , and the associativity of γ , we can write

$$g_2^\mu(\theta) = \{g_1^\mu D_1(\theta), g_1^\mu D_2(\theta)\} = \{g_1^\mu(\gamma(\theta; \text{id}_{\mathcal{P}}, *)), g_1^\mu(\gamma(\theta; *, \text{id}_{\mathcal{P}}))\}$$

$$= \{f_1(\gamma(\mu; \gamma(\theta; \text{id}_{\mathcal{P}}, *))), f_1(\gamma(\mu; \gamma(\theta; *, \text{id}_{\mathcal{P}})))\}$$

$$= \{f_1(\gamma(\gamma(\mu; \theta); \text{id}_{\mathcal{P}}, *)), f_1(\gamma(\gamma(\mu; \theta); *, \text{id}_{\mathcal{P}}))\}.$$

Thus,

$$g_2^\mu = \{f_1 D_1(\gamma(\mu; -)), f_1 D_2(\gamma(\mu; -))\}.$$

Next we need to show that \bar{f}_2 has its image in the wedge $C(X) \vee C(X)$. Since $\alpha = (f_1, f_2, \dots)$ is an element of $C(X)$, it follows that all $f_1 D_i = *$ are the base point, except for at most a single index i . Therefore, indeed, g_2^μ has its image in the wedge. Furthermore, so defined, g_2^μ is S_2 -equivariant. In general, exactly the same steps as for the $r = 2$ case show that the explicit formula for g_r^μ is

$$g_r^\mu(\theta) = \{f_1(\gamma(\gamma(\mu; \theta); \text{id}_{\mathcal{P}}, *, \dots, *)), \dots, f_1(\gamma(\gamma(\mu; \theta); *, \dots, *, \underbrace{\text{id}_{\mathcal{P}}, *, \dots, *}_j)), \dots, f_1(\gamma(\gamma(\mu; \theta); *, \dots, *, \text{id}_{\mathcal{P}}))\}.$$

Above, the j -th component in the wedge has the identity $\text{id}_{\mathcal{P}} \in \mathcal{P}(1)$ at the j -th component.

Step 2 Let us check that $\bar{f}_1 D_i = *$ is the base point for all indexes i except for at most a single one. Recall that for fixed i , the map

$$\bar{f}_1 D_i : \mathcal{P}(r) \rightarrow C(X)$$

evaluated at some operation $\mu \in \mathcal{P}(r)$ is the previously defined sequence

$$\bar{f}_1 D_i(\mu) = (g_1^{D_i(\mu)}, g_2^{D_i(\mu)}, \dots).$$

First, observe that for any $\theta \in \mathcal{P}(1)$ and index i , with $1 \leq i \leq r$, we have

$$\gamma(D_i(\mu); \theta) = D_i(\gamma(\mu; \text{id}_{\mathcal{P}}, \dots, \theta, \dots, \text{id}_{\mathcal{P}})).$$

Therefore, the first component of the sequence $\bar{f}_1 D_1(\mu)$ can be written as

$$g_1^{D_1(\mu)} = f_1(D_1(\gamma(\mu; -))).$$

Since the sequence (f_1, f_2, \dots) is an element of the space $C(X)$, it follows that $f_1 D_i$ is the base point for all i except for at most one, and therefore, the same holds for the family $\{g_1^{D_1(\mu)}, \dots, g_1^{D_i(\mu)}, \dots\}$, which implies that $\bar{f}_1 D_i = *$ is the base point for almost all i .

Remark 2.9. In Proposition 2.6, we gave an identification of $C(X)$ as a certain subspace of $\text{Map}(\mathcal{P}(1), X)$. From this point of view, the comultiplication $\Delta = \Delta_X : C(X) \rightarrow CC(X)$ is given as follows. Let $f \in C(X) \subseteq \text{Map}(\mathcal{P}(1), X)$. Then, $\Delta(f)$ is given by:

$$\begin{aligned} \Delta(f) : \mathcal{P}(1) &\longrightarrow C(X) \\ c &\longmapsto \Delta(f)(c) : \mathcal{P}(1) \longrightarrow X \\ d &\longmapsto f(\gamma(c; d)). \end{aligned}$$

That is, given $f \in C(X)$, and $c, d \in \mathcal{P}(1)$, the map $\Delta(f)$ is explicitly given by

$$\Delta(f)(c)(d) = f(\gamma(c; d)).$$

Proposition 2.10. *With the notation before, the triple (C, ε, Δ) is a comonad in Top_* .*

Proof. We prove the coassociativity and counit axioms object-wise. For a pointed space X , these axioms are described by the following diagrams:

$$\begin{array}{ccc} C(X) & \xrightarrow{\Delta_X} & C(C(X)) \\ \Delta_X \downarrow & & \downarrow \Delta_{C(X)} \\ C(C(X)) & \xrightarrow{C(\Delta_X)} & C(C(C(X))) \end{array} \qquad \begin{array}{ccc} C(X) & \xrightarrow{\Delta_X} & C(C(X)) \\ \Delta_X \downarrow & \searrow \text{id} & \downarrow \varepsilon_{C(X)} \\ C(C(X)) & \xrightarrow{C(\varepsilon_X)} & C(X), \end{array}$$

where the left diagram gives the coassociativity condition and the right diagram the counit condition.

Let $\alpha = (f_1, f_2, \dots) \in C(X)$ then we will check that it satisfies the diagrams.

▷ Coassociativity. We must check that

$$(C(\Delta_X) \circ \Delta_X)(\alpha) = (\Delta_{C(X)} \circ \Delta_X)(\alpha). \quad (4)$$

We analyze $\Delta_X(\alpha)$ first, given that it appears on both sides of the equation above, and then look at each of the sides of the equation above. By Lemma 2.5, it suffices to check that the arity one term of the sequences arising from both sides of Equation (4) agree. This will ultimately follow from the associativity of the operadic composition γ of the operad \mathcal{P} .

- Description of $\Delta_X(\alpha)$.

$$\begin{aligned} \Delta_X : C(X) &\longrightarrow C(C(X)) \\ \alpha &\longmapsto \Delta_X(\alpha) = (\bar{f}_1, \bar{f}_2, \dots) \end{aligned}$$

By Lemma 2.5, the sequence $(\bar{f}_1, \bar{f}_2, \dots)$ is determined by its first component \bar{f}_1 . It is given as follows:

$$\begin{aligned} \bar{f}_1 : \mathcal{P}(1) &\longrightarrow C(X) & g_1^\mu : \mathcal{P}(1) &\longrightarrow X \\ \mu &\longmapsto \bar{f}_1(\mu) = (g_1^\mu, g_2^\mu, \dots) & \theta &\longmapsto g_1^\mu(\theta) = f_1(\gamma(\mu; \theta)) \end{aligned}$$

- The left hand side of Equation (4) reads:

$$(C(\Delta_X) \circ \Delta_X)(\alpha) = C(\Delta_X)(\Delta_X(\alpha)) = C(\bar{f}_1, \bar{f}_2, \dots) = (\Delta_X \circ \bar{f}_1, \{\Delta_X, \Delta_X\} \circ \bar{f}_2, \dots).$$

Here, given maps $\varphi_i : X_i \rightarrow Y$, we are denoting the induced map by $\{\varphi_1, \dots, \varphi_n\} : X_1 \vee \dots \vee X_n \rightarrow Y$. We have:

$$\begin{aligned} \Delta_X \circ \bar{f}_1 : \mathcal{P}(1) &\longrightarrow C(X) \longrightarrow C(C(X)) \\ \mu &\longmapsto \bar{f}_1(\mu) = (g_1^\mu, g_2^\mu, \dots) \mapsto (\bar{g}_1^\mu, \bar{g}_2^\mu, \dots) \end{aligned}$$

The map \bar{g}_1^μ above is determined by:

$$\begin{aligned} \bar{g}_1^\mu : \mathcal{P}(1) &\longrightarrow C(X) & h_1 : \mathcal{P}(1) &\longrightarrow X \\ \theta &\longmapsto \bar{g}_1^\mu(\theta) := (h_1, h_2, \dots) & \lambda &\longmapsto h_1(\lambda) = g_1^\mu(\gamma(\theta; \lambda)) \end{aligned}$$

• The right hand side of Equation (4) reads:

$$(\Delta_{C(X)} \circ \Delta_X)(\alpha) = \Delta_{C(X)}(\Delta_X(\alpha)) = \Delta_{C(X)}(\bar{f}_1, \bar{f}_2, \dots) = (\bar{f}_1 \bar{f}_2, \dots).$$

Here,

$$\begin{aligned} \bar{f}_1 : \mathcal{P}(1) &\longrightarrow C(C(X)) & l_1^\mu : \mathcal{P}(1) &\longrightarrow C(X) \\ \mu &\longmapsto \bar{f}_1(\mu) = (l_1^\mu, l_2^\mu, \dots) & \theta &\longmapsto l_1^\mu(\theta) = \bar{f}_1(\gamma(\mu; \theta)) \end{aligned}$$

As mentioned, to check the coassociativity condition it suffices to check that $\bar{f}_1 = \Delta_X \circ \bar{f}_1$. By Lemma 2.5 again, our problem reduces to checking that $l_1^\mu = \bar{g}_1^\mu$. And once more, using the same lemma, this reduces to checking that the sequence $\bar{f}_1(\gamma(\mu; \theta))$ has first term equal to $h_1(\lambda)$ described before. The first term is explicitly given by

$$f_1(\gamma(\gamma(\mu; \theta); \lambda)). \quad (5)$$

On the right hand side, the first nested term of $g_1^\mu(\gamma(\theta; \lambda))$ is explicitly given by

$$f_1(\gamma(\mu; \gamma(\theta; \lambda))). \quad (6)$$

By the associativity of the operadic composition γ , the term inside f_1 in Equation (5) is the same as the term inside f_1 in Equation (6). Thus, these two maps are equal. This proves the coassociativity of the comultiplication.

▷ Counit. We must check two identities:

1. $(C(\varepsilon_X) \circ \Delta_X)(\alpha) = \alpha$.

Indeed,

$$(C(\varepsilon_X) \circ \Delta_X)(\alpha) = C(\varepsilon_X)(\Delta_X(\alpha)) = C(\varepsilon_X)(\bar{f}_1, \bar{f}_2, \dots) = (\varepsilon_X \circ \bar{f}_1, \{\varepsilon_X, \varepsilon_X\} \circ \bar{f}_2, \dots).$$

Let us check that $\varepsilon_X \circ \bar{f}_1 = f_1$ as maps $\mathcal{P}(1) \rightarrow X$. If $\mu \in \mathcal{P}(1)$, then:

$$(\varepsilon_X \circ \bar{f}_1)(\mu) = \varepsilon_X(\bar{f}_1(\mu)) = \varepsilon_X(g_1^\mu, g_2^\mu, \dots) = g_1^\mu(\text{id}_\emptyset) = f_1(\gamma(\mu; \text{id}_\emptyset)) = f_1(\mu).$$

2. $(\varepsilon_{C(X)} \circ \Delta_X)(\alpha) = \alpha$.

In this case,

$$(\varepsilon_{C(X)} \circ \Delta_X)(\alpha) = \varepsilon_{C(X)}(\Delta_X(\alpha)) = \varepsilon_{C(X)}(\bar{f}_1, \bar{f}_2, \dots) = \bar{f}_1(\text{id}_\emptyset).$$

We must check that $\bar{f}_1(\text{id}_\emptyset) = f_1$ as maps $\mathcal{P}(1) \rightarrow X$. Indeed, if $\theta \in \mathcal{P}(1)$, then

$$\bar{f}_1(\text{id})(\theta) = g_1^1(\theta) = f_1(\gamma(\text{id}; \theta)) = f_1(\theta).$$

The proposition is therefore proven. \square

For the sake of completeness, we recall here the well-known fact that comonads explicitly create the cofree coalgebras of the underlying category (see for instance [18, Corollary 5.4.23]).

Theorem 2.11. *Let X be a pointed space. Then, $C(X)$ is the cofree C -coalgebra on X . That is, for any C -coalgebra A in pointed spaces, there is a natural bijection*

$$\text{Hom}_{\text{Top}_*}(A, X) \cong \text{Hom}_{C\text{-Coalg}}(A, C(X)).$$

In Section 2.3 we will give a few explicit examples of how this comonad looks like in the case of the associative operad and the little n -cubes operad.

2.2 Alternative definitions of coalgebra over an operad

Let \mathcal{P} be a unitary operad in Top . The comonad $C = C_{\mathcal{P}}$ constructed in Section 2.1 naturally gives rise to a category of coalgebras in Top_* . The objects in this category are pointed spaces X together with a coalgebra structure map $c : X \rightarrow C(X)$. We call the objects of this category \mathcal{P} -coalgebras. There is an equivalent way of defining a \mathcal{P} -coalgebra by using the coendomorphism operad that does not require the explicit construction of the comonad C . In this alternative definition, the objects are pointed spaces X together with an operad map $\mathcal{P} \rightarrow \text{CoEnd}_X$, where CoEnd_X is the coendomorphism operad associated to the pointed space X . In this section, we present the alternative definition of \mathcal{P} -coalgebra in terms of coendomorphisms, and show that for unitary operads this is equivalent to the comonadic definition. The definition of \mathcal{P} -coalgebras in terms of the coendomorphism operad is much more intuitive and defines the coalgebra structure in terms of explicit cooperations, i.e. maps $X \rightarrow X^{\vee r}$. On the other hand, the comonad definition has the benefit that it will be much easier to compare it to the $\Sigma^n \Omega^n$ -comonad, making it more suitable for proving the approximation and recognition theorems later in this paper.

We start by defining the category of \mathcal{P} -coalgebras using the comonad $C_{\mathcal{P}}$.

Definition 2.12. Let \mathcal{P} be a unitary operad in Top . The category $C_{\mathcal{P}}\text{-Coalg}$ of coalgebras in Top_* associated to the comonad $C_{\mathcal{P}}$ is called the *category of (comonadic) \mathcal{P} -coalgebras*. The objects in this category are triples (X, c, ϵ) , where $c : X \rightarrow C(X)$, called the coalgebra structure map of X and $\epsilon : C_{\mathcal{P}}(X) \rightarrow X$ the counit, are maps of pointed spaces satisfying counit and coassociativity axioms:

$$\begin{array}{ccc} X & \xrightarrow{c} & C(X) \\ & \searrow \text{id} & \downarrow \epsilon_X \\ & & X \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{c} & C(X) \\ \downarrow c & & \downarrow C(c) \\ C(X) & \xrightarrow{\Delta_X} & C(C(X)) \end{array}$$

The morphisms between these objects are pointed maps $X \rightarrow Y$ that make the obvious square commute.

Before giving the alternative definition of \mathcal{P} -coalgebras, we must give the definition of the coendomorphism operad associated with a pointed space.

Definition 2.13. Let X be a pointed space. The *coendomorphism operad* CoEnd_X in pointed topological spaces with the wedge sum, has arity r component

$$\text{CoEnd}_X(r) := \text{Map}_*(X, X^{\vee r}),$$

the based mapping space from X to the r -fold wedge sum of X with itself. For $r = 0$, set $\text{CoEnd}_X(0) = *$. The operadic composition maps are defined as

$$\gamma : \text{Map}_*(X, X^{\vee n}) \times \text{Map}_*(X, X^{\vee m_1}) \times \cdots \times \text{Map}_*(X, X^{\vee m_n}) \rightarrow \text{Map}_*(X, X^{\vee \sum m_i}),$$

$$\gamma(f, g_1, \dots, g_n) := (g_1 \vee \dots \vee g_n) \circ f.$$

The symmetric group action on $\text{CoEnd}_X(r)$ permutes the wedge factors in the output of a map $f : X \rightarrow X^{\vee r}$. The unit $\eta : I \rightarrow \text{CoEnd}_X$ is determined by mapping the base point in $I(1) = \{*\}$ to the identity map in $\text{CoEnd}_X(1) = \text{Map}_*(X, X)$.

It is straightforward to check that CoEnd_X is an operad in pointed spaces and we leave this to the reader. The coendomorphism operad gives an alternative definition of \mathcal{P} -coalgebras.

Definition 2.14. Let \mathcal{P} be a not necessarily unitary operad in Top . A \mathcal{P} -coalgebra is a pointed topological space X together with an operad map $\mathcal{P} \rightarrow \text{CoEnd}_X$. A morphism of \mathcal{P} -coalgebras is a pointed map $f : X \rightarrow Y$ such that the following diagram commutes for all n :

$$\begin{array}{ccc} \mathcal{P}(n) \times X & \xrightarrow{\Delta_n} & X \vee \dots \vee X \\ \downarrow \text{id} \times f & & \downarrow f \vee \dots \vee f \\ \mathcal{P}(n) \times Y & \xrightarrow{\Delta'_n} & Y \vee \dots \vee Y \end{array}$$

Here, Δ_n and Δ'_n are the coalgebra structure maps of X and Y , respectively, which are written arity-wise by using the mapping space-product adjunctions

$$\text{Map}(\mathcal{P}(n) \times Z, Z^{\vee r}) \cong \text{Map}(\mathcal{P}(n), \text{Map}(Z, Z^{\vee r})),$$

where Z is any pointed topological space. Note that since we are mixing pointed and unpointed spaces we are viewing $\text{Map}_*(X, X^{\vee r})$ as a subspace of the unpointed mapping space so that we are able to use the \times – Map-adjunction.

Remark 2.15. Note that this definition of a \mathcal{P} -coalgebra is more general than the one using the comonad from the previous section. In particular, we do not require the operad to be unitary so these coalgebras are defined for a larger class of operads.

By using the mapping space-product adjunction for S_r -spaces, we see that there are several equivalent ways of unpacking the definition of a coendomorphism \mathcal{P} -coalgebra. The definition of a coalgebra as a sequence of coproduct maps

$$\Delta_r : \mathcal{P}(r) \times X \rightarrow X^{\vee r}$$

is also equivalent to a sequence of maps

$$\Delta'_r : X \rightarrow \text{Map}(\mathcal{P}(r), X^{\vee r})^{S_r},$$

satisfying certain conditions. Here $\text{Map}(\mathcal{P}(r), X^{\vee r})^{S_r}$ is the subspace of S_r -invariant maps.

Versions of the coendomorphism operad have been explicitly used before in for example [1] in the category of chain complexes. The notion of coalgebra in the category of pointed spaces with the wedge product has also appeared before in [14], however they do not use the coendomorphism operad or construct an explicit comonad.

The following result asserts that for unitary operads both definitions of \mathcal{P} -coalgebras are equivalent.

Proposition 2.16. *Let \mathcal{P} be a unitary operad in Top . Then the definition of a \mathcal{P} -coalgebra via the comonad from Section 2.1 is equivalent to definition of a \mathcal{P} -coalgebra via the coendomorphism operad from Definition 2.14.*

Proof. Indeed, we can identify operad maps $\rho : \mathcal{P} \rightarrow \text{CoEnd}_X$ with coalgebra structure maps $c : X \rightarrow C(X)$ by the following rule: for any $\theta \in \mathcal{P}(r)$ and $x \in X$,

$$\rho_r(\theta)(x) = f_r^x(\theta).$$

Here, ρ_r is the arity r component of ρ , and f_r^x is the r th-term of the sequence $c(x) = (f_1^x, f_2^x, \dots)$. The formula above turns a coendomorphism coalgebra into a comonad coalgebra and vice versa. It is further straightforward to check that this definition commutes with morphisms. \square

From now on, we always use the shorter notation $\mathcal{P}\text{-Coalg}$ for the category of \mathcal{P} -coalgebras.

Remark 2.17. The \mathcal{P} -coalgebras defined in this section are *canonically counital*. That is, they come equipped with the unique map $\varepsilon : X \rightarrow *$, and this map behaves as a counit with respect to the rest of the structure. This explains the compatibility conditions of Equation (1). Indeed, if X is a \mathcal{P} -coalgebras, then the following diagram commutes:

$$\begin{array}{ccccc} P(n) \times X & \xrightarrow{\Delta_r} & X^{\vee r} & \xrightarrow{\text{id} \vee \dots \vee \varepsilon \vee \dots \vee \text{id}} & X^{\vee(r-1)} \\ d_i \times \text{id} \downarrow & & & & \downarrow \text{id} \\ P(n-1) \times X & \xrightarrow{\Delta_{r-1}} & & & X^{\vee(r-1)} \end{array}$$

In the diagram above, Δ_r is the arity r coalgebra structure map of X , and $\text{id} \vee \dots \vee \varepsilon \vee \dots \vee \text{id}$ is precisely π_i . Note that the counit of a coalgebra is *unique*, i.e. since $*$ is the terminal object there is only one possible map from X to $X^{\vee 0} = *$. This is in high contrast with the (unpointed) algebra case in which there are many possibilities for a unit, i.e. there are many maps from $X^{\times 0} = *$ to X since $*$ is not the initial object in unpointed spaces.

2.3 The comonad associated to the little n -cubes operad

In this section, we take a closer look at the comonad constructed in Section 2.1, in the particular case of $\mathcal{P} = \mathcal{C}_n$ being the little n -cubes operad. Although we assume familiarity with this operad, there are a number of small variations in the literature. We give a brief summary below in order to carefully fix our conventions and establish the notation. We will consistently denote by C_n the comonad in pointed spaces associated to the little n -cubes operad \mathcal{C}_n . In Proposition 2.18, we give a geometric characterization of $C_n(X)$ as an explicit subspace of $\text{Map}(\mathcal{C}_n(1), X)$.

Denote by I^n the unit n -cube of \mathbb{R}^n and by J^n its interior. A *little n -cube* is a rectilinear embedding $h: I^n \rightarrow I^n$ of the form $h = h_1 \times \cdots \times h_n$, where each component h_i is given by

$$h_i(t) = (y_i - x_i)t + x_i, \quad \text{for } 0 \leq x_i < y_i \leq 1. \quad (7)$$

The image $h(J^n)$ of the interior of I^n under a rectilinear embedding h will be denoted \mathring{h} . So although the operad is called the little n -cubes operad it is technically the little n -rectangle operad.

For each $n \geq 1$, the *little n -cubes operad* \mathcal{C}_n is an operad in Top . It was introduced independently by Boardman–Vogt and May [5, 16] for studying iterated loop spaces. A comprehensive modern reference is [9]. We consider the unitary version of this operad, i.e., $\mathcal{C}_n(0) = *$ is the one-point space. For each $r \geq 1$, the arity r component $\mathcal{C}_n(r)$ of \mathcal{C}_n is the subspace of the mapping space

$$\mathcal{C}_n(r) \subseteq \text{Map}\left(\prod_r I^n, I^n\right)$$

given by those rectilinear embeddings for which the images of the interiors of different cubes are pairwise disjoint. That is,

$$\mathcal{C}_n(r) = \{(c_1, \dots, c_r) \mid \text{each } c_i \text{ is a little } n\text{-cube, and } \mathring{c}_i \cap \mathring{c}_j = \emptyset \text{ for all } i \neq j\}.$$

The symmetric group S_r acts on a configuration $c = (c_1, \dots, c_r)$ of little cubes by permuting its components, $(c_1, \dots, c_r) \cdot \sigma = (c_{\sigma^{-1}(1)}, \dots, c_{\sigma^{-1}(r)})$. The operadic unit $1 \in \mathcal{C}_n(1)$ is the identity map $I^n \rightarrow I^n$, and the partial composition products are explicitly given by

$$(c_1, \dots, c_r) \circ_i (d_1, \dots, d_s) = (c_1, \dots, c_{i-1}, c_i \circ d_1, \dots, c_i \circ d_s, c_{i+1}, \dots, c_r).$$

That is: we re-scale and insert the little n -cubes d_1, \dots, d_s in place of the little n -cube c_i , which is removed, and then relabel accordingly.

Recall from Proposition 2.6 that the underlying space of the comonad $C_{\mathcal{P}}(X)$ associated to a unitary topological operad \mathcal{P} and a pointed space X is characterized as a certain subspace of $\text{Map}(\mathcal{P}(1), X)$. In the particular case of the comonad C_n associated to the little n -cubes operad, there is a very geometrical characterization. We need the following preliminary notation. First, recall that

$$D_i = d_1 \cdots \widehat{d}_i \cdots d_r : \mathcal{C}_n(r) \rightarrow \mathcal{C}_n(1)$$

denotes the composition of the restriction operators omitting the i -th term, which evaluated at a configuration $\theta = (c_1, \dots, c_n) \in \mathcal{C}_n(r)$, recovers the i -th little n -cube c_i . Now, let X be a pointed space. Given $f: \mathcal{C}_n(1) \rightarrow X$ any map, define for all $r \geq 2$ and $1 \leq i \leq r$ the collection of maps

$$f_r^i := f \circ D_i : \mathcal{P}(r) \rightarrow X \quad \text{and} \quad f_r := (f_r^1, \dots, f_r^r) : \mathcal{P}(r) \rightarrow X^{\times r}. \quad (8)$$

The mentioned characterization is the following.

Proposition 2.18. *Let X be a pointed space, and C_n the comonad associated to the little n -cubes operad. Then a map $f: \mathcal{C}_n(1) \rightarrow X$ belongs to $C_n(X)$ if, and only if, f satisfies the following property:*

$$(D) \text{ If } c_1, c_2 \in \mathcal{C}_n(1) \text{ are little } n\text{-cubes such that } \mathring{c}_1 \cap \mathring{c}_2 = \emptyset, \text{ then } f(c_1) = * \text{ or } f(c_2) = *.$$

That is, taking $f = f_1$, each map f_r in (8) has its image in the r -fold wedge $X^{\vee r}$, it is S_r -equivariant, and the compatibility conditions $f_{r-1} d_i = \pi_i f_r$ are satisfied for all $r \geq 2$ and $1 \leq i \leq r$.

Proof. Assume $f = f_1 : \mathcal{C}_n(1) \rightarrow X$ satisfies property (D). Fix an arbitrary $r \geq 2$, and some $1 \leq i \leq r$. Define $f_r : \mathcal{C}_n(r) \rightarrow X^{\times r}$ by

$$f_r = (f_1 D_1, \dots, f_1 D_r).$$

Let us check that f_r has its image in the wedge. Indeed, for any $\theta = (c_1, \dots, c_r) \in \mathcal{C}_n(r)$, it follows from the definition of the space $\mathcal{C}_n(r)$ that $\hat{c}_k \cap \hat{c}_j = \emptyset$ for all $j \neq k$. Furthermore, for each index j between 1 and r , we can write

$$c_j = (d_1 \circ \dots \circ \widehat{d_i} \circ \dots \circ d_r)(\theta) = D_i(\theta).$$

Therefore, condition (D) applied to each pair (j, k) with $j \neq k$ implies that at most a single component $f_1(c_j)$ is not the basepoint. Said differently: f_r has its image in the wedge. The map f_r is S_r -equivariant. Indeed, for any $\sigma \in S_r$, one has

$$f_r \cdot \sigma = \{f_1 D_1, \dots, f_1 D_r\} \cdot \sigma = \{f_1 D_1 \cdot \sigma, \dots, f_1 D_r \cdot \sigma\} = \{f_1 D_{\sigma(1)}, \dots, f_1 D_{\sigma(r)}\} = \sigma \cdot \{f_1 D_1, \dots, f_1 D_r\}.$$

Since σ permutes the coordinates of the wedge factors, the claim is proven.

For the converse, assume that $(f_1, f_2, \dots) \in C_n(X)$, and that $c_1, c_2 \in \mathcal{C}_n(1)$ are little n -cubes such that $\hat{c}_1 \cap \hat{c}_2 = \emptyset$. This is precisely the condition needed to ensure that (c_1, c_2) is an element of $C_n(2)$. Consider $f_2(c_1, c_2) \in X \vee X$. From the comonadic compatibility conditions, one has

$$f_1(c_1) = \pi_1 f_2(c_1, c_2)$$

$$f_1(c_2) = \pi_2 f_2(c_1, c_2)$$

and therefore one of $f_1(c_1)$ or $f_1(c_2)$ must be the base point. Therefore f_1 satisfies property (D). \square

In the next remark, we point out the obvious fact that non-trivial strictly coassociative coalgebras do not exist in pointed spaces.

Remark 2.19. Recall that a pointed space X is a co-H-space if it comes equipped with a map $c : X \rightarrow X \vee X$ that is a factorization up to homotopy of the identity map $X \rightarrow X$:

$$\begin{array}{ccc} X & \xrightarrow{c} & X \vee X \\ & \searrow \text{id} & \downarrow q_i \\ & & X \end{array}$$

That is, $q_1 c \simeq \text{id} \simeq q_2 c$, where $q_i : X \vee X \rightarrow X$ is the projection onto the i th factor of the wedge. If we try to strictify this diagram, considering $q_1 c = \text{id} = q_2 c$, then for any $x \in X$ we would have the following situation. The coproduct $c(x)$ is either a point in the first wedge factor, $(x_1, *)$, or it is a point in the second wedge factor, $(*, x_2)$. Without loss of generality, we may assume that it is of the form $c(x) = (x_1, *)$, we would then have

$$q_2 c(x) = q_2(x_1, *) = *.$$

If X has more than one point, we will not have $q_2 c(x) = x$ for $x \neq *$. Thus, the unique strictly coassociative counital coalgebra is the one point space. This is a significant contrast with the algebra case, where for example, the James construction [13] gives a strictly associative monoid in pointed spaces modelling $\Sigma\Omega X$. The classical Moore loop space is another important example of a pointed space endowed with a strictly associative product. We conclude that there is no possible "rectification" of a counital homotopy coassociative-coalgebra into a counital strictly coassociative coalgebra. Aside from the elementary proof given here, the non-existence of strictly coassociative coalgebras in Top_* will also follow from Proposition 2.21, a more general statement asserting that reduced operads produce trivial comonads, leaving no place for non-trivial counital coassociative coalgebras. Remark that it is the counit that is causing all the problems in the discussion above. Since there are non-trivial non-counital strictly coassociative coalgebras, the argument above does not apply. It is therefore not known whether strictly coassociative rectifications exist in the case of non-counital coalgebras, but this is beyond the scope of this paper.

The particular instance of Theorem 2.11 in this case gives the following important observation.

Theorem 2.20. *Let X be a pointed space. Then, $C_n(X)$ is the cofree C_n -coalgebra on X . That is, for any C_n -coalgebra A , there is a natural bijection*

$$\text{Hom}_{\text{Top}_*}(A, X) \cong \text{Hom}_{C_n\text{-Coalg}}(A, C(X)).$$

2.3.1 Reduced topological operads and weak equivalences

In this section, we prove that for *reduced* unitary topological operads (i.e. $\mathcal{P}(1) = \{*\}$), the comonad $C_{\mathcal{P}}$ is always the trivial one-point comonad. Therefore, the associated category of \mathcal{P} -coalgebras is trivial (Proposition 2.21). This is a striking difference with the construction of C_n in the case of the little n -cubes operad \mathcal{C}_n , whose category of coalgebras is rich and interesting. As a consequence, we readily see that the comonad construction does not respect weak equivalences in the Berger–Moerdijk model structure [2] on topological operads. That is, if $\mathcal{P} \rightarrow \mathcal{Q}$ is a morphism of unitary operads in Top_* which is a weak equivalence in each arity, it does not necessarily follow that the induced map $C_{\mathcal{P}}(X) \rightarrow C_{\mathcal{Q}}(X)$ is a weak equivalence for each pointed space X . For example, the associative operad Ass is reduced, producing a trivial category of coalgebras, but there is a well-known weak equivalence of operads $\mathcal{C}_1 \rightarrow \text{Ass}$. Said differently, a weak equivalence of unitary operads does not imply an equivalence of categories of coalgebras (even of up to homotopy algebras)

Proposition 2.21. *If \mathcal{P} is a reduced unitary topological operad, then $C_{\mathcal{P}}$ is the trivial comonad. That is, $C_{\mathcal{P}}(X)$ is the one-point space for all pointed spaces X . In particular, the comonads C_{Ass} and C_{Com} produced respectively from the associative and commutative operads are trivial.*

Proof. Let \mathcal{P} be an operad as in the statement. Fix a pointed space X , and consider an arbitrary sequence $\alpha = (f_1, f_2, \dots) \in C_{\mathcal{P}}(X)$. Then,

$$f_1 : \mathcal{P}(1) \rightarrow X$$

specifies some point $f_1(*) = x_0 \in X$. Recall (Lemma 2.5) that the higher terms f_r in the sequence α are determined by the recursive formula

$$f_r = \{f_1 D_1, \dots, f_1 D_r\}. \quad (9)$$

In particular, for any $\theta \in \mathcal{P}(2)$,

$$f_2(\theta) = \{f_1 d_2(\theta), f_1 d_1(\theta)\} = \{x_0, x_0\}.$$

Therefore, for f_2 to be well-defined (i.e., having its image in the wedge), the point x_0 must be the base point of X . It then follows from the recursive formula (9) that for all $r \geq 2$ and $\theta \in \mathcal{P}(r)$, we have

$$f_r(\theta) = \{f_1 D_1(\theta), \dots, f_1 D_r(\theta)\} = \{x_0, \dots, x_0\}.$$

That is, α is the trivial sequence. □

2.4 Iterated suspensions are coalgebras over the little cubes operad

In this section, we show that the n -fold reduced suspension $\Sigma^n X$ of a pointed space X is a coalgebra over the little n -cubes operad. These are the paradigmatic examples of \mathcal{C}_n -coalgebras. To show our results, we use the coendomorphism version of \mathcal{C}_n -coalgebras. At the end of the section, we explain how the results in this paper allows us to swiftly recover the classical \mathcal{C}_n -algebra structure on n -fold loop spaces as a convolution structure. The \mathcal{C}_n -coaction on S^n that we describe in this section has previously appeared, in the context of the factorization homology, in [11].

Theorem 2.22. *The n -fold reduced suspension of a pointed space X is a \mathcal{C}_n -coalgebra. More precisely, there is a natural and explicit operad map*

$$\nabla : \mathcal{C}_n \rightarrow \text{CoEnd}_{\Sigma^n X}$$

that encodes the homotopy coassociativity and homotopy cocommutativity of the classical pinch map $\Sigma^n X \rightarrow \Sigma^n X \vee \Sigma^n X$. In particular, the pinch map is an operation associated to an element of $\mathcal{C}_n(2)$. Furthermore, for any based map $X \rightarrow Y$, the induced map $\Sigma^n X \rightarrow \Sigma^n Y$ extends to a morphism of \mathcal{C}_n -coalgebras.

The first step in proving the result above consists in showing that the sphere S^n , with $n \geq 1$, is a coalgebra over the little n -cubes operad. That is, we first show that the statement above is true for $X = S^n = \Sigma^n S^0$.

Proposition 2.23. *For every $n \geq 1$, there is a natural and explicit morphism of operads*

$$\nabla : \mathcal{C}_n \rightarrow \text{CoEnd}_{S^n}$$

turning the n -sphere into a \mathcal{C}_n -coalgebra, so that all properties of Theorem 2.22 for $\Sigma^n X = S^n$ hold true.

Proof. Let us define the arity r component of ∇ . This is a map

$$\nabla_r : \mathcal{C}_n(r) \rightarrow \text{CoEnd}_{S^n}(r) = \text{Map}_*(S^n, S^n \vee \dots \vee S^n).$$

For $c = (c_1, \dots, c_r) \in \mathcal{C}_n(r)$ a configuration of little n -cubes, we define the pointed map

$$\begin{aligned} \nabla_r(c) : S^n &\longrightarrow (S^n)^{\vee r} \\ t &\longmapsto \nabla_r(c)(t) \end{aligned}$$

as follows. Identify $S^n = I^n / \partial I^n$. Then $t \in S^n$ is either the base point $t = \{\partial I^n\}$ or else it is an interior point of the n -cube I^n . If t is interior, then it is at most a single cube c_i such that $t \in \mathring{c}_i$. We define

$$\nabla_r(c)(t) = \begin{cases} [c_i^{-1}(t)] & \text{if } t \in \mathring{c}_i, \\ * & \text{otherwise} \end{cases}$$

Here, $[c_i^{-1}(t)]$ denotes the point in the i -th wedge factor of $S^n \vee \dots \vee S^n$ followed by its inclusion as the i -th factor of the wedge. So defined, the maps $\nabla_r(c)$ are pointed, continuous and turn this into a morphism of operads. The fact that this is a morphism of operads is straightforward to check and left to the reader. \square

We prove next that the little n -cubes coalgebra structure on the sphere S^n just described induces the little n -cubes coalgebra structure on an arbitrary n -fold reduced suspension.

Proof of Theorem 2.22: Let $\Sigma^n X$ be the n -fold reduced suspension of the pointed space X . Write $\Sigma^n X = S^n \wedge X$, and recall that for any three pointed spaces X , Y and Z , the wedge and smash product distribute over each other [12, S. 4.F], i.e.

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z).$$

In particular, when we take X to be S^n

$$\Sigma^n (Y \vee Z) \cong \Sigma^n Y \vee \Sigma^n Z.$$

Then, for $c \in \mathcal{C}_n(r)$, define the map $\Sigma^n X \rightarrow (\Sigma^n X)^{\vee r}$ as the composition

$$\Sigma^n X \cong S^n \wedge X \xrightarrow{\nabla_r(c) \wedge \text{id}_X} ((S^n)^{\vee r}) \wedge X \xrightarrow{\cong} (S^n \wedge X)^{\vee r} \cong (\Sigma^n X)^{\vee r},$$

where ∇_r is the arity r component of the map ∇ defined in Proposition 2.23. All these maps are continuous, commute with the symmetric group actions and the operadic composition maps, producing a functorial construction. Alternatively, one can define the operad map

$$\text{CoEnd}_{S^n} \rightarrow \text{CoEnd}_{\Sigma^n X}$$

given (up to isomorphism) by $f \mapsto f \wedge \text{id}_X$, and precompose it with the operad map of Proposition 2.23. Doing this, one ends up with the map we described before. In this sense, the \mathcal{C}_n -coalgebra structure of an n -fold suspension always factors through the \mathcal{C}_n -coalgebra structure of S^n . \square

Remark 2.24. The defined operad map $\nabla : \mathcal{C}_n \rightarrow \text{CoEnd}_{\Sigma^n X}$ is determined by its arity 1 component $\nabla_1 : \mathcal{C}_n(1) \times \Sigma^n X \rightarrow \Sigma^n X$. Being more precise, as a consequence of Proposition 2.5, the following formula holds for all $c \in \mathcal{C}_n(r)$ and $z \in \Sigma^n X$:

$$\pi_i(\nabla_r(c, z)) = \nabla_{r-1}(d_i(c), z),$$

where π_i and d_i are the wedge collapse and restriction operators from Section 2.1.

In the remainder of the section, we explain how the coalgebraic framework introduced in this work let us swiftly recover the classical result by May that iterated loop spaces are algebras over the little n -cubes operad. For this, we first need to define fold algebras in the category of pointed spaces with the wedge product \vee .

Definition 2.25. Let X be a pointed space. The *fold endomorphism operad* End_X^\vee is the operad whose arity r component is given by

$$\text{End}_X^\vee(r) = \text{Map}_*(X^{\vee r}, X),$$

with the composition map given by inserting the output of a map into the input, and the symmetric group action is given by permuting the inputs. If \mathcal{P} is an operad in unpointed spaces, then a fold \mathcal{P} -algebra is a pointed space X together with a morphism of operads $\mathcal{P} \rightarrow \text{End}_X^\vee$.

We leave it to the reader to check that the definition above gives an operad. Every pointed space is canonically a commutative fold-algebra, where the products are given by the canonical fold maps (which explains the name).

Using the definition of a fold \mathcal{P} -algebra, we can now define a convolution algebra between a \mathcal{P} -coalgebra and a fold \mathcal{Q} -algebra, for operads \mathcal{P} and \mathcal{Q} . Denote by $\mathcal{P} \times \mathcal{Q}$ the arity-wise product of \mathcal{P} and \mathcal{Q} . This allows us to define convolution algebras in pointed spaces as follows.

Proposition 2.26. *Let \mathcal{P} and \mathcal{Q} be operads in unpointed spaces. Let X be a \mathcal{P} -coalgebra and Y a fold \mathcal{Q} -algebra. Then the pointed mapping space $\text{Map}_*(X, Y)$ is a $\mathcal{P} \times \mathcal{Q}$ -algebra. The structure maps*

$$\gamma : \mathcal{P}(r) \times \mathcal{Q}(r) \times \text{Map}_*(X, Y)^{\times r} \rightarrow \text{Map}_*(X, Y)$$

applied to pointed maps $f_1, \dots, f_r : X \rightarrow Y$ are explicitly given by

$$\gamma((\theta, \nu); f_1, \dots, f_r) = (\nu \circ (f_1 \vee \dots \vee f_r) \circ \Delta)(\theta).$$

Here, $(\theta, \nu) \in \mathcal{P}(r) \times \mathcal{Q}(r)$, i.e., the canonical map from the r -fold coproduct of X onto X , and $\Delta : \mathcal{P} \rightarrow \text{CoEnd}_X$ is the \mathcal{P} -coalgebra structure map of X .

Proof. This is similar to the construction in Section 1 of [2] and is left to the reader. \square

In particular, n -fold loop spaces fall into the framework described in the previous result. Since every pointed space is canonically a commutative fold algebra, and the arity-wise product of \mathcal{C}_n with the commutative operad is isomorphic to \mathcal{C}_n , we recover May's classical \mathcal{C}_n -algebra structure on loop spaces as follows (see [16]).

Corollary 2.27. *Let $\Omega^n X$ be an n -fold loop space. Then, the \mathcal{C}_n -algebra structure on*

$$\Omega^n X = \text{Map}_*(S^n, X)$$

induced by the \mathcal{C}_n -coalgebra structure of S^n and the fold Com -algebra structure on X as a convolution algebra is exactly the classical \mathcal{C}_n -algebra structure on loop spaces.

Proof. By definition, each map $S^n \rightarrow S^n \vee \dots \vee S^n$ arising from the \mathcal{C}_n coalgebra structure of S^n induces the following convolution product on an n -fold loop space $\Omega^n X$. Given $\alpha_1, \dots, \alpha_r : S^n \rightarrow X$ and $\theta \in \mathcal{C}_n(r)$, define $\gamma(\alpha_1, \dots, \alpha_r)$ as

$$S^n \xrightarrow{\nabla(\theta)} (S^n)^{\vee r} \xrightarrow{\alpha_1 \vee \dots \vee \alpha_r} X^{\vee r} \xrightarrow{\mu_r} X,$$

where $\mu_r \in \text{Com}(r)$ is the r th fold map. Here, Com is the commutative operad. One checks that these maps are exactly the maps described in [16, Section 5]. \square

3 The Approximation Theorem

To prove the recognition principle for n -fold loop spaces, as well as to develop a unified theory of homology operations for them, May proved the *approximation theorem* [16, Theorem 6.1]. This consists of giving a morphism of monads from the monad M_n associated to the little n -cubes operad to the monad $\Omega^n \Sigma^n$, and proving that this natural transformation is a weak equivalence on connected spaces. In this section, we prove an Eckmann–Hilton dual result to approximate the comonad $\Sigma^n \Omega^n$.

Theorem 3.1. *For every $n \geq 1$, there is a natural morphism of comonads*

$$\alpha_n : \Sigma^n \Omega^n \longrightarrow C_n.$$

Furthermore, for every pointed space X , there is an explicit natural homotopy retract of pointed spaces

$$\Sigma^n \Omega^n X \begin{array}{c} \xrightarrow{\quad} \\ \xleftarrow{\quad} \end{array} C_n(X) \quad \begin{array}{c} \curvearrowright \\ \end{array}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

The proof of the result above does not consist of a dualization of the corresponding proof of May's proof in the case of loop spaces. We take a different route which has the advantage that it gives us explicit homotopies and does not require auxiliary spaces as is needed in May's original approach. It is at the moment not clear whether these methods can also be used to give an alternative proof of the loop space approximation theorem.

Let $n \geq 1$ be a fixed integer. The natural transformation $\alpha = \alpha_n : \Sigma^n \Omega^n \rightarrow C_n$ is defined object-wise as the composition

$$\alpha_X : \Sigma^n \Omega^n X \xrightarrow{\gamma} C_n(\Sigma^n \Omega^n X) \xrightarrow{C_n(\eta_X)} C_n(X),$$

where γ is the \mathcal{C}_n -coalgebra structure map of $\Sigma^n \Omega^n X$ (Theorem 2.22), and η_X is the evaluation at X of the counit $\eta : \Sigma^n \Omega^n \rightarrow \text{id}_{\text{Top}_*}$ of the (Σ^n, Ω^n) -adjunction. Unraveling the definitions, we readily see that $\alpha = \alpha_X$ is explicitly given on a point $[t, \ell] \in \Sigma^n \Omega^n X = S^n \wedge \text{Map}_*(S^n, X)$ as the map $\alpha[t, \ell] : \mathcal{C}_n(1) \rightarrow X$ that acts on a little n -cube $c \in \mathcal{C}_n(1)$ by

$$\alpha[t, \ell](c) = \begin{cases} \ell(c^{-1}(t)) & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases}$$

See Proposition A.2 for more details on the definition of α .

Proof of Theorem 3.1: The proof consists of the following two steps.

(i) We must check that α defines a morphism of comonads. This is not complicated, but it is lengthy. Because of this, we postponed this proof to Appendix A (Proposition A.2).

(ii) We must check that for a fixed pointed space X , the space $\Sigma^n \Omega^n X$ is a retract of spaces of $C_n(X)$. To do so, we give a pointed map (of spaces, not comonads) $\Psi = \Psi_n : C_n(X) \rightarrow \Sigma^n \Omega^n X$ and a homotopy $\mathcal{H} : C_n(X) \times I \rightarrow C_n(X)$ such that

$$\Psi \circ \alpha = \text{id}_{\Sigma^n \Omega^n X} \quad \text{and} \quad \alpha \circ \Psi \simeq \text{id}_{C_n(X)}. \quad (10)$$

To define Ψ and the homotopy $\mathcal{H} = \mathcal{H}_n : \alpha \circ \Psi \simeq \text{id}_{C_n(X)}$, we introduce below for each $f \in C_n(X)$ a certain subset of the n -cube I^n which we name the *cubical support of f* and denote $\text{CSupp}(f)$. In the case of interest, the cubical support of a map f will be non-empty and has a well-defined *center*, which is a point

$$\text{Cent}(f) \in \text{CSupp}(f) \subseteq I^n.$$

Theorem 3.1 will then follow from the two items just described. Since the first item is proved in the mentioned appendix, it remains to prove the second one. We do this in what follows.

Definition of Ψ

The pointed map Ψ is defined as follows.

$$\begin{aligned} \Psi : C_n(X) &\longrightarrow \Sigma^n \Omega^n X \\ f &\longmapsto \Psi(f) = [\text{Cent}(f), \ell]. \end{aligned}$$

Here, we need to explain what the two components above are:

$$t := \text{Cent}(f) \in S^n \quad \text{and} \quad \ell : S^n \rightarrow X, \quad s \mapsto \ell(s) := f(c_{s, \text{Cent}(f)}).$$

Since we are identifying $S^n = I^n / \partial I^n$, we are denoting by $\text{Cent}(f)$ a certain point of the n -cube I^n that we are denoting in the same way and is going to be explained below. On the other hand, the little n -cube $c_{s, \text{Cent}(f)}$ that depends both on f and s , follows a certain construction to be explained below too.

Let us start with the following auxiliary definition. The *cubical support* of an arbitrary map $f : \mathcal{C}_n(1) \rightarrow X$ is the intersection of the images of all little n -cubes $c : I^n \rightarrow I^n$ such that f acts non-trivially on c :

$$\text{CSupp}(f) = \bigcap_{\substack{c \in \mathcal{C}_n(1) \\ f(c) \neq *}} \text{Im}(c) \subseteq I^n.$$

If the family over which we are taking the intersection above is empty, then we define $\text{CSupp}(f) = \emptyset$. If f is an element of $C_n(X)$, then this happens only when f is the trivial map. In this case, we define $\Psi(f)$ to be the base point of $\Sigma^n \Omega^n X$. The cubical support of f is closely related to its classical support, namely, the set of points of the domain of f where f acts non-trivially:

$$\text{Supp}(f) = \bigcap_{\substack{c \in \mathcal{C}_n(1) \\ f(c) \neq *}} c \subseteq \mathcal{C}_n(1).$$

Indeed, since each $c \in \mathcal{C}_n(1)$ defines the subset $\text{Im}(c) \subseteq I^n$, the cubical support of f is the subset of I^n determined by the classical support of f . Recall also that an n -rectangle is a subspace of \mathbb{R}^n which is rectilinearly homeomorphic to I^n or a singleton. An n -rectangle that does not reduce to a single point is determined by the set of its 2^n vertices, but also more efficiently by $2n$ numbers that describe the length of the sides and their position. In other words, an n -rectangle R is simply a cartesian product of closed intervals:

$$R = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for all } i = 1, \dots, n\} = [a_1, b_1] \times \dots \times [a_n, b_n],$$

for certain $a_i, b_i \in \mathbb{R}$ satisfying $a_i \leq b_i$.

Claim 1: The cubical support $\text{CSupp}(f)$ of a map $f \in C_n(X)$ is empty if, and only if, f is the trivial map. Furthermore, if f is non-trivial, then its cubical support is a point or an n -rectangle.

Proof of Claim 1: Let $f \in C_n(X)$ be any map. If $\text{CSupp}(f) \neq \emptyset$, then obviously $f \neq *$. Let us check the converse. Assume therefore that $f \neq *$, and let us check that $\text{CSupp}(f) \neq \emptyset$. Indeed: since $f \neq *$, there is some little n -cube d such that $f(d) \neq *$. Thus, the family $\{\text{Im}(c) \mid f(c) \neq *\}$ over which we are taking the intersection in the definition of the cubical support is non-empty. Now, from Proposition 2.18, it follows that if $c_1, c_2 \in \mathcal{C}_n(1)$ are such that both $f(c_1) \neq *$ and $f(c_2) \neq *$, then necessarily $c_1 \cap c_2 \neq \emptyset$. The intersection of the interiors of any two n -rectangles that do not reduce to a point is either empty, or it is again an n -rectangle that does not reduce to a point. From this fact, it follows that $\text{CSupp}(f)$ is non-empty.

To check the furthermore assertion in Claim 1, let $c \in \mathcal{C}_n(1)$ be a little n -cube, and write $c = (g_1, \dots, g_n)$ in terms of its coordinate functions $g_i : I \rightarrow I$. Then, the image of the cube c is the n -rectangle

$$\text{Im}(c) = [g_1(0), g_1(1)] \times \dots \times [g_n(0), g_n(1)] \subseteq I^n.$$

There is an obvious canonical identification between little n -cubes and n -rectangles contained in I^n that do not reduce to a single point. The cubical support of a fixed map $f : \mathcal{C}_n(1) \rightarrow X$ is therefore the n -rectangle

$$\text{CSupp}(f) = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

where for each $i = 1, \dots, n$

$$\begin{aligned} a_i &:= \sup \{g_i(0) \mid c = (g_1, \dots, g_n) \in \mathcal{C}_n(1) \text{ and } f(c) \neq *\}, \\ b_i &:= \inf \{g_i(1) \mid c = (g_1, \dots, g_n) \in \mathcal{C}_n(1) \text{ and } f(c) \neq *\}. \end{aligned}$$

This finishes the proof of Claim 1. \square

Every non-empty n -rectangle R has a *center* $\text{Cent}(R)$. If $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, then its center is the point determined by the midpoint of each of the intervals,

$$\text{Cent}(R) = \left(\frac{a_1 + b_1}{2}, \dots, \frac{a_n + b_n}{2} \right).$$

Observe that, if $R = (x_1, \dots, x_n)$ is a singleton, then $\text{Cent}(R) = (x_1, \dots, x_n)$. Assuming furthermore that $R = \text{CSupp}(f)$ for some f , then we define $\text{Cent}(f)$, the center of f , as

$$\text{Cent}(f) := \text{Cent}(\text{CSupp}(f)) = \text{Cent}(R).$$

Examples 3.2. Let us compute the cubical support $\text{CSupp}(f)$ for several maps f .

1. Let $C_n(*)$ be the cofree C_n -coalgebra on a single point. Then, $C_n(X) = *$ reduces to the trivial one-point space. Thus, the unique map $f : \mathcal{C}_n(1) \rightarrow *$ collapses all little n -cubes to the base point, and therefore, $\text{CSupp}(f) = \emptyset$.
2. Consider the map $f : \mathcal{C}_1(1) \rightarrow I$ given by

$$f(c) = \begin{cases} 0 & \text{if } r \leq 1/2 \\ r - 1/2 & \text{if } r \geq 1/2 \end{cases}$$

Here, $r = c(1) - c(0)$ is the size of the little 1-cube c . By Proposition 2.18, f defines an element in $C_1(I)$, and one readily checks that $\text{Cent}(f) = \text{CSupp}(f) = \{\frac{1}{2}\}$. By varying r , it is possible to construct a map having as center any chosen point in $(0, 1)$.

3. Define $f : \mathcal{C}_1(1) \rightarrow I$ as in the example above replacing $1/2$ by any real number $a \in [\frac{1}{2}, 1)$. By Proposition 2.18, f defines a map in $C_1(I)$. Its cubical support is the interval $[1 - a, a]$. In the case where $a = \frac{1}{2}$, we see again that $\text{Cent}(f) = \frac{1}{2}$.

The examples above can be generalized to higher-dimensional cubes.

Another important example of cubical support is that of n -fold suspensions.

Proposition 3.3. *Let $\Sigma^n X$ be the n -fold reduced suspension of a pointed space X , and let $\gamma : \Sigma^n X \rightarrow C_n(\Sigma^n X)$ be its \mathcal{C}_n -coalgebra structure map. Then, for every non-base point $[t, x] \in \Sigma^n X$, we have that*

$$\text{CSupp}(\gamma[t, x]) = \{t\}.$$

Proof. First, we prove the result for spheres. If $\gamma : S^n \rightarrow C_n(S^n)$ is the \mathcal{C}_n -coalgebra structure map, we explicitly have

$$\gamma(t)(c) = \begin{cases} c^{-1}(t) & \text{if } t \in \mathring{c} \\ * & \text{otherwise,} \end{cases}$$

where $t \in S^n$ and we identify S^n with $I^n / \partial I^n$, the ambient cube of c modulo its boundary. By definition, $\text{CSupp}(\gamma(t))$ is the intersection of the family

$$\{\text{Im}(c) \mid c \in \mathcal{C}_n(1) \text{ and } \gamma(t)(c) \neq *\}.$$

The image $\text{Im}(c)$ of a little n -cube is non-trivial if, and only if, $t \in \text{Im}(c)$. Thus, the cubical support $\text{CSupp}(\gamma(t))$ is the intersection of all non-trivial cubes containing t , and therefore, it is the singleton $\{t\}$.

Now, for an arbitrary n -fold reduced suspension $\Sigma^n X$, factorize its coalgebra structure map as follows:

$$\Sigma^n X = S^n \wedge X \xrightarrow{\gamma_{S^n \wedge \text{id}_X}} C_n(S^n) \wedge X \xrightarrow{F} C_n(S^n \wedge X).$$

The second map F above is given by

$$F(f, x) = [f(-), x], \quad \text{for } f: \mathcal{C}_n(1) \rightarrow S^n \text{ and } x \in X.$$

The final composition is therefore explicitly given by

$$\begin{aligned} \gamma[t, x]: \mathcal{C}_n(1) &\longrightarrow S^n \wedge X \\ c &\longmapsto [\gamma(t)(c), x]. \end{aligned}$$

Here, the cubical support $\text{CSupp}(\gamma[t, x])$ is the intersection of the family

$$\{\text{Im}(c) \mid c \in \mathcal{C}_n(1) \text{ and } [\gamma(t)(c), x] \neq *\}.$$

Similar to the case of the spheres, we have

$$[\gamma(x)(c), x] = \begin{cases} [c^{-1}(t), x] & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases}$$

We readily see from here that a little n -cube c has non-trivial image if, and only if, \mathring{c} contains the component t of the sphere. Thus, the intersection of them all yields the singleton $\{t\}$. \square

We also need the following auxiliary result. It explicitly describes the little n -cube that appears in the loop $\ell: S^n \rightarrow X$ of the second component of Ψ .

Claim 2: For each pair of points $s, t \in I^n - \partial I^n$, there is a unique little n -cube $c = c_{s,t}: I^n \rightarrow I^n$, depending continuously on (s, t) , such that:

1. $c(s) = t$,
2. $\text{Im}(c)$ is the n -rectangle of maximum size contained in I^n and touching all the faces of the boundary ∂I^n . More precisely, we require that for each coordinate at least one side of the embedded rectangle touches a side of the ambient cube.

If s or t lies in the boundary ∂I^n , we will not need to construct the cube $c_{s,t}$. Indeed, in this case Ψ will map the pair $[t, \ell]$ to the base point of $C_n(X)$.

Proof of Claim 2: Let us explicitly construct c . Recall from Equation (7) that the rectilinear embedding c is of the form

$$c(x_1, \dots, x_n) = ((b_1 - a_1)x_1 + a_1, \dots, (b_n - a_n)x_n + a_n),$$

where $0 \leq a_i < b_i \leq 1$ for all i . Thus, each component c_i of c is determined by the numbers a_i and b_i . Imposing that $c(s) = t$, we get the relations

$$(b_i - a_i)s_i + a_i = t_i \quad \text{for each } i.$$

A second constraint on each component i determines the numbers a_i, b_i uniquely. Since c touches each face of ∂I^n , at each component c_i we must have one of the following two options:

1. $c_i(0) = 0$, and then we deduce that

$$c_i(x_i) = \frac{t_i}{s_i} \cdot x_i,$$

or else

2. $c_i(1) = 1$, and then we deduce that

$$c_i(x_i) = 1 - \frac{(1-t_i)(1-x_i)}{1-s_i}.$$

Now, there is no choice to be made here. Rather, the option is determined by the relationship between s and t . That is, we are considering the separate cases where $s_i > t_i$ or $s_i < t_i$. More precisely, if for a fixed i , we have that $0 < t_i/s_i < 1$, then the first formula gives a well-defined affine linear map onto the interval, but the second formula does not (because its image lands outside the unit interval). If, on the other hand, the inequality $0 < t_i/s_i < 1$ does not hold, then it follows that $0 < \frac{(1-t_i)(1-x_i)}{1-s_i} < 1$, and the second formula does define an affine linear map (while the first one does not). To finish, observe that the formulae agree when $s_i = t_i$, which makes the construction of c a continuous function of s and t . Of course, in the case $s_i = t_i$, we are taking the identity map at the i -th coordinate. This finishes the proof of Claim 2. \square

Having explained in full detail what all the items defining Ψ are, the map Ψ is given by:

$$\begin{aligned} \Psi : C_n(X) &\longrightarrow \Sigma^n \Omega^n X \\ f &\longmapsto \Psi(f) = [\text{Cent}(f), \ell], \end{aligned}$$

where ℓ is defined as

$$\begin{aligned} \ell : S^n &\rightarrow X, \\ s &\mapsto \ell(s) := f\left(c_{s, \text{Cent}(f)}\right). \end{aligned}$$

Our arguments so far show that the resulting function is a pointed continuous function of f .

Definition of the homotopy \mathcal{H}

The next step in the proof of the approximation theorem is to construct a homotopy $\mathcal{H} : C_n(X) \times I \rightarrow C_n(X)$ such that

$$\mathcal{H}_0 = \text{id}_{C_n(X)}, \quad \mathcal{H}_1 = \alpha \circ \Psi, \quad \mathcal{H}(*, t) = * \quad \forall t \in I. \quad (11)$$

The following auxiliary construction is a key ingredient for the homotopy \mathcal{H} . Intuitively speaking, the idea is to construct a homotopy from maps whose cubical support is more than a point to maps whose cubical support is exactly a point. We construct this homotopy by enlarging the cubes in $C_n(1)$ until they hit the boundary while also preserving the center. This is made precise in the following auxiliary construction.

Auxiliary construction: The rectilinear expansion of a little n -cube $c \in \mathcal{C}_n(1)$ induced by a map $f \in C_n(X)$ whose center $\text{Cent}(f)$ belongs to \mathring{c} .

Proof and explanations for the auxiliary construction: Let us explain the construction for a little 1-interval $c \in \mathcal{C}_1(1)$; the general case is an application of this construction at each coordinate of a little n -cube. Let $c \in \mathcal{C}_1(1)$, so that

$$c(t) = (b-a)t + a$$

for some a, b with $0 \leq a < b \leq 1$. Let

$$x_1 = \text{dist}(\text{Im}(c), \partial I) = \min\{a, 1-b\}$$

be the distance from $\text{Im}(c)$ to the boundary of the interval.

Definition 3.4. Let $c \in \mathcal{C}_1(1)$. The *rectilinear expansion* of c induced by a map $f \in C_1(X)$ whose center $\text{Cent}(f)$ belongs to \mathring{c} is the unique path $\gamma = \gamma_c^f : I \rightarrow \mathcal{C}_1(1)$ satisfying:

- $\gamma(0) = c$,
- for every $s \in (0, 1]$,

- the cube $\gamma(s)$ is a rectilinear embedding that increases the size of c by $\min\{s, \text{dist}(\text{Im}(c), \partial I)\}$ while keeping the ratios between the sides equal, and
- the center $\text{Cent}(f)$ is fixed by $\gamma(s)$, i.e. if $z = c^{-1}(\text{Cent}(f))$, then $\gamma(s)(z) = \text{Cent}(f)$.

Let us explicitly describe the path above. For each $s \in I$, we have $\gamma(s) \in \mathcal{C}_1(1)$ of the form

$$\gamma(s)(t) = (b(s) - a(s))t + a(s) \quad \forall t \in I.$$

For a fixed $s \in I$, two conditions on $a(s)$ and $b(s)$ determine $\gamma(s)$ uniquely. We impose the two mentioned conditions, namely that

$$\gamma(s)(p) = p,$$

where for simplicity we denote $p = \text{Cent}(f)$, and that the radius of $\gamma(s)$ is that of c increased by $\min\{s, a, 1 - b\}$:

$$(b(s) - a(s)) - (b - a) = \min\{s, a, 1 - b\}.$$

These conditions produce the linear system of equations

$$\begin{cases} (1 - p)a(s) + pb(s) = p \\ -a(s) + b(s) = \alpha(s) \end{cases}$$

where $\alpha(s) = \min\{s, a, 1 - b\}$. The unique solution to the system above is

$$\begin{aligned} a(s) &= p(1 - \alpha(s)) \\ b(s) &= \alpha(s) - \alpha(s)p + p. \end{aligned}$$

Therefore, for a fixed $s \in I$, the little 1 interval $\gamma(s)$ is given by

$$\gamma(s)(t) = \alpha(s)t + p - p\alpha(s) \quad \forall t \in I.$$

This finishes the construction for a little 1-interval. In the general case, given $c \in \mathcal{C}_n(1)$ of the form

$$c(t_1, \dots, t_n) = ((b_1 - a_1)t_1 + a_1, \dots, (b_n - a_n)t_n + a_n)$$

and $f \in C_n(X)$, define $\gamma = \gamma_c^f : I \rightarrow \mathcal{C}_n(1)$ to be the path such that

$$\gamma(s)(t_1, \dots, t_n) = (\alpha_1(s)t_1 + p_1 - p_1\alpha_1(s), \dots, \alpha_n(s)t_n + p_n - p_n\alpha_n(s)) \quad \forall (t_1, \dots, t_n) \in I^n.$$

This finishes the construction of the auxiliary path $\gamma_c^f : I \rightarrow \mathcal{C}_n(1)$, and therefore the proof and explanations for the auxiliary construction. \square

Now, we are ready to define the homotopy $\mathcal{H} : C_n(X) \times I \rightarrow C_n(X)$. For each $(f, t) \in C_n(X) \times I$, this is the map

$$\mathcal{H}(f, t) : \mathcal{C}_n(1) \rightarrow X$$

whose image on a little n -cube $c \in \mathcal{C}_n(1)$ is

$$\mathcal{H}(f, t)(c) = f(\gamma_c^f(t))$$

Here, γ_c^f is the rectilinear expansion of c induced by f . Note that this rectilinear expansion shrinks the cubical support of f to a point. We must check that \mathcal{H} is well-defined, continuous, and satisfies the requirements for being a pointed homotopy from $\text{id}_{C_n(X)}$ to $\alpha\Psi$. To check that \mathcal{H} is well-defined, we must corroborate that for each (f, t) , the map $\mathcal{H}(f, t)$ indeed defines an element in $C_n(X)$. Recall from Proposition 2.18 that given $c_1, c_2 \in \mathcal{C}_n(1)$ with $\dot{c}_1 \cap \dot{c}_2 = \emptyset$, this amounts to checking that

$$\mathcal{H}(f, t)(c_1) = * \quad \text{or} \quad \mathcal{H}(f, t)(c_2) = *.$$

But this is immediate: if $\mathring{c}_1 \cap \mathring{c}_2 = \emptyset$, then $\text{Cent}(f)$ cannot be in both c_1 and c_2 at the same time. Therefore, by definition $\mathcal{H}(f, t)$ vanishes on the little cube c_i not having $\text{Cent}(f)$ in its image. We conclude that \mathcal{H} is well-defined. It is straightforward to check that \mathcal{H} is indeed continuous and we leave this to the reader. Similarly, it follows directly from the definitions that the identities of Equation (11) hold.

We have therefore explained in full detail the definition of \mathcal{H} , and checked it gives a pointed homotopy between $\text{id}_{C_n(X)}$ and $\alpha \circ \Psi$.

Proving the equality $\Psi \circ \alpha = \text{id}_{\Sigma^n \Omega^n X}$

Let $[t, \ell] \in \Sigma^n \Omega^n X$. By definition,

$$\Psi \alpha[t, \ell] = [\text{Cent}(\alpha[t, \ell]), L], \quad (12)$$

where $L: S^n \rightarrow X$ is the loop

$$s \mapsto L(s) = \alpha[t, \ell](c_{s, \text{Cent}(\alpha[t, \ell])}).$$

Assume that X is not the one-point space and that ℓ is not the constant loop; otherwise the result is trivial. We must check the two components in the right hand side of Equation (12) are, respectively, t and ℓ .

1. Let us check that $\text{Cent}(\alpha[t, \ell]) = t$. To do so, it suffices to check that $\text{CSupp}(\alpha[t, \ell])$ reduces to the single point $\{t\}$. Indeed: if $c \in \mathcal{C}_n(1)$ is such that $\alpha[t, \ell](c) \neq *$, it follows from the definition of $\alpha[t, \ell]$ that $t \in \mathring{c}$ (recall Equation (15)). Thus, $t \in \text{Im}(c)$ for all little n -cubes c such that $\alpha[t, \ell](c) \neq *$. Therefore, t is in the intersection of all such images, namely $\text{CSupp}(\alpha[t, \ell])$. Now, if $t_0 \neq t$, then we can always find a little n -cube \tilde{c} such that $t_0 \notin \text{Im}(\tilde{c})$ and $t \in \text{Im}(\tilde{c})$, and furthermore $\ell((\tilde{c})^{-1}(t)) \neq *$ (possibly after reparametrization: it might be the case that the loop ℓ passes through the basepoint of X , but we are assuming ℓ is not the constant loop).
2. Let us check that $L(s) = \ell(s)$ for all $s \in S^n$. Indeed: for $t = \text{Cent}(\alpha[t, \ell])$, the little n -cube $c = c_{s, \alpha[t, \ell]}$ is such that $c(s) = t$. Said differently, $c^{-1}(t) = s$. Therefore, by definition:

$$L(s) = \alpha[t, \ell](c) = \begin{cases} * & \text{if } t \notin \mathring{c} \\ \ell(c^{-1}(t)) & \text{otherwise} \end{cases} = \ell(s).$$

To summarise: we have explained the definition of the map Ψ and the homotopy \mathcal{H} , and have shown the retract requirements of Equation (10) hold. Thus, the proof of Theorem 3.1 is now complete. \square

Remark 3.5. In this section we have chosen to prove the approximation theorem for the little n -cubes (rectangles) operad, but the ideas could easily be modified to other little convex bodies operads, like the little n -disks operad. Here some small modification would be needed to explain what exactly is meant by the center and how the expansion is defined. For simplicity, we have decided to only look at the little cubes operads.

4 The Recognition Principle for n -fold reduced suspensions

In this section, we prove the recognition principle for n -fold reduced suspensions. The precise statement is the following.

Theorem 4.1. *Let X be a \mathcal{C}_n -coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X , together with a weak equivalence of \mathcal{C}_n -coalgebras*

$$\Sigma^n \Gamma^n(X) \xrightarrow{\cong} X,$$

which is a retract in the category of pointed spaces. Therefore, every \mathcal{C}_n -coalgebra has the homotopy type of an n -fold reduced suspension.

The result above is the converse of Theorem 2.22, where it was proven that n -fold reduced suspensions are \mathcal{C}_n -coalgebras. Summarizing, we are providing the following intrinsic characterization of n -fold reduced suspensions as \mathcal{C}_n -coalgebras.

Corollary 4.2. *Every n -fold suspension is a \mathcal{C}_n -coalgebra, and if a pointed space is a \mathcal{C}_n -coalgebra then it is homotopy equivalent to an n -fold suspension.*

Remark 4.3. Compared to other statements in the literature, see for example [14, 4], Theorem 4.1 does not require any additional connectivity assumptions, and it is therefore the sharpest possible result. This follows from the fact that every \mathcal{C}_n -coalgebra is $(n-1)$ -connected. Indeed, let X be a \mathcal{C}_n -coalgebra with structure map $c : X \rightarrow C_n(X)$. By the approximation theorem, the space $C_n(X)$ is homotopic to $\Sigma^n \Omega^n X$, and thus $(n-1)$ -connected. Since the composition $X \xrightarrow{c} C_n(X) \xrightarrow{\varepsilon_X} X$ is the identity on X by the counit axiom, it follows that X is $(n-1)$ -connected.

For readability, we shall give the proof of Theorem 4.1 straightaway, making reference to the results and notation of the following two subsections.

Proof. By Theorem 3.1, there is a natural morphism of comonads $\alpha_n : \Sigma^n \Omega^n \rightarrow C_n$, and $\Sigma^n \Omega^n X$ is a retract of $C_n(X)$. Since $\Sigma^n \Omega^n$ preserves equalizers (Proposition 4.10), it follows from Lemma 4.8 that the counit map $(\alpha_n)_* \alpha_n^!(X) \rightarrow X$ is a C_n -algebra morphism which is a retract of pointed spaces. Since $(\alpha_n)_*$ preserves the underlying topological space, it follows that the $\Sigma^n \Omega^n$ -coalgebra $\alpha_n^!(X)$ is a retract of X as a pointed space. It then follows from Theorem 4.9 together with the approximation theorem that $\alpha_n^!(X)$ is naturally isomorphic to an n -fold suspension, and so the counit map $(\alpha_n)_* \alpha_n^!(X) \rightarrow X$ is an C_n -coalgebra map from a n -fold reduced suspension to X . In particular, Γ^n is the functor $P_n(\alpha_n)_* \alpha_n^!$. \square

We give a second proof of Theorem 4.1 in Section 4.3 using explicit formulae very similar to those appearing in the approximation theorem. This alternative proof is more concrete, and has the further benefit of giving a characterization in terms of a certain C_n -subcoalgebra.

4.1 The change of coalgebra structures induced by a comonad morphism

In this section, we explain how a morphism of comonads $\alpha : C_1 \rightarrow C_2$ induces an adjoint pair

$$\alpha_* : C_1 - \text{Coalg} \rightleftarrows C_2 - \text{Coalg} : \alpha^!$$

between the corresponding categories of coalgebras (under reasonable hypotheses on the underlying ambient category). The final goal is to prove the technical Lemma 4.8, which is an essential ingredient for proving Theorem 4.1.

Suppose that C_1 and C_2 are two comonads over a category \mathcal{M} which admits finite limits, and that $\alpha : C_1 \rightarrow C_2$ is a morphism of comonads. The *change of coalgebra functor*

$$\alpha_* : C_1 - \text{Coalg} \longrightarrow C_2 - \text{Coalg}$$

is given by mapping a C_1 -coalgebra X to the same underlying object of \mathcal{M} equipped with the C_2 -coalgebra structure map given by the composition

$$X \xrightarrow{\gamma_X} C_1(X) \xrightarrow{\alpha_X} C_2(X).$$

On morphisms, α_* is the identity.

Since \mathcal{M} has finite limits, by the dual of Dubuc's adjoint triangle theorem [8], the change of coalgebra functor α_* has a right adjoint $\alpha^!$ which we call the *enveloping coalgebra functor*. The C_1 -coalgebra $\alpha^!(X)$ is explicitly given as the equalizer in $C_1 - \text{Coalg}$ of the following pair of morphisms:

$$\begin{array}{ccc} C_1(X) & \xrightarrow{C_1(\delta_X)} & C_1 C_2(X) \\ & \searrow \Delta_{C_1} & \nearrow C_1(\alpha_X) \\ & & C_1 C_1(X) \end{array}$$

Above, δ_X is the structure map of X as a C_2 -coalgebra. The following proposition, which is the dual of [6, Prop. 4.3.2], gives conditions for this equalizer to be preserved by the forgetful functor to \mathcal{M} .

Proposition 4.4. *Let C be a comonad on \mathcal{M} and let $U : C - \text{Coalg} \rightarrow \mathcal{M}$ be the forgetful functor. Let $G : D \rightarrow C - \text{Coalg}$ be a diagram such that UG has a limit in \mathcal{M} that is preserved by C and $C \circ C$. Then G has a limit in $C - \text{Coalg}$ that is preserved by U .*

Proof. The proof of this result is dual to that of [6, Prop. 4.3.2], and it is left to the reader. \square

We will need the following auxiliary definition.

Definition 4.5. A *cosplit equalizer* in a category is a diagram

$$\begin{array}{ccc} A & \xrightarrow{p} & B & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & C \\ & \swarrow \text{---} & \swarrow \text{---} & \swarrow \text{---} & \\ & \text{---} h & \text{---} s & \text{---} & \end{array}$$

where

$$sg = \text{id}_B, \quad hp = \text{id}_A \quad \text{and} \quad sf = ph. \quad (13)$$

The notion of a cosplit equalizer above is dual to that of split coequalizer, and it plays in comonad theory the analog role of split coequalizers in the theory of monads (see [15, VI. 6]). The following result is elementary but important.

Proposition 4.6. *The cosplit equalizer of two morphisms is always an equalizer of the two morphisms; and any functor preserves cosplit equalizers.*

Proof. Assume we have a cosplit equalizer with the notation from Definition 4.5. To prove the first assertion, assume that φ is any map such that $f\varphi = g\varphi$. Then,

$$\varphi = hp\varphi = sf\varphi = ph\varphi$$

factors through p . Since $hp = \text{id}_A$, this factorization is unique. The second assertion is a straightforward consequence of the fact that functors preserve the associativity of the composition and the identity on objects. \square

Next, we relate cosplit equalizers with coalgebra structures.

Proposition 4.7. *Let C be a comonad in an arbitrary category, and let X be a C -coalgebra. Then, the coalgebra structure map $\gamma : X \rightarrow C(X)$ fits into a cosplit equalizer diagram*

$$X \xrightarrow{\gamma} C(X) \begin{array}{c} \xrightarrow{C(\gamma)} \\ \xrightarrow{\Delta_X} \end{array} CC(X).$$

Proof. Let X be a C -coalgebra with structure map γ . As a consequence of the coassociativity axiom for γ , we have the fork in the statement. By Proposition 4.6, we are done as soon as we give cosplitting maps h, s satisfying the identities of Equation (13), taking $f = C(\gamma)$ and $g = \Delta_X$. These cosplittings h and s are respectively given by the corresponding counits

$$\varepsilon_X : C(X) \rightarrow X \quad \text{and} \quad \varepsilon_{C(X)} : CC(X) \rightarrow C(X).$$

Let us check that the identities in Equation (13) hold. The identity $hp = \text{id}_A$ becomes $\varepsilon_X \circ \gamma = \text{id}_X$, which holds because it is precisely the counital axiom of the C -coalgebra X . Similarly, the identity $sg = \text{id}_B$ becomes $\varepsilon_{C(X)} \circ \Delta_X = \text{id}_{C(X)}$, which is exactly the counit axiom at $C(X)$. It remains to check the identity $sf = ph$, that is, $\varepsilon_{C(X)} \circ C(\gamma) = \gamma \circ \varepsilon_X$. This follows from the fact ε is a natural transformation and so one has the diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{C(\gamma)} & CC(X) \\ \downarrow \varepsilon_X & & \downarrow \varepsilon_{C(X)} \\ X & \xrightarrow{\gamma} & C(X). \end{array}$$

We have checked the three identities of Equation (13). Therefore, the mentioned diagram is a cosplit equalizer, and the proof is complete. \square

Finally, the following technical lemma allows us to directly compare C_1 and C_2 -coalgebras in pointed spaces under certain conditions. It constitutes an essential ingredient in the proof of Theorem 4.1.

Lemma 4.8. *Let $\alpha : C_1 \rightarrow C_2$ be a morphism of comonads in Top_* which is a retract of pointed spaces at each level. If C_1 preserves equalizers, then the counit $\alpha_* \alpha^! \rightarrow \text{id}_{C_2\text{-Coalg}}$ of the $(\alpha_*, \alpha^!)$ adjunction is a retract of pointed spaces at each level. In particular, for every C_2 -coalgebra X , the underlying map of pointed spaces $\alpha_* \alpha^!(X) \rightarrow X$ is a retract.*

Proof. Let X be a C_2 -coalgebra. Let us prove that the underlying map of pointed spaces of the C_2 -coalgebra morphism $\alpha_* \alpha^!(X) \rightarrow X$ is a retract. Since α_* is the identity on the underlying pointed space, this underlying map is $\alpha^!(X) \rightarrow X$. Recall from Proposition 4.7 that the C_2 -coalgebra structure γ on X is given by presenting X as the (cosplit) equalizer of the following diagram:

$$C_2(X) \begin{array}{c} \xrightarrow{C_2(\gamma)} \\ \xrightarrow{\Delta_X} \end{array} C_2 C_2(X).$$

Here, Δ_X is the comultiplication of the C_2 comonad at X . This equalizer is taken in $C_2\text{-Coalg}$, but we can compute the underlying topological space via the same limit in the category of pointed topological spaces. This is because this limit is a cosplit equalizer, and therefore an equalizer which is preserved by the forgetful functor (see Proposition 4.6). Since C_1 is assumed to preserve equalizers, by Proposition 4.4, and using a similar argument, the underlying topological space of $\alpha^!(X)$ may be computed as the equalizer of the diagram

$$C_1(X) \begin{array}{c} \xrightarrow{C_1(\gamma)} \\ \xrightarrow{C_1(\alpha_X) \circ \Delta_{C_1}} \end{array} C_1 C_2(X)$$

in the category of pointed topological spaces. The retract provided by α thus extends to a map (in the category of pointed topological spaces) between the diagram defining $\alpha^!(X)$ and one defining X , namely,

$$\begin{array}{ccc} C_1(X) & \begin{array}{c} \xrightarrow{C_1(\gamma)} \\ \xrightarrow{C_1(\alpha_X) \circ \Delta_{C_1}} \end{array} & C_1 C_2(X) \\ \alpha_X \downarrow & & \downarrow \alpha_{C_2(X)} \\ C_2(X) & \begin{array}{c} \xrightarrow{C_2(\gamma)} \\ \xrightarrow{\Delta_X} \end{array} & C_2 C_2(X) \end{array}$$

The corresponding map of limits is thus precisely the desired map $\alpha^!(X) \rightarrow X$. Since retracts are preserved under limits, we conclude that this map is a retract of pointed spaces. \square

4.2 The $\Sigma^n \Omega^n$ -coalgebras are n -fold reduced suspensions

In this section, we completely characterize the coalgebras over the $\Sigma^n \Omega^n$ -comonad (Theorem 4.9).

A warning on the notation: in other parts of this paper, we have consistently denoted by Δ and ε the comonadic structure maps of the comonad C_n constructed from the little n -cubes operad; while Δ' and ε' were used for the comonadic structure maps of the comonad $\Sigma^n \Omega^n$. Since there will be only a single comonad appearing in this section, namely $\Sigma^n \Omega^n$, we make an exception here and denote by Δ and ε the comonadic structure maps of $\Sigma^n \Omega^n$ to make the reading easier.

Theorem 4.9. *Let X be a $\Sigma^n \Omega^n$ -coalgebra. Then X is naturally isomorphic to the n -fold reduced suspension of a space $P_n(X)$ which can be computed as the equalizer of the following pair of maps:*

$$\Omega^n X \begin{array}{c} \xrightarrow{\Omega^n \gamma} \\ \xrightarrow{\eta_{\Omega^n X}} \end{array} \Omega^n \Sigma^n \Omega^n X.$$

Here, η is the unit of the (Σ^n, Ω^n) adjunction, and γ is the $\Sigma^n \Omega^n$ -coalgebra structure map of X .

Theorem 4.9 is essentially a consequence of the fact that reduced suspensions, despite being left adjoint, preserve equalizers. Next, we give a proof of this elementary fact for completeness.

Proposition 4.10. *The n -fold reduced suspension functor $\Sigma^n : \text{Top}_* \rightarrow \text{Top}_*$ commutes with equalizers. In other words, if $\text{Eq}(f, g) \hookrightarrow X$ is the equalizer of the diagram*

$$X \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} Y,$$

then $\Sigma^n \text{Eq}(f, g) \hookrightarrow \Sigma^n X$ is the equalizer of the diagram

$$\Sigma^n X \begin{array}{c} \xrightarrow{\Sigma^n f} \\ \xrightarrow{\Sigma^n g} \end{array} \Sigma^n Y.$$

Since Ω^n is right adjoint and thus preserves limits, it further follows that $\Sigma^n \Omega^n$ preserves equalizers.

Proof. Recall that, as a set, the equalizer of f and g is given by

$$\text{Eq}(f, g) = \{x \in X \mid f(x) = g(x)\}.$$

Since we tacitly work in the category CGH of compactly generated Hausdorff spaces, the topology on this set is not necessarily the subspace topology, but might be finer. Explicitly, its topology is given by applying the k -ification functor $k(-)$, see for example [17, Chapter 5]. This functor is the right adjoint of the inclusion of CGH into ordinary topological spaces. This change in the underlying topology is not an issue, because taking n -fold reduced suspension commutes with the k -ification functor. Indeed, if X and Y are any compactly generated Hausdorff spaces and X is locally compact, then $X \times Y$ is a compactly generated Hausdorff space ([20, Thm. 4.3]). Since the sphere S^n is locally compact, the product $S^n \times X$ is compactly generated Hausdorff for any compactly generated Hausdorff space X . Since the smash product $S^n \wedge X$ is the pushout of the inclusion $S^n \vee X \hookrightarrow S^n \times X$ along the collapse map $S^n \vee X \rightarrow *$, it follows that $S^n \wedge X = \Sigma^n X$ is compactly generated Hausdorff. Thus,

$$\Sigma^n \text{Eq}(f, g) = S^n \wedge \text{Eq}(f, g).$$

Points in the suspension above are of the form $[t, x]$, with $t \in S^n$ and $x \in X$ such that $f(x) = g(x)$. On the other hand,

$$\text{Eq}(\Sigma^n f, \Sigma^n g) = \{[t, x] \in \Sigma^n X \mid [t, f(x)] = [t, g(x)]\}.$$

Under the two identifications above, the natural map

$$\Sigma^n \text{Eq}(f, g) \rightarrow \text{Eq}(\Sigma^n f, \Sigma^n g)$$

is a homeomorphism. □

Recall from Proposition 4.7 that every coalgebra structure map is characterized as a cosplit equalizer. In particular, we have the following result.

Proposition 4.11. *Let X be a $\Sigma^n \Omega^n$ -coalgebra with structure map γ . Then, as a pointed space, X is the (cosplit) equalizer of the following pairs of maps*

$$\Sigma^n \Omega^n X \begin{array}{c} \xrightarrow{\Sigma^n \Omega^n \gamma} \\ \xrightarrow{\Delta_X} \end{array} \Sigma^n \Omega^n \Sigma^n \Omega^n X.$$

Here, Δ is the comonadic comultiplication of $\Sigma^n \Omega^n$.

Proof. As mentioned, this is a particular case of Proposition 4.7. The following diagram is a cosplit equalizer:

$$X \xrightarrow{\gamma} \Sigma^n \Omega^n X \xrightarrow[\Delta_X]{\Sigma^n \Omega^n \gamma} \Sigma^n \Omega^n \Sigma^n \Omega^n X,$$

where the cosplittings h and s are respectively given by the corresponding counits

$$\varepsilon_X : \Sigma^n \Omega^n X \rightarrow X \quad \text{and} \quad \varepsilon_{\Sigma^n \Omega^n X} : \Sigma^n \Omega^n \Sigma^n \Omega^n X \rightarrow \Sigma^n \Omega^n X.$$

□

Let us finally prove the main result of this section.

Proof of Theorem 4.9. Use, in the order given, Proposition 4.11, that the comonadic coproduct Δ_X is explicitly given by $\Sigma^n \eta_{\Omega^n(X)}$, and Proposition 4.10 to obtain that

$$X = \text{Eq}(\Sigma^n \Omega^n \gamma, \Delta_X) = \text{Eq}(\Sigma^n \Omega^n \gamma, \Sigma^n \eta_{\Omega^n X}) = \Sigma^n \text{Eq}(\Omega^n \gamma, \eta_{\Omega^n X}).$$

This is exactly what we wished to prove. □

4.3 A point-set description of the recognition principle

We give here an alternative proof of the recognition principle mentioned in the introduction to Section 4. This proof has the advantage of explicitly characterizing the n -fold suspension onto which a C_n -coalgebra retracts.

Theorem 4.12. *Let X be a C_n -coalgebra. Then, there is a pointed space Z together with a homotopy equivalence of C_n -coalgebras $X \simeq \Sigma^n Z$.*

The strategy of the proof is the following. First we show that every C_n -coalgebra X contains a C_n -subcoalgebra $S(X)$ which is also a $\Sigma^n \Omega^n$ -coalgebra, and that there is a retract of X onto $S(X)$ (Theorem 4.13 and Theorem 4.14, respectively). Because of Theorem 4.9 this implies that $S(X)$ is an n -fold suspension, proving Theorem 4.12.

In Proposition 3.3, we saw that $\Sigma^n \Omega^n$ -coalgebras considered as C_n -coalgebras have the property that the cubical support at each point is just a single point. In this section, we prove that the converse is also true. That is, every C_n -coalgebra of which the cubical support of every point (other than the base point) is just a single point is not just a C_n -coalgebra, but also a $\Sigma^n \Omega^n$ -coalgebra.

It further turns out that the set of points whose cubical support is just a single point forms a C_n -subcoalgebra.

Theorem 4.13. *Let X be a C_n -coalgebra with coalgebra structure map $c : X \rightarrow C_n(X)$. Then, the subspace*

$$S(X) = \{x \in X \mid |\text{CSupp}(c(x))| = 1\} \cup \{*\} \subseteq X$$

formed by the points of X whose cubical support is a single point, together with the base point, is such that the following assertions hold.

1. *The inclusion $S(X) \hookrightarrow X$ is a homotopy equivalence of pointed spaces.*
2. *The subspace $S(X)$ is a C_n -subcoalgebra, and the inclusion is a morphism of C_n -coalgebras.*

Therefore, the inclusion $S(X) \hookrightarrow X$ is a homotopy equivalence of C_n -coalgebras.

The result above tells us that any C_n -coalgebra X contains a homotopy equivalent C_n -subcoalgebra $S(X)$ with an extra property. Thus, to prove Theorem 4.12, the task has been reduced to showing that $S(X)$ is equivalent to an n -fold suspension as a C_n -coalgebra. It turns out that $S(X)$ is not only equivalent to a suspension, but we can say slightly more. This is the content of the next result.

Theorem 4.14. *Let X be a C_n -coalgebra. Then, the C_n -subcoalgebra $S(X)$ of Theorem 4.13 is a $\Sigma^n \Omega^n$ -coalgebra.*

Since every $\Sigma^n \Omega^n$ -coalgebra is an n -fold suspension (Proposition 4.10), Theorem 4.12 is proven. Thus, it suffices to show the two results mentioned, and we do that next.

Proof of Theorem 4.13. Denote by $i : S(X) \hookrightarrow C_n(X)$ the inclusion and by $c : X \rightarrow C_n(X)$ the coalgebra structure map.

Item 1. Let us give a retraction (of spaces) $r : X \rightarrow S(X)$, that is, a continuous map r such that $ri = \text{id}_{S(X)}$ and a homotopy $H : X \times I \rightarrow X$ between ir and id_X . The map r is given as the composition

$$r : X \hookrightarrow C_n(X) \xrightarrow{\Psi_X} \Sigma^n \Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\varepsilon_X} X.$$

The maps above are, respectively, the coalgebra structure map of X , the natural transformations Ψ and α , and the counit ε from Section 3. Since the map Ψ_X reduces the cubical support of every point to a singleton, then the image of this map is exactly the subspace $S(X)$. It further follows that ri is the identity on the subspace $S(X)$ because the map Ψ_X does not change the cubical support of points whose cubical support was just a single point already.

The homotopy \mathcal{H} from Theorem 3.1 can also be used to induce a homotopy in this case. In particular we get the following homotopy

$$\mathcal{H} : X \times I \hookrightarrow C_n(X) \times I \xrightarrow{\mathcal{H}_X} \Sigma^n \Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\varepsilon_X} X.$$

It is straightforward to check that by exactly the same arguments as in Theorem 3.1 this is indeed a homotopy between ir and id_X . Therefore the inclusion $S(X)$ is a homotopy equivalence of pointed spaces.

Item 2. To show that $S(X)$ is a C_n -subcoalgebra, we must show it is closed under the coproduct. That is, we must check that if $x \in S(X)$ then the image of the map $c(x) : C_n(1) \rightarrow X$ is contained in the subspace $S(X) \subseteq X$.

To show that this is indeed the case we make the following observation. If $d, d' \in C_n(1)$ are two cubes such that $d \subset d'$, then $c(x)(d) \neq *$ implies that $c(x)(d') \neq *$. This is because of the coassociativity of the comonad. Since d is the composition of d' with some other little cube e $d = e \circ d'$ for some little cube e we have that $c(x)(d)$ is equal to

$$C_n(1) \xrightarrow{e} C_n(1) \xrightarrow{c} X,$$

evaluated at d' . So $c(x)(d) = c(x)(d' \circ e) = e(c(x))(d')$, where $e(c(x))$ is first the composition of e in the comonad and then acting with this on the coalgebra. It therefore follows that if $d \subset d'$ then if $c(x)(d) \neq *$ then $c(x)(d') \neq *$. From this it is straightforward to deduce that if the cubical support of $c(x)$ is just a single point then the image of $c(x)$ is contained in $S(X)$, otherwise the previous identity would be violated. Therefore, $S(X)$ is a C_n -subcoalgebra and the inclusion map is a homotopy equivalence of C_n -coalgebras. \square

Proof of Theorem 4.14. To prove Theorem 4.14, we need to define a map $c' : S(X) \rightarrow \Sigma^n \Omega^n S(X)$ and show that it satisfies the comonad identities.

We define $c' : S(X) \rightarrow \Sigma^n \Omega^n S(X)$ by mapping $c'(x) := [t, \ell]$, where $t = \text{Cent}_{c(x)}$ and $\ell : S^n \rightarrow S(X)$ is given by

$$\ell(s) = c(x)(c_{s, \text{Cent}_{c(x)}}) = c(x)(c_{s,t}),$$

where $c_{s, \text{Cent}_{c(x)}}$ is the cube from the proof of Theorem 3.1. Because c' is a C_n -coalgebra map, it follows that it also satisfies the coassociativity axiom to be a $\Sigma^n \Omega^n$ -coalgebra, which completes the proof. \square

Appendix A The map α is a morphism of comonads

In this appendix, we give the necessary definitions and prove in full detail that the natural transformation

$$\alpha_n : \Sigma^n \Omega^n \rightarrow C_n$$

appearing in Theorem 3.1 defines a morphism of comonads.

Definition A.1. A morphism of comonads $\alpha : (C, \Delta, \varepsilon) \rightarrow (C', \Delta', \varepsilon')$ in a category \mathcal{M} is a natural transformation $\alpha : C \rightarrow C'$ such that for every object $X \in \mathcal{M}$, the following two diagrams commute:

$$\begin{array}{ccc}
C(X) & \xrightarrow{\alpha_X} & C'(X) \\
& \searrow \varepsilon_X & \swarrow \varepsilon'_X \\
& & X
\end{array}
\quad
\begin{array}{ccc}
C(X) & \xrightarrow{\Delta_X} & C(C(X)) \\
\alpha_X \downarrow & & \downarrow \alpha_X^2 \\
C'(X) & \xrightarrow{\Delta'_X} & C'(C'(X))
\end{array}$$

$$\varepsilon'_X \circ \alpha_X = \varepsilon_X \qquad \alpha_X^2 \circ \Delta_X = \Delta'_X \circ \alpha_X$$

The morphism α_X^2 is defined by the following diagram, which is commutative because α is a morphism of comonads.

$$\begin{array}{ccc}
C(C(X)) & \xrightarrow{\alpha_{C(X)}} & C'C(X) \\
C(\alpha_X) \downarrow & \dashrightarrow \alpha_X^2 & \downarrow C'(\alpha_X) \\
C(C'(X)) & \xrightarrow{\alpha_{C'(X)}} & C'(C'(X))
\end{array}$$

$$\alpha_X^2 = C'(\alpha_X) \circ \alpha_{C(X)} = \alpha_{C'(X)} \circ C(\alpha_X) \tag{14}$$

Next, we settle the morphism of comonads assertion made in Theorem 3.1.

Proposition A.2. *The natural transformation $\alpha_n : \Sigma^n \Omega^n \rightarrow C_n$ in Theorem 3.1 is a morphism of comonads.*

Proof. Fix an integer $n \geq 1$, and denote α_n by α to simplify the notation. Recall that object-wise, the natural transformation α is explicitly given by

$$\alpha_X : \Sigma^n \Omega^n X \xrightarrow{\gamma} C_n(\Sigma^n \Omega^n X) \xrightarrow{C_n(\eta_X)} C_n(X),$$

where γ is the \mathcal{C}_n -coalgebra structure map of $\Sigma^n \Omega^n X$ (Theorem 2.22), and η_X is the evaluation at X of the counit $\eta : \Sigma^n \Omega^n \rightarrow \text{id}_{\text{Top}_*}$ of the adjunction (Σ^n, Ω^n) . Identify

$$\Sigma^n \Omega^n X \cong S^n \wedge \text{Map}_*(S^n, X).$$

Under this identification, the counit $\eta_X : \Sigma^n \Omega^n X \rightarrow X$ becomes the evaluation map,

$$ev : S^n \wedge \text{Map}_*(S^n, X) \rightarrow X \quad ev : [t, \ell] \mapsto \ell(t).$$

Next, identify $C_n(X)$ as a subspace of $\text{Map}(\mathcal{C}_n(1), X)$. Recall that under this identification, the value of $C_n(g)$ on a map $g : \mathcal{C}_n(1) \rightarrow X$ is the postcomposition with g (Proposition 2.6). Then, the map $\alpha_X : \Sigma^n \Omega^n X \rightarrow C_n(X)$ is explicitly given on a point $[t, \ell]$ as the map

$$\alpha_X[t, \ell] : \mathcal{C}_n(1) \rightarrow X$$

whose image on a little n -cube $c \in \mathcal{C}_n(1)$ is

$$\alpha[t, \ell](c) = \begin{cases} \ell(c^{-1}(t)) & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases} \tag{15}$$

Geometrically, α_X is just re-scaling the evaluation map $ev : S^n \wedge \text{Map}_*(S^n, X)$ by shrinking the points of $S^n = I^n / \partial I^n$ according to the little n -cube c .

We can now check the commutativity of the diagrams in Definition A.1.

$$\boxed{\varepsilon'_X \circ \alpha_X = \varepsilon_X}$$

Let $[t, \ell] \in \Sigma^n \Omega^n X$. Since ε'_X plugs the identity operation $\text{id} \in \mathcal{C}_n(1)$, we have:

$$\begin{array}{ccc}
\varepsilon'_X \circ \alpha_X : \Sigma^n \Omega^n X & \xrightarrow{\alpha_X} & C_n(X) \xrightarrow{\varepsilon'_X} X \\
[t, \ell] & \longmapsto & \alpha_X[t, \ell] \longmapsto \alpha_X[t, \ell](\text{id}) = \ell(c(t))
\end{array}$$

The composition above is exactly the definition of $\varepsilon_X[t, \ell]$.

$$\boxed{\alpha_X^2 \circ \Delta_X = \Delta'_X \circ \alpha_X}$$

The map α_X^2 can be written as two different compositions, see Diagram (14). Here, we prove that

$$\alpha_{C'(X)} \circ C(\alpha_X) \circ \Delta_X = \Delta'_X \circ \alpha_X, \quad (16)$$

where $C = \Sigma^n \Omega^n \xrightarrow{\alpha_n} C' = C_n$. The left hand side of Equation (16) is the composition

$$\Sigma^n \Omega^n X \xrightarrow{\Delta_X} \Sigma^n \Omega^n (\Sigma^n \Omega^n X) \xrightarrow{\Sigma^n \Omega^n (\alpha_X)} \Sigma^n \Omega^n (C_n(X)) \xrightarrow{\alpha_{C_n(X)}} C_n(C_n(X)).$$

The maps in the composition above are given as follows.

- Denote by $\eta_X : X \rightarrow \Omega^n \Sigma^n X$ the unit of the (Σ^n, Ω^n) adjunction. Then $\Delta_X = \Sigma^n \circ \eta_X \circ \Omega^n$. Thus, a point $[t, \ell] \in \Sigma^n \Omega^n X = S^n \wedge \text{Map}_*(S^n, X)$ maps to the point $[t, \bar{\ell}] \in S^n \wedge \text{Map}_*(S^n, \Sigma^n \Omega^n X)$, where

$$\bar{\ell} : S^n \rightarrow \Sigma^n \Omega^n X \quad s \mapsto [s, \ell].$$

- The second map $\Sigma^n \Omega^n (\alpha_X)$ maps the point $[t, \bar{\ell}]$ to the point $[t, \alpha_X \circ \bar{\ell}]$.
- The last map takes a point $[t, \ell']$, where $\ell' : S^n \rightarrow C_n(X)$ is a loop, to the evaluation

$$\begin{aligned} \alpha_{C_n(X)}[t, \ell'] : \mathcal{C}_n(1) &\longrightarrow C_n(X) \\ c &\longmapsto \ell'(c^{-1}(t)) \end{aligned}$$

Therefore, with the notation above, the full composition applied to a point $[t, \ell]$ yields

$$[t, \ell] \mapsto [t, \bar{\ell}] \mapsto [t, \alpha_X \circ \bar{\ell}] \mapsto \alpha_{C_n(X)}[t, \alpha \circ \bar{\ell}].$$

The resulting map

$$\alpha_{C_n(X)}[t, \alpha \circ \bar{\ell}] : \mathcal{C}_n(1) \rightarrow C_n(X)$$

acts on a little n -cube $c \in \mathcal{C}_n(1)$ by producing

$$c \mapsto (\alpha_X \circ \bar{\ell})(c^{-1}(t)) = \alpha[c^{-1}(t), \ell] : \mathcal{C}_n(1) \rightarrow X,$$

where $c_2 \in \mathcal{C}_n(1)$ gets mapped to

$$\alpha[c^{-1}(t), \ell](c_2) = \ell(c_2^{-1}(c^{-1}(t))).$$

The right hand side of Equation (16) is the composition

$$\Sigma^n \Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\Delta'_X} C_n(C_n(X))$$

The first map in the composition above was given in Equation (15). The map Δ'_X , described in Proposition 2.10, applies an arbitrary map $h : \mathcal{C}_n(1) \rightarrow X$ to the map $\bar{h} : \mathcal{C}_n(1) \rightarrow C_n(X)$ given by

$$\mu \in \mathcal{C}_n(1) \mapsto \bar{h}(\mu) : \mathcal{C}_n(1) \rightarrow X, \quad \bar{h}(\mu)(\theta) := h(\gamma(\mu; \theta)).$$

In particular, Δ'_X applies the map $\alpha_X[t, \ell]$ to the map

$$\Delta'_X(\alpha_X[t, \ell]) : \mathcal{C}_n(1) \longrightarrow C_n(X)$$

$$c \longmapsto \Delta'_X(\alpha_X[t, \ell])(c) = \overline{\alpha[t, \ell]}(c) : \mathcal{C}_n(1) \longrightarrow X$$

$$c_2 \longmapsto \ell(\gamma(c; c_2)^{-1}(t))$$

Since, by definition of the composition in the little cubes operad,

$$\ell(c_2^{-1}(c^{-1}(t))) = \ell(\gamma(c; c_2)^{-1}(t))$$

for all little cubes c, c_2 , the claim is proven. \square

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