The spectrum of local random Hamiltonians

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The spectrum of a local random Hamiltonian can be represented generically by the so-called ε -free convolution of its local terms' probability distributions. We establish an isomorphism between the set of ε -noncrossing partitions and permutations to study its spectrum. Moreover, we derive some lower and upper bounds for the largest eigenvalue of the Hamiltonian.

I. INTRODUCTION

In general, the Hamiltonian of a many-body system is local; namely, the Hamiltonian can be derived from a sum of local terms which describe the interactions between the local systems. For instance, if the particles of a quantum spin only correlate with the short-range, then the total Hamiltonian of the spin is local. Understanding their Hamiltonians' spectrum is essential for their theoretical and experimental aspects [1–3]. Furthermore, if the interactions are disordered, many striking phenomena appear. For example, the upper bound of the speed of information propagation through a quantum spin chain with disordered interactions may be significantly lower than the famous Lieb-Robinson's bound [4], which indicates the system exhibits the phenomenon of Anderson localization [5].

In this paper, we consider the following general model: let *n* be the number of local systems and let *d* be the local dimension. Thus, the Hilbert space, which describes the total system, is given by an *n*-fold tensor product $(\mathbb{C}^d)^{\otimes n}$. Let $\mathscr{K}_t, t = 1, ..., m$ be subsets of the integer set $\{1, ..., n\}$ and suppose that the interactions only occur between the sites in \mathscr{K}_t . As a result, the system's total Hamiltonian *H* of the system is given by the sum of local terms, namely,

$$H = \sum_{l=1}^{m} H_l, \tag{1}$$

where $H_t \in \bigotimes_{s \in \mathscr{K}_t} \mathbb{M}_d(\mathbb{C}) \otimes \bigotimes_{s \notin \mathscr{K}_t} (\mathbb{C} \mathbb{1}_d)$. To express the disorderliness of the interactions, we can ideally assume that H_t 's are sampled by some given random ensembles. The above general framework covers a lot of explicit models, such as the Heisenberg, Ising, XY, XXZ, AKLT models, etc., which are commonly studied in quantum many-body physics. Thus, a natural question arises: can one figure out the spectrum of given local random Hamiltonians? Unfortunately, obtaining an analytical result is not easy since, generally, the local terms do not commute. For quantum spin models (d = 2), the Jordan-Wigner transformation can be used to reduce the problem to an equivalent single Hamiltonian [6]. However, a large amount of work relies on numerical analysis.

Another line of investigation is carried out in the asymptotic picture $(d \to \infty)$. One can consider the random Hamiltonian H in the framework of random matrix theory and study its spectrum in the large d limit. The idea may date back to the 1950s when Wigner studied a model of complicated nuclei. The Hamiltonian of the system can be equivalently considered as a large dimensional random matrix, and the distribution of its eigenvalues converges to the celebrated Wigner's semicircular law [7]. Later, when studying the O(d) and U(d) quantum field theories, 't Hooft discovered that remarkable simplifications occur in the large d limit [8, 9]. Occasionally but surprisingly, the abovementioned theory satisfies Voiculescu's free probability theory [10]. Indeed, the limit law as $d \to \infty$ can be described using random variables in the framework of free probability [11–14].

In [15], Morampudi and Laumann proposed a systematic pictorial method to calculate the correlators based on randomly interacting spin systems with spatial locality. According to 't Hooft's idea, only the so-called monochromatic stacked planar diagrams survive in the large d limit. Consequently, in the large d limit, an algebraic relation known as "heap-freeness" emerges among H_i 's. This phenomenon was also independently observed by Charlesworth and Collins [16]. Due to their theory, heap-freeness is equivalently called ε -freely independence, a mixture of classical and free independence [17, 18]. Here ε is a symmetric matrix given by \mathcal{K}_i 's in our setting.

Therefore, motivated by the free probability theory, we show that the limit of the spectrum of *H* can be given by the so-called ε -free convolution (see Definition III.1) of distributions of local terms for some given random ensembles (see Theorem III.1 and III.2). We remark that this conclusion was also implicitly indicated in [15]. The key point is to establish an isomorphism

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between the set of all ε -noncrossing partitions [17–19] (called monochromatic dependency partitions in [15]) and the set of all ε -noncrossing permutations. Moreover, the Möbius functions of the corresponding set of partitions should also be considered (see Definition II.3 and Theorem II.1). Our second result is that the convergence of the spectrum distribution holds almost surely. To this end, we study the fluctuation of the correlators and obtain a stronger result of the concentration of measure than the one in [15] (see Proposition III.7 and III.8). Finally, by some rough enumerations of ε -noncrossing partitions, we derive some bounds for the largest eigenvalue of *H* (see Proposition IV.1 and IV.3).

II. PRELIMINARIES

A. ε -noncrossing partition and permutation

In this paper, we introduce a symmetric matrix ε , reflecting the particles' interactions, given by the following definition. We use [n] to denote the integer set $\{1, \ldots, n\}$.

Definition II.1. Let $\mathscr{K}_{\iota}, \iota = 1, ..., m$ be subsets of the integer set [n], we define $\varepsilon = (\varepsilon_{i,j})_{i,j=1}^{m}$ by a $m \times m$ symmetric matrix with entries $\varepsilon_{i,j}$ satisfies

$$\boldsymbol{\varepsilon}_{i,j} = \begin{cases} 1 & \text{if } \mathcal{K}_i \cap \mathcal{K}_j = \boldsymbol{\emptyset}; \\ 0 & \text{otherwise.} \end{cases}$$
(2)

We will use a bunch of notions in combinatorics and refer to Appendix A for more details. Let $\mathscr{P}(k)$ be the set of all partitions of [k], and let $\iota : [k] \to [m]$ be a map given by

$$\boldsymbol{\iota} = (\iota_1, \iota_2, \ldots, \iota_k), \iota_i \in [m], i = 1, \ldots, k.$$

We denote ker ι by a partition in $\mathscr{P}(k)$ such that i, j belongs to the same block of ker ι whenever $\iota_i = \iota_j$.

Thus, for the symmetric matrix ε given in (2) and a given map $\iota : [k] \to [m]$, one can define the (ε, ι) -noncrossing partitions as follows:

Definition II.2. [18, 19]. A partition $\alpha \in \mathscr{P}(k)$ is called (ε, ι) -noncrossing if there exist $1 \le i such that <math>i \sim_{\alpha} j \nsim_{\alpha} p \sim_{\alpha} q$, then we must have $\varepsilon_{\iota_i, \iota_p} = 1$. Note that $i \sim_{\alpha} j$ means the elements i, j are in the same block of α . We define

$$\mathscr{NC}^{(\varepsilon,\iota)}(k) := \{ \alpha \in \mathscr{P}(k) : \alpha \leq \ker \iota \text{ and is } (\varepsilon,\iota) \text{-noncrossing} \}.$$

It is known that $\mathcal{NC}^{(\varepsilon,\iota)}(k)$ is a lattice, and the poset order is defined by the refinement of partitions.

Denote $\mathcal{NC}(k)$ by the set of all noncrossing partitions of [k], here are some extreme cases.

- (i) If \mathscr{K}_{l} 's are pairwise disjoint, then every non-diagonal entry of ε is 1. Then $\mathscr{NC}^{(\varepsilon,\iota)}(k) = \{\alpha \in \mathscr{P}(k) : \alpha \leq \ker \iota\}$.
- (ii) If $\mathscr{K}_1 = \mathscr{K}_2 = \cdots = \mathscr{K}_m$, then every entry of ε is 0. Then $\mathscr{NC}^{(\varepsilon,\iota)}(k) = \{\alpha \in \mathscr{NC}(k) : \alpha \leq \ker \iota\}$.
- (iii) If ker $\iota \in \mathscr{NC}(k)$, then $\mathscr{NC}^{(\varepsilon,\iota)}(k) = \{\alpha \in \mathscr{NC}(k) : \alpha \leq \ker \iota\}.$

Example II.1. Let m = 4, and take the entries of ε as follows: $\varepsilon_{1,3} = \varepsilon_{1,4} = \varepsilon_{2,4} = 1$, and other entries are 0's.

- (i) Let k = 8 and $\iota = (\iota_1, \iota_2, \dots, \iota_8) = (1, 3, 3, 1, 3, 2, 4, 2)$, then ker $\iota = \{\{1, 4\}, \{2, 3, 5\}, \{6, 8\}, \{7\}\};$
- (ii) Let $\alpha = \{\{1,4\},\{2\},\{3,5\},\{6,8\},\{7\}\},$ then $\alpha \in \mathcal{NC}^{(\varepsilon,\iota)}(8)$ for the above given ι .

We denote the set of permutations on *n* elements by S_n . For a permutation $\pi \in S_n$, we denote $|\pi|$ by the minimal number of transpositions needed to decompose π . Let $\gamma_n \in S_n$ be the full cycle $\gamma_n = (1, 2, ..., n)$. For any $\pi \in S_n$, we always have

$$|\pi|+|\pi^{-1}\gamma_n|\leq n-1$$

We call π noncrossing if the equality holds, or equivalently we can say that π lies on the geodesic path $1_n - \pi - \gamma_n$ in S_n .

Notation II.1. For given ε and $\iota : [k] \to [m]$, we use the following notations introduced in [16]:

(i) $S_k^{(\iota)}$ is the group of permutations σ with $\iota_j = \iota_{\sigma(j)}$ for all j. Hence each $\sigma \in S_k^{(\iota)}$ stabilize ker ι . In particular, we denote $\mathscr{P}_2^{(\iota)}(k)$ by the set of pair partitions of [k] which stabilize ker ι .

- (ii) For each $s \in [n]$, we denote $J_s := \{j \in [k] : s \in \mathscr{K}_{t_j}\}$, and moreover we denote $k_s := \#J_s$. We denote γ_s by the full cycle of J_s and by 1_s the identity of permutation on J_s .
- (iii) Any $\sigma \in S_k^{(\iota)}$ induces a permutation $\sigma_s \in S_{J_s} \simeq S_{k_s}$, since it preserves each J_s .
- (iv) For any $\alpha \in \mathscr{P}(k)$, we denote $\alpha_s := \alpha|_{J_s}$ by the restriction of α on J_s , and by $\alpha_B := \alpha|_B$ for any block of ker ι .
- (v) We use $\#_{k_s}(\cdot)$ to stress the permutation in the bracket is viewed as an element of S_{k_s} and not as the induced permutation in S_k which is constant on J_s .

Note that J_s is the union of some blocks of ker ι and it might be an empty set for some $s \in [n]$ which depends on the map ι . It is clear that

$$\sum_{s=1}^{n} k_{s} = \sum_{\ell=1}^{k} \#(\mathscr{K}_{\iota_{\ell}}),$$
(3)

where $\#(\cdot)$ means the cardinality of a given set. Now we are ready to introduce the notion of (ε, ι) -noncrossing permutations.

- **Definition II.3.** *For given* ε *and* $\iota : [k] \rightarrow [m]$ *,*
 - (i) We define the set of all (ε, ι) -noncrossing permutations of [k], denoted by $S_{NC}^{(\varepsilon, \iota)}(\gamma_k)$, as follows:

$$S_{NC}^{(\varepsilon,\iota)}(\gamma_k) := \{ \sigma \in S_k^{(\iota)} : 1_s - \sigma_s - \gamma_s \text{ for all } s \in [n] \}.$$

In other words, the restriction to any J_s , σ_s is a noncrossing permutation.

(ii) For $\sigma, \tau \in S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$ we say that $\sigma \leq \tau$, if for any $s \in [n]$, σ_s and τ_s lie on the same geodesic and if σ_s comes before τ_s , i.e.,

$$1_s - \sigma_s - \tau_s - \gamma_s$$
, for all $s \in [n]$.

Example II.2. Let m = 4 and n = 5, let $\mathscr{K}_{\iota} = \{\iota, \iota+1\}, \iota = 1, \ldots, 4$, therefore $\varepsilon_{1,3} = \varepsilon_{1,4} = \varepsilon_{2,4} = 1$ and other entries of ε are 0's. Moreover, let k = 8 and $\iota = \{\iota_1, \iota_2, \ldots, \iota_8\} = (1, 3, 3, 1, 3, 2, 4, 2)$. Then $S_8^{(\iota)}$ is generated by the transpositions (1, 4), (6, 8), (2, 3) and (2, 5). So that we have

$$J_1 = \{1,4\}, J_2 = \{1,4,6,8\}, J_3 = \{2,3,5,6,8\}, J_4 = \{2,3,5,7\}, J_5 = \{7\}.$$

If $\sigma = (1,4)(3,5)(6,8) \in S_8$ we have

$$\sigma_1 = (1,4), \sigma_2 = (1,4)(6,8), \sigma_3 = (3,5)(6,8), \sigma_4 = (3,5), \sigma_5 = \mathbb{1}_{J_5}$$

It is easy to check that $\sigma \in S_{NC}^{(\varepsilon,\iota)}(\gamma_8)$ for given ε and ι .

In [20], Biane showed that there is an isomorphism (of posets) between noncrossing partitions and permutations, which plays an essential role in the aspect of combinatoric free probability theory. Likewise, we can obtain the following isomorphism between ε -noncrossing partitions and permutations. We refer to Example II.1 and II.2 for the isomorphism.

Theorem II.1. With the relation " \leq ", $S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$ becomes a poset. Moreover, there is a bijection between $\mathcal{NC}^{(\varepsilon,\iota)}(k)$ and $S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$ which preserves the poset structure.

Proof. It is elementary to check that $S_{NC}^{(\varepsilon)}(\gamma_k)$ is a poset with the given relation. Let α be a partition of [k]; we will denote P_{α} by the permutation $\pi \in S_k$, which is determined by the following properties:

(a) α is π -invariant, i.e., π stabilize the blocks of α ;

(b) if $B = \{i_1, i_2, \dots, i_s\}$ is a block of α , with $1 \le i_1 < i_2 < \dots < i_s \le k$, then we have $\pi(i_1) = i_2, \dots, \pi(i_{s-1}) = i_s, \pi(i_s) = i_1$.

It induces a map $P : \mathscr{P}(k) \to S_k$ given by $P(\alpha) := P_\alpha = \pi$ for $\alpha \in \mathscr{P}(k)$. It is known [20, 21] that the map P is a bijection between $\mathscr{NC}(k)$ and $S_{NC}(\gamma_k)$ which preserves the poset structure.

Claim. *P* is also an isomorphism between $\mathscr{NC}^{(\varepsilon,\iota)}(k)$ and the set $S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$.

Firstly, by the definition of *P*, it is clear that for any $\alpha \leq \ker \iota$, $P_{\alpha} \in S_k^{(\iota)}$. Now for any $\alpha \in \mathscr{NC}^{(\varepsilon,\iota)}(k)$, we will show that π_s is noncrossing on J_s for every $s \in [n]$, where $\pi = P_{\alpha}$. Hence $P(\alpha) \in S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$. Suppose this is not ture, i.e., there exists a $s \in [n]$ such that π_s is crossing on J_s . Accordingly there exists $1 \leq i such that$

(i) $i, p, j, q \in J_s$;

(ii) $i \sim_{B_{1,s}} j, p \sim_{B_{2,s}} q$, where $B_{1,s}$ and $B_{2,s}$ are two different blocks of α_s .

By (i), we have $s \in \mathscr{K}_{l_i}$ and $s \in \mathscr{K}_{l_p}$. Therefore $\mathscr{K}_{l_i} \cap \mathscr{K}_{l_p} \neq \emptyset$. By the definition of ε we can deduce that $\varepsilon_{l_i,l_p} = 0$. However, by (ii), there must have two different blocks B_1 and B_2 of α such that $i \sim_{B_1} j, p \sim_{B_2} q$. In fact $B_{i,s} = B_i|_{J_s}$. Thus $\iota_i = \iota_j$ and $\iota_p = \iota_q$. Since $\alpha \in \mathscr{NC}^{(\varepsilon,\iota)}(k)$, one has $\varepsilon_{\iota_i,\iota_p} = 1$, which is a contradiction.

Conversely, suppose that $\alpha \notin \mathscr{NC}^{(\varepsilon,\iota)}(k)$, then there exist a crossing $1 \le i with <math>\varepsilon_{\iota_i,\iota_p} = 0$. By the definition of ε we have $\mathscr{K}_{\iota_i} \cap \mathscr{K}_{\iota_p} \ne \emptyset$. Hence there exists a $s \in [n]$ such that $i, p \in J_s$. Moreover, recall that $\iota_i = \iota_j$ and $\iota_p = \iota_q$, we have $i, p, j, q \in J_s$. So, finally, we find a crossing partition in J_s . Thus the restriction of P_{α} on J_s is crossing, which induces that $P_{\alpha} \notin S_{NC}^{(\varepsilon,\iota)}(\gamma_k)$.

Now we turn to show that *P* preserves the poset structure. Let $\alpha, \beta \in \mathscr{NC}^{(\varepsilon,\iota)}(k)$ and denote $\pi = P_{\alpha}, \sigma = P_{\beta}$. Suppose that $\alpha \leq \beta$. Obviously, we have $\alpha_s \leq \beta_s$ for any $s \in [n]$. Now consider the restriction of *P* on $\mathscr{P}(k_s)$, denote by P_s . Certainly, we have P_s maps α_s (resp. β_s) to π_s (resp. σ_s). Hence by Biane's isomorphism [20, 21], P_s is a bijection between $\mathscr{NC}(k_s)$ and $S_{NC}(\gamma_s)$ which preserves the poset structure. It induces that $1_s - \pi_s - \sigma_s - \gamma_s$ for any $s \in [n]$, which implies $\pi \leq \sigma$.

Notation II.2. For given $\iota : [k] \to m$, we denote $\mathscr{NC}_2^{(\varepsilon,\iota)}(k)$ by the set of all pair partitions of $\mathscr{NC}^{(\varepsilon,\iota)}(k)$.

Since each pair partition in $\mathscr{P}_2(k)$ is identified with a transposition of S_k via the map P, then we have

$$\mathscr{P}_{2}^{(\iota)}(k) = \{ \sigma \in S_{k}^{(\iota)} : \sigma \text{ is a transposition} \}.$$

Moreover, by the above proposition, we have

$$\mathscr{NC}_{2}^{(\varepsilon,\iota)}(k) = \{ \alpha \in \mathscr{P}_{2}^{(\iota)}(k) : 1_{s} - \sigma_{s} - \gamma_{s} \text{ for all } s \in [n] \},$$
(4)

where $\sigma = P_{\alpha}$.

Let us remark that for the above-mentioned extreme cases (ii) and (iii), one can reduce to Biane's isomorphism between the noncrossing partitions and permutations [20] (see Proposition A.1).

B. Convolution operations and Möbius inversion

Let \mathscr{P} be a finite poset, denote

$$\mathscr{P}^{(2)} := \{ (\alpha, \beta) : \alpha, \beta \in \mathscr{P}, \alpha \leq \beta \}$$

For $F, G: \mathscr{P}^{(2)} \to \mathbb{C}$, their convolution F * G is given by:

$$(F * G)(\alpha, \beta) := \sum_{\substack{\eta \in \mathscr{P}, \\ \alpha \le \eta \le \beta}} F(\alpha, \eta) G(\eta, \beta).$$

The zeta function $\zeta : \mathscr{P}^{(2)} \to \mathbb{C}$ is defined by

$$\zeta(\alpha,\beta) = 1, \ \forall (\alpha,\beta) \in \mathscr{P}^{(2)}.$$

And the unit $\delta : \mathscr{P}^{(2)} \to \mathbb{C}$ of the convolution is given by

$$\delta(\alpha,\beta) = \begin{cases} 1 & \text{if } \alpha = \beta; \\ 0 & \text{if } \alpha < \beta. \end{cases}$$

Definition II.4. [21]. For given poset \mathscr{P} , the Möbius function of \mathscr{P} , denoted it by μ , is the inverse of ζ under convolution, i.e., $\mu * \zeta = \delta$. Equivalently, the Möbius function is uniquely determined by the following equations

$$\sum_{\substack{\boldsymbol{\eta}\in\mathscr{P},\\\boldsymbol{\alpha}\leq\boldsymbol{\eta}\leq\boldsymbol{\beta}}}\mu(\boldsymbol{\eta},\boldsymbol{\beta}) = \begin{cases} 1 & \text{if } \boldsymbol{\alpha}=\boldsymbol{\beta};\\ 0 & \text{if } \boldsymbol{\alpha}<\boldsymbol{\beta}. \end{cases}$$
(5)

To avoid confusion, let us denote $\mu_{\mathscr{P}}(\alpha,\beta)$ by the Möbius function of the given poset \mathscr{P} .

Möbius function of $\mathscr{NC}^{(\varepsilon,\iota)}(k)$ -Note that the Möbius function of $\mathscr{NC}(k)$ can be explicitly computed by the products of Catalan numbers [21]. Here we will give a direct method to compute the Möbius function of $\mathscr{NC}^{(\varepsilon,\iota)}(k)$ by using Theorem II.1.

Proposition II.2. Fix ε and $\iota : [k] \to [m]$, the Möbius function of $\mathcal{NC}^{(\varepsilon,\iota)}(k)$ can be computed as follows:

$$\mu_{\mathscr{NC}^{(\varepsilon,\iota)}(k)}(\alpha,\beta) = \prod_{B \in \ker\iota} \mu_{\mathscr{NC}(\#(B))}(\alpha_B,\beta_B),\tag{6}$$

for all $\alpha, \beta \in \mathscr{NC}^{(\varepsilon,\iota)}(k)$.

Proof. It sufficies to check Equation (5) for the poset $\mathscr{NC}^{(\varepsilon,\iota)}(k)$. Firstly, suppose that $\alpha = \beta$, thus $\alpha_B = \beta_B$ for every $B \in \ker \iota$, then we have

$$\prod_{B \in \ker \iota} \mu_{\mathscr{NC}(\#(B))}(\alpha_B, \beta_B) = 1$$

since $\mu_{\mathscr{NC}(\#(B))}(\alpha_B, \beta_B) = 1$ for every *B*.

Now suppose that $\alpha < \beta$, then there should exist a $s \in [n]$ such that such that α_s is a strict refinement of β_s , i.e., $\alpha_s < \beta_s$. Denote $S := \{B \in \ker \iota : B \subseteq J_s\}$ we have

$$\sum_{\substack{\eta \in \mathscr{NC}^{(\varepsilon,\iota)}(k), B \in \ker \iota \\ \alpha \leq \eta \leq \beta}} \prod_{\substack{B \in \ker \iota \\ \alpha \leq \eta \leq \beta}} \mu_{\mathscr{NC}(\#(B))}(\eta_B, \beta_B) = \sum_{\substack{\eta \in \mathscr{NC}^{(\varepsilon,\iota)}(k), B \in S \\ \alpha \leq \eta \leq \beta}} \prod_{\substack{B \in S \\ \alpha \leq \eta \leq \beta}} \mu_{\mathscr{NC}(k_s)}(\eta_B, \beta_B) \cdot \prod_{\substack{B \notin S \\ B \notin S}} \mu_{\mathscr{NC}(\#(B))}(\eta_B, \beta_B)$$
$$= \sum_{\substack{\eta \in \mathscr{NC}^{(\varepsilon,\iota)}(k), \\ \alpha \leq \eta \leq \beta}} \mu_{\mathscr{NC}(k_s)}(\eta_s, \beta_s) \cdot \sum_{\substack{B \notin S, \\ \alpha_B \leq \eta_B \leq \beta_B}} \prod_{\substack{B \notin S \\ B \notin S}} \mu_{\mathscr{NC}(\#(B))}(\eta_B, \beta_B)$$
$$= 0.$$

where we used the fact that

$$\sum_{lpha_s\leq \eta_s\leq eta_s} \mu_{\mathscr{NC}(k_s)}(\eta_s,eta_s)=0.$$

Remark II.1. For any $\alpha, \beta \in \mathcal{NC}^{(\varepsilon,\iota)}(k)$, we have

$$[\alpha,\beta] \cong \prod_{B \in \ker \iota} [\alpha_B,\beta_B],\tag{7}$$

where $[\alpha, \beta] := \{\eta \in \mathcal{NC}^{(\varepsilon,\iota)}(k) : \alpha \leq \eta \leq \beta\}$ and $[\alpha_B, \beta_B] := \{\eta_B \in \mathcal{NC}(\#(B)) : \alpha_B \leq \eta_B \leq \beta_B\}$. Therefore, Proposition II.2 is a natural corollary of (7). We refer to [21] for more details.

C. Framework of noncommutative probability

Noncommutative probability space–A noncommutative probability space [21] (\mathscr{A}, ϕ) is an unital algebra \mathscr{A} over \mathbb{C} , with a unital linear functional

$$\phi: \mathscr{A} \to \mathbb{C}; \ \phi(\mathbf{1}_{\mathscr{A}}) = 1$$

An element $a \in \mathscr{A}$ is called a noncommutative random variable, and it is called centered if $\phi(a) = 0$. Here are two examples of noncommutative probability spaces.

(i) $(L^{\infty}(\mathbb{R},\mu),\mathbb{E})$, where μ is a probability measure supporting on \mathbb{R} , and \mathbb{E} is the expectation defined by $\mathbb{E}[f] := \int_{\mathbb{R}} f(t) d\mu(t)$.

(ii) $(\mathbb{M}_d(\mathbb{C}), \mathrm{tr})$, where $\mathbb{M}_d(\mathbb{C})$ is the algebra of $d \times d$ matrices, and $\mathrm{tr} := \mathrm{Tr}/d$ is the normalized trace.

 ε -Independence–Independence plays a central role in probability theory. Roughly speaking, it provides a methodology to calculate the joint moments of random variables. The notation of ε -freely independence was firstly introduced by Mlotkowski [17] and later recovered by Speicher, and Wysoczanski [18]. Here ε is a symmetric $m \times m$ matrix with entries to be 0 or 1,, and we follow the convention that the diagonal entries of ε are 0's. Fix a symmetric matrix ε , we denote $I_k^{(\varepsilon)}$ by the set of all *k*-tuples of indices $(\iota_1, \ldots, \iota_k)$ from the integer set [m] such that whenever $\iota_i = \iota_j$ with $1 \le i < j \le k$ there is an ℓ with $i < \ell < j$, $\iota_i \neq \iota_\ell$, and $\varepsilon_{\iota_i, \iota_\ell} = 0$. Given a noncommutative probability space (\mathscr{A}, ϕ) , let $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m$ be a sequence of subalgebras of \mathscr{A} . We call $\mathscr{A}_1, \mathscr{A}_2, \ldots, \mathscr{A}_m$ are ε -freely independent, if the following holds:

- (i) \mathscr{A}_i and \mathscr{A}_j commute whenever $\varepsilon_{i,j} = 1$;
- (ii) If for any centred elements $a_j \in \mathscr{A}_{i_j}$, $j = 1, \ldots, k$,

$$\phi(a_1 \cdots a_k) = 0 \tag{8}$$

whenever we have $(\iota_1, \ldots, \iota_k) \in I_k^{(\varepsilon)}$.

A sequence of noncommutative random variables a_1, \ldots, a_m are said to be ε -freely independent if the subalgebras they generate are ε -freely independent. Note that if the non-diagonal entries of ε are all 0's (resp. 1's), then we reduce to Voiculescu's free independence (resp. classical independence).

Moment-cumulant formulas for independent random variables–Let (\mathscr{A}, φ) be a noncommutative probability space. Given a family of random variables $a_1, \ldots, a_m \in (\mathscr{A}, \varphi)$, the mixed moments of the random variables a_1, \ldots, a_m are given by $\varphi(a_{\iota_1} \cdots a_{\iota_k})$, where $\iota = (\iota_1, \ldots, \iota_k) : [k] \to [m]$.

Denote

$$\phi_m(a_1,\ldots,a_m):=\phi(a_1\cdots a_m).$$

Then for any partition $\alpha \in \mathscr{P}(m)$, one can define [21]

$$\phi_{\alpha}[a_1,\ldots,a_m] := \prod_{V \in \alpha} \phi(V)[a_1,\cdots,a_m], \tag{9}$$

where *V* is the block of α and

$$\phi(V)[a_1, \cdots, a_m] := \phi_{\#(V)}(a_{i_1}, \cdots, a_{i_r}) \text{ for } V = (i_1, \cdots, i_r).$$

Hence $(\phi_{\alpha})_{\alpha \in \mathcal{NC}(m)}$ are multiplicative functionals on \mathscr{A}^m , namely, they factorize in a product according to the block structure of partitions [21].

Suppose that $\mathscr{A}_1, \ldots, \mathscr{A}_m$ are ε -freely independent subalgebras of \mathscr{A} , then for arbitrary $\iota : [k] \to [m]$, the mixed moment can be represented as follows [18, 19]:

$$\phi(a_1 \cdots a_k) = \sum_{\alpha \in \mathscr{NC}^{(\varepsilon,\iota)}(k)} \kappa_{\alpha}^{(\varepsilon)}[a_1, \dots, a_k],$$
(10)

where $a_j \in \mathscr{A}_{i_j}, j = 1, ..., k$. The multilinear function $\kappa_{\alpha}^{(\varepsilon)}[\cdots]$ is called ε -free cumulant [19]. Note that if every non-diagonal entry of ε is 0, Equation (10) reduces to the following free cumulant-moment formula [21]

$$\phi(a_1 \cdots a_k) = \sum_{\substack{\alpha \in \mathscr{NC}(k), \\ \alpha < \ker \iota}} \kappa_{\alpha}[a_1, \dots, a_k],$$
(11)

where $\{\kappa_{\alpha}\}_{\alpha \in \mathscr{NC}(k)}$ are called free cumulants. And if every non-diagonal entry of ε is 1, we have the following classical cumulant-moment formula

$$\phi(a_1 \cdots a_k) = \sum_{\substack{\alpha \in \mathscr{P}(k), \\ \alpha \le \ker \iota}} k_{\alpha}[a_1, \dots, a_k],$$
(12)

where $\{k_{\alpha}\}_{\alpha \in \mathscr{P}(k)}$ are called classical cumulants. We note that a_1, \ldots, a_m are freely independent in Equation (11) and classical independent in Equation (12), respectively. Indeed, the cumulants reflect the independence of random variables [21], and the above moment-cumulant formulas indicate that the varieties of partitions yield different types of independence in the context of noncommutative probability space (see Table I).

TABLE I: Partitions and Independence

Independence	Partitions	Cumulants
Classical	All partitions	Classical cumulants
Free	Noncrossing partitions	Free cumulants
€-Free	E-Noncrossing partitions	ε -Free cumulants

Moreover, since we have made the convention that the diagonal terms of ε are 0's, only free cumulants contributes in the sum (10) (see [18][Theorem 5.2, Remark 5.4]), namely, for each $\alpha \in \mathcal{NC}^{(\varepsilon,\iota)}(k)$

$$\kappa_{\alpha}^{(\varepsilon)}[a_1,\ldots,a_k] := \kappa_{\alpha}[a_1,\ldots,a_k] = \prod_{V \in \alpha} \kappa_{\#(V)}[a_1,\ldots,a_k],$$
(13)

where $\kappa_{\alpha}[a_1, \ldots, a_k]$ is the product of the free cumulants for each block.

Finally, it is known [19] that $\mathcal{NC}^{(\varepsilon,\iota)}(k)$ is a lattice, and the ε -free cumulants are multiplicative. Therefore, due to the standard theory of convolution, one can obtain the following so-called Möbius inversion of Equation (10)

$$\kappa_{\alpha}^{(\varepsilon)}[a_1,\ldots,a_k] = \sum_{\substack{\beta \in \mathscr{NC}^{(\varepsilon,\iota)}(k), \\ \beta \le \alpha}} \phi_{\beta}[a_1,\ldots,a_k] \cdot \mu_{\mathscr{NC}^{(\varepsilon,\iota)}(k)}(\beta,\alpha)$$
(14)

for every $\alpha \in \mathscr{NC}^{(\varepsilon,\iota)}(k)$.

III. EMPIRICAL DISTRIBUTION OF THE EIGENVALUES

Empirical distribution of the eigenvlues–Let λ_i , $i = 1, ..., d^n$ denote the eigenvalues of H, and define [22] the empirical distribution of the eigenvalues as the probability measure on \mathbb{R} by

$$\mu_H = \frac{1}{d^n} \sum_{i=1}^{d^n} \delta_{\lambda_i}.$$
(15)

Let (\mathscr{A}, ϕ) be a noncommutative probability space. A random variable $a \in \mathscr{A}$ has a probability distribution μ on \mathbb{R} if the following condition holds

$$\phi(a^k) = \int_{\mathbb{R}} t^k \, d\mu(t), \text{ for all } k \in \mathbb{N}.$$
(16)

If we consider *H* in the framework of noncommutative probability space $(\mathbb{M}_{d^n}(\mathbb{C}), \mathrm{tr})$, then we say μ_H converges weakly, almost surely, to a probability measure μ on \mathbb{R} if almost surely,

$$\lim_{d \to \infty} \operatorname{tr} H^k = \int_{\mathbb{R}} t^k \, d\mu(t), \text{ for all } k \in \mathbb{N}.$$
(17)

Define the semicircle distribution μ_{sc} as the probability distribution $\sigma(t)dt$ on \mathbb{R} with density

$$\sigma(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \cdot \mathbf{1}_{|t| \le 2}$$

Suppose that *H* is globally sampled by Gaussian unitary ensemble (GUE); it is well known that μ_H converges weakly, almost surely, to μ_{sc} as the dimension *d* goes to large [7].

 ε -free convolution-convolution is an operation of probability measures. Suppose $a_1, \ldots, a_m \in \mathscr{A}$ associate with probability distribution μ_1, \ldots, μ_m , respectively. If a_1, \ldots, a_m are classical (resp. freely) independent, then the classical (resp. freely) convolution of μ_1, \ldots, μ_m , denoted by $\mu_1 * \cdots * \mu_m$ (resp. $\mu_1 \boxplus \cdots \boxplus \mu_m$), can be defined as follows [21]:

$$\phi\left(\sum_{i=1}^{m} a_i\right)^k = \int_{\mathbb{R}} t^k d(\mu_1 \ast \cdots \ast \mu_m)(t) \text{ (resp.} = \int_{\mathbb{R}} t^k d(\mu_1 \boxplus \cdots \boxplus \mu_m)(t))$$

for all $k \in \mathbb{N}$. Motivated by the above definition, we have the following definition

Definition III.1. Let (\mathscr{A}, ϕ) be a noncommutative probability space, let $a_1 \in \mathscr{A}$ with probability distribution $\mu_1, \iota = 1, ..., m$. Suppose that $a_1, ..., a_m$ are ε -freely independent for given ε . We define the ε -free convolution of $\mu_1, ..., \mu_m$, denoted by $\mu_1 \boxplus_{\varepsilon} \cdots \boxplus_{\varepsilon} \mu_m$, be the jointly distribution of $a_1 + \cdots + a_m$, i.e.,

$$\phi\left(\sum_{i=1}^{m} a_{i}\right)^{k} = \int_{\mathbb{R}} t^{k} d\mu_{a_{1}+\dots+a_{m}}(t)$$

$$:= \int_{\mathbb{R}} t^{k} d\left(\mu_{1} \boxplus_{\varepsilon} \dots \boxplus_{\varepsilon} \mu_{m}\right)(t)$$
(18)

for all $k \in \mathbb{N}$.

It is clear that if every non-diagonal entry of ε is 1 (resp. every entry of ε is 0) then the ε -free convolution reduces to the classical (resp. free) convolution.

Main results-Case 1: Suppose that the local terms H_{ι} , $\iota = 1, ..., m$ are independently sampled by GUE. Namely, for every $\iota = 1, ..., m$ we suppose

$$H_{\iota} = G_{\iota} \otimes \bigotimes_{s \notin \mathscr{K}_{\iota}} (\mathbb{C} 1\!\!1_d), \tag{19}$$

where $\{G_{\iota}, \iota = 1, ..., m\}$ is a family of independent GUEs with G_{ι} in $\bigotimes_{s \in \mathscr{K}_{\iota}} \mathbb{M}_{d}(\mathbb{C})$. Note that $\mu_{H_{\iota}}$ converges weakly, almost surely, to μ_{sc} for each $\iota = 1, ..., m$.

Theorem III.1. As $d \to \infty$, μ_H converges weakly, almost surely, to $\boxplus_{\varepsilon}^{(m)} \mu_{sc}$, if H_i 's are independently sampled by GUE.

Case 2: Suppose that H_t 's are independently sampled via Haar unitary invariant ensembles. Namely, for every t = 1, ..., m we suppose

$$H_{l} = U_{l}A_{l}U_{l}^{*} \otimes \bigotimes_{s \notin \mathscr{K}_{l}} (\mathbb{C}1\!\!1_{d}),$$
⁽²⁰⁾

where $\{U_{\iota}, \iota = 1, ..., m\}$ is a family of independent unitaries with U_{ι} Haar-distributed in $\mathscr{U}(\bigotimes_{s \in \mathscr{K}_{\iota}} \mathbb{M}_{d}(\mathbb{C}))$ and $\{A_{\iota}, \iota = 1, ..., m\}$ a family of deterministic Hermitian matrices with $A_{\iota} \in \bigotimes_{s \in \mathscr{K}_{\iota}} \mathbb{M}_{d}(\mathbb{C})$.

Assumption III.1. There are a sequence of compactly supported measures μ_1, \ldots, μ_m on \mathbb{R} such that μ_{A_1} weakly converges to μ_1 for each $\iota = 1, \ldots, m$. Hence μ_{H_1} converges weakly, almost surely, to μ_1 for each ι .

Theorem III.2. As $d \to \infty$, μ_H converges weakly, almost surely, to $\mu_1 \boxplus_{\varepsilon} \cdots \boxplus_{\varepsilon} \mu_m$, if H_ι 's are independently sampled by the Haar unitary invariant ensemble which obeys Assumption III.1.

We sketch the proofs as follows and refer to Subsections III A, III B and III C for the details. Firstly, we calculate the expectation of *k*th-moment tr(H^k) by using Wick's formula and the Weingarten formula for Cases 1 and 2, respectively. Indeed, it had been done independently in [15], and [16] for Case 2, and we only need to adapt their proofs in our setting. Secondly, we show that the leading order of the variance of tr(H^k) is $O(1/d^2)$. As a result, the Borel-Cantelli lemma can be applied, resulting in almost sure convergence. Finally, using the isomorphism between the ε -noncrossing partition and permutation, μ_H can be approximately represented by the ε -free convolution of the large *d* limit of μ_{H_1} 's.

A. Mixed moments of H_1, \ldots, H_m

In this subsection, we calculate the mixed moments of H_1, \ldots, H_m . The proofs follow [16], where the authors study the mixed moments of some matrix models that are asymptotically ε -free. However, to be self-contained, we adapt their proofs to our settings. Before we start, we recall Wick's formula and Weingarten's formula, which enable us to calculate the mixed moments of GUE and Haar unitary ensemble, respectively.

Definition III.2. [21, 23]. Let G be a $d \times d$ matrix with entries $g_{i,j}$ where $\sqrt{d}g_{i,j}$ is a standard complex Gaussian random variable, i.e., $\mathbb{E}(g_{i,j}) = 0, \mathbb{E}(|g_{i,j}|^2) = 1/d$ and

- (i) $g_{i,j} = \overline{g_{i,j}};$
- (ii) $\{\operatorname{Re}(g_{i,j})\}_{i\geq j} \cup \{\operatorname{Im}(g_{i,j})\}_{i\geq j}$ are independent.

Proposition III.3. [21, 23] (Wick's formula). Let $G = (g_{i,j})_{i,j=1}^d$ be a $d \times d$ GUE random matrix, then the expectation of the product of its entries can be calculated by the following Wick's formula:

$$\mathbb{E}\left(g_{i_1,j_1}\cdots g_{i_kj_k}\right) = \begin{cases} \sum_{\alpha\in\mathscr{P}_2(k)}\prod_{p\in[k]}\delta_{i_p,j_{\alpha(p)}} & \text{for } k \text{ is even;} \\ 0 & \text{for } k \text{ is odd.} \end{cases}$$
(21)

Proposition III.4. [24, 25] (Weingarten's formula). Let $U = (u_{i,j})_{i,j=1}^d$ be a $d \times d$ unitary random matrix distributed according to the Haar measure of the group of $d \times d$ matrices $\mathcal{U}(d)$, then the expectation of the product of its entries can be calculated by the following Weingarten's formula:

$$\mathbb{E}\left(u_{i_1,j_1}\cdots u_{i_kj_k}\bar{u}_{i'_1,j'_1}\cdots \bar{u}_{i'_k,j'_k}\right) = \sum_{\sigma,\tau\in S_k} \operatorname{Wg}(\sigma\tau^{-1},d) \cdot \prod_{p\in[k]} \delta_{i_p,i'_{\sigma(p)}} \delta_{j_p,j'_{\tau(p)}},\tag{22}$$

where $Wg(\sigma, d) : S_k \to \mathbb{C}$ is called a Weingarten function.

The following asymptotics of the Weingarten function is useful [26]

$$Wg(\sigma,d) = \mu(\sigma)d^{-(k+|\sigma|)} \left(1 + O\left(d^{-2}\right)\right),$$
(23)

where $\mu(\sigma)$ is a well defined function on S_k and it can be shown [21] that

$$\mu(\sigma\tau^{-1}) = \mu_{\mathscr{N}\mathscr{C}}(\alpha,\beta),$$

where $\sigma = P_{\alpha}$ and $\tau = P_{\beta}$.

Notation III.1. *For given* $\iota : [k] \to [m]$ *and* $\sigma \in S_k^{(\iota)}$ *,*

- (i) We denote $\gamma := \gamma_k = (1, \ldots, k)$.
- (ii) We denote $\iota_B := \iota_\ell$ for all $\ell \in B$, and denote $\tilde{Wg}(\sigma) := \prod_{B \in \ker \iota} Wg\left(\sigma|_B, d^{\#(\mathscr{K}_{\iota_B})}\right)$.
- (iii) $\operatorname{tr}_{\sigma}[A_1,\ldots,A_k] := \prod_{c \in \sigma} \operatorname{tr}(\prod_{i \in c} A_i)$, where *c* is the cycle of σ .

Proposition III.5. For the symmetric matrix ε given by (2), if H_1 's are independently sampled GUEs, then the mixed moments of H_1, \ldots, H_m is given as follows:

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = \sum_{\alpha \in \mathscr{P}_{2}^{(\iota)}(k)} \prod_{s=1}^{n} d^{\#(\gamma \cdot \alpha_{s}) - \frac{1}{2}k_{s} - 1},$$
(24)

for each $\iota = (\iota_1, \ldots, \iota_k) : [k] \to [m]$.

Proof. Note that the entries of $H_i = (H_{i,j}^{(i)})$ is given by

$$H_{i,j}^{(\iota)} := g_{i,j}^{(\iota)} \cdot \prod_{s \notin \mathscr{K}_{\iota}} \delta_{i[s],j[s]},$$

where *i*, *j* are *n*-tuples described as i = (i[1], ..., i[n]), and $g_{i,j}^{(t)}$ are the coefficients of G_t . We first suppose for convenience that $\mathbb{E}\left(\left|g_{i,j}^{(t)}\right|^2\right) = 1$. Now, for any $k \ge 1$ we have

$$\mathbb{E} \cdot \operatorname{tr}(H_{l_1}H_{l_2}\cdots H_{l_k}) = d^{-n}\mathbb{E} \cdot \operatorname{Tr}(H_{l_1}H_{l_2}\cdots H_{l_k})$$
$$= d^{-n}\sum_{i_1,\dots,i_k} \mathbb{E}\left(H_{i_1,i_2}^{(i_1)}H_{i_2,i_3}^{(i_2)}\cdots H_{i_ki_1}^{(i_k)}\right)$$
$$= d^{-n}\sum_{i_1,\dots,i_k} \prod_{B \in \ker \iota} \mathbb{E}\left(\prod_{p \in B} H_{i_p,i_{\gamma(p)}}^{(i_p)}\right)$$

where $\gamma = (1, 2, ..., k)$ is the full cycle of [k]. Note that $\iota_p = \iota_q$ for $p, q \in B$, we denote $\iota_p = \iota_B$ for $p \in B$. Thus it follows that

$$\mathbb{E} \cdot \operatorname{tr}(H_{\iota_1}H_{\iota_2}\cdots H_{\iota_k}) = d^{-n} \sum_{i_1,\dots,i_k} \prod_{B \in \ker \iota} \mathbb{E}\left(\prod_{p \in B} g_{i_p,i_{\gamma(p)}}^{(\iota_B)}\right) \cdot \prod_{p \in B} \prod_{s \notin \mathscr{K}_{\iota_B}} \delta_{i_p[s],i_{\gamma(p)}[s]}.$$

By Wick's formula, $\mathbb{E}\left(\prod_{p \in B} g_{i_p, i_{\gamma(p)}}^{(\iota_B)}\right) = 0$ whenever #(B) is odd, and otherwise

$$\mathbb{E}\left(\prod_{p\in B}g_{i_p,i_{\gamma(p)}}^{(\mathbf{I}_B)}\right) = \sum_{\alpha\in\mathscr{P}_2(\#(B))}\prod_{p\in B}\prod_{s\in\mathscr{K}_{i_p}}\delta_{i_p[s],i_{\gamma\cdot\alpha(p)}[s]}.$$

Since we need #(B) is even for every block $B \in \ker \iota \in \mathscr{P}(k)$, therefore $\mathbb{E} \cdot \operatorname{tr}(\cdots) = 0$ for *k* is odd. Now for even *k*, we are choosing independently pairings on each block $B \in \ker \iota$ we may sum over all $\alpha \in \mathscr{P}_2^{\iota}(k)$ outside of the product. Thus for even *k*, we have

$$\mathbb{E} \cdot \operatorname{tr}(H_{i_1}H_{i_2}\cdots H_{i_k}) = d^{-n} \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \sum_{i_1,\dots,i_k} \prod_{B \in \ker \iota} \prod_{p \in B} \left(\prod_{s \in \mathscr{K}_{i_p}} \delta_{i_p[s], i_{\gamma \alpha(p)}[s]} \right) \cdot \left(\prod_{s \notin \mathscr{K}_{i_p}} \delta_{i_p[s], i_{\gamma(p)}[s]} \right)$$
$$= d^{-n} \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \sum_{i_1,\dots,i_k} \prod_{\ell=1}^k \left(\prod_{s \in \mathscr{K}_{i_\ell}} \delta_{i_\ell[s], i_{\gamma \alpha(\ell)}[s]} \right) \cdot \left(\prod_{s \notin \mathscr{K}_{i_\ell}} \delta_{i_\ell[s], i_{\gamma(\ell)}[s]} \right).$$

Let us consider the sum $\sum_{i_1,\ldots,i_k} \prod_{\ell=1}^k \left(\prod_{s \in \mathscr{K}_{l_\ell}} \delta_{i_\ell[s],i_{\gamma:\alpha(\ell)}[s]} \right) \cdot \left(\prod_{s \notin \mathscr{K}_{l_\ell}} \delta_{i_\ell[s],i_{\gamma(\ell)}[s]} \right)$. In fact, these Dirac functions give the conditions for the indices that contribute to the sum. Similar to the proof in [16], we will look at the conditions for each $s \in [n]$. We need $i_\ell[s] = i_{\gamma:\alpha(\ell)}[s]$ when $s \in \mathscr{K}_{l_\ell}$, while $i_\ell[s] = i_{\gamma(\ell)}[s]$ when $s \notin \mathscr{K}_{l_\ell}$. That is, $i[s] = (\gamma \cdot \alpha_s) \cdot i[s]$, where we note that the action of permutation $\pi \in S_k$ on the indices $i[s] = (i_1[s], \ldots, i_k[s])$ is defined as $\pi \cdot i[s] := (i_{\pi(1)}[s], \ldots, i_{\pi(k)}[s])$. So we have

$$\sum_{i_1,\dots,i_k} \prod_{\ell=1}^k \left(\prod_{s \in \mathscr{K}_{l_\ell}} \delta_{i_\ell[s],i_{\alpha\cdot\gamma(\ell)}[s]} \right) \cdot \left(\prod_{s \notin \mathscr{K}_{l_\ell}} \delta_{i_\ell[s],i_{\gamma(\ell)}[s]} \right) = \prod_{s=1}^n d^{\#(\gamma \cdot \alpha_s)}.$$
(25)

To summarize, we have

$$\mathbb{E} \cdot \operatorname{tr}(H_{\iota_1}H_{\iota_2}\cdots H_{\iota_k}) = d^{-n} \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \prod_{s=1}^n d^{\#(\gamma \cdot \alpha_s)}$$
$$= \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \prod_{s=1}^n d^{\#(\gamma \cdot \alpha_s) - 1}.$$

Now we return to the normalized entries such that $\mathbb{E}\left(\left|g_{i,j}^{(t)}\right|^{2}\right) = d^{-\frac{1}{2}\#(\mathcal{K}_{t})}$. For normalized H_{t} and even k, we have

$$\mathbb{E} \cdot \operatorname{tr}(H_{\iota_1} H_{\iota_2} \cdots H_{\iota_k}) = \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \prod_{\ell=1}^k d^{-\frac{1}{2} \#(\mathscr{K}_{\iota_\ell})} \cdot \prod_{s=1}^n d^{\#(\gamma \cdot \alpha_s) - 1}$$

$$= \sum_{\alpha \in \mathscr{P}_2^{(\iota)}(k)} \prod_{s=1}^n d^{\#(\gamma \cdot \alpha_s) - \frac{1}{2}k_s - 1},$$
(26)

where we have used the fact that

$$\sum_{\ell=1}^{k} \#(\mathscr{K}_{\iota_{\ell}}) = \sum_{B \in \ker \iota} \sum_{p \in B} \#(\mathscr{K}_{\iota_{p}}) = \sum_{s=1}^{n} k_{s}.$$

The following proposition is one of the main results in [16]. We keep the proof for the convenience of the readers after adapting the notations and adding a few details.

Proposition III.6. [16][Theorem 6]. For the symmetric matrix ε given by (2), if H_1 's are independently sampled by a Haar unitary invariant ensemble, then the mixed moments of H_1, \ldots, H_m are given as follows:

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = \sum_{\boldsymbol{\sigma}, \tau \in S_{k}^{(\boldsymbol{\iota})}} \widetilde{\operatorname{Wg}}(\boldsymbol{\sigma}\tau^{-1}) \cdot \operatorname{tr}_{\tau}\left[A_{\iota_{1}}, \dots, A_{\iota_{k}}\right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\boldsymbol{\sigma}_{s}^{-1}\gamma) - 1},\tag{27}$$

for each $\iota = (\iota_1, \ldots, \iota_k) : [k] \to [m]$.

Proof. Note that the entries of $H_t = (H_{i,j}^{(t)})$ are given by

$$H_{i,j}^{(\iota)} := \sum_{p,q} a_{p,q}^{(\iota)} u_{i,p}^{(\iota)} \bar{u}_{j,q}^{(\iota)},$$

where i, j, p, q are *n*-tuples described as i = (i[1], ..., i[n]), and $a_{p,q}^{(i)}$ and $u_{p,q}^{(i)}$ are the coefficients of A_i and U_i , respectively. For any $k \ge 1$, we have

$$\mathbb{E} \cdot \operatorname{tr} \left(H_{\iota_1} \cdots H_{\iota_k} \right) = d^{-n} \sum_{i_1, \dots, i_k} \mathbb{E} \left(H_{i_1, i_2}^{(\iota_1)} H_{i_2, i_3}^{(\iota_2)} \cdots H_{i_k i_1}^{(\iota_k)} \right)$$
$$= d^{-n} \sum_{i_1, \dots, i_k} \prod_{B \in \ker \iota} \mathbb{E} \left(\prod_{t \in B} H_{i_t, i_\gamma(t)}^{(\iota_t)} \right).$$

We use the Weingarten formula to calculate the term $\mathbb{E}\left(\prod_{t\in B}H_{i_t,i_{\gamma(t)}}^{(l_t)}\right)$. We denote $\iota_t = \iota_B$ for all $t \in B$.

$$\begin{split} \mathbb{E}\left(\prod_{t\in B}H_{i_{t},i_{\gamma(t)}}^{(\iota_{t})}\right) &= \mathbb{E}\left(\prod_{t\in B}\sum_{p_{t},q_{t}}a_{p_{t},q_{t}}^{(\iota_{t})}u_{i_{t},p_{t}}^{(\iota_{t})}\bar{u}_{i_{\gamma(t)},q_{t}}^{(\iota_{t})}\right) \\ &= \sum_{p_{t},q_{t},t\in B}\mathbb{E}\left(\prod_{t\in B}a_{p_{t},q_{t}}^{(\iota_{t})}u_{i_{t},p_{t}}^{(\iota_{t})}\bar{u}_{i_{\gamma(t)},q_{t}}^{(\iota_{t})}\right) \\ &= \sum_{p_{t},q_{t},t\in B}\left(\prod_{t\in B}a_{p_{t},q_{t}}^{(\iota_{t})}\right)\cdot\left(\sum_{\sigma,\tau\in S_{B}}\operatorname{Wg}\left(\sigma\tau^{-1},d^{\#(\mathscr{K}_{lB})}\right)\times\right) \\ &\times \prod_{t\in B}\prod_{s\in\mathscr{K}_{t_{t}}}\delta_{i_{t}}[s]_{i_{\sigma}-1}\gamma_{(t)}[s]}\delta_{p_{t}}[s]_{q_{\tau}-1}(t)}[s]\cdot\prod_{s\notin\mathscr{K}_{t_{t}}}\delta_{i_{t}}[s]_{p_{\tau}(t)}[s]\right) \\ &= \sum_{\sigma,\tau\in S_{B}}\operatorname{Wg}\left(\sigma\tau^{-1},d^{\#(\mathscr{K}_{lB})}\right)\sum_{p_{t},t\in B}\prod_{t\in B}\prod_{s\in\mathscr{K}_{t_{t}}}a_{p_{t}}^{(\iota_{t})}[s]\delta_{i_{t}}[s]_{i_{\sigma}-1}\gamma_{(t)}[s]}\cdot\prod_{s\notin\mathscr{K}_{t_{t}}}\delta_{i_{t}}[s]_{p_{\tau}(t)}[s]\right) \end{split}$$

Since we are choosing permutations on each *B* independently, we may instead sum over all $\sigma, \tau \in S_k^{(\iota)}$ outside of the product. Therefore it follows that

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = d^{-n} \sum_{\sigma,\tau \in S_{k}^{(\iota)}} \tilde{\operatorname{Wg}}(\sigma\tau^{-1}) \cdot \sum_{\substack{i_{1},\dots,i_{k}; \\ p_{1},\dots,p_{k}}} \prod_{B \in \ker \iota} \prod_{t \in B} \prod_{s \in \mathscr{K}_{\iota_{t}}} a_{p_{t}[s],p_{\tau(t)}[s]}^{(\iota_{t})} \delta_{i_{t}[s],i_{\sigma^{-1}\gamma(t)}[s]} \cdot \prod_{s \notin \mathscr{K}_{\iota_{t}}} \delta_{i_{t}[s],i_{\gamma(t)}[s]}$$
$$= d^{-n} \sum_{\sigma,\tau \in S_{k}^{(\iota)}} \tilde{\operatorname{Wg}}(\sigma\tau^{-1}) \cdot \sum_{\substack{i_{1},\dots,i_{k}; \\ p_{1},\dots,p_{k}}} \prod_{s \in \mathscr{K}_{\iota_{\ell}}} a_{p_{\ell}[s],p_{\tau(\ell)}[s]}^{(\iota_{\ell})} \delta_{i_{\ell}[s],i_{\sigma^{-1}\gamma(\ell)}[s]} \cdot \prod_{s \notin \mathscr{K}_{\iota_{\ell}}} \delta_{i_{\ell}[s],i_{\gamma(\ell)}[s]}.$$

Again, let us consider the conditions for the indices that contributes in the sum $\sum_{\substack{i_1,...,i_k \\ p_1,...,p_k}} [\cdots]$. Firstly, similar to Equation (25), we have

$$\sum_{i_1,\ldots,i_k}\prod_{\ell=1}^k\prod_{s\in\mathscr{K}_{l_\ell}}\delta_{i_\ell[s],i_{\sigma^{-1}\gamma(\ell)}[s]}\cdot\prod_{s\notin\mathscr{K}_{l_\ell}}\delta_{i_\ell[s],i_{\gamma(\ell)}[s]}=\prod_{s=1}^n d^{\#(\sigma_s^{-1}\gamma)}.$$

Next, we turn to consider the sum $\sum_{p_1,...,p_k} \prod_{\ell=1}^k \prod_{s \in \mathscr{K}_{l_\ell}} a_{p_\ell[s],p_{\tau(\ell)}[s]}^{(i_\ell)}$. For each block $B \in \ker \iota$, if we look at the sum based on $\tau|_B$, then we have a trace (over $\otimes_{s \in \mathscr{K}_1} \mathbb{M}_d(\mathbb{C})$) of A_i 's respect to $\tau|_B$. Namely,

$$\sum_{p_1,\dots,p_k} \prod_{\ell=1}^k \prod_{s \in \mathscr{K}_{l_\ell}} a_{p_\ell[s], p_{\tau(\ell)}[s]}^{(\iota_\ell)} = \prod_{B \in \ker \iota} \operatorname{Tr}_{\tau|_B}[A_1, \dots, A_k]$$
$$= \prod_{B \in \ker \iota} d^{\#_B(\tau|_B) \cdot \#(\mathscr{K}_{l_B})} \operatorname{tr}_{\tau|_B}[A_1, \dots, A_k]$$
$$= \operatorname{tr}_{\tau}[A_1, \dots, A_k] \cdot \prod_{s=1}^n d^{\#_{k_s}(\tau_s)},$$

where we have used the fact that

$$\sum_{s=1}^n \#_{k_s}(\tau_s) = \sum_{B \in \ker \iota} \#_B(\tau|_B) \cdot \#(\mathscr{K}_{\iota_B})$$

since $\#(\mathscr{K}_{t_B})$ is the number of occurrences that the block *B* appears in J_s . To sum up, we have

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = d^{-n} \sum_{\sigma,\tau \in S_{k}^{(\iota)}} \tilde{Wg}(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\gamma)}$$

$$= \sum_{\sigma,\tau \in S_{k}^{(\iota)}} \tilde{Wg}(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\gamma) - 1}.$$
(28)

We end this subsection with the following remark.

Remark III.1. By Proposition III.6, the local terms H_1, \ldots, H_m that satisfies conditions in Case II are asymptotic ε -free (see [16][Theorem 7]). Moreover, because the GUE random matrix is Haar unitary invariant, Proposition III.6 could be used to induce the GUE model's asymptotic ε -freeness. However, we would like to derive an explicit formula for the moments of the GUE model to access the almost sure convergence, which is one of the main results of our paper.

B. Almost sure convergence

Notation III.2. For given $\tilde{\iota} = (\iota_1, \ldots, \iota_k, \iota_{k+1}, \ldots, \iota_{2k}) : [2k] \to m$.

- (i) Denote $\gamma_1 = (1, 2, ..., k)$, $\gamma_2 = (k + 1, k + 2, ..., 2k)$, and $\delta = \gamma_1 \times \gamma_2 \in S_{2k}$.
- (ii) For each string $s \in [n]$, we denote $J_{1,s} := J_s \cap [1,k]$ and $J_{2,s} := J_s \cap [k+1,2k]$. Moreover, $k_{i,s} := #J_{i,s}$, i = 1,2.
- (iii) Any $\sigma \in S_{2k}^{(\tilde{\iota})}$ induces permutations $\sigma_{i,s} \in S_{J_{i,s}}$, i = 1, 2.
- (iv) Denote $\alpha_{i,s} := \alpha|_{J_{i,s}}, i = 1.2$.

A partition $\alpha \in \mathscr{P}(2k)$ is called connected if there is a block $B \in \alpha$ such that $B \cap [1,k] \neq \emptyset$ and $B \cap [k+1,2k] \neq \emptyset$. A partition $\alpha \in \mathscr{P}(2k)$ is connected if and only if $P_{\alpha} \in S_{2k}$ is connected, i.e., P_{α} and δ generates a transitive subgroup in S_{2k} .

Notation III.3. For given $\tilde{\iota}: [2k] \to m$, we denote $\mathscr{P}_{2,c}^{(\tilde{\iota})}(2k)$ by the set of all connected pair partitions of $\mathscr{P}_{2}^{(\tilde{\iota})}(2k)$, and by $S_{2k,c}^{(\tilde{\iota})}$ the set of all connected permutations of $S_{2k}^{(\tilde{\iota})}$.

Proposition III.7. For the symmetric matrix ε given by (2), if H_1 's are independently sampled by GUE, then almost sure we have

$$\lim_{d \to \infty} \operatorname{tr} \left(H_{\iota_1} \cdots H_{\iota_k} \right) = \sum_{\alpha \in \mathscr{NC}_2^{(\varepsilon, \iota)}(k)} 1$$
(29)

for each $\iota = (\iota_1, \ldots, \iota_k) : [k] \to [m]$.

Proof. The proof follows the ideas in [27]. Firstly, note that for any pair partition α_s , we always have $\#(\alpha_s \gamma) = \#_{k_s}(\alpha_s \gamma_s) \le \frac{k_s}{2} + 1$, the equality holds whenever α_s is non-crossing. Therefore for each $\iota = (\iota_1, \ldots, \iota_k) : [k] \to [m]$ we have

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = \sum_{\alpha \in \mathscr{N}\mathscr{C}_{2}^{(\varepsilon,\iota)}(k)} 1 + O\left(\frac{1}{d}\right),\tag{30}$$

where we have used Equation (4). Hence

$$\lim_{d\to\infty} \mathbb{E} \cdot \operatorname{tr}\left(H_{i_1}\cdots H_{i_k}\right) = \sum_{\alpha\in\mathscr{NC}_2^{(\mathcal{E},\iota)}(k)} 1,\tag{31}$$

For the almost sure convergence, by using the Borel-Cantelli lemma, it suffices to show that

$$\sum_{d=1}^{\infty} \mathbb{E}\left(\operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) - \mathbb{E}\cdot\operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right)\right)^{2} < \infty.$$
(32)

To this end, let us consider the following expectation of the product of normalized traces:

$$\begin{split} \mathbb{E} \cdot \operatorname{tr}^{2} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) &= d^{-2n} \sum_{\substack{j_{1}, \dots, j_{k}, \\ j_{k+1}, \dots, j_{2k}}} \mathbb{E} \left(H_{j_{1}, j_{2}}^{(\iota_{1})} \cdots H_{j_{k}, j_{1}}^{(\iota_{k})} \cdot H_{j_{k+1}, j_{k+2}}^{(\iota_{k+1})} \cdots H_{j_{2k}, j_{k+1}}^{(\iota_{2k})} \right) \\ &= d^{-2n} \sum_{j_{1}, \dots, j_{2k}} \mathbb{E} \left(\prod_{\ell=1}^{2k} H_{j_{\ell}, j_{\delta(\ell)}}^{(\iota_{\ell})} \right). \end{split}$$

Similar to the derivation of Equation (26), we have

$$\mathbb{E} \cdot \operatorname{tr}^{2} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) = \sum_{\alpha \in \mathscr{P}_{2}^{(\tilde{\iota})}(2k)} \prod_{\ell=1}^{2k} d^{-\frac{1}{2}\mathscr{K}_{\iota_{\ell}}} \prod_{s=1}^{n} d^{\#(\delta \cdot \alpha_{s})-2}$$
$$= \sum_{\alpha \in \mathscr{P}_{2}^{(\tilde{\iota})}(2k)} \prod_{s=1}^{n} d^{\#(\delta \cdot \alpha_{s}) - \frac{1}{2}k_{s}-2}.$$

Let ι_1 and ι_2 be the restrictions of $\tilde{\iota}$ to [1,k] and to [k+1,2k], respectively. It follows that

$$\mathbb{E} \cdot \operatorname{tr}^{2} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) - \left(\mathbb{E} \cdot \operatorname{tr} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) \right)^{2} = \sum_{\alpha \in \mathscr{P}_{2}^{(\tilde{\iota})}(2k)} \prod_{s=1}^{s=1} d^{\#(\delta \cdot \alpha_{s}) - \frac{1}{2}k_{s} - 2} - \sum_{\substack{\alpha_{1} \in \mathscr{P}_{2}^{(\iota_{1})}(k), \\ \alpha_{2} \in \mathscr{P}_{2}^{(\iota_{2})}(k)}} \prod_{s=1}^{n} d^{\#(\gamma_{1} \cdot \alpha_{1,s}) + \#(\gamma_{2} \cdot \alpha_{2,s}) - \frac{1}{2}k_{s} - 2} = \sum_{\alpha \in \mathscr{P}_{2,c}^{(\tilde{\iota})}(2k)} \prod_{s=1}^{n} d^{\#(\delta \cdot \alpha_{s}) - \frac{1}{2}k_{s} - 2} = \sum_{\alpha \in \mathscr{P}_{2,c}^{(\tilde{\iota})}(2k)} \prod_{s=1}^{s=1} d^{(\cdots)} \cdot \prod_{s \in S_{2}} d^{(\cdots)},$$

where we denote $S_1 := \{s \in [n] : \alpha_s \text{ is connected}\}$ and $S_2 := \{s \in [n] : \alpha_s \text{ is disconnected}\}$. We note that $S_1 \neq \emptyset$, otherwise α could not be connected.

For any $s \in S_1$, by Formula (A3) we have

$$\begin{aligned} \#(\alpha_s \delta) &= \#_{k_s}(\alpha_s \delta_s) = k_s + 2(1-g) - \#_{k_s}(\alpha_s) - \#_{k_s}(\delta_s) \\ &= k_s + 2(1-g) - \frac{1}{2}k_s - 2 \\ &= \frac{1}{2}k_s - 2g. \end{aligned}$$

And for any $s \in S_2$, we have $\alpha_s = \alpha_{1,s} \times \alpha_{2,s}$ and

$$\#(\alpha_s \delta) - \frac{1}{2}k_s - 2 = \#(\alpha_{1,s}\gamma_1) + \#(\alpha_{2,s}\gamma_2) - \frac{1}{2}k_s - 2 \le 0$$

The equality holds when π_s is noncrossing.

In summary, the leading order of terms in the above sum is $O\left(d^{-2\#(S_1)}\right)$ (by letting g = 0), thus finally we have

$$\mathbb{E} \cdot \operatorname{tr}^{2} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) - \left(\mathbb{E} \cdot \operatorname{tr} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) \right)^{2} = O(d^{-2}),$$

which implies (32).

Proposition III.8. For the symmetric matrix ε given by (2), if H_1 's are independently sampled by a Haar unitary invariant ensemble which obeys Assumption III.1, then almost surely we have

$$\lim_{d\to\infty} \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = \sum_{\substack{\sigma,\tau\in S_{NC}^{(\varepsilon,\iota)}(\gamma);\\\tau\leq\sigma}} \mu(\sigma\tau^{-1}) \cdot \lim_{d\to\infty} \operatorname{tr}_{\tau}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right].$$
(33)

Proof. Recall that for $\sigma \in S_k^{(\iota)}$ we have

$$\begin{split} \tilde{\mathsf{Wg}}(\sigma) &= \prod_{B \in \ker \iota} \mu(\sigma|_B) d^{-\#(\mathscr{K}_{\mathfrak{l}_B}) \cdot (\#(B) + |\sigma|_B)} \left(1 + O\left(d^{-2\#(\mathscr{K}_{\mathfrak{l}_B})} \right) \right) \\ &= \mu(\sigma) \prod_{s=1}^n d^{-(k_s + |\sigma_s|)} (1 + O(d^{-2})), \end{split}$$

where $\mu(\sigma) := \prod_{B \in \ker \iota} \mu(\sigma_B)$. Combining Equation (28), we obtain

$$\mathbb{E} \cdot \operatorname{tr}\left(H_{\iota_{1}}\cdots H_{\iota_{k}}\right) = \sum_{\sigma,\tau \in S_{k}^{(L)}} \mu(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\gamma) - k_{s} - |\sigma_{s}\tau_{s}^{-1}| - 1}(1 + O(d^{-2})).$$
(34)

Given $\sigma, \tau \in S_k^{(\boldsymbol{\iota})}$ we have

$$\begin{aligned} \#_{k_s}(\tau_s) + \#(\sigma_s^{-1}\gamma) - k_s - |\sigma_s\tau_s^{-1}| - 1 &= k_s - 1 - (|\tau_s + |\sigma_s\tau_s^{-1}| + |\sigma_s^{-1}\gamma_s|) \\ &\leq k_s - 1 - |\gamma_s| \leq 0. \end{aligned}$$

The equality holds whenever σ_s , τ_s is on the geodesic path $1_s - \tau_s - \sigma_s - \gamma_s$ in S_{k_s} . Thus Equation (34) induces that

$$\lim_{d \to \infty} \mathbb{E} \cdot \operatorname{tr} \left(H_{\iota_{1}} \cdots H_{\iota_{k}} \right) = \sum_{\substack{\sigma, \tau \in S_{k}^{(\iota)}; \\ \iota_{s} - \tau_{s} - \sigma_{s} - \gamma_{s} \text{ for all } s \\ = \sum_{\substack{\sigma, \tau \in S_{k}^{(\iota, \iota)} \\ \sigma, \tau \in S_{k}^{(\iota, \iota)}(\gamma); \\ \tau \leq \sigma}} \mu(\sigma \tau^{-1}) \cdot \lim_{d \to \infty} \operatorname{tr}_{\tau} \left[A_{\iota_{1}}, \dots, A_{\iota_{k}} \right].$$
(35)

Note that due to Assumption III.1, the limit $\lim_{d\to\infty} \operatorname{tr}_{\tau} [A_{i_1}, \ldots, A_{i_k}]$ always exists.

Now we turn to show the almost sure convergence. Similar to the proof in Proposition III.7, we have

$$\begin{split} \mathbb{E} \cdot \operatorname{tr}^{2} \left(H_{l_{1}} \cdots H_{l_{k}} \right) &- \left(\mathbb{E} \cdot \operatorname{tr} \left(H_{l_{1}} \cdots H_{l_{k}} \right) \right)^{2} \\ &= \sum_{\sigma, \tau \in S_{2k}^{(i)}} \tilde{Wg}(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau} \left[A_{l_{1}}, \dots, A_{l_{2k}} \right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\delta) - 2} \\ &- \sum_{\sigma_{1}, \tau_{1} \in S_{k}^{(\iota_{1})}; \\ \sigma_{2}, \tau_{2} \in S_{k}^{(\iota_{2})}} \tilde{Wg}(\sigma_{1}\tau_{1}^{-1}) \tilde{Wg}(\sigma_{2}\tau_{2}^{-1}) \times \\ &\times \operatorname{tr}_{\tau_{1} \times \tau_{2}} \left[A_{l_{1}}, \dots, A_{l_{2k}} \right] \cdot d^{\#_{k_{1,s}}(\tau_{1}) + \#(\sigma_{1}^{-1}\gamma_{1}) + \#_{k_{2,s}}(\tau_{2}) + \#(\sigma_{2}^{-1}\gamma_{2}) - 2} \\ &= \sum_{\sigma, \tau \in S_{2k,c}^{(i)}} \tilde{Wg}(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau} \left[A_{l_{1}}, \dots, A_{l_{2k}} \right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma^{-1}\delta) - 2} \\ &= \sum_{\sigma, \tau \in S_{2k,c}^{(i)}} \mu(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau} \left[A_{l_{1}}, \dots, A_{l_{2k}} \right] \cdot \prod_{s=1}^{n} d^{\#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\delta) - |\sigma_{s}\tau_{s}^{-1}| - k_{s} - 2} \\ &= \sum_{\sigma, \tau \in S_{2k,c}^{(i)}} \mu(\sigma\tau^{-1}) \cdot \operatorname{tr}_{\tau} \left[A_{l_{1}}, \dots, A_{l_{2k}} \right] \cdot \prod_{s \in S_{1}} d^{(\cdots)} \cdot \prod_{s \in S_{2}} d^{(\cdots)} \cdot \prod_{s \in S_{3}} d^{(\cdots)} \cdot \prod_{s \in S_{4}} d^{(\cdots)}, \end{split}$$

where we denote $S_1 := \{s \in [n] : \sigma_s, \tau_s \text{ are connected}\}, S_2 := \{s \in [n] : \sigma_s \text{ is connected and } \tau_s \text{ is disconnected}\}, S_3 := \{s \in [n] : \tau_s \text{ is connected and } \sigma_s \text{ is disconnected}\}, S_4 := \{s \in [n] : \sigma_s, \tau_s \text{ are disconnected}\}.$

For any $s \in S_1$ we have

$$\begin{aligned} \#_{k_{s}}(\tau_{s}) + \#(\sigma_{s}^{-1}\delta) - |\sigma_{s}\tau_{s}^{-1}| - k_{s} - 2 &= k_{s} - 2 - (|\tau_{s}| + |\sigma_{s}\tau_{s}^{-1}| + |\sigma_{s}^{-1}\delta_{s}|) \\ &\leq k_{s} - 2 - (|\sigma_{s}| + |\sigma_{s}^{-1}\delta_{s}|) \\ &\leq k_{s} - 2 - k_{s} = -2, \end{aligned}$$

where for the first inequality we have used the triangle inequality $|\sigma_s| \le |\tau_s| + |\sigma_s \tau_s^{-1}|$, and for the second one we have used the fact that $k_s \le |\sigma_s| + |\sigma_s^{-1} \delta_s|$ since σ_s is connected.

For any $s \in S_2$ we have

$$\begin{aligned} \#_{k_s}(\tau_s) + \#(\sigma_s^{-1}\delta) - |\sigma_s\tau_s^{-1}| - k_s - 2 &= k_s - 2 - (|\tau_s| + |\sigma_s\tau_s^{-1}| + |\sigma_s^{-1}\delta_s|) \\ &< k_s - 2 - (|\sigma_s| + |\sigma_s^{-1}\delta_s|) \\ &\leq k_s - 2 - k_s = -2. \end{aligned}$$

We noted that for $s \in S_2$, $|\sigma_s| < |\tau_s| + |\sigma_s \tau_s^{-1}|$. Since the connectedness of σ_s and τ_s is different, they can not lies in a geodesic path. The case for $s \in S_3$ is similar.

For any $s \in S_4$, we have $\sigma_s = \sigma_{1,s} \times \sigma_{2,s}$, $\tau_s = \tau_{1,s} \times \tau_{2,s}$ and

$$\begin{aligned} \#_{k_s}(\tau_s) + \#(\sigma_s^{-1}\delta) - |\sigma_s\tau_s^{-1}| - k_s - 2 &= k_s - 2 - (|\tau_s| + |\sigma_s\tau_s^{-1}| + |\sigma_s^{-1}\delta_s|) \\ &\leq k_s - 2 - (|\sigma_s| + |\sigma_s^{-1}\delta_s|) \\ &= k_s - 2 - (|\sigma_{1,s}| + |\sigma_{2,s}| + |\sigma_{1,s}^{-1}\gamma_s| + |\sigma_{2,s}^{-1}\gamma_s|) \\ &\leq k_s - 2 - (k_s - 2) = 0. \end{aligned}$$

The equality holds whenever $1_s - \tau_{i,s} - \sigma_{i,s} - \gamma_s$ for i = 1, 2.

In summary, the leading order of terms in the above sum is $O\left(d^{-2\#(S_1)-3\#(S_2)-3\#(S_3)}\right)$. Hence the leading order of the variance is $O(d^{-2})$, which leads to the almost sure convergence.

C. Empirical eigenvalue distribution of H

Let (\mathscr{A}, ϕ) be a noncommutative probability space and a_1, \ldots, a_m be a sequence of ε -freely independent random variables in (\mathscr{A}, ϕ) such that the law of a_i is μ_i for each $i = 1, \ldots, m$. Because the local terms H_1, \ldots, H_m are asymptotic ε -free, the sum

 $H = \sum_{t=1}^{m} H_t$ converges in moment to the sum $\sum_{t=1}^{m} a_t$ [16][Theorem 4]. We will provide an alternate combinatoric proof which will use the moment-cumulant formulas (10) and (14). Furthermore, due to the results in Subsection IIIB, we have an almost sure convergence for H.

Theorem III.9. (*Restatement of Theorem III.1*). As $d \to \infty$, μ_H converges weakly, almost surely, to $\boxplus_{\varepsilon}^{(m)} \mu_{sc}$, if H_1 's are independently sampled by GUE.

Proof. Let (\mathscr{A}, ϕ) be a noncommutative probability space and s_1, \ldots, s_m be a sequence of ε -freely independent random variables in (\mathscr{A}, ϕ) which satisfies the semicircular law μ_{sc} . For any even $k \ge 1$, the *k*-th moment of $\sum_{t=1}^{m} s_t$ is given by

$$\phi\left(\sum_{i=1}^{m} s_{i}\right)^{k} = \sum_{\iota_{1},\ldots,\iota_{k}=1}^{m} \phi\left(s_{\iota_{1}}s_{\iota_{2}}\cdots s_{\iota_{k}}\right)$$

$$= \sum_{\iota:[k]\to[m]} \sum_{\alpha\in\mathscr{NC}^{(\varepsilon,\iota)}(k)} \kappa_{\alpha}^{(\varepsilon)}[s_{\iota_{1}},s_{\iota_{2}},\ldots,s_{\iota_{k}}]$$

$$= \sum_{\iota:[k]\to[m]} \sum_{\alpha\in\mathscr{NC}^{(\varepsilon,\iota)}(k)} \kappa_{\alpha}[s_{\iota_{1}},s_{\iota_{2}},\ldots,s_{\iota_{k}}]$$

$$= \sum_{\iota:[k]\to[m]} \sum_{\alpha\in\mathscr{NC}^{(\varepsilon,\iota)}(k)} 1,$$
(36)

where we have used the following fact about the free cumulant of semicircular elements [21]:

$$\kappa_{\alpha}[s_{l_1}, s_{l_2}, \dots, s_{l_k}] = \begin{cases} 1 & \text{if } \alpha \text{ is a pairing;} \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, note that for odd k, $\phi \left(\sum_{l=1}^{m} s_{l}\right)^{k} = 0$. Hence by combining Proposition III.5 and III.7, almost surely we have

$$\begin{split} \lim_{d \to \infty} \mathrm{tr} H^k &= \lim_{d \to \infty} \sum_{\iota: [k] \to [m]} \mathrm{tr} \left(H_{\iota_1} \cdots H_{\iota_k} \right) \\ &= \sum_{\iota: [k] \to [m]} \sum_{\pi \in \mathscr{NC}_2^{(\varepsilon, \iota)}(k)} 1 \\ &= \phi \left(\sum_{\iota=1}^m s_\iota \right)^k. \end{split}$$

It follows that μ_H converges weakly, almost surely to $\boxplus_{\varepsilon}^{(m)} \mu_{sc}$ as $d \to \infty$.

Theorem III.10. (*Restatement of Theorem III.2*). As $d \to \infty$, μ_H converges weakly, almost surely, to $\mu_1 \boxplus_{\varepsilon} \cdots \boxplus_{\varepsilon} \mu_m$, if H_ι 's are independently sampled by the Haar unitary invariant ensemble which obeys Assumption III.1.

Proof. Let (\mathscr{A}, ϕ) be a noncommutative probability space and a_1, \ldots, a_m be a sequence of ε -freely independent random variables in (\mathscr{A}, ϕ) such that the law of a_i is μ_i for each $\iota = 1, \ldots, m$. By Assumption III.1, for every $\alpha \in \mathscr{P}(k)$ and $\iota : [k] \to [m]$ we have

$$\lim_{d\to\infty}\operatorname{tr}_{\sigma}\left[A_{\iota_1},\ldots,A_{\iota_k}\right]=\phi_{\alpha}\left[a_{\iota_1},\ldots,a_{\iota_k}\right],$$

where $\sigma = P_{\alpha}$.

Then for any $k \ge 1$, the *k*-th moment of $\sum_{i=1}^{m} a_i$ is given by

$$\phi\left(\sum_{i=1}^{m} a_{i}\right)^{k} = \sum_{\iota_{1},\ldots,\iota_{k}=1}^{m} \phi(a_{\iota_{1}}a_{\iota_{2}}\cdots a_{\iota_{k}})$$

$$= \sum_{\iota:[k]\to[m]} \sum_{\alpha\in\mathscr{NC}^{(\varepsilon,\iota)}(k)} \kappa_{\alpha}^{(\varepsilon)}[a_{\iota_{1}},\ldots,a_{\iota_{k}}]$$

$$= \sum_{\iota:[k]\to[m]} \sum_{\alpha,\beta\in\mathscr{NC}^{(\varepsilon,\iota)}(k),\atop \beta<\alpha} \phi_{\beta}[a_{\iota_{1}},\ldots,a_{\iota_{k}}] \cdot \mu_{\mathscr{NC}^{(\varepsilon,\iota)}(k)}(\beta,\alpha).$$

By Theorem II.1 and Proposition II.2, it follows that

$$\phi\left(\sum_{\iota=1}^{m}a_{\iota}\right)^{k}=\sum_{\iota:[k]\to[m]}\sum_{\substack{\sigma,\tau\in S_{NC}^{(\varepsilon,\iota)}(\gamma);\\\tau\leq\sigma}}\mu(\sigma\tau^{-1})\cdot\lim_{d\to\infty}\operatorname{tr}_{\tau}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right],$$

where $\sigma = P_{\alpha}$ and $\tau = P_{\beta}$. Note we have used the fact that

$$\mu(\sigma\tau^{-1}) = \prod_{B \in \ker\iota} \mu(\sigma_B \tau_B^{-1}) = \prod_{B \in \ker\iota} \mu_{\mathscr{NC}(\#(B))}(\beta_B, \alpha_B)$$
$$= \mu_{\mathscr{NC}(\varepsilon,\iota)(k)}(\beta, \alpha).$$

Hence by combining Proposition III.6 and III.8, almost surely we have

$$\begin{split} \lim_{d \to \infty} \mathrm{tr} H^k &= \lim_{d \to \infty} \sum_{\iota: [k] \to [m]} \mathrm{tr} \left(H_{\iota_1} \cdots H_{\iota_k} \right) \\ &= \sum_{\iota: [k] \to [m]} \sum_{\substack{\sigma, \tau \in S_{NC}^{(\varepsilon, \iota)}(\gamma); \\ \tau \leq \sigma}} \mu(\sigma \tau^{-1}) \cdot \lim_{d \to \infty} \mathrm{tr}_{\tau} \left[A_{\iota_1}, \dots, A_{\iota_k} \right] \\ &= \phi \left(\sum_{\iota=1}^m s_\iota \right)^k. \end{split}$$

It follows that μ_H converges weakly, almost surely to $\mu_1 \boxplus_{\varepsilon} \cdots \boxplus_{\varepsilon} \mu_m$ as $d \to \infty$.

IV. BOUNDS FOR THE LARGEST EIGENVALUE

Let λ_{\max} be the largest eigenvalue of *H*, we have the following bounds for λ_{\max} .

Proposition IV.1. If H_1 's are independently sampled by GUE, then with probability one, we have

$$\lambda_{\max} \ge 2\sqrt{m}, as d \to \infty.$$
 (37)

Proof. Suppose that ker ι is noncrossing, then by the definition of $\mathscr{NC}^{(\varepsilon,\iota)}(k)$ we have

$$\mathscr{NC}^{(\varepsilon,\iota)}(k) = \{ \alpha \in \mathscr{NC}(k) : \alpha \leq \ker \iota \}.$$

By Proposition III.7 almost surely we have

$$\begin{split} \lim_{d \to \infty} \operatorname{tr} \left(H^{2k} \right) &= \sum_{\iota: [2k] \to [m]} \sum_{\alpha \in \mathscr{NC}_{2}^{\mathscr{C}(\varepsilon,\iota)}(2k)} 1 \\ &= \sum_{\beta \in \mathscr{P}(2k)} \sum_{\iota: [2k] \to [m], \alpha \in \mathscr{NC}_{2}^{(\varepsilon,\iota)}(2k)} 1 \\ &\geq \sum_{\beta \in \mathscr{NC}(2k)} \sum_{\iota: [2k] \to [m], \alpha \in \mathscr{NC}_{2}(2k),, 1} \\ &= \sum_{\beta \in \mathscr{NC}(2k)} \sum_{\alpha \in \mathscr{NC}_{2}(2k), \iota: [2k] \to [m], 1} \\ &\geq \sum_{\beta \in \mathscr{NC}(2k)} \sum_{\alpha \in \mathscr{NC}_{2}(2k), \iota: [2k] \to [m], 1} \\ &\geq \sum_{\beta \in \mathscr{NC}_{2}(2k)} \sum_{\substack{\iota: [2k] \to [m], \alpha \in \beta}} 1 \\ &\geq m^{k} \cdot C_{\iota}. \end{split}$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$ is the Catalan number. Note that we have by Stirling's formula

$$\lim_{k \to \infty} C_k^{\frac{1}{k}} = 4.$$
(38)

Then for any $\delta > 0$, and sufficient large d

$$\mathbb{P}[\lambda_{\max} < 2\sqrt{m} - \delta] \le \mathbb{P}[\operatorname{tr}\left(H^{2k}
ight) < (2\sqrt{m} - \delta)^{2k}]$$

 $\le \mathbb{P}[\lim_{k \to \infty} \left(\operatorname{tr}\left(H^{2k}
ight)
ight)^{rac{1}{2k}} \le 2\sqrt{m} - \delta]$
 $= \mathbb{P}[2\sqrt{m} \le 2\sqrt{m} - \delta] = 0.$

Hence $\mathbb{P}[\lambda_{\max} < 2\sqrt{m} - \delta] \to 0$ as $d \to \infty$, which completes our proof.

Proposition IV.2. Suppose that H_1 's are independently sampled by a Haar unitary invariant ensemble which obeys Assumption III.1, and for each ι , μ_1 is distributed with mean 0 and variance 1, then almost surely we have

$$\lambda_{\max} \ge 2\sqrt{m}, \, as \, d \to \infty. \tag{39}$$

Proof. Let (\mathscr{A}, ϕ) be a noncommutative probability space and a_1, \ldots, a_m be a sequence of ε -freely independent random variables in (\mathscr{A}, ϕ) such that the law of a_i is μ_i for each $\iota = 1, \ldots, m$. By our assuption, for every $\alpha \in \mathscr{P}(k)$ and $\iota : [k] \to [m]$ we have

$$\lim_{d\to\infty}\operatorname{tr}_{\sigma}\left[A_{\iota_{1}},\ldots,A_{\iota_{k}}\right]=\phi_{\alpha}\left[a_{\iota_{1}},\ldots,a_{\iota_{k}}\right],$$

where $\sigma = P_{\alpha}$. Moreover, $\phi(a_{\iota}) = 0$ and $\phi(a_{\iota}^2) = 1$ for each $\iota = 1, ..., m$.

By Proposition III.8, almost surely we have

$$\lim_{d \to \infty} \operatorname{tr} \left(H^{2k} \right) = \sum_{\iota:[2k] \to [m]} \sum_{\substack{\alpha \in \mathscr{N} \mathscr{C}^{(\varepsilon,\iota)}(2k)}} \kappa_{\alpha}^{(\varepsilon)} [a_{\iota_{1}}, \dots, a_{\iota_{2k}}] \\
\geq \sum_{\iota:[2k] \to [m]} \sum_{\substack{\alpha \in \mathscr{N} \mathscr{C}(2k), \\ \alpha \leq \ker \iota}} \kappa_{\alpha} [a_{\iota_{1}}, \dots, a_{\iota_{2k}}] \\
= \sum_{\substack{\beta \in \mathscr{N} \mathscr{C}(2k)}} \sum_{\substack{\alpha \in \mathscr{N} \mathscr{C}(2k), \\ \alpha \leq \beta}} \kappa_{\alpha} [a_{\iota_{1}}, \dots, a_{\iota_{2k}}] \\
\geq \sum_{\substack{\beta \in \mathscr{N} \mathscr{C}_{2}(2k)}} \sum_{\substack{\iota:[2k] \to [m], \\ \ker \iota = \beta}} \kappa_{\beta} [a_{\iota_{1}}, \dots, a_{\iota_{2k}}] \\
\geq \sum_{\substack{\beta \in \mathscr{N} \mathscr{C}_{2}(2k)}} \sum_{\substack{\iota:[2k] \to [m], \\ \ker \iota = \beta}} 1 \\
\simeq m^{k} \cdot C_{k},$$
(40)

where we have used the fact that $\kappa_{\alpha}^{(\varepsilon)}[a_{\iota_1},\ldots,a_{\iota_{2k}}] = \kappa_{\alpha}[a_{\iota_1},\ldots,a_{\iota_{2k}}]$ for $\alpha \in \mathscr{NC}(2k)$, and

 $\kappa_{\beta}[a_{\iota_1},\ldots,a_{\iota_{2k}}]=1$

for $\beta \in \mathcal{NC}_2(2k)$ and ker $\iota = \beta$. The rest of the proof is the same as the proof of Proposition IV.1.

Remark that the above lower bound is universal for all ε and it becomes optimal for the extreme case whenever every nondiagonal entry of ε is 0. For example, let $\mathscr{K}_{t} = [n]$ for every $t \in [m]$, then almost surely we have [28–30]

$$\lambda_{\max} \leq 2\sqrt{m}$$
, as $d \to \infty$.

On the other hand, there is a trivial upper bound for λ_{\max} without considering the interactions between H_t 's. For simplicity, we only consider the case of H_t being GUE. Note that it is well-known [31] that $||H_t||_{\infty} = 2$ almost surely as $d \to \infty$, then we have

$$\lambda_{\max} \le \|H\|_{\infty} \le \sum_{\iota=1}^{m} \|H_{\iota}\|_{\infty}$$

$$\le 2m,$$
(41)

as $d \rightarrow \infty$. The above upper bound can be improved by considering the interactions.

Now let us consider the following XY-model: Let m = n - 1 and assume that

$$\mathscr{K}_{1} = \{\iota, \iota+1\}, \iota = 1, \dots, n-1.$$
(42)

Thus the entries of ε are given by $(i \ge j)$

$$\varepsilon_{i,j} = \begin{cases} 0 & \text{if } i = j, j+1; \\ 1 & \text{others.} \end{cases}$$

Moreover, for given $\iota : [k] \rightarrow [n-1]$ we have $J_1 = \{j \in [k] : \iota_j = 1\}, J_s = \{j \in [k] : \iota_j = s - 1, s\}, s = 2, ..., n-1$, and $J_n = \{j \in [k] : \iota_j = n - 1\}$.

Proposition IV.3. Let m = n - 1, and consider the Hamiltonian H with the interactions given by (42), if H_t 's are independently sampled by GUE, then with probability one, we have

$$\lambda_{\max} \le 2(n-3) + 2\sqrt{2}, \ as \ d \to \infty.$$
(43)

Proof. For given $\iota : [k] \to [n-1]$, by the definition of $\mathscr{NC}^{(\varepsilon,\iota)}(k)$ we have the following iteration relation

$$\mathscr{NC}^{(\varepsilon,\iota)}(k) \subseteq \mathscr{NC}(J_1) \times \mathscr{NC}^{(\varepsilon,\iota_1)}([k]/J_1),$$

where ι_1 is the restriction of ι on $[k]/J_1$.

Starting with Equation (30) we have

$$\begin{split} \mathbb{E} \cdot \operatorname{tr} \left(H \right)^{k} &= \sum_{\iota:[k] \to [n-1]} \sum_{\alpha \in \mathscr{NC}_{2}^{(\varepsilon,\iota)}(k)} 1 + O\left(\frac{1}{d}\right) \\ &\leq \sum_{\iota:[k] \to [n-1]} \sum_{\alpha_{1} \in \mathscr{NC}_{2}(J_{1})} \sum_{\alpha_{2} \in \mathscr{NC}_{2}^{(\varepsilon,\iota_{1})}([k]/J_{1})} 1 + O\left(\frac{1}{d}\right) \\ &= \sum_{\ell_{1}=0}^{k} \binom{k}{\ell_{1}} \#(\mathscr{NC}_{2}(\ell_{1})) \cdot \sum_{\iota:[k-\ell_{1}] \to [2,n-1]} \sum_{\alpha_{2} \in \mathscr{NC}_{2}^{(\varepsilon,\iota)}(k-\ell_{1})} 1 + O\left(\frac{1}{d}\right) \\ &= \sum_{\ell_{1}=0}^{k} \binom{k}{\ell_{1}} \#(\mathscr{NC}_{2}(\ell_{1})) \cdot \operatorname{Etr} \left(H_{2} + \dots + H_{n-1}\right)^{k-\ell} + O\left(\frac{1}{d}\right). \end{split}$$

By an inductive argument, we have

$$\begin{split} \mathbb{E} \cdot \operatorname{tr}(H)^{k} &\leq \sum_{\ell_{1}=0}^{k} \binom{k}{\ell_{1}} \#(\mathscr{NC}_{2}(\ell_{1})) \cdot \mathbb{E}\operatorname{tr}(H_{2} + \dots + H_{n-1})^{k-\ell_{1}} + O\left(\frac{1}{d}\right) \\ &\leq \sum_{\ell_{1}=0}^{k} \sum_{\ell_{2}=0}^{k-\ell_{1}} \binom{k}{\ell_{1}} \binom{k-\ell_{1}}{\ell_{2}} \#(\mathscr{NC}_{2}(\ell_{1})) \cdot \#(\mathscr{NC}_{2}(\ell_{2})) \cdot \mathbb{E}\operatorname{tr}(H_{3} + \dots + H_{n-1})^{k-\ell} + O\left(\frac{1}{d}\right) \\ &\dots \\ &\leq \sum_{\ell_{1}=0}^{k} \sum_{\ell_{2}=0}^{k-\ell_{1}} \cdots \sum_{\ell_{n-3}=0}^{k-(\ell_{1} + \dots + \ell_{n-4})} \binom{k}{\ell_{1}} \binom{k-\ell_{1}}{\ell_{2}} \cdots \binom{k-(\ell_{1} + \dots + \ell_{n-4})}{\ell_{n-3}} \prod_{i=1}^{n-3} \#(\mathscr{NC}_{2}(\ell_{i})) \times \\ &\times \mathbb{E}\operatorname{tr}(H_{n-2} + H_{n-1})^{k-(\ell_{1} + \dots + \ell_{n-3})} + O\left(\frac{1}{d}\right) \\ &= \sum_{\ell_{1}=0}^{k} \sum_{\ell_{2}=0}^{k-\ell_{1}} \cdots \sum_{\ell_{n-3}=0}^{k-(\ell_{1} + \dots + \ell_{n-4})} \binom{k}{\ell_{1}} \binom{k-\ell_{1}}{\ell_{2}} \cdots \binom{k-(\ell_{1} + \dots + \ell_{n-4})}{\ell_{n-3}} \prod_{i=1}^{n-3} \#(\mathscr{NC}_{2}(\ell_{i})) \times \\ &\times \#(\mathscr{NC}_{2}(k - (\ell_{1} + \dots + \ell_{n-3})) \cdot 2^{\frac{k-(\ell_{1} + \dots + \ell_{n-3})}{2}} + O\left(\frac{1}{d}\right) \\ &\leq \sum_{\ell_{1}=0}^{k} \sum_{\ell_{2}=0}^{k-\ell_{1}} \cdots \sum_{\ell_{n-3}=0}^{k-(\ell_{1} + \dots + \ell_{n-4})} \binom{k}{\ell_{1}} \binom{k-\ell_{1}}{\ell_{2}} \cdots \binom{k-(\ell_{1} + \dots + \ell_{n-4})}{\ell_{n-3}} 4^{\frac{k}{2}} \cdot 2^{\frac{k-(\ell_{1} + \dots + \ell_{n-3})}{2}} + O\left(\frac{1}{d}\right) \\ &\leq M_{k} + O\left(\frac{1}{d}\right), \end{split}$$

where we have used the fact that $\#(\mathscr{NC}_2(k)) = C_{k/2} \leq 4^{k/2}$. Note that we have by Stirling's formula

$$\lim_{k \to \infty} M_k^{\frac{1}{k}} = 2(n-3) + 2\sqrt{2}.$$
(44)

Choose a sequence $k(d) \rightarrow \infty$ as $d \rightarrow \infty$ such that

$$M_{k(d)} \leq \left(2(n-3)+2\sqrt{2}\right)^{k(d)}$$
, and $k(d)/\log d \to 0$ as $d \to \infty$.

Then for any $\delta > 0$, and sufficient large *d*,

$$\begin{split} \mathbb{P}[\lambda_{\max} > 2(n-3) + 2\sqrt{2} + \delta] &\leq \mathbb{P}[\operatorname{Tr}\left(H^{2k(d)}\right) > (2(n-3) + 2\sqrt{2} + \delta)^{2k(d)}] \\ &\leq \frac{\mathbb{E} \cdot \operatorname{Tr}\left(H^{2k(d)}\right)}{(2(n-3) + 2\sqrt{2} + \delta)^{2k(d)}} \\ &\leq \frac{d^n \cdot (2(n-3) + 2\sqrt{2})^{2k(d)}}{(2(n-3) + 2\sqrt{2} + \delta)^{2k(d)}}. \end{split}$$

Hence $\mathbb{P}[\lambda_{\max} > 2(n-3) + 2\sqrt{2} + \delta] \to 0$ as $d \to \infty$, which completes our proof.

V. CONCLUSION AND DISCUSSION

In this paper, we consider the spectrum of local Hamiltonians given by some random ensembles in the large *d* limit. Since the Hamiltonian is local, the correlations between the local terms are somewhat complicated. However, as pointed out in [15] and [16], the local terms of the Hamiltonian are asymptotically ε -free (or equivalently heap-free) in the limit, namely, it is appropriate to carry out our investigation in the framework of noncommutative probability theory. Therefore, it is natural to say that the spectrum distribution of the total Hamiltonian is some convolution of its local terms' probability distributions. Our work has two main contributions: (i) by proper variance estimations, the asymptotic ε -freeness of the local terms holds generically. As a byproduct, our proof may shed some light on considering the second order ε -freeness of the corresponding random matrix models. Such a problem is inspired by the work of second order freeness [27, 32–34]. Moreover, the almost sure convergence of the matrix model may have further applications in operator algebra. (ii) the isomorphism between the ε -noncrossing partitions and permutations brings us a new bridge to connect the matrix model and its large *d* limit, just like what Biane had done in the free probability theory. And it allows us to provide a direct combinatoric proof for the convergence of the total Hamiltonian *H*, though the convergence in moments can be induced by the results in [16]. Since the enumeration of the ε -noncrossing partitions/permutations is unclear; our results are only formally meaningful. Nevertheless, we can derive bounds for the largest eigenvalue of the total Hamiltonian via some rough estimation of the enumeration. We end our conclusion with the following questions.

(i) Is it possible to explicitly calculate the ε -free convolution for some given non-trivial ε ? For the GUE case, the expectation of *k*th-moment of *H* is given by

$$\mathbb{E} \cdot \operatorname{tr}(H^k) = \sum_{\iota:[k] \to [m]} \sum_{\alpha \in \mathscr{NC}_2^{(\mathcal{E},\iota)}(k)} 1$$

Can one connect the above combinatorial summation to some transformations of μ_{sc} , e.g., the Cauchy transform of μ_{sc} ?

- (ii) Assume that H_i is given by independent projection, i.e., $H_i = U_i P_i U_i^* \otimes \bigotimes_{s \notin \mathscr{H}_i} (\mathbb{Cl}_d)$, where P_i is a rank p projection. Then μ_i converges to the Bernoulli distribution for each ι . Is it possible to figure out the spectrum gap of H in the thermal dynamical limit $(m \to \infty)$? The XY-model given by (42) is of great interest. Since it can be shown that the Hamiltonian of this model is generically gapless [2]. There is some evidence that it is possible to recover this conclusion by estimating the density of ε -free convolution. Consider the extreme case when $\mathscr{H}_i = [n]$, the density of μ_H becomes continuous when $m \to \infty$ [35, 36]. Is it possible to make any similar qualitative estimation for other non-trivial interactions (hence for some non-trivial ε)?
- (iii) In [37], the authors considered the dynamical correlator under the Hamiltonian to be given by the sum of two independent random matrices. So it is natural to consider the correlator under the local random Hamiltonians. Moreover, it is interesting to consider the convergence to equilibrium under a local random Hamiltonian. For the global case, see [38].

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Appendix A: Some notions of combinatorics

In this appendix, we will recall some combinatoric notations and facts frequently appearing in this paper, which mainly come from [21, 23] and the references therein.

For natural numbers $m, n \in \mathbb{N}$ with m < n, we denote by [m, n] the interval of natural numbers between m and n,

$$[m,n] := \{m,m+1,\ldots,n-1,n\}$$

Moreover, we use [n] to denote [1, n].

Permutations–We denote the set of permutations on *n* elements by S_n . For a permutation $\pi \in S_n$ we denote $\#(\pi)$ by the number of cycles of π and by $|\pi|$ the minimal number of transpositions needed to decompose π . There is a nice relation between $\#(\pi)$ and $|\pi|$, namely,

$$|\pi| + \#(\pi) = n, \text{ for all } \pi \in S_n.$$
(A1)

Let $\gamma_n \in S_n$ be the full cycle $\gamma_n = (1, 2, ..., n)$. For any $\pi \in S_n$ we always have

$$|\pi|+|\pi^{-1}\gamma_n|\geq n-1$$

We call π noncrossing if the equality holds, or equivalently we can say that π lies on the geodesic path $1_n - \pi - \gamma_n$ in S_n . We denote the set of all noncrossing permutations in S_n by $S_{NC}(\gamma_n)$. For $\sigma, \pi \in S_{NC}(\gamma_n)$. We say that $\sigma \leq \pi$ if σ and π lie on the same geodesic path and σ comes before π , i.e., $1_n - \sigma - \pi - \gamma_n$. The set $S_{NC}(\gamma_n)$ endowed with " \leq " becomes a poset.

Fix $m, n \in \mathbb{N}$ and denote by $\gamma_{m,n}$ the product of the two cycles

$$\gamma_{m,n} = (1, 2, \dots, m) \cdot (m+1, m+2, \dots, m+n).$$

A permutation $\pi \in S_{m+n}$ is called connected if the pair π and $\gamma_{m,n}$ generates a transitive subgroup in S_{m+n} . We note that a connected permutation $\pi \in S_{m+n}$ always satisfies

$$m+n \le |\pi| + |\pi^{-1}\gamma_{m,n}|.$$
 (A2)

If $\pi \in S_{m+n}$ is connected, and if we have equality, then we call π annular noncrossing. More general, suppose $\pi \in S_{m+n}$ is connected, then we have the following formula

$$\#(\pi) + \#(\pi^{-1}\gamma_{m,n}) + \#(\gamma_{m,n}) = m + n + 2(1 - g), \tag{A3}$$

where $g \ge 0$ is the genus of π relative to $\gamma_{m,n}$.

Partitions–We say $\alpha = \{A_1, A_2, \dots, A_k\}$ is a partition of [n] if the sets A_i are disjoint and non-empty and their union is equal to [n]. We call A_1, \dots, A_k the blocks of partition α . The set of all partitions of [n] is denoted by $\mathscr{P}(n)$. If there are only two elements in every block of $\alpha \in \mathscr{P}(n)$, we call α a pair partition and the set of all pair partitions of [n] is denoted by $\mathscr{P}_2(n)$. If $\alpha = \{A_1, \dots, A_k\}$ and $\beta = \{B_1, \dots, B_s\}$ are partitions of [n], we say that $\alpha \leq \beta$ if for every block A_i there exists some block B_j such that $A_i \subseteq B_j$. With the relation " \leq ", the set $\mathscr{P}(n)$ is also a poset.

Given two elements $i, j \in [n]$, we write $i \sim_{\alpha} j$ if i and j belongs to the same block of α . A partition α is called a crossing if $i_1 < j_1 < i_2 < j_2 \in [n]$ exist such that $i_1 \sim_{\alpha} i_2 \approx_{\alpha} j_1 \sim_{\alpha} j_2$. We call α a noncrossing partition if α is not crossing, and we denote NC(n) (resp. $NC_2(n)$) by the set of all noncrossing partitions (resp. noncrossing pair partitions) of [n]. It is known that the cardinality of the set of noncrossing partitions is the Catalan number, i.e.,

$$#(\mathscr{NC}(k)) = #(\mathscr{NC}_2(2k)) = C_k, \tag{A4}$$

where $C_k = \frac{1}{k+1} \binom{2k}{k}$.

Permutations vs. Partitions–A permutation can always be written as a product of cycles, so one can identify a permutation $\pi \in S_n$ with a partition $\alpha \in \mathscr{P}(n)$ by omitting the order on the cycles. In particular, there is a one-to-one identification between the transpositions and the pair partitions. We remark that now *n* should be even. In this sense, we can multiply a pair partition $\alpha \in \mathscr{P}_2(n)$ with a permutation $\pi \in S_n$, and their product $\alpha \cdot \pi$ is viewed as an element in S_n . Moreover, we always have

$$\#(\alpha \cdot \gamma_n) \le \frac{n}{2} + 1,\tag{A5}$$

the equality holds whenever α is noncrossing.

For the noncrossing case, the situation is much better; we recall the following fact due to Biane [20]:

Proposition A.1. There is a bijection between NC(n) and the set $S_{NC}(\gamma_n)$, which preserves the poset structure.

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