CONFORMALLY RELATED INVARIANT (α, β) -METRICS ON HOMOGENEOUS SPACES

AZAR FATAHI, MASOUMEH HOSSEINI, AND HAMID REZA SALIMI MOGHADDAM

ABSTRACT. In this paper, we give the flag curvature formula of general (α, β) -metrics of Berwald type. We study conformally related (α, β) -metrics, especially general (α, β) -metrics that are conformally related to invariant (α, β) -metrics. Also, a necessary and sufficient condition for a Finsler metric conformally related to an (α, β) -metric is given and conformally related Douglas Randers metrics are studied. Finally, we present some examples of conformally related (α, β) -metrics.

1. Introduction

In the last two decades, many mathematicians worked on invariant Finsler metrics on ho-mogeneous spaces (see [\[4\]](#page-9-0) and [\[5\]](#page-9-1)). Among different types of Finsler metrics, (α, β) -metrics have been paid more attention to because of their simplicity and applications in physics (see [\[1\]](#page-9-2), [\[2\]](#page-9-3) and [\[3\]](#page-9-4)). These Finsler metrics were introduced in [\[8\]](#page-9-5), by Matsumoto. In fact, the Randers metric, the first (α, β) -metric defined by G. Randers in 1941, was introduced because of its application in general relativity (see [\[10\]](#page-9-6)). Other examples of such (α, β) -metrics defined because of their applications in physics are the Matsumoto metric and the Kropina metric (see $[1], [2]$ $[1], [2]$ $[1], [2]$ and $[11]$.

Suppose that g is a Riemannian metric and β is a 1-form on a differentiable manifold M. For a C^{∞} function $\phi: (-b_0, b_0) \longrightarrow \mathbb{R}^+$ satisfying

(1.1)
$$
\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \qquad |s| \le b < b_0,
$$

and $\|\beta\|_{\alpha} < b_0$ (see [\[3\]](#page-9-4)), the Finsler metric $F = \alpha \phi(\frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$) is called an (α, β) -metric, where $\alpha(x, y) = \sqrt{g(y, y)}$. To define the above-mentioned (α, β) -metrics, Randers, Matsumoto, and Kropina metrics, it is sufficient to consider the function ϕ as $\phi(s) = 1 + s$, $\phi(s) = \frac{1}{1-s}$, and $\phi(s) = \frac{1}{s}$, respectively.

In the year 2011, Yu and Zhu generalized the concept of the (α, β) -metric to a more general case which was called the general (α, β) -metric (see [\[12\]](#page-9-8)). A Finsler metric F on a differentiable manifold M is called a general (α, β) -metric if there exists a C^{∞} function ϕ , a Riemannian metric g, and a 1-form β , such that

(1.2)
$$
F = \alpha \phi(x, \frac{\beta}{\alpha}),
$$

Date: October 4, 2022.

Key words and phrases. invariant Riemannain metric, invariant (α, β) -metric, conformally relates, Lie group, homogeneous space

AMS 2010 Mathematics Subject Classification: 53C30, 53C60, 53C25, 22E60.

where $x \in M$ and $\alpha(x, y) = \sqrt{g(y, y)}$.

A special class of general (α, β) -metrics is of the form $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$), where $b^2 := ||\beta||_{\alpha}^2$, $|s| \leq b < b_0$, for some $0 < b_0 \leq +\infty$. This family of general (α, β) -metrics is important because it includes some Bryant Finsler metrics (see [\[12\]](#page-9-8)). It is shown that for $\|\beta\|_{\alpha} < b_0$ the function $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$) is a Finsler metric if and only if ϕ is a positive C^{∞} function such that

• $\phi - s\phi_2 > 0$, $\phi(s) - s\phi_2(s) + (b^2 - s^2)\phi_{22}(s) > 0$, (if dim $M \ge 3$) • $\phi(s) - s\phi_2(s) + (b^2 - s^2)\phi_{22}(s) > 0$, (if dim $M = 2$),

where $|s| \le b < b_0$ (see [\[12\]](#page-9-8)).

It seems that the study of geometric properties of invariant general (α, β) -metrics on homogeneous spaces is interesting. But, we see this is not a good idea, because every invariant general (α, β) -metric, which is defined by an invariant Riemannian metric and an invariant vector field (1-form), is an invariant (α, β) -metric (see proposition 3.1 below). So we study a family of general (α, β) -metrics which are very close to invariant (α, β) -metrics. In this work, we study general (α, β) -metrics that are conformally related to invariant (α, β) -metrics defined by an invariant Riemannian metric and an invariant vector field. Also we give the flag curvature formula of general (α, β) -metrics of Berwald type. A necessary and sufficient condition for a Finsler metric conformally related to an (α, β) -metric is given. Conformally related Douglas Randers metrics are investigated. Finally, some examples of conformally related (α, β) -metrics are given.

2. Conformally related (α, β) -metrics

In this section, firstly we consider the general (α, β) -metrics of Berwald type. Easily, similar to the (α, β) -metrics, we compute the flag curvature formula of the general (α, β) -metrics of Berwald type. Next, we turn our attention to the Finsler metrics that are conformally related to the (α, β) -metrics. We give a necessary and sufficient condition for such metrics to be of Berwald type. Also, we study such Randers metrics that are of Douglas type.

Remark 2.1. Let $F(x,y) = \alpha \phi(b^2, \frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$) be a general (α, β) -metric on a differentiable manifold M. If β is parallel with respect to α , then F is of Berwald type.

Proof. Assume that β is parallel with respect to α i.e. $b_{i,j} = 0$, where $b_{i,j}$ is the covariant derivative of b_i with respect to the Riemannian metric g. Suppose that G^i and G^i_α denote the spray coefficients of F and α , respectively. Now, Proposition 3.4 of [\[12\]](#page-9-8) shows that $G^i = G^i_{\alpha}$ and so F is of Berwald type.

Proposition 2.2. Let M be a Finsler manifold equipped with a general (α, β) -metric $F(x, y)$ = $\alpha\phi(b^2,\frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$). If β is parallel with respect to α , then the flag curvature $K^F(y, P)$ of F is given by

(2.1)
$$
K^{F}(y, P) = \frac{\alpha^{2} ||u||_{\alpha}^{2} - g(y, u)^{2}}{F^{2} g_{y}^{F}(u, u) - g_{y}^{F}(y, u)^{2}} \rho K^{g}(P).
$$

where $P = span\{y, u\}, \rho = \phi(\phi - s\phi_2)$, and $K^g(P)$ is the sectional curvature of the Riemannian metric g.

In a particular case, if $\{y, u\}$ is an orthonormal set with respect to the Riemannian metric g, then

(2.2)
$$
K^{F}(y, P) = \frac{1}{\phi^{2}(1 + g^{2}(X, u)D)} K^{g}(P),
$$

where

$$
D:=\frac{\phi_{22}}{\phi-s\phi_2},
$$

and X is the vector field corresponding to the 1-form β with respect to the Riemannian metric g .

Proof. For a local coordinate system (x^{i}) let $s = \frac{\beta}{\alpha}$ $\frac{\beta}{\alpha}$, where $\alpha(x, y) = \sqrt{g_{ij}y^{i}y^{j}}$ and $\beta(x, y) =$ $b_i y^i$. Using Proposition 3.2 of [\[12\]](#page-9-8) for the Hessian matrix g_{ij}^F of F we have:

$$
g_{ij}^F = \rho g_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s \rho_1 \alpha_{y^i} \alpha_{y^j}.
$$

where

$$
\rho = \phi^2 - s\phi\phi_2 = \phi(\phi - s\phi_2), \quad \rho_0 = \phi\phi_{22} + \phi_2\phi_2, \quad \rho_1 = (\phi - s\phi_2)\phi_2 - s\phi\phi_{22}.
$$

Suppose that β is parallel with respect to α i.e. $b_{i;j} = 0$. According to Remark [\(2.1\)](#page-1-0), we have $G^i = G^i_{\alpha}$. The Riemannian curvature of F is given by

.

,

(2.3)
$$
R_k^i = 2\frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i \partial G^j}{\partial y^j \partial y^k}
$$

The formula of Riemannian curvature [\(2.3\)](#page-2-0) implies that

$$
R_j^i = \,^{\alpha} R_j^i \,.
$$

where R_j^i and ${}^{\alpha}R_j^i$ are the Riemannian curvatures of F and α , respectively. Now, let

$$
R_{ij} := g_{im}^F R_j^m, \, \,^\alpha R_{ij} := g_{im} \, \,^\alpha R_j^m \, .
$$

Using the fact

$$
\alpha_{y^m}{}^{\alpha}R_j^m = \frac{1}{\alpha}g_{im}y^i{}^{\alpha}R_j^m = \frac{1}{\alpha}y^i{}^{\alpha}R_{ij} = 0,
$$

and by a direct computation we have

(2.4)
$$
R_{ij} = g_{im}^F \alpha R_j^m
$$

$$
= (\rho g_{im} + \rho_0 b_i b_m + \rho_1 (b_i \alpha_{y^m} + b_m \alpha_{y^i}) - s \rho_1 \alpha_{y^i} \alpha_{y^m}) \alpha R_j^m
$$

$$
= \rho^{\alpha} R_{ij}.
$$

Now, since $b_{i,j} = 0$, the Ricci identity implies that

$$
b_m{}^{\alpha} R_j^m = b_m{}^{\alpha} R_{ijk}^m = b_{i;j;k} - b_{i;k;j} = 0.
$$

So

$$
b_m{}^{\alpha}R_j^m = b_m{}^{\alpha}R_{ijk}^m y^i y^j = 0.
$$

By the definition, for $P = span\{u, y\}$ and $u = u^i \frac{\partial}{\partial x^i}$, the flag curvature K^F of F and the sectional curvature K^g are given by

(2.5)
$$
K^{F}(y, P) = \frac{g_{y}^{F}(R_{y}(u), u)}{g_{y}^{F}(y, y)g_{y}^{F}(u, u) - (g_{y}^{F}(y, u))^{2}} = \frac{R_{ij}u^{i}u^{j}}{F^{2}g_{y}^{F}(u, u) - (g_{y}^{F}(u, y))^{2}}
$$

and

(2.6)
$$
K^{g}(y,p) = \frac{g({}^{\alpha}R_{y}(u),u)}{g(y,y)g(u,u) - g^{2}(y,u)} = \frac{{}^{\alpha}R_{ij}u^{i}u^{j}}{\alpha^{2}g(u,u) - g^{2}(y,u)}.
$$

The relations $(2.4),(2.5)$ $(2.4),(2.5)$ and (2.6) , imply that (2.1) holds.

It can be shown that, if $\{y, u\}$ is an orthonormal basis of P with respect to the Riemannian metric g , then

$$
g_y^F(u, u) = g_{ij}^F u^i u^j = (\rho g_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i}) - s \rho_1 \alpha_{y^i} \alpha_{y^j}) u^i u^j
$$

= $\rho + \rho_0 g^2(X, u),$

and

$$
g_y^F(y, u) = g_{ij}^F y^i u^j = (\rho g_{ij} + \rho_0 b_i b_j + \rho_1 (b_i \alpha_{y^j} + b_j \alpha_{y^i} - s \rho_1 \alpha_{y^i} \alpha_{y^j}) y^i u^j
$$

= $\rho_0 \beta g(X, u) + \rho_1 \alpha g(X, u) = (s \rho_0 + \rho_1) \alpha g(X, u)$
= $\phi \phi_2 g(X, u)$.

On the other hand, we have

(2.7)
$$
g_{ij}u^i u^j = g(u, u) = 1, \quad b_i u^i = g(X, u), \quad \alpha_{y^i} u^i = \frac{1}{\alpha} g(y, u) = 0.
$$

It follows that

(2.8)
$$
F^{2}g_{y}^{F}(u, u) - (g_{y}^{F}(y, u))^{2} = \phi^{2}(\rho + \rho_{0}g^{2}(X, u)) - (\phi\phi_{2}g(X, u))^{2}
$$

$$
= \phi^{2}\rho + g^{2}(X, u)(\phi^{2}\rho_{0} - (\phi\phi_{2})^{2})
$$

$$
= \phi^{2}\rho + g^{2}(X, u)\phi^{3}\phi_{22}.
$$

Finally, using [\(2.7\)](#page-3-0), [\(2.8\)](#page-3-1) and the relation $D = \frac{\phi_{22}}{\phi - s\phi_2}$, we have

$$
K^{F}(y, P) = \frac{\rho}{\phi^{2} \rho + g^{2}(X, u)\phi^{3} \phi_{22}} K^{g}(P) = \frac{1}{\phi^{2}(1 + g^{2}(X, u)D)} K^{g}(P).
$$

Proposition 2.3. If \tilde{F} is a Finsler metric conformally equivalent to an (α, β) -metric F then \tilde{F} is an (α, β) -metric.

Proof. Suppose that M is an arbitrary differentiable manifold equipped with an (α, β) -metric F, defined by a Riemannian metric g and a vector field X. Let $\tilde{F} = e^{f}F$ be a Finsler metric conformally equivalent to F. Suppose that F is defined by $F = \alpha \phi \left(\frac{\beta}{\alpha} \right)$ $\frac{\beta}{\alpha}$) where $\phi: (b_0, b_0) \to \mathbb{R}$ is a C^{∞} function. Now, we define a vector field \tilde{X} and a Riemannian metric \tilde{g} as follows:

(2.9)
$$
\tilde{X} = e^{-f}X, \quad \tilde{g} = e^{2f}g.
$$

Let $\tilde{\phi} = \phi$, so we have

$$
\tilde{\alpha}\tilde{\phi}\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\right)(x,y) = \sqrt{e^{2f(x)}g(y,y)}\phi\left(\frac{e^{2f(x)}g(e^{-f(x)}X,y)}{\sqrt{e^{2f(x)}g(y,y)}}\right) = e^{f(x)}F(x,y) = \tilde{F}
$$

Also we have $\|\tilde{X}\|_{\tilde{\alpha}} < b_0$.

In the next proposition, we give a necessary and sufficient condition for a Finsler metric \tilde{F} conformally equivalent to an (α, β) -metric F, to be of Berwald type.

Proposition 2.4. Let M be an arbitrary differentiable manifold. Suppose that F, \tilde{F} , g, \tilde{q} , X and \tilde{X} are as the previous proposition. Let ∇ and $\tilde{\nabla}$ be the Levi-Civita connections of the Riemannian metrics q and \tilde{q} , respectively. Then, the Finsler metric \tilde{F} is of Berwald type if and only if for any vector field Y on M we have

$$
\nabla_Y X = g(X, Y)\nabla f - X fY
$$

Proof. We know that the Finsler metric \tilde{F} is of Berwald type if and only if the vector field \tilde{X} is parallel with respect to \tilde{g} , that is, for any vector field Y on $M \tilde{\nabla}_Y \tilde{X} = 0$. On the other hand, by Lemma 1 of $[6]$, we have

(2.11)
$$
\tilde{\nabla}_Y \tilde{X} = \nabla_Y \tilde{X} + (Yf)\tilde{X} + (\tilde{X}f)Y - g(\tilde{X}, Y)\nabla f.
$$

So \tilde{F} is a Berwald metric if and only if

(2.12)
$$
\nabla_Y \tilde{X} + (Yf)\tilde{X} + (\tilde{X}f)Y - g(\tilde{X}, Y)\nabla f = 0.
$$

Now, suppose that $\tilde{X} = e^{-f}X$, then we have,

(2.13)
$$
(Ye^{-f})X + e^{-f}(\nabla_Y X + (Yf)X + (Xf)Y - g(X,Y)\nabla f) = 0.
$$

A direct computation shows that equation [\(2.13\)](#page-4-0) is equivalent to

(2.14)
$$
\nabla_Y X = g(X, Y) \nabla f - X f Y.
$$

In [\[13\]](#page-9-10), Zhu studied general (α, β) -metrics with vanishing Douglas curvature. Also in [\[9\]](#page-9-11), the authors studied two-dimensional conformally related Douglas metrics and showed that such metrics are Randers. In the following proposition, we study conformally related Douglas Randers metrics in arbitrary dimension.

Proposition 2.5. Let $F = \alpha + \beta$ be a Randers metric and \tilde{F} be conformally related to F. Then, \tilde{F} is of Douglas type if and only if

(2.15)
$$
d\beta(Y,Z) + Yfg(X,Z) - Zfg(X,Y) = 0 \quad \forall Y, Z \in \mathcal{X}(M).
$$

Proof. Let $\tilde{F} = \tilde{\alpha} + \tilde{\beta}$ be a Randers metric that is $\tilde{\alpha}(x, y) = \sqrt{\tilde{g}_x(y, y)}$ and $\tilde{\beta}(x, y) = \tilde{g}(\tilde{X}(x), y)$. We know that the Finsler metric \tilde{F} is of Douglas type if and only if the 1-form $\tilde{\beta}$ is closed i.e. $d\tilde{\beta} = 0$. On the other hand, according to Proposition 14.29 of [\[7\]](#page-9-12) we have

(2.16)
$$
d\tilde{\beta}(Y,Z) = Y\tilde{\beta}(Z) - Z\tilde{\beta}(Y) - \tilde{\beta}[Y,Z] \qquad \forall Y, Z
$$

It shows that \tilde{F} is a Douglas metric if and only if

(2.17)
$$
Y\tilde{g}(\tilde{X},Z)-Z\tilde{g}(\tilde{X},Y)-\tilde{g}(\tilde{X},[Y,Z])=0.
$$

We replace \tilde{X} and \tilde{g} with $e^{-f}X$ and $e^{2f}g$ respectively in the above equality obtaining

(2.18)
$$
Y(e^f g(X, Z)) - Z(e^f g(X, Y)) - e^f g(X, [Y, Z]) = 0.
$$

By direct computation, [\(3.4\)](#page-5-0) is equivalent to

(2.19)
$$
d\beta(Y,Z) + Yfg(X,Z) - Zfg(X,Y) = 0.
$$

Corollary 2.6. Let $F = \alpha + \beta$ be a Randers metric of Douglas type and \tilde{F} be conformally related to F, $\tilde{F} = e^f F$, then, \tilde{F} is of Douglas type if and only if $f_i(x) = \frac{\partial f}{\partial x_i}(x)$ proportional to $b_i(x)$ i.e. $f_i b_j - f_j b_i = 0$, where $\beta = b_i(x) y^i$.

Proof. Since F is a Randers metric of Douglas type hence, $d\beta = 0$. Suppose $Y = \frac{\partial}{\partial x_i}$ and $Z = \frac{\partial}{\partial x_j}$. According to Proposition [\(2.5\)](#page-4-1) we have $\frac{\partial f}{\partial x_i} b_j - \frac{\partial f}{\partial x_j} c_j$ $\frac{\partial f}{\partial x_j}b_i=0.$

3. Conformally related invariant (α, β) -metrics

In this short section, we study conformally related invariant (α, β) -metrics on homogeneous spaces. In the following proposition, easily we see that there is no nontrivial G -invariant general (α, β) -metric on a homogeneous space G/H .

Proposition 3.1. Let $F = \alpha \phi(x, \frac{\beta}{\alpha})$ $\frac{\beta}{\alpha}$) be an (α, β) -metric which is defined by a G-invariant vector field X and a G-invariant Riemannian metric g on $M = G/H$. Then, F is a Ginvariant general (α, β) -metric if and only if F is a G-invariant (α, β) -metric. In the special case any left-invariant general (α, β) -metric defined by a left-invariant vector field and a leftinvariant Riemannian metric is a left-invariant (α, β) -metric.

Proof. Let $\tau_a: G/H \to G/H$ be a diffeomorphism that $\tau_a(xH) = axH \quad \forall a, x \in G$. The Riemannian metric g and the vector field X are G -invariant that is

(3.1)
$$
g_{xH}(d\tau_x Y, d\tau_x Z) = g_{eH}(Y, Z)
$$

$$
(3.2) \t\t d\tau_x X = X.
$$

By [\(3.1\)](#page-5-1) and [\(3.2\)](#page-5-2) F is an (α, β) -metric.

Remark 3.2. Let g be a G-invariant Riemannian metric and X be a G-invariant vector field on the homogeneous space $M = G/H$. Suppose that $f : M \to \mathbb{R}$ is a smooth function such that $f(H) = 0$ and the vector field \tilde{X} and the Riemannian metric \tilde{g} are defined as follows

(3.3)
$$
\tilde{X} = e^{-f} X, \tilde{g} = e^{2f} g.
$$

Clearly \tilde{X} and \tilde{g} are not necessarily G-invariant but the two Riemannian metrics g and \tilde{g} on M are conformally related. Suppose that $\tilde{F} = \tilde{\alpha}\tilde{\phi}(\tilde{b}^2, \frac{\tilde{\beta}}{\tilde{\alpha}})$ $\frac{\beta}{\tilde{\alpha}}$) is a general (α, β) -metric on G/H , where $\tilde{\alpha}$ is the norm of the metric \tilde{g} , $\tilde{\beta}$ is the 1-form defined by \tilde{X} , and $\tilde{b}^2 = \tilde{g}(\tilde{X}, \tilde{X})$. Now, we assume $\phi: (-b_0, b_0) \to \mathbb{R}$ such that $\phi(s) := \tilde{\phi}(\tilde{b}^2(H), s)$. Easily, ϕ is a C^{∞} function. It can be shown that $F = \alpha \phi \left(\frac{\beta}{\alpha} \right)$ $\frac{\beta}{\alpha}$) is a G-invariant (α, β) -metric on $M = G/H$, where $X = \tilde{X}(H)$ and $\beta(y) = g(X, y)$. Furthermore \tilde{F} is conformally related to F.

We now turn to the left-invariant metrics on the Lie groups.

Proposition 3.3. Suppose that $F = \alpha + \beta$ is a left-invariant Randers metric defined by a left-invariant vector field X and a left-invariant Riemannian metric q on a Lie group G . Let $\tilde{F} = e^f F$. If \tilde{F} is of Douglas type then for all left-invariant vector fields Y, Z we have

(3.4)
$$
g(X,[Z,Y]) + Yfg(X,Z) - Zfg(X,Y) = 0.
$$

Proof. According to Proposition 14.29 of [\[7\]](#page-9-12) we have

(3.5)
$$
d\beta(Y,Z) = Yg(X,Z) - Zg(X,Y) - g([Y,Z],X).
$$

Since g is a left-invariant Riemannian metric so

$$
(3.6) \t d\beta(Y,Z) = g(X,[Y,Z])
$$

Due to (2.15) and (3.6) we conclude that (3.4) holds and it completes the proof.

4. Examples

In this section, using the results obtained in the previous sections, we give some examples of (α, β) -metrics that are conformally related to Randers metric under which conditions they are of Douglas type. Also for a certain X and f, we show that \tilde{F} , which is the conformally related to an (α, β) -metric, is of Berwald type.

4.1. The Heisenberg group H_3 . The Heisenberg group H_3 can be considered as the Euclidean space \mathbb{R}^3 with the following multiplication

(4.1)
$$
(x', y', z').(x, y, z) = (x' + x, y' + y, z' + z + \frac{1}{2}yx' - \frac{1}{2}y'x).
$$

Let g be the left-invariant Riemannian metric on H_3 such that the left-invariant basis

(4.2)
$$
\{e_1 = \frac{\partial}{\partial x} - \frac{y}{2} \frac{\partial}{\partial z}, e_2 = \frac{\partial}{\partial y} + \frac{x}{2} \frac{\partial}{\partial z}, e_3 = \frac{\partial}{\partial z}\},\
$$

is an orthonormal basis. Easily, we can see

(4.3)
$$
[e_1, e_2] = e_3, [e_1, e_3] = [e_2, e_3] = 0.
$$

Proposition 4.1. Suppose that $G = H_3$ is the Heisenberg Lie group, and $F = \alpha + \beta$ is a Randers metric defined by the left-invariant Riemannian metric g (which is defined above) and a left-invariant vector field on H_3 . Let $\tilde{F} = e^f F$ be conformally related to F. Then \tilde{F} is a Douglas metric if and only if

$$
(4.4) \t\t X = ae_1 + be_2, \t a, b \in \mathbb{R}
$$

$$
\frac{\partial f}{\partial z} = 0,
$$

(4.6)
$$
b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y} = 0.
$$

Proof. If $X = ae_1 + be_2$ and f satisfy the conditions [\(4.5\)](#page-6-1) and [\(4.6\)](#page-6-2), easily it can be seen the relation [\(2.15\)](#page-4-1) holds, and it shows that \tilde{F} is of Douglas type.

Conversely, if $X = ae_1 + be_2 + ce_3$ and \tilde{F} is a Douglas metric, based on Proposition [\(2.5\)](#page-4-1), we have

$$
d\beta(Y,Z) + Yfg(X,Z) - Zfg(X,Y) = 0,
$$

which is satisfied for all left-invariant vector fields Y and Z. In the special case where $Y = e_i$ and $Z = e_j$ $(i < j, i, j \in \{1, 2, 3\})$, we have

(4.7)
$$
b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y} - (by + a\frac{x}{2})\frac{\partial f}{\partial z} = c,
$$

(4.8)
$$
c\frac{\partial f}{\partial x} - (cy + a)\frac{\partial f}{\partial z} = 0,
$$

(4.9)
$$
c\frac{\partial f}{\partial y} + (c\frac{x}{2} - b)\frac{\partial f}{\partial z} = 0.
$$

Then, by [\(4.7\)](#page-7-0), [\(4.8\)](#page-7-1) and [\(4.9\)](#page-7-2), we have $c = 0$, $\frac{\partial f}{\partial z} = 0$ and $b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y} = 0$.

Corollary 4.2. a: If $X = be_2$ then, \tilde{F} is of Douglas type if and only if $f = f(y)$. **b:** If $X = ae_1$ then, \tilde{F} is of Douglas type if and only if $f = f(x)$.

4.2. The Lie group $\mathbb{R} \rtimes \mathbb{R}^+$. Let G be the two-dimensional solvable Lie group $\mathbb{R} \rtimes \mathbb{R}^+$ and g be the left-invariant Riemannian metric on $G = \mathbb{R} \rtimes \mathbb{R}^+$ such that the left-invariant basis

(4.10)
$$
\{e_1 = y\frac{\partial}{\partial y}, e_2 = y\frac{\partial}{\partial x}\},\
$$

is an orthonormal basis. Easily, we can see

$$
(4.11) \t\t\t [e_1, e_2] = e_2.
$$

Suppose that $F = \alpha + \beta$ is a Randers metric defined by the above left-invariant Riemannian metric g and a left-invariant vector field $X = ae_1 + be_2(a, b \in \mathbb{R})$.

Proposition 4.3. Suppose that G is the Lie $\mathbb{R} \rtimes \mathbb{R}^+$, and $F = \alpha + \beta$ is a Randers metric as above. Let $\tilde{F} = e^f F$ be conformally related to F, then \tilde{F} is a Douglas metric if and only if

(4.12)
$$
by \frac{\partial f}{\partial y} - ay \frac{\partial f}{\partial x} = b.
$$

Proof. If $X = ae_1 + be_2$ is a left-invariant vector field, and f satisfies the conditions [\(4.12\)](#page-7-3), easily it can be seen the relation [\(2.15\)](#page-4-1) holds and it is shown that \tilde{F} is of Douglas type. Conversely, if \tilde{F} is a Douglas metric and $X = ae_1 + be_2$, based on Proposition [\(2.5\)](#page-4-1), we have

$$
d\beta(Y,Z) + Yfg(X,Z) - Zfg(X,Y) = 0
$$

which is satisfied for all left-invariant vector fields Y and Z. In the special case where $Y = e_1$ and $Z = e_2$ we have

(4.13)
$$
by \frac{\partial f}{\partial y} - ay \frac{\partial f}{\partial x} = b.
$$

Corollary 4.4. a: If $X = be_2$ then, \tilde{F} is of Douglas type if and only if $f = lny + g(x)$. **b:** If $X = ae_1$ then, \tilde{F} is of Douglas type if and only if $f = f(y)$.

Example 4.5. Let $G = \mathbb{R} \rtimes \mathbb{R}^+$ and F be an (α, β) -metric defined by the above left-invariant Riemannian metric g and a left-invariant vector field X . For the Levi-Civita connection of g, we can see that $\nabla_{e_2}e_1 = -e_2$ and $\nabla_{e_1}e_1 = 0$. Let $X = ae_1(a \in \mathbb{R})$, $f(x, y) = ln y$ and $Y = y_1e_1 + y_2e_2(y_1, y_2 \in C^{\infty}(G))$. Easily we have $\nabla_Y X = -ay_2y\frac{\partial}{\partial x}$, $g(X, Y) = ay_1$, $Xf = a$ and $\nabla f = y \frac{\partial}{\partial y}$. Therefore, according to the proposition[\(2.4\)](#page-4-2), \tilde{F} is of Berwald type.

4.3. The Lie group $\mathbb{R}^2 \rtimes \mathbb{R}^+$. Let $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$ and g be the left-invariant Riemannian metric on G such that the left-invariant basis

(4.14)
$$
\{e_1 = z\frac{\partial}{\partial z}, e_2 = z\frac{\partial}{\partial x}, e_3 = z\frac{\partial}{\partial y}\},\
$$

is an orthonormal basis. Clearly, we can see

(4.15)
$$
[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_2, e_3] = 0.
$$

Suppose that $F = \alpha + \beta$ is a Randers metric defined by g and a left-invariant vector field $X = ae_1 + be_2 + ce_3(a, b, c \in \mathbb{R})$ on $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$.

Proposition 4.6. Let $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$ and $F = \alpha + \beta$ be a Randers metric as above. If $\tilde{F} = e^f F$ is conformally related to F, then \tilde{F} is a Douglas metric if and only if

(4.16)
$$
bz\frac{\partial f}{\partial z} - az\frac{\partial f}{\partial x} = b,
$$

(4.17)
$$
cz\frac{\partial f}{\partial x} - bz\frac{\partial f}{\partial y} = 0,
$$

(4.18)
$$
cz\frac{\partial f}{\partial z} - az\frac{\partial f}{\partial y} = c.
$$

Proof. If $X = ae_1 + be_2 + ce_3$ and f satisfy the conditions [\(4.16\)](#page-8-0), [\(4.17\)](#page-8-1) and [\(4.18\)](#page-8-2), easily it can be seen the relation [\(2.15\)](#page-4-1) holds, and it shows that \tilde{F} is of Douglas type. Conversely, if \tilde{F} is a Douglas metric and $X = ae_1 + be_2 + ce_3$, based on Proposition [\(2.5\)](#page-4-1), we

have

$$
d\beta(Y,Z) + Yfg(X,Z) - Zfg(X,Y) = 0
$$

which is satisfied for all Y and Z in the Lie algebra of G . In the special case where $Y = e_i$ and $Z = e_j$ $(i < j, i, j \in \{1, 2, 3\})$ we have

(4.19)
$$
bz \frac{\partial f}{\partial z} - az \frac{\partial f}{\partial x} = b,
$$

(4.20)
$$
cz\frac{\partial f}{\partial z} - az\frac{\partial f}{\partial y} = c,
$$

(4.21)
$$
cz\frac{\partial f}{\partial x} - bz\frac{\partial f}{\partial y} = 0.
$$

Corollary 4.7. a: If $X = be_2 + ce_3$ then, \tilde{F} is of Douglas type if and only if $f =$ $ln z + g(x, y)$ and $cz\frac{\partial f}{\partial x} - bz\frac{\partial f}{\partial y} = 0.$

b: If $X = ae_1 + ce_3$ then, \tilde{F} is of Douglas type if and only if $f = ln|z| + c'(c' \in \mathbb{R})$. c: If $X = ae_1 + be_2$ then, \tilde{F} is of Douglas type if and only if $f = f(x, z)$ and $bz \frac{\partial f}{\partial z} - az \frac{\partial f}{\partial x} =$ b.

Example 4.8. Let $G = \mathbb{R}^2 \rtimes \mathbb{R}^+$ and F be an (α, β) -metric defined by the above left-invariant Riemannian metric g and a left-invariant vector field X. We see that $\nabla_{e_1} e_3 = \nabla_{e_2} e_3 = 0$ and $\nabla_{e_3}e_3 = e_1$. Suppose that $X = ae_3(a \in \mathbb{R})$, $f(x, y, z) = lnz$ and $Y = y_1e_1 + y_2e_2 +$ $y_3e_3(y_1, y_2, y_3 \in C^{\infty}(G))$. Easily $\nabla_Y X = ay_3z \frac{\partial}{\partial z}$, $g(X,Y) = ay_3$, $Xf = 0$ and $\nabla f = z \frac{\partial}{\partial z}$. Therefore, according to the Proposition (2.4) , \ddot{F} is a Berwald metric.

Acknowledgment. We are grateful to the office of Graduate Studies of the University of Isfahan for their support.

REFERENCES

- [1] Antonelli, P. L. , Ingarden, R. S., Matsumoto, M.: The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology. Kluwer, Dordrecht (1993)
- [2] Asanov, G. S.: Finsler Geometry, Relativity and Gauge Theories. D. Reidel, Dordrecht (1985)
- [3] Chern, S. S., Shen, Z.: Riemann-Finsler Geometry. World Scientific, Singapore (2005)
- [4] Deng, S.: Homogeneous Finsler Spaces. Springer, New York (2012)
- [5] Deng, S., Hosseini, M., Liu, H., Salimi Moghaddam, H. R.: On Left Invariant (α, β) -metrics on Some Lie Groups. Houst. J. Math. 45, 1071-1088 (2019)
- [6] Kuhnel, W.: *Conformal Transformations between Einstein Spaces*. Conformal Geometry: A Publication of the Max-Planck-Institut für Mathematik, Bonn, 105-146 (1988), DOl 10.1007/978-3-322-90616-8
- [7] Lee, J. M.:Introduction to smooth manifolds. Second edition, Springer, (2013)
- [8] Matsumoto, M.: Theory of Finsler spaces with (α, β) -metric. Rep. Math. Phys. **31**, 43-83 (1992)
- [9] Matveev, V. S., Saberali, S.: Conformally related Douglas metrics in dimension two are Randers. Arch. Math. 116, 221-231 (2021)
- [10] Randers, G.:On an asymmetrical metric in the four-space of general relativity. Phys. Rev. 59, 195-199 (1941)
- [11] Yoshikawa, R., Sabau, S. V.: Kropina metrics and Zermelo navigation on Riemannian manifolds. Geom. Dedicata. 171, 119-148 (2014)
- [12] Yu, C., Zhu, H.: On a new class of Finsler metrics. Differential Geom. Appl. 29, 244-254 (2011)
- [13] Zhu, H.: On general (α, β) -metrics with vanishing Douglas curvature. Int. J. Math. **26**, 1550076 (16 pages) (2015)

Azar Fatahi, Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441-Iran.

Masoumeh Hosseini, Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441-Iran.

Email address: hoseini masomeh@ymail.com

Hamid Reza Salimi Moghaddam, Department of Pure Mathematics, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, 81746-73441-Iran., Scopus Author ID: 26534920800, ORCID ID:0000-0001-6112-4259,

Email address: hr.salimi@sci.ui.ac.ir and salimi.moghaddam@gmail.com