STRONG NON-VANISHING OF COHOMOLOGIES AND STRONG NON-FREENESS OF ADJOINT LINE BUNDLES ON n-RAYNAUD SURFACES

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ABSTRACT. We formally give the definition of n-Tango curve and n-Raynaud surface. Then we study the pathologies on n-Raynaud surfaces and as a corollary we give a simple disproof of Fujita's conjecture on surfaces in positive characteristics.

1. INTRODUCTION

In [5] Raynaud constructed a surface X in positive characteristic with an ample line bundle $\mathcal{L}$ such that $H^1(X, \mathcal{L}^{-1}) \neq 0$, a counter example to Kodaira vanishing. Generally speaking, we know that vanishing theorem is a key factor in the proof of the following famous conjecture proposed by T. Fujita in [1] in characteristic zero:

Conjecture 1.1 (Fujita's conjecture). *Let X be a smooth projective variety of dimension n over an algebraically closed field $\mathbf{k}$ and A an ample divisor on X. Then*

(1) *when $m \geq n+1$, the adjoint linear system $|K_X + mA|$ is base point free and*
(2) *when $m \geq n+2$, the adjoint linear system $|K_X + mA|$ is very ample.*

Nevertheless vanishing theorem is not a necessary condition of Fujita's conjecture since there indeed exist some cases where Fujita's conjecture holds while vanishing theorem is false such as quasi-elliptic surfaces. So, many people insisted that Fujita's conjecture should hold in positive characteristic and a lot of work has been done on how to produce global sections of adjoint line bundles in positive characteristic. However, exceeding one's expectations, Fujita's conjecture has been disproved in positive characteristic recently by the author with the cooperators in [3].

In this paper we formally give the definition of n-Tango curves and n-Raynaud surfaces, and find that some pathologies of an n-Raynaud surface are determined by the associated vecter bundle $\mathcal{E}$ on the base n-Tango curve. We will see that the larger the number n is the more pathologies the surface catches, such as strong Kodaira non-vanishing (Theorem 4.2) which is only depended on the degree of the associated divisor $\mathcal{L}$ (or $\mathcal{N}$) on the base n-Tango curve; But the strong non-freeness of adjoint line bundles is even more related with the degree sub-bundles in $\mathcal{E}$ with a fixed quotient line bundle $\mathcal{L}_0$ and a non-trivial parametrise space $\mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}_0))$ (Theorem 3.4). And as a corollary, we give a simple disproof of Fujita's conjecture avoiding so many tedious computations in [3].

2. n-TANGO CURVE AND n-RAYNAUD SURFACE

Through the paper, $\mathbf{k}$ will denote an algebraically closed field with char($\mathbf{k}$) = p > 0.

2.1. n-**Tango curve.** Let C be a smooth projective curve defined over $\mathbf{k}$ and $K(C)$ be the function field of C. Denote by $K(C)^p = \{f^p | f \in K(C)\}$ the subfield of p-th powers. Let F be the absolute Frobenius morphism. In [5,7] the following exact sequence

$$0 \to \mathcal{O}_C \to F_*\mathcal{O}_C \to \mathcal{B}^1 \to 0$$

is used to construct Tango curves, where $\mathcal{B}^1$ is the exact 1-form on C. More generally, we consider the following exact sequence

$$0 \to \mathcal{O}_C \to F^n_* \mathcal{O}_C \to F^n_* \mathcal{O}_C / \mathcal{O}_C \to 0$$

and give the following definition.

Definition 2.1. Let C be a smooth projective curve over $\mathbf{k}$ satisfying the following conditions.

(1) There is a rational function $f \in K(C) \backslash K(C)^p$ such that $(\mathrm{d}f) = p^n D$ for some divisor D on C with $\deg D > 0$ and some integer $n > 0$. Denote the associated line bundle $\mathcal{L} = \mathcal{O}_C(D)$, then $\omega_C \simeq \mathcal{L}^{p^n}$ and we have a nonzero section $s_0 \in H^0(C, F^{n-1}_* \mathcal{B}^1 \otimes \mathcal{L}^{-1})$.

(2) Moveover, we assume that this section can be lift to a section $s \in H^0(C, (F^n_* \mathcal{O}_C / \mathcal{O}_C) \otimes \mathcal{L}^{-1})$ through the quotient $F^n_* \mathcal{O}_C / \mathcal{O}_C \twoheadrightarrow F^{n-1}_* \mathcal{B}^1$.

Then the curve C is called n-**Tango curve** and the triple (C, f, D) is called n-**Tango data**.

Let (C, f, D) be an n-Tango data, then by definition we have a nonzero section $s_0 \in H^0(C, F^{n-1}_* \mathcal{B}^1 \otimes \mathcal{L}^{-1})$. Now let's take an open affine covering $C = U_1 \cap U_2$ such that $\mathcal{L}|_{U_i}$ is free with the generators $\eta_i \in H^0(U_i, \mathcal{L}|_{U_i})$ and the transition relation $\eta_1 = \alpha \eta_2$ for some $\alpha \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)^*$. From the nature morphism

$$F^n_* \mathcal{O}_C \xrightarrow{\psi} F^n_* \mathcal{O}_C / \mathcal{O}_C \xrightarrow{\phi} F^{n-1}_* \mathcal{B}^1,$$

we can find two regular functions $z_i \in \Gamma(\mathcal{O}, U_i)$ such that the section s_0 can be locally written as $s_0|_{U_i} = \phi \circ \psi(\sqrt[p^n]{z_i}) \otimes \frac{1}{\eta_i}$ and then we have the relation $\phi \circ \psi(\sqrt[p^n]{z_1}) = \alpha \phi \circ \psi(\sqrt[p^n]{z_2})$. By (2) this section can be lift to a section $s \in H^0(C, (F^n_* \mathcal{O}_C / \mathcal{O}_C) \otimes \mathcal{L}^{-1})$ through ϕ, so we have $s|_{U_i} = \psi(\sqrt[p^n]{z_i}) \otimes \frac{1}{\eta_i}$ and the relation $\psi(\sqrt[p^n]{z_1}) = \alpha \psi(\sqrt[p^n]{z_2})$. Hence we have the relation

$$\sqrt[p^n]{z_1} = \alpha \sqrt[p^n]{z_2} + \beta$$

for some $\beta \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)$.

Moreover, we get a sub-sheaf $\mathcal{L} \hookrightarrow F^n_* \mathcal{O}_C / \mathcal{O}_C$ and then a locally free sub-sheaf of rank two $\mathcal{E} := \psi^{-1}(\mathcal{L}) \subset F^n_* \mathcal{O}_C$ from the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & F^n_* \mathcal{O}_C & \xrightarrow{\psi} & F^n_* \mathcal{O}_C / \mathcal{O}_C & \longrightarrow & 0 \\
& & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{L} & \longrightarrow & 0.
\end{array}
$$

If it is locally written as

$$\mathcal{E}|_{U_i} = \mathcal{O}_{U_i} \cdot 1 \oplus \mathcal{O}_{U_i} \cdot \sqrt[p^n]{z_i},$$

then the translation relation is $\sqrt[p^n]{z_1} = \alpha \sqrt[p^n]{z_2} + \beta$ for some $\beta \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)$. I.e. the vector bundle $\mathcal{E}$ is defined by the transition matrix $\begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix} \in GL(2, \mathcal{O}_{U_1 \cap U_2})$.

From the above argument we have the following relation of rational functions

$$z_1 = \alpha^{p^n} z_2 + \beta^{p^n}$$

with $z_i \in \mathcal{O}_{U_i}$, $\alpha \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)^*$ and $\beta \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)$. If we take the differential then $\mathrm{d}z_1 = \alpha^{p^n} \mathrm{d}z_2$, which means that ω_C contains a sub-sheaf which is locally generated by $\mathrm{d}z_i$ and isomorphic to $\mathcal{L}^{p^n}$. Note that $\omega_C \simeq \mathcal{L}^{p^n}$, so we see that ω_C is locally generated by $\mathrm{d}z_i$ and z_i is a local parameter on U_i. And hence we see that the nature map $Sym^m(\mathcal{E}) \to F^n_* \mathcal{O}_C$ is an embedding when $m < n$ and isomorphism when $m = n$.

To sum up, we get an equivalent description of n-Tango data: there is a cover $C = U_1 \cap U_2$ and relation of rational functions

$$z_1 = \alpha^{p^n} z_2 + \beta^{p^n}$$

where $z_i \in \mathcal{O}_{U_i}$, $\alpha \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)^*$ and $\beta \in \Gamma(U_1 \cap U_2, \mathcal{O}_C)$ such that ω_C is locally generated by dz_i on U_i.

The following argument is another viewpoint on the motivation of the definition of n-Tango curve.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (F_*^{n-1}\mathcal{O}_C/\mathcal{O}_C) \otimes \mathcal{L}^{-1} & \longrightarrow & (F_*^{n}\mathcal{O}_C/\mathcal{O}_C) \otimes \mathcal{L}^{-1} & \longrightarrow & F_*^{n-1}\mathcal{B}^1 \otimes \mathcal{L}^{-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & F_*^{n-1}\mathcal{O}_C \otimes \mathcal{L}^{-1} & \longrightarrow & F_*^{n}\mathcal{O}_C \otimes \mathcal{L}^{-1} & \longrightarrow & F_*^{n-1}\mathcal{B}^1 \otimes \mathcal{L}^{-1} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & & & \\
& & \mathcal{O}_C \otimes \mathcal{L}^{-1} & = \!\!=\!\!= & \mathcal{O}_C \otimes \mathcal{L}^{-1} & & & &
\end{array}
$$

Let C be an n-Tango curve and $s_0 \in H^0(C, F_*^{n-1}\mathcal{B}^1 \otimes \mathcal{L}^{-1})$ be the associated section. By the second exact column of the above diagram, we have long exact sequence

$$0 \to H^0(C, (F_*^{n-1}\mathcal{O}_C/\mathcal{O}_C) \otimes \mathcal{L}^{-1}) \xrightarrow{\alpha} H^1(C, \mathcal{L}^{-1}) \xrightarrow{F^{*n}} H^1(C, F_*^{n}\mathcal{O}_C \otimes \mathcal{L}^{-1}) \to$$

and get a nonzero element $\alpha(s) \in H^1(C, \mathcal{L}^{-1})$ with $F^{*n}(\alpha(s)) = 0$, which is corresponding to the locally free sheaf $\mathcal{E}$ and will be used to construct the ruled surface in the next section; while by the exact sequence

$$0 \to H^0(C, F_*^{n-1}\mathcal{B}^1 \otimes \mathcal{L}^{-1}) \xrightarrow{\beta} H^1(C, F_*^{n-1}\mathcal{O}_C \otimes \mathcal{L}^{-1}) \xrightarrow{F^*} H^1(C, F_*^{n}\mathcal{O}_C \otimes \mathcal{L}^{-1}) \to$$

obtained from the second exact row of the above diagram, we see that $F^{*n-1}(\alpha(s)) = \beta(s) \neq 0$, i.e. n is the smallest integer such that $F^{*n}(\alpha(s)) = 0$.

Remark 2.2. When $n = 1$, condition (2) is satisfied automatically by (1) and this is the usual definition of Tango curve (see [4, 5, 7]). When $n > 1$ condition (2) is necessary since there are indeed such triples (C, f, D) only satisfying condition (1) without condition (2) (see example 2.4).

As the base curve in [3, section 2.2], the following example is obtained by a slight modification of the example 1.3 in [4], which was first found by Gieseker [2] in the case $e = n = 1$ and $p = 3$.

Example 2.3. *Let $Q(X, Y)$ be a homogeneous polynomial of two variable of degree e with a nonzero coefficient of Y^e and $C \subset \mathbb{P}^2 = \textbf{Proj } k[X, Y, Z]$ be a curve defined by the homogeneous equation of degree $p^n e$:*

$$Q(X^{p^n}, Y^{p^n}) - X^{p^n e - 1} Y = Z^{p^n e - 1} X.$$

Note that C is smooth and intersects with $X = 0$ exactly at one point ∞ with multiplicity $p^n e$. So

$$U_1 := C \setminus \infty = \textbf{Spec } k[y_1, z_1]/(Q(1, y_1^{p^n}) - y_1 - z_1^{p^n e - 1})$$

where $y_1 = \frac{Y}{X}$ and $z_1 = \frac{Z}{X}$. and we have the relation $-dy_1 = -(z_1)^{p^n e - 2} dz_1$ on U. So $\omega_C|_{U_1}$ is generated by dz_1. On the other hand, $\deg(dz_1) = \deg \omega_C = p^n e(p^n e - 3)$, so $(dz_1) = p^n e(p^n e - 3)\infty$. Denote by $D := e(p^n e - 3)\infty$ and $\mathcal{L} = \mathcal{O}_C(D)$, then we get a triple (C, z_1, D), a sub-line bundle $\mathcal{L}^{p^{n-1}} \hookrightarrow \mathcal{B}^1$ and hence a nonzero section $s_0 \in H^0(C, F_*^{n-1}\mathcal{B}^1 \otimes \mathcal{L}^{-1})$.

Next we will check that it also satisfies (2). Let $U_2 = C \cap \{Z \neq 0\} \subset C$ be an open affine subset containing ∞ defined by the equation

$$Q(x^{p^n}, y^{p^n}) - x^{p^n e - 1} y = x$$

where $y = Y/Z$ and $x = X/Z$. By taking differential on this equation we have $-x^{p^n e - 1} dy = (1 - yx^{p^n e - 2}) dx$. Note that the special point ∞ is given by $x = y = 0$ and we could take U_2 to be

a small enough neighborhood of ∞ such that $1 - yx^{p^n e-2} \neq 0$ on U_2, then $\omega_C|_{U_2}$ is generated by $\mathrm{d}y$ and y is a local parameter at $\infty = [0,0,1]$. Note that the ideal $(x) = (y^{p^n e}) \subset \mathcal{O}_{C,\infty}$ and we have

$$v_\infty(x) = p^n e = v_\infty(y^{p^n e}) = v_\infty(Q(x^{p^n}, y^{p^n})).$$

Next we set

$$z_2 := \frac{x^{p^n e-2}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}} \cdot y$$

and then

$$\mathrm{d}z_2 = \frac{x^{p^n e-2} + yx^{2p^n e-4}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}(1 - yx^{p^n e-2})}\mathrm{d}y.$$

It is easy to check that

$$v_\infty\left(\frac{x^{p^n e-2}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}}\right) = v_\infty\left(\frac{x^{p^n e-2} + yx^{2p^n e-4}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}(1 - yx^{p^n e-2})}\right) = 0.$$

So $\mathrm{d}z_2$ is a generator of $\omega_C|_{U_2}$ if we shrink U_2 suitably, for example, such that

$$\frac{x^{p^n e-2}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}} \quad and \quad \frac{x^{p^n e-2} + yx^{2p^n e-4}}{Q(x^{p^n}, y^{p^n})y^{p^n e(p^n e-3)}(1 - yx^{p^n e-2})} \in \mathcal{O}_C(U_2)^*.$$

And fortunately we get the following relation

$$z_1 - \frac{1}{Q(x^{p^n}, y^{p^n})} = \frac{1}{x} - \frac{1}{Q(x^{p^n}, y^{p^n})} = \frac{x^{p^n e-1}y}{xQ(x^{p^n}, y^{p^n})} = y^{p^n e(p^n e-3)}z_2,$$

or equivalently,

$$z_1 = (y^{e(p^n e-3)})^{p^n}z_2 + (Q^{-1}(x,y))^{p^n}.$$

We shrink U_2 again such that $y^{e(p^n e-3)} \in \mathcal{O}(U_1 \cap U_2)^*$ and $Q^{-1}(x,y) \in \mathcal{O}(U_1 \cap U_2)$. Finally we obtain a cover $C = U_1 \cap U_2$ and the above relation of rational functions and ω_C is locally generated by $\mathrm{d}z_1$ and $\mathrm{d}z_2$ on U_1 and U_2 respectively. Therefore, triple (C, z_1, D) is an n-Tango data.

Next we will give a special example which just satisfies (1) but does not satisfies (2).

Example 2.4. *Let $C \subset \mathbb{P}^2 = \text{\textbf{Proj}}\ k[X, Y, Z]$ be a curve defined by a homogeneous equation of degree p^2:*

$$X^{p^2-p}Y^p + Y^{p^2} - X^{p^2-1}Y = Z^{p^2-1}X.$$

Note that $X^{p^2-p}Y^p + Y^{p^2}$ is not of the form $Q(X^{p^2}, Y^{p^2})$ for some polynomial Q.

With the same argument as above, we see that C is smooth and it intersects with $X = 0$ at exactly one point ∞ with multiplicity p^2. Set $U_1 := C \setminus \infty = \text{\textbf{Spec}}\ k[y_1, z_1]/(y_1^p + y_1^{p^2}) - y_1 - z_1^{p^2-1})$ where $y_1 = \frac{Y}{X}$ and $z_1 = \frac{Z}{X}$. and we have the following relation $-\mathrm{d}y_1 = -(z_1)^{p^2-2}\mathrm{d}z_1$ on U_1. So $\omega_C|_{U_1}$ is generated by $\mathrm{d}z_1$. On the other hand, $\deg(\mathrm{d}z_1) = \deg \omega_C = p^2(p^2 - 3)$, so $(\mathrm{d}z_1) = p^2(p^2 - 3)\infty$. Denote by $D := (p^2 - 3)\infty$ and $\mathcal{L} = \mathcal{O}_C(D)$, then we get a triple (C, z_1, D), a sub-line bundle $\mathcal{L}^p \hookrightarrow \mathcal{B}^1$ and hence a nonzero section $s_0 \in H^0(C, F_*\mathcal{B}^1 \otimes \mathcal{L}^{-1})$.

Next we will check that it does not satisfies (2). Let $U_2 = C \cap \{Z \neq 0\} \subset C$ be an open affine subset containing ∞ defined by the equation

$$x^{p^2-p}y^p + y^{p^2} - x^{p^2-1}y = x$$

where $y = Y/Z$ and $x = X/Z$. By taking differential on this equation we have $-x^{p^2-1}\mathrm{d}y = (1 - yx^{p^2-2})\mathrm{d}x$. Note that the special point ∞ is given by $x = y = 0$ and we could take U_2 to be a small enough affine neighborhood of ∞ such that $1 - yx^{p^2-2} \neq 0$ on U_2, then $\omega_C|_{U_2}$ is generated by $\mathrm{d}y$ and y is a local parameter at $\infty = [0,0,1]$. Note that the ideal $(x) = (y^{p^2}) \subset \mathcal{O}_{C,\infty}$ and we have

$$v_\infty(x) = v_\infty(y^{p^2}) = p^2 = v_\infty(x^{p^2-p}y^p + y^{p^2}).$$

Next we set

$$z_2 := \frac{x^{p^2-2}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}} \cdot y$$

and then we have

$$\mathrm{d}z_2 = \frac{x^{p^2-2} + yx^{2p^2-4}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}(1 - yx^{p^2-2})}\mathrm{d}y.$$

It is easy to check that

$$v_\infty\Big(\frac{x^{p^2-2}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}}\Big) = v_\infty\Big(\frac{x^{p^2-2} + yx^{2p^2-4}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}(1 - yx^{p^2-2})}\Big) = 0.$$

So $\mathrm{d}z_2$ *is a generator of* $\omega_C|_{U_2}$ *if we shrink* U_2 *suitably, for example, such that*

$$\frac{x^{p^2-2}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}} \text{ and } \frac{x^{p^2-2} + yx^{2p^2-4}}{(x^{p^2-p}y^p + y^{p^2})y^{p^2(p^2-3)}(1 - yx^{p^2-2})} \in \mathcal{O}_C(U_2)^*.$$

On the other hand, we have the following relation

$$z_1 - \frac{1}{(x^{p^2-p}y^p + y^{p^2})} = \frac{1}{x} - \frac{1}{(x^{p^2-p}y^p + y^{p^2})} = \frac{x^{p^2-1}y}{x(x^{p^2-p}y^p + y^{p^2})} = y^{p^2(p^2-3)}z_2,$$

or equivalently,

$$z_1 = y^{(p^2-3)p^2}z_2 + (x^{p^2-p}y^p + y^{p^2})^{-1}.$$

We shrink U_2 *again such that* $y \in \mathcal{O}(U_1 \cap U_2)^*$ *and* $(x^{p^2-p}y^p + y^{p^2})^{-1} \in \mathcal{O}(U_1 \cap U_2)$*. Finally we obtain a cover* $C = U_1 \cup U_2$ *and the above relation of rational functions, and* ω_C *is locally generated by* $\mathrm{d}z_1$ *and* $\mathrm{d}z_2$ *on* U_1 *and* U_2 *respectively.*

Suppose $\mathcal{L}$ *is locally written as* $\mathcal{L}|_{U_i} \simeq \mathcal{O}_{U_i} \cdot \eta_i$ *with generators* $\eta_i \in H^0(U_i, \mathcal{L}|_{U_i})$*, then the transition equation is* $\eta_1 = y^{p^2-3}\eta_2$*. From the above argument and the nature morphism* $F^2_*\mathcal{O}_C \overset{\psi}{\twoheadrightarrow} F^2_*\mathcal{O}_C/\mathcal{O}_C \overset{\phi}{\twoheadrightarrow} F_*\mathcal{B}^1$*, the section* s_0 *can be locally written as* $s_0|_{U_i} = \phi \circ \psi(\sqrt[p^2]{z_i}) \otimes \frac{1}{\eta_i}$*. Considering the connected morphism* $\delta : H^0(C, F_*\mathcal{B}^1 \otimes \mathcal{L}^{-1}) \to H^1(C, F_*\mathcal{O}_C/\mathcal{O}_C \otimes \mathcal{L}^{-1})$*, we use the Čech cohomology to obtain* $\delta(s_0) = \psi(x^{p^2-p}y^p + y^{p^2})^{-1} \otimes \frac{1}{\eta_i} \neq 0$ *since* $x^{p^2-p}y^p + y^{p^2}$ *does not have a* p^2*-th root in* $\mathcal{O}_{U_1 \cap U_2}$*. Therefore, this section can't be lift to a section in* $H^0(C, (F^n_*\mathcal{O}_C/\mathcal{O}_C) \otimes \mathcal{L}^{-1})$ *and the triple* (C, z_1, D) *is not a 2-Tango data.*
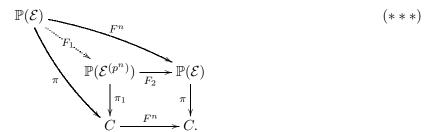
2.2. **Ruled surface over n-Tango curve.** Let C be an n-Tango curve with an associated divisor D on C and $\mathcal{L} = \mathcal{O}(D)$. And let $s_0 \in H^0(C, F^{n-1}_*\mathcal{B}^1 \otimes \mathcal{L}^{-1})$ be the associated section, which can be lift to a section $s \in H^0(C, (F^n_*\mathcal{O}_C/\mathcal{O}_C) \otimes \mathcal{L}^{-1})$. By the argument before Remark 2.2, we get an element $0 \neq \alpha(s) \in H^1(C, \mathcal{L}^{-1})$ with $F^{*n}(\alpha(s)) = 0$ and $F^{*n-1}(\alpha(s)) \neq 0$. So $\alpha(s) \in H^1(C, \mathcal{L}^{-1})$ gives a non-trivial extension

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0. \tag{$*$}$$

and n is the smallest integer such that

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F^{n*}\mathcal{E} \overset{\tau}{\underset{\longleftarrow}{\longrightarrow}} F^{n*}\mathcal{L} \longrightarrow 0 \tag{$**$}$$

splits under the pull back by F^{n*}. Write $\mathcal{E}^{(p^n)} = F^{n*}\mathcal{E}$, then we have the following diagram

$$(***)$$

Let $S \subseteq \mathbb{P}(\mathcal{E})$ be the section corresponding to the exact sequence $(*)$. And the splitting of $(**)$ yields a section $T' \subseteq \mathbb{P}(\mathcal{E}^{(p^n)})$ determined by τ with

$$\mathcal{O}(T') = \mathcal{O}(1)_{\mathbb{P}(\mathcal{E}^{(p^n)})} \otimes \pi_1^* \mathcal{L}^{-p^n},$$

which is disjoint from section $S' = F_2^{-1}(S)$. Let $T = F_1^{-1}(T')$ be the (scheme-theoretic) inverse image of T' by the relative n-th Frobenius morphism F_1, which is a smooth curve by a local calculation. Then T is disjoint with S and

$$\mathcal{O}(T) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(p^n) \otimes \pi^* \mathcal{L}^{-p^n}.$$

2.3. n-**Raynaud surface.** With the same notations as above subsecion, we have $\mathcal{O}(S + T) = \mathcal{O}(p^n + 1) \otimes \pi^* \mathcal{L}^{-p^n}$. Suppose that there is a positive integer l satisfying $l \mid p^n + 1$ and $l \mid \deg \mathcal{L}$, and let $\mathcal{L} = \mathcal{O}(lN)$ for some divisor N on C. Denote by

$$d = \frac{p^n + 1}{l}.$$

Then we can write

$$\mathcal{M}^l = \mathcal{O}(p^n + 1) \otimes \pi^* \mathcal{L}^{-p^n}$$

for the line bundle

$$\mathcal{M} = \mathcal{O}(d) \otimes \pi^* \mathcal{O}(-p^n N)$$

on $\mathbb{P}(\mathcal{E})$, and the global section

$$S + T \in \Gamma(\mathbb{P}(\mathcal{E}), \mathcal{M}^l)$$

defines an l-cyclic cover over $\mathbb{P}(\mathcal{E})$ branched along the divisor $S + T$:

$$\psi : X = \operatorname{Spec} \bigoplus_{i=0}^{l-1} \mathcal{M}^{-i} \longrightarrow \mathbb{P}(\mathcal{E}).$$

Then X is called n-**Raynaud surface**, and when $n = 1$ it is the usual Raynaud surface.

Let $\tilde{S}$ and $\tilde{T}$ be the reduced pre-images of the ramification curves S and T respectively, then

(1) $\psi^*(T) = l\tilde{T}, \ \psi^*(S) = l\tilde{S}, \ \mathcal{O}(\tilde{S} + \tilde{T}) = \psi^*(\mathcal{M})$ and $\mathcal{O}(\tilde{T}) = \mathcal{O}(p^n \tilde{S}) \otimes \psi^* \mathcal{O}(-p^n N)$.

Next, we list some properties of a Raynaud surface X. Write the composition

$$\phi : X \xrightarrow{\psi} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C,$$

then we have

Proposition 2.5. *Let X be an n-Raynaud surface over an n-Tango curve C, with the same notion in this section then we have*

- *$(\tilde{S}^2) = \frac{2g-2}{p^n l}$;*
- *$\omega_X = \mathcal{O}_X((p^n l - l - p^n - 1)\tilde{S}) \otimes \phi^* \mathcal{O}_C((p^n + l)N)$;*
- *when $(p, n, l) = (2, 1, 3)$ or $(3, 1, 2)$, X is a quasi-elliptic surface and ω_X is ample in the other cases;*

- $\phi : X \to C$ is a singular fibration with every fibre F having a cuspidal singularity at $F \cap \tilde{G}$ locally of the form $x^l = y^{p^n}$.

The following lemma is from [8] and we will give a proof for reader's convenience.

Lemma 2.6. [8, Prop3.3] *For any integer $m = ql + r \geq 0$ with $0 \leq r \leq l - 1$, we have*

(1) $\psi_* \mathcal{O}_X(-m\tilde{S}) = \left(\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}(-(q+1)S) \right) \oplus \left(\bigoplus_{i=r}^{l-1} \mathcal{M}^{-i}(-qS) \right)$ *and*

(2) $\psi_* \mathcal{O}_X(m\tilde{S}) = \left(\bigoplus_{i=0}^{l-r-1} \mathcal{M}^{-i}(qS) \right) \oplus \left(\bigoplus_{i=l-r}^{l-1} \mathcal{M}^{-i}((q+1)S) \right)$

Proof. First, by definition of l-cyclic cover we know that $\pi_* \mathcal{O}_X = \mathcal{M}^0 \oplus, \cdots, \oplus \mathcal{M}^{-l+1}$ is an $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-algebra with the multiplication described as:

$$\mathcal{M}^{-i_1} \otimes \mathcal{M}^{-i_2} \to \mathcal{M}^{-i_1-i_2} \text{ and } \mathcal{M}^{-l} = \mathcal{M}^0(-S - T) \hookrightarrow \mathcal{M}^0.$$

Let's push down the following diagram by ψ

By (1) and projection formula we get

where all the morphisms are $\psi_* \mathcal{O}_X$-mod homomorphisms.

In fact, the map $\hbar$ can be described as below:

So we obtain

$$\psi_* O_X|_{m(\tilde{S}+\tilde{T})} = \left(\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}|_{(q+1)(S+T)} \right) \oplus \left(\bigoplus_{i=r}^{l-1} \mathcal{M}^{-i}|_{q(S+T)} \right).$$

On the other hand, since $S \cap T = \emptyset$ we get the two direct summands of $\psi_* O|_{m(\tilde{S}+\tilde{T})}$:

$$\psi_* O_X|_{m\tilde{S}} = \left(\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}|_{(q+1)S} \right) \oplus \left(\bigoplus_{i=r}^{l-1} \mathcal{M}^{-i}|_{qS} \right)$$

and

$$\psi_* O_X|_{m\tilde{T}} = \left(\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}|_{(q+1)T} \right) \oplus \left(\bigoplus_{i=r}^{l-1} \mathcal{M}^{-i}|_{qT} \right).$$

Then by the second column in the second diagram, we get

$$\psi_* \mathcal{O}_X(-m\tilde{S}) = \left(\bigoplus_{i=0}^{r-1} \mathcal{M}^{-i}(-(q+1)S) \right) \oplus \left(\bigoplus_{i=r}^{l-1} \mathcal{M}^{-i}(-qS) \right).$$

For the second equality, note that $\psi_* \mathcal{O}_X(m\tilde{S}) = \psi_* \mathcal{O}_X(((q+1)l - (l-r))\tilde{S}) = \mathcal{O}_X((q+1)S) \otimes \psi_* \mathcal{O}_X(-(l-r)\tilde{S})$, then it follows from the first equality.

$\square$

3. Base point of adjoint line bundle

We will use all the notations in the last section and won't say more than needed. Let C be an n-Tango curve with an associated effective divisor $\mathcal{L} = \mathcal{O}(D) = \mathcal{O}(lN)$ and X is an n-Raynaud surface over C.

We will consider the linear system of the form $|m\tilde{S} + \phi^* Q|$ in this section, where $m \in \mathbb{N}^+$ and $\deg Q > 0$ is an ample divisor on C.

Suppose that $m = lq + r$ with $0 \leq r < l$, then by Lemma 2.6 it follows that

$$\psi_* \mathcal{O}_X(m\tilde{S} + \phi^* Q) = \psi_* \mathcal{O}_X(m\tilde{S}) \otimes \pi^*(Q)$$

$$\cong \left((\bigoplus_{i=0}^{l-r-1} \mathcal{M}^{-i}(qS)) \oplus (\bigoplus_{i=l-r}^{l-1} \mathcal{M}^{-i}((q+1)S)) \right) \otimes \pi^*(Q)$$

$$\triangleq \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_{l-1},$$

where $\mathcal{M}_i = \mathcal{M}^{-i}((q + [\frac{i+r}{l}])S) \otimes \pi^*(Q)$.

Note that it is a $\psi_* \mathcal{O}_X$-module where $\psi_* \mathcal{O}_X = \mathcal{M}^0 \oplus \cdots \oplus \mathcal{M}^{-l+1}$. And by the proof of Lemma 2.6 we see that the module structure can be described as:

$$\mathcal{M}_{i_2} \otimes \mathcal{M}^{-1} \to \mathcal{M}_{i_2+1}$$

with the inclusion

$$\mathcal{M}^{-l} = \mathcal{M}^0(-S - T) \subset \mathcal{M}^0.$$

Consider the natural decomposition

$$H^0(X, \mathcal{O}_X(m\tilde{S} + \phi^* Q)) \cong \bigoplus_{i=0}^{l-1} H^0(\mathbb{P}(\mathcal{E}), \mathcal{M}_i)$$

then we have the following observation.

Lemma 3.1. *With the above decomposition, the sections*

$$s \in \bigoplus_{i=1}^{l-1} H^0(\mathbb{P}(\mathcal{E}), \mathcal{M}_i) \subset H^0(X, \mathcal{O}_X(m\tilde{S} + \phi^* Q))$$

can not generate $\psi_* \mathcal{O}_S(m\tilde{S} + \phi^* Q)$ as $\psi_* \mathcal{O}_X$-module along the divisor T. Moreover, if the line bundle $\mathcal{M}_0$ has a base point $x \in T$ as $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-module, then all the sections

$$s \in \bigoplus_{i=0}^{l-1} H^0(\mathbb{P}(\mathcal{E}), \mathcal{M}_i) = H^0(X, \mathcal{O}_X(m\tilde{S} + \phi^* Q))$$

can not generate $\psi_* \mathcal{O}_X(m\tilde{S} + \phi^* Q)$ as $\psi_* \mathcal{O}_X$-module at the point $x \in T$, i.e. $\psi^{-1}(x)$ is a base point of the line bundle $\mathcal{O}_X(m\tilde{S} + \phi^* Q)$ on X.

Proof. By the above argument, we see that

$$\psi_*\mathcal{O}_X(m\tilde{S} + \phi^*Q) \cong \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_{l-1}$$

is a $\psi_*\mathcal{O}_X = \bigoplus_{i=0}^{l-1} \mathcal{M}^{-i}$-module and the action of the $\mathcal{O}_{\mathbb{P}(\mathcal{E})}$-algebra $\psi_*\mathcal{O}_X$ on the components of $\psi_*\mathcal{O}_S(m\tilde{S} + \phi^*Q)$ into the first term can be described as follows:

- $\mathcal{M}_i \otimes \mathcal{M}^{i-l} = \mathcal{O}(-S-T) \otimes \mathcal{M}_0 \subset \mathcal{M}_0$ as a sub-sheaf determine by tensor with the ideal sheaf $\mathcal{O}(-S-T)$, when $0 \leq i \leq l-r-1$;
- $\mathcal{M}_i \otimes \mathcal{M}^{i-l} = \mathcal{O}(-T) \otimes \mathcal{M}_0 \subset \mathcal{M}_0$ as a sub-sheaf determined by tensor with the ideal sheaf $\mathcal{O}(-T)$, when $l-r \leq i \leq l-1$.

So the sections in any component of $\psi_*\mathcal{O}_S(m\tilde{S} + \phi^*Q)$ but the first one can't generate $\psi_*\mathcal{O}_S(m\tilde{S} + \phi^*Q)$ as $\psi_*\mathcal{O}_X$-module along the divisor T. $\qquad\square$

Lemma 3.2. *Let C be a smooth projective curve over an algebraically closed field $\mathbf{k}$ with dualizing sheaf ω_C and $\mathcal{E}$ a vector bundle of rank 2 on C. Suppose that there is a surjective morphism $\sigma_0 : \mathcal{E} \twoheadrightarrow \mathcal{L}_0$ with $\mathcal{L}_0$ being a line bundle on C satisfying*

(1) $\dim H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}_0) \geq 2$ *and*

(2) $H^0(C, \omega_C \otimes \mathcal{L}_0^{-q}(-Q)) \neq 0$ *for some divisor Q of positive degree on C and some integer q.*

Then there exists a nonempty open subset $C_0 \subset C$ such that the base locus of $\mid \mathcal{O}_{\mathbb{P}(\mathcal{E})}(q) \otimes \pi^*\mathcal{O}(Q + P) \mid$ contains the fibre $F = \pi^{-1}(P)$ for any closed point $P \in C_0$, where $\pi : \mathbb{P}(\mathcal{E}) \to C$ is the projection from the ruled surface $\mathbb{P}(\mathcal{E})$ to C.

Proof. Note that $\mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}))$ parametrises all the morphisms $Mor_C(\mathcal{E}, \mathcal{L})$ up to scalar isomorphisms of $\mathcal{L}$. Since the surjectivity is an open condition, there is a non-empty open subset $U_0 \subset \mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}_0))$ such that the corresponding morphisms are surjective. So those sections σ of $\pi : \mathbb{P}(\mathcal{E}) \to C$ with $\sigma^*\mathcal{O}(1) \simeq \mathcal{L}_0$ are parametrised by U_0. Since $\dim U_0 > 0$, we take another section $(\sigma_0 \neq)\sigma_1 \in U_0$ and let $\widetilde{U_0}$ be the intersection of U_0 with the line $\mathbb{P}^1 \subset \mathbb{P}(H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}))$ generated by σ_0 and σ_1. Then $\widetilde{U_0}$ is an affine curve or $\mathbb{P}^1$. Since all the sections in $\widetilde{U_0}$ are of the form $k_0\sigma_0 + k_1\sigma_1$ with $k_0, k_1 \in \mathbf{k}$, the intersection of any two sections in $\widetilde{U_0}$ on the ruled surface is the set $\sigma_0 \cap \sigma_1$ by a local calculation on the ruled surface.

Now Let's consider the restriction $\sigma^*(\mathcal{O}(q) \otimes \pi^*(\mathcal{O}(Q))) \simeq \mathcal{L}_0^q(Q)$ onto those sections in $\widetilde{U_0}$. Since $H^0(C, \omega_C \otimes \mathcal{L}_0^{-q}(-Q)) \neq 0$, by Lemma 3.3 there exists a nonempty open subset $C_0 \subset C$ such that the base locus of $\mid \mathcal{L}_0^q(Q + P) \mid$ contains the point P for any $P \in C_0$. Hence there exists a nonempty open subset $C_0 \subset C$ such that the base locus of $\mid \mathcal{O}(q) \otimes \pi^*\mathcal{O}(Q+P) \mid$ contains the point $\pi^{-1}(P) \cap \sigma(C)$ for any $P \in C_0$ and any $\sigma \in \widetilde{U_0}$. Note that any two sections in $\widetilde{U_0}$ do not intersect out of the set $\sigma_0 \cap \sigma_1$ on the ruled surface. We may shrink C_0 such that $\pi(\sigma_0 \cap \sigma_1) \cap C_0 = \emptyset$, so the base locus of $\mid \mathcal{O}(q) \otimes \pi^*(Q + P) \mid$ contains infinite many points $\{\pi^{-1}(P) \cap \sigma(C) \mid \sigma \in \widetilde{U_0}\}$ on the fibre $\pi^{-1}(P)$ and hence the entire fibre $\pi^{-1}(P)$. $\qquad\square$

Lemma 3.3. *Let C be a smooth projective curve over an algebraically closed field with a line bundle $\mathcal{L}$ on it.*

(1) *If $H^0(C, \mathcal{L}) \neq 0$, then there exists a nonempty open subset $U \subset C$ such that*

$$h^0(C, \mathcal{L} \otimes \mathcal{O}(-x)) = h^0(C, \mathcal{L}) - 1$$

for any closed point $x \in U$.

(2) *If $H^0(C, \omega_C \otimes \mathcal{L}^{-1}) \neq 0$, then there exists a nonempty open subset $U \subset C$ such that*

$$h^0(C, \mathcal{L} \otimes \mathcal{O}(x)) = h^0(C, \mathcal{L})$$

for any closed point $x \in U$.

Proof. Note that $H^0(C, \mathcal{L}) \neq 0$ and $H^0(C, \omega_C \otimes \mathcal{L}^{-1}) \neq 0$ imply that the base locus $\mathrm{Bs}(|\mathcal{L}|) \subsetneqq C$ and $\mathrm{Bs}(|\omega_C \otimes \mathcal{L}^{-1}|) \subsetneqq C$ respectively. Let $U = C \setminus \mathrm{Bs}(|\mathcal{L}|)$ and $U = C \setminus \mathrm{Bs}(|\omega_C \otimes \mathcal{L}^{-1}|)$ in (1) and (2) respectively, then the two statements follow from Riemann-Roch formula immediately. $\qquad\square$

Theorem 3.4. *Let C be an n-Tango curve, $\mathcal{E}$ an associated vector bundle of rank 2 on it and $\psi : X \xrightarrow{l:1} \mathbb{P}(\mathcal{E})$ an n-Raynaud surface as described in last section. Set $m = lq + r \in \mathbb{N}^+$ with $0 \leq r < l$. Suppose that there is a surjective morphism $\sigma_0 : \mathcal{E} \twoheadrightarrow \mathcal{L}_0$ with $\mathcal{L}_0$ being a line bundle on C satisfying*

(1) $\dim H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}_0) \geq 2$ *and*

(2) $H^0(C, \omega_C \otimes \mathcal{L}_0^{-q}(-Q)) \neq 0$ *for some divisor Q of positive degree on C.*

Then $q < p^n$ and there exists a nonempty open subset $C_0 \subset C$ such that $\phi^{-1}(P) \cap \widetilde{T}$ is a base point of the ample line bundle $\mathcal{O}_X(m\tilde{S} + \phi^*(Q + P))$ on X for any point $P \in C_0$. In particular, if the condition (2) is replaced by

(2*) $H^0(C, \omega_C \otimes \mathcal{L}_0^{-(p^n - 1 - d)}(-(p^n + l)N - Q)) \neq 0$ *for some divisor Q of positive degree,*

then there exists a nonempty open subset $C_0 \subset C$ such that $\phi^{-1}(P) \cap \widetilde{T}$ is a base point of the adjoint line bundle $\mathcal{O}_X(K_X + r\tilde{S} + \phi^*(Q + P))$ on X for any point $P \in C_0$.

Proof. First we claim that the surjectivity implies that $\deg \mathcal{L}_0 \geq \deg \mathcal{L}$. Indeed, by pulling back the non-split sequence

$$0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L} \to 0$$

obtained in subsection 2.2 by n-th iterated Frobenius map we get the splitting sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow F^{n*}\mathcal{E} \overset{\tau}{\underset{\longleftarrow}{\longrightarrow}} \mathcal{L}^{p^n} \longrightarrow 0 \ .$$

Composing τ with the quotient $F^{n*}(\sigma_0) : F^{n*}\mathcal{E} \twoheadrightarrow \mathcal{L}_0^{p^n}$ we get a map $\mathcal{L}^{p^n} \to \mathcal{L}_0^{p^n}$. If $\deg \mathcal{L}_0 < \deg \mathcal{L}$ then it must be a zero map and $\mathcal{L}^{p^n} = \mathrm{Ker}(F^{n*}(\sigma_0))$ by the saturation of $\mathcal{L}^{p^n}$, hence $\mathcal{L}_0 \simeq \mathcal{O}_C$. So this sequence already splits which leads to a contradiction. Then note that $\omega_C \simeq \mathcal{L}^{p^n}$, and by condition (2) we get $q < p^n$.

For any closed point $P \in C$, let's push down $\mathcal{O}_X(m\tilde{S} + \phi^*(Q + P))$ onto the ruled surface $\mathbb{P}(\mathcal{E})$

$$\psi_*\mathcal{O}_X(m\tilde{S} + \phi^*(Q + P)) \cong \mathcal{M}_0 \oplus \mathcal{M}_1 \oplus \cdots \oplus \mathcal{M}_{l-1}.$$

Note that the first term is $\mathcal{M}_0 = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(q) \otimes \pi^*\mathcal{O}(Q+P)$ and by Lemma 3.2 there exists a nonempty open subset $C_0 \subset C$ such that the base locus of $|\ \mathcal{O}_{\mathbb{P}(\mathcal{E})}(q) \otimes \pi^*\mathcal{O}(Q + P)\ |$ contains the fibre $F = \pi^{-1}(P)$ for any closed point $P \in C_0$, where $\pi : \mathbb{P}(\mathcal{E}) \to C$ is the projection from the ruled surface $\mathbb{P}(\mathcal{E})$ to C. Then by Lemma 3.1, $\psi^{-1}(F \cap T) = \phi^{-1}(P) \cap \widetilde{T}$ is a base point of the ample line bundle $\mathcal{O}_X(m\tilde{S} + \phi^*(Q + P))$ on X.

For the last statement, we just note that $K_X = (p^n l - l - p^n - 1)\tilde{S} + \phi^*(p^n + l)N$. $\qquad\square$

Remark 3.5. One may wonder how to check those conditions in Theorem 3.4 for an associated vector bundle $\mathcal{E}$ on an n-Tango curve. Here we introduce a method how to find a line bundle $\mathcal{L}_0$ satisfying those conditions in Theorem 3.4.

Let C be an n-Tango curve, $\mathcal{E}$ an associated vector bundle on it and $\psi : X \xrightarrow{l:1} \mathbb{P}(\mathcal{E})$ an n-Raynaud surface as those notations in last section. First, by $\mathcal{O}_C \subset \mathcal{E} \subset F_*^n \mathcal{O}_C$ we see that $H^0(C, \mathcal{E}) = H^0(C, \mathcal{O}_C) = \mathbf{k}$.

As described in subsection 2.1 , $\mathcal{E}|_{U_i} = \mathcal{O}_{U_i} \cdot 1 \oplus \mathcal{O}_{U_i} \cdot \sqrt[p^n]{z_i}$ and the transition matrix of $\mathcal{E}$ is $\begin{pmatrix} 1 & \beta \\ 0 & \alpha \end{pmatrix} \in GL(2, \mathcal{O}_{U_1 \cap U_2})$. Let $(\beta) = \sum\limits_{D_j \notin U_1} a_j D_j + \mathtt{others}$, $(\alpha) = \sum\limits_{D_j \notin U_1} b_j D_j + \mathtt{others}$ and $D_0 = -\sum\limits_{D_j \notin U_1} \min\{a_j, b_j, 0\} D_j > 0$. Suppose the divisor $D_0 \in \Gamma(X, \mathcal{O}(D_0))$ is locally defined by the functions $1 \in \Gamma(U_2, \mathcal{O}_{U_2})$ and $\gamma \in \Gamma(U_2, \mathcal{O}_{U_2})$. Then $\mathcal{O}(D_0)|_{U_1} = \mathcal{O}_{U_1} \cdot 1$ and $\mathcal{O}(D_0)|_{U_2} = \mathcal{O}_{U_2} \cdot \frac{1}{\gamma}$.

And hence $\mathcal{L}(D_0)|_{U_1} = \mathcal{O}_{U_1} \cdot \eta_1$ and $\mathcal{L}(D_0)|_{U_2} = \mathcal{O}_{U_2} \cdot \frac{1}{\gamma}\eta_2$, where η_i are local bases of $\mathcal{L}$ as in subsection 2.1. Therefore there is a section $\overline{s} \in \Gamma(X, \mathcal{L}(D_0))$, which is locally written as $\overline{s}|_{U_1} = 1 \cdot \eta_1$ and $\overline{s}|_{U_2} = \gamma\alpha \cdot \frac{1}{\gamma}\eta_2$, since $\gamma\alpha \in \Gamma(U_2, \mathcal{O}_{U_2})$ by the construction of D_0. Moreover we see this section can be lift to a section $s \in \Gamma(X, \mathcal{E}(D_0))$, which is locally written as $s|_{U_1} = 1 \cdot \sqrt[p^n]{z_1}$ and $s|_{U_2} = \gamma\alpha \cdot \frac{1}{\gamma} \sqrt[p^n]{z_2} + \gamma\beta \cdot \frac{1}{\gamma}$, since $\gamma\alpha, \gamma\beta \in \Gamma(U_2, \mathcal{O}_{U_2})$ by the construction of D_0. So we have $\dim H^0(C, \mathcal{E}^\vee \otimes \mathcal{L}(D_0)) = \dim H^0(C, \mathcal{E}(D_0)) \geq \dim H^0(C, \mathcal{O}_C(D_0)) + 1 \geq 2$. Moreover, by the construction of D_0, we see that $(\gamma\alpha, \gamma\beta) = 1$ in the local ring $\mathcal{O}_{C,x}$ for any closed point $x \notin U_1$. So the inclusion $\mathcal{O}_C(-D_0) \hookrightarrow \mathcal{E}$ determined by s is saturated and there is a quotient $\sigma_0 : \mathcal{E} \twoheadrightarrow \mathcal{L}(D_0)$. Next, in order to obtain such an $\mathcal{L}_0 := \mathcal{L}(D_0)$ that also satisfies (2) or (2*) in Theorem 3.4, we should find such a divisor D_0 of degree as small as possible by chosen of U_1.

At last of this section, let's consider Example 2.3 again to show the strong non-freeness of adjoint bundles on n-Raynaud surfaces.

Corollary 3.6. [3, Theorem 1.2] *For any $r > 0$, there exist a smooth projective surface X with an ample divisor A on it such that the adjoint linear system*

$$|K_X + rA|$$

has base points on X.

Proof. For simplicity, we set $Q(X,Y) = Y^e$ in Example 2.3. Then $\alpha = y^{e(qe-3)}$ and $\beta = Q^{-1}(x,y) = y^{-e}$. Note that $C \setminus U_1 = \{\infty\}$. By the argument in Remark 3.5, set $D_0 = e\infty$ and $\mathcal{L}_0 = \mathcal{L}(D_0)$, then there is a surjective morphism $\sigma_0 : \mathcal{E} \twoheadrightarrow \mathcal{L}_0$ satisfying the condition (1) in Theorem 3.4. For the convenience of calculation, let $l = p^n + 1$ and $e = kl$ for some integer k, so $\omega_C \otimes \mathcal{L}_0^{-(p^n-1-d)}(-(p^n+l)N) = \mathcal{O}(k(q(q+1)k - 3 - (q-2)(q+1))\infty)$. For any $r > 0$, we can take $n \gg 0$ such that $l = p^n + 1 > r$ and take $k \gg 0$ such that $k(q(q+1)k - 3 - (q-2)(q+1)) > r$. Let $Q = (r-1)\infty$ then $H^0(C, \omega_C \otimes \mathcal{L}_0^{-(p^n-1-d)}(-(p^n+l)N - Q)) = H^0(C, (k(q(q+1)k - 3 - (q-2)(q+1)) - r + 1)\infty) \neq 0$. By Theorem 3.4, there exists a nonempty open subset $C_0 \subset C$ such that $\phi^{-1}(P) \cap \tilde{T}$ is a base point of the adjoint line bundle $\mathcal{O}_X(K_X + r\tilde{S} + \phi^*(Q+P))$ for any point $P \in C_0$. If we write $Q + P = rQ_0$ for some divisor Q_0 of degree 1 on C and let $A = \tilde{S} + \phi^*(Q_0)$ which is ample, then we get our result. $\square$

4. STRONG KODAIRA NON-VANISHING

Although the Kodaira vanishing does not hold in positive characteristic, there is a weak version of vanishing theorem.

Theorem 4.1. [6, Corollary 17] *Let $\mathcal{H}$ be a nef and big line bundle on a smooth projective surface X over $\mathbf{k}$, then $H^0(X, \mathcal{H}^{-p^m}) = 0$ for all $m \gg 0$.*

However, The strong non-vanishing claims that there is no universal bound for this m.

Theorem 4.2. *For any $m > 0$, there exist a smooth projective surface X and an ample line bundle $\mathcal{H}$ on X such that $H^0(X, \mathcal{H}^{-p^m}) \neq 0$*

Proof. In this proof, we will use all those notations in section 2. Let X be an n-Raynaud surface over an n-Tango curve C

$$\phi : X \xrightarrow{\psi} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} C$$

and we consider the ample line bundle $\mathcal{H} = \mathcal{O}(\tilde{S} + \phi^*Q)$ where Q is a divisor on C of positive degree.

By the Leray spectral sequence $E_2^{p,q} = H^p(C, R^q\phi_*\mathcal{H}^{-p^m}) \Rightarrow H^{p+q}(X, \mathcal{H}^{-p^m})$, we have the following exact sequence

$$0 \to H^1(C, \phi_*\mathcal{H}^{-p^m}) \to H^1(X, \mathcal{H}^{-p^m}) \to H^0(C, R^1\phi_*\mathcal{H}^{-p^m}) \to H^2(C, \phi_*\mathcal{H}^{-p^m}) = 0.$$

Write $p^m = lq + r$ for some $0 \leq r \leq l-1$. By Lemma 2.6, we have

$$\phi_* \mathcal{H}^{-p^m} = \pi_* \psi_* \mathcal{O}_X(-p^m(\tilde{S} + \phi^* Q))$$

$$\cong \left(\bigoplus_{i=0}^{r-1} \pi_* \mathcal{M}^{-i}((-q-1)S) \otimes \mathcal{O}(-p^m Q) \right) \oplus \left(\bigoplus_{i=r}^{l-1} \pi_* \mathcal{M}^{-i}(-qS) \otimes \mathcal{O}(-p^m Q) \right)$$

$$\cong \left(\bigoplus_{i=0}^{r-1} Sym^{-id-q-1}(\mathcal{E}) \otimes \mathcal{O}(ip^n N - p^m Q) \right) \oplus \left(\bigoplus_{i=r}^{l-1} Sym^{-id-q}(\mathcal{E}) \otimes \mathcal{O}(ip^n N - p^m Q) \right) = 0$$

since the exponents of symmetric powers are negative. So the leftmost term $H^1(C, \phi_* \mathcal{H}^{-p^m}) = 0$ and it is reduced to compute $H^0(C, R^1\phi_* \mathcal{H}^{-p^m})$.

Note that

$$\omega_{X/C} = \mathcal{O}_X((p^n l - l - p^n - 1)\tilde{S}) \otimes \phi^* \mathcal{O}_C((p^n + l - p^n l)N),$$

then by the relative Serre duality, we have

$$(R^1\phi_* \mathcal{H}^{-p^m})^\vee \simeq \phi_*(\mathcal{H}^{p^m} \otimes \omega_{X/C})$$

$$\simeq \pi_* \psi_* \mathcal{O}_X((p^m + p^n l - l - p^n - 1)\tilde{S} + \phi^*(p^m Q + (p^n + l - p^n l)N))$$

$$\simeq (\bigoplus_{i=0}^{l-r-1} \pi_* \mathcal{M}^{-i}((p^n - d + q - 1)S) \otimes \mathcal{O}(p^m Q + (p^n + l - p^n l)N)) \oplus$$

$$(\bigoplus_{i=l-r}^{l-1} \pi_* \mathcal{M}^{-i}((p^n - d + q)S) \otimes \mathcal{O}(p^m Q + (p^n + l - p^n l)N))$$

$$\simeq (\bigoplus_{i=0}^{l-r-1} Sym^{-id + p^n - d + q - 1}(\mathcal{E}) \otimes \mathcal{O}(p^m Q + (p^n + l + ip^n - p^n l)N)) \oplus$$

$$(\bigoplus_{i=l-r}^{l-1} Sym^{-id + p^n - d + q}(\mathcal{E}) \otimes \mathcal{O}(p^m Q + (p^n + l + ip^n - p^n l)N)).$$

For the first direct summand, there is a quotient

$$Sym^{p^n - d + q - 1}(\mathcal{E}) \otimes \mathcal{O}(p^m Q + (p^n + l - p^n l)N)) \twoheadrightarrow \mathcal{O}(p^m Q + (lq-1)N))$$

by the construction of $\mathcal{E}$. For any $m > 0$, we take $n = mm_0$ for some odd integer $m_0 > 0$ and we set $l = p^m + 1$ then $l \mid p^n + 1$ and $q = 0$. Moreover we could take an n-Tango curve C with $\deg N > p^m$, then there is a divisor Q of degree 1 such that $N - p^m Q > 0$. For instance, the n-Tango curves in example 2.3 satisfy those conditions by taking $e = l$ and $Q = \infty$. So there is a nonzero section in $H^0(X, R^1\phi_* \mathcal{H}^{-p^m})$, hence $H^0(X, \mathcal{L}^{-p^m}) \neq 0$ $\qquad \square$

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