# Extended commonality of paths and cycles via Schur convexity

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#### Abstract

A graph H is *common* if the number of monochromatic copies of H in a 2-edge-colouring of the complete graph  $K_n$  is asymptotically minimised by the random colouring, or equivalently,  $t_H(W) + t_H(1 - W) \ge 2^{1 - e(H)}$  holds for every graphon  $W : [0, 1]^2 \to [0, 1]$ , where  $t_H(.)$  denotes the homomorphism density of the graph  $H$ . Paths and cycles being common is one of the earliest cornerstones in extremal graph theory, due to Mulholland and Smith (1959), Goodman (1959), and Sidorenko (1989).

We prove a graph homomorphism inequality that extends the commonality of paths and cycles. Namely,  $t_H(W) + t_H(1 - W) \ge t_{K_2}(W)^{e(H)} + t_{K_2}(1 - W)^{e(H)}$  whenever H is a path or a cycle and  $W : [0, 1]^2 \to \mathbb{R}$  is a bounded symmetric measurable function.

This answers a question of Sidorenko from 1989, who proved a slightly weaker result for even-length paths to prove the commonality of odd cycles. Furthermore, it also settles a recent conjecture of Behague, Morrison, and Noel in a strong form, who asked if the inequality holds for graphons W and odd cycles  $H$ . Our proof uses Schur convexity of complete homogeneous symmetric functions, which may be of independent interest.

### 1 Introduction

Given a bounded measurable symmetric function  $W : [0, 1]^2 \to \mathbb{R}$  and a graph H, let

$$
t_H(W) := \int_{[0,1]^{V(H)}} \prod_{ij \in E(H)} W(x_i,x_j) \prod_{i \in V(H)} dx_i,
$$

where the integration is taken with respect to the Lebesgue measure. This functional  $t_H(.)$  is often called the *(weighted) homomorphism density of* H, which generalises normalised homomorphism counts from  $H$  to another graph  $G$ .

Various results in extremal graph theory can be interpreted by using homomorphism densities, especially by using *graphons* W, i.e., measurable symmetric functions  $W : [0, 1]^2 \to [0, 1]$ , although extensions to general real-valued functions [\[7,](#page-8-0) [16\]](#page-8-1) or even to complex-valued functions [\[14,](#page-8-2) [22\]](#page-9-0) are certainly possible. We refer the reader to the modern theory of graph limits [\[24\]](#page-9-1) for more examples.

One of the central concepts that can be rephrased conveniently by using homomorphism densities is the *commonality* of graphs. A graph H is *common* if the number of monochromatic H-copies in a 2-edge-colouring of the complete graph  $K_n$  is asymptotically minimised by the random colouring. The modern language rewrites the commonality of  $H$  as the simple inequality

$$
t_H(W) + t_H(1 - W) \ge 2^{1 - e(H)}
$$

for every graphon W, where  $e(H)$  denotes the number of edges in H.

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Since Goodman's formula  $[12]$  and the famous conjectures of Erdős  $[9]$  and of Burr–Rosta  $[5]$ , later disproved by Thomason [\[33\]](#page-9-2) and by Sidorenko [\[27\]](#page-9-3), respectively, common graphs have been extensively studied [\[11,](#page-8-6) [13,](#page-8-7) [15,](#page-8-8) [18,](#page-9-4) [20,](#page-9-5) [30\]](#page-9-6). Amongst many, perhaps the most fundamental examples of common graphs are paths and cycles. Our main result is to prove a new homomorphism density inequality for paths and cycles, which extends their commonality. For brevity, a bounded symmetric measurable function  $W : [0, 1]^2 \to \mathbb{R}$  is said to be a *kernel*.

<span id="page-1-0"></span>Theorem 1.1. *Let* H *be a path or a cycle and let* W *be a kernel. Then*

$$
t_H(W) + t_H(1 - W) \ge t_{K_2}(W)^{e(H)} + t_{K_2}(1 - W)^{e(H)}.
$$
\n(1)

<span id="page-1-1"></span>As an immediate consequence, Theorem [1.1](#page-1-0) extends the commonality of paths and cycles to kernels.

Corollary 1.2. *Let* H *be a path or a cycle and let* W *be a kernel. Then*

<span id="page-1-3"></span><span id="page-1-2"></span>
$$
t_H(W) + t_H(1 - W) \ge 2^{1 - e(H)}.
$$
\n(2)

As  $t_{K_2}(1-W) = 1-t_{K_2}(W)$ , the substitution  $t_{K_2}(W) = x+1/2$  gives  $t_{K_2}(W)^m + t_{K_2}(1-W)^m =$  $(1/2 + x)^m + (1/2 - x)^m$ . For  $m > 1$ , this polynomial attains its global minimum at  $x = 0$  and hence, the corollary follows. In [\[27,](#page-9-3) [30\]](#page-9-6), Sidorenko proved Corollary [1.2](#page-1-1) for cycles, even-length paths, and paths of length  $m = r2^t + 1$ ,  $r \leq 9$ , but left the general odd-length paths case as a question. Corollary [1.2](#page-1-1) thus completes the result of Sidorenko and answer his question in the affirmative. Furthermore, the proof technique allows us to obtain stability results for both Theorem [1.1](#page-1-0) and Corollary [1.2;](#page-1-1) see Section [4](#page-6-0) for more details.

Theorem [1.1](#page-1-0) can also be interpreted as a 'convexity-type' homomorphism inequality, as the proof uses convexity of certain functions and deduction of commonality from it also uses convexity. The inequality [\(2\)](#page-1-2) for kernels was even called 'convexity' by Sidorenko [\[27,](#page-9-3) [30\]](#page-9-6). More generally, both local and global convexity of the functional  $t_H(.)$  has been extremely useful in proving various graph homomorphism inequalities including instances for Sidorenko's conjecture [\[8,](#page-8-9) [29\]](#page-9-7), commonality of graphs [\[13\]](#page-8-7), graph norms [\[21\]](#page-9-8), and density increment argument for the celebrated regularity lemma [\[31\]](#page-9-9). Hence, Theorem [1.1](#page-1-0) adds a new example to the encyclopedia of fundamental homomorphism inequalities.

In particular, when H is the m-edge path  $P_m$ , [\(1\)](#page-1-3) can be seen as a partial extension of the socalled Blakley–Roy inequality [\[3\]](#page-8-10), also obtained by Mulholland and Smith [\[26\]](#page-9-10) and by London [\[23\]](#page-9-11), which proves  $t_{P_m}(W) \ge t_{K_2}(W)^m$  for every graphon W. In fact, it is impossible to fully extend the Blakley–Roy inequality to kernels W, as  $t_{P_m}(-W) = -t_{P_m}(W)$  for odd m. For grahons W, even stronger generalisations are known; see, for example, [\[4\]](#page-8-11).

For cycles H, Theorem [1.1](#page-1-0) settles a conjecture of Behague, Morrison, and Noel [\[2,](#page-8-12) Conjec-ture 9.7, which states that the inequality [\(1\)](#page-1-3) holds for all odd cycles  $H$  and graphons  $W$ . They proposed the conjecture as a natural extension of the commonality of cycles and proved it for the 5 cycle H. We remark that some cases of the Behague–Morrison–Noel conjecture or the inequality [\(1\)](#page-1-3) for kernels and some cycles have been well-known for decades, although the conjecture appeared only very recently. For example, the smallest case when  $H$  is a triangle is essentially Goodman's formula [\[12\]](#page-8-3) and the case when H is an even cycle follows from the fact that even cycles are *norming*, observed by Chung, Graham, and Wilson [\[6\]](#page-8-13) and later rephrased by Hatami [\[16\]](#page-8-1).

Our proof uses Schur convexity of complete homogeneous symmetric functions. The study of complete homogeneous symmetric functions is a central area in algebraic combinatorics, although their Schur convexity received attention only recently [\[19,](#page-9-12) [32\]](#page-9-13). On the other hand, in extremal graph theory, the theory of symmetric functions rarely appears to be useful, to the best of our knowledge. Our method therefore bridges between the seemingly distant areas in a novel way, which may be of independent interest.

### 2 Preliminaries

When considering kernels U and W, the notation  $U = W$  always means the equality holds almost everywhere. We suppress the expression 'almost everywhere' in what follows for brevity.

Denote by  $\mathcal{E}^+(H)$  the set of all subgraphs F of H on  $V(H)$  with positive even number of edges. For a kernel W, let  $U := 2W - 1$ . Then U is again a kernel. By the standard multilinear expansion of  $t_H(1 + U)$  and  $t_H(1 - U)$ ,

<span id="page-2-2"></span>
$$
t_H(W) + t_H(1 - W) = 2^{-e(H)} \left( t_H(1 + U) + t_H(1 - U) \right)
$$
  
= 
$$
2^{1 - e(H)} \left( 1 + \sum_{F \in \mathcal{E}^+(H)} t_F(U) \right).
$$
 (3)

Analogously, one can also expand  $t_{K_2}$ . to obtain

$$
t_{K_2}(W)^{e(H)} + t_{K_2}(1 - W)^{e(H)} = 2^{-e(H)} \left( t_{K_2}(1 + U)^{e(H)} + t_{K_2}(1 - U)^{e(H)} \right)
$$
  

$$
= 2^{1 - e(H)} \sum_{k=0}^{\lfloor e(H)/2 \rfloor} {e(H) \choose 2k} t_{K_2}(U)^k
$$
  

$$
= 2^{1 - e(H)} \left( 1 + \sum_{F \in \mathcal{E}^+(H)} t_{K_2}(U)^{e(F)} \right).
$$
 (4)

Thus, we obtain the following statement equivalent to Theorem [1.1.](#page-1-0)

Proposition 2.1. *Let* H *be a path or a cycle and let* U *be a kernel. Then*

<span id="page-2-3"></span><span id="page-2-1"></span><span id="page-2-0"></span>
$$
\sum_{F \in \mathcal{E}^+(H)} \left( t_F(U) - t_{K_2}(U)^{e(F)} \right) \ge 0. \tag{5}
$$

For an integer  $d > 0$ , let  $\mathcal{E}_{2a}^+$  $C_{2d}^+(H)$  be the set of even subgraphs with exactly 2d edges. Then

$$
\sum_{F \in \mathcal{E}^+(H)} (t_F(U) - t_{K_2}(U)^{e(F)}) = \sum_{d=1}^{\lfloor e(H)/2 \rfloor} \sum_{F \in \mathcal{E}_{2d}^+(H)} (t_F(U) - t_{K_2}(U)^{2d}). \tag{6}
$$

When H is a path, we shall prove [\(5\)](#page-2-0) directly by showing that  $\sum_{F \in \mathcal{E}_{2d}^+(H)} (t_F(U) - t_{K_2}(U)^{2d}) \geq 0$ for each  $d = 1, 2, \ldots, |e(H)/2|$ .

Now consider the case  $H = C_m$ , a cycle of length m. Suppose first that  $e(H) = m$  is odd. Then each  $F \in \mathcal{E}^+(C_m)$  must be a proper subgraph; consider each  $F \in \mathcal{E}^+(C_m)$  as a subgraph of  $C_m \setminus e$ for every choice of  $e \in E(H) \setminus E(F)$ . By doing so, each F counts exactly  $e(H) - e(F)$  times. Thus,

$$
\sum_{F \in \mathcal{E}^{+}(H)} (t_F(U) - t_{K_2}(U)^{e(F)}) = \sum_{e \in E(H)} \sum_{F \in \mathcal{E}^{+}(H \setminus e)} \frac{1}{e(H) - e(F)} (t_F(U) - t_{K_2}(U)^{e(F)})
$$
\n
$$
= \sum_{e \in E(H)} \sum_{d=1}^{\lfloor (e(H) - 1)/2 \rfloor} \frac{1}{e(H) - 2d} \sum_{F \in \mathcal{E}_{2d}^{+}(H \setminus e)} (t_F(U) - t_{K_2}(U)^{2d}). \tag{7}
$$

If  $e(H) = m$  is even, then one extra term  $(t_H(U) - t_{K_2}(U)^{e(H)})$  adds to [\(7\)](#page-2-1).

<span id="page-3-1"></span>If H is a cycle of length  $m + 1$ , then  $H \ \epsilon$  is always a path of length m. Therefore, the following theorem, which will be shown in the next section, implies Theorem [1.1](#page-1-0) for odd cycles  $H$ .

**Theorem 2.2.** Let U be a kernel. Then for all integers m and d with  $1 \le d \le m/2$ ,

$$
\sum_{F \in \mathcal{E}_{2d}^+(P_m)} t_F(U) \ge \binom{m}{2d} t_{K_2}(U)^{2d}.
$$

For even cycles  $H$ , we need an extra inequality

<span id="page-3-0"></span>
$$
t_H(U) \ge t_{K_2}(U)^{e(H)}\tag{8}
$$

for each kernel  $U$  to deduce Theorem [1.1.](#page-1-0) This is reminiscent of Sidorenko's conjecture, which states that  $(8)$  holds for every bipartite graph H and every graphon U. Even cycles are well-known to satisfy Sidorenko's conjecture [\[28\]](#page-9-14), but  $(8)$  for kernels U is slightly stronger than this fact. Even so, it is not hard to verify it and a short proof will be given at the end of this section. In fact, the inequality  $(8)$  for kernels U is well-known since Chung, Graham, and Wilson's quasirandomness characterisation [\[6\]](#page-8-13); also see [\[16\]](#page-8-1) for its modern interpretation in terms of graph limits.

We shall use some spectral properties of kernels. Following  $[24, Section 7.5]$  $[24, Section 7.5]$ , a kernel U can be seen as a Hilbert–Schmidt operator

$$
(Uf)(x) := \int_0^1 U(x, y) f(y) dy,
$$

on  $L^2[0,1]$ . This operator then has countable real eigenvalues  $(\lambda_i)_{i=1}^{\infty}$ , where  $|\lambda_i| \geq |\lambda_j|$  whenever  $i < j$ . Let  $f_i$  be the orthonormal eigenfunction corresponding to nonzero  $\lambda_i$ , i.e.,  $\langle f_i, f_j \rangle = \delta_{i,j}$  and  $Uf_i = \lambda_i f_i$ . Then U admits the spectral decomposition  $U(x, y) = \sum_{i=1}^{\infty} \lambda_i f_i(x) f_i(y)$ . Hence,

$$
t_{P_m}(U) = \sum_{i=1}^{\infty} \lambda_i^m \left( \int_0^1 f_i(x) dx \right)^2
$$

and moreover, by the Parseval identity,

$$
\sum_{i=1}^{\infty} \left( \int_0^1 f_i(x) dx \right)^2 = \left\| \sum_{i=1}^{\infty} \left( \int_0^1 f_i(x) dx \right) f_i \right\|_2^2 = \left\| \sum_{i=1}^{\infty} \langle f_i, 1 \rangle f_i \right\|_2^2 \le \|1\|_2^2 = 1,
$$

which was also observed in [\[20,](#page-9-5) (13)]. The inequality above becomes an equality if and only if the constant function 1 can be expressed as a linear combination of  $f_i$ 's, i.e.,  $1 = \sum_{i\geq 1} \langle f_i, 1 \rangle f_i$ . Let  $p_i := (\int_0^1 f_i(x)dx)^2$  for each  $i \geq 1$ ,  $p_0 := 1 - \sum_{i \geq 1} p_i$ , and  $\lambda_0 := 0$ . Then for each integer  $m \geq 0$ ,  $t_{P_m}(U) = \sum_{i \geq 0} p_i \lambda_i^m$ . This rephrases as

<span id="page-3-2"></span>**Lemma 2.3.** Let U be a kernel. Then there exists a discrete random variable  $X_U$  such that  $\mathbb{P}[X_{U} = \lambda_{i}] = p_{i}, i = 0, 1, ..., and hence, t_{P_{m}}(U) = \mathbb{E}[X_{U}^{m}].$ 

We remark that a 'discrete' analogue of this lemma was already observed by Erdős and Simonovits [\[10,](#page-8-14) Theorem 4]. The spectral technique is also useful in proving the inequality [\(8\)](#page-3-0) for even cycles H and a kernel U. Indeed, as  $t_{C_{2m}}(U) = \sum_i \lambda_i^{2m} [24, (7.22)],$  $t_{C_{2m}}(U) = \sum_i \lambda_i^{2m} [24, (7.22)],$  $t_{C_{2m}}(U) = \sum_i \lambda_i^{2m} [24, (7.22)],$ 

$$
t_{C_{2m}}(U)^{1/2m} \ge |\lambda_1| \ge \left|\sum_{i\ge 0} p_i \lambda_i\right| = |t_{K_2}(U)|,
$$

which is  $(8)$  for even cycles H. Thus, Theorem [2.2](#page-3-1) implies Theorem [1.1](#page-1-0) for even cycles H too, although the result is already known due to the fact that even cycles are *norming*. For more discussions about the norming property, we refer the reader to [\[24,](#page-9-1) Chapter 14].

#### 3 Proof of the main theorem

Our goal in this section is to prove Theorem [2.2,](#page-3-1) which implies Theorem [1.1.](#page-1-0) For a kernel  $U$ , let  $q_{m,d}(U)$  denote the left-hand side of the inequality in Theorem [2.2,](#page-3-1) i.e.,

$$
q_{m,d}(U) := \sum_{F \in \mathcal{E}_{2d}^+(P_m)} t_F(U).
$$

Let us first have a look at a small example that illustrates what  $q_{m,d}(U)$  is. If  $d = 1$ , then the corresponding  $\mathcal{E}^+_{2a}$  $2d^{+}_{2d}(P_m)$  consists of the subgraphs of  $P_m$  with two edges and  $m+1$  vertices. That is, either a 2-edge path or a matching of size two plus isolated vertices. Hence,

$$
q_{m,1}(U) = (m-1)\mathbb{E}[X_U^2] + {m-1 \choose 2} \mathbb{E}[X_U]^2,
$$

where  $X_U$  is defined in Lemma [2.3.](#page-3-2) This can be rewritten as

$$
q_{m,1}(U) = \mathbb{E}\left[\sum_{i=1}^{m-1} X_i^2 + \sum_{1 \le i < j \le m-1} X_i X_j\right],
$$

where  $X_i$ 's are i.i.d. copies of  $X_U$ .

Let  $h_d(x_1, \ldots, x_k)$  be the *k*-variable complete homogeneous symmetric function of degree d. That is,

$$
h_d(x_1,\ldots,x_k)=\sum x_1^{\ell_1}\cdots x_k^{\ell_k},
$$

where the sum is taken over all the nonnegative integer solutions of  $\ell_1 + \cdots + \ell_k = d$ . For example,  $h_2(x_1, \ldots, x_k) = \sum_{i=1}^k x_i^2 + \sum_{1 \le i < j \le k} x_i x_j$  and hence,  $q_{m,1}(U) = \mathbb{E}[h_2(X_1, \ldots, X_{m-1})].$ 

<span id="page-4-0"></span>By generalising this observation, we express  $q_{m,d}(U)$  in terms of the expectation of the homogeneous polynomials  $h_{2d}(X_1, \ldots, X_k)$  of degree  $2d$ , where  $X_i$  is an i.i.d. copy of  $X_U$  in Lemma [2.3.](#page-3-2)

**Lemma 3.1.** Let U be a kernel. For all integers m and d with  $1 \leq d \leq m/2$ ,

$$
q_{m,d}(U) = \mathbb{E}[h_{2d}(X_1, \ldots, X_{m-2d+1})],
$$

where  $X_i$ 's are *i.i.d.* copies of  $X_U$  given in Lemma [2.3.](#page-3-2)

*Proof.* Let  $F \in \mathcal{E}_{2d}^+(P_m)$ . Recall that  $V(F) = V(P_m)$ . Enumerate the  $m-2d$  edges in  $E(P_m) \setminus E(F)$ by  $e_1, \ldots, e_{m-2d}$  from left to right in the m-edge path  $P_m$ . Let  $\ell_i$  be the number of edges in the component of  $F$  that contains the leftmost vertex of  $e_i$ . In particular, if the left-intersecting component to  $e_i$  is an isolated vertex, then  $\ell_i = 0$ . Denote by  $\ell_{m-2d+1}$  the number of edges in the component of F that contains the rightmost vertex of  $P_m$ . Clearly,  $\sum_{i=1}^{m-2d+1} \ell_i = 2d$ .

Conversely, every nonnegative integer solution to the equation  $\sum_{i=1}^{m-2d+1} \ell_i = 2d$  uniquely determines the corresponding  $F \in \mathcal{E}_{2d}^+(P_m)$ , which satisfies

$$
t_F(U) = \prod_{i=1}^{m-2d+1} t_{P_{\ell_i}}(U) = \prod_{i=1}^{m-2d+1} \mathbb{E}[X_U^{\ell_i}] = \mathbb{E}\left[\prod_{i=1}^{m-2d+1} X_i^{\ell_i}\right],
$$

where  $X_i$ 's are i.i.d. copies of  $X_U$ . Therefore,

$$
q_{m,d}(U) = \sum_{F \in \mathcal{E}_{2d}^+(P_m)} t_F(U) = \sum \mathbb{E}\left[\prod_{i=1}^{m-2d+1} X_i^{\ell_i}\right] = \mathbb{E}\left[\sum \prod_{i=1}^{m-2d+1} X_i^{\ell_i}\right]
$$

,

 $\Box$ 

where the last two sums are taken over all nonnegative integers  $\ell_i$ 's such that  $\ell_1 + \cdots + \ell_{m-2d+1} = 2d$ . Thus,  $q_{m,d}(U) = \mathbb{E}[h_{2d}(X_1,\ldots,X_{m-2d+1})].$ 

Note that  $h_{2d}(x_1, \ldots, x_k) = {k+2d-1 \choose 2d}$  $\left( \frac{2d-1}{2d} \right) x^{2d}$  if  $x_i = x$  for all i. Letting  $x = \mathbb{E}[X_U] = t_{K_2}(U)$ and  $k = m - 2d + 1$  then gives  $h_{2d}(\mathbb{E}[X_1], \dots, \mathbb{E}[X_{m-2d+1}]) = {m \choose 2d} t_{K_2}(U)^{2d}$ , which, together with Lemma [3.1,](#page-4-0) suggests that some convexity of  $h_{2d}$  may prove Theorem [2.2.](#page-3-1)

To formalise this idea, we need an easy consequence of *Schur convexity* of  $h_{2d}$ . A real k-tuple  $(x_1, \ldots, x_k)$  majorises another k-tuple  $(y_1, \ldots, y_k)$  if  $\sum_{i=1}^j x_i \geq \sum_{i=1}^j y_i$  for every  $j = 1, \ldots, k$  with equality for  $j = k$ . A k-variable real polynomial h is *Schur convex* if  $h(x_1, \ldots, x_k) \ge h(y_1, \ldots, y_k)$ whenever  $(x_1, \ldots, x_k)$  majorises  $(y_1, \ldots, y_k)$ . One can deduce from the classical Schur–Ostrowski theorem [\[25,](#page-9-15) Chapter 3, A.4. Theorem] that  $h_{2d}(x_1, \ldots, x_k)$  is Schur convex (see, e.g., [\[32\]](#page-9-13)), whose immediate consequence is the following lemma. For self-containedness, we give a brief probabilistic proof which essentially rephrases Barvinok's argument [\[1,](#page-8-15) Lemma 3.1] and pushes it slightly further; see also [\[19,](#page-9-12) Remark 6.4] and an anonymous comment in [\[32\]](#page-9-13).

<span id="page-5-1"></span>**Lemma 3.2.** Let  $d, k > 0$  be integers. Then for all real numbers  $x_1, \ldots, x_k$ ,

$$
h_{2d}(x_1,\ldots,x_k) \ge h_{2d}(\overbrace{\overline{x},\ldots,\overline{x}}^k) = {k+2d-1 \choose 2d} \overline{x}^{2d},
$$

*where*  $\bar{x} = (x_1 + \cdots + x_k)/k$  *and the equality holds if and only if*  $x_1 = \cdots = x_k$ .

*Proof.* Let  $Z_i$ ,  $i = 1, ..., k$ , be i.i.d. exponential random variables with rate parameter  $\lambda = 1$ . We shall use the well-known fact that  $\mathbb{E}[Z_i^t] = t!$ . Let  $S_{2d}(x_1,\ldots,x_k) := (\sum_{i=1}^k x_i Z_i)^{2d} / (2d)!$ . Then

<span id="page-5-0"></span>
$$
\mathbb{E}[S_{2d}(x_1,\ldots,x_k)] = \mathbb{E}\left[\sum_{\ell_1+\cdots+\ell_k=2d} \prod_{i=1}^k \frac{x_i^{\ell_i} Z_i^{\ell_i}}{\ell_i!}\right] = h_{2d}(x_1,\ldots,x_k). \tag{9}
$$

Let  $S_{2d}^{+j}$  $\chi_{2d}^{+j}(x_1,\ldots,x_n) := (\sum_{i=1}^k x_{i+j} Z_i)^{2d}/(2d)!$  be the function obtained by a cyclic permutation of the variables in  $S_{2d}$ , where the addition in the index of  $x_{i+j}$  is taken modulo k. As [\(9\)](#page-5-0) is symmetric in  $x_1, \ldots, x_n$ , we have  $\mathbb{E}[S_{2d}^{+j}]$  $\mathbb{E}[S_{2d}] = \mathbb{E}[S_{2d}] = h_{2d}$ . By convexity of the function  $x \mapsto x^{2d}$ ,

$$
\frac{(2d)!}{k} \sum_{j=1}^{k} S_d^{+j}(x_1, \dots, x_k) = \frac{1}{k} \sum_{j=1}^{k} \left( \sum_{i=1}^{k} x_{i+j} Z_i \right)^{2d}
$$

$$
\ge \left( \frac{1}{k} \sum_{j=1}^{k} \sum_{i=1}^{k} x_{i+j} Z_i \right)^{2d} = \left( \sum_{i=1}^{k} \overline{x} Z_i \right)^{2d} = (2d)! S_{2d}(\overline{\overline{x}}, \dots, \overline{\overline{x}}).
$$

Taking expectation on both sides then concludes the proof.

We are now ready to prove Theorem [2.2.](#page-3-1)

*Proof of Theorem [2.2.](#page-3-1)* Let  $\overline{X} := \frac{1}{k} \sum_{i=1}^{k} X_i$ , where  $k = m - 2d + 1$  and  $X_i$ 's are i.i.d. copies of  $X_U$ in Lemma [2.3.](#page-3-2) Then by Lemmas [3.1](#page-4-0) and [3.2,](#page-5-1)

$$
q_{m,d}(U) = \mathbb{E}[h_{2d}(X_1,\ldots,X_{m-2d+1})] \geq \mathbb{E}\left[\binom{m}{2d}\overline{X}^{2d}\right].
$$

By Jensen's inequality and the fact  $\mathbb{E}[\overline{X}] = \mathbb{E}[X_i] = \mathbb{E}[X_U] = t_{K_2}(U)$  from Lemma [2.3,](#page-3-2)

$$
\mathbb{E}\left[\binom{m}{2d}\overline{X}^{2d}\right] \ge \binom{m}{2d} \mathbb{E}[\overline{X}]^{2d} = \binom{m}{2d} t_{K_2}(U)^{2d}.
$$

Combining the two inequalities then completes the proof.

Without relying on Theorem [1.1,](#page-1-0) one may also directly prove Corollary [1.2](#page-1-1) by using the nonnegativity of complete homogeneous symmetric polynomials. Namely,

<span id="page-6-1"></span>
$$
q_{m,d}(U) = \sum_{F \in \mathcal{E}_{2d}^+(P_m)} t_F(U) \ge 0
$$
\n<sup>(10)</sup>

for each kernel U and  $1 \leq d \leq m/2$ , a weaker inequality than Theorem [2.2,](#page-3-1) is enough. The global nonnegativity of  $h_{2d}(x_1, \ldots, x_k)$ , a classical result of Hunter [\[17\]](#page-8-16) and also an easy consequence of Lemma [3.2,](#page-5-1) together with Lemma [3.1](#page-4-0) therefore proves Corollary [1.2](#page-1-1) directly.

#### <span id="page-6-0"></span>4 Stability

One advantage of our proofs in the previous sections is that they also give a stability analysis, which has not been known for odd cycles  $H$  other than the triangle in Sidorenko's theorem. Roughly speaking, if the inequality in Theorem [1.1](#page-1-0) is 'close' to be an equality, then the graphon  $W$  must be 'almost' regular. We begin by showing a stability result corresponding to Corollary [1.2.](#page-1-1)

<span id="page-6-4"></span>Theorem 4.1. *Let* H *be a path with at least* 2 *edges or a cycle and let* W *be a kernel. For any*  $\varepsilon \geq 0$ , *if* 

<span id="page-6-2"></span>
$$
t_H(W) + t_H(1 - W) \le 2^{1 - e(H)}(1 + \varepsilon),\tag{11}
$$

*then*

<span id="page-6-5"></span><span id="page-6-3"></span>
$$
t_{P_2}(2W - 1) \le \frac{\varepsilon}{e(H) - 1}.\tag{12}
$$

*Proof.* Let  $U = 2W - 1$ . First, suppose  $H = P_m$ . By [\(3\)](#page-2-2) and [\(10\)](#page-6-1), the assumption [\(11\)](#page-6-2) implies

$$
\varepsilon \geq \sum_{F \in \mathcal{E}^+(P_m)} t_F(U) = \sum_{d=1}^{\lfloor m/2 \rfloor} q_{m,d}(U) \geq q_{m,1}(U) \geq (m-1)t_{P_2}(U),
$$

which gives [\(12\)](#page-6-3). Indeed,  $q_{m,1}(U) = (m-1)t_{P_2}(U) + {m-1 \choose 2}t_{K_2}(U)^2$  proves the last inequality.

Suppose now that  $H = C_m$ . By [\(3\)](#page-2-2), [\(10\)](#page-6-1), and the argument for [\(7\)](#page-2-1), the assumption [\(11\)](#page-6-2) implies

$$
\varepsilon \ge \sum_{F \in \mathcal{E}^+(C_m)} t_F(U) = \tau + \sum_{e \in E(C_m)} \sum_{d=1}^{\lfloor (m-1)/2 \rfloor} \frac{q_{m-1,d}(U)}{m-2d} \ge \frac{mq_{m-1,1}(U)}{m-2} \ge m \cdot t_{P_2}(U),\tag{13}
$$

where  $\tau = t_{C_m}(U)$  if m is even and  $\tau = 0$  otherwise. This proves [\(12\)](#page-6-3).

 $\Box$ 

Theorem [4.1](#page-6-4) concludes that  $t_{P_2}(2W - 1)$  is 'small' whenever the inequality in Corollary [1.2](#page-1-1) is close to be an equality. To elaborate on the meaning of  $t_{P_2}(2W - 1)$  being small, suppose that W is the indicator graphon of an *n*-vertex graph G and recall that  $U = 2W - 1$ . As

$$
t_{P_2}(U) = \int_{[0,1]^3} U(x,y)U(y,z)dxdydz = \int_0^1 d_U(y)^2 dy,
$$

where  $d_U(y) := \int_0^1 U(x, y) dx$ , the inequality  $t_{P_2}(U) \leq \varepsilon$  together with Markov's inequality gives

$$
\sqrt{\varepsilon} \cdot \mathbb{P}[d_U(y)^2 \ge \sqrt{\varepsilon}] \le \int_0^1 d_U(y)^2 dy \le \varepsilon.
$$

That is, all but  $\sqrt{\varepsilon}n$  vertices in G have degree between  $(1 - \varepsilon^{1/4})n/2$  and  $(1 + \varepsilon^{1/4})n/2$ .

If  $H = C_m$  with m even in Theorem [4.1,](#page-6-4) the conclusion becomes even stronger. Namely, instead of the lower bound  $m \cdot t_{P_2}(U)$  in [\(13\)](#page-6-5), one may use  $\tau = t_{C_m}(U)$  to simply obtain  $t_{C_m}(U) \leq \varepsilon$ . It is well-known, e.g., [\[6,](#page-8-13) [24\]](#page-9-1), that this implies  $||U||_{\square} \leq \varepsilon^{1/m}$ , where  $||.||_{\square}$  is the *cut norm*. For the other cases, one cannot expect such a result, as the inequality in Theorem [1.1](#page-1-0) attains the equality whenever W is a 'regular' graphon with density  $1/2$ , i.e.,  $d_W(x) = 1/2$  almost everywhere.

An analogous stability result for Theorem [1.1](#page-1-0) can also be obtained.

Theorem 4.2. *Let* H *be a path with at least* 2 *edges or a cycle and let* W *be a kernel. For any*  $\varepsilon \geq 0$ , *if* 

<span id="page-7-0"></span>
$$
t_H(W) + t_H(1 - W) \le t_{K_2}(W)^{e(H)} + t_{K_2}(1 - W)^{e(H)} + 2^{1 - e(H)}\varepsilon.
$$
 (14)

*then for*  $U = 2W - 1$ ,

<span id="page-7-1"></span>
$$
t_{P_2}(U) \le t_{K_2}(U)^2 + \frac{\varepsilon}{e(H) - 1}.\tag{15}
$$

*Proof.* Suppose that  $H = C_m$ . By [\(3\)](#page-2-2), [\(4\)](#page-2-3), [\(7\)](#page-2-1), and Theorem [2.2,](#page-3-1) the assumption [\(14\)](#page-7-0) implies

$$
\varepsilon \geq \sum_{F \in \mathcal{E}^+(H)} \left( t_F(U) - t_{K_2}(U)^{e(F)} \right) \geq \frac{m}{m-2} \left( q_{m-1,1}(U) - \binom{m-1}{2} t_{K_2}(U)^2 \right)
$$
  
=  $m \left( t_{P_2}(U) - t_{K_2}(U)^2 \right),$ 

which gives [\(15\)](#page-7-1). The case  $H = P_m$  follows in an analogous way.

The inequality  $(15)$  again implies that W is 'almost' regular with respect to the edge density  $t_{K_2}(W)$  instead of  $1/2$ , as  $t_{P_2}(U) - t_{K_2}(U)^2$  translates to the variance of  $d_U$ . That is, if W is the indicator graphon of an *n*-vertex graph G, then all but  $\sqrt{\varepsilon}n$  vertices of G have degree between  $(p - \varepsilon^{1/4})n$  and  $(p + \varepsilon^{1/4})n$ , where  $p = t_{K_2}(W)$ . If  $H = C_m$  with m even, then we have a stronger conclusion  $||U - t_{K_2}(U)||_{\square} \leq \varepsilon^{1/2m}$ , i.e., U is  $\varepsilon^{1/2m}$ -close to be quasirandom.

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 $\Box$ 

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