NILPOTENCE AND DUALITY IN THE COMPLETE COHOMOLOGY OF A MODULE

JON F. CARLSON

ABSTRACT. Suppose that G is a finite group and k is a field of characteristic p > 0. We consider the complete cohomology ring $\mathcal{E}_M^* = \sum_{n \in \mathbb{Z}} \widehat{\operatorname{Ext}}_{kG}^n(M, M)$. We show that the ring has two distinguished ideals $I^* \subseteq J^* \subseteq \mathcal{E}_M^*$ such that I^* is bounded above in degrees, \mathcal{E}_M^*/J^* is bounded below in degree and J^*/I^* is eventually periodic with terms of bounded dimension. We prove that if M is neither projective nor periodic, then the subring of all elements in negative degrees in \mathcal{E}_M^* is a nilpotent algebra.

1. INTRODUCTION

Let G be a finite group and k a field of characteristic p > 0. In [1], Dave Benson and this author defined products in the negative cohomology $\widehat{H}^*(G, k)$ and showed that products of elements in negative degrees often vanish. For G an elementary abelian p-groups, the product of any two elements with negative degrees is zero as well as the product of any element in a positive degree with another in a negative degree. In general, for any group, the negative cohomology ring is a nilpotent kalgebra. This fact was not actually proved in [1], though it follows directly from the statement about negative degree products in the cohomology of elementary abelian groups and Theorem 2.5 of [4].

In this paper we extend some of the results of [1] to the complete cohomology ring $\widehat{\operatorname{Ext}}_{kG}^*(M, M)$ of a finitely generated kG-module M. In particular, we prove that the k-algebra of elements in negative cohomology is nilpotent. In the course of the proof we introduce two graded ideals I^* and J^* , with $I^* \subseteq J^*$, whose membership for an element is determined by the polynomial rate of growth of the quotient of $H^*(G, k)$ by the annihilator of the element. The quotient J^*/I^* has bounded dimension and is periodic in high degrees. There is another graded ideal \mathfrak{J}^* generated by elements in arbitrarily high negative degrees having the properties that $I^* \subseteq \mathfrak{J}^* \subseteq J^*$ and that \mathfrak{J}^*/I^* is truly periodic. When J^*/I^* is nontrivial, its annihilator determines a zero dimensional subvariety of the spectrum $\operatorname{Proj}(\operatorname{H}^*(G, k))$. Both it and the subvariety

²⁰²⁰ Mathematics Subject Classification. 20C20 (primary), 20J06, 18G80.

Research partially supported by Simons Foundation grant 054813-01.

corresponding to \mathfrak{J}^*/I^* are invariants of the module. The last section of the paper contains some examples of modules over small elementary abelian groups, showing that the J^*/I^* is not trivial.

The cohomology ring $\operatorname{Ext}_{kG}^*(M, M)$ of a finitely generated module M, is known to be a finitely generated algebra over its center and finitely generated as a module over $\operatorname{H}^*(G, k) \cong \operatorname{Ext}_{kG}^*(k, k)$ [3]. It is a PI algebra, but is not graded commutative. Some of the constructions that we develop in this paper, in particular the nontriviality of the ideal J^* , show up also in examples of the cohomology rings given in [1].

The last section of the paper contains several examples for small elementary abelian subgroups, showing that the J^*/I^* is not trivial.

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2. NOTATION AND DEFINITIONS

Here we recall and quickly sketch the definitions of the products in complete cohomology. The notation introduced here is also very useful in the examples in Section 7.

Throughout the paper, G is a finite group and k is a field of characteristic p > 0. All kG-modules are assumed to be finitely generated. The stable category **stmod**(kG) is the category whose objects are finitely generated kG-modules. If M and N are kG-modules, then the set of morphisms from M to N in the stable category is given as

$$\underline{\operatorname{Hom}}_{kG}(M, N) = \operatorname{Hom}_{kG}(M, N) / \operatorname{PHom}_{kG}(M, N)$$

where $\operatorname{PHom}_{kG}(M, N)$ is the subset of homomorphisms from M to N consisting of those that factor through projective modules.

For M a kG-module, let $P_* = P_*(M)$ be a complete projective resolution of M. That is, P_* is an acyclic complex (exact sequence) of projective modules:

$$\dots \longrightarrow P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} P_{-1} \xrightarrow{\partial_{-1}} P_{-2} \longrightarrow \dots$$

such that the image of $\partial_0 : P_0 \to P_{-1}$ is isomorphic to M. Let $\Omega^n(M)$ be the image of $\partial_n : P_n \to P_{n-1}$. The module $\Omega^n(M)$ depends on the projective resolution. However, its class in the stable category is a well defined object. This means that it is well defined up to isomorphism and direct sum with a projective module in the module category. Note that the operator Ω^{-1} is the translation functor or shift functor on the stable category stmod(kG). For any n, and modules M and N, the cohomology $\widehat{\operatorname{Ext}}_{kG}^{n}(M,N)$ is defined to be $\widehat{\operatorname{H}}^{n}(\operatorname{Hom}_{kG}(P_{*}(M),N) \cong \operatorname{Hom}_{kG}(\Omega^{n}(M),N)$. We note that, by translation, $\operatorname{Hom}_{kG}(\Omega^{n}(M),N) \cong \operatorname{Hom}_{kG}(\Omega^{n+s}(M),\Omega^{s}(N))$. Hence, the products in the cohomology can be defined as the composition

$$\widehat{\operatorname{Ext}}_{kG}^{n}(M,N) \otimes \widehat{\operatorname{Ext}}_{kG}^{m}(L,M) \longrightarrow \underline{\operatorname{Hom}}_{kG}(\Omega^{n}(M),N) \otimes \underline{\operatorname{Hom}}_{kG}(\Omega^{n+m}(L),\Omega^{n}(M)) \longrightarrow \underline{\operatorname{Hom}}_{kG}(\Omega^{n+m}(L),N) \longrightarrow \widehat{\operatorname{Ext}}_{kG}^{m+n}(L,N)$$

But notice that any homomorphism $\psi : \Omega^n(M) \to N$, can be lifted to chain maps $\{\psi_j\}_{j\geq n}$ and $\{\psi_j\}_{j\leq n}$:

$$\dots \to P_{n+1}(M) \to P_n(M) \to \Omega^n(M) \to 0 \qquad 0 \to \Omega^n(M) \to P_{n-1}(M) \to \dots$$
$$\downarrow \psi_{n+1} \qquad \downarrow \psi_n \qquad \downarrow \psi \qquad \qquad \downarrow \psi \qquad \downarrow \psi_{n-1} \\ \dots \to P_1(N) \longrightarrow P_0(N) \longrightarrow N \longrightarrow 0, \qquad 0 \longrightarrow N \longrightarrow P_{-1}(N) \longrightarrow \dots$$

This works because kG is a self-injective ring. Hence, ψ defines a chain map of degree n from the complete resolution of M to that of N, and the chain map is well defined up to homotopy.

Thus, we conclude that $\widehat{\operatorname{Ext}}_{kG}^n(M,N)$ is isomorphic to the space of homotopy classes of chain maps of degree n from a complete projective resolution of M to that of N. And, importantly, the product of two cohomology elements is the homotopy class of the composition of their corresponding chain maps.

3. The basic ideals

Throughout the paper, we let k be a field of characteristic p > 0 and let G be a finite group. For convenience we assume that k is algebraically closed. We also assume that p divides the order of G as otherwise the results of this paper are vacuous.

We fix a finitely generated kG-module M. The purpose of this section is to introduce two ideals in $\widehat{\operatorname{Ext}}_{kG}^*(M, M)$ that play an essential role in our study. But first some notation.

For the sake of notational economy, let $\mathrm{H}^n_G = \mathrm{H}^n(G, k)$ and $\mathrm{H}^*_G = \mathrm{H}^*(G, k)$. Let $\mathcal{E}^*_+ = \mathrm{Ext}^*_{kG}(M, M)$ be the cohomology ring of M in nonnegative degrees and let $\widehat{\mathcal{E}}^* = \widehat{\mathrm{Ext}}^*_{kG}(M, M) = \sum_{n \in \mathbb{Z}} \widehat{\mathrm{Ext}}^n_{kG}(M, M)$ be the complete cohomology ring. Both \mathcal{E}^*_+ and $\widehat{\mathcal{E}}^*$ are modules over H^*_G , and $\widehat{\mathcal{E}}^*$ is a module over \mathcal{E}^*_+ . Moreover, \mathcal{E}^*_+ is finitely generated over H^*_G , and the homomorphism $\mathrm{H}^*_G \to \mathcal{E}^*_+$ given by $\zeta \mapsto \zeta \operatorname{Id}_M$ sends H^*_G to the center (in the sense of graded commutative rings) of \mathcal{E}^*_+ (for example see [3, Lemma 2.6]).

Definition 3.1. For any $n \in \mathbb{Z}$, let I^n be the k-subspace consisting of all $m \in \widehat{\mathcal{E}}^n$ such that \mathcal{E}^*_+m has finite dimension in $\widehat{\mathcal{E}}^*$. Let $I^* = I^*_M = \sum_{n \in \mathbb{Z}} I^n$.

Lemma 3.2. Let \tilde{I}^n be the subspace of I^n consisting of all $m \in \widehat{\mathcal{E}}^n$ such that $\mathrm{H}^*_G m$ has finite dimension. Let $\tilde{I}^* = \sum_{n \in \mathbb{Z}} \tilde{I}^n$. Then $\tilde{I}^* = I^*$.

Proof. Because the action of \mathcal{H}_{G}^{*} on $\widehat{\mathcal{E}}^{*}$ factors through the action on \mathcal{E}_{+} , it is clear that $I^{*} \subseteq \widetilde{I}^{*}$. Now suppose that $m \in \widetilde{I}^{n}$ for some n and $\mu \in \mathcal{E}_{+}^{t}$ for some t. Then because the action of \mathcal{H}_{G}^{*} commutes with that of \mathcal{E}_{+}^{*} , $\mu m \in \widetilde{I}^{n+t}$. There exist homogeneous elements μ_{1}, \ldots, μ_{s} such that $\mathcal{E}_{+}^{*} = \sum \mathcal{H}_{G} \mu_{i}$. Thus $\mathcal{E}_{+}^{*} m = \sum \mathcal{H}_{G}^{*} \mu_{i} m$ which has finite dimension. Hence, $m \in I^{*}$.

Proposition 3.3. The subspace I^* is an ideal in $\widehat{\mathcal{E}}^*$.

Proof. It is not difficult to see that I^* is closed under addition. In fact, I^* can be characterized as the collection of elements m in $\widehat{\mathcal{E}}^*$ having the property that the \mathcal{E}^*_+ -submodule generated by m is finite dimensional. That is, an element m has this finite generation property if and only if each of its homogeneous pieces has the property. In addition, I^* is easily seen to be closed under multiplication by elements of \mathcal{E}^*_+ . The only thing remaining to show is that if $\mu \in \widehat{\mathcal{E}}^*$ has negative degree and m is in I^* then μm is in I^* . However, the action of μ commutes with the action of $\mathrm{H}^*_{\mathrm{G}}$. Hence, $\mathrm{H}^*_{\mathrm{G}} \, \mu m = \mu \, \mathrm{H}^*_{\mathrm{G}} \, m$ has finite dimension and is in I^* by the last lemma. \Box

The second ideal that we introduce is somewhat similar, but the construction depends heavily on the fact that H_G^* is a finitely generated noetherian k-algebra. Thus we can use ideas and results from (graded-) commutative algebra. For an element m in $\widehat{\mathcal{E}}^*$, let Ann(m) denote the annihilator of m in H_G^* . Note that if m is a homogeneous element, then Ann(m) is a graded ideal.

Definition 3.4. For $n \in \mathbb{Z}$, let J^n be the k-subspace of $\widehat{\mathcal{E}}^n$ consisting of all elements m with the property that $\mathrm{H}^*_G / \mathrm{Ann}(m)$ has Krull dimension at most one. This is equivalent to the condition that there is some bound B, such that for any i, $\mathrm{H}^i_G m$ has dimension at most B. Let $J^* = \sum_{n \in \mathbb{Z}} J^m$.

As with I^* , the subspace J^* has a characterization in terms of the action of \mathcal{E}^*_+ .

Lemma 3.5. Let \tilde{J}^n be the set of all elements m in $\widehat{\mathcal{E}}^n$ such that there exists a number B = B(m) with the property that for any t > 0 the dimension of $\mathcal{E}^t_+ m$ is at most B. Let $\tilde{J}^* = \sum_{n \in \mathbb{Z}} \tilde{J}^n$. Then $J^* = \tilde{J}^*$.

Proof. From the definition we deduce that $\tilde{J}^* \subseteq J^*$. For the reverse inclusion, suppose that $m \in J^n$ for some n. Then there exists B such that the dimension of $\mathrm{H}^t_G m$ is at most B for any t. Suppose that μ_1, \ldots, μ_s is a complete collection

of generator for \mathcal{E}^*_+ as a module over H^*_G . Then $\mathcal{E}^*_+ = \sum_{i=1}^{s} \mathrm{H}^*_G \mu_i$ and we see that $\mathcal{E}^*_+ m = \sum_{i=1}^{s} \mu_i(\mathrm{H}^G_G m)$. It follows that the dimension of $\mathcal{E}^t_+ m$ is at most sB for any t. Hence, m is in \tilde{J}^* .

Proposition 3.6. The subspace J^* is an ideal in $\widehat{\mathcal{E}}^*$.

Proof. The subspace J^* is clearly closed under addition. Moreover, we have seen that if $m \in J^n$ for some n and $\mu \in \widehat{\mathcal{E}}^t$ for any t, then μm is in J^{m+t} .

Remark 3.7. Suppose that the module M is periodic. This means that $\Omega^n(M) \cong M \oplus P$ for some $n \neq 0$ and some projective module P. It implies also that there exist $\zeta \in \mathrm{H}^n_G$ for some n such that multiplication by ζ induces an isomorphism $\widehat{\mathcal{E}}^j \to \widehat{\mathcal{E}}^{j+m}$ for all j. Then provided M is not projective, $J^* = \widehat{\mathcal{E}}^*$ and $I^* = \{0\}$. Conversely, if $J^* = \widehat{\mathcal{E}}^*$, then M is periodic.

Remark 3.8. In the event that \mathcal{E}^*_+ has a regular element (as does $\mathrm{H}^*(G, k)$ [7]), then $I^n = \{0\}$ for all $n \geq 0$.

4. Bounds on the ideals

In this section, we show that the submodule I^* is bounded above in degrees while the quotient $\widehat{\mathcal{E}}^*/J^*$ is bounded below. The kG-module M is fixed and we continue the same notation as in the previous section. The first statement is easy to prove.

Lemma 4.1. There exists a number $B = B_I$ such that $I^n = \{0\}$ for all n > B.

Proof. For $n \geq 0$ let $U^n = I^n$ and $U^* = \sum_{n\geq 0} U^n = I^* \cap \mathcal{E}_+$. We know that U^* is an ideal in \mathcal{E}^*_+ and is an H^*_G -submodule of \mathcal{E}^*_+ . As such it is finitely generated, since H^*_G is noetherian. Because the submodule generated by each generator is finite dimensional, there is an upper bound on the degrees of these submodules and a bound on the degrees of the whole of U^* .

The proof for the bound on $\widehat{\mathcal{E}}^*/J^*$ requires a deeper analysis that uses an idea from [1]. The bound we obtain is likely to be far from optimal.

Theorem 4.2. Let ℓ be the least common multiple of the degrees of a complete set of generators for the k-algebra H_{G}^{*} . Let d denote the maximum of the dimensions of $\mathrm{Ext}_{kG}^{j}(M, M)$ for $0 \leq j \leq \ell$. Then for $n > d\ell$, we have that $\widehat{\mathcal{E}}^{-n} = J^{-n}$.

Proof. Suppose that $m \in \widehat{\mathcal{E}}^{-n}$ with $-n < -d\ell$ and m not in J^{-n} . Then, $\mathrm{H}_{G}^{*} / \mathrm{Ann}(m)$ has Krull dimension at least two. As a consequence, there exist elements ζ_{1} and ζ_{2} in H_{G}^{ℓ} such that subalgebra $A = k[\zeta_{1}, \zeta_{2}] \subseteq \mathrm{H}_{G}^{*}$ is a polynomial ring with two generators and $A \cap \mathrm{Ann}(m) = \{0\}$. Hence, the collection of all elements $\zeta_{1}^{i}\zeta_{2}^{j}m$ are k-linearly independent for all $i, j \geq 0$. This means that, for any u, the subspace $\mathrm{H}_{G}^{u\ell}m$ has dimension at least u + 1. Such a subspace is in $\widehat{\mathcal{E}}^{u\ell-n}$.

So if we write $n = q\ell + r$ with $0 \le r < \ell$, then $q \ge d$ and $\operatorname{H}_{G}^{(q+1)\ell} m \subseteq \mathcal{E}_{+}^{\ell-r}$ has dimension at least q+2 > d. The contradiction proves the theorem.

For any number b, let $\mathcal{D}_b^* = \sum_{n \geq b} \widehat{\mathcal{E}}^n$. Then \mathcal{D}_b^* is a finitely generated module over both \mathcal{E}_+^* and \mathcal{H}_G^* . Let $\mathcal{J}^* = J^* \cap \mathcal{D}_b^*$ and $\mathcal{I}^* = I^* \cap \mathcal{D}_b^*$. Thus, $\mathcal{J}^n = J^n$ if $n \geq b$ and $\mathcal{J}^n = \{0\}$ otherwise. It is similar for \mathcal{I}^n . We may assume that $\mathcal{J}^*/\mathcal{I}^* \neq 0$ as otherwise there is nothing to prove. Let Ann_b be the annihilator of $\mathcal{J}^*/\mathcal{I}^*$ in \mathcal{H}_G^* . Then $\mathcal{H}_G^*/\operatorname{Ann}_b$ has Krull dimension one because of the finite generation.

Now choose an element ζ in H^s_G for some s such that $\mathrm{H}^*_G / \mathrm{Ann}_b$ is a finitely generated module over $k[\zeta]$. That is, we want that ζ is a one-element homogeneous system of parameters $\mathrm{H}^*_G / \mathrm{Ann}_b$. In particular, if U is the ideal generated by Ann_b and ζ , or by Ann_b and any power of ζ , then H^*_G / U has finite dimension.

With this notation we can prove the following.

Proposition 4.3. For any $n \ge b$, the map $J^n/I^n \to J^{n+s}/I^{n+s}$ induced by multiplication by ζ is injective.

Proof. Suppose that $m \in J^n$ has the property that $\zeta m \in I^n$. Because $H^*_G \zeta m$ has finite dimension, ζ^t annihilates ζm for some t. Thus m is annihilated by ζ^{t+1} as well as by Ann_b. Hence, by the choice of ζ , $H^*_G m$ has finite dimension and $m \in I_n$. \Box

Because the choice of b is arbitrary, the following corollary is immediate.

Corollary 4.4. for any n, $Dim(J^n/I^n) \leq Dim(J^{n+s}/I^{n+s})$, where s is the degree of ζ .

In addition we note that the choice of the number b is irrelevant.

Corollary 4.5. Suppose that b and c are any numbers. Then $Ann_b = Ann_c$ is the annihilator of J^*/I^* .

Proof. Suppose that b > c. Clearly, $\operatorname{Ann}_c \subseteq \operatorname{Ann}_b$, since $\mathcal{D}_c \supseteq \mathcal{D}_b$. So suppose that $x \in \operatorname{Ann}_b$ and $m \in \mathcal{D}_c \cap J$. Then for n sufficiently large $\zeta^n m \in J \cap \mathcal{D}_b$. So $\zeta^n xm = x\zeta^n m \in I^*$. Because multiplication by ζ^n on J^*/I^* is injective, we must have that $xm \in I^*$. The fact that Ann_b is the annihilator of J^*/I^* is clear from what we have proved.

5. DUALITY

In this section we recall the definition of Tate duality and reveal some of its consequences for the structure of cohomology rings. For background on Tate duality see one of the books [2] (see particularly Problem 4 on page 148), [6] (XII.6) and the discussion in [1]. As before, the module M is fixed and we continue the notation of previous sections.

For a kG-module N, let $N^* = \operatorname{Hom}_k(N, k)$ be its k-dual. Tate duality, as applied to $\widehat{\mathcal{E}}^*$, for an integer n, is a nondegenerate pairing

$$\widehat{\operatorname{Ext}}_{kG}^{n}(M,M)\otimes \widehat{\operatorname{Ext}}_{kG}^{-n-1}(M,M) \to k$$

Using the adjointness and the standard isomorphism

$$\widehat{\operatorname{Ext}}_{kG}^{n}(M,M) \cong \widehat{\operatorname{Ext}}_{kG}^{n}(k,M^{*} \otimes M) \cong \widehat{\operatorname{H}}^{n}(G,\operatorname{Hom}_{k}(M,M))$$

the duality is induced by the composition and is the usual cup product.

$$\widehat{\operatorname{H}}^{n}(G, \operatorname{Hom}_{k}(M, M)) \otimes \widehat{\operatorname{H}}^{-n-1}(G, \operatorname{Hom}_{k}(M, M)) \longrightarrow$$
$$\widehat{\operatorname{H}}^{-1}(G, \operatorname{Hom}_{k}(M, M)) \longrightarrow \widehat{\operatorname{H}}^{-1}(G, k) \cong k$$

where the first map is the product and the second is induced by the trace map $\operatorname{Hom}_k(M, M) \to k$. The upshot of this is that if α , β and γ are three homogeneous elements of $\operatorname{Ext}^*_{kG}(M, M)$ whose degrees sum to -1, then

$$\langle \alpha \beta, \gamma \rangle = \langle \alpha, \beta \gamma \rangle.$$

All of this leads us to the following.

Proposition 5.1. Suppose that α is a homogeneous element of $\widehat{\mathcal{E}}^*$ having negative degree. Then there exists an element $\beta \in \widehat{\mathcal{E}}$ with negative degree such that $\alpha\beta \neq 0$, if and only if there exists an element γ such that $\gamma\alpha \neq 0$ has positive degree. In either case we may assume that $\deg(\alpha) + \deg(\beta) + \deg(\gamma) = -1$.

Proof. Suppose that there exists β with $\deg(\beta) < 0$ such that $\alpha\beta \neq 0$. Then there is an element γ such that $\deg(\gamma) = -\deg(\alpha) - \deg(\beta) - 1$ and $\langle \gamma, \alpha\beta \rangle \neq 0$. Thus $\langle \gamma\alpha, \beta \rangle \neq 0$, and $\gamma\alpha \neq 0$. The reverse statement is proved by the reverse argument. The statement about the degrees is obvious from the construction.

Corollary 5.2. Let B_I be the bound such that $I^n = 0$ if $n > B_I$ from Lemma 4.1. Suppose that $\alpha \in I^n$ with $n < -B_I - 1$. Then $\beta \alpha = 0$ for every element $\beta \in I^*$.

Proof. Choose any homogeneous $\beta \in I$. Suppose that $\beta \alpha \neq 0$. Let γ be as in the proposition. That is, $\gamma \beta \neq 0$ and $\deg(\gamma) = -\deg(\alpha) - \deg(\beta) - 1$. Then $\deg(\gamma\beta) = -\deg(\alpha) - 1 > B_I$ implying that $\gamma\beta = 0$. The contradiction proves the corollary.

Theorem 5.3. The ideal I^* is a nilpotent ideal in $\widehat{\mathcal{E}}^*$.

Proof. Let $\gamma_1, \gamma_2, \ldots$ be a sequence of homogeneous elements in I^* . Our object is to show that for any n, sufficiently large, the product $\gamma_1 \gamma_2 \cdots \gamma_n = 0$. For $i \ge 2$, let a_i be the degree of $\mu_i = \gamma_2 \cdots \gamma_i$. Note that if for any $i, a_i > B_I$, then $\mu_i = 0$ by

Lemma 4.1. Likewise, if $a_i < -B_I - 1$ then $\gamma_1 \mu_i = 0$ by Corollary 5.2. Consequently,

if $\gamma_1 \mu_i \neq 0$ for all *i* we must have that $B_I \leq a_i \leq -B_I - 1$. The ring $\widehat{\mathcal{E}}_+^{0} = \widehat{\operatorname{Ext}}_{kG}^{0}(M, M)$ is local, because *M* is indecomposable. Let *N* be the nilpotence degree of its radical. That is, any product of at least *N* elements in the radical of $\widetilde{\mathcal{E}}_{+}^{0}$ vanishes. We claim the the nilpotence degree of I is at most $2(N+1)(B_I+1)$. For suppose that $n > 2(N+1)(B_I+1)$. Then by the pigeonhole principle, at least N + 1 of the a_i 's for $2 \le i \le n$ must be the same. That is, there exist $1 < i_0 < i_1 < \cdots < i_N$ such that $a_{i_0} = a_{i_1} = \cdots = a_N$. This means that the elements $\vartheta_j = \gamma_{i_j+1} \cdots \gamma_{i_{j+1}}$ all have degree 0 and hence are elements in the radical of $\widehat{\mathcal{E}}^0$. Consequently, $\vartheta_1 \cdots \vartheta_N = 0$ and also $\mu_n = 0$. This proves the theorem.

Theorem 5.4. Suppose that M is an indecomposable, nonprojective kG-module that is not periodic. Let $\mathcal{K}^* = \sum_{n < 0} J^n$ be the negative cohomology that is in the ideal J^* . Then \mathcal{K}^* is a nilpotent algebra.

Proof. Let N denote the nilpotence degree of the radical of $\mathcal{E}^0_+ = \operatorname{Ext}^0_{kG}(M, M)$. Let $\zeta \in H_G^*$ be an element as in Proposition 4.3. Let $n = \deg(\zeta)$. Suppose that $\gamma_1, \gamma_2, \ldots$ is a sequence of homogeneous elements in \mathcal{K}^* . Let $a_i = \deg(\gamma_1 \cdots \gamma_i)$ and write $a_i = q_i n + r_i$ where $0 \le r_i < n$. Assume that $m \ge n(N+1)$.

By the pigeonhole principle there exist $i_1 < i_2 < \cdots < i_{N+1}$ such that $r_{i_1} =$ $\cdots = r_{i_{N+1}}$. Let $\mu_j = \gamma_{i_j+1} \cdots \gamma_{i_{j+1}}$. Then $\deg(\mu_j) = s_j n$ for some $s_j < 0$. Also, $\vartheta_i = \zeta^{-s_j} \mu_i$ has degree 0 and is an element of $\widehat{\mathcal{E}}^0$. Because M in neither projective nor periodic, ϑ_j factors through some $\Omega^t(M) \not\cong M$ for some t, and ϑ_j is in the radical of $\widehat{\mathcal{E}}^0$ for every *j*. Let $s = \sum s_i$. Then

$$\zeta^s \gamma_1 \cdots \gamma_{i_{N+1}} = \gamma_1 \dots \gamma_{i_1} \vartheta_1 \dots \vartheta_N \in I^*$$

by Proposition 4.3. By Theorem 5.3 for m sufficiently large, $\gamma_1 \dots \gamma_m = 0$.

Theorem 5.5. Suppose that M is a kG-module that is neither projective nor periodic. Let $\widehat{\mathcal{E}}_{-}^{*} = \sum_{n < 0}^{n} \widehat{\operatorname{Ext}}_{kG}^{n}(M, M)$ be the algebra of negative cohomology of M. Then $\widehat{\mathcal{E}}^*$ is a nilpotent algebra.

Proof. By Theorem 4.2, the product of any sufficiently large number of elements on $\widehat{\mathcal{E}}_{-}^{*}$ lies in J^{*} and in negative degrees. Hence, the proof follows from Theorem 5.4.

6. The periodic streak

In the case that M = k, there are examples of groups where multiplication by the element ζ as in Proposition 4.3 is also surjective on J^*/I^* . These include the case that G is a semidihedral 2-group (see Section 4 of [1]). However, we see no reason for ζ to be an isomorphism on J^*/I^* in general.

In this section, we show that with a modification \mathfrak{J}^* of the ideal J^* , there is a periodic streak \mathfrak{J}^*/I^* that runs through the entire cohomology ring. Moreover, if M is not a periodic module, then the ideal \mathfrak{J}^* is nilpotent also in positive degrees. In the next section we show that this streak may be nonzero even for modules over elementary abelian p-groups. First we give the definition. For the fixed kG-module M, let I^* and J^* be as in Definitions 3.1 and 3.4 for the complete cohomology ring $\widehat{\mathcal{E}}^*$.

Definition 6.1. For any integers t and n, let

$$\mathfrak{J}_t^n = I^n + \widehat{\mathcal{E}}^n \cap (\widehat{\mathcal{E}}^* \sum_{m \leq t} \widehat{\mathcal{E}}^m).$$

That is, \mathfrak{J}_t^* is the $\widehat{\mathcal{E}}^*$ -submodule (left ideal) of $\widehat{\mathcal{E}}^*$ that is the sum of I^* and the submodule generated by elements in degrees at most t. Let $\mathfrak{J}^n = \bigcap_{t < 0} \mathfrak{J}_t^n$.

Then we have the following.

Lemma 6.2. For any $n, \mathfrak{J}^n \subseteq J^n$. In addition, \mathfrak{J}^* is a left ideal.

Proof. Choose t sufficiently large negatively that $J^s = \widehat{\mathcal{E}}^s$ for all $s \leq t$. Such a t exists by Theorem 4.2. Then $\mathfrak{J}^n \subseteq \mathfrak{J}^n_t \subseteq J^n$. This proves the first statement. For the second, note that \mathfrak{J}^*_t is a left ideal for any t. So \mathfrak{J}^* , which is the intersection, is also a left ideal.

Theorem 6.3. Assume that M is neither periodic nor projective. Let $\zeta \in H^s_G$ be as in Proposition 4.3. Then, multiplication by ζ is an isomorphism on \mathfrak{J}^*/I^* . In addition, \mathfrak{J}^* is a nilpotent two-sided ideal.

Proof. From the lemma we see that multiplication by ζ is injective on $\mathfrak{J}^*/I^* \subseteq J^*/I^*$. For any n, the dimension of J^n/I^n is smaller than that of J^{n+s}/I^{n+s} , where s is the degree of ζ . Hence for some t sufficiently large, negatively, it must be that $\text{Dim}(J^n/I^n) = \text{Dim}(J^{n-s}/I^{n-s})$ for all n < t. Hence, if n < t, then multiplication by ζ is an isomorphism $J^{n-s}/I^{n-s} \to J^n/I^n$. The implication is that for n < t, $\mathfrak{J}^n = J^n$, and moreover, $\mathfrak{J}^* = \mathfrak{J}^*_t$.

So suppose that γ is in \mathfrak{J}^n for some *n*. Then by the definition, $\gamma = \mu + \nu$ where μ is in I^n and ν is a finite sum of elements of the form $\sigma\tau$ for $\sigma \in \widehat{\mathcal{E}}^{n-r}$ and $\tau \in \mathfrak{J}^r_t$ for some r < t. However, by the above argument, $\tau = \mu' + \zeta \tau'$ for $\mu' \in I^r$ and τ' in \mathfrak{J}^{r-s} . Consequently, the class of γ modulo I^* is the same as the class of $\sigma \zeta \tau'$ which is the same as that of $\zeta(\sigma\tau')$. So the class of γ in \mathfrak{J}^*/I^* is a multiple of ζ and multiplication by ζ is surjective on \mathfrak{J}^*/I^* .

Suppose that $\gamma \in \mathfrak{J}^n$ and $\sigma \in \widehat{\mathcal{E}}^t$ for some *n* and *t*. Then for any m > 0, there exists $\gamma = \mu + \zeta^m \nu$ with $\mu \in I^n$ and $\nu \in \mathfrak{J}^{n-sm}$ by what we have just proved. Then

 $\gamma \sigma = \mu \sigma + \zeta^m(\nu \sigma)$, where $\nu \sigma \in J^{n-sm+t}$. Assuming that *m* is sufficiently large, $J^{n-sm+t} = \mathfrak{J}^{n-sm+t}$ and $\gamma \sigma \in \mathfrak{J}^*$. Hence, \mathfrak{J}^* is a two-sided ideal.

The nilpotence of \mathfrak{J}^* is evident. For suppose that x_1, \ldots, x_m are elements of \mathfrak{J}^* . Write each as $x_i = \mu_i + \zeta^{n_i} y_i$ for $\mu_i \in I^*$ and n_i large enough that $y_i \in \mathfrak{J}^*$ has negative degree. Then the product $x_1 \cdots x_m$ equal to $\zeta^a(y_i \cdots y_m)$ modulo I^* , where $a = \sum n_i$. So, by Theorem 5.5, if m is large enough the product will lie in I^* which we know to be nilpotent.

7. Examples

The aim of this section is to show by examples that the ideal \mathfrak{J}^* is not trivial, *i. e.* not equal to I^* , even in some very simple cases. Two examples are presented, one in characteristic 2 and another in odd characteristics. The second is presented with somewhat less detail.

7.1. A characteristic 2 example. Suppose that $G = \langle x, y, z \rangle$ is an elementary abelian group of order 8, and let k be a field of characteristic 2. Then $kG \cong k[X, Y, Z]/(X^2, Y^2, Z^2)$ where X = x - 1, Y = y - 1 and Z = z - 1. A $k\langle x \rangle$ -projective resolution of k has the form

$$\dots \longrightarrow P_2^X \xrightarrow{X} P_1^X \xrightarrow{X} P_0^X \xrightarrow{\varepsilon} k \longrightarrow 0$$

where for every $j \ge 0$, $P_j^X \cong k\langle x \rangle \cong k[X]/(X^2)$, and the boundary maps $P_{j+1}^X \to P_j^X$ are multiplication by X. Substituting Y for X and y for x, we get a projective $k\langle y \rangle$ resolution P_*^Y of k. Then a projective kG-resolution of k is the tensor product $P_* = P_*^X \otimes P_*^Y \otimes P_*^Z$ since $kG \cong k\langle x \rangle \otimes k\langle y \rangle \otimes k\langle z \rangle$. To obtain a complete resolution we take the k-dual of P_* , shift by one degree and

To obtain a complete resolution we take the k-dual of P_* , shift by one degree and splice. The dual of P_*^X has the form

$$0 \longrightarrow k \xrightarrow{\varepsilon^{\sharp}} P_{-1}^X \xrightarrow{X} P_{-2}^X \xrightarrow{X} P_{-3}^X \longrightarrow \dots$$

Here, since $k\langle x \rangle$ is self-dual, we have that $P_{-j-1}^X = (P_j^X)^*$ for all $j \geq 0$ and the boundary maps are the duals of the maps in the projective resolution. Thus, a complete projective kG-resolution of k has the form

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{XYZ} P_{-1} \longrightarrow P_{-2} \longrightarrow \dots$$

where the boundary map $P_0 \to P_{-1}$ is the splice $\varepsilon^{\sharp} \varepsilon : P_0^X \otimes P_0^Y \otimes P_0^Z \to P_{-1}^X \otimes P_{-1}^Y \otimes P_{-1}^Z$, taking $1 \otimes 1 \otimes 1$ to $X \otimes Y \otimes Z$.

We define the module M as follows. Let $F = kGu \oplus kGv$ be the free kG-module with free basis $\{u, v\}$. Then M = F/L where $L = \langle Xu, ZYu, Yu - Xv \rangle$. Then

M has a basis consisting of the classes modulo L of the elements of the set $B = \{Zu, u, Yu, v, Zv, Yv, ZYv\}$. A diagram for the module looks like



where the vertices are a k-basis and the arrows indicate multiplications by the designated elements.

In what follows, we use the isomorphism $\widehat{\operatorname{Ext}}_{kG}^*(M, M) \cong \widehat{\operatorname{H}}^*(G, \operatorname{Hom}_k(M, M))$. In addition, for convenience of notation, we view kG as a truncated polynomial ring in variables X, Y, Z. This has the Hopf algebra structure we would get by regarding kGas the restricted enveloping algebra of a three dimensional commutative restricted Lie algebra. In particular, for $f \in \operatorname{Hom}_k(M, M)$, $m \in M$ and U any k-linear combination of X, Y, Z, we have that (Uf)(m) = Uf(m) - f(Um). We see from the results of [5] that changing the coalgebra structure makes essentially no difference in the example that we compute. That is, the action of the subring of $H^*(G, k)$ generated by the Bocksteins of the degree-one elements on $\operatorname{Ext}_{kG}^*(M, M)$ is the same regardless of which Hopf algebra structure is chosen.

For notation, let $u_{a,b,c}$ denote the element $1 \otimes 1 \otimes 1$ in $P_a^X \otimes P_b^Y \otimes P_c^Z$. Then for $n \geq 0$, P_n is generated by all $u_{a,b,c}$ with a + b + c = n and a, b, c all nonnegative. for n < 0, P_n is generated by all $u_{a,b,c}$ with a + b + c = n - 2 and a, b, c all negative.

Define $f, g \in \operatorname{Hom}_k(M, M) = \operatorname{End}_k(M)$ by f(Zu) = v and g(Zu) = u and f(w) = 0 = g(w) for w any element in the basis B other than Zu. For n < -1, we define a map $\alpha_n : P_n \to \operatorname{End}_k(M)$ by

$$\alpha_n(u_{a,b,c}) = \begin{cases} f & \text{for } (a,b,c) = (n,-1,-1), \\ g & \text{for } (a,b,c) = (n+1,-2,-1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that, with n < -1, $\partial(u_{n+1,-1,-1}) = Xu_{n,-1,-1} + Yu_{n+1,-2,-1} + Zu_{n+1,-1,-2}$. So that $\alpha_n(\partial(u_{n+1,-1,-1})) = Xf + Yg = 0$. Thus, with some additional effort, it can be seen that α_n is a cocycle.

For $n \geq -1$, define $\alpha_n : P_n \to \operatorname{End}_k(M)$ as follows:

$$\alpha_n(u_{a,b,c}) = \begin{cases} f & \text{for } n = -1 \text{ and } (a,b,c) = (-1,-1,-1), \\ YZf & \text{for } n \ge 0 \text{ and } (a,b,c) = (n,0,0), \\ 0 & \text{otherwise} \end{cases}$$

The function YZf has values (YZf)(u) = Yv, (YZf)(Zu) = YZv and (YZf)(w) = 0 for w any other element of the basis B. Notice that YZf is a kG-homomorphism. The fact that every α_n is a cocycle is a consequence of the next result.

Proposition 7.1. Let $\zeta \in H_G^1$ be the element represented by the cocycle $\zeta : P_1(k) \to k$ by $\zeta(u_{1,0,0}) = 1$ and $\zeta(u_{0,1,0}) = 0 = \zeta(u_{0,0,1})$. Let $\tilde{\alpha}_n$ denote the cohomology class in $\widehat{\mathcal{E}}_n$ of α_n . Then $\tilde{\alpha}_n \zeta = \tilde{\alpha}_{n+1} \neq 0$ for all n. In particular, $\tilde{\alpha}_n \in \mathfrak{J}^n$ and is not in I^n .

Proof. The trick is to write out the chain map corresponding to ζ . For specific n assume that if $n \ge 0$, then a + b + c = n and that a, b, c are all nonnegative while if n < 0 then a + b + c = n - 2 and a, b, c are all negative. Then the chain map for ζ is given by

$$\zeta_n(u_{a,b,c}) = \begin{cases} u_{a-1,b,c} & \text{if } n > 0 \text{ and } a \neq 0, \\ 0 & \text{if } n > 0 \text{ and } a = 0, \\ YZu_{-1,-1,-1} & \text{if } n = 0, \\ u_{a-1,b,c} & \text{if } n < 0. \end{cases}$$

Note that in degree zero we have that

$$\partial_{-1}\zeta_0(u_{0,0,0}) = \partial_{-1}(YZu_{-1,-1,-1}) = XYZu_{-2,-1,-1} = XYZ\zeta_{-1}(u_{-1,-1,-1}) = \zeta_{-1}\partial_0(u_{0,0,0}).$$

The remaining checks that this is a chain map are even easier. Verifying that $\tilde{\alpha}_n \zeta = \tilde{\alpha}_{n+1}$ is straightforward.

For n > 0 notice that $\partial_n(u_{n,0,0}) = X u_{n-1,0,0}$. So if α_n is a coboundary, then $YZf = \alpha_n(u_{n,0,0}) = \beta \partial_n(u_{n,0,0}) = X\beta(u_{n-1,0,0})$ for some β . However, we can check that this does not happen, since YZf is not in $X \operatorname{End}_k(M)$. Thus $\tilde{\alpha}_n \neq 0$.

7.2. An odd characteristic example. In this example we assume that k has prime characteristic p > 2 and that $G = \langle x, y \rangle$ is a elementary abelian group of order p^2 . So $kG \cong k[X,Y]/(X^p,Y^p)$ is a truncated polynomial ring, where X = x - 1 and Y = y - 1. Note this time that the projective resolution of k as a $k\langle x \rangle$ -module has the form

$$\ldots \longrightarrow P_2^X \xrightarrow{X^{p-1}} P_1^X \xrightarrow{X} P_0^X \xrightarrow{\varepsilon} k \longrightarrow 0.$$

That is, the even dimensional boundary maps are multiplication by X^{p-1} rather than X.

We use a variant of the notation of the last example. In particular, $u_{i,j}$ denotes the element $1 \otimes 1$ generating $P_i^X \otimes P_j^Y$. If $n \geq 0$, then P_n is a direct sum of all $P_i^X \otimes P_j^Y$ with *i* and *j* nonnegative and i + j = n. If n < 0, then P_n is a direct sum of all $P_i^X \otimes P_j^Y$ with *i* and *j* both negative and i + j = n - 1.

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The module is $M \cong (kGu \oplus kGv)/L$ where L is generated by X^2u , Yu - Xv, XYu, Y^2u . Hence, M has a basis consisting of the classes modulo L of the elements in the set $B = \{Xu, u, Yu = Xv, v, Yv, \dots, Y^{p-1}v\}$. Let $f : M \to M$ be given by the rule that f(Xu) = v and f(w) = 0 for w any element of B other than Xu. Notice that $X^3f = 0$.

Assume first that $p \geq 5$. Define $\alpha_n : P_{2n-1} \to \operatorname{End}_k(M)$ by

$$\alpha_n(u_{a,b}) = \begin{cases} f & \text{if } (a,b) = (2n-1,-1) \text{ and } n \le 0\\ Y^{p-1}f & \text{if } (a,b) = (2n-1,0) \text{ and } n > 0\\ 0 & \text{otherwise} \end{cases}$$

In the case that p = 3 some adjustment must be made because of the fact that $X^2 f \neq 0$. Let $g, h \in \text{End}_k(M)$ be the functions given by $g(Yu) = Yu, h(Y^2v) = Yu$, and g(w) = 0 = h(t) for $w, t \in B, w \neq Yu, t \neq Y^2u$. Then, in case p = 3, let α_n be given by

$$\alpha_n(u_{a,b}) = \begin{cases} f & \text{if } (a,b) = (2n-1,-1) \text{ and } n \leq 0, \\ g & \text{if } (a,b) = (2n,-2) \text{ and } n \leq -1, \\ h & \text{if } (a,b) = (2n+1,-3) \text{ and } n \leq -1, \\ Y^{p-1}f & \text{if } (a,b) = (2n-1,0) \text{ and } n > 0, \\ 0 & \text{otherwise} \end{cases}$$

We emphasize that α_n is a cocycle in degree 2n-1. The result is the following.

Proposition 7.2. Let $\zeta \in H_G^2$ be the element represented by the cocycle $\zeta : P_2(k) \rightarrow k$ by $\zeta(u_{2,0}) = 1$ and $\zeta(u_{1,1}) = 0 = \zeta(u_{0,2})$. Let $\tilde{\alpha}_n$ denote the cohomology class in $\widehat{\mathcal{E}}_{2n-1}$ of α_n . Then $\tilde{\alpha}_n \zeta = \tilde{\alpha}_{n+1} \neq 0$ for all n. In particular, $\tilde{\alpha}_n \in \mathfrak{J}^n$ and $\tilde{\alpha}_n$ is not in I^n .

It can be computed that a chain map of ζ on the complete resolution of k is given by the following. Here $\zeta_n : P_n \to P_{n-2}$. For any n and a, b, let

$$\zeta(u_{a,b}) = \begin{cases} u_{a-2,b} & \text{if } n > 1 \text{ and } a > 1, \\ 0 & \text{if } n > 0, a = 0, 1 \text{ and } b > 0, \\ Y^{p-1}u_{-1,-1} & \text{if } (a,b) = (1,0), \\ Y^{p-1}u_{-2,-1} & \text{if } n = 0 \text{ and } (a,b) = (0,0), \\ u_{a-2,b} & \text{if } n < 0. \end{cases}$$

Again, we leave it to the reader to prove that $\{\zeta_n\}$ is a chain map and that the proposition holds.

7.3. One more example. Finally, we present an example of a kG-module M and an element of γ of $\widehat{\operatorname{Ext}}_{kG}^*(M, M)$ in degree -1 where $\gamma \operatorname{H}^*(G, k)$ is more than periodic meaning that $J^{-1} \neq \widehat{\mathcal{E}}^{-1}$. The group and notation for the cohomology are the same as in the first example 7.1. In particular, p = 2 and the projective resolution of k is the tensor product $P_* = P_*^X \otimes P_*^Y \otimes P_*^Z$.

Let M be the quotient module M = F/L, where $F = kGu \oplus kGv$ is the free kGmodule on generators u and v, and L is generated by the elements Zu, Yu - Xv, and YZv. It has basis consisting of the classes modulo L of the elements B = $\{Xu, u, Yu, v, Yv, Zv\}$. Define $f \in \operatorname{End}_k(M)$ by setting f(Xu) = v and f(w) = 0for w any other element of B. Let $\gamma \in \widehat{\mathcal{E}}^{-1}$ be the element represented by the cocycle $\gamma(u_{-1,-1,-1}) = f$.

Let $\zeta_Y \in H^1(G, k)$ be represented by the cocyle $\zeta_Y : P_1 \to k$, such that $\zeta_Y(u_{0,1,0}) = 1$, $\zeta_Y(u_{1,0,0}) = 0 = \zeta_Y(u_{0,0,1})$. Let ζ_Z be given by $\zeta_Z(u_{0,0,1}) = 1$, $\zeta_Z(u_{1,0,0}) = 0 = \zeta_Z(u_{0,1,0})$. Notice that $\gamma\zeta_Y(u_{0,0,0}) = XZf = g_1$ and $\gamma\zeta_Z(u_{0,0,0}) = XYf = g_2$, where $g_1(u) = Zv$, $g_2(u) = Yv$ and $g_1(w) = 0 = g_2(w)$ for $w \in B$, $w \neq u$. Note that g_1 and g_2 are kG-homomorphisms.

We can prove the following.

Proposition 7.3. With the given module M and the above notation, we have that homomorphism $H^*(G, k) \to \widehat{\mathcal{E}}^*$ given by multiplication by γ is injective on the polynomial subring $k[\zeta_Y, \zeta_Z]$. In particular, $\gamma \in \widehat{\mathcal{E}}^{-1}$, and $\gamma \notin J^{-1}$.

Proof. We leave most of the details to the reader. As in the other examples, one proceeds by finding representing chain maps on a complete resolution for the cohomology elements ζ_Y and ζ_Z . This is very similar to the chain map for the element ζ in the proof of Proposition 7.1. One important item to notice is that if $\alpha : P_0 \to \operatorname{End}_k(M)$ is given by $\alpha(u_{0,0,0}) = Xf$, then $\alpha\partial(u_{0,1,0}) = g_1$ and $\alpha\partial(u_{0,0,1}) = g_2$. This means that the cocycles β_1 and β_2 given by $\beta_1(u_{0,1,0}) = g_1$, $\beta_2(u_{0,0,1}) = g_2$, $\beta_1(u_{1,0,0}) = \beta_1(u_{0,0,1}) = \beta_2(u_{1,0,0}) = \beta_1(u_{0,1,0}) = 0$, differ by a coboundary. These cocycles both represent $\gamma\zeta_Y\zeta_Z$.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GEORGIA, ATHENS, GA 30602, USA *Email address*: jfc@math.uga.edu