

Bilinear θ -type Calderón-Zygmund operators and its commutator on generalized weighted Morrey spaces over RD-spaces

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Abstract: An RD-space \mathcal{X} is a space of homogeneous type in the sense of Coifman and Weiss with the additional property that a reverse doubling property holds in \mathcal{X} . In this setting, the authors establish the boundedness of bilinear θ -type Calderón-Zygmund operator T_θ and its commutator $[b_1, b_2, T_\theta]$ generated by the function $b_1, b_2 \in BMO(\mu)$ and T_θ on generalized weighted Morrey space $\mathcal{M}^{p,\phi}(\omega)$ and generalized weighted weak Morrey space $W\mathcal{M}^{p,\phi}(\omega)$ over RD-spaces.

Keywords: RD-space · Bilinear θ -type Calderón-Zygmund operator · Commutator · Generalized weighted Morrey spaces

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1 Introduction

The space of homogeneous type, first introduced by Coifman and Weiss [2, 3], is a general framework for studying the Calderón-Zygmund operators and functions spaces. Around 1970s, Coifman and Weiss began to investigate the some classical harmonic analysis problems on the metric space, which is called space of homogeneous type (\mathcal{X}, d, μ) , equipped with a metric d and a regular Borel measure μ satisfying the doubling condition, if there exists a positive constant $C_\mu > 1$ such that, for any ball $B(x, r) := \{y \in \mathcal{X} : d(x, y) < r\}$ with $x \in \mathcal{X}$ and $r > 0$,

$$(1.1) \quad \mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$

holds. Since then, many classical results were extended to the spaces of homogeneous type in the sense of Coifman and Weiss. However, some results have so far obtained only on the RD-spaces, which means that (\mathcal{X}, d, μ) is a space of homogeneous type if there exists positive constants $a, b > 1$ such that,

$$(1.2) \quad b\mu(B(x, r)) \leq \mu(B(x, ar));$$

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holds for all $x \in \mathcal{X}$ and $r \in (0, \text{diam}(\mathcal{X})/\alpha)$. On the development and research of the operators over RD-spaces, we refer readers see, e.g., [11, 13, 17, 28, 29].

In recent years, solving numerous important problems in harmonic analysis and PDEs became possible due to the progress reached in the weighted theory, generally speaking, in new function spaces; see, for instance, [15, 16]. Moreover, the A_p weight theory, which was first studied in [20], is one of the cores of the weighted theory. It should be pointed out that Morrey space, which were introduced by Morrey in 1938 (see [19]) in order to study regularity questions which appear in the calculus of variations, describe local regularity more precisely than Lebesgue spaces and widely use not only harmonic analysis but also partial differential equations (see [8, 9]). In 2009, Komori and Shirai [14] introduced the weighted Morrey spaces and study the several properties of classical operators on the classical Euclidean space. The further research and development about the Morrey spaces and weighted Morrey spaces over different settings, the readers can see [1, 5, 6, 7, 12, 21, 22, 23, 24, 25] and the references therein. In 2020, Chou et al. [4] introduced the generalized weighted Morrey spaces over RD-spaces, as applications, the boundedness of the classical operator was established. Very recently, Li et al. [18] established that the boundedness of the commutators generalized by the θ -Calderón-Zygmund operator and the BMO functions in generalized weighted Morrey spaces over RD-spaces.

Motivated by the above research, in this article, we will mainly study the boundedness of bilinear θ -type Calderón-Zygmund operator T_θ and its commutator $[b_1, b_2, T_\theta]$ associated with function $b_1, b_2 \in BMO(\mu)$ on generalized weighted Morrey space $\mathcal{M}^{p,\phi}(\omega)$ and generalized weighted weak Morrey space $W\mathcal{M}^{p,\phi}(\omega)$ over (\mathcal{X}, d, μ) . Before present the mainly results of this paper, we first recall some necessary definitions and notion.

The definition of the generalized weighted Morrey space introduced by Chou et al. [4].

Definition 1.1. Let $p \in [1, \infty)$, $\phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing functions and ω be a weight on \mathcal{X} . Then the generalized weighted Morrey space $\mathcal{M}^{p,\phi}(\omega)$, equipped with the norm

$$\|f\|_{\mathcal{M}^{p,\phi}(\omega)} := \sup_B \left\{ \frac{1}{\phi(\omega(B))} \int_B |f(x)|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}} < \infty.$$

We also denote by $W\mathcal{M}^{p,\phi}(\omega)$ the generalized weighted weak Morrey space of all locally integrable functions f satisfying

$$\|f\|_{W\mathcal{M}^{p,\phi}(\omega)} := \sup_B \sup_{t>0} \frac{1}{[\phi(\omega(B))]^{\frac{1}{p}}} t \omega(\{x \in B : |f(x)| > t\})^{\frac{1}{p}} < \infty.$$

Remark 1.2. (i) When $\phi(x) = 1$, $\mathcal{M}^{p,\phi}(\omega) = L^p(\omega)$ and $W\mathcal{M}^{p,\phi}(\omega) = WL^p(\omega)$. Consequently, the generalized weighted (weak) Morrey space is an extension of the weighted (weak) Lebesgue space.

(ii) When $\phi(x) = x^k$ with $0 < k < 1$, then $\mathcal{M}^{p,\phi}(\omega) = \mathcal{M}^{p,k}(\omega)$ and $W\mathcal{M}^{1,\phi}(\omega) = W\mathcal{M}^{1,k}(\omega)$. Hence, the generalized weighted (weak) Morrey space is an extension of the weighted (weak) Morrey space.

In what follows, let $V(x, y) := \mu(B(x, d(x, y)))$, now we state the definition of bilinear θ -type Calderón-Zygmund operator as follows.

Definition 1.3. Let θ be a non-negative nondecreasing functions on $(0, \infty)$ satisfy the following Dini condition:

$$(1.3) \quad \int_0^1 \frac{\theta(t)}{t} dt < \infty.$$

A kernel $K(\cdot, \cdot, \cdot) \in L^1_{loc}((\mathcal{X})^3 \setminus \{(x, y_1, y_2) : x = y_1 = y_2\})$ is called the bilinear θ -type Calderón-Zygmund kernel if there exists a positive constant C , such that

(1) for all $(x, y_1, y_2) \in \mathcal{X}^3$ with $x \neq y_j$ for $j \in \{1, 2\}$,

$$(1.4) \quad |K(x, y_1, y_2)| \leq C \frac{1}{[\sum_{j=1}^2 V(x, y_j)]^2};$$

(2) there exists a positive constants C , such that, for all x, x', y_1, y_2 with $d(x, x') \leq c \max_{1 \leq j \leq 2} d(x, y_j)$,

$$(1.5) \quad |K(x, y_1, y_2) - K(x', y_1, y_2)| \leq C\theta\left(\frac{d(x, x')}{\sum_{j=1}^2 d(x, y_j)}\right) \frac{1}{[\sum_{j=1}^2 V(x, y_j)]^2}.$$

Remark 1.4. Let $\delta \in (0, 1]$, $\theta(t) = t^\delta$ with $t > 0$, then $K(x, y_1, y_2)$ defined as in Definition 1.3 is just the standard Calderón-Zygmund kernel.

Let $L_b^\infty(\mu)$ be the space of all $L^\infty(\mu)$ functions with bounded support. An operator T_θ is called a bilinear θ -type Calderón-Zygmund operator with K satisfying (1.4) and (1.5) if, for all $f_1, f_2 \in L_b^\infty(\mu)$ and $x \notin \bigcap_{j=1}^2 \text{supp } f_j$,

$$(1.6) \quad T_\theta(f_1, f_2)(x) = \int_{\mathcal{X}} \int_{\mathcal{X}} K(x, y_1, y_2) f_1(y_1) f_2(y_2) d\mu(y_1) d\mu(y_2).$$

Given $b_1, b_2 \in BMO(\mu)$, the commutators $[b_1, b_2, T_\theta]$ associated with the bilinear θ -type Calderón-Zygmund operator T_θ is respectively defined by

$$\begin{aligned} [b_1, b_2, T_\theta](f_1, f_2)(x) &= b_1(x)b_2(x)T_\theta(f_1, f_2)(x) - b_1(x)T_\theta(f_1, b_2 f_2)(x) \\ &\quad - b_2(x)T_\theta(b_1 f_1, f_2)(x) + T_\theta(b_1 f_1, b_2 f_2)(x). \end{aligned}$$

Also, $[b_1, T_\theta]$ and $[b_2, T_\theta]$ are defined as follows,

$$(1.7) \quad [b_1, T_\theta](f_1, f_2)(x) := b_1(x)T_\theta(f_1, f_2)(x) - T_\theta(b_1 f_1, f_2)(x),$$

$$(1.8) \quad [b_2, T_\theta](f_1, f_2)(x) := b_2(x)T_\theta(f_1, f_2)(x) - T_\theta(f_1, b_2 f_2)(x).$$

Now let's recall the definition of $BMO(\mu)$ space.

Definition 1.5. A function $b \in L^1_{loc}(\mu)$ is said to be in the space $BMO(\mu)$, equipped with the norm

$$\|b\|_{BMO(\mu)} = \sup_B \frac{1}{\mu(B)} \int_B |b(x) - b_B| d\mu(x) < \infty,$$

where the supremum is taken over all balls $B \subset \mathcal{X}$ and

$$b_B = \frac{1}{\mu(B)} \int_B b(y) d\mu(y).$$

Definition 1.6. A weight ω is said to belong to the class A_p for $p > 1$ if for any $B \subset \mathcal{X}$,

$$\sup_B \left(\frac{1}{\mu(B)} \int_B \omega(y) d\mu(y) \right) \left(\frac{1}{\mu(B)} \int_B \omega^{1-p'}(x) d\mu(y) \right)^{p-1} < \infty,$$

and ω belongs to the class A_1 , if there is a constant C such that for any $B \subset \mathcal{X}$,

$$\frac{1}{\mu(B)} \int_B \omega(y) d\mu(y) \leq C \inf_{y \in B} \omega(x).$$

We denote A_∞ class in the natural way by $A_\infty = \bigcup_{p>1} A_p$.

To get the boundedness of $[b, T_\theta]$ with $b \in BMO(\mu)$ on the generalized weighted Morrey space $\mathcal{M}^{p,\phi}(\omega)$, Wang [27] also suppose that the function ϕ in Definition 1.1 need following condition: there exist a positive constants $k \in [0, 1)$ and C such that

$$(1.9) \quad \frac{\phi(r)}{r^k} \leq C \frac{\phi(s)}{s^k} \quad \text{for all } 0 < s \leq r < +\infty.$$

In addition, to obtain the boundedness of T_θ with $\theta = t^\delta$, that is to say $K(x, y)$ is standard Calderón-Zygmund kernel, the following two conditions are used by Chou et al. [4]: there exist two positive constants C_1 and C_2 such that

$$(1.10) \quad \frac{\phi(r)}{r} \leq C_1 \frac{\phi(s)}{s} \quad \text{for all } 0 < s \leq r < +\infty.$$

$$(1.11) \quad \int_t^\infty \frac{\phi(s)}{s} \frac{ds}{s} \leq C \frac{\phi(t)}{t} \quad \text{for all } t \in (0, \infty).$$

Remark 1.7. The condition (1.9) implies the conditions (1.10) and (1.11), see e.g.[4].

The following statements are our main results.

Theorem 1.8. Assume that T_θ is a bilinear θ -type Calderón-Zygmund operator with kernel K satisfying (1.4) and (1.5). Let $p_1, p_2 \in [1, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\omega \in A_p(\mu)$. Let $\theta : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function satisfying (1.3) and $\phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing, continuous function satisfying conditions (1.10) and (1.11), we have the following:

(i) when all $p_i > 1$, there exists a constant C such that

$$\|T_\theta(f_1, f_2)\|_{\mathcal{M}^{p,\phi}(\omega)} \leq C \|f_1\|_{\mathcal{M}^{p_1,\phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2,\phi}(\omega)},$$

(ii) when some $p_i = 1$, there exists a constant C such that

$$\|T_\theta(f_1, f_2)\|_{W\mathcal{M}^{p,\phi}(\omega)} \leq C \|f_1\|_{\mathcal{M}^{p_1,\phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2,\phi}(\omega)}.$$

Theorem 1.9. Let $p_1, p_2 \in [1, \infty)$ with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\omega \in A_p(\mu)$ and $b_1, b_2 \in BMO(\mu)$. Let $\theta : [0, \infty) \rightarrow [0, \infty)$ be a nondecreasing function satisfying (1.3) and $\phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing, continuous function satisfying conditions (1.10) and (1.11), we have the following:

(i) when all $p_j > 1$, there exists a constant C such that

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{\mathcal{M}^{p, \phi}(\omega)} \leq C \prod_{j=1}^2 \|b_j\|_{BMO(\mu)} \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)},$$

(ii) when some $p_j = 1$, there exists a constant C such that

$$\|[b_1, b_2, T_\theta](f_1, f_2)\|_{W\mathcal{M}^{p, \phi}(\omega)} \leq C \prod_{j=1}^2 \|b_j\|_{BMO(\mu)} \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}.$$

Finally, we make some conventions on notation. Throughout the paper, C represents a positive constant being independent of the main parameters involved, but may vary from line to line. For a μ -measurable set E , χ_E denotes its characteristic function. The symbol $f \lesssim g$ means that there exists a positive constant C such that $f \leq Cg$. Given a ball $B \subseteq \mathcal{X}$ and $\lambda > 0$, λB denote the ball which has the same center of B and the radius is t times of B . For any exponent $p > 1$, we denote by p' its conjugate index, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$.

2 Preliminaries

To prove the main theorems of this paper, in this section, we give some auxiliary lemmas.

Lemma 2.1. [26] Let $p \in (1, \infty)$ and $\omega \in A_p(\mu)$. There exist positive constants C_1 and C_2 such that for any ball $B \subset \mathcal{X}$ and each measurable set $E \subseteq B$,

$$\frac{\omega(E)}{\omega(B)} \leq C_1 \left[\frac{\mu(E)}{\mu(B)} \right]^{\frac{1}{p}} \quad \text{and} \quad \frac{\omega(E)}{\omega(B)} \geq C_2 \left[\frac{\mu(E)}{\mu(B)} \right]^p.$$

Lemma 2.2. [4] Let (\mathcal{X}, d, μ) be an RD-space, if $\omega \in A_p(\mu)$, $p \in (1, \infty)$, then there exist positive constants $C_3, C_4 > 1$ such that for any ball $B \subset \mathcal{X}$,

$$(2.1) \quad \omega(2B) \geq C_3 \omega(B),$$

$$(2.2) \quad \omega(2B) \leq C_4 \omega(B).$$

Lemma 2.3. [18] Let $p \in (1, \infty)$, $\omega \in A_p(\mu)$ and $\phi : (0, \infty) \rightarrow (0, \infty)$ be an increasing, continuous function satisfying (1.10) and (1.11). Then there exists a positive constant C such that for any ball $B \subset X$,

$$\sum_{k=1}^{\infty} k \left[\frac{\phi(\omega(2^k B))}{\omega(2^k B)} \right]^{\frac{1}{p}} \leq C \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}.$$

Lemma 2.4. [18] Let $b \in BMO(\mu)$ and $\omega \in A_p(\mu)$ with $p \in [1, \infty)$, then

- (1) there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$ and $k \in \mathbb{Z}^+$,

$$|b_{2^{k+1}B} - b_B| \leq C(k+1)\|b\|_{BMO(\mu)};$$

- (2) there exists a positive constant C such that, for any ball $B \subset \mathcal{X}$

$$\left\{ \int_B |b(x) - b_B|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}} \leq C \|b\|_{BMO(\mu)} [\omega(B)]^{\frac{1}{p}}.$$

Finally, we need to recall the following boundedness of the bilinear θ -type Calderón-Zygmund T_θ and commutator $[b_1, b_2, T_\theta]$ on the weighted Lebesgue space $L^p(\omega)$.

Lemma 2.5. Let $K(\cdot, \cdot, \cdot)$ satisfy (1.4) and (1.5), $\omega \in A_p(\mu)$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$ with $1 < p_1, p_2 < \infty$. Then T_θ can be extended to a bounded operator from $L^{p_1}(\omega) \times L^{p_2}(\omega)$ to $L^p(\omega)$.

Proof. From [10], it is not hard to get that the T_θ is bounded from weighted Lebesgue space $L^{p_1}(\omega) \times L^{p_2}(\omega)$ to $L^p(\omega)$. Hence, here we omit the proof.

By an argument analogous to the proof of [27] with a slight modification, we obtain the following results, for briefly, we omit the details here.

Lemma 2.6. Let $b_1, b_2 \in BMO(\mu)$, $\omega \in A_p(\mu)$. Then commutators $[b_1, b_2, T_\theta]$ is bounded from the product of space $L^{p_1}(\omega) \times L^{p_2}(\omega)$ to space $L^p(\omega)$ with $1 < p_1, p_2 < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$.

3 Proof of the main theorems

In this section, We will mainly give out the proof of Theorem 1.8. and Theorem 1.9.

Proof of Theorem 1.8. We just point out that the estimate for the strong type is almost the same as the weak type. Here we only present the proof of the strong type estimate. For any fixed ball $B = B(x_0, r_B) \subset \mathcal{X}$ and $2B := B(x_0, 2r_B)$. We decompose f_j as

$$f_j = f_j^0 + f_j^\infty = f_j \chi_{2B} + f_j \chi_{\mathcal{X} \setminus 2B}, j = 1, 2.$$

By the linearity of T_θ and the Minkowski inequality, we have

$$\begin{aligned} & \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1, f_2)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\ & \leq \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^0, f_2^0)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\ & \quad + \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^0, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^\infty, f_2^0)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\
& + \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^\infty, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\
& := I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

For the term I_1 . By Lemma 2.2 and 2.5, we can deduce that

$$\begin{aligned}
I_1 & \leq \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_{\mathcal{X}} |T_\theta(f_1^0, f_2^0)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\
& \lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} \left[\frac{\phi(\omega(2B))}{\phi(\omega(B))} \right]^{\frac{1}{p}} \\
& \lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} \left[\frac{\omega(2B)}{\omega(B)} \right]^{\frac{1}{p}} \\
& \lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}.
\end{aligned}$$

To estimates I_2 , we first consider $|T_\theta(f_1^0, f_2^\infty)(x)|$, for any $x \in B$ and $y_1 \in 2B, y_2 \in (2B)^c$, then $d(x, y_2) \sim d(x_0, y_2)$ and $V(x, y_2) \sim V(x_0, y_2)$. By applying (1.4) and (1.6) with Hölder inequality, we can get

$$\begin{aligned}
& |T_\theta(f_1^0, f_2^\infty)(x)| \\
& \leq \int_{\mathcal{X}} \int_{\mathcal{X}} |K(x, y_1, y_2)| |f_1^0(y_1)| |f_2^\infty(y)| d\mu(y_1) d\mu(y_2) \\
& \lesssim \int_{2B} \int_{\mathcal{X} \setminus 2B} \frac{|f_1(y_1)| |f_2(y)|}{[\sum_{j=1}^2 V(x, y_j)]^2} d\mu(y_1) d\mu(y_2) \\
& \lesssim \int_{2B} |f_1(y_1)| [\omega(y_1)]^{\frac{1}{p_1} - \frac{1}{p_1}} d\mu(y_1) \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} d\mu(y_2) \right\} \\
& \lesssim [\phi(\omega(2B))]^{\frac{1}{p_1}} \left(\frac{1}{\phi(\omega(2B))} \int_{2B} |f_1(y_1)|^{p_1} \omega(y_1) \right)^{\frac{1}{p_1}} \left(\int_{2B} [\omega(y_1)]^{-\frac{p'_1}{p_1}} \right)^{\frac{1}{p'_1}} \\
& \quad \times \left\{ \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |f_2(y_2)| [\omega(y_2)]^{\frac{1}{p_2} - \frac{1}{p_2}} d\mu(y_2) \right\} \\
& \lesssim \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \left(\int_{2B} [\omega(y_1)]^{-\frac{p'_1}{p_1}} \right)^{\frac{1}{p'_1}} \left\{ \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \phi(\omega(2^{k+1}B))^{\frac{1}{p_2}} \right. \\
& \quad \times \left. \left(\frac{1}{\phi(\omega(2^{k+1}B))} \int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \left(\int_{2^{k+1}B} [\omega(y_2)]^{-\frac{p'_2}{p_2}} d\mu(y_2) \right)^{\frac{1}{p'_2}} \right\}
\end{aligned}$$

$$\begin{aligned} &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \left\{ \sum_{k=1}^{\infty} \frac{\phi(\omega(2^{k+1}B))^{\frac{1}{p_2}}}{[\mu(2^k B)]^2} \frac{\mu(2^{k+1}B)}{\omega(2^{k+1}B)^{\frac{1}{p_2}}} \right\} \\ &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega_j)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}. \end{aligned}$$

Furthermore, we can deduce that

$$\begin{aligned} I_2 &\leq \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^0, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\ &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}. \end{aligned}$$

Since the estimates for and I_2 and I_3 are similar, hence

$$I_3 \lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}.$$

For the term I_4 , we first estimate $T_\theta(f_1^\infty, f_2^\infty)(x)$ with $x \in B$. By (1.4) and Hölder inequality, Definition 1.1 and 1.6, we obtain

$$\begin{aligned} &|T_\theta(f_1^\infty, f_2^\infty)(x)| \\ &\leq \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} |K(x, y_1, y_2)| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\ &\lesssim \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|f_1(y_1)| |f_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} d\mu(y_1) d\mu(y_2) \\ &\lesssim \prod_{j=1}^2 \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_j(y_j)|}{V(x_0, y_j)} d\mu(y_j) \right\} \\ &\lesssim \prod_{j=1}^2 \left\{ \sum_{k=1}^{\infty} \frac{1}{\mu(2^k B)} \int_{2^{k+1}B} |f_j(y_j)| [\omega(y_j)]^{\frac{1}{p_j}} [\omega(y_j)]^{-\frac{1}{p_j}} d\mu(y_j) \right\} \\ &\lesssim \prod_{j=1}^2 \left\{ \sum_{k=1}^{\infty} \frac{\phi(\omega(2^{k+1}B))^{\frac{1}{p_j}}}{\mu(2^k B)} \left(\frac{1}{\phi(\omega(2^{k+1}B))} \int_{2^{k+1}B} |f_j(y_j)|^{p_j} \omega(y_j) d\mu(y_j) \right)^{\frac{1}{p_j}} \right. \\ &\quad \times \left. \left(\int_{2^{k+1}B} [\omega(y_j)]^{-\frac{p'_j}{p_j}} d\mu(y_j) \right)^{\frac{1}{p'_j}} \right\} \\ &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} \left\{ \sum_{k=1}^{\infty} \frac{\phi(\omega(2^{k+1}B))^{\frac{1}{p_j}}}{\mu(2^k B)} \cdot \frac{\mu(2^{k+1}B)}{(\omega(2^{k+1}B))^{\frac{1}{p_j}}} \right\} \\ &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}. \end{aligned}$$

Furthermore, together with Definition 1.1, implies that

$$\begin{aligned} I_4 &\leq \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left[\int_B |T_\theta(f_1^\infty, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right]^{\frac{1}{p}} \\ &\lesssim \prod_{j=1}^2 \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}. \end{aligned}$$

Combining the estimate for I_1, I_2, I_3 and I_4 , the theorem 1.8 is proved.

Proof of Theorem 1.9. We just point out that the estimate for the strong type is almost the same as the weak type. Here we omit the proof details of weak type estimate. For any fixed ball $B = B(x_0, r_B) \subset \mathcal{X}$ and $2B := B(x_0, 2r_B)$. We decompose f_j as

$$f_j = f_j^0 + f_j^\infty = f_j \chi_{2B} + f_j \chi_{\mathcal{X} \setminus 2B}, j = 1, 2.$$

Then, write

$$\begin{aligned} &\|[b_1, b_2, T_\theta](f_1, f_2)\|_{\mathcal{M}^{p, \phi}(\omega)} \\ &\leq \|[b_1, b_2, T_\theta](f_1^0, f_2^0)\|_{\mathcal{M}^{p, \phi}(\omega)} + \|[b_1, b_2, T_\theta](f_1^0, f_2^\infty)\|_{\mathcal{M}^{p, \phi}(\omega)} \\ &\quad + \|[b_1, b_2, T_\theta](f_1^\infty, f_2^\infty)\|_{\mathcal{M}^{p, \phi}(\omega)} + \|[b_1, b_2, T_\theta](f_1^\infty, f_2^\infty)\|_{\mathcal{M}^{p, \phi}(\omega)} \\ &:= J_1 + J_2 + J_3 + J_4. \end{aligned}$$

The estimates for J_1 goes as follows. By applying Definition 1.1 and Lemma 2.5, we obtain that

$$\begin{aligned} J_1 &= \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left\{ \int_B |[b_1, b_2, T_\theta](f_1^0, f_2^0)(x)|^p \omega(x) d\mu(x) \right\}^{\frac{1}{p}} \\ &\leq \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \|[b_1, b_2, T_\theta](f_1^0, f_2^0)\|_{L^p(\omega)} \\ &\lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_{2B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \\ &\quad \times \left(\int_{2B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \\ &\lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sup_B \left(\frac{\phi(\omega(2B))}{\phi(\omega(B))} \right)^{\frac{1}{p}} \\ &\lesssim \prod_{j=1}^2 \|b_j\|_{BMO(\mu)} \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)} \left(\frac{(\omega(2B))}{(\omega(B))} \right)^{\frac{1}{p}} \\ &\lesssim \prod_{j=1}^2 \|b_j\|_{BMO(\mu)} \|f_j\|_{\mathcal{M}^{p_j, \phi}(\omega)}. \end{aligned}$$

For any $x \in B$, write

$$|[b_1, b_2, T_\theta](f_1^0, f_2^\infty)(x)|$$

$$\begin{aligned}
&\leq \int_{2B} \int_{\mathcal{X} \setminus 2B} |b_1(x) - b_1(y_1)| |b_2(x) - b_2(y_2)| |K(x, y_1, y_2)| |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\lesssim \int_{2B} \int_{\mathcal{X} \setminus 2B} \frac{|b_1(x) - b_1(y_1)| |b_2(x) - b_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\lesssim \int_{2B} |b_1(x) - b_1(y_1)| |f_1(y_1)| d\mu(y_1) \left(\int_{\mathcal{X} \setminus 2B} \frac{|b_2(x) - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&\lesssim |b_1(x) - (b_1)_{2B}| \int_{2B} |f_1(y_1)| d\mu(y_1) \left(\int_{\mathcal{X} \setminus 2B} \frac{|b_2(x) - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&\quad + \int_{2B} |b_1(y_1) - (b_1)_{2B}| |f_1(y_1)| d\mu(y_1) \left(\int_{\mathcal{X} \setminus 2B} \frac{|b_2(x) - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&:= J_{21} + J_{22}.
\end{aligned}$$

By virtue of Hölder's inequality and lemma 2.2-2.4, it follows that

$$\begin{aligned}
J_{21} &= |b_1(x) - (b_1)_{2B}| \int_{2B} |f_1(y_1)| d\mu(y_1) \left(\int_{\mathcal{X} \setminus 2B} \frac{|b_2(x) - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&\leq |b_1(x) - (b_1)_{2B}| \left(\int_{2B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \left(\int_{2B} \omega(y_1)^{-\frac{p'_1}{p_1}} d\mu(y_1) \right)^{\frac{1}{p'_1}} \\
&\quad \times \left(\sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|b_2(x) - (b_2)_{2B} + (b_2)_{2B} - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&\leq |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
&\quad \times \left(|b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_2(y_2)|}{[V(x, y_2)]^2} d\mu(y_2) \right) \\
&\quad + |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
&\quad \times \left(\sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|(b_2)_{2B} - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
&\leq |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
&\quad \times \left\{ |b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \cdot \frac{\mu(2^{k+1}B)}{\omega(2^{k+1}B)^{\frac{1}{p_2}}} \right\} \\
&\quad + |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
&\quad \times \left\{ \sum_{k=1}^{\infty} \frac{|(b_2)_{2B} - (b_2)_{2^{k+1}B}|}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |f_2(y_2)| d\mu(y_2) \right\}
\end{aligned}$$

$$\begin{aligned}
& + \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |b_2(y_2) - (b_2)_{2^{k+1}B}| |f_2(y_2)| d\mu(y_2) \Big\} \\
& \leq |b_1(x) - (b_1)_{2B}| |b_2(x) - (b_2)_{2B}| \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \\
& \quad + |b_1(x) - (b_1)_{2B}| \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \\
& \quad \times \left\{ \sum_{k=1}^{\infty} (k+1) \|b_2\|_{BMO(\mu)} \left[\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right]^{\frac{1}{p_2}} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \right\} \\
& \quad + |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
& \quad \times \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \\
& \quad \times \left(\int_{2^{k+1}B} |b_2(y_2) - (b_2)_{2^{k+1}B}|^{p'_2} [\omega(y_2)]^{-\frac{p'_2}{p_2}} d\mu(y_2) \right)^{\frac{1}{p'_2}} \\
& \leq |b_1(x) - (b_1)_{2B}| |b_2(x) - (b_2)_{2B}| \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \\
& \quad + |b_1(x) - (b_1)_{2B}| \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \\
& \quad \times \left\{ \sum_{k=1}^{\infty} (k+1) \|b_2\|_{BMO(\mu)} \left[\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right]^{\frac{1}{p_2}} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \right\} \\
& \quad + |b_1(x) - (b_1)_{2B}| \phi(\omega(2B))^{\frac{1}{p_1}} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \frac{\mu(2B)}{\omega(2B)^{\frac{1}{p_1}}} \\
& \quad \times \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \|b_2\|_{BMO(\mu)} \frac{\mu(2^{k+1}B)}{[\omega(2^{k+1}B)]^{\frac{1}{p_2}}} \\
& \lesssim |b_1(x) - (b_1)_{2B}| |b_2(x) - (b_2)_{2B}| \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\
& \quad + |b_1(x) - (b_1)_{2B}| \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}.
\end{aligned}$$

As for the term J_{22} , by applying Hölder's inequality and lemma 2.3, we have

$$\begin{aligned}
J_{22} & = \int_{2B} |b_1(y_1) - (b_1)_{2B}| |f_1(y_1)| d\mu(y_1) \left(\int_{\mathcal{X} \setminus 2B} \frac{|b_2(x) - b_2(y_2)|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \\
& \leq \left(\int_{2B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \left(\int_{2B} |b_1(y_1) - (b_1)_{2B}|^{p'_1} [\omega(y_1)]^{-\frac{p'_1}{p_1}} d\mu(y_1) \right)^{\frac{1}{p'_1}}
\end{aligned}$$

$$\begin{aligned}
& \times \left\{ |b_2(x) - (b_2)_{2B}| \int_{\mathcal{X} \setminus 2B} \frac{|f_2(y_2)|}{[V(x, y_2)]^2} d\mu(y_2) \right. \\
& \quad \left. + \int_{\mathcal{X} \setminus 2B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right\} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \frac{\mu(2B)}{[\omega(2B)]^{\frac{1}{p_1}}} \\
& \quad \times \left\{ |b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_2(y_2)|}{[V(x, y_2)]^2} d\mu(y_2) \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right\} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \frac{\mu(2B)}{[\omega(2B)]^{\frac{1}{p_1}}} \\
& \quad \times \left\{ |b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |f_2(y_2)| d\mu(y_2) \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |b_2(y_2) - (b_2)_{2B}| |f_2(y_2)| d\mu(y_2) \right\} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \frac{\mu(2B)}{[\omega(2B)]^{\frac{1}{p_1}}} \\
& \quad \times \left\{ |b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \frac{\mu(2^{k+1}B)}{[\omega(2^{k+1}B)]^{\frac{1}{p_2}}} \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \frac{|(b_2)_{2B} - (b_2)_{2^{k+1}B}|}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \frac{\mu(2^{k+1}B)}{[\omega(2^{k+1}B)]^{\frac{1}{p_2}}} \right. \\
& \quad \left. + \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \right. \\
& \quad \left. \times \left(\int_{2^{k+1}B} |b_2(x) - (b_2)_{2^{k+1}B}|^{p'_2} [\omega(y_2)]^{-\frac{p'_2}{p_2}} d\mu(y_2) \right)^{\frac{1}{p'_2}} \right\} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} [\phi(\omega(2B))]^{\frac{1}{p_1}} \frac{\mu(2B)}{[\omega(2B)]^{\frac{1}{p_1}}} \\
& \quad \times \left\{ \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} |b_2(x) - (b_2)_{2B}| \sum_{k=1}^{\infty} \left[\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right]^{\frac{1}{p_2}} \right. \\
& \quad \left. + \|b_2\|_{BMO(\mu)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sum_{k=1}^{\infty} (k+1) \left[\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right]^{\frac{1}{p_2}} \right\}
\end{aligned}$$

$$\lesssim (|b_2(x) - (b_2)_{2B}| + \|b_2\|_{BMO(\mu)}) \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}},$$

which, together with the estimates of J_{21} and lemma 2.4, we can deduce that

$$\begin{aligned} & \| [b_1, b_2, T_\theta](f_1^0, f_2^\infty) \|_{\mathcal{M}^{p, \phi}(\omega)} \\ &= \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |[b_1, b_2, T_\theta](f_1^0, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\ &\lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\omega(B)}{\phi(\omega(B))} \right]^{\frac{1}{p}} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\ &\quad + \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\ &\quad \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_1(x) - (b_1)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\ &\quad + \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\ &\quad \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_2(x) - (b_2)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\ &\quad + \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\ &\quad \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_1(x) - (b_1)_{2B}|^p |b_2(x) - (b_2)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\ &\lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \\ &\quad + \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sup_B \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \\ &\quad \times \left(\int_B |b_1(x) - (b_1)_{2B}|^{p_1} \omega(x) d\mu(x) \right)^{\frac{1}{p_1}} \left(\int_B |b_2(x) - (b_2)_{2B}|^{p_2} \omega(x) d\mu(x) \right)^{\frac{1}{p_2}} \\ &\lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)}. \end{aligned}$$

Similarly, we get

$$J_3 \lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)}.$$

For any $x \in B$, we write

$$\begin{aligned} & |[b_1, b_2, T_\theta](f_1^\infty, f_2^\infty)(x)| \\ &\lesssim \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|b_1(x) - b_1(y_1)| |b_2(x) - b_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \end{aligned}$$

$$\begin{aligned}
&\lesssim |b_1(x) - (b_1)_{2B}| |b_2(x) - (b_2)_{2B}| \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|f_1(y_1)| |f_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} d\mu(y_1) d\mu(y_2) \\
&\quad + |b_1(x) - (b_1)_{2B}| \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[\sum_{j=1}^2 V(x, y_j)]^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\quad + |b_2(x) - (b_2)_{2B}| \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|b_1(y_1) - (b_1)_{2B}|}{[\sum_{j=1}^2 V(x, y_j)]^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&\quad + \int_{\mathcal{X} \setminus 2B} \int_{\mathcal{X} \setminus 2B} \frac{|(b_1)_{2B} - b_1(y_1)| |(b_2)_{2B} - b_2(y_2)|}{[\sum_{j=1}^2 V(x, y_j)]^2} |f_1(y_1)| |f_2(y_2)| d\mu(y_1) d\mu(y_2) \\
&:= J_{41} + J_{42} + J_{43} + J_{44}.
\end{aligned}$$

By applying Hölder's inequality and Lemma 2.3, we deduce that

$$\begin{aligned}
J_{41} &= \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| \right. \\
&\quad \times \left. \left(\sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f_2(y_2)|}{[V(x, y_1) + V(x, y_2)]^2} d\mu(y_2) \right) d\mu(y_1) \right\} \\
&= \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| \right. \\
&\quad \times \left. \left(\sum_{j=1}^k \int_{2^{j+1}B \setminus 2^j B} \frac{|f_2(y_2)|}{[V(x, y_1) + V(x, y_2)]^2} d\mu(y_2) \right. \right. \\
&\quad \left. \left. + \sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f_2(y_2)|}{[V(x, y_1) + V(x, y_2)]^2} d\mu(y_2) \right) d\mu(y_1) \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_1(y_1)|}{[V(x, y_1)]^2} \left(\sum_{j=1}^k \int_{2^{j+1}B \setminus 2^j B} |f_2(y_2)| d\mu(y_2) \right) \right. \\
&\quad \left. + \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| \left(\sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f_2(y_2)|}{[V(x, y_2)]^2} d\mu(y_2) \right) d\mu(y_1) \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_1(y_1)|}{[V(x, y_1)]^2} \left(\int_{2^{k+1}B} |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \right. \\
&\quad \left. + \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|f_2(y_2)|}{[V(x, y_2)]^2} \left(\sum_{k=1}^{j-1} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| d\mu(y_1) \right) d\mu(y_2) \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |f_1(y_1)| \left(\int_{2^{k+1}B} |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \right. \\
&\quad \left. + \sum_{j=1}^{\infty} \frac{1}{[\mu(2^j B)]^2} \int_{2^{j+1}B} |f_2(y_2)| \left(\int_{2^j B} |f_1(y_1)| d\mu(y_1) \right) d\mu(y_2) \right\}
\end{aligned}$$

$$\begin{aligned}
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \int_{2^{k+1}B} |f_i(y_i)| d\mu(y_i) \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \left\{ \sum_{k=1}^{\infty} \left(\int_{2^{k+1}B} |f_i(y_i)|^{p_i} \omega(y_i) d\mu(y_i) \right)^{\frac{1}{p_i}} \frac{1}{[\omega(2^{k+1}B)]^{\frac{1}{p_i}}} \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \|f_i\|_{\mathcal{M}^{p_i, \phi}(\omega)} \left\{ \sum_{k=1}^{\infty} \left[\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right]^{\frac{1}{p_i}} \right\} \\
&\lesssim \prod_{i=1}^2 |b_i(x) - (b_i)_{2B}| \|f_i\|_{\mathcal{M}^{p_i, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p_i}}.
\end{aligned}$$

For any $x \in B$, by Definition 1.1, Hölder's inequality and Lemma 2.3, we can get

$$\begin{aligned}
J_{42} &= |b_1(x) - (b_1)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| \right. \\
&\quad \times \left. \left(\sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[V(x, y_1) + V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \right\} \\
&\leq |b_1(x) - (b_1)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_1(y_1)|}{[V(x, y_1)]^2} d\mu(y_1) \right. \\
&\quad \times \left. \left(\sum_{j=1}^k \int_{2^{j+1}B \setminus 2^j B} |b_2(y_2) - (b_2)_{2B}| |f_2(y_2)| d\mu(y_2) \right) \right\} \\
&\quad + |b_1(x) - (b_1)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| d\mu(y_1) \right. \\
&\quad \times \left. \left(\sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) \right\} \\
&\leq |b_1(x) - (b_1)_{2B}| \left\{ \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|f_1(y_1)|}{[V(x, y_1)]^2} d\mu(y_1) \right. \\
&\quad \times \left. \left(\int_{2^{k+1}B} |b_2(y_2) - (b_2)_{2B}| |f_2(y_2)| d\mu(y_2) \right) \right\} \\
&\quad + |b_1(x) - (b_1)_{2B}| \left\{ \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|b_2(y_2) - (b_2)_{2B}|}{[V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right. \\
&\quad \times \left. \left(\sum_{k=1}^{j-1} \int_{2^{k+1}B \setminus 2^k B} |f_1(y_1)| d\mu(y_1) \right) \right\} \\
&\leq |b_1(x) - (b_1)_{2B}| \left\{ \sum_{k=1}^{\infty} \frac{1}{[\mu(2^k B)]^2} \left(\int_{2^{k+1}B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \frac{\mu(2^{k+1}B)}{[\omega(2^{k+1}B)]^{\frac{1}{p_1}}} \right\}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{2^{k+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \left(\int_{2^{k+1}B} |b_2(y_2) - (b_2)_{2B}|^{p'_2} [\omega(y_2)]^{-\frac{p'_2}{p_2}} d\mu(y_2) \right)^{\frac{1}{p'_2}} \Big\} \\
& + |b_1(x) - (b_1)_{2B}| \left\{ \sum_{j=1}^{\infty} \frac{1}{[\mu(2^j B)]^2} \left(\int_{2^{j+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \right. \\
& \quad \times \left. \left(\int_{2^{j+1}B} |b_2(y_2) - (b_2)_{2B}|^{p'_2} [\omega(y_2)]^{-\frac{p'_2}{p_2}} d\mu(y_2) \right)^{\frac{1}{p'_2}} \right. \\
& \quad \times \left. \left(\int_{2^j B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \frac{\mu(2^j B)}{[\omega(2^j B)]^{\frac{1}{p_1}}} \right\} \\
& \lesssim |b_1(x) - (b_1)_{2B}| \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}.
\end{aligned}$$

Similarty, we also get

$$J_{43} \lesssim |b_2(x) - (b_2)_{2B}| \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}.$$

For the last term J_{44} , by Hölder's inequality and Lemma 2.3, it follows that

$$\begin{aligned}
J_{44} &= \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |(b_1)_{2B} - b_1(y_1)| |f_1(y_1)| \\
&\quad \times \left(\sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|(b_2)_{2B} - b_2(y_2)|}{[V(x, y_1) + V(x, y_2)]^2} |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \\
&\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|(b_1)_{2B} - b_1(y_1)| |f_1(y_1)|}{[V(x, y_1)]^2} \\
&\quad \times \left(\sum_{j=1}^k \int_{2^{j+1}B \setminus 2^j B} |(b_2)_{2B} - b_2(y_2)| |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \\
&\quad + \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} |(b_1)_{2B} - b_1(y_1)| |f_1(y_1)| \\
&\quad \times \left(\sum_{j=k+1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|(b_2)_{2B} - b_2(y_2)| |f_2(y_2)|}{[V(x, y_2)]^2} d\mu(y_2) \right) d\mu(y_1) \\
&\leq \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^k B} \frac{|(b_1)_{2B} - b_1(y_1)| |f_1(y_1)|}{[V(x, y_1)]^2} \\
&\quad \times \left(\int_{2^{k+1}B} |(b_2)_{2B} - b_2(y_2)| |f_2(y_2)| d\mu(y_2) \right) d\mu(y_1) \\
&\quad + \sum_{j=1}^{\infty} \int_{2^{j+1}B \setminus 2^j B} \frac{|(b_2)_{2B} - b_2(y_2)| |f_2(y_2)|}{[V(x, y_2)]^2}
\end{aligned}$$

$$\begin{aligned}
& \times \left(\int_{2^{j+1}B} |(b_1)_{2B} - b_1(y_1)| |f_1(y_1)| d\mu(y_1) \right) d\mu(y_2) \\
& \lesssim \|b_2\|_{BMO(\mu)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sum_{k=1}^{\infty} \frac{k}{\mu(2^k B)} \int_{2^{k+1}B} |(b_1)_{2B} - b_1(y_1)| |f_1(y_1)| d\mu(y_1) \\
& \quad \times \left(\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right)^{\frac{1}{p_2}} \\
& \quad + C \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \sum_{j=1}^{\infty} \frac{j}{\mu(2^j B)} \int_{2^{j+1}B} |(b_2)_{2B} - b_2(y_2)| |f_2(y_2)| d\mu(y_2) \\
& \quad \times \left(\frac{\phi(\omega(2^{j+1}B))}{\omega(2^{j+1}B)} \right)^{\frac{1}{p_1}} \\
& \lesssim \|b_2\|_{BMO(\mu)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sum_{k=1}^{\infty} \frac{k}{\mu(2^k B)} \left(\int_{2^{k+1}B} |f_1(y_1)|^{p_1} \omega(y_1) d\mu(y_1) \right)^{\frac{1}{p_1}} \\
& \quad \times \left(\int_{2^{k+1}B} |(b_1)_{2B} - b_1(y_1)|^{p'_1} [\omega(y_1)]^{-\frac{p'_1}{p_1}} \right)^{\frac{1}{p'_1}} \left(\frac{\phi(\omega(2^{k+1}B))}{\omega(2^{k+1}B)} \right)^{\frac{1}{p_2}} \\
& \quad + C \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \sum_{j=1}^{\infty} \frac{j}{\mu(2^j B)} \left(\int_{2^{j+1}B} |f_2(y_2)|^{p_2} \omega(y_2) d\mu(y_2) \right)^{\frac{1}{p_2}} \\
& \quad \times \left(\int_{2^{j+1}B} |(b_2)_{2B} - b_2(y_2)|^{p'_2} [\omega(y_2)]^{-\frac{p'_2}{p_2}} \right)^{\frac{1}{p'_2}} \left(\frac{\phi(\omega(2^{j+1}B))}{\omega(2^{j+1}B)} \right)^{\frac{1}{p_1}} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}},
\end{aligned}$$

which, combining the estimates of J_{41} , J_{42} and J_{43} , Lemma 2.4, implies that

$$\begin{aligned}
& \| [b_1, b_2, T_\theta](f_1^\infty, f_2^\infty) \|_{\mathcal{M}^{p, \phi}(\omega)} \\
& = \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |[b_1, b_2, T_\theta](f_1^\infty, f_2^\infty)(x)|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\omega(B)}{\phi(\omega(B))} \right]^{\frac{1}{p}} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\
& \quad + \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\
& \quad \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_1(x) - (b_1)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\
& \quad + \|b_1\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}}
\end{aligned}$$

$$\begin{aligned}
& \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_2(x) - (b_2)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\
& + \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \\
& \times \sup_B \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \left(\int_B |b_1(x) - (b_1)_{2B}|^p |b_2(x) - (b_2)_{2B}|^p \omega(x) d\mu(x) \right)^{\frac{1}{p}} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \\
& + \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)} \sup_B \left[\frac{\phi(\omega(B))}{\omega(B)} \right]^{\frac{1}{p}} \frac{1}{\phi(\omega(B))^{\frac{1}{p}}} \\
& \times \left(\int_B |b_1(x) - (b_1)_{2B}|^{p_1} \omega(x) d\mu(x) \right)^{\frac{1}{p_1}} \left(\int_B |b_2(x) - (b_2)_{2B}|^{p_2} \omega(x) d\mu(x) \right)^{\frac{1}{p_2}} \\
& \lesssim \|b_1\|_{BMO(\mu)} \|b_2\|_{BMO(\mu)} \|f_1\|_{\mathcal{M}^{p_1, \phi}(\omega)} \|f_2\|_{\mathcal{M}^{p_2, \phi}(\omega)}.
\end{aligned}$$

Hence, the proof of the Theorem 1.9 is completed.

Conflict of interest

The authors declare that there is no conflict of interests.

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