THE ANSWER TO BAGGETT'S PROBLEM IS AFFIRMATIVE

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Dedicated to Yuan Qing Lan

Abstract

Let ψ be a Parceval wavelet in $L^2(\mathbb{R})$ with the space of negative dilates $V(\psi)$. The intersection of the dilates $V(\psi)$ is the zero space. In other words, we have

$$\bigcap_{n \in \mathbb{Z}} D^n \overline{\operatorname{span}} \{ D^{-m} T^{\ell} \psi \mid m \ge 0, m, \ell \in \mathbb{Z} \} = \{ 0 \}.$$

1. Introduction

Denote $L^2(\mathbb{R})$ as \mathbb{H} . Let $B(\mathbb{H})$ denote the space of bounded linear operators acting on \mathbb{H} . Let T, D and \mathcal{F} be the translation, dilation and Fourier transform operators defined as follows. For $f \in \mathbb{H}$,

$$\begin{aligned} (Tf)(t) &= f(t-1).\\ (Df)(t) &= \sqrt{2}f(2t).\\ (\mathcal{F}f)(s) &= \widehat{f}(s) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}e^{-its}f(t)dt \end{aligned}$$

The operators T, D and \mathcal{F} are unitary operators. We have $(\mathcal{F}^{-1}f)(t) = \check{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ist} f(t) dt$. For A in $B(\mathbb{H})$, let $\widehat{A} \equiv \mathcal{F} A \mathcal{F}^{-1}$. We have $\mathcal{F} A = \mathcal{F} A \mathcal{F}^{-1} \mathcal{F} = \widehat{A}\mathcal{F}$. The operator \widehat{T} is the multiplication operator $M_{e^{-is}}$ which maps f(s) to $e^{-is}f(s), f \in \mathbb{H}$. Also $\widehat{D}^n = D^{-n}, n \in \mathbb{Z}$. see [4]

A set $\{\vec{x}_n \mid n \in \mathbb{J}\}$ in \mathbb{H} is a normalized tight frame for \mathbb{H} if for each $\vec{x} \in \mathbb{H}$

$$\|\vec{x}\|^2 = \sum_{n \in \mathbb{J}} |\langle \vec{x}, \vec{x}_n \rangle|^2$$

Here the index set \mathbb{J} is countable infinite. An orthonormal basis for \mathbb{H} is a normalized tight frame for \mathbb{H} , and not vice versa.

A function $\psi \in L^2(\mathbb{R})$ is called a *Parseval wavelet* if the set $\{D^m T^\ell \psi \mid (m, \ell) \in \mathbb{Z}^2\}$ forms a normalized tight frame for $L^2(\mathbb{R})$. In addition, if the set $\{D^n T^\ell \psi \mid (n, \ell) \in \mathbb{Z}^2\}$ is orthogonal, then ψ must be a unit vector and it is an orthonormal wavelet. For $\vec{x} \in L^2(\mathbb{R})$, we will use notation $V(\vec{x})$ as

$$V(\vec{x}) \equiv \overline{\operatorname{span}}\{D^{-m}T^{\ell}x \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z}\}.$$

Some authors called $V(\vec{x})$ the space of negative dilates. [3]

In 1999 Larry Baggett asked the following question.

QUESTION 1.1. (Baggett, 1999) Let $\psi \in L^2(\mathbb{R})$ be a Parseval wavelet. Is the following Equation (1.1) holds?

(1.1)
$$\bigcap_{n \in \mathbb{Z}} D^n \left(V(\psi) \right) = \{ 0 \}$$

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An affirmative answer to the Question 1.1 implies that every Parseval wavelet is associated with the general multiresolution analysis (GMRA). The concept of GMRA is introduced by Baggett, Medina and Merril [1] as a natural generalization to the concept multiresolution analysis (MRA). The question 1.1 is posted as an open question in [2].

In this paper we prove that the answer to Baggett's question is "Yes". Our reasoning is for $L^2(\mathbb{R})$ case. However, the idea for the proof works for the general case $L^2(\mathbb{R}^d)$.

2. Proof

A sequence $\{\vec{x}_n \mid n \in \mathbb{J}\}$ in a (separable) Hilbert space (Banach space) X is called a Schauder basis of X if for every $\vec{x} \in X$ there is a unique sequence of scalars $\{a_n \mid n \in \mathbb{J}\}$ so that

$$\vec{x} = \sum_{n \in \mathbb{J}} a_n \vec{x}_n.$$

The convergence is in the norm of X. We will call the set of numbers $\{a_n \mid n \in \mathbb{J}\}$ the basis coefficients associated with \vec{x} . In addition, if for an arbitrary permutation π of \mathbb{J} we have

$$\vec{x} = \sum_{n \in \mathbb{J}} a_{\pi(n)} \vec{x}_{\pi(n)},$$

we will call the above Schauder basis an *unconditional basis*. An orthonormal basis of a Hilbert space is an unconditional basis. Let ι be a isomorphism (hence continuous, by the Open mapping theorem) from a Hilbert space X onto a Hilbert space Y and let $\{\vec{x}_n \mid n \in \mathbb{J}\}$ be an orthonormal basis of X. Since the isomorphism ι maps a cauchy sequence in X to a cauchy sequence in Y, the sequence $\{\vec{y}_n \equiv \iota(\vec{x}_n) \mid n \in \mathbb{J}\}$ is an unconditional basis in Y. Let $\{\vec{x}_n \mid n \in \mathbb{J}\}$ be a Schauder basis of a Hilbert space X. Then there exist corresponding linear functionals $\vec{x}_n^*, n \in \mathbb{J}$ in X* so that

$$x_n^*(\vec{x}_m) = \delta_{n,m}, n, m \in \mathbb{J}.$$

The notation δ is the Kronecker delta. Let $\mathbb{H} = L^2(\mathbb{R})$ with an orthonormal wavelet η . We will view the orthonormal basis $\{D^j T^\ell \eta \mid (j, \ell) \in \mathbb{Z}^2\}$, since it is an unconditional basis as $\{\vec{e}_n \mid n \in \mathbb{J}\}$ for $\mathbb{J} = \mathbb{Z}^2$ in one stream. For the basis, we refer [7] to the reader.

We will need the followsing Lemma 2.1 by Han and Larson.

LEMMA 2.1. (Han,Larson [6]) Let $\{\vec{x}_n \mid n \in \mathbb{J}\}$ be a normalized tight frame for \mathbb{H} . Then there exists a Hilbert space \mathbb{M} with a normalized tight frame $\{\vec{m}_n \mid n \in \mathbb{J}\}$ for \mathbb{M} such that the set $\{\vec{m}_n \oplus \vec{x}_n \mid n \in \mathbb{J}\}$ forms an orthonormal basis for $\mathbb{M} \oplus \mathbb{H}$.

In above Lemma 2.1 if the set $\{\vec{x}_n \mid n \in \mathbb{J}\}\$ is an orthonormal basis for \mathbb{H} , then \mathbb{M} is a 0 space and $\{\vec{m}_n \mid n \in \mathbb{J}\}\$ is the set of zero vectors.

Let ψ be a given Parseval wavelet for $\mathbb{H} = L^2(\mathbb{R})$. By Lemma 2.1 there exists a separable Hilbert space \mathbb{M} with a normalozed tight frame $\{\vec{m}_{n,\ell} \mid (n,\ell) \in \mathbb{Z}^2\}$ such that the set

$$\{\vec{m}_{n,\ell} \oplus D^n T^\ell \psi \mid (n,\ell) \in \mathbb{Z}^2\}$$

forms an orthonormal basis for $\mathbb{M} \oplus \mathbb{H}$. Denote $\vec{e}_{n,\ell} \equiv \vec{m}_{n,\ell} \oplus D^n T^{\ell} \psi$. Let $\vec{x} \in \mathbb{H}$. We have

$$\begin{split} 0 \oplus \vec{x} &= \sum_{n,\ell \in \mathbb{Z}} \langle 0 \oplus \vec{x}, \vec{m}_{n,\ell} \oplus D^n T^\ell \psi \rangle \vec{e}_{n,\ell} \\ &= \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \psi \rangle \; (0 \oplus D^n T^\ell \psi) \\ &= 0 \oplus \left(\sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \psi \rangle D^n T^\ell \psi \right) \end{split}$$

So, we have the well known equation,

(2.1)
$$\vec{x} = \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \psi \rangle D^n T^\ell \psi, \forall \vec{x} \in \mathbb{H}.$$

This is equivalent to the definition of the Parceval wavelet.

Let η be the function defined as

$$\widehat{\eta} = \frac{1}{\sqrt{2\pi}} \chi_{[-2\pi, -\pi] \cup [\pi, 2\pi]},$$

where χ is the characteristic function. It is well known that the set $\{D^n T^\ell \eta \mid (n,\ell) \in \mathbb{Z}^2\}$ is an orthonormal basis of $L^2(\mathbb{R})$. The function η is the Littlewood-Paley wavelet. An element $\vec{x} \in L^2(\mathbb{R})$ is in the form $\vec{x} = \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \eta$. Define a mapping $U : \mathbb{H} \to \mathbb{M} \oplus \mathbb{H}$ as

$$U\vec{x} = U\left(\sum_{n,\ell\in\mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \eta\right) \equiv \sum_{n,\ell\in\mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle \vec{e}_{n,\ell}$$
$$= \sum_{n,\ell\in\mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle (\vec{m}_{n,\ell} \oplus D^n T^\ell \psi).$$

The operator U maps the orthonormal basis $\{D^n T^\ell \eta \mid (n,\ell) \in \mathbb{Z}^2\}$ of \mathbb{H} to the orthonormal basis $\{\vec{e}_{n,\ell} \mid (n,\ell) \in \mathbb{Z}^2\} = \{\vec{m}_{n,\ell} \oplus D^n T^\ell \psi \mid (n,\ell) \in \mathbb{Z}^2\}$ of $\mathbb{M} \oplus \mathbb{H}$. This is a unitary operator. Let P be the orthogonal projection from $\mathbb{M} \oplus \mathbb{H}$ to the subspace $0 \oplus \mathbb{H}$. Let I_0 denote the mapping sending $0 \oplus f$ in $0 \oplus \mathbb{H}$ to f in \mathbb{H} . Define the operator Ξ as

(2.2)
$$\Xi = I_0 P U.$$

The operator Ξ is a bounded linear operator, i.e $\Xi \in B(\mathbb{H})$. We have

$$\begin{split} \Xi(\vec{x}) &= \Xi \left(\sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \eta \right) \\ &= I_0 P \left(U \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \eta \right) \\ &= I_0 \left(P \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle \; (\vec{m}_{n,\ell} \oplus D^n T^\ell \psi) \right) \end{split}$$

$$\begin{split} &= I_0 \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle \ (0 \oplus D^n T^\ell \psi) \\ &= \sum_{n,\ell \in \mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \psi. \end{split}$$

We obtain

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(2.3)
$$\Xi\left(\sum_{n,\ell\in\mathbb{Z}}\langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \eta\right) = \sum_{n,\ell\in\mathbb{Z}}\langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \psi.$$

By Equation (2.3) when $\vec{x} = \eta$, we have $\Xi(\eta) = \psi$. When $\vec{x} = D^n T^{\ell} \eta$, we have

(2.4)
$$\Xi(D^n T^\ell \eta) = D^n T^\ell \psi = D^n T^\ell \Xi(\eta), \forall (n,\ell) \in \mathbb{Z}^2 .$$

The operator Ξ commute with the unitary system $\{D^nT^\ell \mid (n,\ell) \in \mathbb{Z}^2\}$ at the point η . The collection of all operator with this properties is called the *point commutant* at η , which is denoted as $C_{\eta}(D,T)$ in [4]. $\Xi \in C_{\eta}(D,T)$. Also $(\Xi D)D^{n}T^{\ell}\eta = \Xi D^{n+1}T^{\ell}\eta = D^{n+1}T^{\ell}\Xi\eta = (D\Xi)D^{n}T^{\ell}\eta$. Since $\{D^{n}T^{\ell}\eta\}$ is an orthonormal basis, so we have

(2.5)
$$D\Xi = \Xi D \text{ and } D^{-1}\Xi = \Xi D^{-1}$$

Let \vec{y} be an arbitrarily given element in \mathbb{H} . Let $\vec{x} = \Xi^* \vec{y}$. Then

$$\begin{split} \Xi(\vec{x}) &= \sum_{n,\ell\in\mathbb{Z}} \langle \vec{x}, D^n T^\ell \eta \rangle D^n T^\ell \psi \\ &= \sum_{n,\ell\in\mathbb{Z}} \langle \Xi^* \vec{y}, D^n T^\ell \eta \rangle D^n T^\ell \psi \\ &= \sum_{n,\ell\in\mathbb{Z}} \langle \vec{y}, \Xi D^n T^\ell \eta \rangle D^n T^\ell \psi \\ &= \sum_{n,\ell\in\mathbb{Z}} \langle \vec{y}, D^n T^\ell \psi \rangle D^n T^\ell \psi \\ &= \vec{y}, \end{split}$$

by Equation (2.1). So Ξ is surjective.

$$(2.6) \qquad \qquad \Xi(\mathbb{H}) = \mathbb{H}.$$

By the Open mapping theorem the operator Ξ is an open mapping.

Denote the kernel of Ξ as N and denote the orthogonal projection to N as Q. Denote the orthogonal complement of N as $\mathbb{K} \equiv N^{\perp}$ and denote the projection to \mathbb{K} as Q^{\perp} . It is clear that $Q^2 = Q$ and $(Q^{\perp})^2 = Q^{\perp}$. We have $\mathbb{K} = N^{\perp} =$ $Q^{\perp}\mathbb{H} = Q^{\perp}L^2(\mathbb{R})$. By the Open mapping theorem, the operator $\Xi_{\mid\mathbb{K}}$ is a continuous isomorphism from \mathbb{K} onto \mathbb{H} . We denote $\Xi_{|\mathbb{K}}$ as ι and denote its inverse as κ .

$$\mathbb{K} = Q^{\perp} \mathbb{H}.$$
$$\mathbb{H} = N \oplus \mathbb{K} = N \oplus Q^{\perp} \mathbb{H}.$$
$$\iota = \Xi_{|\mathbb{K}} : \mathbb{K} \to \mathbb{H}.$$
$$\kappa = \iota^{-1} : \mathbb{H} \to \mathbb{K}.$$

$$W_n = \overline{\operatorname{span}} \{ D^n T^\ell \eta \mid \ell \in \mathbb{Z} \} = D^n W_0.$$

Since η is an orthogonal wavelet, the subspaces $\{W_n \mid n \in \mathbb{Z}\}$ are mutually orthogonal to each other. Let

$$V_n = \overline{\operatorname{span}} \{ D^j T^\ell \eta \mid j \in \mathbb{Z}, j \le n; \ell \in \mathbb{Z} \}.$$

We have

$$V_0 = V(\eta) = \bigoplus_{j \le 0, j \in \mathbb{Z}} W_j,$$
$$V_n = D^n V_0 = \bigoplus_{j \le n, j \in \mathbb{Z}} W_j,$$

and

$$\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$$

 N,\mathbb{K} are subspaces of $\mathbb{H}.$ In Lemma 2.2 we will discuss relations of D with N,N^{\perp},Q,ι and $\kappa.$

LEMMA 2.2. (1)
$$DN = N$$
.
(2) $DN^{\perp} = N^{\perp}$ or $D\mathbb{K} = \mathbb{K}$.
(3) $DQ = QD$.
(4) $DQ^{\perp} = Q^{\perp}D$.
(5) $D\iota = \iota D$.
(6) $D\kappa = \kappa D$.

PROOF. (1). Let $\vec{x} \in N$. Then $\Xi(D\vec{x}) = D(\Xi\vec{x}) = 0$, and $\Xi(D^{-1}\vec{x}) = D^{-1}(\Xi\vec{x}) = 0$. So $D\vec{x} \in N$ and $D^{-1}\vec{x} \in N$. We have DN = N.

(2) Let $\vec{y} \in \mathbb{K} \subset \mathbb{H}$. Then $\vec{y} \in \mathbb{K}$ iff $\vec{y} \perp N$. Consider $D\vec{y}$ and $\vec{x} \in N$,

$$\langle D\vec{y}, \vec{x} \rangle = \langle \vec{y}, D^*\vec{x} \rangle = \langle \vec{y}, D^{-1}\vec{x} \rangle = 0,$$

since $\vec{y} \in \mathbb{K}$ by assumption and $D^{-1}\vec{x} \in N$ by (1). So $D\vec{y} \perp N$, or $D\vec{y} \in \mathbb{K}$, $D\mathbb{K} \subset \mathbb{K}$. When we replace $D\vec{y}$ by $D^{-1}\vec{y} (= D^*\vec{y})$, the above reasoning will show that $D^{-1}\mathbb{K} \subset \mathbb{K}$, which equivalent to $\mathbb{K} \subset D\mathbb{K}$. So we have $D\mathbb{K} = \mathbb{K}$.

(3) Let $\vec{x} \in \mathbb{H}$. Then $\vec{x} = f + g$ for $f = Q\vec{x}$ and $g = Q^{\perp}\vec{x}$. So Qg = 0, Qf = f. So

$$DQ\vec{x} = Df.$$

Notice that $Df \in DN = N$ and $Dg \in DN^{\perp} = N^{\perp} = \mathbb{K}$. We have QDg = 0 and QDf = Df.

$$QD\vec{x} = Q\left(Df + Dg\right) = QDf = Df = DQ\vec{x}.$$

So, DQ = QD.

(4) Similar as (3).

(5) By (4)

$$D\iota = D\Xi Q^{\perp} = \Xi D Q^{\perp} = \Xi Q^{\perp} D = \iota D.$$

(6) By (5) we have $D\iota = \iota D$. This is true iff $(D\iota)^{-1} = (\iota D)^{-1}$ iff $\iota^{-1}D^{-1} = D^{-1}\iota^{-1}$ iff $D\kappa = \kappa D$.

Consider the subspace W_0 of \mathbb{H} . It is infinite dimensional. We have

 $W_0 = QW_0 \oplus Q^\perp W_0.$

Lemma 2.3.

(2.7)
$$Q^{\perp}W_0 \neq \{0\}.$$

PROOF. We prove by contradiction. Assume $Q^{\perp}W_0 = \{0\}$. This implies that $W_0 \subset QW_0 \subset N.$

So, for $n \in \mathbb{Z}$,

$$W_n = D^n W_0 \subset D^n Q W_0 \subset Q D^n W_0 = Q W_n \subset N.$$

This implies that $\mathbb{H} \subset N$, or

$$\Xi(\mathbb{H}) = \{0\}.$$

A contradiction to Equation (2.6).

Let

$$\{\vec{a}_{0,i} \mid i \in \mathbb{J}_1\} \text{ and } \{b_{0,i} \mid i \in \mathbb{J}_2\}$$

be orthonormal base for QW_0 and $Q^{\perp}W_0$, respectively. Here \mathbb{J}_1 and \mathbb{J}_2 are subset of \mathbb{N} , the counting numbers. By Lemma 2.3 $\mathbb{J}_2 \neq \emptyset$. Consider the disjoint union $\{\vec{a}_{0,i} \mid i \in \mathbb{J}_1\} \cup \{\vec{b}_{0,i} \mid i \in \mathbb{J}_2\}$. This is a countably infinite set since W_0 is infinite dimensional. Also, this is an orthonormal set. We reorder it and write it as

$$\{\vec{\lambda}_{0,i} \mid i \in \mathbb{N}\} = \{\vec{a}_{0,i} \mid i \in \mathbb{J}_1\} \cup \{\vec{b}_{0,i} \mid i \in \mathbb{J}_2\}.$$

For a point $\vec{x} \in W_0$,

$$\begin{split} \vec{x} &= Q \vec{x} + Q^{\perp} \vec{x} \\ &= \sum_{i \in \mathbb{J}_1} \langle Q \vec{x}, \vec{a}_{0,i} \rangle \vec{a}_{0,i} + \sum_{i \in \mathbb{J}_2} \langle Q^{\perp} \vec{x}, \vec{b}_{0,i} \rangle \vec{b}_{0,i} \\ &= \sum_{i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{0,i} \rangle \vec{a}_{0,i} + \sum_{i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{0,i} \rangle \vec{b}_{0,i}. \end{split}$$

Thus

$$\vec{x} = \sum_{i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{0,i} \rangle \vec{a}_{0,i} + \sum_{i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{0,i} \rangle \vec{b}_{0,i} = \sum_{i \in \mathbb{N}} \langle \vec{x}, \vec{\lambda}_{0,i} \rangle \vec{\lambda}_{0,i}.$$

So the set $\{\vec{\lambda}_{0,i} \mid i \in \mathbb{N}\}$ is an orthonormal basis for W_0 . This follows that the set $D^j\{\vec{\lambda}_{0,i} \mid i \in \mathbb{N}\}$ is an orthonormal basis for $W_j = D^j W_0$. Define

$$\vec{a}_{j,i} \equiv D^j \vec{a}_{0,i}, j \in \mathbb{Z}, i \in \mathbb{J}_1,$$
$$\vec{b}_{j,i} \equiv D^j \vec{b}_{0,i}, j \in \mathbb{Z}, i \in \mathbb{J}_2,$$
$$\vec{\lambda}_{j,i} \equiv D^j \vec{\lambda}_{0,i}, j \in \mathbb{Z}, i \in \mathbb{N}.$$

Define

$$\Lambda = \{ \vec{b}_{j,i} \mid j \in \mathbb{Z}, i \in \mathbb{J}_2 \}.$$
$$\Phi = \{ \vec{\lambda}_{j,i} \mid j \in \mathbb{Z}, i \in \mathbb{N} \}.$$

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It is clear that

$$\vec{a}_{j,i} \in QW_j, j \in \mathbb{Z}, i \in \mathbb{J}_1.$$
$$\vec{b}_{j,i} \in Q^{\perp}W_j, j \in \mathbb{Z}, i \in \mathbb{J}_2.$$
$$\vec{\lambda}_{j,i} \in W_j, j \in \mathbb{Z}, i \in \mathbb{N}.$$

Since D^j is a unitary operator, and the spaces W_j are mutually orthogonal and sum to \mathbb{H} , the set Φ is an orthonormal basis for \mathbb{H} . For a point $\vec{x} \in \mathbb{H}$, we have

(2.8)
$$\vec{x} = \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{j,i} \rangle \vec{a}_{j,i} + \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i} = \sum_{j \in \mathbb{Z}, i \in \mathbb{N}} \langle \vec{x}, \vec{\lambda}_{j,i} \rangle \vec{\lambda}_{j,i}.$$

Let $\vec{x} \in \mathbb{K}$, we have $\langle \vec{x}, \vec{a}_{j,i} \rangle = 0$. By the above Equation (2.8), we have

$$\vec{x} = \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i}, \forall \vec{x} \in \mathbb{K}$$

The set $\Lambda = \{ \vec{b}_{j,i} \mid j \in \mathbb{Z}, i \in \mathbb{J}_2 \}$ is an orthonormal basis for \mathbb{K} . It is also clear that

LEMMA 2.4. Let $\vec{x} \in V_n$. Then

(2.9)
$$\vec{x} = \sum_{j \le n, i \in \mathbb{N}} \langle \vec{x}, \vec{\lambda}_{j,i} \rangle \vec{\lambda}_{j,i} = \sum_{j \le n, i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{j,i} \rangle \vec{a}_{j,i} + \sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i}$$

Define

$$\sigma_{j,i} = \iota(\vec{b}_{j,i}), j \in \mathbb{Z}, i \in \mathbb{J}_2.$$

$$\Theta = \{\sigma_{j,i} \mid j \in \mathbb{Z}, i \in \mathbb{J}_2\}.$$

It is clear that

(2.10)
$$\Xi(\vec{b}_{j,i}) = \Xi_{|\mathbb{K}}(\vec{b}_{j,i}) = \iota(\vec{b}_{j,i}) = \sigma_{j,i}, j \in \mathbb{Z}, i \in \mathbb{J}_2$$

Notice that ι is an continuous isomorphism from \mathbb{K} to \mathbb{H} , the set $\Theta = \iota(\Lambda)$, and Λ is an orthonormal basis of \mathbb{K} . The set Θ is an unconditional basis for \mathbb{H} . An element $\vec{y} \in \mathbb{H}$ has the form

$$\vec{y} = \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_2} \beta_{j,i} \sigma_{j,i}.$$

LEMMA 2.5. Let $\vec{y} \in \Xi(V_n)$ Then \vec{y} has the form

(2.11)
$$\vec{y} = \sum_{j \le n, i \in \mathbb{J}_2} \beta_{j,i} \sigma_{j,i}$$

Other words, for the dual basis $\sigma_{j,i}^*$, we have

$$\sigma_{j,i}^*(\vec{y}) = 0, \forall j > n \text{ and } i \in \mathbb{J}_2.$$

PROOF. Assume $\vec{y} = \Xi(\vec{x})$ for some $\vec{x} \in V_n$. By Equation (2.9),

$$\vec{x} = \sum_{j \leq n, i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{j,i} \rangle \vec{a}_{j,i} + \sum_{j \leq n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i}.$$

Notice that the elements $\vec{a}_{j,i}, j \in \mathbb{Z}, i \in \mathbb{J}_1$ are in the kernel of Ξ ,

$$\vec{y} = \Xi \left(\sum_{j \le n, i \in \mathbb{J}_1} \langle \vec{x}, \vec{a}_{j,i} \rangle \vec{a}_{j,i} + \sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i} \right)$$

$$= \Xi \left(\sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \vec{b}_{j,i} \right)$$
$$= \sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \Xi \left(\vec{b}_{j,i} \right)$$
$$= \sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \sigma_{j,i},$$

by Equation (2.10). We have

$$\vec{y} = \sum_{j \le n, i \in \mathbb{J}_2} \langle \vec{x}, \vec{b}_{j,i} \rangle \ \sigma_{j,i}.$$

Since $\{\sigma_{j,i}\}$ is a unconditional basis for \mathbb{H} , the coefficients $\beta_{j,i}$ for \vec{y} are unique. $\beta_{j,i} = \langle \vec{x}, \vec{b}_{j,i} \rangle, j \leq n$. Also $\beta_{j,i} = 0$ when j > n.

Next we have

Lemma 2.6.

(2.12)
$$\bigcap_{n\in\mathbb{Z}} \Xi(V_n) = \{0\}.$$

PROOF. Let $\vec{y} \in \bigcap_{n \in \mathbb{Z}} \Xi(V_n)$. Then $y \in \Xi(V_n)$ for each $n \in \mathbb{Z}$. Notice that $\{\sigma_{j,i}\}$ is a Schauder basis for \mathbb{H} ,

$$\vec{y} = \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_2} \beta_{j,i} \sigma_{j,i},$$

the coefficients $\{\beta_{j,i}\}\$ are unique for \vec{y} . Let $\beta_{j,i}$ be one of the coefficient for some $j \in \mathbb{Z}, i \in \mathbb{J}_2$. Let n = j - 1. Since $\vec{y} \in \Xi(V_n)$, and j > n, by Lemma 2.5 $\beta_{j,i} = 0$. This implies that $\vec{y} = 0$.

We have

Lemma 2.7.

(2.13)
$$\overline{\Xi(V_0)} = \Xi(V_0)$$

PROOF. Let $\vec{y}_0 \in \mathbb{H} \setminus \Xi(V_0)$. We will show that \vec{y}_0 is an exterior point of $\Xi(V_0)$. It suffices to show that the distance from \vec{y}_0 to $\Xi(V_0)$ is positive. Notice that $\{\sigma_{j,i}\}$ is a Schauder basis for \mathbb{H} . Let $\{\sigma_{j,i}^*\}$ be the associated dual basis. We have

$$\vec{y}_0 = \sum_{j \in \mathbb{Z}, i \in \mathbb{J}_2} \gamma_{j,i} \sigma_{j,i},$$

for some coefficients $\gamma_{j,i}$. Since $\vec{y_0}$ in \mathbb{H} but not in $\Xi(V_0)$, $\gamma_{j_0,i_0} \neq 0$ for some $j_0 > 0, i_0 \in \mathbb{J}_2$. Let σ_{j_0,i_0}^* be element in the dual basis with index j_0, i_0 . Let \vec{y} be an element in $\Xi(V_0)$. By Lemma 2.5, $\sigma_{j_0,i_0}^*(\vec{y}) = 0$ for each $\vec{y} \in \Xi(V_0)$ but $\sigma_{j_0,i_0}^*(\vec{y}_0) = \gamma_{j_0,i_0} \neq 0$.

$$|\gamma_{j_0,i_0}| = \left|\sigma_{j_0,i_0}^*\left(\vec{y}_0 - \vec{y}\right)\right| \le \|\vec{\sigma}_{j_0,i_0}^*\| \cdot \|\vec{y}_0 - \vec{y}\|.$$

This implies

$$\|\vec{y}_0 - \vec{y}\| \ge \frac{|\gamma_{j_0, i_0}|}{\|\sigma^*_{j_0, i_0}\|} > 0, \forall \vec{y} \in \Xi(V_0).$$

So the open ball $B(\vec{y}_0, r)$ with $r = \frac{|\gamma_{j_0, i_0}|}{2\|\sigma_{j_0, i_0}^*\|}$ must be disjoint with $\vec{y} \in \Xi(V_0)$. So $\Xi(V_0)$ is closed.

Now we will prove our conclusion in this paper. Since Ξ is linear, and by Equation (2.4), we have

$$\begin{aligned} \operatorname{span} \{ D^{-m} T^{\ell} \psi \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z} \} \\ &= \operatorname{span} \{ D^{-m} T^{\ell} \Xi \eta \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z} \} \\ &= \Xi \left(\operatorname{span} \{ D^{-m} T^{\ell} \eta \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z} \} \right) \end{aligned}$$

This implies

$$V(\psi) = \overline{\operatorname{span}\{D^{-m}T^{\ell}\psi \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z}\}}$$

= $\overline{\Xi\left(\operatorname{span}\{D^{-m}T^{\ell}\eta \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z}\}\right)}$
 $\subseteq \overline{\Xi\left(\operatorname{span}\{D^{-m}T^{\ell}\eta \mid m \in \mathbb{Z}, m \ge 0, \ell \in \mathbb{Z}\}\right)}$
= $\overline{\Xi(V_0)}.$

Thus

(2.14)
$$V(\psi) \subseteq \overline{\Xi(V_0)}$$

By Equation (2.13) we have

(2.15)
$$V(\psi) \subseteq \overline{\Xi(V_0)} = \Xi(V_0).$$

So for each $n \in \mathbb{Z}$, $D^n V(\psi) \subseteq D^n \Xi(V_0) = \Xi D^n V_0 = \Xi(V_n)$. $D^n V(\psi) \subseteq \Xi(V_n), \forall n \in \mathbb{Z}.$

Therefore, by Equation (2.12)

$$\bigcap_{n\in\mathbb{Z}} D^n V(\psi) \subseteq \bigcap_{n\in\mathbb{Z}} \Xi(V_n) = \{0\}$$

So, Equation (1.1) has been established.

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