REFINED VERLINDE AND SEGRE FORMULA FOR HILBERT SCHEMES

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Abstract. Let $\mathrm{Hilb}_n S$ be the Hilbert scheme of n points on a smooth projective surface S. To a class $\alpha \in K^0(S)$ correspond a tautological vector bundle $\alpha^{[n]}$ on $\mathrm{Hilb}_n S$ and line bundle $L_{(n)} \otimes E^{\otimes r}$ with $L = \det(\alpha)$, $r = \mathrm{rk}(\alpha)$. In this paper we give closed formulas for the generating functions for the Segre classes $\int_{\mathrm{Hilb}_n S} s(\alpha^{[n]})$, and the Verlinde numbers $\chi(\mathrm{Hilb}_n S, L_{(n)} \otimes E^{\otimes r})$, for any surface S and any class $\alpha \in K^0(S)$. In fact we determine a more general generating function for K-theoretic invariants of Hilbert schemes of points, which contains the formulas for Segre and Verlinde numbers as specializations. We prove these formulas in case $K_S^2 = 0$. Without assuming the condition $K_S^2 = 0$, we show the Segre-Verlinde conjecture of Johnson and Marian-Oprea-Pandharipande, which relates the Segre and Verlinde generating series by an explicit change of variables.

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Date: October 4, 2022.

1. Introduction

Let S be a smooth projective surface. Among the most basic moduli spaces associated to S, with connections to many other moduli spaces, are the Hilbert schemes $\mathrm{Hilb}_n S$ of n points on S, which have been a focus of interest for many years. In particular the following two important series of enumerative invariants of the Hilbert schemes of points have been studied for more than 2 decades.

(1) Chern and Segre series of tautological bundles.
Let $Z_n(S) \subset S \times \mathrm{Hilb}_n S$ be the universal subscheme, with projections $p : Z_n(S) \to \mathrm{Hilb}_n S$, $q : Z_n(S) \to S$. For a vector bundle V on S, the corresponding tautological bundle is $V^{[n]} := p_* q^*(V)$, a vector bundle of rank $n\mathrm{rk}(V)$ on $\mathrm{Hilb}_n S$. This extends to a homomorphism $\bullet^{[n]} : K^0(S) \to K^0(\mathrm{Hilb}_n S)$ of Grothendieck groups of vector bundles. We consider the series of Chern integrals

$$I^C_{S,\alpha}(x) := \sum_{n\geq 0} x^n \int_{\mathrm{Hilb}_n S} c_{2n}(\alpha^{[n]}),$$

for any class $\alpha \in K^0(S)$, and the corresponding Segre series

$$I^S_{S,\alpha}(x) := \sum_{n\geq 0} x^n \int_{\mathrm{Hilb}_n S} s_{2n}(\alpha^{[n]}).$$

We will always write $k = \mathrm{rk}(\alpha)$. Because of the obvious identity $I^S_{S,\alpha}(x) = I^C_{S,-\alpha}(x)$ we will concentrate on $I^C_{S,\alpha}(x)$.

(2) The Verlinde series.
Let $\sigma : S^n \to S^{(n)}$ be the quotient map to the symmetric power, and $\pi : \mathrm{Hilb}_n S \to S^{(n)}$ the Hilbert-Chow morphism. For a line bundle $L \in \mathrm{Pic}(S)$ let $L_{(n)} := \pi^*\sigma_*(\otimes_{i=1}^n pr_i^* L)^{\mathfrak{S}_n}$ be the pullback of the symmetrized pushforward. Furthermore for $n \geq 2$ let $E := -\frac{1}{2}D$, where D is the exceptional divisor of π. Then it is well-known that $\mathrm{Pic}(\mathrm{Hilb}_n S) = \mathrm{Pic}(S)_{(n)} \oplus \mathbb{Z}E$. Furthermore for $\beta \in K^0(S)$ we have $\det(\beta^{[n]}) = L_{(n)} \otimes E^{\otimes r}$ with $\det(\beta) = L$ and $\mathrm{rk}(\beta) = r$. The Verlinde series associated to (S, β) is the generating series of the holomorphic Euler

characteristics

$$I^V_{S,\beta}(t) = \sum_{n\geq 0} t^n \chi(\mathrm{Hilb}_n S, \det(\beta^{[n]})),$$

where we always write $r = \mathrm{rk}(\beta)$. Finally when k and r occur together we will always have $r = k-1$.

For $\alpha = L$ a line bundle, the coefficients of the Segre series $I^S_{S,L}(z)$ were already considered in [ES96] in the case $S = \mathbb{P}^2$; the first 8 terms were computed and related to the degrees of the varieties of sums of powers of ternary linear forms, to the counting of Darboux curves and to Donaldson invariants of $\mathbb{P}^2$. The well-known Lehn conjecture [Leh99] is a conjectural formula for $I^S_{S,L}(z)$ for any surface and any line bundle $L \in \mathrm{Pic}(S)$. It was first proven in [MOP17b] in the special case of K-trivial surfaces, and then in general in [Voi19], [MOP19]. Finally, in [MOP17a], the case of $I^C_{S,\alpha}(x) = I^S_{S,-\alpha}(x)$ for an arbitrary class $\alpha \in K^0(S)$ is considered: Applying the cobordism invariance of [EGL01], one can write $I^C_{S,\alpha}(x)$ in the following form

$$I^C_{S,\alpha}(x) = A_0(x)^{c_2(\alpha)} A_1(x)^{\chi(\det(\alpha))} A_2(x)^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3(x)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} A_4(x)^{K_S^2}.$$

Furthermore, by explicitly determining the invariants for $K3$ surfaces, they determined A_0, A_1, A_2 as algebraic functions. Finally they proved explicit formulas as algebraic functions for $A_3(x)$ and $A_4(x)$ for $k=-1$, $k=-2$. In case $k=0$ they showed (in our notation) that $A_4 = 1$ and conjectured $A_3 = 1$, which was recently proved in [Yua22].

Identifying a subscheme $Z \in \mathrm{Hilb}_n S$ with its ideal sheaf I_Z, the Hilbert scheme $\mathrm{Hilb}_n S$ can be viewed as a moduli space of stable rank 1 torsion free sheaves on S with Chern classes $c_1 = 0$ and $c_2 = n$. Thus a formula for the Verlinde series $I^V_{S,\beta}(t)$ can be viewed as the rank 1 case of a general Verlinde formula for surfaces, computing the holomorphic Euler characteristics of all determinant bundles on all moduli spaces of sheaves on S. This is the higher dimensional analogue of the famous Verlinde formula [Ver] for curves, which computes the dimensions of the spaces of sections of determinant bundles on moduli spaces of vector bundles on curves.

By [EGL01] the Verlinde series $I^V_{S,\beta}(t)$ has for all surfaces S and all $\beta \in K^0(S)$ the factorization (recall $L = \det(\beta)$)

$$I^V_{S,\beta}(t) = B_1(t)^{\chi(L)} B_2(t)^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3(t)^{LK_S - \frac{1}{2}K_S^2} B_4(t)^{K_S^2}.$$

With the change of variables $t = -y(1-y)^{r^2-1}$ we have

$$B_1(t) = 1-y, \quad B_2(t) = \frac{(1-y)^{r^2}}{(1-r^2y)};$$

Furthermore $B_3 = B_4 = 1$ for $r = -1, 0, 1$, and replacing r by $-r$ sends $B_3(t)$ to $\frac{1}{B_3(t)}$ and $B_4(t)$ to itself.

Based on Le Potier's strange duality conjecture for surfaces [LP], in [Joh] the Segre and Verlinde series are related to each other by a change of variables, whose explicit form was determined in [MOP19]. This gives the following conjectural Verlinde-Segre correspondence.

Conjecture 1. [Joh], [MOP19]. Fix $k = r + 1$ then

$$A_3(x) = B_3(t), \quad A_4(x) = B_4(t),$$

under the change of variables

$$x = s(1 - rs)^{-r}, \quad t = \frac{s(1 - (r-1)s)^{r^2-1}}{(1 - rs)^{r^2}}.$$

In this paper we prove a closed formula for $A_3(x)$ and $B_3(t)$ and give a conjectural formula for $A_4(x)$ and $B_4(t)$ for arbitrary $k = r + 1$. Thus we obtain complete conjectual Segre and Verlinde formulas for Hilbert schemes of points on any surface S, and we prove them when $K_S^2 = 0$. Furthermore we prove the Segre-Verlinde correspondence in general. In addition our methods give independent proofs for the formulas $A_0(x)$, $A_1(x)$, $A_2(x)$, of [MOP17a], and also express them in a slightly simpler form.

We obtain these formulas as specializations of a considerably more general result. We introduce a K-theoretic invariant of Hilbert schemes of points, which has both the Verlinde and Segre invariants as specializations, and determine its generating function. The general shape of this generating function then implies in particular the Verlinde-Segre correspondence, and to some extend explains it. For $\alpha \in K^0(S)$ we put

$$I_{S,\alpha}(w,z) := \sum_{n\ge 0} (-w)^n \chi\left(\mathrm{Hilb}_n S, \left(\Lambda_{-z}\alpha^{[n]}\right) \otimes \det(\mathcal{O}_S^{[n]})^{-1}\right). \tag{1.1}$$

Here $\Lambda_{-z} : (K^0(S), +) \to (K^0(S)[[z]], \cdot)$ is the homomorphism given by

$$\Lambda_{-z}(W) = \sum_n (-z)^n \Lambda^n W, \qquad \Lambda_{-z}(-W) = \sum_n z^n S^n W,$$

for W a vector bundle. It is easy to see that both $I^C_{S,\alpha}(z)$ and $I^V_{S,\alpha-\mathcal{O}_S}(t)$ are specializations of $I_{S,\alpha}(w,z)$, in fact for $k = \mathrm{rk}(\alpha)$ we have

$$I^C_{S,\alpha}(x) = I_{S,\alpha}\Big(-\epsilon^{2-k}(1+\epsilon)^k x, \frac{1}{1+\epsilon}\Big)\Big|_{\epsilon^0}, \qquad I^V_{S,\alpha-\mathcal{O}_S}(t) = I_{S,\alpha}(-tz^{-k}, -z)|_{z^0}. \tag{1.2}$$

Similar expressions to (1.1) have been considered in [Arb21] and [Arb22], and very similar expressions were studied in [Boj21] for virtual invariants of Quot schemes of curves, surfaces and 4-folds and used to understand the Verlinde-Segre correspondence for these virtual invariants. The following is our main result.

Theorem 1.1. *Fix $k \in \mathbb{Z}$. Then there are power series $G_0, G_1, G_2, G_3, G_4 \in \mathbb{Z}[[w, z]]$, such that for all smooth projective surfaces S and all $\alpha \in K^0(S)$ of rank k we have*

$$I_{S,\alpha}(w,z) = G_0(w,z)^{c_2(\alpha)} G_1(w,z)^{\chi(\det(\alpha))} G_2(w,z)^{\frac{1}{2}\chi(\mathcal{O}_S)} G_3(w,z)^{c_1(\alpha)K_S - \frac{1}{2}K_S^2} G_4(w,z)^{K_S^2}. \tag{1.3}$$

With the change of variables

$$w = \frac{u(1-u)^{k-1}}{v(1-v)^{k-1}}, \qquad z = \frac{v}{(1-u)^{k-1}}, \tag{1.4}$$

we have

$$\begin{aligned} G_0(w,z) &= \frac{(1-u-v)^k}{(1-v)^{k-1}\big((1-u)^{k-1}-v\big)}, \\ G_1(w,z) &= \frac{(1-v)^{k-2}\big((1-u)^{k-1}-v\big)}{(1-u)(1-u-v)^{k-1}}, \\ G_2(w,z) &= \frac{(1-\frac{u}{v})^2(1-v)^{(k-2)^2}\big((1-u)^{k-1}-v\big)^{2(k-1)}}{(1-u-v)^{(k-1)^2}(1-u)^{k^2-2k}\big(1-u-v-(k^2-2k)uv\big)}. \end{aligned}$$

With the further change of variables $y = \frac{uv}{(1-u)(1-v)}$, we have that

$$G_3(w,z) = (1-y)^{-\frac{k-1}{2}} \exp\left(\sum_{n=1}^{\infty} -\frac{y^n}{2n} \left(\frac{x^{k-1} - x^{1-k}}{x - x^{-1}} \right)^{2n} \Bigg|_{x^0} \right),$$

and $G_4(w,z) \in \mathbb{Q}[[y]]$.

Here the product formula (1.3) is a direct application of [EGL01, Thm. 4.2], and the substance of the theorem are the formulas for the universal power series $G_i(w,z)$. Specializing, we obtain the Verlinde-Segre correspondence and the Verlinde and Segre formulas for Hilbert schemes. The Verlinde-Segre correspondence follows, because $B_3(t)$, $A_3(x)$ and $B_4(t)$, $A_4(x)$ are explicit specializations of $G_3(w,z)$, and $G_4(w,z)$, together with the fact that after the changes of variables (1.4) and $y = \frac{uv}{(1-u)(1-v)}$, (or equivalently $u = \frac{-(1-v^{-1})y}{1-(1-v^{-1})y}$), both $G_3(w,z)$ and $G_4(w,z)$ depend only on y.

Corollary 1.2. *Conjecture 1 is true. Explicitly we have the following. Assume $r=k-1$, then under the changes of variables* (1.4), $y=\frac{uv}{(1-u)(1-v)}$, *and*

$$x=-y(1-ry)^{r-1},\quad t=-y(1-y)^{r^2-1} \tag{1.5}$$

we have

$$\begin{aligned}B_1(t)&=A_0(x)A_1(x)=G_0(w,z)G_1(w,z),\\ B_3(t)&=A_3(x)=G_3(w,z),\quad B_4(t)=A_4(x)=G_4(w,z).\end{aligned}$$

Note that Conjecture 1 follows from Corollary 1.2 by putting $s=\frac{-y}{1-ry}$.

Corollary 1.3. *Fix $r=k-1\in\mathbb{Z}$. Under the changes of variables 1.5 we have*

$$A_0(x)=\frac{(1-y)^{r+1}}{1-ry},\quad A_1(x)=\frac{1-ry}{(1-y)^r},\quad A_2(x)=\frac{(1-ry)^{2r}}{(1-y)^{r^2}(1-r^2y)},$$

$$B_1(t)=1-y,\quad B_2(t)=\frac{(1-y)^{r^2}}{1-r^2y},$$

$$A_3(x)=B_3(t)=\frac{1}{(1-y)^{\frac{r}{2}}}\exp\left(-\sum_{n=1}^\infty \frac{y^n}{2n}\left(\frac{x^r-x^{-r}}{x-x^{-1}}\right)^{2n}\Bigg|_{x^0}\right).$$

Finally we give a conjectural formula for the last power series $G_4(w,z)$. It is given in terms of a product decomposition of $G_3(w,z)$. Note that by the results of [EGL01] on B_3, B_4 and the Verlinde-Segre correspondence Corollary 1.2, replacing r be $-r$ replaces $G_3(w,z)$ by $\frac{1}{G_3(w,z)}$ and does not change $G_4(w,z)$. Therefore for the statement we restrict to the case $r\ge 0$. For $i=1,\ldots r-1$, let $\alpha_i(y)$ be the branches of the inverse series of

$$f(x):=\left(\frac{x^{\frac12}-x^{-\frac12}}{x^{\frac r2}-x^{-\frac r2}}\right)^2=x^{r-1}+\ldots.$$

Concretely, let g be the inverse series to $f(x)^{\frac{1}{r-1}}$ and put $\alpha_i(y):=g(\zeta^i y^{\frac{1}{r-1}})$ for ζ a primitive $(r-1)$-th root of unity. Then $G_3(w,z)$ satisfies the following product formula.

Proposition 1.4. *With $r=k-1\ge 0$ and the changes of variables* (1.5), (1.4), $y=\frac{uv}{(1-u)(1-v)}$ *we have*

$$A_3(x)=B_3(t)=G_3(w,z)^2=\frac{y}{(1-y)^r\prod_{i=1}^{r-1}\alpha_i(y)}.$$

Now we can state the conjectural formulas for the power series $G_4(w,z)$ in terms of the factors $\alpha_i(y)$.

Conjecture 2. With $r = k-1 \geq 0$ and the changes of variables (1.5), (1.4), $y = \frac{uv}{(1-u)(1-v)}$ we have

$$\left(G_4(w,z)G_3(w,z)^r\right)^8 = \frac{(1-r^2y)^3}{(1-y)^{3r^2}} \left(\prod_{i,j=1}^{r-1} (1-\alpha_i(y)\alpha_j(y)) \prod_{\substack{i,j=1 \\ i\neq j}}^{r-1} (1-\alpha_i(y)^r\alpha_j(y)^r) \right)^2,$$

and $A_4(x) = B_4(t) = G_4(w,z)$.

As $G_4(w,z)$ is a power series starting with 1, this determines $G_4(w,z)$. A higher rank generalization of the Segre and Verlinde series was studied in [GK20]. Work in progress on the generating functions of these invariants, using the blowup formulas of [Göt21] and a virtual version of strange duality, leads to a series of conjectures about these generating functions. In the rank 1 case, i.e. for the Hilbert schemes of points, these conjectures include the product formula for $B_3(t)$. In addition they suggest that $B_4(t)$ should be expressible in terms of the factors $\alpha_i(y)$ of $B_3(t)$ by means of a product formula. Using also Corollary 1.2, this lead to Conjecture 2.

Below in Proposition 7.3, we also give an (albeit not very attractive) alternative formula for $G_4(w,z)$ in terms of binomial coefficients, which has the advantage of being very efficiently computable. Using this alternative formula, together with computer calculations of the Verlinde numbers for $\mathrm{Hilb}_n S$ for $n < 50$, we get the following.

Proposition 1.5. *Conjecture 2 is true modulo* w^{50}.

Strategy of the proof. By the product formula (1.3) (or equivalently the cobordism invariance of [EGL01]) it is enough to show the result for S a toric surface and α the class of a toric vector bundle. Then the localization formula expresses the generating function $I_{S,\alpha}(w,z)$ in terms of a "master" partition function $\Omega(w,z_1,\ldots,z_k,q,t)$. Identities of modified Macdonald polynomials lead to a functional equation for Ω which we call the symmetry. The symmetry together with some regularity properties leads to enough constraints on the $G_i(w,z)$, to determine them.

It is natural to ask for the geometric meaning of the symmetry and regularity properties (see Definition 4.1). The regularity seems to have the same nature as the following observation, which also implies (1.2) for $I^C_{S,\alpha}(x)$. Let X be a compact complex manifold of dimension d. Let V be a vector bundle of rank k, and L a line bundle on X. Let

$$f(z) = \chi(X, \Lambda_{-z}V \otimes L).$$

By Riemann-Roch one can check that

$$\epsilon^{d-k}(1+\epsilon)^k f\left(\frac{1}{1+\epsilon}\right)$$

is a polynomial in ϵ of degree $\le d$. Its constant term is the Chern integral $(-1)^d \int_X c_d(V)$.

The symmetry property for instance manifests itself in the $(w,z) \leftrightarrow (w^{-1}, wz)$ symmetry of the series $G_0(w,z)G_1(w,z)$, $G_3(w,z)$, and $G_4(w,z)$, or equivalently the $u \leftrightarrow v$ symmetry after the substitution (1.4). It's geometric meaning remains mysterious to us. One consequence is that under the assumption $\chi(\alpha) = \chi(\mathcal{O}_S) = 0$ we have

$$\chi\left(\mathrm{Hilb}_n S, \Lambda^m \alpha^{[n]} \otimes \det(\mathcal{O}_S^{[n]})^{-1}\right) = (-1)^m \chi\left(\mathrm{Hilb}_{m-n} S, \Lambda^m \alpha^{[m-n]} \otimes \det(\mathcal{O}_S^{[m-n]})^{-1}\right).$$

Acknowledgements. L. G. would like to thank Don Zagier for numerous discussions and advice over the course of several years, which were very helpful in finding some of the formulas of this paper. In the course of these discussions he also suggested and helped to write an efficient Pari/GP program to compute the series $B_3(w)$, $B_4(w)$ with r as variable to high order. This data has many times proved to be useful to check ideas about the structure of these power series, and also was used to show Proposition 1.5. He also thanks Emanuel Carneiro for useful discussions.

A. M. thanks Rahul Pandharipande for the suggestion to try to compute the Chern and Verlinde series using localization and Macdonald polynomials, and for continuous encouragement. He also thanks Dragos Oprea for his minicourse on the Segre-Verlinde correspondence and many useful discussions. A. M. is supported by the ERC Consolidator Grant, grant agreement 101001159.

2. Partition functions

We will use the following "master" partition function

$$\Omega(w; z_1, \ldots, z_k; q, t) = \sum_{\lambda} \frac{\prod_{i=1}^k \prod_{\square\in\lambda}(1 - q^{c(\square)} t^{r(\square)} z_i)}{\prod_{\square\in\lambda}(q^{a(\square)+1} - t^{l(\square)})(q^{a(\square)} - t^{l(\square)+1})} w^{|\lambda|}.$$

Here for every partition λ we consider products over the boxes $\square \in \lambda$ and for each box we denote by $c(\square)$, $r(\square)$, $a(\square)$, $l(\square)$ the column index, the row index, the arm length and the leg length respectively. The column and row indices are zero-based.

2.1. **Notation.** We will often use the falling and rising factorial notation

$$a_{(n)} = a(a-1)\cdots(a-n+1), \qquad a^{(n)} = a(a+1)\cdots(a+n-1).$$

When f is a polynomial or a power series involving a variable z then $f|_{z^n}$ denotes the coefficient in front of z^n. On the other hand, when we write $f(z)|_{z=a}$ we mean $f(a)$.

2.2. **Plethysms.** In what follows we will be using *plethystic evaluation*: when F is a symmetric function and A is a formal expression in some variables $x, y, z, \ldots$ then to define $F[A]$ we first express F in terms of the power sum functions

$$F = f(p_1, p_2, p_3, \ldots),$$

and then set

$$F[A] := f(A(x, y, z, \ldots), A(x^2, y^2, z^2, \ldots), A(x^3, y^3, z^3, \ldots), \ldots).$$

When A involves letters X, Y which stand for symmetric function alphabets we agree to treat symbols X, and Y as sums $X = x_1 + x_2 + \cdots$, $Y = y_1 + y_2 + \cdots$. In this way we have for instance

$$F[X] = F(x_1, x_2, \cdots),$$

where on the left we have the plethystic evaluation, and on the right we have the usual evaluation of F as a function on the arguments $x_1, x_2, \ldots$.

Denote by Exp the *plethystic exponential*, which is the infinite series of symmetric functions

$$\mathrm{Exp} = \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n}\right).$$

It satisfies

$$\mathrm{Exp}[A + B] = \mathrm{Exp}[A]\,\mathrm{Exp}[B], \qquad \mathrm{Exp}[-A] = \frac{1}{\mathrm{Exp}[A]}$$

for formal expressions A, B when the corresponding infinite sums make sense, and we have

$$\mathrm{Exp}[x + y + z + \cdots] = \frac{1}{(1 - x)(1 - y)(1 - z)\cdots},$$

so one may think of Exp as a notation for product expansions.

2.3. **Macdonald polynomials toolkit.** Our main reference for Macdonald polynomials and relevant results is [GHT99]. Let $\widetilde{H}_\lambda[X; q, t]$ denote the *modified Macdonald polynomial* for a partition λ. This is a symmetric function in $X = (x_1, x_2, \ldots)$ whose coefficients are polynomials in q, t. Let

$$B_\lambda(q, t) = \sum_{\square \in \lambda} q^{c(\square)} t^{r(\square)}, \qquad D_\lambda(q, t) = -1 + (1 - q)(1 - t) B_\lambda,$$

$$T_\lambda(q,t) = q^{n(\lambda')}t^{n(\lambda)} = \prod_{\square\in\lambda} q^{c(\square)}t^{r(\square)},$$

where $n(\lambda) = \sum_{\square\in\lambda} r(\square) = \sum_i (i-1)\lambda_i$ and λ' denotes the conjugate partition. We also abbreviate

$$N_\lambda(q,t) = \prod_{\square\in\lambda}(q^{a(\square)+1} - t^{l(\square)})(q^{a(\square)} - t^{l(\square)+1}).$$

We have the *Cauchy identity*

$$\operatorname{Exp}\left[-\frac{XY}{(1-q)(1-t)}\right] = \sum_\lambda \frac{\widetilde{H}_\lambda[X;q,t]\widetilde{H}_\lambda[Y;q,t]}{N_\lambda(q,t)},$$

where $X = (x_1, x_2, \ldots)$ and $Y = (y_1, y_2, \ldots)$ are two sets of variables.

Remark 2.1. It should be clear that we can also write the left hand side as an infinite product

$$\prod_{k,l=1}^\infty \prod_{i,j=0}^\infty (1 - x_k y_l q^i t^j),$$

but the infinite product expansion makes it not so apparent that the coefficients in front of monomials in x and y are rational functions in q and t rather then arbitrary power series.

We will need the following identity proved in [GHT99]:

Theorem 2.2. *For any partition μ we have*

$$\widetilde{H}_\mu[X+1;q,t] = \operatorname{Exp}\left[\frac{X}{(1-q)(1-t)}\right] \sum_\lambda (-1)^{|\lambda|} \frac{\widetilde{H}_\lambda[X;q,t]\widetilde{H}_\lambda[D_\mu(q,t);q,t]}{T_\lambda(q,t)N_\lambda(q,t)}. \tag{2.1}$$

This identity can be thought of as a strengthening of the *Macdonald-Koornwinder duality*, which says that for any partitions μ, ν we have the following identity of rational functions in u, q, t:

$$\frac{H_\nu[1+uD_\mu(q,t);q,t]}{\prod_{\square\in\nu}(1-uq^{c(\square)}t^{r(\square)})} = \frac{H_\mu[1+uD_\nu(q,t);q,t]}{\prod_{\square\in\mu}(1-uq^{c(\square)}t^{r(\square)})}. \tag{2.2}$$

Indeed, setting $X = uD_\nu(q,t)$ in (2.1), we obtain

$$\widetilde{H}_\mu[1+uD_\nu(q,t);q,t] = \operatorname{Exp}\left[\frac{-u}{(1-q)(1-t)} + uB_\nu\right] \sum_\lambda (-u)^{|\lambda|} \frac{\widetilde{H}_\lambda[D_\nu(q,t);q,t]\widetilde{H}_\lambda[D_\mu(q,t);q,t]}{T_\lambda(q,t)N_\lambda(q,t)},$$

so the right hand side of (2.2) equals

$$\operatorname{Exp}\left[\frac{-u}{(1-q)(1-t)} + uB_\nu + uB_\mu\right] \sum_\lambda (-u)^{|\lambda|} \frac{\widetilde{H}_\lambda[D_\nu(q,t);q,t]\widetilde{H}_\lambda[D_\mu(q,t);q,t]}{T_\lambda(q,t)N_\lambda(q,t)},$$

which is manifestly symmetric in μ, ν.

For the empty partition $\nu = \varnothing$ we have $B_\varnothing = 0$, $D_\varnothing = -1$, $\widetilde{H}_\varnothing = 1$, and so (2.2) implies

$$H_\mu[1-u;q,t] = \prod_{\square\in\mu}(1-uq^{c(\square)}t^{r(\square)}). \tag{2.3}$$

Both sides of (2.2) are rational functions of u. In the limit $u \to \infty$ we obtain

$$(-1)^{|\nu|}\frac{H_\nu[D_\mu;q,t]}{T_\nu(q,t)} = (-1)^{|\mu|}\frac{H_\mu[D_\nu;q,t]}{T_\mu(q,t)}. \tag{2.4}$$

2.4. **Functional equation.** Using these identities we can now prove the following result about the partition function Ω (see [Mel16] for a more general identity and an application to a conjecture of Hausel-Mereb-Wong [HMW19]):

Theorem 2.3. *We have the following identity of power series in $w, z_1, \ldots, z_k$ with coefficients in $\mathbb{Q}(q,t)$:*

$$\Omega(w;z_1,\ldots,z_k;q,t) = \operatorname{Exp}\left[-\frac{w+\sum z_i}{(1-q)(1-t)}\right] \tag{2.5}$$
$$\cdot\sum_\mu(-1)^{|\mu|}\frac{\tilde{H}_\mu[w+1;q,t]\tilde{H}_\mu[z_1+\cdots+z_k;q,t]T_\mu(q,t)}{N_\mu(q,t)}.$$

Proof. Let $Z = z_1+\cdots+z_k$. We write the term in the definition of Ω as follows:

$$\prod_{i=1}^k\prod_{\square\in\lambda}(1-q^{c(\square)}t^{r(\square)}z_i) = \operatorname{Exp}[-ZB_\lambda(q,t)] = \operatorname{Exp}\left[-\frac{Z(D_\lambda(q,t)+1)}{(1-q)(1-t)}\right]$$
$$= \operatorname{Exp}\left[-\frac{Z}{(1-q)(1-t)}\right]\sum_\mu\frac{\widetilde{H}_\mu[Z;q,t]\widetilde{H}_\mu[D_\lambda(q,t);q,t]}{N_\mu(q,t)}.$$

Collecting the coefficients in front of the terms $\widetilde{H}_\mu[Z;q,t]$, the statement is reduced to showing

$$\sum_\lambda\frac{\widetilde{H}_\mu[D_\lambda(q,t);q,t]}{N_\lambda(q,t)}w^{|\lambda|} = (-1)^{|\mu|}\operatorname{Exp}\left[-\frac{w}{(1-q)(1-t)}\right]\tilde{H}_\mu[w+1;q,t]T_\mu(q,t).$$

Applying (2.4), this reduces to

$$\sum_\lambda\frac{\widetilde{H}_\lambda[D_\mu(q,t);q,t]}{T_\lambda(q,t)N_\lambda(q,t)}(-w)^{|\lambda|} = \operatorname{Exp}\left[-\frac{w}{(1-q)(1-t)}\right]\tilde{H}_\mu[w+1;q,t].$$

Noting that $\widetilde{H}_\mu[w;q,t] = w^{|\mu|}$ we recognize the specialization of (2.1) to $X = w$. $\square$

This implies a functional equation for Ω.

Corollary 2.4. *Let*

$$\widetilde{\Omega}(w; z_1, \dots, z_k; q, t) = \operatorname{Exp}\left[\frac{w + \sum z_i}{(1-q)(1-t)}\right] \Omega(w; z_1, \dots, z_k; q, t).$$

This is a power series in $z_1, \dots, z_k$ whose coefficients are polynomials *in w, and we have*

$$\widetilde{\Omega}(w; z_1, \dots, z_k; q, t) = \widetilde{\Omega}(w^{-1}; wz_1, \dots, wz_k; q, t).$$

2.5. **The logarithm.** Let

$$H(w; z_1, \dots, z_k; q, t) = \log \Omega(w; z_1, \dots, z_k; q, t).$$

In [Mel18] it was shown that series of the form

$$\sum_\lambda \frac{\widetilde{H}_\lambda[X; q, t]\widetilde{H}_\lambda[Y; q, t]\widetilde{H}_\lambda[Z; q, t] \cdots}{N_\lambda},$$

with arbitrary number of Macdonald polynomials in the numerator, have the property that they can be written as

$$\operatorname{Exp}\left[\frac{1}{(1-q)(1-t)}(\cdots)\right],$$

where $(\cdots)$ is a series in the variables from X, Y, Z, ... whose coefficients are *polynomials* in q, t. Specializing $X = w$, $Y = 1 - z_1$, $Z = 1 - z_2$ and so on and applying (2.3) we deduce that Ω can be written in this form. In particular, from the definition of Exp we deduce the following:

Proposition 2.5. *All coefficients of*

$$(1-q)(1-t)H(w; z_1, \dots, z_k; q, t),$$

as a power series in $w, z_1, \dots, z_k$ are regular in a neighborhood of $q = t = 1$.

Thus we can expand

$$H(w; z_1, \dots, z_k; e^{t_1}, e^{t_2}) = \sum_{d_1, d_2 \geq -1} H_{d_1,d_2}(w; z_1, \dots, z_k) t_1^{d_1} t_2^{d_2}. \tag{2.6}$$

for some power series $H_{d_1,d_2}(w, z_1, \dots, z_k)$.

Next we expand

$$\log \operatorname{Exp}\left[\frac{w + \sum z_i}{(1-q)(1-t)}\right] = \sum_{n=1}^\infty \frac{w^n + \sum_{i=1}^k z_i^n}{n(1-q^n)(1-t^n)}$$

and use

$$\frac{1}{e^{tn}-1}=\sum_{d=-1}^{\infty}\frac{n^d B_{d+1}}{(d+1)!}t^d,$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers $1, -1/2, 1/6, \ldots$, to deduce a functional equation for H_{d_1,d_2}.

Theorem 2.6. *For each* $d_1, d_2 \geq -1$ *let*

$$\widetilde{H}_{d_1,d_2}(w; z_1, \ldots, z_k) = H_{d_1,d_2}(w; z_1, \ldots, z_k)+\frac{B_{d_1+1}B_{d_2+1}}{(d_1+1)!(d_2+1)!}\left(\mathrm{Li}_{1-d_1-d_2}(w)+\sum_{i=1}^{k}\mathrm{Li}_{1-d_1-d_2}(z_i)\right),$$

where $\mathrm{Li}_d(z)=\sum_{n=1}^{\infty}\frac{z^n}{n^d}$ *is the polylogarithm function. Then we have*

$$\widetilde{H}_{d_1,d_2}(w; z_1, \ldots, z_k) = \widetilde{H}_{d_1,d_2}(w^{-1}; wz_1, \ldots, wz_k).$$

In other words, expanding $\widetilde{H}_{d_1,d_2}$ as a power series in $z_1, \ldots, z_k$ each coefficient is a *palindromic* polynomial in w.

3. Localization formulas

3.1. **Integrals over the surface.** Assume S is a smooth compact toric surface. Let $T = (\mathbb{C}^*)^2$ which naturally acts on S. We denote by $\mathcal{T}_S$ the tangent bundle of S. Let V be a T-equivariant bundle on S. Denote $H_T^*(\text{point}) = \mathbb{C}[a_1, a_2]$. Suppose there are M fixed points $p_1, \ldots, p_M$. Denote the weights of the tangent space at the i-th fixed point by $t_1^{(i)}(a_1, a_2), t_2^{(i)}(a_1, a_2)$ and the weights of V by $v_1^{(i)}(a_1, a_2), \ldots, v_k^{(i)}(a_1, a_2)$. Each weight is a $\mathbb{Z}$-linear combination of a_1, a_2. The fundamental class of S in the localized equivariant homology is given by

$$[S]=\sum_{i=1}^{M}\frac{1}{t_1^{(i)}t_2^{(i)}}[p_i].$$

Denote by $\pi : S \to \text{point}$ the natural map. By degree reasons the following sums vanish

$$0=\pi_*[S]=\sum_{i=1}^{M}\frac{1}{t_1^{(i)}t_2^{(i)}}, \tag{3.1}$$

$$0=\pi_*([S]\cap c_1(\mathcal{T}_S))=\sum_{i=1}^{M}\frac{t_1^{(i)}+t_2^{(i)}}{t_1^{(i)}t_2^{(i)}}, \tag{3.2}$$

$$0=\pi_*([S]\cap c_1(V))=\sum_{i=1}^{M}\frac{v_1^{(i)}+\cdots+v_k^{(i)}}{t_1^{(i)}t_2^{(i)}}, \tag{3.3}$$

and numerical invariants of the pair S, V are given as follows:

$$\int_S c_2(V) = \pi_*([S] \cap c_2(V)) = \sum_{i=1}^{M} \frac{e_2(v_1^{(i)}, \dots, v_k^{(i)})}{t_1^{(i)} t_2^{(i)}}, \tag{3.4}$$

$$\int_S c_1(V)^2 = \pi_*([S] \cap c_1(V)^2) = \sum_{i=1}^{M} \frac{(v_1^{(i)} + \dots + v_k^{(i)})^2}{t_1^{(i)} t_2^{(i)}}, \tag{3.5}$$

$$\chi(S) = \pi_*([S] \cap c_2(\mathcal{T}_S)) = \sum_{i=1}^{M} 1 = M, \tag{3.6}$$

$$\int_S c_1(V) c_1(\mathcal{T}_S) = \pi_*([S] \cap c_1(\mathcal{T}_S) c_1(V)) = \sum_{i=1}^{M} \frac{(t_1^{(i)} + t_2^{(i)})(v_1^{(i)} + \dots + v_k^{(i)})}{t_1^{(i)} t_2^{(i)}}, \tag{3.7}$$

$$\int_S c_1(\mathcal{T}_S)^2 = \pi_*([S] \cap c_1(\mathcal{T}_S)^2) = \sum_{i=1}^{M} \frac{(t_1^{(i)} + t_2^{(i)})^2}{t_1^{(i)} t_2^{(i)}}, \tag{3.8}$$

where e_2 denotes the second elementary symmetric function. In all these formulas the right hand side does not depend on a_1 and a_2.

Remark 3.1. The right hand sides of (3.4)–(3.8) are apriori rational functions in a_1, a_2, but the identities imply that these functions are constant.

3.2. **Integrals over the Hilbert schemes.** The fixed points of $\mathrm{Hilb}_* S = \bigcup_{n=0}^{\infty} \mathrm{Hilb}_n S$ correspond to M-tuples of partitions $\lambda^{(1)}, \dots, \lambda^{(M)}$. The weights of the tangent space are given by

$$\bigcup_{i=1}^{M} \bigcup_{\square \in \lambda^{(i)}} \{(a(\square)+1)t_1^{(i)} - l(\square)t_2^{(i)},\ (l(\square)+1)t_2^{(i)} - a(\square)t_1^{(i)}\}.$$

The bundle V induces a bundle $V^{[n]}$ on each $\mathrm{Hilb}_n S$ whose weights at a fixed point are given by

$$\bigcup_{i=1}^{M} \bigcup_{\square \in \lambda^{(i)}} \{v_j^{(i)} - t_1^{(i)} c(\square) - t_2^{(i)} r(\square) \mid j = 1, \dots, k\}.$$

We can compute various integrals over Hilb_* as in [MOP17a]. By definition

$$\Omega(w; z_1, \dots, z_k; e^{-t_1}, e^{-t_2}) = \sum_{\lambda} \frac{e^{\sum_{\square\in\lambda} a(\square)t_1 + l(\square)t_2} \prod_{i=1}^{k} \prod_{\square\in\lambda} (1 - e^{-c(\square)t_1 - r(\square)t_2} z_i)}{\prod_{\square\in\lambda} (1 - e^{-(a(\square)+1)t_1 + l(\square)t_2})(1 - e^{a(\square)t_1 - (l(\square)+1)t_2})} (-w)^{|\lambda|}.$$

If $\lambda=(\lambda_0,\ldots,\lambda_s)$ is a partition and $\lambda'=(\lambda'_0,\ldots,\lambda'_t)$ the dual partition, then for $\square=(n,m)\in\lambda$ also $\square_1=(n,\lambda_n-m-1)\in\lambda$, and similarly $\square_2=(\lambda'_m-n-1,m)\in\lambda$, with $a(\square)=c(\square_1)$ and $l(\square)=r(\square_2)$, therefore

$$\sum_{\square\in\lambda}a(\square)=\sum_{\square\in\lambda}c(\square),\quad \sum_{\square\in\lambda}l(\square)=\sum_{\square\in\lambda}r(\square).$$

Therefore we get

$$I_{S,V}(w,z)=\sum_{n=0}^\infty(-w)^n\chi(\Lambda_{-z}V^{[n]}\otimes\det(\mathcal{O}_S^{[n]})^{-1})=\Bigg(\sum_{\lambda^{(1)},\cdots,\lambda^{(M)}}\prod_{i=1}^M(-w)^{|\lambda^{(i)}|}$$
$$\cdot\frac{e^{\sum_{\square\in\lambda^{(i)}}c(\square)t_1^{(i)}+r(\square)t_2^{(i)}}\prod_{j=1}^k\prod_{\square\in\lambda^{(i)}}(1-e^{-c(\square)t_1^{(i)}-r(\square)t_2^{(i)}}e^{v_j^{(i)}}z)}{\prod_{\square\in\lambda^{(i)}}(1-e^{-(a(\square)+1)t_1^{(i)}+l(\square)t_2^{(i)}})(1-e^{a(\square)t_1^{(i)}-(l(\square)+1)t_2^{(i)}})}\Bigg)\Bigg|_{a_1=a_2=0}$$
$$=\left(\prod_{i=1}^M\Omega(w;ze^{v_1^{(i)}},\ldots,ze^{v_k^{(i)}};e^{-t_1^{(i)}},e^{-t_2^{(i)}})\right)\Bigg|_{a_1=a_2=0}.$$

The Chern integrals are given by the generating series

$$I^C_{S,V}(w)=\sum_{n=0}^\infty w^n\int_{\mathrm{Hilb}_n S}c_{2n}(V^{[n]})$$
$$=\left(\sum_{\lambda^{(1)},\cdots,\lambda^{(M)}}\prod_{i=1}^M\prod_{\square\in\lambda^{(i)}}\frac{w\prod_{j=1}^k(1+v_j^{(i)}-t_1^{(i)}c(\square)-t_2^{(i)}r(\square))}{((a(\square)+1)t_1^{(i)}-l(\square)t_2^{(i)})((l(\square)+1)t_2^{(i)}-a(\square)t_1^{(i)})}\right)\Bigg|_{a_1=a_2=0}.$$

Denoting

$$\Omega^C(w;v_1,\ldots,v_k;t_1,t_2)=\sum_\lambda\prod_{\square\in\lambda}\frac{\prod_{j=1}^k(1+v_j-t_1c(\square)-t_2r(\square))}{((a(\square)+1)t_1-l(\square)t_2)((l(\square)+1)t_2-a(\square)t_1)}w^{|\lambda|},$$

we have

$$I^C_{S,V}(w)=\left(\prod_{i=1}^M\Omega^C(w;v_1^{(i)},\ldots,v_k^{(i)};t_1^{(i)},t_2^{(i)})\right)\Bigg|_{a_1=a_2=0}.$$

For the Verlinde series we let

$$\Omega^V(w;v_1,\ldots,v_k;t_1,t_2)=\sum_\lambda\prod_{\square\in\lambda}\frac{e^{\sum_{i=1}^k(v_i-c(\square)t_1-r(\square)t_2)}}{(1-e^{-(a(\square)+1)t_1+l(\square)t_2})(1-e^{-(l(\square)+1)t_2+a(\square)t_1})}w^{|\lambda|},$$

so that

$$I^V_{S,V}(w)=\sum_{n=0}^\infty w^n\chi(S,(\det V)_{(n)}\otimes E^k)=\left(\prod_{i=1}^M\Omega^V(w;v_1^{(i)},\ldots,v_k^{(i)};t_1^{(i)},t_2^{(i)})\right)\Bigg|_{a_1=a_2=0}.$$

Remark 3.2. Similarly to Remark 3.1, before setting $a_1 = a_2 = 0$, the coefficients on the right hand sides of the above identities are guaranteed to be power series in a_1, a_2. Thus setting $a_1 = a_2 = 0$ makes sense.

We want more generally to consider $I_{S,\alpha}(w,z)$, $I^C_{S,\alpha}(w)$, $I^V_{S,\alpha}(w)$ for elements $\alpha = V - W \in K^0(S)$, where V and W are T-equivariant bundles on S. Denote the weights of V and W at the i-th fixpoint by $v_1^{(i)}(a_1,a_2),\ldots,v_k^{(i)}(a_1,a_2)$ and $x_1^{(i)}(a_1,a_2),\ldots,x_m^{(i)}(a_1,a_2)$. We consider a more general version of the partition function Ω. We put

$$\Omega(w;z_1,\ldots,z_k;y_1,\ldots,y_m;q,t) := \sum_\lambda \frac{\prod_{\square\in\lambda} \frac{\prod_{i=1}^k(1-q^{c(\square)}t^{r(\square)}z_i)}{\prod_{j=1}^m(1-q^{c(\square)}t^{r(\square)}y_j)}}{\prod_{\square\in\lambda}(q^{a(\square)+1}-t^{l(\square)})(q^{a(\square)}-t^{l(\square)+1})} w^{|\lambda|},$$

$$\Omega^C(w;z_1,\ldots,z_k;y_1,\ldots,y_m;t_1,t_2) := \sum_\lambda \frac{\prod_{\square\in\lambda}\frac{\prod_{i=1}^k(1+z_i-t_1c(\square)-t_2r(\square))}{\prod_{j=1}^m(1+y_j-t_1c(\square)-t_2r(\square))}}{\prod_{\square\in\lambda}((a(\square)+1)t_1-l(\square)t_2)((l(\square)+1)t_2-a(\square)t_1)} w^{|\lambda|};$$

Then

$$I_{S,\alpha}(w,z) = \left(\prod_{i=1}^M \Omega(w; ze^{v_1^{(i)}},\ldots,ze^{v_k^{(i)}}; ze^{x_1^{(i)}},\ldots,ze^{x_m^{(i)}}; e^{-t_1^{(i)}}, e^{-t_2^{(i)}})\right)\Bigg|_{a_1=a_2=0},$$

$$I^C_{S,\alpha}(w) = \left(\prod_{i=1}^M \Omega^C(w; v_1^{(i)},\ldots,v_k^{(i)}; x_1^{(i)},\ldots,x_m^{(i)}; t_1^{(i)}, t_2^{(i)})\right)\Bigg|_{a_1=a_2=0}.$$

To reduce to the case that α is a vector bundle we use a well-known trick.

Lemma 3.3. *Let $f(z) = \sum_{n=0}^\infty f_n z^n$ be a power series where each f_n is a power series in some other variables and f_0 begins with 1. Then*

$$\prod_{i=1}^k f(z_i)$$

is a power series whose coefficients can be written in a uniform way as polynomials in k and symmetric functions of $z_1, z_2, \ldots$. Moreover, the substitution $k \to m-l$ and $p_n(z_1,z_2,\ldots) \to p_n(y_1,y_2,\ldots) - p_n(x_1,x_2,\ldots)$ produces

$$\frac{\prod_{i=1}^m f(y_i)}{\prod_{j=1}^l f(x_i)}.$$

Proof. Suppose the logarithm of f is given by

$$\log f(z) = \sum_{n=0}^\infty g_n z^n,$$

where g_0 begins with 0. Then

$$\prod_{i=1}^{k} f(z_i) = e^{kg_0+\sum_{n=1}^{\infty} p_n(z_1,z_2,\dots)g_n}, \qquad \frac{\prod_{i=1}^{m} f(y_i)}{\prod_{j=1}^{l} f(x_i)} = e^{(m-l)g_0-\sum_{n=1}^{\infty}(p_n(y_1,y_2,\dots)-p_n(x_1,x_2,\dots))g_n},$$

from which the statement is clear. □

Corollary 3.4. *$\Omega(w;e^{y_1}z,\dots,e^{y_m}z;e^{x_1}z,\dots,e^{x_l}z;q,t)$ is obtained from $\Omega(w;e^{v_1}z,\dots,e^{v_k}z;q,t)$ and $\Omega^C(w;y_1,\dots,y_m;x_1,\dots,x_l;t_1,t_2)$ from $\Omega^C(w;v_1,\dots,v_k;t_1,t_2)$ by the substitutions*

$$k \to m-l, \qquad p_n(v_1,v_2,\dots) \to p_n(y_1,y_2,\dots) - p_n(x_1,x_2,\dots).$$

3.3. **From the master partition function to the specialized ones.** The following specializations are easy to verify term-by-term.

Proposition 3.5. *The Chern and Verlinde functions Ω^C, Ω^V are given by the following term-by-term limits:*

$$\Omega^C(w;v_1,\dots,v_k;t_1,t_2) = \lim_{\varepsilon\to 0} \Omega\left(-w\varepsilon^{2-k}(1+\varepsilon)^k; \frac{e^{-\varepsilon v_1}}{1+\varepsilon},\dots,\frac{e^{-\varepsilon v_k}}{1+\varepsilon}; e^{\varepsilon t_1}, e^{\varepsilon t_2}\right),$$

$$\Omega^V(w;v_1,\dots,v_k;t_1,t_2) = \lim_{\varepsilon\to 0} \Omega\left((-1)^k w\varepsilon^{k+1}; \varepsilon^{-1}e^{v_1},\dots,\varepsilon^{-1}e^{v_k},\varepsilon^{-1}; e^{-t_1}, e^{-t_2}\right).$$

Proof. For the Chern function substitution we have

$$\Omega\left(-w\varepsilon^{2-k}(1+\varepsilon)^k; \frac{e^{-\varepsilon v_1}}{1+\varepsilon},\dots,\frac{e^{-\varepsilon v_k}}{1+\varepsilon}; e^{\varepsilon t_1}, e^{\varepsilon t_2}\right)$$

$$= \sum_{\lambda} \frac{\varepsilon^{(2-k)|\lambda|}\prod_{i=1}^{k}\prod_{\Box\in\lambda}(1+\varepsilon-e^{\varepsilon(c(\Box)t_1+r(\Box)t_2-v_i)})}{\prod_{\Box\in\lambda}(e^{\varepsilon(a(\Box)+1)t_1}-e^{\varepsilon l(\Box)t_2})(e^{\varepsilon(l(\Box)+1)t_2}-e^{\varepsilon a(\Box)t_1})} w^{|\lambda|}, \tag{3.9}$$

from which it is clear that the limit $\varepsilon \to 0$ exists and equals Ω^C. The formula for the Verlinde function is straightforward. □

3.4. **Taking the logarithms.** By the expansion (2.6)

$$\log\Omega(w;z_1,\dots,z_k;e^{t_1},e^{t_2}) = \sum_{d_1,d_2\geq -1} H_{d_1,d_2}(w;z_1,\dots,z_k)t_1^{d_1}t_2^{d_2}$$

we obtain

$$\log\Omega(w;ze_1^{v_1},\dots,ze^{v_k};e^{t_1},e^{t_2}) = \sum_{d_1,d_2\geq -1} H_{d_1,d_2}(w;ze_1^{v_1},\dots,ze^{v_k})t_1^{d_1}t_2^{d_2}.$$

Combining with Proposition 3.5 with the expansion we also obtain

$$\log \Omega^C(w; v_1, \dots, v_k; t_1, t_2) = \sum_{d_1, d_2 \geq -1} t_1^{d_1} t_2^{d_2} H^C_{d_1,d_2}(w; v_1, \dots, v_k),$$

where

$$H^C_{d_1,d_2}(w; v_1, \dots, v_k) = \lim_{\varepsilon \to 0} \varepsilon^{d_1+d_2} H_{d_1,d_2}\left(-w\varepsilon^{2-k}(1+\varepsilon)^k; \frac{e^{-\varepsilon v_1}}{1+\varepsilon}, \dots, \frac{e^{-\varepsilon v_k}}{1+\varepsilon}\right). \tag{3.10}$$

An analogous statement holds for the Verlinde series with

$$H^V_{d_1,d_2}(w; v_1, \dots, v_k) = (-1)^{d_1+d_2} \lim_{\varepsilon \to 0} H_{d_1,d_2}\left((-1)^k w \varepsilon^{k+1}; \varepsilon^{-1} e^{v_1}, \dots, \varepsilon^{-1} e^{v_k}, \varepsilon^{-1}\right). \tag{3.11}$$

Let us expand $H^*_{d_1,d_2}$ where $*$ is either empty or one of C, V as a power series in v_i. Writing the argument $\underline{w}$, we mean that the arguments are w, z for $*$ empty and w otherwise, and we temporarily write $H_{d_1,d_2}(w, z; v_1, \dots, v_k) = H_{d_1,d_2}(w; ze_1^{v_1}, \dots, ze^{v_k})$.

(3.12)

$$H^*_{-1,-1}(\underline{w}; v_1, \dots, v_k) = C^*_0(\underline{w}) + C^*_1(\underline{w}) \sum_{i=1}^k v_i + C^*_2(\underline{w}) e_2(v_1, \dots, v_k) + C^*_{1,1}(\underline{w}) \left(\sum_{i=1}^k v_i\right)^2 + \cdots,$$

$$H^*_{-1,0}(w; v_1, \dots, v_k) = D^*_0(\underline{w}) + D^*_1(\underline{w}) \sum_{i=1}^k v_i + \cdots, \tag{3.13}$$

$$H^*_{-1,1}(\underline{w}; v_1, \dots, v_k) = E^*(\underline{w}) + \cdots, \tag{3.14}$$

$$H^*_{0,0}(\underline{w}; v_1, \dots, v_k) = F^*(\underline{w}) + \cdots, \tag{3.15}$$

where dots mean terms of higher total degree in v_i.

Proposition 3.6. *For $*$ empty or one of C, V we have*

$$\log I^*_{S,V}(\underline{w}) = C^*_2(\underline{w}) \int_S c_2(V) + C^*_{1,1}(\underline{w}) \int_S c_1(V)^2 + F^*(\underline{w}) \chi(S)$$
$$+ D^*_1(\underline{w}) \int_S c_1(V) c_1(T_S) + E^*(\underline{w}) \left(\int_S c_1(T_S)^2 - 2\chi(S)\right).$$

Proof. We write the identities from Section 3.2 as follows:

$$\log I^*_{S,V}(\underline{w}) = \left(\sum_{d_1,d_2 \geq -1} \sum_{i=1}^M \left(t_1^{(i)}\right)^{d_1} \left(t_2^{(i)}\right)^{d_2} H^*_{d_1,d_2}\left(\underline{w}; v_1^{(i)}, \dots, v_k^{(i)}\right)\right)\Bigg|_{a_1=a_2=0}.$$

The terms of negative degree in $t_1^{(i)}, t_2^{(i)}, v_1^{(i)}, \dots, v_k^{(i)}$ after summing over i produce zero by (3.1)–(3.3). Terms of positive degree can be ignored because they vanish after setting $a_1 = a_2 = 0$. Thus we are left with the terms of degree zero, which are precisely given by the series $C_2^*(\underline{w})$, $C_{1,1}^*(\underline{w})$, $F^*(\underline{w})$, $D_1^*(\underline{w})$, $E^*(\underline{w})$. The sum over i is then evaluated using (3.4)–(3.8). □

Recall that we can write

$$I_{S,V}(w,z) = G_0(w,z)^{c_2(V)} G_1(w,z)^{\chi(\det(V))} G_2(w,z)^{\frac{1}{2}\chi(\mathcal{O}_S)} G_3(w,z)^{c_1(V)\cdot K_S - \frac{1}{2}K_S^2} G_4(w,z)^{K_S^2},$$

where we omit $\int_S$ in the exponents. We use $c_1(T_S) = -K_S$, $\chi(S) = 12\chi(\mathcal{O}_S) - K_S^2$ and obtain

$$\begin{aligned} &\log G_0(w,z) = C_2(w,z), \quad \log G_1(w,z) = 2C_{1,1}(w,z),\\ &\log G_2(w,z) = 24(F(w,z) - 2E(w,z)) - 4C_{1,1}(w,z),\\ &\log G_3(w,z) = -D_1(w,z) + C_{1,1}(w,z),\\ &\log G_4(w,z) = -F(w,z) + 3E(w,z) + \frac{1}{2}(C_{1,1}(w,z) - D_1(w,z)). \end{aligned} \tag{3.16}$$

We get the same formulas for the $A_i(w)$, $B_i(w)$ with $C_2(w,z)$, $C_{1,1}(w,z)$, $D_1(w,z)$, $F(w,z)$, $E(w,z)$ replaced by $C_2^C(w)$, $C_{1,1}^C(w)$, $D_1^C(w)$, $F^C(w)$, $E^C(w)$ and $C_2^V(w)$, $C_{1,1}^V(w)$, $D_1^V(w)$, $F^V(w)$, $E^V(w)$, respectively. Note that Ω^V depends only on the sum $\sum_{i=1}^k v_i$. Thus the term $C_2^V(w)$ vanishes.

Remark 3.7. In the product formula (1.3) for $I_{S,\alpha}(z,w)$ with $\alpha \in K^0(S)$, we see by Corollary 3.4, and Proposition 3.6 that $G_0, \dots, G_4$ are power series in w, z whose coefficients are universal polynomials in $k = \mathrm{rk}(\alpha)$. In particular they are determined for all α and k by the cases of vector bundles of all sufficiently high ranks. The analogous statement holds for $I_{S,\alpha}^V(w)$, $I_{S,\alpha}^C(z,w)$. Thus, in future we can assume that α is the class of a vector bundle of rank at least 3.

4. The constraints

4.1. **Regularity and symmetry.** In order to determine the generating functions $G_i(w,z)$, our strategy is to obtain information about the series $H_{-1,-1}$, $H_{-1,0}$, $H_{-1,1}$ and $H_{0,0}$. Besides the functional equation from Theorem 2.6 we need a constraint that stems from the observation that the right hand side of (3.10) before taking the limit has no pole at $\varepsilon = 0$. This is clear from (3.9).

Definition 4.1. Let k be an integer or complex parameter. Let $f(w,z)=\sum_{m,n=0}^{\infty} f_{m,n}w^m z^n$ be a power series.

(i) $f(w,z)$ is *d-regular* (simply *regular* if $d=0$) for some integer $d\geq 0$ if for all m there exists a polynomial $p_m(x)$ of degree at most $2m-d$ such that for all $n\geq 0$ we have
$$f_{m,n}=(-1)^n p_m(n)\binom{km}{n}.$$
d-regularity depends on k, but k will always be clear from the context.

(ii) $f(w,z)$ is *symmetric* if
$$f(w,z)=f(w^{-1},wz)$$
holds.

The regularity condition is motivated by the following

Lemma 4.2. *Suppose $k>0$ is an integer and a series $f(w,z)=\sum_{m,n=0}^{\infty} f_{m,n}w^m z^n$ is such that for each m we have $f_{m,n}=0$ as soon as $n>km$. Form a new series*
$$g(w,\varepsilon)=f\left(w\varepsilon^{2-k}(1+\varepsilon)^k,\frac{1}{1+\varepsilon}\right).$$
Then f is d-regular if and only if $g(w,\varepsilon)\in\varepsilon^d\mathbb{C}[[w,\varepsilon]]$. If f is d-regular, then we have
$$\lim_{\varepsilon\to 0}\varepsilon^{-d}g(w,\varepsilon)=(-1)^d\sum_{m=0}^{\infty}p_m(x)|_{x^{2m-d}}\,(km)_{(2m-d)}w^m, \tag{4.1}$$
where p_m are the polynomials from Definition 4.1.

Proof. Fix m and consider the polynomial $f_m(z)=\sum_{n=0}^{\infty} f_{m,n}z^n$. Then
$$g_m=g(w,\varepsilon)|_{w^m}=f_m\left(\frac{1}{1+\varepsilon}\right)(1+\varepsilon)^{km}\,\varepsilon^{(2-k)m}$$
is a Laurent polynomial that can be written as
$$g_m(\varepsilon)=\sum_{i=0}^{km}c_i\varepsilon^{2m-i}.$$
Performing the inverse substitution $\varepsilon=z^{-1}-1$ we find f_m in terms of g_m:
$$f_m(z)=\sum_{i=0}^{km}c_i z^i(1-z)^{km-i}=\sum_{i=0}^{km}c_i\sum_{n=0}^{km}(-1)^{n-i}\binom{km-i}{n-i}z^n.$$

So we have

$$f_{m,n} = \sum_{i=0}^{km} c_i(-1)^{n-i}\binom{km-i}{n-i} = (-1)^n\binom{km}{n}\sum_{i=0}^{km}(-1)^i\frac{(n)_{(i)}}{(km)_{(i)}}c_i = (-1)^n\binom{km}{n}p_m(n),$$

where

$$p_m(x) = \sum_{i=0}^{km}(-1)^i\frac{(x)_{(i)}}{(km)_{(i)}}c_i.$$

The polynomial $p_m(x)$ is unique among polynomials of degree at most km satisfying this property. The degree of p_m is at most $2m-d$ if and only if $g_m \in \varepsilon^d\mathbb{C}[[\varepsilon]]$, as claimed. If this is the case, then the top degree coefficient of p_m is given by

$$p_m(x)|_{x^{2m-d}} = (-1)^d\frac{c_{2m-d}}{(km)_{2m-d}}.$$

Reversing this we obtain

$$\lim_{\varepsilon\to 0}\varepsilon^{-d}g(w,\varepsilon) = c_{2m-d} = (-1)^d p_m(x)|_{x^{2m-d}}(km)_{2m-d},$$

as claimed. □

Remark 4.3. Notice that if $f(w,z)$ is d-regular for a positive integer value k, then the assumption $f_{m,n}=0$ for $n>km$ holds automatically.

4.2. **The Chern limit.** The above motivates

Definition 4.4. Suppose a series $f(w,z)$ is d-regular. Its *Chern limit* $f_{\mathrm{Chern}}(w)$ is the series on the right hand side of (4.1).

In relation to our series H_{d_1,d_2} we have

Proposition 4.5. *For $d_1, d_2 \geq -1$ such that $d=-d_1-d_2\geq 0$ the series*

$$H_{d_1,d_2,k}(w,z) := H_{d_1,d_2}(w;\underbrace{z,\dots,z}_{k})$$

is d-regular. The series

$$H_{-1,-1,k}(w,z) + (\mathrm{Li}_3(w) + k\,\mathrm{Li}_3(z)),$$
$$H_{-1,0,k}(w,z) - \frac{1}{2}(\mathrm{Li}_2(w) + k\,\mathrm{Li}_2(z)),$$
$$H_{-1,1,k}(w,z) + \frac{1}{12}(\mathrm{Li}_1(w) + k\,\mathrm{Li}_1(z)),$$
$$H_{0,0,k}(w,z) + \frac{1}{4}(\mathrm{Li}_1(w) + k\,\mathrm{Li}_1(z)),$$

are symmetric.

Remark 4.6. These constraints uniquely determine H_{d_1,d_2} when $d = -d_1 - d_2 > 0$. To see this, suppose we want to determine the coefficients of $w^m z^n$ for a fixed m. By the regularity property it is sufficient to determine these coefficients for $n = 0, 1, \ldots, 2m - d$ since these many values completely determine a polynomial of degree at most $2m-d$. But if n is one of those arguments then $n < 2m$. By the symmetry property the coefficient of $w^m z^n$ is determined by the coefficient of $w^{n-m} z^n$ where $n - m < m$, so we can recursively determine all the coefficients. In the case $d = 0$ the only terms whose coefficients are not determined from the recursion are $w^m z^{2m}$, and these coefficients can be arbitrary.

4.3. **Regular symmetric series.** It should be clear from Remark 4.6 that a symmetric d-regular series with $d > 0$ must be zero. Recall that regular means 0-regular.

Notation. We will in future write D_z for $z\frac{\partial}{\partial z}$ and D_w for $w\frac{\partial}{\partial w}$.

Proposition 4.7. *The following series are symmetric and regular:*

$$\mathcal{C}_k(w,z) := D_z D_w \left(D_z - D_w\right) H_{-1,-1,k}(w,z),$$

$$\mathcal{D}_k(w,z) := D_z \left(H_{-1,0,k}(w,z) + \frac{1}{2} D_z H_{-1,-1,k}(w,z)\right),$$

$$\mathcal{E}_k(w,z) := H_{-1,1,k}(w,z) + \frac{1}{12}\left(D_w\left(D_z - D_w\right) - (D_z)^2\right) H_{-1,-1,k}(w,z),$$

$$\mathcal{F}_k(w,z) := H_{0,0,k}(w,z) + \frac{1}{4}\left(D_w\left(D_z - D_w\right) - (D_z)^2\right) H_{-1,-1,k}(w,z).$$

Proof. The operators D_z and $D_w\left(D_z - D_w\right)$ respect symmetry and send d-regular series to $(d-1)$-regular series. It remains to check the polylogarithm corrections to the symmetry of the above linear combinations cancel out. □

4.4. **Verlinde limit.** In order to understand the specialization (3.11) we need

Definition 4.8. Suppose $k > 2$ is an integer and let $f(w,z) = \sum_{m,n=0}^{\infty} f_{m,n} w^m z^n$ be a d-regular series. The *Verlinde limit* is defined by

$$f_{\mathrm{Ver}}(w) = \sum_{m=0}^{\infty} f_{m,km} w^m.$$

We have

Proposition 4.9. *The Verlinde limits of the series $\mathcal{C}_k$, $\mathcal{D}_k$, $\mathcal{E}_k$, $\mathcal{F}_k$ are given as follows:*

$$\mathcal{C}_{k\ \mathrm{Ver}}(w) = 2k(k-1) D_w C^{V}_{1,1}\left((-1)^{k-1} w\right), \tag{4.2}$$

$$\mathcal{D}_{k\text{ Ver}}(w) = -kD_1^V\left((-1)^{k-1}w\right) + k^2C_{1,1}^V\left((-1)^{k-1}w\right), \tag{4.3}$$

$$\mathcal{E}_{k\text{ Ver}}(w) = E^V\left((-1)^{k-1}w\right) - \frac{1}{6}(k^2-k+1)C_{1,1}^V\left((-1)^{k-1}w\right),$$

$$\mathcal{F}_{k\text{ Ver}}(w) = F^V\left((-1)^{k-1}w\right) - \frac{1}{2}(k^2-k+1)C_{1,1}^V\left((-1)^{k-1}w\right),$$

where in $C_{1,1}^V$, D_1^V, E^V, F^V we need to use $k-1$ instead of k.

Proof. The Verlinde series $H_{d_1,d_2}^V(w;v_1,\ldots,v_k)$ depends only on the sum $\sum_{i=1}^k v_k$

$$H_{d_1,d_2}^V(w;v_1,\ldots,v_k) = H_{d_1,d_2,k}^V\left(w;\sum_{i=1}^k v_i\right)$$

and from (3.11) setting $v_i = \frac{v}{k}$ we obtain

$$H_{d_1,d_2,k}^V(w;v) = (-1)^{d_1+d_2}H_{d_1,d_2,k+1}\left((-1)^k we^v, z\right)_{\text{Ver}}.$$

So we have

$$C_{1,1}^V(w) = \frac{1}{2}\left(D_w\right)^2 H_{-1,-1,k+1\text{ Ver}}\left((-1)^k w\right),$$

Using

$$\left(D_w f(w,z)\right)_{\text{Ver}} = D_w\left(f(w,z)_{\text{Ver}}\right), \quad \left(D_z f(w,z)\right)_{\text{Ver}} = kD_w\left(f(w,z)_{\text{Ver}}\right)$$

we obtain

$$\mathcal{C}_{k\text{ Ver}}(w) = k(k-1)\left(D_w\right)^3 H_{-1,-1,k\text{ Ver}}(w),$$

Combining these identities the statement follows. □

Remark 4.10. By Remark 4.6 a symmetric regular series $f(w,z)$ is determined by the coefficients of $w^m z^{2m}$. It is not hard to see that the three pieces of information:

(i) the coefficients of $w^m z^{2m}$,
(ii) the Chern limit,
(iii) the Verlinde limit

must be related by invertible upper-triangular linear transformations independent of f. Thus in particular the 4 Chern series $C_2^C(w), D_1^C(w), E^C(w), F^C(w)$ for k determine the Verlinde series $C_2^V(w), D_1^V(w), E^V(w), F^V(w)$ and vice versa. In Section 5 we determine these linear transformations precisely.

4.5. **One more series.** We add the following series to the consideration:

$$\mathcal{C}'_k(w,z) := \left(z_1\frac{\partial}{\partial z_1}z_2\frac{\partial}{\partial z_2}H_{-1,-1}(w;z_1,\ldots,z_k)\right)\Bigg|_{z_1=\cdots=z_k=z}.$$

By a direct computation one can check that this series is symmetric and regular. By definition (3.12) we have

$$\mathcal{C}'_k(w,z) = C_2(w,z) + 2C_{1,1}(w,z). \tag{4.4}$$

4.6. **Conclusions.** The logarithms of some of the universal functions $G_i(w,z)$ are linear combinations of regular symmetric series and thus symmetric and regular themselves. We find from the definitions (3.12)–(3.15)

$$H_{0,0,k}(w,z) = F(w,z), \quad H_{-1,1,k}(w,z) = E(w,z), \quad D_z H_{-1,0,k}(w,z) = -kD_1(w,z),$$

$$D_z^2 H_{-1,-1,k}(w,z) = 2k^2 C_{1,1}(w,z) + 2\binom{k}{2} C_2(w,z).$$

Therefore the following series are symmetric and regular

$$\mathcal{D}_k(w,z) = -kD_1(w,z) + k^2 C_{1,1}(w,z) + \binom{k}{2} C_2(w,z),$$

$$3\mathcal{E}_k(w,z) - \mathcal{F}_k(w,z) = 3E(w,z) - F(w,z).$$

Thus, by (3.16), the following universal series are symmetric and regular

$$\log(G_0(w,z)G_1(w,z)) = 2C_{1,1}(w,z) + C_2(w,z) = \mathcal{C}'_k(w,z),$$

$$\log G_3(w,z) = -D_1(w,z) + C_{1,1}(w,z) = \frac{1}{k}\mathcal{D}_k(w,z) - \frac{k-1}{2}\mathcal{C}'_k(w,z),$$

$$\log G_4(w,z) = 3\mathcal{E}_k(w,z) - \mathcal{F}_k(w,z) + \frac{1}{2}(C_{1,1}(w,z) - D_1(w,z)).$$

5. Symmetric regular functions

We completely classify symmetric and regular series by the following:

Theorem 5.1. *Let k be an integer or complex parameter. Suppose a series $f(w,z) \in \mathbb{C}[[w,z]]$ is symmetric and regular. Then there exists a unique power series $h(y) \in \mathbb{C}[[y]]$ such that*

$$f\left(\frac{u(1-u)^{k-1}}{v(1-v)^{k-1}}, \frac{v}{(1-u)^{k-1}}\right) = h\left(\frac{uv}{(1-u)(1-v)}\right). \tag{5.1}$$

Conversely, for any $h(y) \in \mathbb{C}[[y]]$ there exists a unique power series f such the above identity holds and this f is symmetric and regular.

Proof. It may be worth pointing out that the main difficulty was to find the suitable two-variable substitution above. Once the substitution was discovered using computer experiments the proof is straightforward.

A term of the form $w^m z^{m+n}$ in f corresponds to the following function in u, v:

$$u^m v^n (1-u)^{-(k-1)n} (1-v)^{-(k-1)m},$$

from which it is clear that for a given h there exists a unique f satisfying (5.1). Moreover, this f is clearly symmetric. Let us verify that it is also regular. It is sufficient to consider the case $h(y) = y^a$ for some $a \geq 0$. Then we have

$$f\left(\frac{u(1-u)^{k-1}}{v(1-v)^{k-1}}, \frac{v}{(1-u)^{k-1}}\right) = \left(\frac{uv}{(1-u)(1-v)}\right)^a.$$

The coefficient of $w^m z^n$ is given by the double residue

$$\begin{aligned}
&\operatorname{res}_{u=v=0} \left(\frac{uv}{(1-u)(1-v)}\right)^a \left(\frac{u}{(1-v)^{k-1}}\right)^{-m} \left(\frac{v}{(1-u)^{k-1}}\right)^{m-n} \\
&\qquad d\log\left(\frac{u}{(1-v)^{k-1}}\right) \wedge d\log\left(\frac{v}{(1-u)^{k-1}}\right) \\
&= \operatorname{res}_{u=v=0} (1-u)^{-a+(n-m)(k-1)} (1-v)^{-a+m(k-1)} u^{a-m} v^{a+m-n} \\
&\qquad \left(1-(k-1)^2 \frac{uv}{(1-u)(1-v)}\right) \frac{du}{u} \wedge \frac{dv}{v}.
\end{aligned}$$

So it is explicitly given as follows:

$$\begin{aligned}
f(w,z)|_{w^m z^n} &= (-1)^n \binom{-a+(n-m)(k-1)}{m-a} \binom{-a+m(k-1)}{n-m-a} \\
&\quad -(k-1)^2(-1)^n \binom{-a-1+(n-m)(k-1)}{m-a-1} \binom{-a-1+m(k-1)}{n-m-a-1} \\
&= \frac{(k-2)a(n(k-1)-ak)(-1)^n}{(-a+(n-m)(k-1))(-a+m(k-1))} \binom{-a+(n-m)(k-1)}{m-a} \binom{-a+m(k-1)}{n-m-a}.
\end{aligned}$$

Suppose $m \geq a$, otherwise the coefficients vanish. Notice that $\binom{-a+(n-m)(k-1)}{m-a}$ is a polynomial in n of degree $m-a$. When we multiply it by $\frac{n(k-1)-ak}{(-a+(n-m)(k-1))}$ it remains a polynomial of the same degree: either $m > a$ and it is clear, or $m = a$ and the factor is 1. We have

$$\binom{-a+m(k-1)}{n-m-a} = \binom{mk}{n} \frac{(n)_{(m+a)}}{(mk)_{(m+a)}},$$

where $(n)_{(m+a)}$ is a polynomial of degree $m+a$. Thus for

(5.2)
$$p_m(x) = \frac{(k-2)a(x(k-1)-ak)}{(-a+(x-m)(k-1))(-a+m(k-1))} \binom{-a+(x-m)(k-1)}{m-a} \frac{(x)_{(m+a)}}{(mk)_{(m+a)}}$$

the condition in Definition 4.1 is satisfied.

So we have an injective map that sends arbitrary power series h to the corresponding symmetric regular series f. It remains to observe that by choosing h we can achieve arbitrary values for $f(w,z)|_{w^m z^{2m}}$ and using Remark 4.6 conclude that this map is also surjective. □

5.1. **Chern and Verlinde series.** Let us compute the relationship between h and the Chern/Verlinde limits of f.

Theorem 5.2. *Suppose $h(y) = \sum_{a=1}^{\infty} h_a y^a$ corresponds to $f(w,z)$ via Theorem 5.1. Then we have*

$$f_{\mathrm{Chern}}\left(\frac{t}{(1+(k-1)t)^{k-1}}\right) = h\left(\frac{t}{1+(k-1)t}\right),$$
$$f_{\mathrm{Ver}}\left((-1)^k \frac{q}{(1+q)^{(k-1)^2}}\right) = h\left(\frac{q}{1+q}\right).$$

Equivalently, we have

$$f_{\mathrm{Chern}}\left(y(1-(k-1)y)^{k-2}\right) = f_{\mathrm{Ver}}\left((-1)^k y(1-y)^{k(k-2)}\right) = h(y).$$

Proof. Suppose $h(y) = y^a$. By (5.2), the top degree coefficient of $p_m(x)$ is given by

$$\frac{(k-2)(k-1)^{m-a} a}{(-a+m(k-1))(m-a)!(mk)_{(m+a)}}.$$

To obtain the coefficient of $f_{\mathrm{Chern}}(w)$ we need to multiply it by $(mk)_{(2m)}$. So we obtain

$$\begin{aligned} f_{\mathrm{Chern}}(w) &= (k-2)a \sum_{m=a}^{\infty} \frac{(k-1)^{m-a}}{mk-m-a} \binom{mk-m-a}{m-a} w^m \\ &= a \sum_{m=a}^{\infty} \frac{(k-1)^{m-a}}{m} \binom{mk-m-a-1}{m-a} w^m. \end{aligned}$$

Using residues, this can be written as follows:

$$\mathrm{res}_{w=0} \left(D_w f_{\mathrm{Chern}}(w)\right) w^{-m} \frac{dw}{w} = a\, \mathrm{res}_{t=0} (1+(k-1)t)^{mk-m-a-1} t^{a-m} \frac{dt}{t}.$$

The left hand side can be written as

$$\mathrm{res}_{w=0}\, w^{-m} df_{\mathrm{Chern}}(w).$$

Using $y(t) = \frac{t}{1+(k-1)t}$ and $w(t) = \frac{t}{(1+(k-1)t)^{k-1}}$ the right hand side can be written as

$$\mathrm{res}_{t=0}\, w(t)^{-m} dy(t)^a.$$

So we have

$$\operatorname{res}_{t=0} w(t)^{-m} df_{\mathrm{Chern}}(w(t)) = \operatorname{res}_{t=0} w(t)^{-m} dy(t)^a,$$

and since this holds for every $m \geq 1$, we conclude $f_{\mathrm{Chern}}(w(t)) = y(t)^a = h(y(t))$. Since this holds for every a, the statement holds for every h.

For the coefficient of $f_{\mathrm{Ver}}(w)$ we simply set $n = km$ in the formula for $f(w,z)|_{w^m z^n}$ and obtain

$$\frac{k(k-2)a(-1)^{km}}{-a+m(k-1)^2}\binom{-a+m(k-1)^2}{m-a}.$$

So

$$\begin{aligned} f_{\mathrm{Ver}}(w) &= k(k-2)a\sum_{m=a}^{\infty}\frac{(-1)^{km}}{m(k-1)^2-a}\binom{m(k-1)^2-a}{m-a}w^m \\ &= a\sum_{m=a}^{\infty}\frac{(-1)^{km}}{m}\binom{m(k-1)^2-a-1}{m-a}w^m. \end{aligned}$$

Then the proof is completed analogously to the Chern case. □

5.2. **The Verlinde-Segre correspondence.** The Verlinde-Segre correspondence Corollary 1.2, now follows quite easily. By the results of Section 4.3, the series $G_0(w,z)G_1(w,z)$, $G_3(w,z)$ and $G_4(w,z)$ are symmetric and regular. Therefore Theorem 5.1 implies that with the variable changes

$$w = \frac{u(1-u)^{k-1}}{v(1-v)^{k-1}}, \quad z = \frac{v}{(1-u)^{k-1}}, \quad y = \frac{uv}{(1-u)(1-v)},$$

we can write

$$G_0(w,z)G_1(w,z) = h_1(y), \quad G_3(w,z) = h_2(y), \quad G_4(w,z) = h_3(y)$$

for power series $h \in \mathbb{C}[[y]]$. By Theorem 5.2 we have with $r = k-1$ and $x = -y(1-(k-1)y)^{k-2}$, $t = -y(1-y)^{r^2-1}$ that

$$B_1(t) = A_0(x)A_1(x) = h_1(y). \quad B_3(t) = A_3(x) = h_2(y), \quad B_4(t) = A_4(x) = h_3(y).$$

This shows Corollary 1.2.

6. Determination of $G_0(w,z)$, $G_1(w,z)$, $G_2(w,z)$, $G_3(w,z)$

6.1. **An approach.** If $\widetilde{f}(w,z)$ is a 1-regular series, then $f(w,z) = D_z\widetilde{f}(w,z)$ is a regular series. Let $p_m(x)$ be the polynomials corresponding to f via Definition 4.1. Then we have

$$\widetilde{f}(w,0) = \sum_{m=1}^{\infty} w^m \lim_{x\to 0}\frac{p_m(x)}{x}.$$

By passing to the limit in (5.2) we obtain

$$\widetilde{f}(w,0) = -\sum_{a,m} \frac{a^2(k-2)}{m} \frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a)_{(2a+1)}} h_a w^m,$$

where $h = \sum_a h_a y^a$ corresponds to f via Theorem 5.1. We will use this identity in reverse: $\widetilde{f}(w,0)$ will be given, and we will check that our candidate for h is correct.

6.2. **Explicit expressions I.**

Proposition 6.1. *(i) The series $\mathcal{C}_k(w,z)$ via Theorem 5.1 corresponds to*

$$h(y) = -\frac{k(k-1)y}{1-(k-1)^2 y},$$

(ii) the series $\mathcal{C}'_k(w,z)$ via Theorem 5.1 corresponds to

$$h(y) = \log(1-y).$$

Proof. (i) We apply Section 6.1 to the situation

$$\widetilde{f}(w,z) = D_w\left(D_z - D_w\right) H_{-1,-1,k}(w,z), \qquad f(w,z) = \mathcal{C}_k(w,z).$$

The series $\widetilde{f}(w,z) - \mathrm{Li}_1(w)$ is symmetric, which implies $\widetilde{f}(w,0) = \mathrm{Li}_1(w)$. So it is sufficient to verify that for every $m > 0$. we have

$$-\sum_{a=1}^{m} \frac{a^2(k-2)}{m} \frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a)_{(2a+1)}} h_a = \frac{1}{m},$$

where $h_a = -k(k-1)^{2a-1}$ is the a-th coefficient of h. So we need to verify

$$k(k-2)\sum_{a=1}^{m} a^2 (k-1)^{2a-1} \frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a)_{(2a+1)}} = 1. \tag{6.1}$$

From

$$(k-1)^{2a-1}\frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a-1)_{(2a-1)}} - (k-1)^{2a+1}\frac{(m+a)_{(2a+1)}}{(m(k-1)+a)_{(2a+1)}}$$
$$= k(k-2)a^2(k-1)^{2a-1}\frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a-1)_{(2a+1)}}$$

we see that the sum in (6.1) is telescoping and the result is precisely 1.

(ii) By Theorem 5.2 part *(i)* implies

$$\mathcal{C}_{k\ \mathrm{Ver}}\left((-1)^k y(1-y)^{k(k-2)}\right) = -\frac{k(k-1)y}{1-(k-1)^2 y}.$$

Using (4.2) we obtain

$$2C^V_{1,1}\left(-y(1-y)^{k(k-2)}\right) = -\int \frac{y}{1-(k-1)^2 y} d\log\left(y(1-y)^{k(k-2)}\right) = \log(1-y).$$

Recall that $C^V_2(w) = 0$. So by (4.4), the above gives the Verlinde specialization of the symmetric regular series $\mathcal{C}'_k(w,z)$. Theorem 5.2 implies that with the changes of variables (1.4) and $y = \frac{uv}{(1-u)(1-v)}$ we have

$$\mathcal{C}'_k(w,z) = \mathcal{C}'_{k\ \mathrm{Ver}}\left((-1)^k y(1-y)^{k(k-2)}\right) = \log(1-y).$$

□

6.3. **Explicit expressions II.**

Proposition 6.2. *Let $k \geq 1$. The series $\mathcal{D}_k(w,z)$ via Theorem 5.1 corresponds to $h(y) = \sum_{a=1}^{\infty} h_a y^a$ where*

$$h_a = -\frac{k}{2a}\left(\frac{x^{k-1}-x^{1-k}}{x-x^{-1}}\right)^{2a}\Bigg|_{x^0}.$$

Proof. First rewrite the constant term as follows:

$$\left(\frac{1-x^{2k-2}}{1-x^2}\right)^{2a}\Bigg|_{x^{2a(k-2)}} = \left(\frac{1-x^{k-1}}{1-x}\right)^{2a}\Bigg|_{x^{a(k-2)}}$$

$$= \sum_{i=0}^{2a}(-1)^i\binom{2a}{i}(1-x)^{-2a}\Big|_{x^{a(k-2)-i(k-1)}} = \sum_{i=0}^{a-1}(-1)^i\binom{2a}{i}\binom{ak-i(k-1)-1}{2a-1}.$$

We have truncated the summation because for $i \geq a$ we have $a(k-2)-i(k-1) \leq -a < 0$. Next we replace i by $a-i$ to obtain

$$h_a = -k\sum_{i=1}^{a}(-1)^{a-i}\frac{(i(k-1)+a-1)_{(2a-1)}}{(a-i)!(a+i)!}.$$

Notice that the latter expression makes sense for arbitrary k, so we may use it to define h_a for all k. In order to apply Section 6.1, we let

$$\widetilde{f}(w,z) = H_{-1,0,k}(w,z) + \frac{1}{2}D_z H_{-1,-1,k}(w,z),$$

so that $f(w,z) = D_z\mathcal{D}_k(w,z)$. We have $\widetilde{f}(w,0) = \frac{1}{2}\operatorname{Li}_2(w)$. So it is sufficient to verify that for every $m > 0$ we have

$$-\sum_{a=1}^{m}\frac{a^2(k-2)}{m}\frac{(m+a-1)_{(2a-1)}}{(m(k-1)+a)_{(2a+1)}}h_a = \frac{1}{2m^2}. \tag{6.2}$$

We claim that for any $0 < i < m$ the following holds:

$$\sum_{a=i}^{m}(-1)^a a^2\frac{(i(k-1)+a-1)_{(2a-1)}(m+a-1)_{(2a-1)}}{(a-i)!(a+i)!(m(k-1)+a)_{(2a+1)}}=0. \tag{6.3}$$

Notice that

$$\frac{(i(k-1)+a-1)_{(2a-1)}(m+a-1)_{(2a-1)}}{(a-i-1)!(a+i-1)!(m(k-1)+a-1)_{(2a-1)}}+\frac{(i(k-1)+a)_{(2a+1)}(m+a)_{(2a+1)}}{(a-i)!(a+i)!(m(k-1)+a)_{(2a+1)}}$$
$$=a^2(m^2-i^2)k(k-2)\frac{(i(k-1)+a-1)_{(2a-1)}(m+a-1)_{(2a-1)}}{(a-i)!(a+i)!(m(k-1)+a)_{(2a+1)}},$$

where for $a = i$ the first summand on the left hand side is understood as 0. Thus the sum (6.3) telescopes to zero. This implies that when we expand (6.2) into a sum over a and i only the term with $a = i = m$ survives. This term equals

$$\frac{m^2(k-2)}{m}\frac{(2m-1)!}{(km)_{(2m+1)}}\cdot k\frac{(km-1)_{2m-1}}{(2m)!}=\frac{1}{2m^2},$$

which is the right hand side of (6.2). □

By Section 4.3 we have $\log G_3(w,z) = \frac{1}{k}\mathcal{D}_k(w,z) - \frac{k-1}{2}\mathcal{C}'_k(w,z)$. Thus by Propositions 6.1, 6.2 we get with the changes of variables (1.4) and $y = \frac{uv}{(1-u)(1-v)}$ that

$$G_3(w,z)=(1-y)^{-\frac{k-1}{2}}\exp\left(\sum_{n=1}^{\infty}-\frac{y^n}{2n}\left(\frac{x^{k-1}-x^{1-k}}{x-x^{-1}}\right)\bigg|_{x^0}\right).$$

6.4. **Solving differential equations.** We start with some preliminaries. Recall that we denote $D_w := w\frac{\partial}{\partial w}$, $D_z = z\frac{\partial}{\partial z}$. We will also write D_w^{-1} and D_z^{-1} for the inverse operations with zero integration constants. We freely use the changes of variables

$$y=\frac{uv}{(1-u)(1-v)},\qquad w=\frac{u(1-u)^{k-1}}{v(1-v)^{k-1}},\qquad z=\frac{v}{(1-u)^{k-1}}. \tag{6.4}$$

We need to compute partial derivatives $D_w f(u,v)$, $D_z f(u,v)$ of power series $f(u,v)$ with the result expressed in terms of u, v. For this we use the following formulas.

Lemma 6.3.

$$D_w=\frac{u(1-u)(1-v)\frac{\partial}{\partial u}-(k-1)uv(1-v)\frac{\partial}{\partial v}}{1-u-v-(k^2-2k)uv},$$
$$D_z=\frac{u(1-u)(1-kv)\frac{\partial}{\partial u}+v(1-v)(1-ku)\frac{\partial}{\partial v}}{1-u-v-(k^2-2k)uv}.$$

Proof. The chain rule gives

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial u} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial u}, \quad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial w}\frac{\partial w}{\partial v} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial v}.$$

Solving for $\frac{\partial f}{\partial w}$ and $\frac{\partial f}{\partial z}$ gives

$$\frac{\partial f}{\partial w} = \frac{(\frac{\partial z}{\partial u})^{-1}\frac{\partial f}{\partial u} - (\frac{\partial z}{\partial v})^{-1}\frac{\partial f}{\partial v}}{(\frac{\partial z}{\partial u})^{-1}\frac{\partial w}{\partial u} - (\frac{\partial z}{\partial v})^{-1}\frac{\partial w}{\partial v}}, \qquad \frac{\partial f}{\partial z} = \frac{(\frac{\partial w}{\partial u})^{-1}\frac{\partial f}{\partial u} - (\frac{\partial w}{\partial v})^{-1}\frac{\partial f}{\partial v}}{(\frac{\partial w}{\partial u})^{-1}\frac{\partial z}{\partial u} - (\frac{\partial w}{\partial v})^{-1}\frac{\partial z}{\partial v}}.$$

and, using (6.4), this gives the lemma by direct computation. □

Remark 6.4. Denote by $\mathcal{S}$ the involution $\mathcal{S} : g(w, z) \mapsto g(w^{-1}, zw)$, which is the identity for symmetric power series. From the relation $w = \frac{u(1-u)^{k-1}}{v(1-v)^{k-1}}$, $z = \frac{v}{(1-u)^{k-1}}$, we see that $\mathcal{S}u = v$ and $\mathcal{S}v = u$, and thus $\mathcal{S}(f(u, v)) = f(v, u)$.

Proposition 6.5.

$$D_w D_z H_{-1,-1,k}(w, z) = -k \log(1 - u), \tag{6.5}$$

$$D_z^2 H_{-1,-1,k}(w, z) = k\big(\log((1 - u)^{k-1} - v) - k \log(1 - u) - \log(1 - v)\big), \tag{6.6}$$

$$D_w^2 H_{-1,-1,k}(w, z) = \log(1 - u/v) - \log(1 - u). \tag{6.7}$$

Proof. We have $wz = u/(1 - v)^{k-1}$, $z = v/(1 - u)^{k-1}$, and so for a power series $h(w, z) \in \mathbb{C}[[w, z]]$, we find that

$$h(0, z) = h(0, v), \quad h(w, 0) = h(u/v, 0). \tag{6.8}$$

To prove (6.5), we use

$$\mathcal{C}_k(w, z) = (D_z - D_w)D_w D_z H_{-1,-1,k}(w, z). \tag{6.9}$$

By Proposition 6.1 we have with the variable change (6.4) that

$$\mathcal{C}_k(w, z) = -\frac{k(k-1)y}{1 - (k-1)^2 y} = -\frac{k(k-1)uv}{1 - u - v - (k^2 - 2k)uv}$$

Writing $D_w D_z H_{-1,-1,k}(w, z) = \sum_{n,m} f_{n,m} w^n z^m$, we get by symmetry of $\mathcal{C}_k(w, z)$ that

$$\mathcal{C}_k(w, z) = \sum_{n,m}(m - n)f_{n,m}w^n z^m = \sum_{n,m}(m - n)f_{n,m}w^{m-n} z^m = \sum_{n,m} n f_{m-n,m} w^n z^m.$$

Using Remark 6.4 we have

$$D_w^{-1}\mathcal{C}_k(w,z)=\sum_{\substack{n,m\\ n\neq 0}} f_{m-n,m}w^nz^m=\sum_{\substack{n,m\\ n\neq m}} f_{n,m}w^{m-n}z^m=\mathcal{S}\Big(\sum_{\substack{n,m\\ n\neq m}} f_{n,m}w^nz^m\Big).$$

As $\mathcal{S}$ is an involution, we get

$$\begin{aligned}\mathcal{S}\Big(D_w^{-1}\mathcal{C}_k(w,z)+\sum_n f_{n,n}z^n\Big)&=\mathcal{S}D_w^{-1}\mathcal{C}_k(w,z)+\sum_n f_{n,n}w^nz^n\\ &=\sum_{n,m} f_{n,m}w^nz^m=D_wD_zH_{-1,-1,k}(w,z).\end{aligned}$$

We write

$$H_{-1,-1,k}(w,z)+\mathrm{Li}_3(w)+k\,\mathrm{Li}_3(z)=\sum_{n,m} g_{n,m}w^nz^m.$$

As this is symmetric, and we have by definition $H_{-1,-1,k}(0,z)=0$, we get

$$\sum_n g_{n,n}w^nz^n=\sum_n g_{0,n}(wz)^n=k\,\mathrm{Li}_3(zw).$$

This gives

$$\sum_n f_{n,n}z^n=kD_z^2\,\mathrm{Li}_3(z)=k\,\mathrm{Li}_1(z)=-k\log(1-z).$$

Thus we obtain

$$D_wD_zH_{-1,-1,k}(w,z)=\mathcal{S}\Big(D_w^{-1}\mathcal{C}_k(w,z)-k\log(1-z)\Big).$$

Thus the proof of (6.5) is reduced to the explicit identity

$$D_w^{-1}\Big(-\frac{(k-1)uv}{1-u-v-(k^2-2k)uv}\Big)=-\mathcal{S}(\log(1-u))+\log(1-z)=-\log(1-v)+\log(1-z).$$

By (6.8) we have $\log(1-z)\big|_{w=0}=\log(1-v)|_{w=0}$. Thus it is enough to see that

$$D_w(\log(1-v))=-\frac{(k-1)uv}{1-u-v-(k^2-2k)uv}.$$

Using Lemma 6.3, this follows by a direct computation.

Now we deal with (6.6). By (6.9) and (6.5), we have

$$\begin{aligned}D_z^2H_{-1,-1,k}(w,z)&=D_w^{-1}\mathcal{C}_k(w,z)+D_wD_zH_{-1,-1,k}(w,z)\\ &=D_w^{-1}\Big(-\frac{k(k-1)uv}{1-u-v-(k^2-2k)uv}\Big)+k\log(1-u)\end{aligned}$$

Thus it is enough to show the explicit identity

$$D_w^{-1}\Big(-\frac{(k-1)uv}{1-u-v-(k^2-2k)uv}\Big)=\log\big((1-u)^{k-1}-v\big)-(k-1)\log(1-u)-\log(1-v).$$

By (6.8) we have the coefficient of w^0 of the right hand side is $\log(1-v)-\log(1-v)=0$. Thus to show (6.6), we only have to see

$$-\frac{(k-1)uv}{1-u-v-(k^2-2k)uv}=D_w\Big(\log\big((1-u)^{k-1}-v\big)-(k-1)\log(1-u)-\log(1-v)\Big),$$

which again, using Lemma 6.3, follows by a direct computation.

Finally we show (6.7). We see that

$$D_w^2H_{-1,-1,k}(w,z)=D_z^{-1}D_w(D_zD_wH_{-1,-1,k}(w,z))-\mathrm{Li}_1(w),$$

thus we are reduced to the proof of the identity

$$D_z^{-1}D_w(-k\log(1-u))=\log(1-u/v)-\log(1-u)-\log(1-w).$$

By (6.8), the left hand side vanishes at $z=0$, and it is enough to see that

$$D_w(-k\log(1-u))=D_z\big(\log(1-u/v)-\log(1-u)\big),$$

which is again a direct computation using Lemma 6.3. □

6.5. **Determining G_0, G_1, and G_2.** By Proposition 6.1 we have

$$\log(G_0(w,z)\,G_1(w,z))=\mathcal{C}_k'(w,z)=\log(1-y)=\log\left(\frac{1-u-v}{(1-u)(1-v)}\right).$$

By (6.6) we have

$$G_0(w,z)=\exp\big(k\mathcal{C}_k'(w,z)-\frac{1}{k}D_z^2H_{-1,-1,k}(w,z)\big)=\frac{(1-u-v)^k}{(1-v)^{k-1}\big((1-u)^{k-1}-v\big)},$$

which also gives $G_1(w,z)=\frac{(1-v)^{k-2}((1-u)^{k-1}-v)}{(1-u)(1-u-v)^{k-1}}$.

Finally we determine $G_2(w,z)$. Putting $K_k(w,z):=(D_w(D_z-D_w)-D_z^2)\,H_{-1,-1,k}(w,z)$, we get by Proposition 6.5 that

$$\exp\left(K_k(w,z)\right)=\frac{(1-v)^k(1-u)^{k^2-k+1}}{(1-\frac{u}{v})\big((1-u)^{k-1}-v\big)^k}. \tag{6.10}$$

By (3.16) and the definition of $\mathcal{E}_k(w,z)$ and $\mathcal{F}_k(w,z)$ in Proposition 4.7, we have

$$\begin{aligned}\log(G_2(w,z)) &= 24\big(H_{0,0,k}(w,z) - 2H_{-1,1,k}(w,z)\big) - 4C_{1,1}(w,z) \\ &= 24\big(\mathcal{F}_k(w,z) - 2\mathcal{E}_k(w,z)\big) - 2K_k(w,z) - 2\log G_1(w,z).\end{aligned} \tag{6.11}$$

From [EGL01] we have

$$B_2\left(-y(1-y)^{k(k-2)}\right) = \frac{(1-y)^{(k-1)^2}}{1-(k-1)^2y}.$$

By the remarks after (3.16), Proposition 6.1, Theorem 5.2 and Proposition 4.9, this gives

$$24(F^V - 2E^V)\left(-y(1-y)^{k(k-2)}\right) = \big((k-1)^2+2\big)\log(1-y) - \log(1-(k-1)^2y),$$

$$24(\mathcal{F}_{k\ \mathrm{Ver}} - 2\mathcal{E}_{k\ \mathrm{Ver}})\left((-1)^k y(1-y)^{k(k-2)}\right) = (1-k^2)\log(1-y) - \log(1-(k-1)^2y)$$

As $(\mathcal{F}_{k\ \mathrm{Ver}} - 2\mathcal{E}_{k\ \mathrm{Ver}})$ is symmetric and regular, it follows that

(6.12)

$$24\big(\mathcal{F}_k(w,z) - 2\mathcal{E}_k(w,z)\big) = (1-k^2)\log\left(\frac{1-u-v}{(1-u)(1-v)}\right) - \log\left(\frac{1-u-v-(k^2-2k)uv}{(1-u)(1-v)}\right)$$

Thus, combining (6.10),(6.11), (6.12) and the formula for $G_1(w,z)$, we get

$$G_2(w,z) = \frac{(1-\frac{u}{v})^2(1-v)^{(k-2)^2}\big((1-u)^{k-1}-v\big)^{2(k-1)}}{(1-u-v)^{(k-1)^2}(1-u)^{k^2-2k}\big(1-u-v-(k^2-2k)uv\big)}.$$

7. The missing power series

We give evidence for the conjectural formula of Conjecture 2 for $G_4(w,z)$. First we show the product formula Proposition 1.4 for $G_3(w,z)$. We begin with a version of the Lagrange inversion formula that we will use.

Proposition 7.1. *Let $f(y) = y^{-1} + \cdots$ be a Laurent series. Let g be the inverse series of $\frac{1}{f(y)}$. Then we have*

$$\frac{g(u)}{u} = \exp\left(\sum_{n=1}^{\infty} \frac{u^n}{n}(f(y)^n)_{y^0}\right).$$

Proof. Take the logarithmic derivative of both sides. For $n \geq 1$ the coefficient of u^{n-1} on the left is

$$\operatorname{res} \frac{g'(u)}{g(u)} u^{-n} du.$$

Perform the substitution $u = \frac{1}{f(y)}$. Then $g(u) = y$ and the expression above equals

$$\operatorname{res} f(y)^n \frac{dy}{y} = (f(y)^n)_{y^0},$$

which is the term on the right. □

Proposition 7.2. *Let $f(y) = y^{-m} + \cdots$ be a Laurent series for $m \geq 1$. Let $g_1, \ldots, g_m$ be the different branches of the inverse series of $\frac{1}{f(y)}$. Then we have*

$$\frac{\prod_{i=1}^m g_i(u)}{u} = \exp\left(\sum_{n=1}^\infty \frac{u^n}{n}(f(y)^n)_{y^0}\right).$$

Proof. Let $f_1(y) = f(y)^{1/m}$. Let g be the inverse series of $\frac{1}{f_1(y)}$. Then the branches g_i are given by $g_i(u) = g(\zeta^i u^{1/m})$, where ζ is a primitive m-th root of unity. By Proposition 7.1 we have

$$\frac{\prod_{i=1}^m g_i(u)}{u} = \exp\left(\sum_{i=1}^m \sum_{n=1}^\infty \frac{\zeta^{in} u^{n/m}}{n}(f_1(y)^n)_{y^0}\right).$$

From

$$\sum_{i=1}^m \zeta^{in} = \begin{cases} m & (n = mn_1,\ n_1 \in \mathbb{Z}), \\ 0 & \text{otherwise} \end{cases}$$

we obtain the statement. □

Proof of Proposition 1.4. Apply Proposition 7.2 to $f(x) := \left(\frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{x^{\frac{r}{2}} - x^{-\frac{r}{2}}}\right)^2$. Note that $f(x)^n\big|_{x^0} = f(x^2)^n\big|_{x^0}$. Thus we get

$$\frac{y}{\prod_{i=1}^{r-1} \alpha_i(y)} = \exp\left(-\sum_{n=1}^\infty \frac{y^n}{n}\left(\frac{x - x^{-1}}{x^r - x^{-r}}\right)^{2n}\Bigg|_{x^0}\right) = (1-y)^r B_3(-y(1-y)^{r^2-1})^2$$

For $G_3(w, z)$ apply Corollary 1.2. □

We can rewrite the conjectural formula for $B_4(t)$ directly in terms of binomial coefficients. The argument involves again Proposition 7.1. We leave the details to the reader as an exercise. We introduce the following expressions, where the sums are over nonnegative integers.

$$\alpha_n = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j+k=n+2i} (-1)^j \binom{kr+n-1}{2n-1}\binom{2n}{j},$$

$$\beta_n = \sum_{\substack{k+l=n\\k,l>0}} \frac{1}{kl} \sum_{i=0}^{k-1} \left(\sum_{j=0}^{l-1} (-1)^{i+j} \binom{2k}{i} \binom{2l}{j} \sum_{e=1}^{2l} e \cdot \binom{(r+1)l - jr}{2l-e} \binom{(r+1)k - ir}{2k+e} \right.$$
$$\left. + \sum_{j=l}^{\min(2l, n-i-1)} (-1)^{i+j} \binom{2k}{i} \binom{2l}{j} \frac{(jk-li)r}{n} \binom{(r+1)n - r(i+j) - 1}{2n-1} \right),$$

$$\gamma_n = \sum_{\substack{k+l=n\\k,l>0}} \frac{1}{kl} \sum_{a=0}^{min(k,l)} a \left(\sum_{i+j=k-a} (-1)^i \binom{rj+k-1}{2k-1} \binom{2k}{i} \right)$$
$$\cdot \left(\sum_{i'+j'=a+l} (-1)^{i'} \binom{rj'+l-1}{2l-1} \binom{2l}{i'} \right).$$

Proposition 7.3. *With the above notation, Conjecture 2 is equivalent to the following formula for B_4.*

$$B_4(-y(1-y)^{r^2-1}) = \exp\left(\sum_{n=1}^{\infty} \frac{y^n}{8n} \left(4r\alpha_n - r^2 - 3r^{2n} - 2n\beta_n - 2nr^2\gamma_n \right) \right).$$

Note that by Corollary 1.2 with the changes of variables (1.4) and $u = \frac{-(1-v^{-1}}{1-(1-v^{-1})y}$, we have $G_4(w,z) = B_4(-y(1-y)^{r^2-1})$.

The formula of Proposition 7.3 for B_4 is not very attractive, but it has the advantage that it can be easily evaluated by computer to high order in w with r as variable. In a previous attempt to understand the power series B_3, B_4, the first named author computed with Don Zagier the power series $B_4(t)$ modulo t^{50} via a Pari/GP program. This program essentially computes the lowest order terms in a_1, a_2 of the specialization $\Omega^V(w; 0, \ldots, 0; t_1, t_2)$ of the power series of Proposition 3.5 modulo w^{50} with $r = k-1$ as variable. Using Proposition 7.3, a direct Pari/GP computation then shows Proposition 1.5.

References

[Arb21] Noah Arbesfeld, *K-theoretic Donaldson-Thomas theory and the Hilbert scheme of points on a surface*, Algebr. Geom. (2021), no. 8, 587–625.

[Arb22] ———, *K-theoretic descendent series for Hilbert schemes of points on surfaces*, arXiv preprint arXiv:2201.07392 (2022).

[Boj21] Arkadij Bojko, *Wall-crossing for punctual Quot-schemes*, arXiv preprint arXiv:2111.11102 (2021).

[EGL01] Geir Ellingsrud, Lothar Göttsche, and Manfred Lehn, *On the cobordism class of the Hilbert scheme of a surface*, J. Algebraic Geom. **10** (2001), no. 1, 81–100. MR 1795551

[ES96] Geir Ellingsrud and Stein Arild Strømme, *Bott's formula and enumerative geometry*, J. Amer. Math. Soc **9** (1996), no. 1, 175–193.

[GHT99] A. M. Garsia, M. Haiman, and G. Tesler, *Explicit plethystic formulas for Macdonald q,t-Kostka coefficients*, Sém. Lothar. Combin. **42** (1999), Art. B42m, 45pp, The Andrews Festschrift (Maratea, 1998). MR 1701592 (2001b:05222)

[GK20] Lothar Göttsche and Martijn Kool, *Virtual Segre and Verlinde numbers of projective surfaces*, arXiv preprint arXiv:2007.11631 (2020).

[Göt21] Lothar Göttsche, *Blowup formulas for Segre and Verlinde numbers of surfaces and higher rank Donaldson invariants*, arXiv:2109.13144 (2021).

[HMW19] Tamás Hausel, Martin Mereb, and Michael Lennox Wong, *Arithmetic and representation theory of wild character varieties*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 10, 2995–3052. MR 3994099

[Joh] Drew Johnson, *Universal series for Hilbert schemes and Strange Duality*, International Mathematics Research Notices **2020**, 3130–3152.

[Leh99] Manfred Lehn, *Chern classes of tautological sheaves on Hilbert schemes of points on surfaces*, Inventiones mathematicae **136** (1999), 157–207.

[LP] Joseph Le Potier, *Dualité étrange sur le plan projectif*, Unpublished 2005.

[Mel16] Anton Mellit, *Plethystic identities and mixed Hodge structures of character varieties*, arXiv preprint arXiv:1603.00193 (2016).

[Mel18] ———, *Integrality of Hausel-Letellier-Villegas kernels*, Duke Math. J. **167** (2018), no. 17, 3171–3205. MR 3874651

[MOP17a] Alina Marian, Dragos Oprea, and Rahul Pandharipande, *Higher rank Segre integrals over the Hilbert scheme of points*, arXiv preprint arXiv:1712.02382 (2017).

[MOP17b] ———, *Segre classes and Hilbert schemes of points*, Annales Scientifiques de l'ENS **501** (2017), 239–267.

[MOP19] ———, *The combinatorics of Lehn's conjecture*, J. Math. Soc. Japan **71** (2019), no. 1, 299–308. MR 3909922

[Ver] Eric Verlinde, *Fusion rules and modular transformations in 2d conformal field theory*, Nucl. Phys. B **300**, 360–376.

[Voi19] Claire Voisin, *Segre classes of tautological bundles on Hilbert schemes of surfaces*, Algebr. Geom **6** (2019), 186–195.

[Yua22] Yao Yuan, *Rank zero Segre integrals on Hilbert schemes of points on surfaces*, arXiv preprint arXiv:2209.06600 (2022).