MULTIFRACTAL ANALYSIS AND ERDÖS-RÉNYI LAWS OF LARGE NUMBERS FOR BRANCHING RANDOM WALKS IN \mathbb{R}^d

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ABSTRACT. We revisit the multifractal analysis of \mathbb{R}^d -valued branching random walks averages by considering subsets of full Hausdorff dimension of the standard level sets, over each infinite branch of which a quantified version of the Erdös-Rényi law of large numbers holds. Assuming that the exponential moments of the increments of the walks are finite, we can indeed control simultaneously such sets when the levels belong to the interior of the compact convex domain I of possible levels, i.e. when they are associated to so-called Gibbs measures, as well as when they belong to the subset $(\partial I)_{\rm crit}$ of ∂I made of levels associated to "critical" versions of these Gibbs measures. It turns out that given such a level of one of these two types, the associated Erdös-Rényi LLN depends on the metric with which is endowed the boundary of the underlying Galton-Watson tree. To extend our control to all the boundary points in cases where $\partial I \neq (\partial I)_{\rm crit}$, we slightly strengthen our assumption on the distribution of the increments to exhibit a natural decomposition of $\partial I \setminus (\partial I)_{\rm crit}$ into at most countably many convex sets J of affine dimension $\leq d-1$ over each of which we can essentially reduce the study to that of interior and critical points associated to some $\mathbb{R}^{\dim J}$ -valued branching random walk.

1. Introduction and main results

Let T be a supercritical Galton-Watson tree jointly constructed with a branching random walk taking values in the Euclidean space \mathbb{R}^d , $d \geq 1$: on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, there exists a random vector $(N, X = (X_i)_{i \in \mathbb{N}}) \in \mathbb{N} \times (\mathbb{R}^d)^{\mathbb{N}}$, as well as $\{(N_u, (X_{ui})_{i \in \mathbb{N}})\}_{u \in \bigcup_{n \geq 0} \mathbb{N}^n}$, a family of independent copies of (N, X) indexed by the finite words over the alphabet \mathbb{N} (with the convention that \mathbb{N}^0 contains only the empty word denoted ϵ) such that: $(i) \mathbb{E}(N) > 1$, $\mathsf{T} = \bigcup_{n=0}^{\infty} \mathsf{T}_n$, where $\mathsf{T}_0 = \{\epsilon\}$, and $\mathsf{T}_n = \{ui : u \in \mathsf{T}_{n-1}, 1 \leq i \leq N_u\}$ for all $n \geq 1$. The boundary of T is then the set $\partial \mathsf{T}$ of infinite words $t_1 \cdots t_n \cdots$ over \mathbb{N} such that $t_1 \cdots t_k \in \mathsf{T}_k$ for all $k \geq 1$. (ii) For each $t = t_1 t_2 \cdots \in \mathbb{N}^{\mathbb{N}}$ and $n \geq 0$, setting

$$S_n X(\omega, t) = \sum_{i=1}^n X_{t_1 \cdots t_n}(\omega),$$

the restriction of $(S_n X)_{n\geq 0}$ to ∂T is the branching random walk on ∂T to be considered in this paper.

When N is a constant interger ≥ 2 and the components of X are identically distributed and non constant, the family $\{(S_nX(t))_{n\in\mathbb{N}}\}_{t\in\partial T}$ provides uncountably many random walks with the same law, and it turns out that the large deviations properties shared by these random walks have a counterpart in ∂T in the sense that if we consider $E_X = \{t \in \partial T : A_X(t) := \lim_{n\to\infty} n^{-1}S_nX(t) \text{ exists}\}$, the set $A_X(E_X)$ is almost surely

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equal to a deterministic non trivial closed convex set. The same property holds if we consider a general branching random walk as defined above. Quantifying geometrically this phenomenon, that is measuring the sizes of the level sets $A_X^{-1}(\alpha)$, $\alpha \in A_X(E_X)$, as well as that of subsets over which the law of large numbers like property $A_X(t) = \alpha$ is refined by Erdös-Rényi law of large numbers, is the purpose of the present paper.

Conditionally on non extinction of T, that is $\partial T \neq \emptyset$, if the set ∂T is endowed with some metric d, the multifractal analysis of the averages of $(S_n X)_{n \in \mathbb{N}}$ consists in computing the Hausdorff dimensions of the level sets

$$E(X,\alpha) = \left\{ t \in \partial \mathsf{T} : \lim_{n \to \infty} \frac{S_n X(t)}{n} = \alpha \right\}, \ \alpha \in \mathbb{R}^d$$

and thus provides a geometric hierarchy between the level sets $E(X, \alpha)$. A general result (Theorem A below) was obtained in [3] for these dimensions in the case that d is the restriction to ∂T of the standard ultrametric distance on $\mathbb{N}^{\mathbb{N}}$ defined by

$$d_1(s,t) = e^{-|s \wedge t|},$$

where $s \wedge t$ is the maximal common prefix of s and t and the length of any word w in $(\bigcup_{n>0} \mathbb{N}^n) \cup \mathbb{N}^{\mathbb{N}}$ is denoted by |w|.

Define the Legendre transform of a function $f: \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that $f \not\equiv \infty$ as

$$f^*: \alpha \in \mathbb{R}^d \mapsto \inf\{f(q) - \langle q | \alpha \rangle : q \in \mathbb{R}^d\}.$$

Also, define the function

(1.1)
$$\widetilde{P}_X : q \in \mathbb{R}^d \mapsto \log \mathbb{E} \sum_{i=1}^N \exp(\langle q | X_i \rangle)$$

as well as

(1.2)
$$I_X = \{ \alpha \in \mathbb{R}^d : \widetilde{P}_X^*(\alpha) \ge 0 \}.$$

From now on we work conditionally on non extinction of T, so without loss of generality we assume that $\mathbb{P}(N \ge 1) = 1$. The authors proved the following result :

Theorem A ([3, Theorem 1.1]) With probability 1,

(1.3)
$$\forall \alpha \in \mathbb{R}^d, \ \dim E(X, \alpha) = \begin{cases} \widetilde{P}_X^*(\alpha) & \text{if } \alpha \in I_X \\ -\infty & \text{otherwise.} \end{cases}$$

In this paper, dim stands for the Hausdorff dimension, and we adopt the convention that for any set $E \subset \mathbb{N}^{\mathbb{N}}$, dim $E = -\infty$ if and only if $E = \emptyset$. This result is a geometric counterpart of the following large deviations properties associated with S_nX ([3, Theorem 1.3]): For $u \in \mathsf{T}_n$ define $S_nX(u)$ as the constant value taken by S_nX restricted to the set of elements of $\partial \mathsf{T}$ with common prefix of generation n equal to u. With probability 1, for all $\alpha \in \mathbb{R}^d$,

$$\lim_{\varepsilon \to 0^+} \liminf_{n \to \infty} \frac{\log \#\{u \in \mathsf{T}_n, \ n^{-1}S_nX(u) \in B(\alpha, \varepsilon)\}}{n}$$

$$= \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{\log \#\{u \in \mathsf{T}_n, \ n^{-1}S_nX(u) \in B(\alpha, \varepsilon)\}}{n} = \begin{cases} \widetilde{P}_X^*(\alpha) & \text{if } \alpha \in I_X \\ -\infty & \text{otherwise} \end{cases},$$

where $B(\alpha, \varepsilon)$ stands for the closed Euclidean ball of radius ε centered at α .

We aim at strengthening these information in two directions: at first quantify how, for $t \in E(X, \alpha)$, the local averages $n^{-1}(S_{j+n}X(t) - S_jX(t))$ $(j \geq 0)$ can deviate from α . This will be done by using quantified Erdös-Rényi laws of large numbers (see the next paragraphs for the definition). Specifically, we will seek for subsets $\widetilde{E}(X,\alpha)$ of $E(X,\alpha)$ of full Hausdorff dimension and for all points of which the local averages of $(S_nX(t))_{n\in\mathbb{N}}$ obey the same quantified Erdös-Rényi law. Also, we will measure the effect of changing the standard metric to a metric associated to some branching random walk, both by providing the new values for the Hausdorff dimensions of the sets $E(X,\alpha)$, and observing how the quantified Erdös-Rényi law invoked in $\widetilde{E}(X,\alpha)$ may vary with the metric.

To begin, let us precise what we mean by quantified Erdös-Rényi law of large numbers. To concretely observe such laws in our context, we naturally select points in ∂T according to some Mandelbrot measures. To define a Mandelbrot measure on ∂T , jointly with $(\partial T, (S_n X)_{n\geq 0})$, consider a family $\{(N_u, (X_{ui}, \psi_{u_i})_{i\geq 1})\}_{u\in\bigcup_{n\geq 0}\mathbb{N}^n}$ of independent copies of a random vector $(N, (X, \psi) = (X_i, \psi_i)_{i\geq 1})$ taking values in $\mathbb{N} \times (\mathbb{R}^d \times \mathbb{R})^{\mathbb{N}}$, still on $(\Omega, \mathcal{A}, \mathbb{P})$.

Assume that

$$\mathbb{E}\sum_{i=1}^{N}\exp(\psi_i)=1,\ \mathbb{E}\sum_{i=1}^{N}\psi_i\exp(\psi_i)<1\ \mathrm{and}\ \mathbb{E}\Big(\sum_{i=1}^{N}\exp(\psi_i)\Big)\log^+\Big(\sum_{i=1}^{N}\exp(\psi_i)\Big)<\infty.$$

Then (see [25, 8, 27]), for each $u \in \bigcup_{n \geq 0} \mathbb{N}^n$, defining $\mathsf{T}(u) = \bigcup_{n=0}^{\infty} \mathsf{T}_n(u)$, where $\mathsf{T}_0(u) = \{u\}$, and $\mathsf{T}_n(u) = \{vi: v \in \mathsf{T}_{n-1}(u), 1 \leq i \leq N_v\}$ for all $n \geq 1$, the sequence

(1.4)
$$Y_n(u) = \sum_{v=v_1\cdots v_n\in\mathsf{T}(u)} \exp(\psi_{uv_1} + \cdots + \psi_{uv_1\cdots v_n})$$

is a positive uniformly integrable martingale of expectation 1 with respect to its natural filtration. We denote by Y(u) its almost sure limit. By construction, the random variables so obtained are identically distributed and almost surely positive.

Now, for each $u \in \bigcup_{n\geq 0} \mathbb{N}^n$, let [u] denote the cylinder $u \cdot \mathbb{N}^{\mathbb{N}}$ and denote by \mathcal{B} the σ -algebra generated by these cylinders in $\mathbb{N}^{\mathbb{N}}$ (which is nothing but the Borel σ -algebra associated with d_1). Then, define

$$\nu([u]) = \mathbf{1}_{\{u \in \mathsf{T}\}} \exp(\psi_{u_1} + \dots + \psi_{u_1 \dots u_n}) Y(u).$$

Due to the branching property $Y(u) = \sum_{i=1}^{N_u} \exp(\psi_{ui}) Y(ui)$, this yields a non-negative additive function of the cylinders, which can be extended into a random measure $\nu_{\omega} = \nu_{\psi,\omega}$ on $(\mathbb{N}^{\mathbb{N}}, \mathcal{B})$, whose topological support is $\partial \mathsf{T}$. Consider the so-called Peyrière probability measure \mathcal{Q} on $(\Omega \times \mathbb{N}^{\mathbb{N}}, \mathcal{A} \otimes \mathcal{B})$, defined by

$$Q(C) = \int_{\Omega} \int_{\mathbb{N}^{\mathbb{N}}} \mathbf{1}_{C}(\omega, t) \, d\nu_{\omega}(t) \, d\mathbb{P}(\omega).$$

The random vectors $\widetilde{X}_n: (\omega, t) \in \Omega \times \mathbb{N}^{\mathbb{N}} \mapsto X_{t_1 \cdots t_n}(\omega), \ n \geq 1$, are independent and identically distributed with respect to \mathcal{Q} , and by definition, given $\omega \in \Omega$, $S_n \widetilde{X}(\omega, \cdot) = \sum_{k=1}^n \widetilde{X}_k(\omega, \cdot)$ coincides with $S_n X(\omega, \cdot)$ on $\partial \mathsf{T}$.

Let

$$\Lambda_{\psi}: \lambda \in \mathbb{R}^d \mapsto \log \mathbb{E}_{\mathcal{Q}}\left(\exp(\langle \lambda | \widetilde{X}_1 \rangle)\right) = \log \mathbb{E}\left(\sum_{i=1}^N \exp(\langle \lambda | X_i \rangle + \psi_i)\right).$$

Suppose that Λ_{ψ} is finite on an open convex subset $\mathcal{D}_{\Lambda_{\psi}}$ of \mathbb{R}^d . Suppose also that $\mathcal{D}_{\Lambda_{\psi}}$ contains 0, so that $\nabla \Lambda_{\psi}(0)$ is well defined, and one has

(1.5)
$$\nabla \Lambda_{\psi}(0) = \mathbb{E}_{\mathcal{Q}}(\widetilde{X}_1) = \mathbb{E}\left(\sum_{i=1}^{N} X_i \exp(\psi_i)\right); \text{ set } \beta_{\Lambda_{\psi}} = \nabla \Lambda_{\psi}(0).$$

By the strong law of large numbers $S_n\widetilde{X}(\omega,t)/n$ tends to $\beta_{\Lambda_{\psi}}$ as $n\to\infty$, \mathcal{Q} -almost surely. In other words, for \mathbb{P} -almost every ω , the measure ν_{ω} is supported on the set $E(X,\beta_{\Lambda_{\psi}})$. Note that $\max(\Lambda_{\psi}^*) = \Lambda_{\psi}^*(\beta_{\Lambda_{\psi}}) = 0$ since $\Lambda_{\psi}(0) = 0$.

The classical Erdös-Rényi law of large numbers [14] applied to $(S_n\widetilde{X})_{n\in\mathbb{N}}$ claims that if \widetilde{X} is real valued (i.e. d=1) and is not \mathcal{Q} -almost surely equal to the constant $\beta_{\Lambda_{\psi}} = \Lambda'_{\psi}(0)$, then for all $\alpha > \beta_{\Lambda_{\psi}}$ in $\mathcal{D}_{\Lambda_{\psi}}$, one has

$$\lim_{N\to\infty} \max_{0\leq j\leq N-\lfloor c_\alpha\log(N)\rfloor} (S_{j+\lfloor c_\alpha\log(N)\rfloor}\widetilde{X} - S_j\widetilde{X})/\lfloor c_\alpha\log(N)\rfloor = \alpha, \text{ where } c_\alpha^{-1} = -\Lambda_\psi^*(\alpha).$$

This can be reformulated as follows: Let $\widetilde{k} = (k(n))_{n \in \mathbb{N}}$ be an increasing sequence of integers. If $\lim_{n \to \infty} n^{-1} \log(k(n)) = -\Lambda_{\psi}^*(\alpha)$, then one has

$$\lim_{n \to \infty} \max_{0 < j < nk(n) - 1} (S_{j+n}\widetilde{X} - S_j\widetilde{X})/n = \alpha.$$

The quantified version of the Erdös-Rényi law of large numbers established in [6] for $(S_n\widetilde{X})_{n\in\mathbb{N}}$ (see specifically [6, section 3.4]) as a consequence of a more general statement valid for some class of weakly correlated processes, corresponds to the following large deviations properties: If $\widetilde{k} = (k(n))_{n\in\mathbb{N}}$ is an increasing sequence of integers, for $t \in \partial T$, B a Borel subset of \mathbb{R}^d , $\lambda \in \mathbb{R}^d$, $n \geq 1$ and $0 \leq j \leq nk(n) - 1$, consider the following two objects: the empirical measure associated with the nk(n) first normalised increments $n^{-1}(S_{j+n}X(t) - S_j(t))$ along the branch t,

(1.6)
$$\mu_{\tilde{k},n}^t = \frac{1}{nk(n)} \sum_{j=0}^{nk(n)-1} \delta_{\frac{S_{j+n}X(t)-S_jX(t)}{n}},$$

as well as the logarithmic moment generating function

(1.7)
$$\Lambda_{\tilde{k},n}^t(\lambda) = \log \int_{\mathbb{R}^d} \exp(n\langle \lambda | x \rangle) \, \mathrm{d}\mu_{\tilde{k},n}^t(x).$$

Theorem B ([6, section 3.4]). Let $\widetilde{k} = (k(n))_{n \in \mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ be increasing. With probability 1, for ν -almost every $t \in \partial \mathsf{T}$, the following large deviations properties $\mathrm{LD}(\Lambda_{\psi}, \widetilde{k})$ hold:

$$LD(\Lambda_{\psi}, \widetilde{k})$$
:

(1) for all
$$\lambda \in \mathcal{D}_{\Lambda_{\psi}}$$
 such that $\liminf_{n \to \infty} \frac{\log(k(n))}{n} > -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, one has

$$\lim_{n \to \infty} \frac{1}{n} \Lambda_{\tilde{k},n}^t(\lambda) = \Lambda_{\psi}(\lambda);$$

Hence, for all $\lambda \in \mathcal{D}_{\Lambda_{\psi}}$ such that $\liminf_{n \to \infty} \frac{\log(k(n))}{n} > -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, due to the Gartner-Ellis theorem [13, 12], one has

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \mu_{\tilde{k},n}^t(B(\nabla \Lambda(\lambda), \epsilon)) = \Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda)).$$

In other words

$$\lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log \frac{\#\left\{0 \le j \le nk(n) - 1 : \frac{S_{j+n}X(t) - S_jX(t)}{n} \in B(\nabla \Lambda_{\psi}(\lambda), \epsilon)\right\}}{k(n)} = \Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda)).$$

- (2) For all $\lambda \in \mathcal{D}_{\Lambda_{\psi}}$ such that $\limsup_{n \to \infty} \frac{\log(k(n))}{n} < -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, there exists $\epsilon > 0$ such that for n large enough, $\left\{ 0 \le j \le nk(n) 1 : \frac{S_{j+n}X(t) S_jX(t)}{n} \in B(\nabla \Lambda_{\psi}(\lambda), \epsilon) \right\} = \emptyset$.
- (3) If $\lambda \in \mathcal{D}_{\Lambda_{\psi}}$ and $\lim_{\substack{n \to \infty \\ \text{convex at 1, then for all } \theta \geq 1 \text{ one has}}} \frac{\log k(n)}{n} = -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, and if $\theta \geq 0 \mapsto \Lambda_{\psi}(\theta \lambda)$ is strictly

$$\lim_{n \to \infty} \frac{1}{n} \Lambda_{\tilde{k},n}^t(\theta \lambda) = \Lambda_{\psi}(\lambda) + (\theta - 1) \langle \lambda | \nabla \Lambda_{\psi}(\lambda) \rangle.$$

Remark 1.1. Notice that by convention the concave Legendre transform defined in this paper, which is convenient to express Hausdorff dimensions of level sets, is the opposite of the more standard convex convention used in [6]).

In fact, in [6] such a large deviation principle is established for the k(n) "disjoint" increments $(S_{(j+1)n}X(t) - S_{jn}X(t))_{0 \le j \le k(n)-1}$, but as we will see in this paper, one can extend this result to a large deviation principle valid for $(S_{j+n}X(t) - S_{j}X(t))_{0 \le j \le nk(n)-1}$, which is more faithful to the spirit of the original Erdös-Rényi law of large numbers. Moreover, the validity of $LD(\Lambda_{\psi}, \widetilde{k})$ for all sequences \widetilde{k} implies the validity of this law.

Thus, with probability 1, ν is in fact supported on the finer set

$$E(X, \beta_{\Lambda_{\psi}}, \mathrm{LD}(\Lambda_{\psi}, \widetilde{k})) = \left\{ t \in \partial \mathsf{T} : \lim_{n \to \infty} \frac{S_n X(t)}{n} = \beta_{\Lambda_{\psi}} \text{ and } \mathrm{LD}(\Lambda_{\psi}, \widetilde{k}) \text{ holds} \right\}.$$

It turns out that if we define

$$\widetilde{\boldsymbol{K}} = \left\{ \widetilde{k} \in \mathbb{N}^{\mathbb{N}} : \ \widetilde{k} \ \text{is increasing and} \ \lim_{n \to \infty} \frac{\log(k(n))}{n} \ \text{exists} \ \right\},$$

and say that $LD(\Lambda_{\psi})$ holds if $LD(\Lambda_{\psi}, \widetilde{k})$ holds for all $k \in \widetilde{K}$, the previous theorem has the following rather direct corollary (see Section 7):

Corollary 1.1. With probability 1, for ν -almost every infinite branch $t \in \partial T$, $LD(\Lambda_{\psi})$ holds. Thus, ν is supported on

$$E(X,\beta_{\Lambda_\psi},\operatorname{LD}(\Lambda_\psi)) = \Big\{t \in \partial \mathsf{T}: \ \lim_{n \to \infty} \frac{S_nX(t)}{n} = \beta_{\Lambda_\psi} \ \ and \ \operatorname{LD}(\Lambda_\psi) \ \ holds \Big\}.$$

Our goal is to refine Theorem A by finding, for a given $\alpha \in I_X$, that is such that $E(X,\alpha) \neq \emptyset$, a differentiable convex function Λ_{α} finite over an open neighborhood $\mathcal{D}_{\Lambda_{\alpha}}$ of 0, such that $\Lambda_{\alpha}(0) = 0$, $\alpha = \nabla \Lambda_{\alpha}(0)$, and the sets $E(X,\alpha,\mathrm{LD}(\Lambda_{\alpha},\widetilde{k}))$ and $E(X,\alpha,\mathrm{LD}(\Lambda_{\alpha}))$

are of maximal Hausdorff dimension in $E(X, \alpha)$. This requires the finiteness of some exponential moments of ||X||, and for the sake of simplicity of the discussion and exposition of our results, we assume that

$$\widetilde{P}_X(q) < \infty, \ \forall \, q \in \mathbb{R}^d.$$

This is equivalent to requiring that $\mathbb{E}\left(\sum_{i=1}^{N} \exp(\lambda ||X_i||)\right) < \infty$ for all $\lambda \geq 0$. We will discuss some possible relaxation of this assumption in Section 8.

Without loss of generality, we also assume the following property about X:

$$(1.9) \exists (q,c) \in (\mathbb{R}^d \setminus \{0\}) \times \mathbb{R}, \langle q | X_i \rangle = c \quad \forall \ 1 \le i \le N \text{ almost surely (a.s.)}.$$

If (1.9) does not hold, either d = 1 and the X_i , $1 \le i \le N$, are equal to the same constant almost surely, which is a trivial case, or $d \ge 2$, and the X_i belong to the same affine hyperplane so that we can reduce our study to the case of \mathbb{R}^{d-1} valued random variables.

Define

$$J_X = \left\{ q \in \mathbb{R}^d : \widetilde{P}_X^*(\nabla \widetilde{P}_X(q)) > 0 \right\}.$$

It turns out that under (1.8) and (1.9), I_X (recall (1.2)) is a compact set with non-empty interior, such that (see Proposition 2.1)

$$\mathring{I}_X = \nabla \widetilde{P}(J_X).$$

When $\alpha \in \mathring{I}_X$, i.e. $\alpha = \nabla \widetilde{P}_X(q)$ for some $q \in J_X$, setting $\psi_\alpha = (\exp(\langle q|X_i\rangle - \widetilde{P}_X(q)))_{i\geq 1}$ and assuming that $\mathbb{E}\left(\sum_{i=1}^N \exp(\psi_{\alpha,i})\right) \log^+\left(\sum_{i=1}^N \exp(\psi_{\alpha,i})\right) < \infty$, one obtains a non degenerate Mandelbrot measure $\nu_\alpha = \nu_{\psi_\alpha}$ associated with the "potential" ψ_α , also called Gibbs measure associated with X at q, and the previous discussion shows that given an increasing sequence of integers \widetilde{k} , with probability 1, the measure ν_α is supported on $E(X,\alpha,\mathrm{LD}(\Lambda_{\psi_\alpha},\widetilde{k}))$; moreover, due to (1.8), we can take $\mathcal{D}_{\Lambda_{\psi_\alpha}} = \mathbb{R}^d$. Moreover, Λ_{ψ_α} is strictly convex due to (1.9). Also, the Hausdorff dimension of ν_α equals $\dim E(X,\alpha)$, which yields $\dim E(X,\alpha,\mathrm{LD}(\Lambda_{\psi_\alpha},\widetilde{k})) = \dim E(X,\alpha)$ almost surely.

Then, several questions arise. Let us state and comment them:

(Q1) Is it possible to get the previous property a.s. simultaneously for all $\alpha \in \mathring{I}_X$?

It is of course closely related to the possibility to estimate almost surely simultaneously the Hausdorff dimensions of the sets $E(X,\alpha)$, $\alpha \in I_X$. For $\alpha \in \mathring{I}_X$, this can be done under slightly stronger assumptions by constructing simultaneously the Gibbs measures ν_{α} (thanks to a uniform convergence result due to Biggins [9]), and simultaneously for all $\alpha \in \mathring{I}_X$ controlling the Haudorff dimension of ν_{α} and showing that this measure is carried by $E(X,\alpha)$ (see [4, 1]). So one may think that an adaptation of this "uniform" approach should give a positive answer to (Q1), since using the Gibbs measure ν_{α} for each individual $\alpha \in \mathring{I}_X$ does give dim $E(X,\alpha,\operatorname{LD}(\Lambda_{\psi_{\alpha}},\widetilde{k})) = \dim E(X,\alpha)$ almost surely. However, this strategy meets an essential difficulty (see Remark 5.1). To overcome it, inspired by techniques used in ergodic theory for the multifractal analysis of Birkhoff averages on hyperbolic attractors [17, 19], we will use a concatenation/approximation method to get inohomogeneous Mandelbrot measures adapted to our problem.

(Q2) When $\alpha \in \partial I_X$, is there some $(\Lambda, \mathcal{D}_{\Lambda})$ (depending on α) such that the equality $\dim E(X, \alpha, \mathrm{LD}(\Lambda, \widetilde{k})) = \dim E(X, \alpha)$ holds?

It will be first answered positively for those α belonging to the subset $(\partial I_X)_{\text{crit}}$ of ∂I_X made of levels that can be associated to "critical" versions of the Gibbs measures mentioned above; for such a level α there is indeed a natural candidate $\Lambda_{\psi_{\alpha}}$ as well. Extending our control to all the boundary points of I_X demands to be able to associate to each α of $\partial I_X \setminus (\partial I_X)_{\text{crit}}$ some quantified Erdös-Rényi LLN, possibly explicit in terms of the parameter (N,X). However, we even do not have any good description of $\partial I_X \setminus (\partial I_X)_{\text{crit}}$ at our disposal yet. We will strengthen a little (1.8) and show that there is a natural decomposition of $\partial I \setminus (\partial I)_{\text{crit}}$ into at most countably many convex sets I of affine dimension I0 over each of which we can essentially reduce the study to that of interior and critical points associated to some I1 over each of which we can essentially reduce the study to that of interior and critical points associated to some I2 of explicit Erdös-Rényi LLN.

(Q3) For $\alpha \in I_X$, would it be possible that there were several couples $(\Lambda, \mathcal{D}_{\Lambda})$ (with distinct Λ) such that dim $E(X, \alpha, LD(\Lambda, \widetilde{k})) = \dim E(X, \alpha)$ holds?

It remains open when $\dim E(X,\alpha) > 0$, and there is no uniqueness in general when $\dim E(X,\alpha) = 0$ (see Remark 1.4(4)).

(Q4) If dim $E(X, \alpha, LD(\Lambda, k)) = \dim E(X, \alpha)$ with respect to some metric, how does Λ depend on the choice of the metric?

As it was said previously, we are going to consider natural metrics obtained from branching random walks. We will compute dim $E(X,\alpha)$, $\alpha \in I_X$, with respect to such a metric and show that for the same levels α as under d₁, one has dim $E(X,\alpha, LD(\Lambda, \widetilde{k})) = \dim E(X,\alpha)$ for some large deviations properties $LD(\Lambda, \widetilde{k})$ which depends both on α and the metric.

The more general metrics we consider on $\partial \mathsf{T}$ are constructed jointly with $(\partial \mathsf{T}, (S_n X)_{n\geq 0})$ as follows: consider a family $\{(N_u, (X_{ui}, \phi_{ui})_{i\geq 1})\}_{u\in \bigcup_{n\geq 0}\mathbb{N}^n}$ of independent copies of a random vector $(N, (X, \phi) = (X_i, \phi_i)_{i\geq 1})$ taking values in $\mathbb{N} \times (\mathbb{R}^d \times \mathbb{R}_+^*)^{\mathbb{N}}$. Again, $(\Omega, \mathcal{A}, \mathbb{P})$ stands for the probability space over which these random variables are defined. Denote by $S_n \phi$ the (positive) branching random walk on $\partial \mathsf{T}$ associated with the family $\{(N_u, (\phi_{ui})_{i\geq 1})\}_{u\in \bigcup_{n\geq 0}\mathbb{N}^n}$. Symmetrically to (1.8), assume that

(1.10)
$$P_{\phi}(t) = \mathbb{E}\left(\sum_{i=1}^{N} \exp(\lambda \phi_i)\right) < \infty, \ \forall \ \lambda \in \mathbb{R}.$$

Then (see Lemma 2.1), with probability 1, $S_n\phi(u)$ tends uniformly in $u \in \mathsf{T}_n$ to ∞ as $n \to \infty$, so that one gets the random ultrametric distance

$$(1.11) d_{\phi}: (s,t) \mapsto \exp(-S_{|s \wedge t|}\phi(t))$$

on ∂T , and $(\partial T, d_{\phi})$ is compact. Such metrics are used to study geometric realization of Mandelbrot measures on random statistically self-similar sets ([22, 15, 31, 29, 4, 28, 10]).

We now define a family of convex functions which will be essential to describe the Hausdorff dimensions of the sets $E(X,\alpha)$ under d_{ϕ} . For all $(q,\alpha,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, let

(1.12)
$$\Sigma_{\alpha}(q,t) = \sum_{i=1}^{N} \exp(\langle q|X_i - \alpha \rangle - t\phi_i).$$

Under (1.8) and (1.10), $\mathbb{E}(\Sigma_{\alpha}(q,t))$ is finite for all $(q,\alpha,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, and since the ϕ_i are positive, for each $q \in \mathbb{R}^d$ and $\alpha \in \mathbb{R}^d$ there exists a unique $t = \widetilde{P}_{X,\phi,\alpha}(q) \in \mathbb{R}$ such that

$$(1.13) \mathbb{E}(\Sigma_{\alpha}(q,t)) = 1$$

(we indicate the dependence on (X,ϕ) in $\widetilde{P}_{X,\phi,\alpha}$ in order to avoid confusion with \widetilde{P}_X or \widetilde{P}_{ϕ}). Moreover, it is direct to see that $(\alpha,q)\mapsto \widetilde{P}_{X,\phi,\alpha}(q)$ is real analytic by using the implicit function theorem and the real analyticity of $(\alpha,q,t)\mapsto \mathbb{E}(\Sigma_{\alpha}(q,t))$.

Notice that $\widetilde{P}_{X,\phi,\alpha}(0)$ does not depend on α ; it turns out that it is the Hausdorff dimention of $\partial \mathsf{T}$ under d_{ϕ} . Notice also that when $\phi_i = 1$ for all $i \geq 1$, one has $\mathrm{d}_{\phi} = \mathrm{d}_1$, and $\widetilde{P}_{X,\phi,\alpha}(q) = \widetilde{P}_X(q) - \langle q | \alpha \rangle$, hence $\widetilde{P}_{X,\phi,\alpha}^*(0) = \widetilde{P}_X^*(\alpha)$.

Set

$$(1.14) J_{X,\phi} = \{ (q,\alpha) \in \mathbb{R}^d \times I_X : \widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) > 0 \}.$$

We assume also that

$$(1.15) \qquad \forall (q,\alpha) \in J_{X,\phi}, \ \exists \ \gamma > 1, \ \mathbb{E}\left(\left(\Sigma_{\alpha}(q,\widetilde{P}_{X,\phi,\alpha}(q))^{\gamma}\right) < \infty,\right)$$

which is automatically satisfied as soon as $\mathbb{E}(N^p) < \infty$ for some p > 1 and both (1.8) and (1.10) hold. This assumption is quite natural in the sense that it is equivalent to requiring that the total mass of the Mandelbrot measure associated with $\psi_{\alpha,q} = \left(\langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i\right)_{i\geq 1}$ does not vanish and belongs to $L^{\gamma}(\Omega,\mathbb{P})$ for some $\gamma > 1$; conditions like (1.15) are required in [9] to construct simultaneously the limits of the martingales (1.4) when ψ varies in the family $\{\psi_{\alpha}\}_{\alpha\in \mathring{I}_X}$.

Under the assumptions adopted in this paper, Theorem A has the following extension.

Theorem 1.1. Assume (1.8), (1.9), (1.10) and (1.15), and suppose that ∂T is endowed with the distance d_{ϕ} .

With probability 1, for all $\alpha \in I_X$ one has $\dim E(X,\alpha) = \widetilde{P}_{X,\phi,\alpha}^*(0)$. More generally, for any compact subset K of \mathbb{R}^d , let

$$E(X,K) = \Big\{ t \in \partial \mathsf{T} : \bigcap_{n \in \mathbb{N}} \overline{\Big\{ \frac{S_n X(t)}{n} : n \ge N \Big\}} = K \Big\},$$

the set of those $t \in \partial T$ such that the set of limit points of $(S_nX(t)/n)_{n\in\mathbb{N}}$ is equal to K. Denote by \mathscr{K} the set of compact connected subsets of \mathbb{R}^d . With probability 1, for all $K \in \mathscr{K}$, one has dim $E(X,K) = \inf_{\alpha \in K} \widetilde{P}^*_{X,\phi,\alpha}(0)$.

Notice that contrarily to what happens when ∂T is endowed with d_1 , in general the mapping $\alpha \in I_X \mapsto \dim E(X,\alpha) = \widetilde{P}_{X,\phi,\alpha}^*(0)$ is not concave when ∂T is endowed with d_{ϕ} . For instance, when $X_i = \phi_i$ (note that in this case d = 1), the distortion induced by d_{ϕ} can be observed by noting that in this case $\widetilde{P}_{X,\phi,\alpha}(q) = q - \widetilde{P}_X^{-1}(\alpha q)$, which implies $\widetilde{P}_{X,\phi,\alpha}^*(0) = \widetilde{P}_X^*(\alpha)/\alpha$ for $\alpha \in I_X$. Also, Theorem 1.1 should be compared to those obtained in [7, 19, 32] in the context of Birkhoff averages on conformal repellers.

Next we state our results on quantified Erdös-Rényi laws. They require to assume the following property:

(1.16)
$$\sup_{q \in \mathbb{R}^d} \mathbb{E}\left(\sum_{i=1}^N \phi_i \exp(\langle q|X_i\rangle - \widetilde{P}_X(q))\right) < \infty,$$

which holds as soon as $\|\sup_{1\leq i\leq N} \mathbb{E}(\phi_i|\sigma(N,X))\|_{\infty} < \infty$. We also need the following proposition. Let

$$(1.17) \widetilde{I}_X = \mathring{I}_X \cup (\partial I_X)_{\text{crit}}, \text{ where } (\partial I_X)_{\text{crit}} = \{\alpha \in \nabla \widetilde{P}_X(\mathbb{R}^d) : \widetilde{P}_X^*(\alpha) = 0\}.$$

Proposition 1.1. Assume (1.8), (1.9), (1.10), (1.15) and (1.16). For all $\alpha \in \widetilde{I}_X$, there exists a unique $q = q_\alpha$ such that $\nabla \widetilde{P}_{X,\phi,\alpha}(q) = 0$, hence $\widetilde{P}_{X,\phi,\alpha}(q_\alpha) = \widetilde{P}_{X,\phi,\alpha}^*(0)$. Moreover, the mapping $\alpha \mapsto q_\alpha$ is real analytic over \mathring{I}_X and continuous over \widetilde{I}_X .

Then, observe that if $\alpha \in \mathring{I}_X$, with probability 1, $E(X,\alpha)$ carries a Mandelbrot measure of maximal Hausdorff dimension $\widetilde{P}_{X,\phi,\alpha}^*(0)$ with respect to d_{ϕ} , namely the non degenerate Mandelbrot measure ν_{α} associated with $\psi_{\alpha} = \psi_{X,\phi,\alpha} = \left(\langle q_{\alpha}|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q_{\alpha})\phi_i\right)_{i\geq 1}$. The domain of the associated strictly convex function

$$\Lambda_{\psi_{\alpha}} : \lambda \in \mathbb{R}^d \mapsto \log \mathbb{E} \Big(\sum_{i=1}^N \exp(\langle \lambda | X_i \rangle + \psi_{\alpha,i}) \Big)$$

is \mathbb{R}^d , and the quantified Erdös-Rényi law of large numbers of Theorem B holds with $\psi = \psi_{\alpha}$. If $\alpha \in (\partial I_X)_{\text{crit}}$, ψ_{α} and $\Lambda_{\psi_{\alpha}}$ can be defined as above as well. There is no associated Mandelbrot measure, but what is called a critical Mandelbrot measure ν_{α}^c associated to α (see [21, 26, 4, 5, 11] for the definition and geometric properties of these objects). However, though ν_{α}^c can be used to show that $E(X,\alpha) \neq \emptyset$, there is no associated Peyrière measure, so using ν_{α}^c to get dim $E(X,\alpha,\text{LD}(\Lambda_{\psi_{\alpha}},\tilde{k})) = \dim E(X,\alpha)(=0)$ is not possible.

Theorem 1.2. Assume (1.8), (1.9), (1.10), (1.15) and (1.16). Let k be an increasing sequence of integers. With probability 1, for all $\alpha \in \widetilde{I}_X$, dim $E(X, \alpha, LD(\Lambda_{\psi_\alpha}, \widetilde{k})) = \dim E(X, \alpha)$.

Corollary 1.2. Assume (1.8), (1.9), (1.10), (1.15) and (1.16). With probability 1, for all $\alpha \in \widetilde{I}_X$, dim $E(X, \alpha, LD(\Lambda_{\psi_{\alpha}})) = \dim E(X, \alpha)$.

As an example, take N deterministic, $\phi = (1)_{i \geq 1}$, and X_1, \ldots, X_N , N Gaussian vectors with at least one of them non degenerate. Then \widetilde{P}_X is strictly convex and quadratic, $\widetilde{I}_X = I_X$, and $\partial I_X = (\partial I_X)_{\text{crit}}$ is an hyperellipsoid. However, in general $I_X \setminus \widetilde{I}_X = \partial I_X \setminus (\partial I_X)_{\text{crit}}$ may be non empty, and even equal to ∂I_X . This is for instance the case when (N, X_1, \ldots, X_N) is deterministic and satisfies (1.9); indeed, it is easily checked that in this case one has $J_X = \mathbb{R}^d$ so that $(\partial I_X)_{\text{crit}} = \emptyset$.

To complete Theorem 1.2, we need to slightly strengthen the assumptions. First, we replace (1.8) and (1.10) by

(1.18)
$$\mathbb{E}\left(\sum_{i=1}^{N} \exp\left(\psi(\|X_i\|)\right) + \exp\left(\psi(\phi_i)\right)\right) < \infty$$

for some convex non decreasing function $\psi: \mathbb{R}_+ \to \mathbb{R}_+$ such that $\lim_{x\to\infty} \psi(x)/x = \infty$. Also, we assume that $\mathbb{E}(N^p) < \infty$ for some p > 1. Note that when the ϕ_i are constant, this condition is implied by (1.15) (consider q = 0); also, we already observed that together with (1.8) and (1.10) it implies (1.15)). Moreover, since we will have to guaranty that (1.16) holds for various branching random walks deduced from $(S_n X, S_n \phi)$ by restriction to some subtrees, we will assume from the outset that

(1.19)
$$\|\sup_{1 \le i \le N} \mathbb{E}(\phi_i | \sigma(N, X)) \|_{\infty} < \infty.$$

Theorem 1.3. (First formulation) Assume that $\mathbb{E}(N^p) < \infty$ for some p > 1, as well as (1.9), (1.18) and (1.19). If $I_X \setminus \widetilde{I}_X \neq \emptyset$, there exists an explicit family of differentiable convex functions $(\Lambda_{\alpha})_{\alpha \in I_X \setminus \widetilde{I}_X}$ with \mathbb{R}^d as domain, such that: for every increasing sequence of integers \widetilde{k} , with probability 1, for all $\alpha \in I_X \setminus \widetilde{I}_X$, one has dim $E(X, \alpha, LD(\Lambda_{\alpha}, \widetilde{k})) = \dim E(X, \alpha)$.

Corollary 1.3. Assume that $\mathbb{E}(N^p) < \infty$ for some p > 1, as well as (1.9), (1.18) and (1.19). Suppose that $I_X \setminus \widetilde{I}_X \neq \emptyset$, and let $(\Lambda_{\alpha})_{\alpha \in I_X \setminus \widetilde{I}_X}$ be as in Theorem 1.3. With probability 1, for all $\alpha \in I_X \setminus \widetilde{I}_X$, one has dim $E(X, \alpha, LD(\Lambda_{\alpha})) = \dim E(X, \alpha)$.

Making explicit the family $(\Lambda_{\alpha})_{\alpha \in I_X \setminus \widetilde{I}_X}$ requires additional definitions. As mentioned above, our approach will exhibit and use a natural decomposition of $I_X \setminus \widetilde{I}_X$, essentially as a union of at most countably many convex subsets of the form \widetilde{I}_Y , where $Y = (Y_i)_{i \in \mathbb{N}}$ defines the increments of some branching random walk, and the components of Y take values in some strict affine subspace of \mathbb{R}^d .

Decomposition of $I_X \setminus \widetilde{I}_X$ and explicitation of $(\Lambda_\alpha)_{\alpha \in I_X \setminus \widetilde{I}_X}$. Denote by \mathcal{C}_X the closure of the convex subset of \mathbb{R}^d defined as the set of vectors α of the form $\mathbb{E}(\sum_{i=1}^N W_i X_i)$, where $(W_i)_{i \geq 1}$ is a non negative random element of $\mathbb{R}_+^{\mathbb{N}}$ jointly defined with $(N, (X_i)_{i \geq 1})$, such that $\mathbb{E}(\sum_{i=1}^N W_i) = 1$. It is easily seen that \mathcal{C}_X is bounded if and only if the X_i , $1 \leq i \leq \|N\|_\infty$, are uniformly bounded $(\|N\|_\infty)$ may be infinite).

If $F \subset \mathcal{B}(\mathbb{R}^d)$, set $N^F = \#\{1 \leq i \leq N: X_i \in F\}$, and if $\mathbb{E}(N^F) > 0$, set

$$\alpha_F = \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_F(X_i)X_i\Big)/\mathbb{E}(N^F).$$

We refer to [35, Ch. 18] for an introduction to the geometric properties of convex sets. Let \mathcal{H}_X be the set of supporting affine hyperplanes of the close convex set \mathcal{C}_X , and $\widetilde{\mathcal{H}}_X$ be the set of those elements H of \mathcal{H}_X such that $\mathbb{E}(N^H) \geq 1$. Also, let \mathcal{F}_X be the set of affine subspaces F of \mathbb{R}^d such that $F \subset H$ for some $H \in \mathcal{H}_X$ and

$$\widehat{\mathcal{F}}_X = \{ F \in \mathcal{F}_X : \mathbb{E}(N^F) > 1 \text{ and } \forall G \in \mathcal{F}_X, G \subseteq F, \mathbb{E}(N^G) < \mathbb{E}(N^F) \}.$$

If $F \in \widehat{\mathcal{F}}_X$ and $\mathbb{E}(N^F) > 1$, to the integers $N_u^F = \#\{1 \leq i \leq N_u : X_{ui} \in F\}$, $u \in \bigcup_{n \geq 0} \mathbb{N}^n$, are naturally associated two trees : the supercritical Galton-Watson tree $\widetilde{\mathsf{T}}^F$ defined as T but with the branching numbers N_u^F instead of the N_u , and the subtree T^F of T defined as $\bigcup_{n \geq 0} \mathsf{T}_n^F$, where $\mathsf{T}_0^F = \{\epsilon\}$ and for $n \geq 1$, $\mathsf{T}_n^F = \{ui : u \in \mathsf{T}_{n-1}^F, 1 \leq i \leq N_u, X_{ui} \in F\}$. Denote by F the vector subspace $F - \alpha_F$, and denote by

 $X_F - \alpha_F$ the random vector $(X_{i_1} - \alpha_F, \dots, X_{i_{NF}} - \alpha_F, 0, \dots, 0, \dots)$, where i_1, \dots, i_{N_F} are the indices $i \in [1, N]$ such that $X_i \in F$, ranked in increasing order; also define $\phi_F = (\phi_{i_1}, \dots, \phi_{i_{NF}}, 0, \dots, 0, \dots)$. Similarly, define $X_{F,u} - \alpha_F = (X_{ui_1} - \alpha_F, \dots, X_{ui_{N_u^F}} - \alpha_F, 0, \dots, 0, \dots)$ and $\phi_{F,u} = (\phi_{ui_1}, \dots, \phi_{ui_{N_u^F}}, 0, \dots, 0, \dots)$ for all $u \in \bigcup_{n \geq 0} \mathbb{N}^{\mathbb{N}}$. The trees $\widetilde{\mathsf{T}}^F$ and T^F are in bijection via the mapping \mathbf{b}_F defined by $\mathbf{b}_F(\epsilon) = \epsilon$ and once \mathbf{b}_F is defined as a bijection between $\widetilde{\mathsf{T}}_n^F$ and T_n^F for some $n \geq 0$, if $u \in \widetilde{\mathsf{T}}_n^F$, one sets $\mathbf{b}_F(uj) = \mathbf{b}_F(u)i_j$ for $1 \leq j \leq N_{\mathbf{b}_F(u)}^F$. Conditionally on non-extinction of $\widetilde{\mathsf{T}}^F$ (and so $\widetilde{\mathsf{T}}^F$), \mathbf{b}_F extends naturally into a bijection between $\partial \widetilde{\mathsf{T}}^F$ and $\partial \mathsf{T}^F$. Also, the branching random walk $(S_n(X_F - \alpha_F))_{n \in \mathbb{N}}$ on ∂T^F is related to the F-valued branching random walk $(S_n(X_F - \alpha_F))_{n \in \mathbb{N}}$ associated with the random vectors $(N_u^F, (X_{F,u} - \alpha_F)), u \in \bigcup_{n \geq 0} \mathbb{N}^\mathbb{N}$, on $\partial \widetilde{\mathsf{T}}^F$ via the equality $S_nX(t) - n\alpha_F = S_n(X_F - \alpha_F)(\mathbf{b}_F^{-1}(t))$, and \mathbf{b}_F^{-1} is an isometry between ∂T^F and $\partial \widetilde{\mathsf{T}}^F$ endowed with the restriction of d_ϕ and the metric d_{ϕ_F} respectively.

By construction, the function $\widetilde{P}_{X_F-\alpha_F}$ associated to $(N^F, X_F - \alpha_F)$ as \widetilde{P}_X is to (N, X) in (1.1) satisfies

$$\widetilde{P}_{X_F - \alpha_F}(q) = \log \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_F(X_i) \exp(\langle q|X_i - \alpha_F \rangle)\Big)$$
 for all $q \in \vec{F}$.

If, moreover, dim $F \geq 1$, it easily seen that under the assumptions of Theorem 1.3, $X_F - \alpha_F$ satisfies the same assumptions as X since $F \in \widehat{\mathcal{F}}_X$ and $\mathbb{E}(N^F) > 1$. One associates to $\widetilde{P}_{X_F - \alpha_F}$ the sets $I_{X_F - \alpha_F}$ and $\widetilde{I}_{X_F - \alpha_F}$ in the same way as I_X and \widetilde{I}_X are associated to \widetilde{P}_X , that is $I_{X_F - \alpha_F} = \{\beta \in F : (\widetilde{P}_X^F)^*(\beta) \geq 0\}$, and $\widetilde{I}_{X_F - \alpha_F} = \{\nabla \widetilde{P}_{X_F - \alpha_F}(q) : q \in F, (\widetilde{P}_{X_F - \alpha_F})^*(\nabla \widetilde{P}_{X_F - \alpha_F}(q)) \geq 0\}$. Then set $I_X^F = \alpha_F + I_{X_F - \alpha_F}$ and $\widetilde{I}_X^F = \alpha_F + \widetilde{I}_{X_F - \alpha_F}$. The set I_X^F has non-empty relative interior in F, that we denote by \mathring{I}_X^F , and $\alpha_F \in \mathring{I}_X^F$.

One also defines $\mathcal{C}_{X,F}$, the closure of the convex subset of F defined as the set of vectors α of the form $\mathbb{E}(\sum_{i=1}^N W_i X_i)$, where $(W_i)_{i \in \mathbb{N}}$ is a non negative random element of $\mathbb{R}_+^{\mathbb{N}}$ jointly defined with $(N, (X_i)_{i \in \mathbb{N}})$, such that $\mathbb{E}(\sum_{i=1}^N W_i) = 1$ and $\mathbb{E}(\sum_{i=1}^N W_i) = \mathbb{E}(\sum_{i=1}^N \mathbf{1}_F(X_i) W_i)$. Note that $\mathcal{C}_X = \mathcal{C}_{X,\mathbb{R}^d}$. To $\mathcal{C}_{X,F}$ are associated the sets $\mathcal{H}_{X,F}$, $\widetilde{\mathcal{H}}_{X,F}$ and $\widehat{\mathcal{F}}_{X,F}$ as $\mathcal{H}_X = \mathcal{H}_{X,\mathbb{R}^d}$, $\widetilde{\mathcal{H}}_X = \widetilde{\mathcal{H}}_{X,\mathbb{R}^d}$ and $\widehat{\mathcal{F}}_X = \widehat{\mathcal{F}}_{X,\mathbb{R}^d}$ are to \mathcal{C}_X . Note that if dim F = 0, then, conditionally on $\partial \mathsf{T}^F \neq \emptyset$, the branching random walk $(S_n X)_{n \in \mathbb{N}}$ restricted to $\partial \mathsf{T}^F \neq \emptyset$ equals $(n\alpha_F)_{n \in \mathbb{N}}$.

If $F \in \widehat{\mathcal{F}}_X$, for β and q in \vec{F} , define $\widetilde{P}_{X_F - \alpha_F, \phi_F, \beta}(q)$ in the same way as $\widetilde{P}_{X,\phi,\alpha}(q)$, that is as the unique solution of the equation

$$\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) \exp(\langle q|X_{i} - \alpha_{F} - \beta \rangle - t\phi_{i})\Big) = 1.$$

If, moreover, $\mathbb{E}(N^F) > 1$ and dim $F \geq 1$, according to Proposition 1.1 for all $\beta \in \widetilde{I}_{X-\alpha_F}^F$ there exists a unique $q_{\beta}^F \in \vec{F}$ such that $\widetilde{P}_{X_F-\alpha_F,\phi_F,\beta}(q_{\beta}^F) = (\widetilde{P}_{X_F-\alpha_F,\phi_F,\beta})^*(0)$. For all $\alpha \in \widetilde{I}_X^F$, we then set for all $\lambda \in \mathbb{R}^d$

$$\Lambda_{\alpha}^{F}(\lambda) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) \exp\left(\langle \lambda | X_{i} \rangle + \langle q_{\alpha-\alpha_{F}}^{F} | X_{i} - \alpha \rangle - \widetilde{P}_{X_{F}-\alpha_{F},\phi_{F},\alpha-\alpha_{F}}(q_{\alpha-\alpha_{F}}^{F})\phi_{i}\right)\Big).$$

Theorems 1.1 and 1.2 yield that given an increasing sequence of integers k, with probability 1, conditionally on $\partial \mathsf{T}_F \neq \emptyset$, for all $\alpha \in I_X^F$, one has:

 $\dim(E(X,\alpha)\cap\partial\mathsf{T}_F)=(\widetilde{P}_{X_F-\alpha_F,\phi_F,\alpha-\alpha_F})^*(0), \text{ and for all }\alpha\in\widetilde{I}_X^F, \text{ one has }\dim(E(X,\alpha)\cap\mathcal{T}_F)$ $\partial \mathsf{T}_F) = \dim(E(X, \alpha, \mathrm{LD}(\Lambda_\alpha^F, \widetilde{k})) \cap \partial \mathsf{T}_F).$

Remark 1.2. One checks that choosing another reference point $\widetilde{\alpha}_F \in F$ and considering the branching random walk associated with $X_F - \widetilde{\alpha}_F$ in \vec{F} on $\partial \mathsf{T}^F$ yields the same functions Λ_{α}^{F} . This is due to the fact that if $\alpha = \alpha_{F} + \beta = \widetilde{\alpha}_{F} + \widetilde{\beta}$, then for all $q \in \overrightarrow{F}$ one has $\widetilde{P}_{X_F-\alpha_F,\phi_F,\beta}(q) = \widetilde{P}_{X_F-\widetilde{\alpha}_F,\phi_F,\widetilde{\beta}}(q).$

If
$$F \in \widehat{\mathcal{F}}_X$$
 is the singleton $\{\alpha_F\}$, set $I_X^F = \widetilde{I}_X^F = \{\alpha_F\}$ and $\Lambda_{\alpha_F}^F : \lambda \in \mathbb{R}^d \mapsto \langle \lambda | \alpha_F \rangle$.

Note that in this case $(\Lambda_{\alpha_F}^F)^* = -\infty \cdot \mathbf{1}_{\mathbb{R}^d \setminus \{\alpha_F\}}$, so that for any increasing sequence of integers \widetilde{k} , if property $LD(\Lambda_{\alpha_F}^F, \widetilde{k})$ holds for $(S_n X)_{n \in \mathbb{N}}$ on some infinite branch, it takes a trivial form, and does not depend on ϕ .

If $F \in \widehat{\mathcal{F}}_X$ and $\mathbb{E}(N^F) = 1$, set $I_X^F = \widetilde{I}_X^F = \{\alpha_F\}$ and

$$\Lambda_{\alpha_F}^F: \lambda \in \mathbb{R}^d \mapsto \log \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_F(X_i) \exp\big(\langle \lambda | X_i \rangle\big)\Big).$$

Finally, we select a subcollection of $\widehat{\mathcal{F}}_{X,\mathbb{R}}$. First, define for any affine subspace E of \mathbb{R} $\widetilde{\mathcal{F}}_{X,E} = \{ F \in \widehat{\mathcal{F}}_{X,E} : \dim F \ge 1, \text{ and } \mathbb{E}(N^F) = \mathbb{E}(N^H) > 1 \text{ for all } H \in \mathcal{H}_{X,E}, F \subset H \},$ $\overline{\mathcal{F}}_{X,E} = \{ F \in \widehat{\mathcal{F}}_{X,E} : \mathbb{E}(N^F) = \mathbb{E}(N^H) \text{ for all } H \in \mathcal{H}_{X,E}, F \subset H \} \setminus \widetilde{\mathcal{F}}_{X,E}.$

Note that for $\widetilde{\mathcal{F}}_{X,E}$ to be non empty it is necessary that dim $E \geq 2$, and if $F \in \overline{\mathcal{F}}_{X,E}$ then either $\mathbb{E}(N^F) = 1$, or $\mathbb{E}(N^F) > 1$ and dim F = 0.

Then, define $\widetilde{\mathcal{F}}_X^1 = \widetilde{\mathcal{F}}_{X,\mathbb{R}^d}$, $\overline{\mathcal{F}}_X^1 = \overline{\mathcal{F}}_{X,\mathbb{R}^d}$, and for $2 \leq i \leq d$, $\widetilde{\mathcal{F}}_X^i = \widetilde{\mathcal{F}}_X^{i-1} \bigcup_{F \in \widetilde{\mathcal{F}}_X^{i-1}} \widetilde{\mathcal{F}}_{X,F}$ and $\overline{\mathcal{F}}_X^i = \overline{\mathcal{F}}_X^{i-1} \bigcup \bigcup_{F \in \widetilde{\mathcal{F}}_X^{i-1}} \overline{\mathcal{F}}_{X,F}$. Note that with the convention $\widetilde{\mathcal{F}}_X^0 = \emptyset$, the elements of $\widetilde{\mathcal{F}}_X^i \setminus \widetilde{\mathcal{F}}_X^{i-1}$ have dimension at most d-i, and $\widetilde{\mathcal{F}}_X^{d-1} = \widetilde{\mathcal{F}}_X^d$.

Note that $\widetilde{\mathcal{F}}_X^d \cup \overline{\mathcal{F}}_X^d$ is at most countable. Indeed, the mapping $\mu : F \in \mathcal{B}(\mathbb{R}^d) \mapsto \mathbb{E}(N^F)$ is a finite positive Borel measure, so it cannot assign positive mass to uncountably many affine subspaces F of \mathbb{R}^d such that $\mu(G) < \mu(F)$ for all affine subspaces G of F.

Theorem 1.3. Assume that $\mathbb{E}(N^p) < \infty$ for some p > 1, as well as (1.9), (1.18) and (1.19). The following properties hold:

- (1) $\widetilde{\mathcal{H}}_X = \{ H \in \mathcal{H}_X : H \cap I_X \neq \emptyset \}.$ (2) $\widetilde{I}_X \subset \mathring{\mathcal{C}}_X$ and $I_X \setminus \widetilde{I}_X = \bigcup_{H \in \widetilde{\mathcal{H}}_X} H \cap I_X$. In particular, $I_X = \widetilde{I}_X$ if and only if $\widetilde{\mathcal{H}}_X = \emptyset$. Also, $I_X \setminus \widetilde{I}_X = \bigsqcup_{F \in \mathcal{F}_x^d \cup \overline{\mathcal{F}}_X^d} \widetilde{I}_X^F$.
- (3) One has $I_X \setminus \widetilde{I}_X = \partial I_X$, i.e. $(\partial I_X)_{\text{crit}} = \emptyset$, if and only if $\widetilde{\mathcal{H}}_X = \mathcal{H}_X$. Moreover, in this case \mathcal{C}_X is a convex polytope and $\mathcal{C}_X = I_X$, which is equivalent to saying that for any exposed point P of C_X one has $\mathbb{E}(N^{\{\vec{P}\}}) \geq 1$.
- (4) With probability 1, for all $F \in \widetilde{\mathcal{F}}_X^d \cup \overline{\mathcal{F}}_X^d$:

- If $\mathbb{E}(N^F) = 1$, then $F \cap I_X = I_X^F = \widetilde{I}_X^F = \{\alpha_F\}$, and dim $E(X, \alpha_F) = 0$. If $\mathbb{E}(N^F) > 1$ then $F \cap I_X = I_X^F$ and for all $\alpha \in F \cap I_X$ one has dim $E(X, \alpha) = \widetilde{I}_X^F = \widetilde{I$ $P_{X,\phi,\alpha}^*(0) = (P_{X_F-\alpha_F,\phi_F,\alpha-\alpha_F})^*(0).$
- (5) Suppose that $I_X \setminus \widetilde{I}_X \neq \emptyset$. Let \widetilde{k} be an increasing sequence of integers. With probability 1, for all $F \in \widetilde{\mathcal{F}}_X^d \cup \overline{\mathcal{F}}_X^d$, for all $\alpha \in \widetilde{I}_X^F$, one has dim $E(X, \alpha, LD(\Lambda_\alpha^F, \widetilde{k})) =$

Remark 1.3. Except the claim about the partition of $I_X \setminus \widetilde{I}_X$, the properties stated in items (1) to (3) of the previous statement hold under (1.8) and (1.9) only. See Proposition 6.1.

Some examples. To illustrate the previous result, we focus on examples where \mathcal{C}_X is compact. Let K be a compact convex subset of \mathbb{R}^d , with non-empty interior. Fix μ a Borel probability measure either fully supported on ∂K or on the set of extremal points of K. Suppose that $(N, (X_i)_{i \in \mathbb{N}})$ is chosen so that the X_i are identically distributed with law μ , and independent of N. It is clear that $\mathcal{C}_X \subset K$, since for any half-space $V = \{\beta \in \mathbb{R}^d :$ $\langle q|\beta\rangle \leq c\}$ $((q,c)\in\mathbb{S}^{d-1}\times\mathbb{R})$ which contains K and any $\alpha=\mathbb{E}(\sum_{i=1}^N W_iX_i)\in\mathcal{C}_X$, the fact that almost surely for all $1\leq i\leq N$ one has $\langle q|X_i\rangle\leq c$ hence $\langle q|W_iX_i\rangle\leq W_ic$, implies $\langle q|\alpha\rangle\leq\mathbb{E}(\sum_{i=1}^N W_i)c=c$, that is $\alpha\in V$. To see the reverse inclusion, fix $\alpha\in K$ and a Borel probability measure on ν_{α} supported on ∂K such that $\alpha = \int_{\partial K} \beta \, d\nu_{\alpha}(\beta)$. Fix $\varepsilon > 0$ and a finite partition $\mathcal{A}_{\varepsilon} = \{A_j\}_{j=1}^{p_{\varepsilon}}$ of ∂K into Borel subsets of positive μ -measure and diameter less than ε . Define the sequence of identically distributed nonnegative random variables $W_{\mathcal{A}_{\varepsilon},i} = (\mathbb{E}(N))^{-1} \sum_{j} \mathbf{1}_{A_{j}}(X_{i}) \frac{\nu_{\alpha}(A_{i})}{\mu(A_{i})}, i \geq 1$. Note that $\mathbb{E}(\sum_{i=1}^{N} W_{\mathcal{A}_{\varepsilon},i}) = 1$. As ε tends to 0, by construction $\mathbb{E}(\sum_{i=1}^{N} W_{A_{\varepsilon},i} X_{i})$ tends to α . Hence $\alpha \in \mathcal{C}_{X}$.

In the following examples we pick special examples corresponding to the situation just described.

- (1) Suppose that K is a convex polytope with n vertices P_1, \ldots, P_n . Take $\mu = \sum_{j=1}^n p_j \delta_{P_j}$, where (p_1, \ldots, p_n) is a positive probability vector. Point (3) of Theorem 1.3 shows that it is necessary and sufficient that $\mathbb{E}(N) \ge \max_j p_j^{-1}$ so that $I_X = K$.
- (2) We now give an example to illustrate the fact that the elements of $\widetilde{\mathcal{F}}_X^1 \cup \overline{\mathcal{F}}_X^1$ can have any integer dimension between 0 and d-1 when $d \ge 2$ (the case d=1 is trivial). Suppose that K as the following properties: ∂K is C^{∞} smooth; also, there exist a convex polytope \widetilde{K} such that $K \subset \widetilde{K}$, for each face Q_j of \widetilde{K} of dimension ≥ 1 , $K_j = K \cap Q_j = \partial K \cap Q_j$ is included in the relative interior of Q_j and is itself open relative to Q_j , and $\partial K \setminus \bigcup_j K_j$ has positive curvature (recall that a convex subset Q of dimension ≥ 1 is a face of P if $Q = P \cap H$ where H is a supporting hyperplane of P). The previous properties imply that the sets K_j are pairwise disjoint and each K_j is contained in a unique supporting hyperplane of K (the existence of such configurations is quite intuitive and it turns out that such a K can be associated to any convex polytope K with non empty interior [20]. Note also that any convex polytope does possess faces of dimension k for all integers kbetween 0 and d-1).

Suppose, moreover, that μ has a topological support equal to ∂K , that its restriction to each K_i does not vanish and has topological support equal to K_i , and that μ has a unique atom, at a point $\alpha_0 \in \partial K \setminus \bigcup_j K_j$. Denote by F_j the smallest affine subset containing K_j . Taking $\mathbb{E}(N) \geq \max(\mu(\{\alpha_0\})^{-1}, \max_j \mu(K_j)^{-1})$ implies that $F_j \in \widetilde{\mathcal{F}}_X^1$ or $F_j \in \overline{\mathcal{F}}_X^1$ according to whether $\mathbb{E}(N^{F_j}) = \mathbb{E}(N)\mu(K_j) > 1$ or $\mathbb{E}(N^{F_j}) = \mathbb{E}(N)\mu(K_j) = 1$. Similarly, since K possesses a tangent space at $\{\alpha_0\}$, one has $\{\alpha_0\} \in \widetilde{\mathcal{F}}_X^1$ or $\{\alpha_0\} \in \overline{\mathcal{F}}_X^1$ according to whether $\mathbb{E}(N^{\{\alpha_0\}}) = \mathbb{E}(N)\mu(\{\alpha_0\}) > 1$ or $\mathbb{E}(N^{\{\alpha_0\}}) = \mathbb{E}(N)\mu(\{\alpha_0\}) = 1$. Also, by construction and due to Theorem 1.3(1)(2), $(\partial K \setminus (\{\alpha_0\} \cup \bigcup_j K_j)) \cap I_X = \emptyset$.

(3) The last example provides a situation where $\widetilde{\mathcal{F}}_X^1$ is infinite (note that d has to be ≥ 3), as well as $\widetilde{\mathcal{F}}_X^i \setminus \widetilde{\mathcal{F}}_X^{i-1}$ for all $2 \leq i \leq d-1$.

Let $(\theta_n)_{n\in\mathbb{N}}$ be an increasing sequence in $[0,2\pi)$ converging to 2π and such that $\theta_1=0$. Define K_2 as the convex hull of the set $\{\alpha_n=(\cos(\theta_n),\sin(\theta_n),0,\dots,0):n\geq 1\}$. Then, pick for $3\leq i\leq d,\ \beta_i\in\mathbb{R}^i\times\{0\}^{d-i}\setminus(\mathbb{R}^{i-1}\times\{0\}^{d-i+1})$ and define recursively K_i as the convex hull of $K_{i-1}\cup\{\alpha_i\}$. Let $(p_n)_{n\in\mathbb{N}}$ and $(q_i)_{i=3}^d$ be positive sequences, such that $\sum_{n\in\mathbb{N}}p_n+\sum_{i=3}^dq_i=1$ and set $\mu=\sum_{n\in\mathbb{N}}p_n\delta_{\alpha_n}+\sum_{i=3}^dq_i\delta_{\beta_i}$.

For $n \geq 1$, denote $\{\alpha_n\}$ by F_0^n , the line containing $[\alpha_n, \alpha_{n+1}]$ by F_1^n , and for $2 \leq i \leq d-1$, the i-dimensional affine subspace generated by $\{\alpha_n, \alpha_{n+1}, \beta_3, \ldots, \beta_{i+1}\}$ by F_i^n . We get an increasing sequence of convex polytopes, which are faces of K_d . If $q_3\mathbb{E}(N) > 1$, then for every $2 \leq i \leq d-1$, the affine subspaces F_i^n , $n \geq 1$, belong to $\widetilde{\mathcal{F}}_X^1$ if i = d-1, and to $\widetilde{\mathcal{F}}_X^{d-i} \setminus \widetilde{\mathcal{F}}_X^{d-i-1}$ otherwise, with the convention $\widetilde{\mathcal{F}}_X^0 = \emptyset$. Moreover, for all $n \geq 1$, denoting by G_1^n the line supporting the segments $[\alpha_n, \beta_3]$, one has $G_1^n \in \widetilde{\mathcal{F}}_X^{d-1} \setminus \widetilde{\mathcal{F}}_X^{d-2}$. Also, the line supporting $[\alpha_n, \alpha_{n+1}]$ belongs to $\widetilde{\mathcal{F}}_X^{d-1} \setminus \widetilde{\mathcal{F}}_X^{d-2}$ or $\overline{\mathcal{F}}_X^{d-1} \setminus \overline{\mathcal{F}}_X^{d-2}$ according to whether $p_n + p_{n+1} > 1$ of $p_n + p_{n+1} = 1$, which can happen for finitely many n only, so that $(\partial I_X)_{\text{crit}} \neq \emptyset$. The point β_3 belongs to $\overline{\mathcal{F}}_X^d \setminus \overline{\mathcal{F}}_X^{d-1}$. Note that the previous observations do not exhaust the description of $\widetilde{\mathcal{F}}_X^d \cup \overline{\mathcal{F}}_X^d$.

Let us finish this section with some comments and remarks.

Remark 1.4. (1) In the case d=1, a partial result regarding the Hausdorff dimensions of the sets $E(X,\alpha)$ is presented in [2], where dim $E(X,\alpha)$ is computed under d_{ϕ} for each individual α of the interval \mathring{I}_X , almost surely, by using the Gibbs measure ν_{α} .

- (2) A concatenation/approximation method, somehow more elaborate than that used in the present paper, was used in [3] to construct some family of inhomogeneous Mandelbrot measures particularly adapted to get the general Theorem A, and more generally to compute dim E(X,K) under the metric d_1 (where K may belong to a larger classe of closed connected sets when I_X is not bounded); $0-\infty$ laws are also established for the Hausdorff and packing measures of the sets $E(X,\alpha)$. This method was considered in particular to deal with those α belonging to $I_X \setminus \overline{\nabla P_X}(J_X)$ whenever this set is non empty, a situation which occurs in presence of so-called first order phase transitions but that we will not meet here due to (1.8). Adapting this method when working with a metric d_{ϕ} more general than d_1 seems quite delicate; this is related to the loss of concavity of the mapping $\alpha \mapsto \dim E(X,\alpha)$. Moreover, even when working with d_1 , quantified Erdös-Rényi laws of large numbers associated with $\alpha \in I_X \setminus \overline{\nabla P_X}(J_X)$ seem out of reach by using the techniques currently at our disposal.
- (3) With respect to [3], appart from the fact that we modify the metric, in our study of the Hausdorff dimensions of the sets $E(X,\alpha)$ or E(X,K), a difference will come from the way we simultaneously estimate from below the Hausdorff dimensions of the inhomogeneous Mandelbrot measures coming into play. We use the fact that the elements of this uncountable family are simultaneously not killed by the actions of some percolations processes, adapting an original idea of Kahane for the action of a given multiplicative chaos

on a fixed Radon measure [23, 24]. In [3], we directly estimated the dimensions of such measures by using large deviations estimates, with different technicalities as a counterpart.

(4) Let us come back to (Q3). When $\alpha \in (\partial I_X)_{crit}$, for each choice of ϕ such that the assumptions of Theorem 1.2 hold, there is a function $\Lambda_{X,\phi,\alpha}$, depending on ϕ , such that the conclusions of Corollary 1.2 hold for α . Moreover, for each such ϕ , one has $\dim E(X,\alpha,\operatorname{LD}(\Lambda_{\psi_{X,\phi,\alpha}}))=0=\dim E(X,\alpha)$ both under d_1 and d_{ϕ} due to Lemma 2.1. If ϕ is not a multiple of $(1)_{i\in\mathbb{N}}$, then $\Lambda_{\psi_{X,\phi,\alpha}}$ differs in general from $\Lambda_{\psi_{X,1,\alpha}}$, hence the choice of (Λ,\mathbb{R}) is non unique. Also, when (X,ϕ) is bounded, it is not hard to see that when $\alpha \in \mathring{I}_X$ there is a unique Mandelbrot measure whose dimension equals $\dim E(X,\alpha)$, but this observation is far from being sufficient to answer the uniqueness question we raise.

The paper is organised as follows. In Section 2 we justify some properties used in the introduction. In Section 3 we construct inhomogeneous Mandelbrot measures and compute their Hausdorff dimensions; these measures will be used to get the sharp lower bound for $\dim E(X,K)$ and $\dim E(X,\alpha,\operatorname{LD}(\Lambda_{\psi_{\alpha}},\widetilde{k}))$ in the proofs of our three main results. Section 4 establishes Theorem 1.1, while Theorems 1.2 and 1.3 are proved in Sections 5 and 6 respectively, and Corollaries 1.1, 1.2 and 1.3 are proved in Section 7.

2. Justification of some properties claimed in the introduction

The following proposition was invoked in the previous section.

Proposition 2.1. [3, Proposition 2.2 and Remark 2.1] Assume (1.8).

- (1) \widetilde{P}_X is strictly convex and I_X is convex, compact with non-empty interior.
- (2) $I_X = \overline{\{\nabla \widetilde{P}_X(q) : q \in J_X\}}, \text{ and } \mathring{I}_X = \nabla \widetilde{P}_X(J_X).$

The fact that d_{ϕ} is a metric is a direct consequence of the third inequality of the following elementary lemma, whose proof is left to the reader.

Lemma 2.1. Assume (1.10). There exist $0 < \widetilde{\beta} \le \beta < 1$ such that, with probability 1, for n large enough,

$$\widetilde{\beta}^n \le \min\{\exp(-S_n\phi(u)) : u \in \mathsf{T}_n\} \le \max\{\exp(-S_n\phi(u)) : u \in \mathsf{T}_n\} \le \beta^n.$$

Proof of Proposition 1.1. Fix $q \in \mathbb{R}^d$ and $\alpha \in \widetilde{I}_X$. Then for $t \in \mathbb{R}$ define

$$\ell(t) = \log \mathbb{E}\Big(\sum_{i=1}^{N} \exp\left(\langle q|X_i - \alpha\rangle - t\phi_i\right)\Big).$$

Note that $\ell(0) = \widetilde{P}_X(q) - \langle q | \alpha \rangle$ and $\ell'(0) = -\mathbb{E}\left(\sum_{i=1}^N \phi_i \exp\left(\langle q | X_i \rangle - \widetilde{P}_X(q)\right)\right) \geq -\lambda$, where $\lambda = \sup_{q \in \mathbb{R}^d} \mathbb{E}\left(\sum_{i=1}^N \phi_i \exp(\langle q | X_i \rangle - \widetilde{P}_X(q))\right) \in (0, \infty)$ due to (1.16). Moreover ℓ is convex, so for all $t \geq 0$ one has $\ell(t) \geq \ell(0) - \lambda t$. Since, by definition of $\widetilde{P}_{X,\phi,\alpha}(q)$, $\ell(\widetilde{P}_{X,\phi,\alpha}(q)) = 0$, it follows that $\widetilde{P}_{X,\phi,\alpha}(q) \geq \lambda^{-1}(\widetilde{P}_X(q) - \langle q | \alpha \rangle)$.

Now, observe that due to the convexity of $\widetilde{P}_{X,\phi,\alpha}(\cdot)$, as well as the strict convexity of $\widetilde{P}_{X}(q) - \langle q | \alpha \rangle$ and the fact that $\widetilde{P}_{X}(q) - \langle q | \alpha \rangle$ reaches its infimum (at q' such that $\nabla \widetilde{P}_{X}(q') = \alpha$), the mapping $\widetilde{P}_{X,\phi,\alpha}(\cdot)$ reaches its infimum at some $q_{\alpha} \in \mathbb{R}^{d}$. If there are two distinct such q_{α} and q'_{α} , then, setting $v = q_{\alpha} - q'_{\alpha}$, one has $\langle v | \nabla \widetilde{P}_{X,\phi,\alpha}(q) \rangle = 0$ over $[q_{\alpha}, q'_{\alpha}]$, that

is $\mathbb{E}\left(\sum_{i=1}^{N}\langle v|X_i-\alpha\rangle\exp(\langle q|X_i-\alpha\rangle-\widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\right)=0$, in view of (3.1). Differentiating again over $[q_{\alpha},q'_{\alpha}]$ yields $\mathbb{E}\left(\sum_{i=1}^{N}\langle v|X_i-\alpha\rangle^2\exp(\langle q|X_i-\alpha\rangle-\widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\right)=0$ over $[q_{\alpha},q'_{\alpha}]$, hence $\langle v|X_i\rangle=\langle v|\alpha\rangle$ almost surely for all $1\leq i\leq N$. But this contradicts (1.9). By construction, $\nabla\widetilde{P}_{X,\phi,\alpha}(q_{\alpha})=0$ and q_{α} is the unique q at which $\nabla\widetilde{P}_{X,\phi,\alpha}$ vanishes.

To see that $\alpha \in \mathring{I}_X \mapsto q_\alpha$ is real analytic, one first observes that the differential of $f: q \mapsto \nabla \widetilde{P}_{X,\phi,\alpha}(q)$ at q_α is invertible. Indeed, a calculation shows that there exists c>0 such that $\frac{\partial f}{\partial q_k}(q_\alpha) = c\mathbb{E}\Big(\sum_{i=1}^N (X_i - \alpha)_k (X_i - \alpha) \exp(\langle q_\alpha | X_i - \alpha \rangle - \widetilde{P}_{X,\phi,\alpha}(q_\alpha)\phi_i)\Big)$. This implies that if the differential of f at q_α vanishes at some $v \in \mathbb{R}^d \setminus \{0\}$, then once again $\mathbb{E}\Big(\sum_{i=1}^N \langle v | (X_i - \alpha) \rangle^2 \exp(\langle q_\alpha | X_i - \alpha \rangle - \widetilde{P}_{X,\phi,\alpha}(q_\alpha)\phi_i)\Big) = 0$, hence the same contradiction as above. It is now possible to apply the implicit function theorem to $(\alpha,q)\mapsto (\alpha,\nabla \widetilde{P}_{X,\phi,\alpha}(q))$ at (α,q_α) . To see that $\alpha\in \widetilde{I}_X\mapsto q_\alpha$ is continuous, note that if $\alpha\in \widetilde{I}_X\setminus \mathring{I}_X$, then $\alpha=\nabla \widetilde{P}_X(q_0)$ for some q_0 such that $\widetilde{P}_X^*(\nabla \widetilde{P}_X(q_0))=0$. Also, for such an α , there exists a neighborhood U of q_0 and a neighborhood V of α such that $V=\nabla \widetilde{P}_X(U)$ and for all $\beta\in V$ one has $\widetilde{P}_X^*(\beta)=\widetilde{P}_X(q')-\langle q'|\beta\rangle>-\infty$ as well $(q\mapsto \nabla \widetilde{P}_X(q))$ is a local diffeomorphism). By the same argument as above, one has $\widetilde{P}_{X,\phi,\beta}(q)\geq \lambda^{-1}(\widetilde{P}_X(q)-\langle q|\beta\rangle)$, so $\inf_{q\in\mathbb{R}^d}\widetilde{P}_{X,\phi,\beta}(q)$ is attained at a unique q_β , and q_β depends analytically on β over V.

3. Some inhomogeneous Mandelbrot measures on ∂T and simultaneous calculation of their Hausdorff dimensions using percolation

We construct our main tool to get the simultaneous calculation of the Hausdorff dimensions of the sets we are interested in. It consists of a family of non degenerate inhomogeneous Mandelbrot measures on ∂T , of which we provide a lower bound of the lower Hausdorff dimension. This requires several steps. In Section 3.1, we gather useful preliminary observations to determine a good set of parameters \mathcal{R} to be used to define the measures. In Section 3.2 we introduce families of inhomogeneous Mandelbrot real valued martingales indexed both by \mathcal{R} and the set (0,1] of parameters involved in fractal percolation processes on ∂T . The non degenerate character of some of these martingales is established and used in Sections 3.3 and 3.4 to define the inhomogeneous Mandelbrot measures and estimate their Hausdorff dimension from below thanks to a "uniform" version of the percolation argument developed by Kahane in [23, 24].

3.1. Preliminary observations, and definition of a set of parameters. A calculation shows that for $(q, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d$ one has

(3.1)
$$\nabla \widetilde{P}_{X,\phi,\alpha}(q) = \frac{\mathbb{E}\left(\sum_{i=1}^{N} X_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\right) - \alpha}{\mathbb{E}\left(\sum_{i=1}^{N} \phi_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\right)}.$$

Also, for each $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$ a Mandelbrot measure $\mu_{q,\alpha}$ on $\partial \mathsf{T}$ is associated with the vectors $(N_u, \exp(\langle q|X_{u1} - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_{u1}), \exp(\langle q|X_{u2} - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_{u2}), \ldots), u \in \bigcup_{n \geq 0} \mathbb{N}^n$, and this measure is non degenerate if and only if, after setting

$$\psi(q,\alpha) = (\psi_i(q,\alpha) = \langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i)_{i>1},$$

the "entropy"

(3.2)
$$h(q,\alpha) = -\mathbb{E}\left(\sum_{i=1}^{N} \psi_i(q,\alpha) \exp(\psi_i(q,\alpha))\right)$$

is positive, and $\mathbb{E}\left(\left(\sum_{i=1}^{N} \exp(\psi_i(q,\alpha)) \log_+ \sum_{i=1}^{N} \exp(\psi_i(q,\alpha))\right) < \infty.\right)$

Define the "Lyapounov exponent"

(3.3)
$$\lambda(q,\alpha) := \mathbb{E}\Big(\sum_{i=1}^{N} \phi_i \exp(\psi_i(q,\alpha))\Big) \in (0,\infty).$$

An identification shows that

$$(3.4) \widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) = \widetilde{P}_{X,\phi,\alpha}(q) - \langle q | \nabla \widetilde{P}_{X,\phi,\alpha}(q) \rangle = \frac{h(q,\alpha)}{\lambda(q,\alpha)}.$$

Consequently, since we assumed (1.15), the measure $\mu_{q,\alpha}$ is non degenerate if and only if $\widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) > 0$, that is $(q,\alpha) \in J_{X,\phi}$. Also, using the definition of $\beta_{\Lambda_{\psi}}$ introduced in (1.5) and setting $\beta(q,\alpha) = \beta_{\Lambda_{\psi(q,\alpha)}}$, one gets

(3.5)
$$\beta(q,\alpha) = \mathbb{E}\Big(\sum_{i=1}^{N} X_i \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\Big),$$

and with probability 1,

(3.6)
$$\lim_{n \to \infty} \frac{S_n X(t)}{n} = \beta(q, \alpha) \quad \mu_{q, \alpha}\text{-a.e.}$$

Note that when the assumptions of Proposition 1.1 hold, if $\alpha \in \mathring{I}_X$ and $q = q_\alpha$, one has $\nabla \widetilde{P}_{X,\phi,\alpha}(q) = 0$, hence (3.1) yields $\beta(q,\alpha) = \alpha$ and $\widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) = \widetilde{P}_{X,\phi,\alpha}^*(0) = \widetilde{P}_{X,\phi,\alpha}(q_\alpha)$. Moreover, for all $(q,\alpha) \in J_{X,\phi}$, with probability 1, $\lambda(q,\alpha) = \lim_{n \to \infty} \frac{S_n\phi(t)}{n}$ at $\mu_{q,\alpha}$ -almost every point t. Consequently, for all $(q,\alpha) \in J_{X,\phi}$ one has

$$(3.7) -\log(\beta) \le \lambda(q,\alpha) \le -\log(\widetilde{\beta}),$$

where β and $\widetilde{\beta}$ are taken as in Lemma 2.1.

Recall that for $(q, \alpha, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ we set

$$\Sigma_{\alpha}(q,t) = \sum_{i=1}^{N} \exp(\langle q|X_i - \alpha \rangle - t\phi_i).$$

Define

$$L_{\alpha}(q,t) = \log \mathbb{E}(\Sigma_{\alpha}(q,t)).$$

One checks that

(3.8)
$$\frac{\partial L_{\alpha}}{\partial q}(q, \widetilde{P}_{X,\phi,\alpha}(q)) = \beta(q,\alpha) - \alpha, \quad \frac{\partial L_{\alpha}}{\partial t}(q, \widetilde{P}_{X,\phi,\alpha}(q)) = -\lambda(q,\alpha),$$

and

(3.9)
$$\frac{\mathrm{d}L_{\alpha}((1+u)q,(1+u)\widetilde{P}_{X,\phi,\alpha}(q))}{\mathrm{d}u}(0) = -h(q,\alpha).$$

Recall the definitions (1.2) and (1.17) of I_X and \widetilde{I}_X respectively.

Lemma 3.1. Let D be a dense subset of $J_{X,\phi}$. For all $\alpha \in I_X$ there exists a sequence $(q_n, \alpha_n)_{n \in \mathbb{N}}$ of elements of D such that $\lim_{n \to \infty} \beta(q_n, \alpha_n) = \alpha$ and $\lim_{n \to \infty} \widetilde{P}_{X,\phi,\alpha_n}(q_n) - \langle q_n | \nabla \widetilde{P}_{X,\phi,\alpha_n}(q_n) \rangle = \widetilde{P}^*_{X,\phi,\alpha}(0)$.

Moreover, if (1.16) holds, one can choose D so that it contains a sequence of the form $(q_{\alpha_m}, \alpha_m)_{m\geq 1}$ such that $\{\alpha_m : m \geq 1\}$ is dense in \widetilde{I}_X , and for all $\alpha \in \widetilde{I}_X$ the previous sequence can be chosen so that $q_n = q_{\alpha_n}$. In particular, $\lim_{n\to\infty} (q_n, \alpha_n) = (q_\alpha, \alpha)$.

Proof. Let $\alpha \in I_X$. It is not hard to adapt the proof of [3, Proposition 2.2] to show that due to (1.9), the mapping $\widetilde{P}_{X,\phi,\alpha}$ is strictly convex, and the set $I_{X,\phi,\alpha} = \{\beta \in \mathbb{R}^d : \widetilde{P}_{X,\phi,\alpha}^*(\beta) \geq 0\}$ is a convex compact set with non-empty interior equal to $\mathring{I}_{X,\phi,\alpha} = \{\nabla \widetilde{P}_{X,\phi,\alpha}(q) : q \in \mathbb{R}^d, \widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) > 0\}$. Now, note that due to Propositions 4.2 and 4.3, one has $\widetilde{P}_{X,\phi,\alpha}(0) \geq 0$. Let $\beta \in \mathring{I}_{X,\phi,\alpha}$. The sequence β/n belongs to $\mathring{I}_{X,\phi,\alpha}$ and converges to 0. Since $\widetilde{P}_{X,\phi,\alpha}^*$ is upper-semi-continuous and concave, one has $\lim_{n\to\infty} \widetilde{P}_{X,\phi,\alpha}^*(\beta/n) = \widetilde{P}_{X,\phi,\alpha}^*(0)$ Moreover, there exists a sequence $(q_n)_{n\in\mathbb{N}}$ such that $\nabla \widetilde{P}_{X,\phi,\alpha}(q_n) = \beta/n$ and $\widetilde{P}_{X,\phi,\alpha}(q_n) - \langle q_n|\nabla \widetilde{P}_{X,\phi,\alpha}(q_n) \rangle > 0$ so that $(q_n,\alpha) \in J_{X,\phi}$. Also, due to (3.1), (3.7), and the fact that $\lim_{n\to\infty} \nabla \widetilde{P}_{X,\phi,\alpha}(q_n) = 0$, one has $\lim_{n\to\infty} \beta(q_n,\alpha) = \alpha$ (remember (3.5)). Now, the mappings $(q,\alpha) \mapsto \beta(q,\alpha)$ and $(q,\alpha) \mapsto \widetilde{P}_{X,\phi,\alpha}(q) - \langle q|\nabla \widetilde{P}_{X,\phi,\alpha}(q) \rangle$ being continuous, the property claimed about D follows. The second one is clear due to Proposition 1.1.

We can now start the construction of a set of parameters that will be used to define inhomogeneous Mandelbrot measures. Fix a dense subset D of $J_{X,\phi}$, so that if (1.16) holds, the property claimed in the second assertion of Lemma 3.1 holds. Let $(D_j)_{j\geq 1}$ be a non decreasing sequence of non-empty subsets of D such that $D = \bigcup_{j\geq 1} D_j$. Let $(N_j)_{j\geq 0}$ be a sequence of integers such that $N_0 = 0$, and that we will specify at the end of this section. Then let $(M_j)_{j\geq 0}$ be the increasing sequence defined as

(3.10)
$$M_0 = 0$$
 and $M_j = \sum_{k=1}^{j} N_k$ for all $j \ge 1$.

For $n \in \mathbb{N}$, let j_n denote the unique integer satisfying

$$M_{j_n} + 1 \le n \le M_{j_n+1}.$$

We will construct a family of random measures indexed by the set

$$\mathcal{R} = \{((q_k, \alpha_k))_{k \geq 1} : \forall j \geq 0, \ \exists (q, \alpha) \in D_{j+1}, \ \forall M_j + 1 \leq k \leq M_{j+1}, \ (q_k, \alpha_k) = (q, \alpha)\}.$$

Since each D_i is finite, the set \mathcal{R} is compact, once endowed with the natural metric

$$d(\varrho, \varrho') = \sum_{k>1} 2^{-k} \frac{|q_k - q'_k| + |\alpha'_k - \alpha_k|}{1 + |q_k - q'_k| + |\alpha'_k - \alpha_k|}.$$

For $\varrho = ((q_k, \alpha_k))_{k \geq 1} \in \mathcal{R}$ and $n \geq 1$ we will denote by $\varrho_{|n}$ the sequence $((q_k, \alpha_k))_{1 \leq k \leq n}$.

Specification of the sequences $(D_j)_{j\geq 1}$ and $(N_j)_{j\geq 1}$. The following tuning of the sequences $(D_j)_{j\geq 1}$ and $(N_j)_{j\geq 1}$ can be skipped at first reading.

At first, assume that

(3.11)
$$\forall j \ge 1, \begin{cases} \#D_j \le j \\ \max_{(q,\alpha) \in D_j} \mathbb{E}(\mathbf{1}_{\{N=1\}} \exp(\langle q|X_1 - \alpha \rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_1)) \le 1 - \frac{c_0}{j} \end{cases}$$

for some constant $c_0 > 0$. This is possible since $\mathbb{E}(N) > 1$ and $\mathbb{E}\left(\sum_{i=1}^N \exp(\langle q|X_i - \alpha\rangle - \widetilde{P}_{X,\phi,\alpha}(q)\phi_i)\right) = 1$ for all $(q,\alpha) \in J_{X,\phi}$.

For each $\alpha \in I$ the function L_{α} is analytic. Denote by $H_{L_{\alpha}}$ its Hessian matrix. Also, simply denote $\widetilde{P}_{X,\phi,\alpha}$ by \widetilde{P}_{α} in (3.12) and (3.13) below. For each $j \geq 1$, both

$$(3.12) \quad m_{j} = \max_{t \in [0,1]} \max_{v \in \mathbb{S}^{d-1}} \max_{(q,\alpha) \in D_{j}} {}^{t} {v \choose 0} H_{L_{\alpha}}(q+tv, \widetilde{P}_{\alpha}(q)) {v \choose 0}$$

$$+ \max_{t \in [0,1]} \max_{v \in \mathbb{S}^{d-1}} \max_{(q,\alpha) \in D_{j}} \frac{\partial^{2}}{\partial t^{2}} L_{\alpha}(q, \widetilde{P}_{\alpha}(q) + tv)$$

and

$$(3.13) \quad \widetilde{m}_{j} = \max_{t \in [0,1]} \max_{p \in [1,2]} \max_{(q,\alpha) \in D_{j}} {}^{t}V_{q,\alpha}H_{L_{\alpha}}\left(q + t(p-1)q, \widetilde{P}_{\alpha}(q) + t(p-1)\widetilde{P}_{\alpha}(q)\right)V_{q,\alpha}$$

are finite, where $V_{q,\alpha} = \begin{pmatrix} q \\ \widetilde{P}_{\alpha}(q) \end{pmatrix}$. Let

$$\widehat{m}_j = \max(m_j, \widetilde{m}_j)$$

and $(\gamma_i)_{i>1} \in (0,1]^{\mathbb{N}}$ be a positive sequence such that

(3.15)
$$\gamma_j^2 \widehat{m}_j \le 1/j^2 \quad \text{(note that } \lim_{j \to \infty} \gamma_j = 0\text{)}.$$

Let $(\widetilde{p}_j)_{j\geq 1}$ be a sequence taking values in (1,2) such that

$$\lim_{j \to \infty} (\widetilde{p}_j - 1)\widetilde{m}_j = 0.$$

Due to (1.15) we can also suppose that \widetilde{p}_j is small enough so that we also have

$$\sup_{(q,\alpha)\in D_j} \mathbb{E}(\Sigma_{\alpha}(q,\widetilde{P}_{X,\phi,\alpha}(q))^{\widetilde{p}_j}) < \infty.$$

For each $(q, \alpha) \in J_{X,\phi}$ there exists $p_{q,\alpha} \in (1,2)$ such that $L_{\alpha}(pq, p\widetilde{P}_{X,\phi,\alpha}(q)) < 0$ for all $p \in (1, p_{q,\alpha})$. Indeed, $\widetilde{P}_{X,\phi,\alpha}^*(\nabla \widetilde{P}_{X,\phi,\alpha}(q)) > 0$ if and only if $\frac{\mathrm{d}}{\mathrm{d}p}(L_{\alpha}(pq, p\widetilde{P}_{X,\phi,\alpha}(q)))(1^+) < 0$, and $L_{\alpha}(q, \widetilde{P}_{X,\phi,\alpha}(q)) = 0$ by definition of $\widetilde{P}_{X,\phi,\alpha}(q)$.

For all $j \geq 1$, set

$$p_j = \min\left(\widetilde{p}_j, \inf_{(q,\alpha) \in D_{j+1}} p_{q,\alpha}\right) \quad \text{and} \quad L_j = \sup_{(q,\alpha) \in D_j} L_{\alpha}(p_j q, p_j \widetilde{P}_{X,\phi,\alpha}(q)).$$

By construction, one has $a_i < 0$. Then let

$$(3.16) \quad s_j = \max\left\{\left\|\Sigma_\alpha(q,\widetilde{P}_{X,\phi,\alpha}(q)))\right\|_{p_j}: (q,\alpha) \in D_j\right\} \quad \text{ and } \quad r_j = \max\left(\frac{L_j}{p_j},\frac{1-p_j}{2jp_j}\right).$$

Now set $N_0 = 0$ and for $j \ge 1$ choose an integer N_j big enough so that

(3.17)
$$\frac{(j+1)!s_{j+1}}{1-\exp(r_{j+1})}\exp(N_jr_{j+1}) \le j^{-2},$$

(3.18)
$$\frac{(j+1)!s_{j+1}}{(1-\exp(r_{j+1}))} + \frac{(j+2)!s_{j+2}}{(1-\exp(r_{j+2}))} \le C_0 \exp(N_j \gamma_{j+1}^2 m_{j+1}),$$

with
$$C_0 = \frac{2s_1}{1 - \exp(r_1)} + \frac{2s_2}{1 - \exp(r_2)}$$
,

(3.19)
$$N_j \ge \max \left(\log((j+1)!)^3, (\gamma_{j+1}^2 \widehat{m}_{j+1})^{-2} \right),$$

and if $j \geq 2$,

(3.20)
$$\sum_{k=1}^{j-1} N_k \le \frac{N_j}{j} \frac{\min(1, \{\|\beta(q, \alpha)\| : (q, \alpha) \in D_j\})}{\max(1, \max\{\|\beta(q, \alpha)\| : (q, \alpha) \in D_{j-1}\})}.$$

3.2. Inhomogeneous Mandelbrot martingales indexed by fractal percolation parameters and by \mathcal{R} . For each $\beta \in (0,1]$, let \widetilde{W}_{β} be a random variable distributed according to $\beta \delta_{\beta^{-1}} + (1-\beta)\delta_0$. Consider $\{\widetilde{W}_{\beta,u}\}_{u \in \bigcup_{n \geq 0} \mathbb{N}^n}$ be a family of independent copies of \widetilde{W}_{β} define on a probability space $(\Omega_{\beta}, \mathcal{A}_{\beta}, \mathbb{P}_{\beta})$.

Each random variable $\widetilde{W}_{\beta,u}$ and the random vector $(N_u, (X_{ui}, \phi_{ui})_{i \geq 1})$ extends to $(\Omega_{\beta} \times \Omega, \mathcal{A}_{\beta} \otimes \mathcal{A}, \mathbb{P}_{\beta} \otimes \mathbb{P})$ as

$$\widetilde{W}_{\beta,u}(\omega_{\beta},\omega) = \widetilde{W}_{\beta,u}(\omega_{\beta})$$
 and $(N_u, (X_{ui}, \phi_{ui})_{i>1})(\omega_{\beta},\omega) = (N_u(\omega), (X_{ui}(\omega), \phi_{ui}(\omega))_{i>1}),$

and the families $\{\widetilde{W}_{\beta,u}\}_{u\in\bigcup_{n\geq 0}\mathbb{N}^n}$ and $\{(N_u,(X_{ui},\phi_{ui})_{i\geq 1})\}_{u\in\bigcup_{n\geq 0}\mathbb{N}^n}$ are $\mathbb{P}_{\beta}\otimes\mathbb{P}$ -independent.

We adopt the convention that $\mathbb{E}_{\mathbb{P}_{\beta}\otimes\mathbb{P}}$ is denoted by \mathbb{E} .

For each $\beta \in ((\mathbb{E}(N))^{-1}, 1]$, the random integers $N_{\beta,u}(\omega_{\beta}, \omega) = \sum_{i=1}^{N_u(\omega)} \mathbf{1}_{\{\beta^{-1}\}}(\widetilde{W}_{\beta,ui}(\omega_{\beta}))$ define a new supercritical Galton-Watson process to which are associated trees $\mathsf{T}_{\beta,n} \subset \mathsf{T}_n$ and $\mathsf{T}_{\beta,n}(u) \subset \mathsf{T}_n(u), \ u \in \bigcup_{n \geq 0} \mathbb{N}^n, \ n \geq 1$, as well as the infinite tree $\mathsf{T}_{\beta} \subset \mathsf{T}$ and the boundary $\partial \mathsf{T}_{\beta} \subset \partial \mathsf{T}$ conditionally on non extinction of T_{β} .

For
$$u \in \bigcup_{n \geq 0} \mathbb{N}^n$$
, $1 \leq i \leq N_u$, $\beta > \mathbb{E}(N)^{-1}$, and $\varrho = (q_k, \alpha_k)_{k \geq 1} \in \mathcal{R}$ set

$$\begin{cases} W_{\varrho,ui} = \exp\left(\langle q_{|u|+1}|X_{ui} - \alpha_{|u|+1}\rangle - \widetilde{P}_{\alpha_{|u|+1}}(q_{|u|+1})\phi_{ui}\right), \\ W_{\beta,\varrho,ui} = \widetilde{W}_{\beta,ui} \cdot W_{\varrho,ui}. \end{cases}$$

Also, for $\varrho = (q_k, \alpha_k)_{k \geq 1} \in \mathcal{R}$, $u \in \bigcup_{n \geq 0} \mathbb{N}^n$, $\beta > \mathbb{E}(N)^{-1}$, and $n \geq 0$ define

$$Y_n(\varrho, u) = \sum_{v_1 \cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\varrho, u \cdot v_1 \cdots v_k} \text{ and } Y_n(\beta, \varrho, u) = \sum_{v_1 \cdots v_n \in \mathsf{T}_n(u)} \prod_{k=1}^n W_{\beta, \varrho, u \cdot v_1 \cdots v_k}.$$

When $u = \epsilon$, those quantities will be denoted by $Y_n(\varrho)$ and $Y_n(\beta, \varrho)$ respectively, and when n = 0, their values equal 1.

Recall the definition of $h(q,\alpha)$ given in (3.2). For $\beta \in (\mathbb{E}(N)^{-1},1], \ell \in \mathbb{N}$ and $\varepsilon > 0$, set

(3.21)
$$\mathcal{R}(\beta, \ell, \varepsilon) = \left\{ \varrho \in \mathcal{R} : \frac{1}{n} \sum_{k=1}^{n} h(q_k, \alpha_k) \ge -\log \beta + \varepsilon, \forall n \ge \ell \right\},$$

which is a compact subset of \mathcal{R} .

Notice that $h(q_k, \alpha_k) > 0$, and this number is the opposite of the derivative at 1 of the convex function $f: \lambda \geq 0 \longmapsto \log \mathbb{E}(\sum_{i=1}^{N} W_i^{\lambda})$, with $W_i = \exp(\langle q_k | X_i - \alpha_k \rangle - \widetilde{P}_{\alpha_k}(q_k)\phi_i)$, so that f(1) = 0 and $f(0) = \log \mathbb{E}(N) > 0$. Thus $h(q_k, \alpha_k) \in (0, \log \mathbb{E}(N)]$. Consequently,

(3.22)
$$\left\{\varrho \in \mathcal{R} : \liminf_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} h(q_k, \alpha_k) > 0\right\} = \bigcup_{\beta \in (\mathbb{E}(N)^{-1}, 1], \ell \geq 1, \varepsilon > 0} \mathcal{R}(\beta, \ell, \varepsilon).$$

For $n \geq 1$ and $\beta \in (0,1]$, set $\mathcal{F}_n = \sigma((N_u, (X_{u1}, \phi_{u1}, (X_{u2}, \phi_{u2}), \dots) : u \in \bigcup_{k=0}^n \mathbb{N}^{n-1})$ and $\mathcal{F}_{\beta,n} = \sigma(\widetilde{W}_{\beta,u1}, (\widetilde{W}_{\beta,u2}, \ldots) : u \in \bigcup_{k=0}^n \mathbb{N}^{n-1})$. Set $\mathcal{F}_0 = \mathcal{F}_{\beta,0} = \{\emptyset, \Omega\}$.

3.3. Construction of inhomogeneous Mandelbrot measures indexed by \mathcal{R}_{\bullet} . The following statement about the simultaneous construction of inhomogeneous Mandelbrot measures is similar to that obtained in [3] for a different family. We include the proof, as the estimates to follow are important to derive Proposition 3.2, which will be applied in the study of the action of percolation processes on these measures. Also, these estimates yield Lemma 3.4 which will be useful in the proof of Proposition 5.1.

Proposition 3.1.

- (1) For all $u \in \bigcup_{n>0} \mathbb{N}^n$, the sequence of continuous functions $(Y_n(\cdot, u))_{n \in \mathbb{N}}$ converges uniformly on \mathbb{R} , almost surely and in L^1 norm, to a positive limit $Y(\cdot, u)$.
- (2) With probability 1, for all $\varrho \in \mathcal{R}$, the mapping defined on the cylinders of \mathbb{N}^N by

$$\mu_{\varrho}([u]) = Y(\varrho, u) \cdot \prod_{k=1}^{|u|} W_{\varrho, u_1 \cdots u_k}, \ u \in \bigcup_{n>0} \mathbb{N}^n$$

extends to a positive Borel measure on $\mathbb{N}^{\mathbb{N}}$ supported on ∂T .

Lemma 3.2. [36] Let $(X_j)_{j\geq 1}$ be a sequence of centered independent real random variables. For every finite $I \subset \mathbb{N}$ and $p \in (1,2]$, one has $\mathbb{E}(\left|\sum_{i \in I} X_i\right|^p) \leq 2^{p-1} \sum_{i \in I} \mathbb{E}(\left|X_i\right|^p)$.

Lemma 3.3. Let $\varrho \in \mathcal{R}$ and $\beta \in (0,1]$. Define $Z_n(\beta,\varrho) = Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)$ for $n \geq 0$. Recall the definition (1.12) of Σ_{α} . For every $p \in (1,2)$ one has (writing P_{α_n} for P_{X,ϕ,α_n})

$$(3.23) \quad \mathbb{E}(|Z_n(\beta,\varrho)|^p) \leq (2\beta^{-1})^p \mathbb{E}\left(\Sigma_{\alpha_n}(q_n,\widetilde{P}_{\alpha_n}(q_n))^p\right) \prod_{k=1}^{n-1} \beta^{1-p} \exp\left(L_{\alpha_k}(pq_k,p\widetilde{P}_{\alpha_k}(q_k))\right).$$

Proof. Fix $p \in (1,2)$. Setting $A_{\beta,\varrho,u} = \sum_{i=1}^{N_u} \widetilde{W}_{\beta,ui} W_{\varrho,ui}$ and using the branching property

we can write $Z_n(\beta, \varrho) = \sum_{u \in T_{n-1}} \Big(\prod_{k=1}^{n-1} \widetilde{W}_{\beta, u_1 \cdots u_k} W_{\varrho, u_1 \cdots u_k} \Big) (A_{\beta, \varrho, u} - 1)$. By construction, the random variables $(A_{\beta, \varrho, u} - 1)$, $u \in \mathbb{N}^{n-1}$, are centered and i.i.d., and independent of

 $\mathcal{F}_{\beta,n-1} \otimes \mathcal{F}_{n-1}$. Consequently, conditionally on $\mathcal{F}_{\beta,n-1} \otimes \mathcal{F}_{n-1}$, we can apply Lemma 3.2 to the $\{A_{\beta,\varrho,u} \prod_{k=1}^{n-1} \widetilde{W}_{\beta,u_1\cdots u_k} W_{\varrho,u_1\cdots u_k}\}_{u\in \mathsf{T}_{n-1}}$. Since the $A_{\beta,\varrho,u}$, $u\in \mathbb{N}^{n-1}$, have the same distribution, this yields, with $A_{\beta,\varrho} = A_{\beta,\varrho,\varepsilon}$:

$$\mathbb{E}(|Z_n(\beta,\varrho)|^p |\mathcal{F}_{\beta,n-1} \otimes \mathcal{F}_{n-1}) \leq 2^{p-1} \mathbb{E}(|A_{\beta,\rho}-1|^p) \sum_{u \in \mathsf{T}_{n-1}} \prod_{k=1}^{n-1} \widetilde{W}_{\beta,u_1 \cdots u_k}^p W_{\varrho,u_1 \cdots u_k}^p.$$

Also, $\mathbb{E}(|A_{\beta,\rho}-1|^p) \leq 2\mathbb{E}(A_{\beta,\rho}^p)$, since $\mathbb{E}(A_{\beta,\rho}) = 1$ and $p \geq 1$, and as $0 \leq \widetilde{W}_{\beta,i} \leq \beta^{-1}$, $A_{\beta,\rho} \leq \beta^{-1} \Sigma_{\alpha_n}(q_n, \widetilde{P}_{X,\phi,\alpha_n}(q_n))$. Thus, $2^{p-1}\mathbb{E}(|A_{\beta,\rho}-1|^p) \leq (2\beta^{-1})^p\mathbb{E}(\Sigma_{\alpha_n}(q_n, \widetilde{P}_{X,\phi,\alpha_n}(q_n))^p)$. Moreover, a recursive using of the branching property and the independence of the random vectors $(N_u, X_{u1}, \phi_{u1}, \ldots)$ and random variables $\widetilde{W}_{\beta,u}$ used in the constructions yields, setting $W_{q_k,i} = \exp(\langle q_k | X_i - \alpha_k \rangle - \widetilde{P}_{X,\phi,\alpha_k}(q_k)\phi_i)$:

$$\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n-1}}\prod_{k=1}^{n-1}\widetilde{W}_{\beta,u_1\cdots u_k}^pW_{\varrho,u_1\cdots u_k}^p\Big) = \prod_{k=1}^{n-1}\mathbb{E}(\widetilde{W}_{\beta}^p)\mathbb{E}\Big(\sum_{i=1}^NW_{q_k,i}^p\Big)$$
$$= \prod_{k=1}^{n-1}\beta^{1-p}\exp\big(L_{\alpha_k}(pq_k,p\widetilde{P}_{X,\phi,\alpha_k}(q_k))\big).$$

Collecting the previous estimates one gets the desired conclusion.

Proof of Proposition 3.1. (1) It is similar to the proof of [3, Proposition 2.8(1)]. We detail it for reader's convenience.

First, consider the case $u = \epsilon$. Observe that for all $n \geq 1$, by construction the function $Y_n(\cdot) = Y_n(\cdot, \emptyset)$ is continuous and constant over the set of those sequences ϱ having the same first terms. Given $n \geq 1$ and $\varrho \in \mathcal{R}$, since $M_{j_n} + 1 \leq n \leq M_{j_n+1}$, applying Lemma 3.3 with $p = p_{j_n+1}$ and $\beta = 1$, one obtains

$$\begin{aligned} &\|Y_{n}(\varrho) - Y_{n-1}(\varrho)\|_{p_{j_{n}+1}}^{p_{j_{n}+1}} \\ &\leq 2^{p_{j_{n}+1}} \mathbb{E} \left(\sum_{\alpha_{n}} (q_{n}, \widetilde{P}_{X,\phi,\alpha_{n}}(q_{n}))^{p_{j_{n}+1}} \right) \prod_{k=1}^{n-1} \exp \left(L_{\alpha_{k}}(p_{j_{n}+1}q_{k}, p\widetilde{P}_{X,\phi,\alpha_{k}}(q_{k})) \right) \\ &\leq 2^{p_{j_{n}+1}} s_{j_{n}+1}^{p_{j_{n}+1}} \prod_{k=1}^{n-1} \exp \left(\sup_{(q,\alpha) \in D_{j_{n}+1}} L_{\alpha}(p_{j_{n}+1}q, p\widetilde{P}_{X,\phi,\alpha_{k}}(q)) \right) \\ & (\text{since } \{(q_{k}, \alpha_{k}) : 1 \leq k \leq n\} \subset D_{j_{n}+1} \\ &\leq 2^{p_{j_{n}+1}} s_{j_{n}+1}^{p_{j_{n}+1}} \exp((n-1)p_{j_{n}+1}r_{j_{n}+1}) \text{ (due } (3.16)); \end{aligned}$$

this bound is independent of ϱ . Note that by definition of \mathcal{R} , $\#\{\varrho_{|n}: \varrho \in \mathcal{R}\} = \prod_{j=1}^{j_n+1} \#D_j \leq (j_n+1)!$; also $Y_n(\varrho) - Y_{n-1}(\varrho)$ only depends on $\varrho_{|n} = ((q_1, \alpha_1), \cdots, (q_n, \alpha_n))$. Consequently,

$$\left\| \|Y_n(\cdot) - Y_{n-1}(\cdot)\|_{\infty} \right\|_1 \le \sum_{\varrho|_n} \|Y_n(\varrho) - Y_{n-1}(\varrho)\|_{p_{j_n+1}} \le 2(j_n+1)! s_{j_n+1} \exp((n-1)r_{j_n+1}).$$

This yields

$$\sum_{n \in \mathbb{N}} \|\|Y_n(\cdot) - Y_{n-1}(\cdot)\|_{\infty}\|_{1} \leq \sum_{j \geq 0} \sum_{M_j + 1 \leq n \leq M_{j+1}} 2(j+1)! s_{j+1} \exp((n-1)r_{j+1})
\leq \sum_{j \geq 0} 2(j+1)! s_{j+1} \frac{\exp(M_j r_{j+1})}{1 - \exp(r_{j+1})} < \infty,$$

where we used (3.17). If follows that $(Y_n)_{n\in\mathbb{N}}$ converges uniformly, almost surely and in L^1 norm, to a function Y, as $n \to \infty$.

Let us show that Y does not vanish on \mathcal{R} almost surely. For each $n \geq 1$, let $\mathcal{R}_{X,\phi|n} =$ $\{\varrho_{\mid n}:\varrho\in\mathcal{R}\},\ \text{and for }\gamma\in\mathcal{R}_{X,\phi\mid n}\ \text{define the event}\ \mathcal{N}_{\gamma}=\{\omega\in\Omega:\exists\varrho\in\mathcal{R},\ Y(\varrho)=1\}\}$ $0, \varrho_{|n} = \gamma$. Let $\mathcal{N} = \{\omega \in \Omega : \exists \varrho \in \mathcal{R}, Y(\varrho) = 0\}$. Since the functions Y_n are almost surely positive, this event is a tail event, and it has probability 0 or 1. The same property holds for the events \mathcal{N}_{γ} , $\gamma \in \bigcup_{n \in \mathbb{N}} \mathcal{R}_{X,\phi|n}$. Suppose that \mathcal{N} has probability 1. Since $\mathcal{N} = \bigcup_{\varrho_1 \in \mathcal{R}_1} \mathcal{N}_{(\gamma_1)}$, necessarily, there exists $\gamma_1 \in \mathcal{R}_1$ such that $\mathbb{P}(\mathcal{N}_{(\gamma_1)}) > 0$, and so $\mathbb{P}(\mathcal{N}_{(\gamma_1)}) = 1$. Iterating this remark we can build an infinite deterministic sequence $\gamma = (\gamma_k)_{k \geq 1} \in \mathcal{R}$ such that $\mathbb{P}(\mathcal{N}_{(\gamma_1, \dots, \gamma_n)}) = 1$ for all $n \geq 1$. This means that almost surely, for all $n \geq 1$, there exists $\varrho^{(n)} \in \mathcal{R}$ such that $\varrho_{|n}^{(n)} = (\gamma_1, \dots, \gamma_n)$ and $Y(\varrho^{(n)}) = 0$. But $\varrho_{|n}^{(n)}=(\gamma_1,\ldots,\gamma_n)$ implies that $\varrho^{(n)}$ converges to γ as $n\to\infty$. Hence, by continuity of Yat γ , we get $Y(\gamma) = 0$ almost surely. However, a consequence of our convergence result for Y_n is that the martingale $Y_n(\gamma)$ converges in L^1 to $Y(\gamma)$, so that $\mathbb{E}(Y(\gamma)) = 1$. This is a contradiction. Thus $\mathbb{P}(\mathcal{N}) = 0$.

Now fix any $u \in \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$. Mimicking what was done for $u = \epsilon$, for all $n \geq 1$ one gets

$$\left\| \|Y_n(\cdot,u) - Y_{n-1}(\cdot,u)\|_{\infty} \right\|_1 \le 2(j_{|u|+n}+1)! s_{j_{|u|+n}+1} \exp\left((n-1)r_{j_{|u|+n}+1}\right).$$

Consequently, setting $a_{j,n}(u) = 2(j_{|u|+n} + 1)! s_{j_{|u|+n}+1} \exp((n-1)r_{j_{|u|+n}+1})$, we can get

$$\begin{split} & \sum_{n \in \mathbb{N}} \left\| \|Y_n(\cdot, u) - Y_{n-1}(\cdot, u)\|_{\infty} \right\|_1 \\ & \leq \sum_{n=1}^{M_{j_{|u|}+1}-|u|} a_{j_{|u|},n}(u) + \sum_{j \geq j_{|u|}+1} \sum_{M_j+1 \leq |u|+n \leq M_{j+1}} a_{j,n}(u) \\ & \leq \sum_{j_{|u|} \leq j \leq j_{|u|}+1} \frac{2(j+1)! s_{j+1}}{1 - \exp(r_{j+1})} + \sum_{j \geq j_{|u|}+2} 2(j+1)! s_{j+1} \frac{\exp((M_j - |u|) r_{j+1})}{1 - \exp(r_{j+1})}. \end{split}$$

Note that due to the inequalities $M_{j_{|u|}} + 1 \leq |u| \leq M_{j_{|u|}+1}$, for $j \geq j_{|u|} + 2$ one has $M_j - |u| \ge N_j$, so

$$\sum_{n \in \mathbb{N}} \|\|Y_n(\cdot, u) - Y_{n-1}(\cdot, u)\|_{\infty}\|_{1}$$
(3.24)
$$\leq \sum_{j_{|u|} \leq j \leq j_{|u|}+1} \frac{2(j+1)! s_{j+1}}{1 - \exp(r_{j+1})} + \sum_{j \geq j_{|u|}+2} 2(j+1)! s_{j+1} \frac{\exp(N_j r_{j+1})}{1 - \exp(r_{j+1})}$$

$$\leq 2C_0 \exp(N_{j_{|u|}} \gamma_{j_{|u|}+1}^2 m_{j_{|u|}+1}) + 2 \sum_{j \geq j_{|u|}+2} j^{-2},$$
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where (3.17) and (3.18) have been used. This implies the desired convergence to a limit $Y(\cdot,u)$. The set $\bigcup_{k\geq 0} \mathbb{N}^k$ being countable, the convergence holds also almost surely, simultaneously for all u, and the almost sure positivity of $Y(\cdot,u)$ is proven by using the same argument as for $u = \epsilon$.

Finally, recall the definition (3.14) of $(\widehat{m}_j)_{j\geq 1}$ and set $\varepsilon_k = \gamma_{j_k+1}^2 \widehat{m}_{j_k+1}$ for all $k\geq 0$. The previous calculations, together with the fact that $Y_0(\cdot, u) = 1$ for all $u \in \bigcup_{k \ge 0} \mathbb{N}^k$ and the inequality $|u| \ge N_{j_{|u|}}$ imply the existence of a constant $C_{X,\phi}$ such that:

$$(3.25) \qquad \|\sup_{\varrho \in \mathcal{R}} Y(\varrho, u)\|_1 \le C_{X,\phi} \exp(\varepsilon_{|u|} N_{j_{|u|}}) \le C_{X,\phi} \exp(\varepsilon_{|u|} |u|) \quad (\forall \ u \in \bigcup_{k>0} \mathbb{N}^k).$$

(2) This follows from the branching property.

Estimates similar to the previous ones yield the following lemma, which will be used in the proof of Proposition 5.1.

Lemma 3.4. Let K be a compact subset of $J_{X,\phi}$ containing the unique element of D_1 . There exists $p_K \in (1,2)$ such that $\sup_{j\geq 1} \sup_{(q,\alpha)\in D_j\cap K} L_{\alpha}(p_K q, p_K P_{X,\phi,\alpha}(q)) < 0$ and $\sup\nolimits_{j\geq 1}\sup\nolimits_{(q,\alpha)\in D_j\cap K}\mathbb{E}\big(\Sigma_\alpha(q,\widetilde{P}_{X,\phi,\alpha}(q))^{p_K}\big)<\infty.\ \ \textit{Set}\ \mathcal{R}(K)=\{\varrho\in\mathcal{R}:\ \forall\ k\geq 1,\ (q_k,\alpha_k)\in \mathcal{R}\}$ K}. One has $\|\sup_{\varrho \in \mathcal{R}(K)} Y(\varrho, u)\|_{p_K} = O((j_{|u|} + 2)!).$

3.4. Lower bounds for the Hausdorff dimensions of the measures μ_{ρ} via percolation. Let us recall the definition of the lower Hausdorff dimension of a measure and its characterisation in terms of lower local dimension (see [16] for instance).

Definition 3.1. Let (\mathcal{Z},d) be a compact metric space and μ a finite Borel measure on \mathcal{Z} . Then, the lower Hausdorff dimension of μ is defined as

$$\underline{\dim}(\mu) = \inf \{ \dim E : E \in \mathcal{B}(\mathcal{Z}), \, \mu(E) > 0 \}.$$

Lemma 3.5. Let (\mathcal{Z}, d) be a compact metric space. Then

$$\underline{\dim}(\mu) = \operatorname{ess\,inf}_{\mu} \liminf_{r \to 0^{+}} \frac{\log(\mu(B(z,r)))}{\log(r)}.$$

The goal of this section is to prove the following result.

Theorem 3.1. With probability 1, for all $\varrho \in \mathcal{R}$,

$$\underline{\dim}(\mu_{\varrho}) \ge \liminf_{n \to \infty} \frac{\sum_{k=1}^{n} h(q_k, \alpha_k)}{\sum_{k=1}^{n} \lambda(q_k, \alpha_k)}.$$

We need the following two propositions. Recall the definition (3.21) of the set of parameters $\mathcal{R}(\beta, \ell, \varepsilon)$.

Proposition 3.2. Let $\beta \in ((\mathbb{E}(N))^{-1}, 1]$. Conditionally on non extinction of $(\mathsf{T}_{\beta,n}(u))_{n \in \mathbb{N}}$, for all $\ell \geq 1$ and $\varepsilon \in \mathbb{Q}_+^*$,

- (1) the sequence of continuous functions $(Y_n(\cdot,\beta))_{n\in\mathbb{N}}$ converges uniformly, almost surely and in L^1 norm, to a positive limit $Y(\beta, \cdot)$ on $\mathcal{R}(\beta, \ell, \varepsilon)$;
- (2) the sequence of continuous functions

$$\left(\varrho \mapsto \widetilde{Y}_n(\beta,\varrho) = \sum_{u \in \mathsf{T}_n} (\prod_{k=1}^n \widetilde{W}_{\beta,u_1 \cdots u_k}) \mu_{\varrho}([u]) \right)_{n \in \mathbb{N}}$$

converges uniformly, almost surely and in L^1 norm, to $Y(\beta, \cdot)$ on $\mathcal{R}(\beta, \ell, \varepsilon)$.

Proposition 3.3.

(1) With probability 1, for all $\varrho = (q_k, \alpha_k)_{k \geq 1} \in \mathcal{R}$, for μ_{ϱ} -almost all $t \in \partial \mathsf{T}$, for n large enough, one has

$$\lim_{n\to\infty} n^{-1} \Big| \log \Big(\prod_{k=1}^n W_{\varrho,t_1\cdots t_n} \Big) - \sum_{k=1}^n h(q_k,\alpha_k) \Big| = 0 = \lim_{n\to\infty} n^{-1} \Big| S_n \phi(t) - \sum_{k=1}^n \lambda(q_k,\alpha_k) \Big|.$$

(2) With probability 1, for all $\varrho \in \mathcal{R}$, for μ_{ϱ} -almost every $t \in \partial \mathsf{T}$, one has

$$\lim_{n \to \infty} \frac{\log(\operatorname{diam}([t_{|n}]))}{-S_n \phi(t)} = 1,$$

where the diameter is measured with respect to the metric d_{ϕ} .

Proof of Theorem 3.1. Let $\beta \in (0,1]$ such that $\beta \mathbb{E}(N) > 1$. Let $\ell \geq 1$ and $\varepsilon \in \mathbb{Q}_+^*$. For every $t \in \partial T$ and $\omega_{\beta} \in \Omega_{\beta}$ set

$$Q_{\beta,n}(t,\omega_{\beta}) = \prod_{k=1}^{n} \widetilde{W}_{\beta,t_{|k}},$$

so that for $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$, $\widetilde{Y}_n(\beta, \varrho)$ is the total mass of the measure $Q_{\beta,n}(t, \omega_\beta) \cdot d\mu_\rho^\omega(t)$.

There exists a measurable subset $\Omega(\beta,\ell,\varepsilon)$ of Ω , such that $\mathbb{P}(\Omega(\beta,\ell,\varepsilon))=1$ and for all $\omega\in\Omega(\beta,\ell,\varepsilon)$, there exists $\Omega^\omega_\beta\subset\Omega_\beta$ of positive probability such that for all $\omega\in\Omega(\beta,\ell,\varepsilon)$, for all $\varrho\in\mathcal{R}(\beta,\ell,\varepsilon)$, for all $\omega_\beta\in\Omega^\omega_\beta$, $\widetilde{Y}_n(\beta,\varrho)$ does not converge to 0. In terms of the multiplicative chaos theory developed in [23], this means that for all $\omega\in\Omega(\beta,\ell,\varepsilon)$ and $\varrho\in\mathcal{R}(\beta,\ell,\varepsilon)$, the set of those ω_β such that the multiplicative chaos $(Q_{\beta,n}(\cdot,\omega))_{n\in\mathbb{N}}$ has not killed the measure μ_ϱ on the compact set ∂T has a positive \mathbb{P}_β -probability. Moreover, under the metric d_1 , for any ball B in ∂T , there exists $n\geq 0$ and $u\in T_n$ such that $B=[u]\cap\partial T,\,Q_{\beta,n}(t)$ is constant over B, and denoting by $|B|_{d_1}$ the diameter of B under d_1 , for any $h\in(0,1)$ we have

$$\mathbb{E}_{\beta} \left(\sup_{t \in B} (Q_{\beta,n}(t))^h \right) = e^{n(1-h)\log(\beta)} = (|B|_{d_1})^{-(1-h)\log(\beta)},$$

where \mathbb{E}_{β} stands for the expectation with respect to \mathbb{P}_{β} . Thus, one can apply [23, Theorem 3] and obtain that for all $\omega \in \Omega(\beta, \ell, \varepsilon)$ and all $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$, the measure μ_{ϱ} is not carried by a Borel set of Hausdorff dimension less than $-\log(\beta)$.

Let
$$\Omega' = \bigcap_{\beta \in (\mathbb{E}(N)^{-1},1] \cap \mathbb{Q}_+^*, \ell \geq 1, \varepsilon \in \mathbb{Q}^*} \Omega(\beta,\ell,\varepsilon)$$
. This set is of \mathbb{P} -probability 1.

Let $\varrho \in \mathcal{R}$ and set $D_{\varrho} = \liminf_{n \to \infty} n^{-1} \sum_{k=1}^{n} h(q_{k}, \alpha_{k})$. If $D_{\varrho} > 0$, by (3.22) there exists a sequence of points $(\beta_{n}, \ell_{n}, \varepsilon_{n}) \in (\mathbb{E}(N)^{-1}, 1] \times \mathbb{N} \times \mathbb{Q}_{+}^{*}$ such that $D_{\varrho} \geq -\log(\beta_{n}) + \varepsilon_{n}/2$ for all $n \geq 1$, $\lim_{n \to \infty} -\log(\beta_{n}) = D_{\varrho}$, $\lim_{n \to \infty} \varepsilon_{n} = 0$, and $\varrho \in \bigcap_{n \in \mathbb{N}} \mathcal{R}(\beta_{n}, \ell_{n}, \varepsilon_{n})$. Consequently, the previous paragraph implies that, with respect to the metric d_{1} , for all $\omega \in \Omega'$, $\dim(\mu_{\varrho}^{\omega}) \geq \limsup_{n \to \infty} -\log(\beta_{n}) = D_{\varrho}$. In particular, due to Proposition 3.3(2) applied to $\phi = (1)_{n \in \mathbb{N}}$ (it is valid in this case as well, due to the proof of this part of the proposition) and Lemma 3.5 one has $\liminf_{n \to \infty} \frac{\log \mu_{\varrho}([t_{|n}])}{-n} \geq D_{\varrho}$, μ_{ϱ} -almost everywhere.

Note now that since $\mu_{\varrho}([t_{|n}]) = Y(\rho, t_1 \cdots t_n) \prod_{k=1}^n W_{\varrho, t_1 \cdots t_n}$, we can deduce from the first limit in Proposition 3.3(1) that $\limsup_{n \to \infty} n^{-1} \log(Y(\rho, t_1 \cdots t_n)) \leq 0$, μ_{ϱ} -almost

everywhere. Due to the second limit in Proposition 3.3(1), this implies that under d_{ϕ} one has $\liminf_{n\to\infty}\frac{\log\mu_{\varrho}([t_{|n}])}{\log(\mathrm{diam}([t_{|n}]))}\geq \liminf_{n\to\infty}\frac{\sum_{k=1}^nh(q_k,\alpha_k)}{\sum_{k=1}^n\lambda(q_k,\alpha_k)}$ (an inequality which holds as well trivially if $D_{\varrho}=0$).

Proof of Proposition 3.2. (1) Let $\ell \geq 1$ and $\varepsilon > 0$. For $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$ and $n \geq 1$, Lemma 3.3 applied with $p = p_{j_n+1}$ provides us with the inequality (where \widetilde{P}_{α_k} stands for $\widetilde{P}_{X,\phi,\alpha_k}$)

$$||Y_n(\beta, \varrho) - Y_{n-1}(\beta, \varrho)||_{p_{j_n+1}}^{p_{j_n+1}}$$

$$\leq (2\beta^{-1})^{p_{j_{n+1}}} \mathbb{E}\left(\Sigma_{\alpha_n}(q_n, \widetilde{P}_{\alpha_n}(q_n))^{p_{j_{n+1}}}\right) \prod_{k=1}^{n-1} \beta^{1-p_{j_{n+1}}} \exp\left(L_{\alpha_k}(p_{j_{n+1}}q_k, p_{j_{n+1}}\widetilde{P}_{\alpha_k}(q_k))\right).$$

Let $(q, \alpha) \in D_{j_n+1}$ and set $g_{q,\alpha} : \lambda \in \mathbb{R} \mapsto L_{\alpha}(pq, p\widetilde{P}_{X,\phi,\alpha}(q))$. By construction we have $g_{q,\alpha}(1) = 0$ so for $p \in [1, 2]$

$$g_{q,\alpha}(p) = (p-1)g'_{q,\alpha}(1) + (p-1)^2 \int_0^1 (1-t)g''_{q,\alpha}(1+t(p-1)) dt,$$

with $g'_{q,\alpha}(1) = -h(q,\alpha)$ (see (3.2) for the definition of $h(q,\alpha)$) and

$$g_{q,\alpha}''(1+t(p-1)) = t \binom{q}{\widetilde{P}_{\alpha}(q)} H_{L_{\alpha}}(q+t(p-1)q, \widetilde{P}_{\alpha}(q)+t(p-1)\widetilde{P}_{\alpha}(q)) \binom{q}{\widetilde{P}_{\alpha}(q)}$$

$$\leq \widetilde{m}_{j_{n}+1},$$

where $(\widetilde{m}_j)_{j\geq 1}$ is defined in (3.13). Let $\eta_j=(p_j-1)\widetilde{m}_j$ for $j\geq 1$. By construction of $(p_j)_{j\geq 1}$, one has $\lim_{j\to\infty}\eta_j=0$ and specifying $p=p_{j_n+1}$ one obtains

$$L_{\alpha}(p_{j_n+1}q, p_{j_n+1}\widetilde{P}_{X,\phi,\alpha}(q)) \le (1 - p_{j_n+1})h(q,\alpha) + \eta_{j_n+1}(p_{j_n+1} - 1).$$

We can insert this upper bound in our estimation of $Y_n(\beta, \varrho) - Y_{n-1}(\beta, \varrho)$ and get, remembering that $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$, for $n \ge \ell + 1$

$$\begin{aligned} & \|Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)\|_{p_{j_n+1}}^{p_{j_n+1}} \\ & \leq & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\left((1-p_{j_n+1}) \sum_{k=1}^{n-1} \log(\beta) + h(q_k,\alpha_k) - \eta_{j_n+1}\right) \\ & \leq & (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\left((n-1)(1-p_{j_n+1})(\varepsilon - \eta_{j_n+1})\right). \end{aligned}$$

Let $j(\varepsilon) = \min\{j \geq \lfloor \varepsilon^{-1} \rfloor + 1 : \eta_j \leq \varepsilon/2\}$ and $n_{\varepsilon} = \min\{n \geq \ell + 1 : j_{n+1} \geq j(\varepsilon)\}$. For $n \geq n_{\varepsilon}$ on has, remembering (3.16),

$$||Y_n(\beta,\varrho) - Y_{n-1}(\beta,\varrho)||_{p_{j_n+1}}^{p_{j_n+1}} \le (2\beta^{-1})^{p_{j_n+1}} s_{j_n+1}^{p_{j_n+1}} \exp\left((n-1)p_{j_n+1}r_{j_n+1}\right).$$

Consequently, using the estimates as in the proof of Proposition 3.1 one gets

$$\sum_{n \geq n_{\varepsilon}} \left\| \sup_{\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)} |Y_n(\beta, \varrho) - Y_{n-1}(\beta, \varrho)| \right\|_1 < \infty.$$

This yields the conclusion about the uniform convergence. The fact that the limit $Y(\beta, \cdot)$ does not vanish almost surely, conditionally on non extinction of $(\mathsf{T}_{\beta,n})_{n\geq 1}$, follows the same lines as in the study of $Y(\cdot)$, combined with the fact that for a fixed $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$, the probability that the limit of $Y_n(\beta, \varrho)$ be 0 equals that of the extinction of $(\mathsf{T}_{\beta,n})_{n\in\mathbb{N}}$. This comes from the fact that conditionally on non extinction, the event $\{Y(\beta, \varrho) = 0\}$ is asymptotic so has probability 0 or 1, and it has probability 0 since the convergence of

 $Y_n(\beta, \varrho)$ to $Y(\beta, \varrho)$ holds in L^1 . Thus, we have the desired result for a given couple (ℓ, ε) ; but it holds simultaneously for all $\ell \geq 1$ and $\varepsilon \in \mathbb{Q}_+^*$ since $\mathbb{N} \times \mathbb{Q}_+^*$ is countable.

(2) The approach to follow can be interpreted as a uniform version of the "decomposition" principle of multiplicative cascades on homogeneous trees due Kahane (see [24], as well as [37] for a proof and [18] for general multiplicative chaos).

Fix $\ell \geq 1$ and $\varepsilon > 0$. Denote by E the separable Banach space of real valued continuous functions over the compact set $\mathcal{R}(\beta, \ell, \varepsilon)$ endowed with the supremum norm $\|\cdot\|_{\infty}$.

For $n \geq m \geq 1$ and $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$ let

$$\widetilde{Y}_{m,n}(\beta,\varrho) = \sum_{u \in \mathsf{T}_m} Y_{n-m}(\varrho,u) \prod_{k=1}^m \widetilde{W}_{\beta,u_{|k}} W_{\varrho,u_{|k}}.$$

Notice that $\widetilde{Y}_{n,n}(\beta,\varrho) = Y_n(\beta,\varrho)$. Moreover, since $Y_n(\beta,\cdot)$ converges uniformly, almost surely and in L^1 norm to $Y(\beta,\cdot)$ as $n\to\infty$, $Y_n(\beta,\cdot)$ belongs to $L^1_E=L^1_E(\Omega_\beta\times\Omega,\mathcal{A}_\beta\times\mathcal{A},\mathbb{P}_\beta\times\mathbb{P})$ (where we use the notations of [30, Section V-2]), so that the continuous random function $\mathbb{E}(\widetilde{Y}_{n,n}(\beta,\varrho)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n)$ is well defined by [30, Proposition V-2-5]. Also, given $\varrho\in\mathcal{R}(\beta,\ell,\varepsilon)$, we can deduce from the definitions and the independence assumptions that

$$\widetilde{Y}_{m,n}(\beta,\varrho) = \mathbb{E}(\widetilde{Y}_{n,n}(\beta,\varrho)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n)$$

almost surely. Consequently, by [30, Proposition V-2-5] again, since $e \in E \mapsto e(\varrho)$ is a continuous linear form over E, we obtain $\widetilde{Y}_{m,n}(\beta,\varrho) = \mathbb{E}(\widetilde{Y}_{n,n}(\beta,\cdot)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n)(\varrho)$ almost surely. Since given any dense countable subset \mathcal{D} of $\mathcal{R}(\beta,\ell,\varepsilon)$ this holds simultaneously for all $\varrho \in \mathcal{D}$, we can conclude that the random continuous functions $\widetilde{Y}_{m,n}(\beta,\cdot)$ and $\mathbb{E}(\widetilde{Y}_{n,n}(\beta,\cdot)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n)$ are equal almost surely.

Similarly, since for each $\varrho \in \mathcal{R}(\beta, \ell, \varepsilon)$ the martingale $(Y_n(\beta, \varrho), \mathcal{F}_{\beta,n} \otimes \mathcal{F}_n)_{n \in \mathbb{N}}$ converges to $Y(\beta, \varrho)$ almost surely and in L^1 , and $Y(\beta, \cdot) \in L^1_E$, by using [30, Proposition V-2-5] once more we can get

$$(3.26) \qquad \widetilde{Y}_{n,n}(\beta,\cdot) = \mathbb{E}(Y(\beta,\cdot)|\mathcal{F}_{\beta,n}\otimes\mathcal{F}_n), \text{ hence } \widetilde{Y}_{m,n}(\beta,\cdot) = \mathbb{E}(Y(\beta,\cdot)|\mathcal{F}_{\beta,m}\otimes\mathcal{F}_n),$$

almost surely. Moreover, it follows from Proposition 3.1(1) and the definition of $\mu_{\varrho}([u])$ that $\widetilde{Y}_{m,n}(\beta,\cdot)$ converges uniformly, almost surely and in L^1 norm, as $n\to\infty$, to $\widetilde{Y}_m(\beta,\cdot)$. This and (3.26) yield, using [30, Proposition V-2-6],

$$\widetilde{Y}_m(\beta,\cdot) = \lim_{n \to \infty} \widetilde{Y}_{m,n}(\cdot) = \mathbb{E}(Y(\beta,\cdot)|\mathcal{F}_{\beta,m} \otimes \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)),$$

and finally

$$\lim_{m\to\infty}\widetilde{Y}_m(\beta,\cdot)=\mathbb{E}\big(Y(\beta,\cdot)|\sigma(\bigcup_{m\geq 1}\mathcal{F}_{\beta,m})\otimes\sigma(\bigcup_{n\in\mathbb{N}}\mathcal{F}_n)\big)=Y(\beta,\cdot)$$

almost surely (since by construction $Y(\beta, \cdot)$ is $\sigma(\bigcup_{m \geq 1} \mathcal{F}_{\beta,m}) \otimes \sigma(\bigcup_{n \in \mathbb{N}} \mathcal{F}_n)$ -measurable), where the convergences hold in the uniform norm.

Proof of Proposition 3.3. (1) This will be established simultaneously with Proposition 4.4 below, which deals with \mathbb{R}^d -valued branching random walks. (2) Recall (3.7). Observe that by construction, for all $t \in \partial \mathbb{T}$ one has $\operatorname{diam}([t_{|n}])) = \exp(-S_{n+k_n(t)}\phi(t))$, where $k_n(t) = \inf\{k \geq 0 : N_{t_{|n+k}} > 1\}$. Consequently, if follows from the previous observation

and part (1) of the proposition that the property we have to establish will follow if we show that with probability 1, for all $\varrho \in \mathcal{R}$ one has $k_n(t) = o(n)$ for μ_{ϱ} -almost every t.

Fix
$$\eta \in (0,1)$$
. Denoting by 1^k the word $\underbrace{1\cdots 1}_k$, for all $\varrho \in \mathcal{R}$ one has

$$\begin{split} \mu_{\varrho}(\{t \in \partial \mathsf{T}: \, k_n(t) > \lfloor n\eta \rfloor\}) &\leq \sum_{|u|=n} \mu_{\varrho}([u \cdot 1^{\lfloor n\eta \rfloor}]) \mathbf{1}_{\{N_u = N_{u1} = \ldots = N_{u \cdot 1} \lfloor n\eta \rfloor - 1 = 1\}} \\ &= \sum_{|u|=n} \mu_{\varrho|_n}([u]) \Big(\prod_{k=0}^{\lfloor n\eta \rfloor - 1} \mathbf{1}_{\{N_{u \cdot 1} k = 1\}} W_{\varrho, u \cdot 1^{k+1}} \Big) Y(\varrho, u \cdot 1^{\lfloor n\eta \rfloor}). \end{split}$$

Thus

$$\sup_{\varrho \in \mathcal{R}} \mu_{\varrho}(\{t \in \partial \mathsf{T} : k_n(t) > \lfloor n\eta \rfloor\})$$

$$\leq \sum_{\varrho_{|n+|n\eta|}:\varrho\in\mathcal{R}}\sum_{|u|=n}\mu_{\varrho_{|n}}([u])\Big(\prod_{k=0}^{\lfloor n\eta\rfloor-1}\mathbf{1}_{\{N_{u\cdot 1^k}=1\}}W_{\varrho,u\cdot 1^{k+1}}\Big)\sup_{\varrho\in\mathcal{R}}Y(\varrho,u\cdot 1^{\lfloor n\eta\rfloor}).$$

Consequently, using (3.11) and (3.25), we get

$$\mathbb{E}(\sup_{\varrho \in \mathcal{R}} \mu_{\varrho}(\{t \in \partial \mathsf{T} : k_{n}(t) \geq \lfloor n\eta \rfloor\}))$$

$$\leq (\#\{\varrho_{|n+\lfloor n\eta \rfloor} : \varrho \in \mathcal{R}\}) \left(1 - \frac{c_{0}}{j_{n+\lfloor n\eta \rfloor}}\right)^{\lfloor n\eta \rfloor} C_{X,\phi} e^{(n+\lfloor n\eta \rfloor)\varepsilon_{n+\lfloor n\eta \rfloor}}$$

$$\leq C_{X,\phi}(j_{n+\lfloor n\eta \rfloor})! \exp\left(-c_{0} \frac{\lfloor n\eta \rfloor}{j_{n+\lfloor n\eta \rfloor}} + (n+\lfloor n\eta \rfloor)\varepsilon_{n+\lfloor n\eta \rfloor}\right).$$
(3.27)

Note that due to (3.15) and the fact that $\varepsilon_k = \gamma_{j_k+1}^2 \widehat{m}_{j_k+1}$, for n large enough we have $-c_0 \frac{\lfloor n\eta \rfloor}{j_{n+\lfloor n\eta \rfloor}} + (n + \lfloor n\eta \rfloor) \varepsilon_{n+\lfloor n\eta \rfloor} \leq -c_0 \frac{n\eta}{2j_n}$. Moreover, $n \geq N_{j_n} \geq (j_n)!$ so $n \geq \frac{n+\lfloor n\eta \rfloor}{1+\eta} \geq \frac{(j_{n+\lfloor n\eta \rfloor})!}{1+\eta}$, and $j_n = o(\log(n))$ as $n \to \infty$. Consequently, (3.27) implies that

$$\sum_{n\in\mathbb{N}} \mathbb{E}(\sup_{\varrho\in\mathcal{R}} \mu_{\varrho}(\{t\in\partial\mathsf{T}: k_n(t)\geq \lfloor n\eta\rfloor\}) < \infty,$$

This holds for all $\eta \in (0,1)$ from which it follows that, with probability 1, for all positive rational number $\eta > 0$, one has $\sum_{n \in \mathbb{N}} \mathbb{E}(\sup_{\varrho \in \mathcal{R}} \mu_{\varrho}(\{t \in \partial \mathsf{T} : k_n(t) \geq \lfloor n\eta \rfloor\}) < \infty$. By the Borel-Cantelli lemma, this implies that with probability 1, for all $\varrho \in \mathcal{R}$ one has $k_n(t) = o(n)$ for μ_{ϱ} -almost every t, which is what had to be established.

4. Proof of Theorem 1.1

Sections 4.1 and 4.2 are respectively dedicated to establish the sharp upper bound and lower bound for dim E(X, K), almost surely for all $K \in \mathcal{K}$.

4.1. Upper bounds for the Hausdorff dimensions of the sets E(X,K). For each $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$, recall the definition (1.13) of $\widetilde{P}_{X,\phi,\alpha}(q)$ and define

$$P_{X,\phi,\alpha}(q) = \inf \Big\{ t \in \mathbb{R} : \limsup_{n \to \infty} \frac{1}{n} \log \Big(\sum_{u \in \mathsf{T}_n} \exp(\langle q | S_n(X - \alpha)(u) \rangle - t S_n \phi(u)) \Big) \le 0 \Big\}.$$

The following proposition is a direct consequence of the log-convexity, of the mappings $(q,t) \mapsto \sum_{u \in \mathsf{T}_n} \exp(\langle q | S_n(X-\alpha)(u) \rangle - t S_n \phi(u))$ and $(\alpha,t) \mapsto \sum_{u \in \mathsf{T}_n} \exp(\langle q | S_n(X-\alpha)(u) \rangle - t S_n \phi(u))$ given $\alpha \in \mathbb{R}^d$ and $q \in \mathbb{R}^d$ respectively.

Proposition 4.1. The mappings $q \mapsto P_{X,\phi,\alpha}(q)$ and $\alpha \mapsto P_{X,\phi,\alpha}(q)$ are convex.

Proposition 4.2. With probability 1, $P_{X,\phi,\alpha}(q) \leq \widetilde{P}_{X,\phi,\alpha}(q)$ for all $(q,\alpha) \in \mathbb{R}^d \times \mathbb{R}^d$.

Proof. Due to Proposition 4.1, we only need to prove the inequality for each $(q, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d$, almost surely. Fix $(q, \alpha) \in \mathbb{R}^d \times \mathbb{R}^d$. For $t > \widetilde{P}_{\alpha}(q)$ we have

$$\mathbb{E}(\sum_{n\geq 1}\sum_{u\in\mathsf{T}_n}\exp(\langle q|S_n(X-\alpha)(u)\rangle-tS_n\phi(u)) = \sum_{n\geq 1}\mathbb{E}(\sum_{i=1}^N\exp(\langle q|X_i-\alpha\rangle-t\phi_i))^n < \infty.$$

Consequently, $\sum_{u \in \mathsf{T}_n} \exp\left(\langle q | S_n(X - \alpha)(u) \rangle - t S_n \phi(u)\right)$ is bounded almost surely, so $t \geq P_{X,\phi,\alpha}(q)$ almost surely. Since $t > \widetilde{P}_{X,\phi,\alpha}(q)$ is arbitrary, we get the desired conclusion. \square

For $\alpha \in \mathbb{R}^d$, set

$$\widehat{E}(X,\alpha) = \Big\{ t \in \partial \mathsf{T} : \alpha \in \bigcap_{n \in \mathbb{N}} \overline{\Big\{ \frac{S_n X(t)}{n} : n \ge N \Big\}} \Big\}.$$

The following proposition and its corollary extend [3, Proposition 2.5 and Corollary 2.3], valid for d_1 , to the case of the more general metric d_{ϕ} .

Proposition 4.3. With probability 1, for all $\alpha \in \mathbb{R}^d$, dim $\widehat{E}(X,\alpha) \leq P_{X,\phi,\alpha}^*(0)$, a negative dimension meaning that $\widehat{E}(X,\alpha)$ is empty. In particular $P_{X,\phi,\alpha}^*(0) \geq 0$ for all $\alpha \in I_X$.

Proof. The argument is largely inspired by the approach developed for the multifractal analysis of \mathbb{R}^d -values Birkhoff averages on conformal repellers (see [7]; for a general multifractal formalism for vector-valued functions, see [33]). Recall that for any $E \subset \partial \mathcal{T}$, dim $E = -\infty$ if $E = \emptyset$ and dim $E = \inf\{s \in \mathbb{R}_+ : \lim_{\delta \to 0^+} \mathcal{H}^s_{\delta}(E) = 0\}$ otherwise, where

$$\mathcal{H}^{s}_{\delta}(E) = \inf \Big\{ \sum_{i \in \mathbb{N}} \operatorname{diam}(E_{i})^{s} : E \subset \bigcup_{i \in \mathbb{N}} E_{i}, \operatorname{diam}(E_{i}) \leq \delta \Big\}.$$

For every $n \ge 1$ let us denote $\widetilde{r}_n = \max\{\operatorname{diam}([u]) : u \in \mathsf{T}_n\}$. To begin, note that

$$\widehat{E}(X,\alpha) = \bigcap_{\varepsilon>0} \bigcap_{n\in\mathbb{N}} \bigcup_{n\geq N} \{t \in \partial \mathsf{T} : \|S_nX(t) - n\alpha\| \le n\varepsilon\}$$

$$\subset \bigcap_{q\in\mathbb{R}} \bigcap_{d} \bigcap_{\varepsilon>0} \bigcap_{n\in\mathbb{N}} \bigcup_{n>N} \{t \in \partial \mathsf{T} : |\langle q|S_nX(t) - n\alpha\rangle| \le n\|q\|\varepsilon\}.$$

Fix $q \in \mathbb{R}^d$ and $\varepsilon > 0$. For $N \ge 1$, the set $E(q, N, \varepsilon, \alpha) = \bigcup_{n \ge N} \{t \in \partial \mathsf{T} : |\langle q|S_nX(t) - n\alpha\rangle| \le n\|q\|\varepsilon\}$ is covered by the union of those [u] such that $u \in \mathsf{T}_n$ and $\langle q|S_nX(u) - n\alpha\rangle + n\|q\|\varepsilon \ge 0$. Consequently, noting that by construction of the metric d_{ϕ} , for all

 $u \in \mathsf{T}_n$ we have diam([u]) $\leq \exp(-S_n\phi(u))$, for any $s \geq 0$ we can write

$$\mathcal{H}_{\tilde{r}_{N}}^{s}(E(q, N, \varepsilon, \alpha)) \leq \sum_{n \geq N} \sum_{u \in \mathsf{T}_{n}} \operatorname{diam}([u])^{s} \exp(\langle q | S_{n}X(u) - n\alpha \rangle + n \|q\|\varepsilon)$$

$$\leq \sum_{n \geq N} \sum_{u \in \mathsf{T}_{n}} \exp(\langle q | S_{n}(X - \alpha)(u) \rangle - sS_{n}\phi(u) + n \|q\|\varepsilon).$$

Hence, if $\eta > 0$ and $s > P_{X,\phi,\alpha}(q) + \eta + \|q\|\varepsilon$, by definition of $P_{X,\phi,\alpha}(q)$, for N large enough one has $\mathcal{H}^s_{\widetilde{r}_N}(E(q,N,\varepsilon,\alpha)) \leq \sum_{n \geq N} e^{-n\eta/2}$. Since $\widetilde{r}_N \leq r_n = \max\{\exp(-S_n\phi(u)) : u \in \mathsf{T}_n\}$, \widetilde{r}_N tends to 0 almost surely as N tends to ∞ , and we conclude that $\dim E(q,N,\varepsilon,\alpha) \leq s$. As this holds for all $\eta > 0$, we get $\dim E(q,N,\varepsilon,\alpha) \leq P_{X,\phi,\alpha}(q) + \|q\|\varepsilon$. It follows that $\dim \widehat{E}(X,\alpha) \leq \inf_{q \in \mathbb{R}} \inf_{g \in \mathbb{R}} \inf_{g \in \mathbb{R}} \inf_{g \in \mathbb{R}} \inf_{g \in \mathbb{R}} P_{X,\phi,\alpha}(g) = P^*_{X,\phi,\alpha}(0)$. If $\inf_{q \in \mathbb{R}} P_{X,\phi,\alpha}(q) < 0$, we necessarily have $\widehat{E}(X,\alpha) = \emptyset$. Since for $\alpha \in I_X$ one has $\emptyset \neq E(X,\alpha) \subset \widehat{E}(X,\alpha)$, we get $P^*_{X,\phi,\alpha}(0) \geq 0$.

Corollary 4.1. With probability 1, for all compact connected subset K of \mathbb{R}^d , one has $E(X,K) = \emptyset$ if $K \not\subset I_X$, and dim $E(X,K) \leq \inf_{\alpha \in K} \widetilde{P}^*_{X,\phi,\alpha}(0)$ otherwise.

Proof. This follows directly from Propositions 4.2 and 4.3.

4.2. Lower bounds for the Hausdorff dimensions of the set E(X,K). The sharp lower bound estimates for the Hausdorff dimensions of the set E(X,K) are direct consequences of Theorem 3.1, the fact that $\liminf_{n\to\infty} \frac{\sum_{k=1}^n h(q_k,\alpha_k)}{\sum_{k=1}^n \lambda(q_k,\alpha_k)} \ge \liminf_{n\to\infty} \frac{h(q_n,\alpha_n)}{\lambda(q_n,\alpha_n)}$ for all $\varrho \in \mathcal{R}$, (3.4), and the following two propositions. Recall the definition (3.5) of $\beta(q,\alpha)$.

Proposition 4.4. With probability 1, for all $\varrho = ((q_k, \alpha_k))_{k \geq 1} \in \mathcal{R}$, for μ_{ϱ} -almost all $t \in \partial \Gamma$, one has

$$\lim_{n \to \infty} n^{-1} \left\| S_n X(t) - \sum_{k=1}^n \beta(q_k, \alpha_k) \right\| = 0.$$

Proposition 4.5. For all compact connected subset K of I_X , there exists $\varrho \in \mathcal{R}$ such that

$$\left\{ \bigcap_{n \in \mathbb{N}} \overline{\left\{ n^{-1} \sum_{k=1}^{n} \beta(q_k, \alpha_k) : n \ge N \right\}} = K \\ \liminf_{n \to \infty} \widetilde{P}_{X, \phi, \alpha_n}^* (\nabla \widetilde{P}_{X, \phi, \alpha_n}(q_n)) \ge \inf \left\{ P_{X, \phi, \alpha}^*(0) : \alpha \in K \right\}$$

Proofs of Proposition 4.4 and Proposition 3.3(1). We will prove slightly stronger results by controlling uniformly the speed of convergence to 0.

Fix $(\delta_n)_{n\in\mathbb{N}}$ converging to 0, and to be specified later. Let v be a vector of the canonical basis \mathcal{B} of \mathbb{R}^d . For $\varrho \in \mathcal{R}$, $v \in \mathcal{B}$, $\lambda \in \{-1,1\}$ and $n \geq 1$, we set :

$$\begin{cases} E_{\varrho,n,\delta_n}^{\lambda}(v) = \left\{t \in \partial \mathsf{T} : \lambda \Big\langle v \Big| S_n X(t) - \sum_{k=1}^n \beta(q_k,\alpha_k) \Big\rangle \geq n \delta_n \right\} \\ F_{\varrho,n,\delta_n}^{\lambda} = \left\{t \in \partial \mathsf{T} : \lambda \Big(S_n \phi(t) - \sum_{k=1}^n \lambda(q_k,\alpha_k) \Big) \geq n \delta_n \right\} \\ G_{\varrho,n,\delta_n}^{\lambda} = \left\{t \in \partial \mathsf{T} : \lambda \Big(\log \Big(\prod_{k=1}^n W_{\varrho,t_1\cdots t_k} \Big) - \sum_{k=1}^n h(q_k,\alpha_k) \Big) \geq n \delta_n \right\} \end{cases}$$

It is enough to specify $(\delta_n)_{n\in\mathbb{N}}$ such that for $\lambda\in\{-1,1\}$ and $v\in\mathcal{B}$ one has

$$(4.1) \qquad \mathbb{E}\Big(\sup_{\varrho \in \mathcal{R}} \sum_{n \geq 1} \mu_{\varrho}(E_{\varrho,n,\delta_n}^{\lambda}(v)) + \mu_{\varrho}(F_{\varrho,n,\delta_n}^{\lambda}) + \mu_{\varrho}(G_{\varrho,n,\delta_n}^{\lambda})\Big) < \infty.$$

Indeed, if (4.1) holds then, with probability 1, for all $\varrho \in \mathcal{R}$, $\lambda \in \{-1,1\}$ and $v \in \mathcal{B}$, one has $\sum_{n\geq 1} \mu_{\varrho}(E_{\varrho,n,\delta_n}^{\lambda}(v)) + \mu_{\varrho}(F_{\varrho,n,\delta_n}^{\lambda}) + \mu_{\varrho}(G_{\varrho,n,\delta_n}^{\lambda}) < \infty$. Consequently, by the Borel-Cantelli lemma, for μ_{ϱ} -almost every t, for n large enough, for all $v \in \mathcal{B}$,

$$\begin{cases}
\left| \left\langle v \middle| \left(S_n X(t) - \sum_{k=1}^n \beta(q_k, \alpha_k) \right) \right\rangle \right| \leq n \delta_n \\
\max \left(\left| S_n \phi(t) - \sum_{k=1}^n \lambda(q_k, \alpha_k) \middle|, \left| \log \left(\prod_{k=1}^n W_{\varrho, t_1 \cdots t_k} \right) - \sum_{k=1}^n h(q_k, \alpha_k) \middle| \right) \leq n \delta_n,
\end{cases}$$

which yields the desired result.

Now we prove (4.1) when $\lambda = 1$ (the case $\lambda = -1$ is similar). Let $\varrho \in \mathcal{R}$. For $n \geq 1$ and $u \in \mathbb{N}^{\mathbb{N}}$, and $\gamma > 0$, set

$$\Pi_{n,\gamma}^{e}(\varrho,u) = \prod_{k=1}^{n} \exp\left(\langle q_k + \gamma v | X_{u_{|k}} - \alpha_k \rangle - \widetilde{P}_{\alpha_k}(q_k) \phi_{u_{|k}} - \langle \gamma v | \beta(q_k,\alpha_k) - \alpha_k \rangle - \gamma \delta_n\right),$$

where \widetilde{P}_{α_k} stands for $\widetilde{P}_{X,\phi,\alpha_k}$. For every $\gamma > 0$, using Chernov's inequality one can get $\mu_{\varrho}(E^1_{\varrho,n,\delta_n}(v)) \leq e_{n,\gamma}(\varrho)$,

where

$$e_{n,\gamma}(\varrho) = \sum_{u \in \mathsf{T}_n} \mu_{\varrho}([u]) \prod_{k=1}^n \exp\left(\gamma \langle v | X_{u_{|k}} - \beta(q_k, \alpha_k) \rangle - \gamma \delta_n\right) = \sum_{u \in \mathsf{T}_n} \Pi_{n,\gamma}^e(\varrho, u) Y(\varrho, u).$$

Note that $\Pi_{n,\gamma}^e(\cdot,u)$ only depends on $\varrho_{|n}$, so

$$\sup_{\varrho \in \mathcal{R}} e_{n,\gamma}(\varrho) \le \sum_{u \in \mathsf{T}_n} \sup_{\varrho \mid n : \varrho \in \mathcal{R}} \Pi_{n,\gamma}^e(\varrho, u) \cdot \sup_{\varrho \in \mathcal{R}} Y(\varrho, u).$$

Consequently, since $\mathbb{E}(\sup_{\varrho \in \mathcal{R}} Y(\varrho, u)) \leq C_{X,\phi} \exp(\varepsilon_{|u|}|u|)$ by (3.25), one obtains (taking into account the independences)

$$\begin{split} &\mathbb{E}(\sup_{\varrho \in \mathcal{R}} e_{n,\gamma}(\varrho)) \\ &\leq C_{X,\phi} \, e^{n\varepsilon_n} \mathbb{E}\Big(\sum_{u \in \mathsf{T}_n} \sup_{\varrho|_n: \varrho \in \mathcal{R}} \Pi_{n,\gamma}^e(\varrho,u)\Big) \\ &\leq C_{X,\phi} \, e^{n\varepsilon_n} \mathbb{E}\Big(\sum_{u \in \mathsf{T}_n} \sum_{\varrho|_n: \varrho \in \mathcal{R}} \Pi_{n,\gamma}^e(\varrho,u)\Big) \\ &= C_{X,\phi} \, e^{n\varepsilon_n} \sum_{\varrho|_n: \varrho \in \mathcal{R}} \exp\Big(\sum_{k=1}^n L_{\alpha_k}(q_k + \gamma v, \widetilde{P}_{\alpha_k}(q_k)) - \langle \gamma v | \beta(q_k, \alpha_k) - \alpha_k \rangle - \gamma \delta_n\Big). \end{split}$$

Similarly, setting

$$\Pi_{n,\gamma}^{f}(\varrho,u) = \prod_{k=1}^{n} \exp\left(\langle q_k | X_{u_{|k}} - \alpha_k \rangle - (\widetilde{P}_{\alpha_k}(q_k) + \gamma)\phi_{u_{|k}} - \gamma\lambda(q_k,\alpha_k) - \gamma\delta_n\right),\,$$

a new application of Chernov's inequality yields

$$\mu_{\varrho}(F_{\varrho,n,\delta_n}^1(v)) \le f_{n,\gamma}(\varrho),$$

where

$$f_{n,\gamma}(\varrho) = \sum_{u \in \mathsf{T}_n} \mu_{\varrho}([u]) \prod_{k=1}^n \exp\left(\gamma(\phi_{u_{|k}} - \lambda(q_k, \alpha_k)) - \gamma \delta_n\right) = \sum_{u \in \mathsf{T}_n} \Pi_{n,\gamma}^f(\varrho, u) Y(\varrho, u).$$

It follows that

$$\mathbb{E}(\sup_{\varrho \in \mathcal{R}} f_{n,\gamma}(\varrho)) \le C_{X,\phi} e^{n\varepsilon_n} \sum_{\varrho|_n: \varrho \in \mathcal{R}} \exp\Big(\sum_{k=1}^n L_{\alpha_k}(q_k, \widetilde{P}_{\alpha_k}(q_k) + \gamma) - \gamma \lambda(q_k, \alpha_k) - \gamma \delta_n\Big).$$

Also, setting $p = 1 + \gamma$ and

$$\Pi_{n,\gamma}^{g}(\varrho,u) = \prod_{k=1}^{n} \exp\left(p\langle q_k|X_{u_{|k}} - \alpha_k\rangle - p\widetilde{P}_{\alpha_k}(q_k)\phi_{u_{|k}} - \gamma h(q_k,\alpha_k) - \gamma \delta_n\right),$$

it holds that

$$\mu_{\varrho}(G^1_{\varrho,n,\delta_n}(v)) \le g_{n,\gamma}(\varrho),$$

where

$$g_{n,\gamma}(\varrho) = \sum_{u \in \mathsf{T}_n} \mu_{\varrho}([u]) \prod_{k=1}^n \exp\left(\gamma(\log(W_{\varrho,u_1 \cdots u_k}) - h(q_k, \alpha_k)) - \gamma \delta_n\right)$$
$$= \sum_{u \in \mathsf{T}_n} \Pi_{n,\gamma}^g(\varrho, u) Y(\varrho, u).$$

This time, one has the upper bound

$$\mathbb{E}(\sup_{\varrho \in \mathcal{R}} g_{n,\gamma}(\varrho)) \leq C_{X,\phi} e^{n\varepsilon_n} \sum_{\rho_{1n}: \rho \in \mathcal{R}} \exp\Big(\sum_{k=1}^n L_{\alpha_k}(pq_k, p\widetilde{P}_{\alpha_k}(q_k)) - \gamma h(q_k, \alpha_k) - \gamma \delta_n\Big).$$

For each $\varrho \in \mathcal{R}$, one has $q_k \in D_{j_n+1}$ for all $1 \leq k \leq n$. Thus, reasoning as in the proof of Proposition 3.2(1) and writing for each $1 \leq k \leq n$ the Taylor expansion with integral rest of order 2 of $\gamma \mapsto L_{\alpha_k}(q_k + \gamma v, \widetilde{P}_{\alpha_k}(q_k)) - \langle \gamma v | \beta(q_k, \alpha_k) - \alpha_k \rangle$ at 0, taking $\gamma = \gamma_{j_n+1}$, and using (3.8) and (3.12) one gets

$$\sum_{k=1}^{n} L_{\alpha_k}(q_k + \gamma_{j_n+1}v, \widetilde{P}_{\alpha_k}(q_k)) - \langle \gamma_{j_n+1}v | \beta(q_k, \alpha_k) - \alpha_k \rangle - \gamma_{j_n+1}\delta_n$$

$$\leq n\gamma_{j_n+1}^2 m_{j_n+1} - n\gamma_{j_n+1}\delta_n$$

uniformly in $\varrho \in \mathcal{R}$. Similarly, using (3.8) and (3.12) again one gets

$$\sum_{k=1}^{n} L_{\alpha_k}(q_k, \widetilde{P}_{\alpha_k}(q_k) + \gamma) - \gamma \lambda(q_k, \alpha_k) - \gamma \delta_n \le n \gamma_{j_n+1}^2 m_{j_n+1} - n \gamma_{j_n+1} \delta_n,$$

while using (3.9) and (3.13) (as in the proof of Proposition 3.2) yields

$$\sum_{k=1}^{n} L_{\alpha_k} \left(pq_k, p\widetilde{P}_{X,\phi,\alpha_k}(q_k) \right) - \gamma h(q_k,\alpha_k) - \gamma \delta_n \le n\gamma_{j_n+1}^2 \widetilde{m}_{j_n+1} - n\gamma_{j_n+1} \delta_n.$$

Consequently, since $\varepsilon_n = 2\gamma_{j_n+1}^2 \widehat{m}_{j_n+1}$, $\max(m_{j_n+1}, \widetilde{m}_{j_n+1}) \leq \widehat{m}_{j_n+1}$, and $\operatorname{card}(\{\varrho_{|n} : \varrho \in \mathcal{O}\})$ \mathcal{R}) $\leq (j_n + 1)!$, one obtains the following upper bound:

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{R}}e_{n,\gamma_{j_{n+1}}}(\varrho) + \sup_{\varrho\in\mathcal{R}}f_{n,\gamma_{j_{n+1}}}(\varrho) + \sup_{\varrho\in\mathcal{R}}g_{n,\gamma_{j_{n+1}}}(\varrho)\Big)$$

$$\leq 3C_{X,\phi}(j_n+1)!\exp\big((-n\gamma_{j_{n+1}}(\delta_n - 3\gamma_{j_{n+1}}^2\widehat{m}_{j_{n+1}})\big).$$

Let $\delta_n = 4\gamma_{j_n+1}^2 \widehat{m}_{j_n+1}$. Note that $\lim_{n\to\infty} \delta_n = 0$. Now we use (3.19): $(j_n+1)! \leq 1$ $\exp(N_{j_n}^{1/3}) \le \exp(n^{1/3})$ and $\gamma_{j_{n+1}}^2 \widehat{m}_{j_n+1} \ge N_{j_n}^{-1/2} \ge n^{-1/2}$. Thus

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{R}}e_{n,\gamma_{j_{n+1}}}(\varrho)+\sup_{\varrho\in\mathcal{R}}f_{n,\gamma_{j_{n+1}}}(\varrho)+\sup_{\varrho\in\mathcal{R}}g_{n,\gamma_{j_{n+1}}}(\varrho)\Big)\leq 3\,C_{X,\phi}\,\exp(n^{1/3})\exp(-n^{1/2}).$$

This yields (4.1).

Proof of Proposition 4.5. For every integer $m \geq 1$, let $B(\widetilde{\alpha}_{m,\ell}, 1/m)_{1 \leq \ell \leq L_m}$ be a finite covering of K by balls centered on K, with $L_m \geq 2$. Since K is connected, without loss of generality we can assume that $B(\widetilde{\alpha}_{m,\ell}, 1/m) \cap B(\widetilde{\alpha}_{m,\ell+1}, 1/m) \neq \emptyset$ for all $1 \leq \ell \leq L_m - 1$, and $B(\widetilde{\alpha}_{m+1,1}, 1/(m+1)) \cap B(\widetilde{\alpha}_{m,L_m}, 1/m) \neq \emptyset$.

Now, applying Lemma 3.1, for each $\widetilde{\alpha}_{m,\ell}$ let $(q_{m,\ell}, \alpha_{m,\ell}) \in D$ such that $\|\beta(q_{m,\ell}, \alpha_{m,\ell}) - \beta(q_{m,\ell}, \alpha_{m,\ell})\|$ $\widetilde{\alpha}_{m,\ell} \| \leq 1/m \text{ and } |\widetilde{P}_{X,\phi,\alpha_{m,\ell}}^*(\nabla \widetilde{P}_{X,\phi,\alpha_{m,\ell}}(q_{m,\ell})) - \widetilde{P}_{X,\phi,\widetilde{\alpha}_{m,\ell}}^*(0)| \leq 1/m.$

Let $j_{1,1} = \min\{j \geq 1 : (q_{1,1}, \alpha_{1,1}) \in D_i\}$. Then, define recursively for each $m \geq 1$ and $j_{m,L_m}:(q_{m+1,1},\alpha_{m+1,1})\in D_j$.

The sequence ϱ is constructed as follows. For $1 \leq k \leq M_{j_{1,1}-1}$, let (q_k, α_k) be equal to the unique element of D_1 . Then, for all $m \ge 1$, let $(q_k, \alpha_k) = (q_{m,\ell}, \alpha_{m,\ell})$ for $k \in [M_{j_{m,\ell}-1} +$ $[1, M_{j_{m,\ell+1}-1}]$ if $1 \leq \ell \leq L_m - 1$ and $(q_k, \alpha_k) = (q_{m,L_m}, \alpha_{m,L_m})$ for $k \in [M_{j_{m,L_m}-1} + 1]$ $1, M_{j_{m+1,1}-1}$].

Now let $n \ge M_{j_{2,1}} + 1$. There is an integer $m_n \ge 2$ such that either $n \in [M_{j_{m_n,\ell_n-1}} + 1, M_{j_{m_n,\ell_n+1}-1}]$ for some $1 \le \ell_n \le L_{m_n} - 1$ or $n \in [M_{j_{m_n,L_{m_n}-1}} + 1, M_{j_{m_n+1,1}-1}]$.

In the first case, let us write $\sum_{k=1}^{n} \beta(q_k, \alpha_k) = S_1 + S_2 + S_3$, where

$$S_1 = \sum_{k=1}^{M_{j_{m_n,\ell_n}-2}} \beta(q_k, \alpha_k), \ S_2 = \sum_{k=M_{j_{m_n,\ell_n}-2}+1}^{M_{j_{m_n,\ell_n}-1}} \beta(q_k, \alpha_k), \ S_3 = \sum_{k=M_{j_{m_n,\ell_n}-1}+1}^n \beta(q_k, \alpha_k).$$

Setting $(q, \alpha) = (q_{m_n, \ell_n - 1}, \alpha_{m_n, \ell_n - 1})$ if $\ell_n \ge 2$ and $(q, \alpha) = (q_{m_n - 1, L_{m_n - 1}}, \alpha_{m_n - 1, L_{m_n - 1}})$ otherwise, one has $S_2 = (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})\beta(q,\alpha)$. Thus, by construction of $(q_{m,\ell},\alpha_{m,\ell})$, setting $\widetilde{\alpha} = \widetilde{\alpha}_{m_n,\ell_{n}-1}$ if $\ell_n \geq 2$ and $\widetilde{\alpha} = \widetilde{\alpha}_{m_n-1,L_{m_n-1}}$ otherwise, one has

$$||S_2 - (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})\widetilde{\alpha}|| \le (M_{j_{m_n,\ell_n}-1} - M_{j_{m_n,\ell_n}-2})/(m_n - 1).$$

Also, setting $q = q_{m_n,\ell_n}$, $\alpha' = \alpha_{m_n,\ell_n}$ and $\widetilde{\alpha}' = \widetilde{\alpha}_{m_n,\ell_n}$, one has $S_3 = (n - M_{j_{m_n,\ell_n}-1})\beta(q,\alpha)$, SO

$$||S_3 - (n - M_{j_{m_n,\ell_n}-1})\widetilde{\alpha}'|| \le (n - M_{j_{m_n,\ell_n}-1})/m_n.$$

Moreover, due to (3.20), one has $||S_1|| \leq (j_{m_n,\ell_n} - 1)^{-1} N_{j_{m_n,\ell_n} - 1} ||\beta(q,\alpha')|| \leq (j_{m_n,\ell_n} - 1)^{-1} N_{j_{m_n,\ell_n} - 1} ||\beta(q,\alpha')||$ $1)^{-1}n\|\beta(q,\alpha')\|, \text{ so}$

$$||S_1|| \le (j_{m_n,\ell_n} - 1)^{-1} n(||\widetilde{\alpha}'|| + 1/m_n);$$

also, due to (3.20) again, $||M_{j_{m_n,\ell_n}-2}\widetilde{\alpha}|| \leq (j_{m_n,\ell_n}-1)^{-1}n||\widetilde{\alpha}'||$. Moreover, the choice of the balls $B(\widetilde{\alpha}_{m,\ell},1/m)$ implies that $||\widetilde{\alpha}-\widetilde{\alpha}'|| \leq 1/(m_n-1)$. Consequently, putting the previous estimates together one gets

$$\left\| \sum_{k=1}^{n} \beta(q_k, \alpha_k) - n\alpha_{m_n, \ell_n} \right\| \le n \left(\frac{3}{m_n - 1} + \frac{2\|\alpha_{m_n, \ell_n}\| + 1/m_n}{j_{m_n, \ell_n} - 1} \right).$$

The same estimate holds if $n \in [M_{j_{m_n,L_{m_n}-1}}+1,M_{j_{m_n+1,1}-1}]$. Consequently, since as n tends to ∞ the sequence α_{m_n,ℓ_n} describes all the $\alpha_{m,\ell}$, the set of limit points of $n^{-1}\sum_{k=1}^n \beta(q_k,\alpha_k)$ is the same as that of the sequence $((\alpha_{m,\ell})_{1\leq \ell\leq L_m})_{m\geq 1}$, that is K.

The fact that $\liminf_{n\to\infty} n^{-1}\widetilde{P}_{X,\phi,\alpha_n}^*(\nabla \widetilde{P}_{X,\phi,\alpha_n}(q_n)) \geq \inf\{\widetilde{P}_{X,\phi,\alpha}^*(0) : \alpha \in K\}$ is a direct consequence of the choice of the vectors $q_{m,\ell}$ and $\alpha_{m,\ell}$, since $\widetilde{P}_{\alpha_{m,\ell}}^*(\nabla \widetilde{P}_{\alpha_{m,\ell}}(q_{m,\ell})) \geq \inf\{\widetilde{P}_{X,\phi,\alpha}^*(0) : \alpha \in K\} - 1/m$.

5. Proof of Theorem 1.2

We need to slightly modify the set \mathcal{R} by requiring, in addition to the initial conditions on $(N_j)_{j\geq 0}$, that for all $j\geq 1$

(5.1)
$$N_{j+1} > M_j(k(M_j) + 1)$$
 and $((j+3)!)^2 \exp(-N_j/j) \le j^{-2}$.

Since the sequence $(k(n))_{n\in\mathbb{N}}$ is increasing and $M_{j_n}+1 \le n \le M_{j_n+1}$, one has $n(k(n)+1) \le M_{j_n+1}(k(M_{j_n+1})+1) < N_{j_n+2} < M_{j_n+2}$, so $j_{n(k(n)+1)} \le j_n+1$.

For each integer $m \geq 1$, define the compact set

$$K_m = \{(q, \alpha) \in J_{X, \phi} \cap B(0, m) : d((q, \alpha), \partial J_{X, \phi}) \ge 1/m\} \cup \{(q_1, \alpha_1)\},\$$

where B(0,m) is the Euclidean ball of radius m centered at 0 in \mathbb{R}^{2d} and (q_1,α_1) is the unique element of D_1 . Then, recalling that D was chosen so that the second claim of Lemma 3.1 holds, for $\ell, m \geq 1$, let

(5.2)
$$\mathcal{R}(m) = \left\{ \varrho = (q_k, \alpha_k)_{k \ge 1} \in \mathcal{R} \cap K_m^{\mathbb{N}} : \exists \ \alpha \in \widetilde{I}_X, \ \lim_{k \to \infty} (q_k, \alpha_k) = (q_\alpha, \alpha) \right\}$$

and

$$\widetilde{I}_X(m) = \left\{ \alpha \in \widetilde{I}_X : (q_\alpha, \alpha) \in \left\{ \lim_{k \to \infty} \varrho_k = (q_k, \alpha_k) : \varrho \in \mathcal{R}(m) \right\} \right\}.$$

Note that if $\varrho \in \mathcal{R}(m)$, then there is a unique $\alpha \in \widetilde{I}_X$ such that $\lim_{k \to \infty} (q_k, \alpha_k) = (q_\alpha, \alpha)$. By construction, $\widetilde{I}_X = \bigcup_{m \geq 1} \widetilde{I}_X(m)$. Note also that in the statement of Theorem 1.2 the vector $\psi(q_\alpha, \alpha)$ is simply denoted by ψ_α , a notation that we adopt in this section as well.

Let $\kappa = \liminf_{n \to \infty} \log(k(n))/n$ and $\kappa' = \limsup_{n \to \infty} \log(k(n))/n$. For all integers $\ell, m \ge 1$ and closed dyadic cube Q in \mathbb{R}^d , define the sets

$$\begin{cases} \mathcal{R}(m,\ell,Q) = \left\{ \varrho \in \mathcal{R}(m) : \forall \ \lambda \in \widetilde{Q}, \ -\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}(\lambda)) < \min(\ell,\kappa - 1/\ell) \right\}, \\ \mathcal{R}'(m,\ell,Q) = \left\{ \varrho \in \mathcal{R}(m) : \forall \ \lambda \in Q, \ -\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}(\lambda)) > \kappa' + 1/\ell \right\}, \end{cases}$$

where \widetilde{Q} stands for the union of Q and the closed dyadic cubes of the same generation as Q and neighboring Q. The usefulness of considering the cubes \widetilde{Q} will appear in the proof of the following proposition. Recall the definitions (1.6) and (1.7) of μ_n^t and Λ_n^t respectively.

Proposition 5.1. With probability 1, for all integers $\ell, m \geq 1$ and all dyadic cubes Q,

- (1) for all $\varrho \in \mathcal{R}(m,\ell,Q)$, for μ_{ϱ} -almost every t, one has $\lim_{n\to\infty} \frac{1}{n} \Lambda_{\tilde{k},n}^t(\lambda) = \Lambda_{\psi_{\alpha}}(\lambda)$ for all $\lambda \in Q$.
- (2) For all $\varrho \in \mathcal{R}'(m,\ell,Q)$, for μ_{ϱ} -almost every t, for all $\lambda \in Q$, there exists $\varepsilon > 0$ such that for n large enough, one has $\mu_{\tilde{k},n}^t(B(\nabla \Lambda_{\psi_{\alpha}}(\lambda),\varepsilon)) = 0$.

We assume this proposition for the time being and prove Theorem 1.2.

Proof of Theorem 1.2. Let \mathcal{C} stand for the set of closed dyadic cubes in \mathbb{R}^d . One has

$$\begin{split} &\{(\alpha,\lambda)\in \widetilde{I}_X\times\mathbb{R}^d:\ -\Lambda_{\psi_\alpha}^*(\nabla\Lambda_{\psi_\alpha}(\lambda))<\kappa\}\\ &=\bigcup_{m\geq 1}\bigcup_{\ell\geq 1}\{(\alpha,\lambda)\in \widetilde{I}_X\times\mathbb{R}^d:\alpha\in \widetilde{I}_X(m),\ -\Lambda_{\psi_\alpha}^*(\nabla\Lambda_{\psi_\alpha}(\lambda))<\min(\ell,\kappa-1/\ell)\}\\ &=\bigcup_{m\geq 1}\bigcup_{\ell\geq 1}\bigcup_{Q\in\mathcal{C}}\{\alpha\in \widetilde{I}_X(m):\forall\,\lambda\in \widetilde{Q},\ -\Lambda_{\psi_\alpha}^*(\nabla\Lambda_{\psi_\alpha}(\lambda))<\min(\ell,\kappa-1/\ell)\}\times\widetilde{Q}, \end{split}$$

where one used the continuity in (α, λ) of $-\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}(\lambda))$. Consequently, due to Proposition (5.1)(1), with probability 1, for all $\alpha \in \widetilde{I}_X$, if m is large enough so that $\alpha \in \widetilde{I}_X(m)$, for all $\varrho \in \mathcal{R}(m)$ such that $\lim_{k \to \infty} \varrho_k = (q_{\alpha}, \alpha)$, since each $\lambda \in \mathbb{R}^d$ such that $-\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}(\lambda)) < \kappa$ belongs, for ℓ large enough, to a dyadic cube Q such that one has $-\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}) < \min(\ell, \kappa - 1/\ell)$ over Q, one has μ_{ϱ} -almost everywhere, $\lim_{n \to \infty} n^{-1} \Lambda_{\widetilde{k},n}^t(\lambda) = \Lambda_{\psi_{\alpha}}$ for all $\lambda \in \mathbb{R}^d$ such that $-\Lambda_{\psi_{\alpha}}^*(\nabla \Lambda_{\psi_{\alpha}}(\lambda)) < \kappa$, i.e. the part (1) of the large deviations properties $\mathrm{LD}(\Lambda_{\psi_{\alpha}}, \widetilde{k})$. One uses a similar argument to derive part (2) of $\mathrm{LD}(\Lambda_{\psi_{\alpha}}, \widetilde{k})$ from part (2) of Proposition (5.1). Part (3) of $\mathrm{LD}(\Lambda_{\psi_{\alpha}}, \widetilde{k})$ is established as [6, Theorem 2.3(3)]).

To get the desired lower bound for dim $E(X, \alpha, LD(\Lambda_{\psi_{\alpha}}, \widetilde{k}))$, it is enough to pick ϱ such that $\lim_{k\to\infty} \varrho_k = (q_{\alpha}, \alpha)$ and $\lim_{k\to\infty} \widetilde{P}_{X,\phi,\alpha_k}(q_k) - \langle q_k | \nabla \widetilde{P}_{X,\phi,\alpha_k}(q_k) \rangle = \widetilde{P}_{X,\phi,\alpha}^*(0)$, as in Lemma 3.1 (then Proposition (3.1) yields the result).

Proof of Proposition 5.1. Fix m, ℓ, Q . We cut the sets $\mathcal{R}(m, \ell, Q)$ and $\mathcal{R}'(m, \ell, Q)$ into countably many pieces as follows: for every integer $L \geq 1$, setting $\Lambda_k = \Lambda_{\psi(q_k, \alpha_k)}$, let

$$\mathcal{R}(m,\ell,L,Q) = \left\{ \varrho \in \mathcal{R}(m,\ell,Q) : \begin{cases} \forall \ k \geq L, \ \forall \ \lambda \in \widetilde{Q}, \\ -\Lambda_k^*(\nabla \Lambda_k(\lambda)) < \min(2\ell,\kappa - 1/2\ell) \end{cases} \right\},$$

and

$$\mathcal{R}'(m,\ell,L,Q) = \left\{ \varrho \in \mathcal{R}(m,\ell,Q) : \begin{cases} \forall \ k \ge L, \ \forall \ \lambda \in Q, \\ |\Lambda_k^*(\nabla \Lambda_k(\lambda)) - \Lambda_{\psi_\alpha}^*(\nabla \Lambda_{\psi_\alpha}(\lambda))| < 1/2\ell \end{cases} \right\}.$$

The mappings $\Lambda_{\psi(q,\alpha)}$ and $\nabla \Lambda_{\psi(q,\alpha)}$ are continuous as functions of (q,α) taking values in $\mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$ and $\mathcal{C}(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R}^d)$ respectively (these spaces being endowed with the topology of the uniform convergence over compact sets). Thus, since \widetilde{Q} is compact, we have $\mathcal{R}(m,\ell,Q) = \bigcup_{L>1} \mathcal{R}(m,\ell,L,Q)$ and $\mathcal{R}'(m,\ell,Q) = \bigcup_{L>1} \mathcal{R}'(m,\ell,L,Q)$.

Let us prove part (1) of the proposition. Fix $L \geq 1$. For $\varrho \in \mathcal{R}(m,\ell,L,Q), \ \lambda \in \widetilde{Q}, \ n \geq 1, 1 \leq j \leq k(n), \ 0 \leq i \leq n-1 \ \text{and} \ t \in \partial \mathsf{T}, \ \text{set}$

$$s_{n,j}^{(i)}(\varrho,\lambda) = \sum_{k=i+(j-1)n+1}^{i+jn} \Lambda_{\psi(q_k,\alpha_k)}(\lambda)$$

and

(5.3)
$$Z_{n,j}^{(i)}(\varrho,\lambda,t) = \exp(\langle \lambda | (S_{i+jn}X(t) - S_{i+(j-1)n}X(t)) - s_{n,j}^{(i)}(\varrho,\lambda)).$$

It is enough to prove that for every $\lambda \in \widetilde{Q}$ and $\varepsilon > 0$, one has

(5.4)
$$\mathbb{E}\Big(\sum_{n\in\mathbb{N}}\sup_{\varrho\in\mathcal{R}(m,\ell,L,Q)}\mu_{\varrho}(E(n,\varrho,\lambda))\Big)<\infty,$$

where

$$E(n,\varrho,\lambda) = \Big\{ t \in \partial \mathsf{T} : \exists \, 0 \le i \le n-1, \, \left| k(n)^{-1} \sum_{j=1}^{k(n)} (Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1) \right| > \varepsilon \Big\}.$$

Indeed, suppose that (5.4) holds true. Then, for every $\lambda \in \widetilde{Q}$ and $\varepsilon \in (0,1)$, with probability 1, for all $\varrho \in \mathcal{R}(m,\ell,L,Q)$, applying the Borel-Cantelli lemma to μ_{ϱ} yields, for μ_{ϱ} -almost every t, an integer $n_{\varrho} \geq 1$ such that for all $n \geq n_{\varrho}$, for all $0 \leq i \leq n-1$,

$$1 - \varepsilon \le k(n)^{-1} \sum_{j=1}^{k(n)} Z_{n,j}^{(i)}(\varrho, \lambda, t) \le 1 + \varepsilon.$$

Moreover, given $\varrho \in \mathcal{R}(m,\ell,L,Q)$, there exists $k_{\varrho} > 1$ such that for all $k \geq k_{\varrho}$, one has $|\Lambda_{\psi(q_k,\alpha_k)}(\lambda) - \Lambda_{\psi_{\alpha}}(\lambda)| \leq \varepsilon$, hence for $n \geq k_{\varrho}$ and $0 \leq i \leq n-1$ one has

$$|s_{n,j}^{(i)}(\varrho,\lambda) - n\Lambda_{\psi_{\alpha}}(\lambda)| \le n\varepsilon + C_{\varrho}(\lambda),$$

where $C_{\varrho}(\lambda) = \sum_{k=1}^{k_{\varrho}} |\Lambda_{\psi(q_k,\alpha_k)}(\lambda) - \Lambda_{\psi_{\alpha}}(\lambda)|$. Consequently, setting

$$\mathcal{I}_n(t,\lambda) = \sum_{i=0}^{n-1} \sum_{j=1}^{k(n)} \exp(\langle \lambda | (S_{i+jn}X(t) - S_{i+(j-1)n}X(t) \rangle),$$

for $n \ge \max(k_{\rho}, n_{\rho})$, one obtains

$$(1-\varepsilon)\exp(n\Lambda_{\psi_{\alpha}}(\lambda)-n\varepsilon-C_{\varrho}(\lambda)) \leq \frac{\mathcal{I}_{n}(t,\lambda)}{nk(n)} \leq (1+\varepsilon)\exp(n\Lambda_{\psi_{\alpha}}(\lambda)+n\varepsilon+C_{\varrho}(\lambda)).$$

By definition (1.7) of $\Lambda_{\tilde{k},n}^t(\lambda)$, one also has $\exp\left(\Lambda_{\tilde{k},n}^t(\lambda)\right) = (nk(n))^{-1}\mathcal{I}_n(t,\lambda)$. Letting ε go to 0 along a discrete family, this gives that almost surely, for all $\varrho \in \mathcal{R}(m,\ell,L,Q)$, for μ_{ϱ} -almost every t, $\lim_{n\to\infty} n^{-1}\Lambda_{\tilde{k},n}^t(\lambda) = \Lambda_{\psi_{\alpha}}(\lambda)$. Then, this convergence holds almost surely for a countable and dense subset of elements λ of \widetilde{Q} , and finally the convexity of the functions $\Lambda_{\tilde{k},n}^t$ gives the convergence for all $\lambda \in Q$, since Q is included in the interior of \widetilde{Q} .

To prove (5.4), we need the following lemma, in which \mathcal{Q}_{ϱ} stands for the Peyrière measure associated with μ_{ϱ} , that is the measure on $(\Omega \times \mathbb{N}^{\mathbb{N}}, \mathcal{A} \otimes \mathcal{B})$, defined by

$$\mathcal{Q}_{\varrho}(C) = \int_{\Omega} \int_{\mathbb{N}^{\mathbb{N}}} \mathbf{1}_{C}(\omega, t) \, \mathrm{d}\mu_{\varrho, \omega}(t) \, \mathrm{d}\mathbb{P}(\omega).$$

- **Lemma 5.1.** (1) Let $\varrho \in \mathcal{R}(m,\ell,L,Q)$, $\lambda \in \widetilde{Q}$ and $n \in \mathbb{N}$. For each $0 \le i \le n-1$, the random variables $(\omega,t) \mapsto Z_{n,j}^{(i)}(\varrho,\lambda,t) 1$, $1 \le j \le k(n)$, defined on $\Omega \times \mathbb{N}^{\mathbb{N}}$ are centered and independent with respect to \mathcal{Q}_{ϱ} .
 - (2) There exists $p(m, \ell, L, Q) \in (1, 2]$ and $C(m, \ell, L, Q) > 0$ such that for all $p \in (1, p(m, \ell, L, c)]$, for all $\varepsilon > 0$, for all $n \in \mathbb{N}$ and $0 \le i \le n 1$, one has

$$\mathcal{Q}_{\varrho}\Big(\Big|k(n)^{-1}\sum_{i=1}^{k(n)} \left(Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1\right)\Big| > \varepsilon\Big) \leq C(m,\ell,L,Q)\exp(-n(p-1)\ell/4)$$

independently of $\varrho \in \mathcal{R}(m, \ell, L, Q)$ and $\lambda \in \widetilde{Q}$.

We postpone the proof of this lemma to the end of the section.

Now, for $\varrho \in \mathcal{R}(m,\ell,L,Q)$, $\lambda \in \widetilde{Q}$, $n \in \mathbb{N}$, $0 \le i \le n-1$ and $t \in \partial \mathsf{T}$ let

$$V_n^{(i)}(\varrho,\lambda,t) = \left| k(n)^{-1} \sum_{i=1}^{k(n)} (Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1) \right|,$$

and notice that by construction $V_n^{(i)}(\varrho, \lambda, t)$ is constant over each cylinder [u] of generation nk(n) + i, so that we also denote it by $V_n^{(i)}(\varrho, \lambda, u)$. One has

$$\mu_{\varrho}(\{t\in\partial\mathsf{T}:\,\exists\,0\leq i\leq n-1,\,V_n^{(i)}(\varrho,\lambda,t)>\varepsilon\})\leq\sum_{i=0}^{n-1}\sum_{u\in\mathsf{T}_{nk(n)+i}}\mathbf{1}_{\{V_n^{(i)}(\varrho,\lambda,u)>\varepsilon\}}\mu_{\varrho}([u]).$$

Recall that by definition $\mu_{\varrho}([u]) = (\prod_{k=1}^n W_{\varrho,u_1\cdots u_k})Y(\varrho,u)$, with $\mathbb{E}(Y(\varrho,u)) = 1$, and due to Lemma 3.4, since $\mathcal{R}(m,\ell,L,Q) \subset \mathcal{R}(K_m)$, one has $\|\sup_{\varrho \in \mathcal{R}(m,\ell,L,Q)} Y(\varrho,u)\|_1 = O((j_{|u|}+2)!)$. Fix $u^{n,i} \in \mathbb{N}^{nk(n)+i}$ and denote $\|\sup_{\varrho \in \mathcal{R}(m,\ell,L,Q)} Y(\varrho,u^{n,i})\|_1$ by $B_{n,i}$. Using the independence between $(\prod_{k=1}^{nk(n)+i} W_{\varrho,u_1\cdots u_k})_{\varrho \in \mathcal{R}(m,\ell,L,Q)}$ and $Y(\cdot,u)$ for all $u \in \mathbb{N}^{nk(n)+i}$, one gets

$$\mathbb{E}\Big(\sup\Big\{\mu_{\varrho}(\{t\in\partial\mathsf{T}:\,\exists\,0\leq i\leq n-1,\,V_{n}^{(i)}(\varrho,\lambda,t)>\varepsilon\}):\varrho\in\mathcal{R}(m,\ell,L,Q)\Big\}\Big)$$

$$\leq \sum_{i=0}^{n-1}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)+i}}\sup_{\varrho\in\mathcal{R}(m,\ell,L,Q)}\mathbf{1}_{\{V_{n}^{(i)}(\varrho,\lambda,u)>\varepsilon\}}\prod_{k=1}^{nk(n)+i}W_{\varrho,u_{1}\cdots u_{k}}\Big)\,B_{n,i}.$$

From this inequality one can obtain

$$\mathbb{E}\Big(\sup\Big\{\mu_{\varrho}(\{t\in\partial\mathsf{T}:\exists\,0\leq i\leq n-1,\,V_{n}(\varrho,\lambda,t)>\varepsilon\}):\varrho\in\mathcal{R}(m,\ell,L,Q)\Big\}\Big)$$

$$\leq \sum_{i=0}^{n-1}\sum_{\substack{\varrho\mid nk(n)+i:\\\varrho\in\mathcal{R}(m,\ell,L,Q)}}\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)+i}}\mathbf{1}_{\{V_{n}^{(i)}(\varrho,\lambda,u)>\varepsilon\}}\prod_{k=1}^{nk(n)+i}W_{\varrho,u_{1}\cdots u_{k}}\Big)\,B_{n,i}.$$

Then, noting that

$$\mathbb{E}\Big(\sum_{u\in\mathsf{T}_{nk(n)+i}}\mathbf{1}_{\{V_n^{(i)}(\varrho,\lambda,u)>\varepsilon\}}\Big(\prod_{k=1}^{nk(n)+i}W_{\varrho,u_1\cdots u_k}\Big)\Big)=\mathcal{Q}_{\varrho}(V_n^{(i)}(\varrho,\lambda,t)>\varepsilon)$$

yields

$$\mathbb{E}\Big(\sup\Big\{\mu_{\varrho}(\{t\in\partial\mathsf{T}:\exists\,0\leq i\leq n-1,\,V_{n}^{(i)}(\varrho,\lambda,t)>\varepsilon\}):\varrho\in\mathcal{R}(m,\ell,L,Q)\Big\}\Big)$$

$$\leq\sum_{i=0}^{n-1}\sum_{\varrho\mid nk(n)+i:\varrho\in\mathcal{R}(m,\ell,L,Q)}\mathcal{Q}_{\varrho}(V_{n}^{(i)}(\varrho,\lambda,t)>\varepsilon)\,\,B_{n,i}$$

$$\leq\sum_{i=0}^{n-1}(\#\{\varrho\mid nk(n)+i:\varrho\in\mathcal{R}\})\Big(\sup_{\varrho\in\mathcal{R}(m,\ell,L,Q)}\mathcal{Q}_{\varrho}(V_{n}^{(i)}(\varrho,\lambda,t)>\varepsilon)\Big)\,\,B_{n,i}$$

$$\leq n(j_{n(k(n)+1)}+1!)C(m,\ell,L,Q)\exp(-n(p-1)\ell/4)O((j_{n(k(n)+1)}+2)!),$$

where Lemma 5.1(2) was used. Now recall that due to (5.1) one has $j_{n(k(n)+1)} \leq j_n + 1$, hence

$$\mathbb{E}\Big(\sup\Big\{\mu_{\varrho}(\{t\in\partial\mathsf{T}:\,\exists\,0\leq i\leq n-1,\,V_n^{(i)}(\varrho,\lambda,t)>\varepsilon\}):\varrho\in\mathcal{R}(m,\ell,L,Q)\Big\}\Big)$$
$$=O(1)C(m,\ell,L,Q)n(j_n+3)!^2\exp(-n(p-1)\ell/4).$$

It follows that

$$\begin{split} & \sum_{n \in \mathbb{N}} \mathbb{E} \Big(\sup \Big\{ \mu_{\varrho}(\{t \in \partial \mathsf{T} : \exists 0 \le i \le n-1, \, V_n^{(i)}(\varrho, \lambda, t) > \varepsilon\}) : \varrho \in \mathcal{R}(m, \ell, L, Q) \Big\} \Big) \\ &= O\Big(\sum_{j \ge 0} \sum_{M_j + 1 \le n \le M_{j+1}} ((j+3)!)^2 n \exp(-n(p-1)\ell/4) \Big) \\ &= O\Big(\sum_{j \ge 0} \frac{((j+3)!)^2}{1 - \exp(-(p-1)\ell/8)} \exp(-M_j(p-1)\ell/8) \Big) \\ &= O\Big(\sum_{j \ge 0} ((j+3)!)^2 \exp(-N_j(p-1)\ell/8) \Big). \end{split}$$

Due to (5.1) the above series converges.

Remark 5.1. The reason why we did not succeed in proving that with probability 1, the Gibbs measure ν_{α} is carried by $E(X, \alpha, \mathrm{LD}(\Lambda_{\psi_{\alpha}}, \tilde{k}))$ simultaneously for all $\alpha \in \mathring{I}_X$, is the following. As said in the introduction, it is tempting to adapt to the present problem what was done in [4, 1] to get that with probability 1, the measure ν_{α} is carried by $E(X, \alpha)$ simultaneously for all $\alpha \in \mathring{I}_X$. To do so, notice first that for $\alpha \in \mathring{I}_X$, the measure ν_{α} is the homogeneous Mandelbrot measure μ_{ϱ} , where $\varrho = \varrho^{(\alpha)}$ is the constant sequence $((q_{\alpha}, \alpha))_{n \in \mathbb{N}}$. Then, what we would need is to have at our disposal a suitable almost sure upper bound g_n for the mapping $\alpha \mapsto \mu_{\varrho(\alpha)}(\{t \in \partial \mathsf{T} : V_n^{(i)}(\varrho^{(\alpha)}, \lambda, t) > \varepsilon\})$, with the following properties: g_n possesses almost surely an holomorphic extension on a deterministic complex neiborhood V_{α} of any $\alpha \in \mathring{I}_X$; $\sum_{n \geq 0} \mathbb{E}(|g_n(z)|)$ converges uniformly on any compact subset of $\bigcup_{\alpha \in \mathring{I}_X} V_{\alpha}$, so that an application of the Cauchy-formula yields the almost surely simultaneous convergence of $\sum_{n \geq 0} g_n(\alpha)$ for all $\alpha \in \mathring{I}_X$; but we could not find any way to apply such a strategy.

Now we prove part (2) of the proposition. This situation is not empty only if $\kappa' = \limsup_{n\to\infty} \log(k(n))/n < \infty$.

Fix L>0. Given $\varrho\in\mathcal{R}'(m,\ell,L,Q),\ \lambda\in Q,\ \varepsilon>0,\ n\geq 1,\ 1\leq j\leq k(n)$ and $0\leq i\leq n-1,$ set

$$\widetilde{V}_{j}^{(i)}(\varrho,\lambda,t) = \mathbf{1}_{B(0,n\varepsilon)}(S_{i+jn}X(t) - S_{i+(j-1)n}X(t) - n\nabla\Lambda_{\psi_{\alpha}}(\lambda)).$$

Mimicking what was done above, one can get

$$\mathbb{E}\Big(\sup_{\varrho\in\mathcal{R}'(m,\ell,L,Q)}\mu_{\varrho}\Big(\Big\{t\in\mathsf{T}:\,\exists\,0\leq i\leq n-1,\,\sum_{j=1}^{k(n)}\widetilde{V}_{j}^{(i)}(\varrho,\lambda,t)\geq1\Big\}\Big)\Big)$$

$$\leq\sum_{i=0}^{n-1}(\#\varrho_{|nk(n)+i}:\varrho\in\mathcal{R})\Big(\sup_{\varrho\in\mathcal{R}'(m,\ell,L,Q)}\mathcal{Q}_{\varrho}\Big(\Big\{\sum_{j=1}^{k(n)}\widetilde{V}_{j}^{(i)}(\varrho,\lambda,t)\geq1\Big\}\Big)\Big)\,B_{n,i}$$

$$=O((j_{n}+3)!^{2})\sum_{i=0}^{n-1}\sup_{\varrho\in\mathcal{R}'(m,\ell,L,Q)}\mathcal{Q}_{\varrho}\Big(\Big\{\sum_{j=1}^{k(n)}\widetilde{V}_{j}^{(i)}(\varrho,\lambda,t)\geq1\Big\}\Big).$$

For $\varrho \in \mathcal{R}'(m,\ell,L,Q)$, the fact that the inclusion $S_{i+jn}X(t) - S_{i+(j-1)n}(t) - n\nabla\Lambda_{\psi_{\alpha}}(\lambda) \in B(0,n\varepsilon)$ implies $\langle \lambda | S_{i+jn}X(t) - S_{i+(j-1)n}(t) - n\nabla\Lambda_{\psi_{\alpha}}(\lambda) \rangle \geq -n\varepsilon ||\lambda||$, yields

$$\mathcal{Q}_{\varrho}\left(\left\{\sum_{j=1}^{k(n)} \widetilde{V}_{j}^{(i)}(\varrho, \lambda, t) \geq 1\right\}\right)$$

$$\leq \sum_{j=1}^{k(n)} \mathcal{Q}_{\varrho}\left(\langle \lambda | S_{i+jn}X(t) - S_{i+(j-1)n}(t) - n\nabla \Lambda_{\psi_{\alpha}}(\lambda)\rangle \geq -n\varepsilon \|\lambda\|\right).$$

Then, applying Markov inequality gives

$$\mathcal{Q}_{\varrho}\left(\left\{\sum_{j=1}^{k(n)} \widetilde{V}_{j}^{(i)}(\varrho, \lambda, t) \geq 1\right\}\right) \\
\leq \sum_{j=1}^{k(n)} \exp\left(n\varepsilon \|\lambda\| - n\langle\lambda|\nabla\Lambda_{\psi_{\alpha}}(\lambda)\rangle\right) \mathbb{E}_{\mathcal{Q}_{\varrho}}\left(\exp(\langle\lambda|S_{i+jn}X(t) - S_{i+(j-1)n}(t)\rangle)\right) \\
= \sum_{j=1}^{k(n)} \exp\left(n\varepsilon \|\lambda\| - n\langle\lambda|\nabla\Lambda_{\psi_{\alpha}}(\lambda)\rangle\right) \exp\left(\sum_{k=i+(j-1)n+1}^{i+jn} \Lambda_{\psi(q_{k},\alpha_{k})}(\lambda)\right).$$

Now, using the definition of $\mathcal{R}'(m, \ell, L, Q)$ and setting

$$A = \sup_{\varrho \in \mathcal{R}'(m,\ell,L,Q)} \sup_{\lambda \in Q} \sum_{k=1}^{L} |\Lambda_{\psi_{\alpha}}(\lambda) - \Lambda_{\psi(q_{k},\alpha_{k})}(\lambda)|,$$

one obtains

$$\mathcal{Q}_{\varrho}\left(\left\{\sum_{j=1}^{k(n)} \widetilde{V}_{j}^{(i)}(\varrho,\lambda,t) \geq 1\right\}\right) \leq e^{A} \sum_{j=1}^{k(n)} \exp\left(n\varepsilon \|\lambda\| - n\langle\lambda|\nabla\Lambda_{\psi_{\alpha}}(\lambda)\rangle\right) \exp\left(n/2\ell + n\Lambda_{\psi_{\alpha}}(\lambda)\right) \\
= e^{A} \sum_{j=1}^{k(n)} \exp\left(n\varepsilon \|\lambda\| + n/2\ell + n\Lambda_{\psi_{\alpha}}^{*}(\nabla\Lambda_{\psi_{\alpha}}(\lambda))\right) \\
\leq e^{A} k(n) \exp\left(n\varepsilon \|\lambda\| - n/2\ell - n\kappa'\right).$$

Thus, taking $0 < \varepsilon = \varepsilon_Q$ small enough so that $\varepsilon ||\lambda|| \le \ell/8$, since $\log(k(n)) < n(\kappa' + \ell/8)$ for n large enough, one gets

$$\sup_{\varrho \in \mathcal{R}'(m,\ell,L,Q)} \mathcal{Q}_{\varrho} \Big(\Big\{ \sum_{i=1}^{k(n)} \widetilde{V}_{j}^{(i)}(\varrho,\lambda,t) \ge 1 \Big\} \Big) = O(\exp(-n/4\ell)).$$

Mimicking the end of the proof of part (1) of this proposition, we can get that given $\lambda \in Q$, with $\varepsilon = \varepsilon_Q$

$$\mathbb{E}\Big(\sum_{n\in\mathbb{N}}\sup_{\varrho\in\mathcal{R}'(m,\ell,L,Q)}\mu_{\varrho}\Big(\Big\{t:\,\exists\,0\leq i\leq n-1,\,\sum_{j=1}^{k(n)}\widetilde{V}_{j}^{(i)}(\varrho,\lambda,t)\geq1\Big\}\Big)\Big)<\infty,$$

hence, with probability 1, by the Borel-Cantelli Lemma applied to each μ_{ϱ} , one has that for all $\varrho \in \mathcal{R}'(m,\ell,L,Q)$, for μ_{ϱ} -almost every t, for n large enough, for all $0 \le i \le n-1$, $E_{n,i}(t) := \{1 \le j \le k(n) : n^{-1}(S_{i+jn}X(t) - S_{i+(j-1)n}X(t)) \in B(\nabla \Lambda_{\psi_{\alpha}}(\lambda), \varepsilon_{Q})\} = \emptyset$.

Now, for each $\lambda \in Q$, there exists $\eta_{\lambda} > 0$ such that for all $\lambda' \in B(\lambda, \eta_{\lambda})$, for all $\varrho \in \mathcal{R}'(m, \ell, L, Q)$ one has $B(\nabla \Lambda_{\psi_{\alpha}}(\lambda'), \varepsilon_{Q}/2) \subset B(\nabla \Lambda_{\psi_{\alpha}}(\lambda), \varepsilon_{Q})$. One can extract from $(B(\lambda, \eta_{\lambda}))_{\lambda \in Q}$ a finite subfamily $(B(\lambda_{s}, \eta_{\lambda_{s}}))_{1 \leq s \leq r}$ which covers Q. Since this family is finite, with probability 1, for all $\varrho \in \mathcal{R}'(m, \ell, L, Q)$, for μ_{ϱ} -almost every t, for n large enough, for all $1 \leq s \leq r$ and $0 \leq i \leq n-1$, one has $E_{n,i}(t) = \emptyset$; consequently, by construction of $(B(\lambda_{s}, \eta_{\lambda_{s}}))_{1 \leq i \leq r}$, for all $\lambda' \in Q$ and $0 \leq i \leq n-1$, $E_{n,i}(t) = \emptyset$. This finishes the proof of the proposition.

Proof of Lemma 5.1. (1) This is elementary, but we detail it for reader's convenience. Let $n \ge 1$ and $(f_1, \ldots, f_{k(n)})$ be k(n) positive Borel functions defined on \mathbb{R}_+ . Fix $0 \le i \le n-1$. One has

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\Big(\prod_{j=1}^{k(n)} f_j\big(Z_{n,j}^{(i)}(\varrho,\lambda,t)\big)\Big) = \mathbb{E}\Big(\int \prod_{j=1}^{k(n)} f_j(Z_{n,j}^{(i)}(\varrho,\lambda,t)) \,\mathrm{d}\mu_{\varrho}(t)\Big).$$

For each word u of generation nk(n), we denote by $Z_{n,j}^{(i)}(\varrho,\lambda,u)$ the constant value of $Z_{n,j}^{(i)}(\varrho,\lambda,t)$ over the cylinder [u]. Using the fact that $Z_{n,j}^{(i)}(\varrho,\lambda,t)$ is $\sigma(N_u,X_{us}:u\in\bigcup_{k=i+(j-1)n}^{i+jn-1}\mathbb{N}^k,s\in\mathbb{N})\otimes\mathcal{C}$ -measurable, as well as the definition of μ_{ϱ} , the independence between generations, and the branching property yields

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\left(\prod_{j=1}^{k(n)} f_{j}\left(Z_{n,j}^{(i)}(\varrho,\lambda,t)\right)\right)$$

$$= \mathbb{E}\left(\sum_{u \in \mathsf{T}_{nk(n)}} Y(\varrho,u) \left[\prod_{k=(j-1)n+1}^{jn} W_{\varrho,u_{1}\cdots u_{k}}\right] \cdot \left[\prod_{j=1}^{k(n)} f_{j}\left(Z_{n,j}^{(i)}(\varrho,\lambda,u)\right)\right]\right)$$

$$= \mathbb{E}\left(\sum_{u \in \mathsf{T}_{nk(n)}} \prod_{j=1}^{k(n)} \left[f_{j}\left(Z_{n,j}^{(i)}(\varrho,\lambda,u)\right) \prod_{k=i+(j-1)n+1}^{i+jn} W_{\varrho,u_{1}\cdots u_{k}}\right]\right).$$

Recall that $W_{\varrho,u_1\cdots u_k}=\exp\left(\langle q_k|X_{u_1\cdots u_k}-\alpha_k\rangle-\widetilde{P}_{X,\phi,\alpha_k}(q_k)\phi_{u_1\cdots u_k}\right)$, and set

$$U_{n,j}(u) = f_j(Z_{n,j}^{(i)}(\varrho,\lambda,u)) \prod_{k=i+(j-1)n+1}^{i+jn} \exp(\langle q_k | X_{u_1\cdots u_k} - \alpha_k \rangle - \widetilde{P}_{X,\phi,\alpha_k}(q_k)\phi_{u_1\cdots u_k}).$$

Note that this random variable is measurable with respect to the σ -algebra $\mathcal{G}_{n,j}$ defined as $\mathcal{G}_{n,j} = \sigma((N_w, (X_{w1}, \phi_{w1}), \dots) : w \in \bigcup_{k=i+j(n-1)}^{i+jn-1} \mathbb{N}^k)$. Now, the above equality rewrites

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\Big(\prod_{j=1}^{k(n)} f_{j}(Z_{n,j}^{(i)}(\varrho,\lambda,t))\Big) = \mathbb{E}\Big(\sum_{u \in \mathsf{T}_{n(k(n)-1)}} \sum_{v \in \mathsf{T}_{n}(u)} \prod_{j=1}^{k(n)} U_{n,j}(uv)\Big).$$

Conditioning on $\sigma(\mathcal{G}_{n,j}: 1 \leq j \leq k(n)-1)$ and using the independences and identity in distribution between the random variables of the construction one gets

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\Big(\prod_{j=1}^{k(n)}f_{j}(Z_{n,j}^{(i)}(\varrho,\lambda,t))\Big) = \mathbb{E}\Big(\sum_{u\in\mathsf{T}_{n(k(n)-1)}}\prod_{j=1}^{k(n)-1}U_{n,j}(uv)\Big)\widetilde{U}_{n,k(n)},$$

where for $1 \le j \le k(n)$ one set

$$\widetilde{U}_{n,j} = \mathbb{E}\Big(\sum_{u \in \mathsf{T}_n} f_j\Big(\exp(\langle \lambda | S_n X(u) \rangle - s_{n,j}^{(i)}(\varrho, \lambda)\Big)$$

$$\cdot \prod_{k=i+1}^{i+n} \exp(\langle q_{(j-1)n+k} | X_{u_1 \cdots u_k} - \alpha_{(j-1)n+k} \rangle - \widetilde{P}_{X,\phi,\alpha_{(j-1)n+k}}(q_{(j-1)n+k})\phi_{u_1 \cdots u_k})\Big).$$

Iterating the previous calculation yields

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}^{(i)}(\varrho,\lambda,t))\Big) = \prod_{j=1}^{k(n)} \widetilde{U}_{n,j},$$

and applying this with $f_{j'} = 1$ for $j' \neq j$ one naturally obtains $\widetilde{U}_{n,j} = \mathbb{E}_{\mathcal{Q}_{\varrho}} (f_j(Z_{n,j}^{(i)}(\varrho, \lambda, t)))$, hence

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}\Big(\prod_{j=1}^{k(n)} f_j(Z_{n,j}^{(i)}(\varrho,\lambda,t))\Big) = \prod_{j=1}^{k(n)} \mathbb{E}_{\mathcal{Q}_{\varrho}}\big(f_j(Z_{n,j}^{(i)}(\varrho,\lambda,t))\big),$$

which is the desired independence. Then, taking $f_j(z) = z$ and $f_{j'}(z) = 1$ for $j' \neq j$ yields, writing k' for i + (j-1)n + k and dropping X, ϕ in $\widetilde{P}_{X,\phi,\alpha}$:

$$\begin{split} &\mathbb{E}_{\mathcal{Q}_{\varrho}}\big(Z_{n,j}(\varrho,\lambda,t)\big) \\ &= \mathbb{E}\Big(\sum_{u \in \mathsf{T}_n} \prod_{k=1}^n \exp\big(\langle \lambda | X_{u_1 \cdots u_k} \rangle - \Lambda_{\psi(q_{k'},\alpha_{k'})}(\lambda) + \langle q_{k'} | X_{u_1 \cdots u_k} - \alpha_{k'} \rangle - \widetilde{P}_{\alpha_{k'}}(q_{k'})\phi_{u_1 \cdots u_k}\big)\Big) \\ &= \prod_{k=1}^n \mathbb{E}\Big(\sum_{i=1}^N \exp\big(\big(\langle \lambda | X_i \rangle + \langle q_{k'} | X_i - \alpha_{k'} \rangle - \widetilde{P}_{\alpha_{k'}}(q_{k'})\phi_i - \Lambda_{\psi(q_{k'},\alpha_{k'})}(\lambda)\big)\Big) = 1 \end{split}$$

by definition of $\Lambda_{\psi(q_{k'},\alpha_{k'})}$. Finally, the random variables $Z_{n,j}^{(i)}(\varrho,\lambda,t)-1$ are \mathcal{Q}_{ϱ} -independent and centered.

(2) Thanks to (1), we can apply Lemma 3.2 and get

$$\begin{aligned} & \mathcal{Q}_{\varrho} \Big(\Big| k(n)^{-1} \sum_{j=1}^{k(n)} (Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1) \Big| > \varepsilon \Big) \leq (\varepsilon k(n))^{-p} \mathbb{E}_{\mathcal{Q}_{\varrho}} \Big(\Big| \sum_{j=1}^{k(n)} (Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1) \Big|^{p} \Big) \\ & \leq 2^{p-1} (\varepsilon k(n))^{-p} \sum_{j=1}^{k(n)} \mathbb{E}_{\mathcal{Q}_{\varrho}} (|Z_{n,j}^{(i)}(\varrho,\lambda,t) - 1|^{p}) \leq 2^{2p-1} (\varepsilon k(n))^{-p} \sum_{j=1}^{k(n)} \mathbb{E}_{\mathcal{Q}_{\varrho}} (Z_{n,j}^{(i)}(\varrho,\lambda,t)^{p}) \end{aligned}$$

since $\mathbb{E}_{\mathcal{Q}_{\varrho}}(Z_{n,j}(\varrho,\lambda,t)) = 1$. Moreover, calculations similar to those used to establish part (1) of this lemma yield

$$\mathbb{E}_{\mathcal{Q}_{\varrho}}(Z_{n,j}^{(i)}(\varrho,\lambda,t)^{p}) = \exp\Big(\sum_{k=i+(j-1)n+1}^{i+jn} \Lambda_{\psi(q_{k},\alpha_{k})}(p\lambda) - p\Lambda_{\psi(q_{k},\alpha_{k})}(\lambda)\Big).$$

Since $\mathcal{R}(m,\ell,L,Q) \subset \mathcal{R}(K_m)$, and K_m is a compact subset of $J_{X,\phi}$, using Taylor's expansion one gets $\Lambda_{\psi(q_k,\alpha_k)}(p\lambda) - p\Lambda_{\psi(q_k,\alpha_k)}(\lambda) = (1-p)\Lambda_{\psi(q_k,\alpha_k)}^*(\nabla \Lambda_{\psi(q_k,\alpha_k)}\lambda) + O((p-1)^2)$ uniformly in $\varrho \in \mathcal{R}(m,\ell,L,Q)$, $\lambda \in \widetilde{Q}$ and p-1 small enough. Consequently, by definition of $\mathcal{R}(m,\ell,L,Q)$, for $k \geq L$ one obtains

$$\Lambda_{\psi(q_k,\alpha_k)}(p\lambda) - p\Lambda_{\psi(q_k,\alpha_k)}(\lambda) \le (p-1)\min(2\ell,\kappa-1/2\ell) + O((p-1)^2),$$

hence for all $1 \le j \le k(n)$

$$\sum_{k=i+(j-1)n+1}^{i+jn} \Lambda_{\psi(q_k,\alpha_k)}(p\lambda) - p\Lambda_{\psi(q_k,\alpha_k)}(\lambda) \leq A + n\left((p-1)\min(2\ell,\kappa-1/2\ell) + O((p-1)^2)\right)$$

uniformly in $\varrho \in \mathcal{R}(m,\ell,L,Q), \lambda \in \widetilde{Q}$ and p-1 small enough, where

$$A = \sup_{p \in [1,2]} \sup_{\varrho \in \mathcal{R}(m,\ell,L,Q)} \sup_{\lambda \in \widetilde{Q}} \sum_{k=1}^{L} |\Lambda_{\psi(q_k,\alpha_k)}(p\lambda) - p\Lambda_{\psi(q_k,\alpha_k)}(\lambda)|.$$

The previous estimates yield

$$\begin{aligned} & \mathcal{Q}_{\varrho} \Big(\Big| k(n)^{-1} \sum_{j=1}^{k(n)} (Z_{n,j}^{(i)}(\varrho, \lambda, t) - 1) \Big| > \varepsilon \Big) \\ & \leq e^{A} \varepsilon^{-p} (k(n))^{1-p} \exp \left(n \left((p-1) \min(2\ell, \kappa - 1/2\ell) + O((p-1)^2) \right) \right) \end{aligned}$$

in the same uniform manner as above. Take p close enough to 1 so that $O((p-1)^2) \le (p-1)/8\ell$.

Now, for n large enough, one has $k(n) \ge \exp(n(\min(2\ell, \kappa - 1/8\ell)))$, so that

$$\mathcal{Q}_{\varrho}\left(\left|k(n)^{-1}\sum_{j=1}^{k(n)}(Z_{n,j}^{(i)}(\varrho,\lambda,t)-1)\right|>\varepsilon\right)\leq e^{A}\varepsilon^{-p}\exp\left(n\left(1-p\right)\ell/4\right)$$

uniformly in $\varrho \in \mathcal{R}(m, \ell, L, Q)$, $\lambda \in \widetilde{Q}$ and $0 \le i \le n - 1$.

6. Proof of Theorem 1.3

Let us start by stating the following proposition.

Proposition 6.1. *Assume* (1.8) *and* (1.9).

- (1) $\widetilde{I}_X \subset \mathring{\mathcal{C}}_X$ and $I_X \setminus \widetilde{I}_X \subset \partial \mathcal{C}_X$.
- (2) If $H \in \mathcal{H}_X \setminus \widetilde{\mathcal{H}}_X$ then $H \cap I_X = \emptyset$.
- (3) If $\mathcal{H}_X \setminus \widetilde{\mathcal{H}}_X \neq \emptyset$, then $(\partial I_X)_{\text{crit}} = \widetilde{I}_X \setminus \mathring{I}_X \neq \emptyset$. Assume now that $\mathbb{E}(N^p) < \infty$ for some p > 1.
- (4) If $H \in \widetilde{\mathcal{H}}_X$, then $H \cap I_X \neq \emptyset$. Assume, in addition, (1.18) and (1.19).
- (5) For all $F \in \widetilde{\mathcal{F}}_X^1 \cup \overline{\mathcal{F}}_X^1$, for all $H \in \widetilde{H}_X$ such that $F \subset H$, one has $H \cap I_X = F \cap I_X = I_X^F$. Moreover, the conclusions of items (4) and (5) of Theorem 1.3 hold true for $\alpha \in I_X^F$ and $\alpha \in \widetilde{I}_X^F$ respectively.
- (6) If $F, F' \in \widetilde{\mathcal{F}}_X^1 \cup \overline{\mathcal{F}}_X^1$ with $F \neq F'$ then $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} = \emptyset$.

Proof of Theorem 1.3. The properties $\widetilde{\mathcal{H}}_X = \{H \in \mathcal{H}_X : H \cap I_X \neq \emptyset\}$, $\widetilde{I}_X \subset \mathring{\mathcal{C}}_X$, and $I_X \setminus \widetilde{I}_X = \bigcup_{H \in \widetilde{\mathcal{H}}_X} H \cap I_X$, that is point (1) and the first part of point (2) of the theorem follow directly from Proposition 6.1(1)-(4). This is also the case of the first part of point (3), that is the property that $(\partial I_X)_{\text{crit}} = \emptyset$ if and only if $\widetilde{\mathcal{H}}_X = \mathcal{H}_X$.

To see why the second part of point (3) holds, recall that the set of exposed points is dense in the set of extremal points of \mathcal{C}_X (see [35, Theorem 18.6]). If $\widetilde{\mathcal{H}}_X = \mathcal{H}_X$, the previous properties imply that any supporting hyperplane H of \mathcal{C}_X does intersect ∂I_X . In particular, for each exposed point P of \mathcal{C}_X , choosing $H \in \mathcal{H}_X$ which intersects C_X only at P, we get $P \in I_X$, and the condition $\mathbb{E}(N^H) \geq 1$ is equivalent to $\mathbb{E}(N^{\{P\}}) \geq 1$. This implies that the set of exposed points of \mathcal{C}_X is finite (since $\mathbb{E}(N) < \infty$) and coincides with its set of extremal points. It follows that the set \mathcal{C}_X is a polytope and equals I_X . Again due to the properties previously established, this is equivalent to the fact that every exposed point P of \mathcal{C}_X satisfies $\mathbb{E}(N^{\{P\}}) \geq 1$.

Suppose that $I_X \setminus \widetilde{I}_X \neq \emptyset$. It follows from the first part of point (5) of Proposition 6.1 that $I_X \setminus \widetilde{I}_X = \bigcup_{F \in \widetilde{\mathcal{F}}_X^1 \cup \overline{\mathcal{F}}_X^1} I_X^F$. Then, the second part of this fifth point combined with point (6) of the same proposition and a recursion on the dimension of I_X yields both that $I_X \setminus \widetilde{I}_X = \bigsqcup_{F \in \widetilde{\mathcal{F}}_X^d \cup \overline{\mathcal{F}}_X^d} \widetilde{I}_X^F$ and points (4) and (5) of Theorem 1.3.

If $H \in \mathcal{H}_X$, there is a unique couple $(e, c) \in \mathbb{S}^{d-1} \times \mathbb{R}$ such that

(6.1)
$$H = L_{e,c}^{-1}(\{0\})$$
 and $\mathcal{C}_X \subset L_{e,c}^{-1}(\mathbb{R}_-)$, where $L_{e,c} : \beta \in \mathbb{R}^d \mapsto \langle e|\beta \rangle - c$.

We set

$$H_+ = L_{e,c}^{-1}(\mathbb{R}_+^*).$$

The following preliminary observation will be useful.

Lemma 6.1. Let
$$H \in \mathcal{H}_X$$
. One has $\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{H_+}(X_i)\right) = 0$.

Proof. Suppose that $\theta = \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{H_{+}}(X_{i})\right) > 0$ and set $W_{i} = \theta^{-1}\mathbf{1}_{H_{+}}(X_{i})\mathbf{1}_{[1,N]}(i)$ for all $i \geq 1$. By construction of $(W_{i})_{i \in \mathbb{N}}$ and by definition of \mathcal{C}_{X} and \mathcal{H}_{X} , $\beta = \mathbb{E}\left(\sum_{i=1}^{N} W_{i}X_{i}\right) \in \mathcal{C}_{X} \subset \mathbb{R}^{d} \setminus H_{+}$, so $\langle e|\beta\rangle \leq c$. However, $\langle e|\beta\rangle = \theta^{-1}\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{H_{+}}(X_{i})\langle e|X_{i}\rangle\right) > c$. This contradiction implies that $\theta = 0$.

We also define, for $\varepsilon > 0$,

(6.2)
$$U_{\varepsilon} = L_{e,c}^{-1}((-\infty, -\varepsilon)) \text{ and } V_{\varepsilon} = \mathbb{R}^d \setminus U_{\varepsilon}.$$

Proof of Proposition 6.1. (1) The fact that $\widetilde{I}_X \subset \mathring{\mathcal{C}}_X$ follows from the inclusion $\nabla \widetilde{P}_X(\mathbb{R}^d) \subset \mathcal{C}_X$ and the fact that the mapping $q \in \mathbb{R}^d \mapsto \nabla \widetilde{P}_X(q)$ is open.

Suppose, by contradiction, that there exists $\alpha \in (I_X \setminus \widetilde{I}_X) \cap \mathring{\mathcal{C}}_X$. Given such an α , there exists a sequence $(q_n)_{n \in \mathbb{N}}$ of elements of J_X such that $\alpha = \lim_{n \to +\infty} (\alpha_n = \nabla \widetilde{P}_X(q_n))$. Moroever, $\lim_{n \to \infty} ||q_n|| = +\infty$, for otherwise it is easily seen that $\alpha \in \widetilde{I}_X$.

Without loss of generality, let us assume that $e_n = q_n/||q_n||$ converges to $e \in \mathbb{S}^{d-1}$. Recall that

$$\alpha_n = \nabla \widetilde{P}_X(q_n) = \mathbb{E}\Big(\sum_{i=1}^N \exp(\|q_n\|\langle e_n|X_i\rangle - \widetilde{P}_X(q_n))X_i\Big).$$

Since we assumed that α is an interior point of \mathcal{C}_X , there exists $c > \langle e | \alpha \rangle$ such that $L_{e,c}^{-1}(\{0\}) \cap \mathring{\mathcal{C}}_X \neq \emptyset$. This implies that for all $\varepsilon > 0$, one has $\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon}}(X_i)\right) > 0$; indeed, otherwise there would exist $\varepsilon_1 > 0$ such that $\mathbb{P}\left(X_i \in U_{\varepsilon_1}, \forall 1 \leq i \leq N\right) = 1$, hence for each non negative random vector $(W_i)_{i \in \mathbb{N}}$ jointly defined with $(N, (X_i))_{i \in \mathbb{N}}$ and such that $\mathbb{E}(\sum_{i=1}^N W_i) = 1$, one would have $\langle e | \mathbb{E}(\sum_{i=1}^N W_i X_i) \rangle \leq c - \varepsilon_1$, contradicting $L_{e,c}^{-1}(\{0\}) \cap \mathcal{C}_X \neq \emptyset$.

Consequently, since $(e_n)_{n\in\mathbb{N}}$ converges to e, for all $\varepsilon>0$ there exist $n_{\varepsilon}\in\mathbb{N}$ and $A_{\varepsilon}>0$ such that if $n\geq n_{\varepsilon}$, setting $U_{n,\varepsilon}=L_{e_n,c}^{-1}((-\infty,-\varepsilon))$ and $V_{n,\varepsilon}=\mathbb{R}^d\setminus U_{n,\varepsilon}$:

$$\mathbb{E}\Big(\sum_{i=1}^{N}\mathbf{1}_{V_{n,\varepsilon}}(X_i)\Big)\geq \mathbb{E}\Big(\sum_{i=1}^{N}\mathbf{1}_{V_{\varepsilon/2}}(X_i)\mathbf{1}_{[0,A_{\varepsilon}]}(\|X_i\|)\Big)>0.$$

Fix $\varepsilon > 0$ such that $\alpha \in U_{4\varepsilon}$. Noting that $\mathbb{E}\left(\sum_{i=1}^N \exp(\langle q_n | X_i \rangle - \widetilde{P}_X(q_n))\right) = 1$ for all $n \in \mathbb{N}$, if $n \geq n_{\varepsilon}$ one has

$$\begin{split} \left\| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{n,3\varepsilon}}(X_i) X_i e^{\langle q_n | X_i \rangle - \widetilde{P}_X(q_n)} \right) \right\| &\leq \frac{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{n,3\varepsilon}}(X_i) \| X_i \| \exp(\|q_n\| \langle e_n | X_i \rangle) \right)}{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{V_{n,\varepsilon}}(X_i) \exp(\|q_n\| \langle e_n | X_i \rangle) \right)} \\ &\leq \frac{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{n,3\varepsilon}}(X_i) \| X_i \| \exp(\|q_n\| \langle e_n | X_i \rangle) \right)}{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{V_{n,\varepsilon}}(X_i) \exp(\|q_n\| \langle e_n | X_i \rangle) \right)} \\ &\leq \frac{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{V_{n,\varepsilon}}(X_i) \exp(\|q_n\| \langle e_n | X_i \rangle) \right)}{\mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon/2}}(X_i) \mathbf{1}_{[0,A_{\varepsilon}]}(\|X_i\|) \right)} e^{-2\|q_n\| \varepsilon}. \end{split}$$

It follows that $\lim_{n\to\infty} \|\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{U_{n,3\varepsilon}}(X_i)X_ie^{\langle q_n|X_i\rangle-\widetilde{P}_X(q_n)}\right)\| = 0$. This yields that

$$\langle e | \alpha \rangle = \lim_{n \to \infty} \langle e_n | \alpha_n \rangle = \lim_{n \to \infty} \mathbb{E} \Big(\sum_{i=1}^{N} \mathbf{1}_{V_{n,3\varepsilon}}(X_i) \langle e_n | X_i \rangle e^{\langle q_n | X_i \rangle - \widetilde{P}_X(q_n)} \Big) \ge c - 3\varepsilon,$$

since the previous estimates imply that $\lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{n,\varepsilon'}}(X_i)e^{\langle q_n|X_i\rangle -\widetilde{P}_X(q_n)}\right) = 1$ for all $\varepsilon' > 0$. However, $\langle e|\alpha\rangle \leq c - 4\varepsilon$ since $\alpha \in U_{4\varepsilon}$, so we meet a contradiction.

(2) Fix $H \in \mathcal{H}_X \setminus \widetilde{\mathcal{H}}_X$. Recall that $\mathbb{E}(N^H) < 1$. Suppose that $H \cap I_X \neq \emptyset$. Let $\alpha \in H \cap I_X$ and $(q_n)_{n \in \mathbb{N}} \in J_X^{\mathbb{N}}$ such that $\lim_{n \to \infty} \nabla \widetilde{P}_X(q_n) = \alpha$.

For $\varepsilon > 0$, set

$$\begin{cases} G_{X,\varepsilon}: (q,\beta) \in J_X \times [0,1] \mapsto \mathbb{E} \big(\sum_{i=1}^N \mathbf{1}_{U_{\varepsilon}}(X_i) e^{\beta(\langle q|X_i \rangle - \widetilde{P}_X(q))} \big) \\ \widetilde{G}_{X,\varepsilon}: (q,\beta) \in J_X \times [0,1] \mapsto \mathbb{E} \big(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon}}(X_i) e^{\beta(\langle q|X_i \rangle - \widetilde{P}_X(q))} \big). \end{cases}$$

Note that given $q \in J_X$, the mapping $G_X(q,\cdot) = G_{X,\varepsilon}(q,\cdot) + \widetilde{G}_{X,\varepsilon}(q,\cdot)$, which neither depends on ε nor on H, is convex, takes values $\mathbb{E}(N) > 1$ and 1 at $\beta = 0$ and $\beta = 1$ respectively, and has $-\widetilde{P}_X^*(\nabla \widetilde{P}_X(q)) < 0$ as left derivative at $\beta = 1$. Thus, $G_X(q,\beta) > 1$ for all $\beta \in (0,1)$. Below we prove that the existence of α contradicts this fact.

Fix $\beta \in (0,1)$ and $\rho > 0$ such that $\beta + (1-\beta)\mathbb{E}(N^H) + \rho < 1$. For any $\eta \in (0,1]$ and $q \in J_X$ one has, setting $W_{q,i} = \exp(\langle q|X_i\rangle - \widetilde{P}_X(q))$:

$$G_{X,\varepsilon}(q,\beta) \leq \eta^{\beta} \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon}}(X_{i}) \mathbf{1}_{\{W_{q,i} \leq \eta\}}\Big) + \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon}}(X_{i}) \mathbf{1}_{\{W_{q,i} > \eta\}} W_{q,i}^{\beta-1} W_{q,i}\Big)$$
$$\leq \eta^{\beta} \mathbb{E}(N) + \eta^{\beta-1} \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon}}(X_{i}) \mathbf{1}_{\{W_{q,i} > \eta\}} W_{q,i}\Big) \leq \eta^{\beta} \mathbb{E}(N) + \eta^{\beta-1} G_{X,\varepsilon}(q,1).$$

Fix $\eta > 0$ such that $\eta^{\beta}\mathbb{E}(N) \leq \rho/4$ and then $n = n(\rho, \varepsilon) \in \mathbb{N}$ such that $\eta^{\beta-1}G_{X,\varepsilon}(q_n, 1) \leq \rho/4$, and consequently $G_{X,\varepsilon}(q_n, \beta) \leq \rho/2$. This is possible since $\lim_{n \to \infty} G_{X,\varepsilon}(q_n, 1) = 0$. Indeed, $\left\langle e | \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon}}(X_i)W_{q_n,i}^{\beta}X_i\right)\right\rangle \leq (c-\varepsilon)G_{X,\varepsilon}(q_n, 1)$, so that due to Lemma 6.1, $\langle e | \nabla \widetilde{P}_X(q_n) \rangle \leq c \widetilde{G}_{X,\varepsilon}(q_n, 1) + (c-\varepsilon)G_{X,\varepsilon}(q_n, 1) \leq c - \varepsilon G_{X,\varepsilon}(q_n, 1)$, and consequently $c = \langle e | \alpha \rangle \leq c - \varepsilon \lim \sup_{n \to \infty} G_{X,\varepsilon}(q_n, 1)$.

Now, note that $\widetilde{G}_{X,\varepsilon}(q_n,0)$ tends to $\mathbb{E}(N^H)<1$ as $\varepsilon\to 0$ (due to Lemma 6.1 again), and $\widetilde{G}_{X,\varepsilon}(q_n,1)=G_X(q_n,1)-G_{X,\varepsilon}(q_n,1)\leq 1$. It follows from the convexity of $\widetilde{G}_{X,\varepsilon}(q_n,\cdot)$ that if ε is chosen small enough one has $\widetilde{G}_{X,\varepsilon}(q_n,\beta)\leq \beta+(1-\beta)\mathbb{E}(N_H)+\rho/2$. Finally,

$$G_X(q_n, \beta) = G_{X,\varepsilon}(q_n, \beta) + \widetilde{G}_{X,\varepsilon}(q_n, \beta) \le \beta + (1 - \beta)\mathbb{E}(N^H) + \rho < 1,$$

which is the expected contradiction.

- (3) Suppose that $\mathcal{H}_X \setminus \widetilde{\mathcal{H}}_X \neq \emptyset$ and fix $H \in \mathcal{H}_X \setminus \widetilde{\mathcal{H}}_X$. Fix also $\beta \in H \cap \mathcal{C}_X$ and $\alpha \in \mathring{I}_X$. Since $\beta \notin I_X$, there exists $\alpha' \in [\alpha, \beta)$ such that $\alpha' \in \partial I_X$ and $\alpha' \notin \partial \mathcal{C}_X$. By point (1) of this proposition, this implies that $\alpha' \in \widetilde{I}_X \setminus \mathring{I}_X$.
- (4) For $\varepsilon \in (0,1)$, define $\gamma_{\varepsilon} = \mathbb{E}(N^H) + \varepsilon \mathbb{E}(N^{H^c})$. Then, for $1 \leq i \leq N$ set $W_{\varepsilon,i} = \frac{1}{\gamma_{\varepsilon}} \mathbf{1}_{H}(X_i) + \frac{\varepsilon}{\gamma_{\varepsilon}} \mathbf{1}_{H^c}(X_i)$ and set $W_{\varepsilon,i} = 0$ for i > N. Note that since $\mathbb{E}(N^H) \geq 1$, and (1.9) implies that $\mathbb{E}(N^{H^c}) > 0$, one has $\sup_{i \geq 1} W_{\varepsilon,i} < 1$. Consequently $-\mathbb{E}\left(\sum_{i=1}^{N} W_{\varepsilon,i} \log W_{\varepsilon,i}\right)$

is positive; also, since $\mathbb{E}(N^p) < \infty$ for somme p > 1, one has $\mathbb{E}((\sum_{i=1}^N W_{\varepsilon,i})^p) < \infty$. In addition, $\mathbb{E}\left(\sum_{i=1}^{N} W_{\varepsilon,i} X_i\right)$ converges to $\alpha_H = \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_H(X_i) X_i\right) / \mathbb{E}(N^H)$ as ε tends to 0.

For each $u \in \bigcup_{n\geq 0} \mathbb{N}^{\mathbb{N}}$, set $W_{\varepsilon,i}(u) = \frac{1}{\gamma_{\varepsilon}} \mathbf{1}_H(X_{ui}) + \frac{\varepsilon}{\gamma_{\varepsilon}} \mathbf{1}_{H^c}(X_{ui})$ for $1\leq i\leq N_u$ and $W_{\varepsilon,i}(u) = 0$ for $i > N_u$. Mimicking what was done to construct the measures μ_{ϱ} ($\varrho \in \mathcal{R}$) and determine the behavior of $(S_nX)_{n\in\mathbb{N}}$ almost everywhere with respect to such a measure (Proposition 4.4), we can find a non increasing positive sequence $(\varepsilon_n)_{n>0}$ converging to 0 such that the inhomogeneous Mandelbrot martingale associated with the random vectors $(W_{\varepsilon_{|u|},i}(u))_{i\in\mathbb{N}},\ u\in\bigcup_{n\geq 0}\mathbb{N}^{\mathbb{N}},$ yields almost surely a positive measure μ^H supported on $E(X,\alpha_H)$.

(5) Let $F \in \widetilde{\mathcal{F}}_X^1 \cup \overline{\mathcal{F}}_X^1$. Note that the fact that $F \cap I_X \neq \emptyset$ can be obtained directly in the same way as $H \cap I_X \neq \emptyset$ when $H \in \widetilde{H}_X$.

Fix $H \in \widetilde{\mathcal{H}}_X$ such that $F \subset H$.

We distinguish the cases $F \in \widetilde{\mathcal{F}}_X^1$ and $F \in \overline{\mathcal{F}}_X^1$.

Case 1: $F \in \widetilde{\mathcal{F}}_X^1$. Remember that in this case $\mathbb{E}(N^F) > 1$.

We first prove that $H \cap I_X \subset F \cap I_X \subset I_X^F$.

Let $\alpha \in H \cap I_X$. According to Lemma 3.1, we can take a sequence $(q_n, \alpha_n)_{n \in \mathbb{N}}$ of elements of D such that $\lim_{n\to\infty} \beta(q_n,\alpha_n) = \alpha$ and $\lim_{n\to\infty} \widetilde{P}_{X,\phi,\alpha_n}(q_n) - \langle q_n | \nabla \widetilde{P}_{X,\phi,\alpha_n}(q_n) \rangle = \widetilde{P}_{X,\phi,\alpha}^*(0)$. Set $W_{n,i} = \mathbf{1}_{[1,N]}(i)e^{\langle q_n | X_i - \alpha_n \rangle - \widetilde{P}_{X,\phi,\alpha_n}(q_n)}$ for $i \geq 1$. With the notations (3.2), (3.3) and (3.5), one has $h(q_n, \alpha_n) = -\mathbb{E}\left(\sum_{i=1}^N W_{n,i} \log W_{n,i}\right), \ \lambda(q_n, \alpha_n) = \mathbb{E}\left(\sum_{i=1}^N W_{n,i}\phi_i\right), \ \widetilde{P}_{X,\phi,\alpha_n}(q_n) - \langle q_n | \nabla \widetilde{P}_{X,\phi,\alpha_n}(q_n) \rangle = \frac{h(q_n,\alpha_n)}{\lambda(q_n,\alpha_n)}, \text{ and } \beta(q_n,\alpha_n) = \mathbb{E}\left(\sum_{i=1}^N W_{n,i}X_i\right).$

For any Borel subset V of \mathbb{R}^d and any $n \in \mathbb{N}$, set

$$p_{V,n} = \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V}(X_i)W_{n,i}\Big).$$

Then, define the sequence of vectors $(\widetilde{W}_{n,i})_{i\geq 1}$, $n\in\mathbb{N}$, by $\widetilde{W}_{n,i}=\mathbf{1}_{[1,N]}(i)p_{F,n}^{-1}\mathbf{1}_{F}(X_{i})W_{n,i}$.

Lemma 6.2.

mma 6.2. (1)
$$\lim_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} X_i\right) = \alpha$$
. In particular $\alpha \in F$.
(2) $\liminf_{n\to\infty} \frac{\sum_{k=1}^{n} -\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{k,i} \log \widetilde{W}_{k,i}\right)}{\sum_{k=1}^{n} \mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{k,i} \phi_i\right)} \geq \widetilde{P}_{X,\phi,\alpha}^{*}(0)$.

Assume this lemma for a while. Let $(W_i^F)_{i\geq 1}$ be defined by $W_i^F = \mathbf{1}_{[1,N]}(i)\mathbf{1}_F(X_i)/\mathbb{E}(N^F)$. For $\theta \in (0,1]$, $n \geq 1$ and $i \geq 1$ set $\widetilde{W}_{n,i}(\theta) = \theta W_i^F + (1-\theta)\widetilde{W}_{n,i}$. By convexity of the function $x \log(x)$, noting that $\mathbb{E}\left(\sum_{i=1}^N W_i^F \log W_i^F\right) = -\log(\mathbb{E}(N^F))$, one has

$$(6.3) \qquad \mathbb{E}\Big(\sum_{i=1}^{N} \widetilde{W}_{n,i}(\theta) \log \widetilde{W}_{n,i}(\theta)\Big) \leq -\theta \log(\mathbb{E}(N^{F})) + (1-\theta)\mathbb{E}\Big(\sum_{i=1}^{N} \widetilde{W}_{n,i} \log \widetilde{W}_{n,i}\Big).$$

Then, Lemma 6.2(2) together with $\mathbb{E}(N^F) > 1$ yields $-\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i}(\theta) \log \widetilde{W}_{n,i}(\theta)\right) > 0$. Let now $(\theta_n)_{n\in\mathbb{N}}$ be a positive sequence converging to 0. One has

$$\liminf_{n\to\infty} \frac{\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i}(\theta_n) \log \widetilde{W}_{n,i}(\theta_n)\right)}{\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} \log \widetilde{W}_{n,i}\right)} \ge 1 \text{ and } \lim_{n\to\infty} \frac{\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i}(\theta_n)\phi_i\right)}{\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i}\phi_i\right)} = 1,$$

where the inequality comes from (6.3) and the equality is direct. Then, Lemma 6.2(2) again implies $\liminf_{n\to\infty}\frac{\sum_{k=1}^n-\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)\log\widetilde{W}_{n,i}(\theta_k)\right)}{\sum_{k=1}^n\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)\phi_i\right)}\geq\widetilde{P}_{X,\phi,\alpha}^*(0).$ Now, note that the random vectors $(\widetilde{W}_{n,i}(\theta_n))_{n\in\mathbb{N}}$ with positive "entropy" can be used to construct an inhomogeneous Mandelbrot martingale which, conditionally on $\partial\mathsf{T}^F\neq\emptyset$, converges to a positive limit, and makes it possible to define a positive measure μ^α fully supported on $\partial\mathsf{T}^F$. Moreover, the martingale can be adjusted so that the conclusions of Propositions 3.1 hold, as well as that of Proposition 4.4 applied to μ^α and $(S_nX)_{n\in\mathbb{N}}$ restricted to ∂T^F : $\underline{\dim}(\mu^\alpha)\geq \underline{\liminf}_{n\to\infty}\frac{\sum_{k=1}^n-\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)\log\widetilde{W}_{n,i}(\theta_k)\right)}{\sum_{k=1}^n\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)\phi_i\right)}\geq \widetilde{P}_{X,\phi,\alpha}^*(0),$ and for μ^α -a.e. $t\in\partial\mathsf{T}^F$, $\underline{\lim}_{n\to\infty}n^{-1}S_nX(t)=\underline{\lim}_{n\to\infty}n^{-1}\sum_{k=1}^n\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)\phi_i\right)$ a.e. $t\in\partial\mathsf{T}^F$, $\underline{\lim}_{n\to\infty}n^{-1}S_nX(t)=\underline{\lim}_{n\to\infty}n^{-1}\sum_{k=1}^n\mathbb{E}\left(\sum_{i=1}^N\widetilde{W}_{k,i}(\theta_k)X_i\right)=\alpha$, where Lemma 6.2(1) was used to get the second equality. In other words, recalling the definitions introduced before the statement of Theorem 1.3, the F-valued branching random walk $(S_n(X_F-\alpha_F))_{n\in\mathbb{N}}$ on $\partial\widetilde{\mathsf{T}}^F$ satisfies that $\alpha-\alpha_F\in I_{X_F-\alpha_F}$, so $\alpha\in I_X^F$ by definition of I_X^F .

Thus, we proved that $H \cap I_X = F \cap I_X \subset I_X^F$. Moreover, for all $\alpha \in F \cap I_X$, conditional on $\partial \mathsf{T}^F \neq \emptyset$, one has $(\widetilde{P}_{X_F - \alpha_F, \phi_F, \alpha - \alpha_F})^*(0) \geq \dim(E(X, \alpha) \cap \partial \mathsf{T}^F) \geq \widetilde{P}_{X, \phi, \alpha}^*(0)$, where the second inequality was just proved and for the first inequality one uses the fact that by definition $E(X, \alpha) \cap \partial \mathsf{T}^F = \mathbf{b}_F(E(X_F - \alpha_F, \alpha - \alpha_F))$, where \mathbf{b}_F is an isometry between $(\partial \widetilde{\mathsf{T}}^F, \mathrm{d}_{\phi_F})$ and $(\partial \mathsf{T}^F, \mathrm{d}_{\phi})$, and the fact that Proposition 4.3 holds for $(S_n(X_F - \alpha_F))_{n \in \mathbb{N}}$ on $(\partial \widetilde{\mathsf{T}}^F, \mathrm{d}_{\phi_F})$.

Let us now prove that $I_X^F \subset F \cap I_X$, as well as Theorem 1.3(4) and (5).

This time, we use what we know about the \vec{F} -valued branching random walk $(S_nX - n\alpha_F)_{n\in\mathbb{N}}$ on $\partial \mathsf{T}^F$, conditionally on $\partial \mathsf{T}^F \neq \emptyset$, thanks to Theorems 1.1 and 1.2 and their proofs applied to $(S_n(X_F - \alpha_F))_{n\in\mathbb{N}}$ on $\partial \widetilde{\mathsf{T}}^F$ and the isometry \boldsymbol{b}_F between $(\partial \widetilde{\mathsf{T}}^F, \mathrm{d}_{\phi_F})$ and $(\partial \mathsf{T}^F, \mathrm{d}_{\phi})$. We can consider a family $(\mu_{\varrho}^F)_{\varrho\in\mathcal{R}^F}$ of inhomogeneous Mandelbrot measures simultaneously constructed and fully supported on $\partial \mathsf{T}^F$, and dedicated to the study of $(S_nX - n\alpha_F))_{n\in\mathbb{N}}$ on $\partial \mathsf{T}^F$. For each sequence $\rho \in \mathcal{R}^F$, the measure μ_{ρ}^F is constructed by using, at each generation $n \geq 1$ of the associated multiplicative cascade, independent copies of non negative random vectors $(\widetilde{W}_{\varrho_n})_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$ simultaneously defined with $(N, (\phi_i)_{i\in\mathbb{N}})$ such that $\mathbb{E}(\sum_{i=1}^N W_{\varrho_n,i}) = 1 = \mathbb{E}(\sum_{i=1}^N \mathbf{1}_F(X_i)W_{\varrho_n,i})$. Let us denote these copies by $((W_{\varrho_n,ui})_{i\in\mathbb{N}})_{u\in\mathbb{N}^{n-1}}$. For any positive sequence $\theta = (\theta_n)_{n\in\mathbb{N}}$ converging to 0 define

$$W_{\varrho_n,ui}(\theta) = (\lambda_{\varrho,\theta}(n))^{-1} \Big(\mathbf{1}_{[1,N_u]}(i) \cdot (\mathbf{1}_F(X_{ui})W_{\varrho_n,ui} + \theta_n \mathbf{1}_{F^c}(X_{ui})) \Big)_{i>1},$$

where $\lambda_{\varrho,\theta}(n) = \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_F(X_i) W_{\varrho_n,i} + \theta_n \mathbf{1}_{F^c}(X_i)\Big)$. It is easily seen that following the lines of the proof of Theorem 1.1, one can choose the small perturbation θ so that these new weights make it possible to define almost surely a family $(\mu_\varrho)_{\varrho\in\mathcal{R}^F}$ of inhomogeneous Mandelbrot measures, all fully supported on $\partial \mathsf{T}$, and such that with probability 1: (1) for all $\rho \in \mathcal{R}^F$, $\dim(\mu_\rho)$ equals the value taken by $\dim(\mu_\rho^F)$ almost surely, conditionally on $\partial \mathsf{T}^F \neq \emptyset$; (2) μ_ρ is supported on $E(X,\alpha)$ if and only if conditionally on $\partial \mathsf{T}^F \neq \emptyset$, μ_ρ^F is supported on $E(X-\alpha_F,\alpha-\alpha_F)\cap \partial \mathsf{T}^F$. This implies that if $\alpha\in I_X^F$, then $\alpha\in I_X\cap F$, and $\widetilde{P}_{X,\phi,\alpha}^*(0)=\dim E(X,\alpha)\geq (\widetilde{P}_{X_F-\alpha_F,\phi_F,\alpha-\alpha_F}^F)^*(0)$ almost surely. This, together with previous estimates shows that with probability 1, for all $\alpha\in I_X^F=F\cap I_X$, one has

 $\dim E(X,\alpha) = \widetilde{P}_{X,\phi,\alpha}^*(0) = (\widetilde{P}_{X_F-\alpha_F,\phi_F,\alpha-\alpha_F}^F)^*(0), \text{ hence the conclusion of Theorem 1.3(4)}$ for F.

To get the conclusion of Theorem 1.3(5) for F, we proceed just as above, but this time we use a small perturbation of the martingales generating the family of measures $(\mu_{\varrho}^F)_{\varrho \in \bigcup_{n \in \mathbb{N}} \mathcal{R}^F(m)}$ that we would use in the proof of Theorem 1.2 to treat the case of the \vec{F} -valued branching random walk $(S_n X - n\alpha_F)_{n \in \mathbb{N}}$ on $\partial \mathsf{T}^F$. This is enough to conclude.

Proof of Lemma 6.2. We begin with preliminary observations.

At first, for all $n \in \mathbb{N}$, the fact that $(q_n, \alpha_n) \in J_{X,\phi}$ implies that the entropy $h(q_n, \alpha_n) = -\mathbb{E}(\sum_{i=1}^N W_{n,i} \log W_{n,i})$ is positive. Consequently,

$$\mathbb{E}\left(\sum_{i=1}^{N} W_{n,i} \log^{+} W_{n,i}\right) \leq \mathbb{E}\left(\sum_{i=1}^{N} W_{n,i} \log^{-} W_{n,i}\right) \leq e^{-1} \mathbb{E}(N).$$

This uniform bound plays a crucial rôle in the estimates below and justifies the introduction of the assumption (1.18). Also, let ψ such that (1.18) holds. Without loss of generality we can assume that $\psi(x)=0$. Define the non negative convex function $\Psi:x\geq 0\mapsto \exp(\psi(x))-1$. Due to our assumptions on ψ , Ψ satisfies $\lim_{x\to\infty}\Psi(x)/x=\infty$ as well as $\Psi(0)=0$. In the language of Orlicz spaces theory [34, 30], both ψ and Ψ are the restrictions to \mathbb{R}_+ of strict Young functions. The convex conjugate of Ψ is the function Φ defined as $\Phi:y\in\mathbb{R}\mapsto\sup\{x|y|-\Psi(x):x\geq 0\}$. It is a strict Young function as well, and it is not difficult to see that $\lim_{x\to\infty}\psi(x)/x=\infty$ implies that $\lim_{x\to\infty}\Phi(x)/(x\log(x))=0$. Let $G:x\geq 0\mapsto \Phi^{-1}(x\log^+(x))$, where Φ^{-1} stands for the right-continuous inverse of Φ . One has $\lim_{x\to\infty}G(x)/x=\infty$. Set $g:x\geq 1\mapsto\sup\{z/G(z):z\geq x\}$. For all $a\geq 1$,

$$\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{(a,\infty)}(W_{n,i})W_{n,i}||X_i||\Big) \le g(a)\mathbb{E}\Big(\sum_{i=1}^{N} G(W_{n,i})||X_i||\Big).$$

Moreover, Hölder's inequality for Orlicz spaces yields

$$\mathbb{E}\Big(\sum_{i=1}^{N} G(W_{n,i}) \|X_i\|\Big) \le 2 \|(G(W_{n,i}))_{1 \le i \le N}\|_{\Phi} \cdot \|(X_i)_{1 \le i \le N}\|_{\Psi},$$

where $\|(Z_i)_{1\leq i\leq N}\|_{\Upsilon}=\inf\{k>0:\mathbb{E}\left(\sum_{i=1}^N\Upsilon(Z_i/k)\right)\leq 1\}$. However, it is clear that $\|(X_i)_{1\leq i\leq N}\|_{\Psi}<\infty$ and since $\Phi(z/k)\leq \Phi(z)/k$ for all $z\geq 0$ and $k\geq 1$ (by convexity and the fact that $\Phi(0)=0$), one has $\mathbb{E}\left(\sum_{i=1}^N\Phi(G(W_{n,i})/k)\right)\leq k^{-1}\mathbb{E}\left(\sum_{i=1}^NW_{n,i}\log^+W_{n,i}\right)\leq k^{-1}e^{-1}\mathbb{E}(N)$ for $k\geq 1$, so $\|(G(W_{n,i}))_{1\leq i\leq N}\|_{\Phi}\leq \lfloor e^{-1}\mathbb{E}(N)\rfloor+1$ for all $n\geq 1$. Finally, there exists $C_{N,X,\psi}>0$ depending on (N,X,ψ) only such that

$$\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{(a,\infty)}(W_{n,i})W_{n,i}\|X_i\|\Big) \le C_{N,X,\psi}g(a) \quad \text{for all } a \ge 1 \text{ and } n \in \mathbb{N},$$

Now we prove assertion (1). Since $\beta(q_n, \alpha_n) = \mathbb{E}(\sum_{i=1}^N W_{n,i} X_i)$ converges to α as $n \to \infty$, it is enough to prove that

(6.4)
$$\lim_{n\to\infty} \left\| \mathbb{E}\left(\sum_{i=1}^{N} W_{n,i} X_i\right) - p_{F,n} \mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} X_i\right) \right\| = 0 \text{ and } \lim_{n\to\infty} p_{F,n} = 1.$$

Recall the notations (6.1) and (6.2). Due to Lemma 6.1, for all $\varepsilon > 0$ one has $\lim_{n \to \infty} \left(p_{V_{\varepsilon},n} = \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon}}(X_i)W_{n,i}\right) \right) = 1$. Also, for any $\eta \in (0,1)$ there exists $\varepsilon = \varepsilon_{\eta} > 0$ such that $\mathbb{E}\left(\sum_{i=1}^{N} \left(\mathbf{1}_{V_{\varepsilon}}(X_i) - \mathbf{1}_{F}(X_i)\right) \|X_i\|\right) \le \eta^2$. It is so since $\mathbb{E}(N^H) = \mathbb{E}(N^F)$, $\lim_{\varepsilon \to 0} 1_{V_{\varepsilon}} = 1_H$ and $\mathbb{E}\left(\sum_{i=1}^{N} \|X_i\|\right) < \infty$. It follows that

$$\begin{split} & \left\| \mathbb{E} \left(\sum_{i=1}^{N} W_{n,i} X_{i} \right) - p_{F,n} \mathbb{E} \left(\sum_{i=1}^{N} \widetilde{W}_{n,i} X_{i} \right) \right\| \\ & \leq \left\| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}} W_{n,i} X_{i} \right) \right\| + \eta^{-1} \mathbb{E} \left(\sum_{i=1}^{N} \left(\mathbf{1}_{V_{\varepsilon_{\eta}}} (X_{i}) - \mathbf{1}_{F}(X_{i}) \right) \mathbf{1}_{[0,1/\eta]} (W_{n,i}) \|X_{i}\| \right) \\ & + \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{(1/\eta,\infty)} (W_{n,i}) W_{n,i} \|X_{i}\| \right) \leq \left\| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}} W_{n,i} X_{i} \right) \right\| + \eta + C_{N,X,\psi} g(1/\eta). \end{split}$$

Fix $B_{\eta} > 0$ such that $\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{(B_{\eta},\infty)}(\|X_i\|)\|X_i\|\right) \leq \eta^2$. One also has

$$\left\| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}} W_{n,i} X_{i} \right) \right\| \leq \eta^{-1} \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}} \mathbf{1}_{[0,1/\eta]} (W_{n,i}) \mathbf{1}_{(B_{\eta},\infty)} (\|X_{i}\|) \|X_{i}\| \right) + B_{\eta} \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}} W_{n,i} \mathbf{1}_{[0,B_{\eta}]} (\|X_{i}\|) \right) + \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{(1/\eta,\infty)} (W_{n,i}) W_{n,i} \|X_{i}\| \right).$$

so $\|\mathbb{E}\left(\sum_{i=1}^{N}\mathbf{1}_{U_{\varepsilon_{\eta}}}W_{n,i}X_{i}\right)\| \leq \eta + B_{\eta}(1-p_{V_{\varepsilon_{\eta},n}}) + C_{N,X,\psi}g(1/\eta)$. Finally, for any $\rho > 0$, if we fix $\eta \in (0,1)$ such that $3\eta + 2C_{N,X,\psi}g(1/\eta) \leq \rho$ and $n_{\rho} \in \mathbb{N}$ such that for all integers $n \geq n_{\rho}$ one has $B_{\eta}(1-p_{V_{\varepsilon_{\eta},n}}) \leq \eta$, we get $\|\mathbb{E}\left(\sum_{i=1}^{N}W_{n,i}X_{i}\right) - p_{F,n}\mathbb{E}\left(\sum_{i=1}^{N}\widetilde{W}_{n,i}X_{i}\right)\| \leq \rho$ for all $n \geq n_{\rho}$, hence the first assertion of (6.4).

The fact that $\lim_{n\to\infty} p_{F,n} = 1$ follows from the properties $\lim_{n\to\infty} p_{V_{\varepsilon},n} = 1$ for all $\varepsilon > 0$, $\lim_{\varepsilon\to 0} \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon}}(X_i) - \mathbf{1}_F(X_i)\right) = 0$, and the inequality $\mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{(a,\infty)}(W_{n,i})W_{n,i}\right) \leq (\log(a))^{-1}\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\log^+ W_{n,i}\right) \leq (\log(a))^{-1}e^{-1}\mathbb{E}(N)$ for all a > 1.

For assertion (2), assume that $\widetilde{P}_{X,\phi,\alpha}^*(0) > 0$, for otherwise the result is direct. Since we know that $\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\phi_i\right) \geq -\log(\beta) > 0$, where β is as in Lemma 2.1, one has $h := \inf_{n \in \mathbb{N}} -\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\log W_{n,i}\right) > 0$. Also, mimicking what was done above yields $\lim_{n \to \infty} \left|\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\phi_i\right) - \mathbb{E}\left(\sum_{i=1}^N \widetilde{W}_{n,i}\phi_i\right)\right| = 0$, hence $\lim_{n \to \infty} \frac{\mathbb{E}\left(\sum_{i=1}^N \widetilde{W}_{n,i}\phi_i\right)}{\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\phi_i\right)} = 1$.

Since $\lim_{n\to\infty} \frac{-\mathbb{E}\left(\sum_{i=1}^N W_{n,i} \log W_{n,i}\right)}{\mathbb{E}\left(\sum_{i=1}^N W_{n,i}\phi_i\right)} = \widetilde{P}_{X,\phi,\alpha}^*(0)$, to conclude we only need to prove that

(6.5)
$$\liminf_{n \to \infty} \frac{\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} \log \widetilde{W}_{n,i}\right)}{\mathbb{E}\left(\sum_{i=1}^{N} W_{n,i} \log W_{n,i}\right)} \ge 1.$$

To see this, write

(6.6)
$$-\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} \log \widetilde{W}_{n,i}\right) = \log(p_{F,n}) - \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) W_{n,i} \log W_{n,i}\right),$$

and setting $Z_{n,i} = W_{n,i} \log W_{n,i}$, note that

(6.7)
$$\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) Z_{n,i}\right) \leq \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) \mathbf{1}_{[0,1]}(W_{n,i}) Z_{n,i}\right) + \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{(1,\infty)}(W_{n,i}) Z_{n,i}\right),$$

and that for all $\varepsilon > 0$,

$$\left| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{F}(X_{i}) \mathbf{1}_{[0,1]}(W_{n,i}) Z_{n,i} \right) - \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{[0,1]}(W_{n,i}) Z_{n,i} \right) \right|$$

$$(6.8) \qquad \leq \left| \mathbb{E} \left(\sum_{i=1}^{N} \left(\mathbf{1}_{V_{\varepsilon}} - \mathbf{1}_{F} \right) (X_{i}) \mathbf{1}_{[0,1]} (W_{n,i}) Z_{n,i} \right) \right| + \left| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon}} (X_{i}) \mathbf{1}_{[0,1]} (W_{n,i}) Z_{n,i} \right) \right|.$$

Since $|W_{n,i} \log W_{n,i}|$ is bounded by 1/e over $\{W_{n,i} \in [0,1]\}$, for every $\eta \in (0,h)$ there exists $\varepsilon_{\eta} > 0$ such that

(6.9)
$$\left| \mathbb{E} \left(\sum_{i=1}^{N} \left(\mathbf{1}_{V_{\varepsilon_{\eta}}}(X_i) - \mathbf{1}_{F}(X_i) \right) \mathbf{1}_{[0,1]}(W_{n,i}) Z_{n,i} \right) \right| \leq \eta/2.$$

Set $a_{\eta,n} = (1 - p_{V_{\varepsilon_{\eta},n}})^{-1/2}$ and $r_{\eta,n} = \left| \mathbb{E} \left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}}(X_i) \mathbf{1}_{[0,1]}(W_{n,i}) Z_{n,i} \right) \right|$. Since by construction $\lim_{n \to \infty} a_{\eta,n} = \infty$, for n large enough one has $e^{-a_{\eta,n}} \le 1/e$, whence $|Z_{n,i}| \le a_{\eta,n} e^{-a_{\eta,n}}$ when $W_{n_k,i} \in [0, e^{-a_{\eta,n}}]$. Consequently, for n large enough

$$r_{\eta,n} \le a_{\eta,n} e^{-a_{\eta,n}} \mathbb{E}(N) + a_{\eta,n} \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{U_{\varepsilon_{\eta}}}(X_i) W_{n,i}\right) = a_{\eta,n} e^{-a_{\eta,n}} \mathbb{E}(N) + a_{\eta,n}^{-1}$$

which implies that $\lim_{n\to\infty} r_{\eta,n} = 0$. Putting this together with (6.6)–(6.9) yields that for any $\eta > 0$, for n large enough, $-\mathbb{E}\left(\sum_{i=1}^{N} \widetilde{W}_{n,i} \log \widetilde{W}_{n,i}\right) \geq -\mathbb{E}\left(\sum_{i=1}^{N} W_{n,i} \log W_{n,i}\right) - \eta$. This yields (6.5).

Case 2: $F \in \overline{\mathcal{F}}_X^1$. We start by proving that $I_X \cap F \subset \{\alpha_F\}$ (we already proved that $\{\alpha_F\} \subset I_X \cap F$), which will establish that $I_X \cap F = I_X^F = \widetilde{I}_X^F$ by definition of I_X^F and \widetilde{I}_X^F .

If $\mathbb{E}(N^F) > 1$ then $\dim F = 0$, and the conclusion is direct. So we assume that $\mathbb{E}(N^F) = 1$ (and implicitly $d \geq 2$ for otherwise the discussion is trivial). Let $H \in \widetilde{H}_X$ such that $F \subset H$. By definition of $\overline{\mathcal{F}}_X^1$ one has $\alpha_F = \alpha_H$, and it remains to prove that $I_X \cap H \subset \{\alpha_H\}$. To do so, fix $\alpha \in I_X \cap H$ and a sequence $(q_n)_{n \in \mathbb{N}} \in J_X^{\mathbb{N}}$ such that $\nabla \widetilde{P}_X(q_n)$ converges to α as $n \to \infty$. Set $W_{n,i} = e^{\langle q_n | X_i \rangle - \widetilde{P}_X(q_n)}$, $i \geq 1$. For $\varepsilon > 0$ and $n \in \mathbb{N}$, define

$$P_{n,\varepsilon}: \theta \in [0,1] \mapsto \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon}}(X_i) W_{n,i}^{\theta}\Big).$$

Lemma 6.3. For all $r \in (0,1)$, there exist a positive sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ converging to 0 and an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} \|P_{n_k,\varepsilon_k} - 1\|_{\infty} = 0$, as well as a sequence $(\theta_k)_{k \in \mathbb{N}} \in [0,1]^{\mathbb{N}}$ such that for all $k \in \mathbb{N}$:

(6.10)
$$\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i}) \mathbf{1}_{\{W_{n_{k}}, i \in [1-r, 1+r]^{c}\}} W_{n_{k}, i}^{\theta_{k}}\right) \leq r.$$

Assume this lemma for a while. We know that $\lim_{k\to\infty} \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon_k}}(X_i)X_i\right) = \alpha_H$ and $\lim_{k\to\infty} \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon_k}}(X_i)W_{n_k,i}X_i\right) = \alpha$ (using the same arguments as in the proof of Lemma 6.2(1)). Proving that $\lim_{k\to\infty} \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon_k}}(X_i)|W_{n_k,i}-1|\|X_i\|\right) = 0$ will give us the conclusion. To do so, note that for any $r \in (0,1)$, for all $k \in \mathbb{N}$, due to (6.10) and the fact that $1-\theta_k \in [0,1]$, one has

$$\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})\mathbf{1}_{[1-r,1+r]}(W_{n_{k},i})W_{n_{k},i}\Big) \geq (1-r)^{1-\theta_{k}}\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})\mathbf{1}_{[1-r,1+r]}(W_{n_{k},i})W_{n_{k},i}^{\theta_{k}}\Big)$$
$$\geq (1-r)^{2}.$$

Consequently, since $\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_k}}(X_i) W_{n_k,i}\right) \leq 1$, one obtains

(6.11)
$$\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_k}}(X_i) \mathbf{1}_{[1-r,1+r]^c}(W_{n_k,i}) W_{n_k,i}\right) \le 1 - (1-r)^2 \le 2r.$$

For $\eta \in (0,1/2)$, let B_{η} as in the proof of Lemma 6.2(1). Assume without loss of generality that $B_{\eta} > 1$ and set $r = r_{\eta} = \eta/B_{\eta}$; one has $1 + r_{\eta} < \eta^{-1}$. For $k \in \mathbb{N}$, set $A_{r,k} = \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_k}}(X_i)\mathbf{1}_{[0,1-r]}(W_{n_k,i})|W_{n_k,i} - 1|\|X_i\|\right)$. One has $A_{r,k} \leq B_{\eta}B_{r,k} + \mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{(B_{\eta},\infty)}(\|X_i\|)\|X_i\|\right)$, where

$$B_{r,k} = \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_k}}(X_i)\mathbf{1}_{[0,1-r]}(W_{n_k,i})(1-W_{n_k,i})\Big)$$

$$\leq |P_{n_k,\varepsilon_k}(0) - P_{n_k,\varepsilon_k}(1)| + \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon_k}}(X_i) \big(\mathbf{1}_{[1-r,1+r]}(W_{n_k,i})r + \mathbf{1}_{[1-r,1+r]^c}(W_{n_k,i})W_{n_k,i}\big)\Big).$$

Moreover,

$$\mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})|W_{n_{k},i} - 1|\|X_{i}\|\Big) \leq A_{r,k} + \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{[1-r,1+r]}(W_{n_{k},i})r\|X_{i}\|\Big) \\
+ \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})\mathbf{1}_{[1-r,1+r]^{c}}(W_{n_{k},i})W_{n_{k},i}\mathbf{1}_{[0,B_{\eta}]}(\|X_{i}\|)B_{\eta}\Big) \\
+ \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})\mathbf{1}_{[0,1/\eta]}(W_{n_{k},i})\eta^{-1}\mathbf{1}_{(B_{\eta},\infty)}(\|X_{i}\|)\|X_{i}\|\Big) \\
+ \mathbb{E}\Big(\sum_{i=1}^{N} \mathbf{1}_{(1/\eta,\infty)}(W_{n_{k},i})W_{n_{k},i}\|X_{i}\|\Big).$$

This, together with the estimates obtained in the proof of Lemma 6.2 as well as (6.11) yield, after setting $\delta_k = 2\|P_{n_k,\varepsilon_k} - 1\|_{\infty}$:

$$\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_{k}}}(X_{i})|W_{n_{k},i} - 1|\|X_{i}\|\right) \leq B_{\eta}\delta_{k} + \left(3 + \mathbb{E}(N) + \mathbb{E}\left(\sum_{i=1}^{N} \|X_{i}\|\right)\right)\eta + C_{N,X,\psi}g(1/\eta),$$

which is enough to conclude.

Proof of Lemma 6.3. Let us now prove the claim. Note first that $P_{n,\varepsilon}(0) = \mathbb{E}(N^{V_{\varepsilon}})$ tends to $\mathbb{E}(N^H) = 1$ as $\varepsilon \to 0$. Moreover, arguing like in the proof of (2) shows that for any fixed $\varepsilon > 0$, one has $\lim_{n \to \infty} P_{n,\varepsilon}(1) = 1$. Consider a positive sequence $(\varepsilon_k)_{k \in \mathbb{N}}$ converging to 0 and an increasing sequence of integers $(n_k)_{k \in \mathbb{N}}$ such that $\lim_{k \to \infty} P_{n_k,\varepsilon_k}(1) = 1$. Since the functions P_{n_k,ε_k} are convex, the previous properties imply that to prove that $\lim_{k \to \infty} \|P_{n_k,\varepsilon_k} - 1\|_{\infty} = 0$, it is sufficient to prove that $\lim_{k \to \infty} P'_{n_k,\varepsilon_k}(1) = 0$.

Suppose, by contradiction, that this is not the case. Without loss of generality, fix $\eta > 0$ such that for all $k \in \mathbb{N}$ one has $P'_{n_k,\varepsilon_k}(1) = \mathbb{E}\Big(\sum_{i=1}^N \mathbf{1}_{V_{\varepsilon_k}}(X_i)W_{n_k,i}\log W_{n_k,i}\Big) \geq \eta$. Define two sequences $(R_k = \mathbb{E}\big(\sum_{i=1}^N \mathbf{1}_{U_{\varepsilon_k}}(X_i)\mathbf{1}_{[0,1]}(W_{n_k,i})W_{n_k,i}\log W_{n_k,i}\big)\Big)_{k\in\mathbb{N}}$ and $(a_k = (1-P_{n_k,\varepsilon_k}(1))^{-1/2} = (\mathbb{E}\big(\sum_{i=1}^N \mathbf{1}_{U_{\varepsilon_k}}(X_i)W_{n_k,i}\big)\Big)^{-1/2}\Big)_{k\in\mathbb{N}}$. Using the same argument as in the proof of Lemma 6.2(2) one can get $|R_k| \leq a_k e^{-a_k} \mathbb{E}(N) + a_k^{-1}$ for k large enough, so $\lim_{k\to\infty} R_k = 0$, and finally $\lim\inf_{k\to\infty} \mathbb{E}\big(\sum_{i=1}^N W_{n_k,i}\log W_{n_k,i}\big) \geq \liminf_{k\to\infty} P'_{n_k,\varepsilon_k}(1) \geq \eta$. However, for all $k\in\mathbb{N}$ one has $\mathbb{E}\big(\sum_{i=1}^N W_{n_k,i}\log W_{n_k,i}\big) = -\widetilde{P}_X^*(\nabla\widetilde{P}_X(q_{n_k})) < 0$, since $q_{n_k} \in J_X$. This is the desired contradiction.

Next, suppose that there exists $r \in (0,1)$ such that for k large enough, for all $\theta \in [0,1]$, one has $\mathbb{E}\left(\sum_{i=1}^{N} \mathbf{1}_{V_{\varepsilon_k}}(X_i) \mathbf{1}_{\{W_{n_k,i} \in [1-r,1+r]^c\}} W_{n_k,i}^{\theta}\right) > r$. This implies that for k large enough, $P''_{n_k,\varepsilon_k}(\theta) \geq r(\log(1+r))^2$ for all $\theta \in [0,1]$, which contradicts the fact that $\lim_{k\to\infty} \|P_{n_k,\varepsilon_k} - 1\|_{\infty} = 0$. This new contradiction yields the claim.

Now we come to the validity of the conclusions of Theorem 1.3(4) and (5) for F.

Suppose again that $\mathbb{E}(N^F) = 1$. The previous arguments and calculations show that $\lim_{k \to \infty} P'_{n_k, \varepsilon_k}(1) = 0$ so $\liminf_{k \to \infty} \mathbb{E}\left(\sum_{i=1}^N W_{n_k, i} \log W_{n_k, i}\right) \ge \liminf_{k \to \infty} P'_{n_k, \varepsilon_k}(1) = 0$. This implies that $\widetilde{P}_X^*(\alpha_H) = \lim_{k \to \infty} \widetilde{P}_X^*(\nabla \widetilde{P}_X(q_{n_k})) = 0$. Since under d_{ϕ} the Hausdorff dimension of $E_X(\alpha)$ is bounded by $(|\log(\beta)|)^{-1}$ times its Hausdorff dimension under d_1 (where β is as in (3.7)), we conclude that $\dim E(X, \alpha_H) = 0$, hence Theorem 1.3(4) for F.

If $F = \{\alpha_F\}$ and $\mathbb{E}(N^F) > 1$, the same argument as when $\dim F \geq 1$ shows that $\widetilde{P}_{X,\phi,\alpha_F}^*(0) \leq \widetilde{P}_{X_F-\alpha_F,\phi_F,0}(0)$, where $\widetilde{P}_{X_F-\alpha_F,\phi_F,0}(0)$ is the Hausdorff dimension of $\partial \mathsf{T}^F$ conditionally on non extinction of T^F . Now, consider a positive sequence $\theta = (\theta_n)_{n \in \mathbb{N}}$ converging to 0, as well as $\widetilde{W} = (\widetilde{W}_i = \mathbf{1}_{[0,N]}(i)\mathbf{1}_F(X_i)\exp(-\widetilde{P}_{X_F-\alpha_F,\phi_F,0}(0)\phi_i))_{i\geq 1}$. Then, for $n \geq 1$ and $(u,i) \in \mathbb{N}^{n-1} \times \mathbb{N}$, set $W_{ui}(\theta) = (\lambda_{\theta}(n))^{-1}(\mathbf{1}_{[1,N_u]}(i)\cdot(\widetilde{W}_{ui}+\theta_n\mathbf{1}_{F^c}(X_{ui})))_{i\geq 1}$, where $\lambda_{\theta}(n) = \mathbb{E}(\sum_{i=1}^N \widetilde{W}_i + \theta_n\mathbf{1}_{F^c}(X_i))$. We leave the reader check that this family of random vectors defines almost surely a non degenerate inhomogeneous Mandelbrot measure μ^F supported on $E(X,\alpha_F)$ and of Hausdorff dimension $\widetilde{P}_{X_F-\alpha_F,\phi_F,0}(0)$, so $\widetilde{P}_{X_F-\alpha_F,\phi_F,0}(0) \leq \dim E(X,\alpha_F) \leq \widetilde{P}_{X,\phi,\alpha_F}^*(0)$. This yields Theorem 1.3(4) for F.

Finally, Theorem 1.3(5) for F follows from calculations similar to those done in the proof of Theorem 1.2, by considering μ^F instead of μ_{ϱ} .

(6) Suppose that $F \in \widetilde{\mathcal{F}}_X^1$ and let H be an element of \mathcal{H}_X such that $F' \subset H$ and $F \not\subset H$. Suppose that $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} \neq \emptyset$. Due to point (1) of this proposition applied to the branching random walk $(S_nX)_{n\in\mathbb{N}}$ restricted to $\partial \mathsf{T}^F$, one has $\mathring{\mathcal{C}}_{X,F} \cap \mathcal{C}_{X,F'} \neq \emptyset$. This implies that for any point $\alpha \in \mathring{\mathcal{C}}_{X,F} \cap \mathcal{C}_{X,F'}$ there are necessarily points α' and α'' in a neighbourhood of α relative to F such that $L_{e,c}(\alpha') < 0$ and $L_{e,c}(\alpha'') > 0$, for otherwise some neighbourhood

of α relative to F would be included in F', hence $F \subset F'$, which would contradict the assumption that $F \in \widetilde{\mathcal{F}}_X^1$. However, the inequality $L_{e,c}(\alpha'') > 0$ contradicts the fact that $\mathcal{C}_X \subset L_{e,c}^{-1}(\mathbb{R}_-)$. So $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} = \emptyset$.

By symmetry, we can now suppose that both F and F' belong to $\overline{\mathcal{F}}_X^1$. If $\mathbb{E}(N^F) > 1$ one has $F = \{\alpha_F\} = \widetilde{I}_X^F$. So if $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} \neq \emptyset$, then $F \subset F'$, which contradicts the fact that $F \in \overline{\mathcal{F}}_X^1$. Still by symmetry, the only case which remains to be studied is $\mathbb{E}(N^F) = 1 = \mathbb{E}(N^{F'})$. Then $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} \neq \emptyset$ means that $\alpha_F = \alpha_{F'} \in H$. Since $\alpha_F = \mathbb{E}\left(\sum_{i=1}^N \mathbf{1}_F(X_i)X_i\right)$ and $F \subset L_{e,c}^{-1}(\mathbb{R}_-)$, this implies that with probability 1, for all $1 \leq i \leq N$, $\mathbf{1}_F(X_i) = 1$ implies $\mathbf{1}_H(X_i) = 1$. This holds for any element H of \mathcal{H}_X containing F', from which we conclude that $\mathbb{E}(N^{F \cap F'}) = 1$. Consequently, $F \cap F' = F$ because $F \in \widehat{\mathcal{F}}_X$, and we get $F \subset F'$, new contradiction. Thus $\widetilde{I}_X^F \cap \widetilde{I}_X^{F'} = \emptyset$.

7. Proofs of Corollaries 1.1, 1.2 and 1.3

Proof of Corollary 1.1. For each $r \in \mathbb{Q} \cap (0, \infty)$, fix $\widetilde{k}_r \in \widetilde{K}$ such that $\lim_{n \to \infty} \frac{\log(k_r(n))}{n} = r$. It follows from Theorem B that there exists $\Omega^* \subset \Omega$ of \mathbb{P} -probability 1, such that for all $\omega \in \Omega^*$, there exists a set E^{ω} of full ν -measure such that for all $t \in E^{\omega}$ and all $r \in \mathbb{Q} \cap (0, \infty)$ the large deviation principle $\mathrm{LD}(\Lambda_{\psi}, \widetilde{k}_r)$ holds.

Now fix $\omega \in \Omega^*$ and $t \in E^{\omega}$. Then fix $\widetilde{k} \in \widetilde{K}$. Set $\ell = \lim_{n \to \infty} \frac{\log(k(n))}{n}$. If $\ell = 0$ there is nothing to prove. If $\ell > 0$, and $\lambda \in \mathcal{D}_{\Lambda_{\psi}}$ such that $\ell > -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, for any $r_1 < \ell < r_2$ with $r_1, r_2 \in \mathbb{Q} \cap (0, \infty)$ and $r_1 > -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$, for n large enough one has $k_{r_1}(n) \leq k(n) \leq k_{r_2}(n)$ so that

$$n^{-1}\log(k(n)/k_{r_1}(n)) + n^{-1}\Lambda_{\tilde{k}_{r_1},n}^t(\lambda) \leq n^{-1}\Lambda_{\tilde{k},n}^t(\lambda) \leq n^{-1}\log(k_{r_2}(n)/k(n)) + n^{-1}\Lambda_{\tilde{k}_{r_2},n}^t(\lambda),$$
hence $\ell - r_1 + \Lambda_{\psi}(\lambda) \leq \liminf_{n \to \infty} n^{-1}\Lambda_{\tilde{k},n}^t(\lambda) \leq \limsup_{n \to \infty} n^{-1}\Lambda_{\tilde{k},n}^t(\lambda) \leq r_2 - \ell + \Lambda_{\psi}(\lambda),$
where we used that $LD(\Lambda_{\psi}, \tilde{k}_r)$ holds at (ω, t) for $r \in \{r_1, r_2\}$. Since r_1 and r_2 are arbitrary, we get that part (1) of $LD(\Lambda_{\psi}, \tilde{k})$ holds as well at (ω, t) .

To get part (2) of $\mathrm{LD}(\Lambda_{\psi}, \widetilde{k})$, suppose that $\ell < -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$ and take $r_2 \in \mathbb{Q} \cap (0, \infty)$ such that $\ell < r_2 < -\Lambda_{\psi}^*(\nabla \Lambda_{\psi}(\lambda))$. The fact that $\mathrm{LD}(\Lambda_{\psi}, \widetilde{k}_{r_2})$ holds directly implies that there exists $\epsilon > 0$ such that $\left\{0 \le j \le nk(n) - 1 : \frac{S_{j+n}X(t) - S_jX(t)}{n} \in B(\nabla \Lambda_{\psi}(\lambda), \epsilon)\right\} = \emptyset$. for n large enough.

Finally, the fact that part (3) of $LD(\Lambda_{\psi}, \tilde{k})$ holds follows from the fact that part (1) holds and the arguments developed to derive [6, Theorem 2.3(3)].

Corollaries 1.2 and 1.3 are proved similarly.

8. Possible relaxation of the assumptions in Theorems 1.1 and 1.2

Set $\mathcal{D}_X = \text{dom}(\widetilde{P}_X)$ and note that \mathcal{D}_X is closed (as shows a simple application of Fatou's lemma).

Assume that $\mathcal{D}_X \neq \mathbb{R}^d$ and \mathcal{D}_X contains an open neighbourhood of 0, denoted by V. We can assume that $\widetilde{P}_X^*(\nabla \widetilde{P}_X(q)) > 0$ for all $q \in \mathcal{V}$ as well.

Trying to mimick what was done when \mathcal{D}_X equals \mathbb{R}^d , set $\widehat{I}_X = \{\nabla \widetilde{P}_X(q) : q \in \mathring{\mathcal{D}}_X, \widetilde{P}_X^*(\nabla \widetilde{P}_X(q)) \geq 0\}$. In (1.14), replace $\mathbb{R}^d \times I_X$ by $\mathring{\mathcal{D}}_X \times \widehat{I}_X$ in the definition of $J_{X,\phi}$, and in (1.16) take the supremum over $q \in \mathcal{D}_X$. The other assumptions remain the same. Then, as when $\mathcal{D}_X = \mathbb{R}^d$, for all $\alpha \in \widehat{I}_X$ there exists a unique point $q_\alpha \in \mathcal{D}_X$ at which $\inf_{q \in \mathcal{D}_X} \widetilde{P}_{X,\phi,\alpha}(q)$ is reached. Moreover, if $\partial \mathcal{D}_X$ is compact, then $\alpha \in \widehat{I}_X \mapsto q_\alpha$ is continuous. But it is not clear that in general q_α belongs to $\mathring{\mathcal{D}}_X$.

Denote by \widetilde{I}_X the set of those $\alpha \in \widehat{I}_X$ such that $q_\alpha \in \mathring{\mathcal{D}}_X$. The points of \widetilde{I}_X of the form $\nabla \widetilde{P}_X(q)$ such that $\widetilde{P}_X^*(\nabla \widetilde{P}_X(q)) > 0$ are interior points of \widetilde{I}_X . Then, the part of Theorem 1.1 regarding $\dim E(X,\alpha)$ is still valid if one replaces I_X by \widetilde{I}_X and that about $\dim K$ is valid if $K \subset \widetilde{I}_X$. Moreover, Theorem 1.2 is valid if for every $\alpha \in \widetilde{I}_X$ the domain $\mathcal{D}_{\Lambda_{\psi_\alpha}}$ of definition of Λ_{ψ_α} is taken equal to $\mathring{\mathcal{D}}_X - q_\alpha$. Now, we wish \widetilde{I}_X to be not empty. The set \widetilde{I}_X is not easy to understand in general. It is obvious that if $\mathrm{d}_\phi = \mathrm{d}_1$, then $\widetilde{I}_X = \widehat{I}_X$ and $q_\alpha = q$ if $\alpha = \nabla \widetilde{P}_X(q)$. The same properties hold if more generally the components of $\phi = (\phi_i)_{i \geq 1}$ are identically distributed and ϕ is independent of $(N, X = (X_i)_{i \in \mathbb{N}})$, or if the components of X are identically distributed and X is independent of (N, ϕ) . Also, \widetilde{I}_X is not empty if ϕ is a small perturbation of $(1)_{i \in \mathbb{N}}$.

However, let $\alpha_0 = \mathbb{E}\left(\sum_{i=1}^N X_i \exp(-\widetilde{P}_{X,\phi,\alpha}(0)\phi_i)\right)$, where α is arbitrary in \mathbb{R}^d (we already saw that $\widetilde{P}_{X,\phi,\alpha}(0)$ does not depend on α). By construction one has $q_{\alpha_0} = 0$ since $\nabla \widetilde{P}_{X,\phi,\alpha_0}(0) = \mathbb{E}\left(\sum_{i=1}^N (X_i - \alpha_0) \exp(-\widetilde{P}_{X,\phi,\alpha_0}(0)\phi_i)\right) = 0$. Moreover, it is easily seen that the differential of $\alpha \mapsto \nabla \widetilde{P}_{X,\phi,\alpha}(0)$ at α_0 is invertible (precisely, it is a nontrivial multiple of the identity). Consequently, one can apply the implicit function theorem to $f: (q,\alpha) \mapsto (q,\nabla \widetilde{P}_{X,\phi,\alpha}(q))$ at $(0,\alpha_0)$ where f takes the value (0,0), and obtain a neighborhood of α_0 over which $q_{\alpha} \in \mathring{\mathcal{D}}_X$, and $\widetilde{P}_{X,\phi,\alpha}(q_{\alpha}) \geq 0$.

It is thus natural to consider the open set $\widetilde{I}_{X,\phi}$ of those $\alpha \in I_X$ such that $q_\alpha \in \mathring{\mathcal{D}}_X$ and f is a local diffeomorphism at (q_α, α) . Then, the extensions of Theorems 1.1 and 1.2 given in the penultimate paragraph hold as well if one replaces \widetilde{I}_X by $\widetilde{I}_{X,\phi}$. Moreover, when $I_X = \overline{\{\nabla \widetilde{P}_X(q) : q \in J_X\}}$, one has $\widetilde{I}_{X,\phi} \subset \widetilde{I}_X$.

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