

Rearranged Stochastic Heat Equation

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Abstract The purpose of this work is to provide an explicit construction of a strong Feller semigroup on the space of probability measures over the real line that additionally maps bounded measurable functions into Lipschitz continuous functions, with a Lipschitz constant that blows up in an integrable manner in small time. Our construction relies on a rearranged version of the stochastic heat equation on the circle driven by a coloured noise. Formally, this stochastic equation writes as a reflected equation in infinite dimension. Under the action of the rearrangement, the solution is forced to live in a space of quantile functions that is isometric to the space of probability measures on the real line. We prove the equation to be solvable by means of an Euler scheme in which we alternate flat dynamics in the space of random variables on the circle with a rearrangement operation that projects back the random variables onto the subset of quantile functions. A first challenge is to prove that this scheme is tight. A second one is to provide a consistent theory for the limiting reflected equation and in particular to interpret in a relevant manner the reflection term. The last step in our work is to establish the aforementioned Lipschitz property of the semigroup by adapting earlier ideas from the Bismut-Elworthy-Li formula.

Keywords: Measure-valued Diffusions, Wasserstein Diffusions, Reflected SPDE, Common Noise Mean Field Models, Rearrangement Inequalities, Bismut-Elworthy-Li formula.

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1 Introduction

Mean-field models with common noise. Our work is motivated by recent developments in the theory of mean-field models, at the intersection of stochastic analysis, calculus of variations and control and game theories. Although mean field models have a long history, stemming from statistical mechanics (see the pioneering work [40]), the problems studied in recent years are, in comparison, of an increasing complexity. For example, the solutions of control or game problems give rise, in the mean-field regime, to partial differential equations posed on the space of probability measures, whose understanding remains an active area of research in the case of control and with even more open questions in the case of games (see [17, 18, 20, 37] and the references therein for a recent state of the art on these questions).

Stochastic mean-field models lead, as soon as they evolve with time, to the study of dynamics with values in the space of probability measures. Although the latter are understood as evolutions of the law of a typical particle, representative of the mean-field continuum, these probability measures remain most often deterministic. For example, they may be governed by non-linear Fokker-Planck equations or, depending on the terminology, may obey nonlinear Markovian dynamics, see for instance the seminal work by McKean [57] and the monograph [41]. Nevertheless, many recent works have underlined the interest in considering random dynamics on the space of probability measures. From a modelling point of view, the nonlinear Fokker-Planck equations become stochastic when the particles composing the mean-field continuum are subject to common noise, see for instance the earlier works [25, 49, 50, 73] and also the more recent monographs [18, 19] within the framework of control and games. The presence of a common noise also raises interesting mathematical challenges; although it is possible in some cases to adapt the usual techniques of mean-field models, the understanding of the impact of common noise is in fact rather limited. In particular, there is currently no catalogue listing the varying effects of common noise on the statistical behaviour of solutions, unlike the theory of finite-dimensional diffusion processes, in which the impact of noise has been widely studied.

Models with a smoothing effect. Typically - and this is the framework of this paper - it may be relevant to ask about the possible regularisation properties of the semigroup induced by a stochastic Fokker-Planck equation or by a mean-field model with a common noise. Although the expected properties are certainly limited when the common noise is of finite dimension (since the ambient space is of infinite dimension), the situation is different when the common noise is allowed to be infinite-dimensional. In other words, it is reasonable to imagine that a sufficiently “large” common noise could indeed provide regularisation phenomena. There is an example in the literature. The Fleming-Viot process with mutations induced by diffusions is a probability measure valued process whose semigroup is strong Feller and maps bounded functions into Lipschitz continuous functions, see [70]. The generator, which acts on functionals of probability measures, contains two parts: a first-order term that coincides with the operator coming from a deterministic linear Fokker-Planck equation and a second-order term (that should be regarded as being induced by a form of common noise) yielded by the “sampling replacement” rule characterising

the Moran and Fleming-Viot models. However, it must be stressed that the small-time smoothing property is rather poor, as the Lipschitz constant of the functions returned by the semigroup may blow up exponentially fast in small time. This may seem anecdotal, yet such a limitation renders this noise almost impossible to use to establish regularisation by noise results.

1.1 Diffusions with values in the space of probability measures

Wasserstein diffusions. Searching for common noise(s) able to force some practicable smoothing properties on the space of probability measures is connected to a *distinct* question addressed by a series of authors for almost fifteen years: what should be a Brownian motion on the space of probability measures? Whilst there has not yet been an answer to this question that may be called canonical, the existing candidates are usually referred to as “Wasserstein diffusions” (we emphasise that we do not propose a candidate Wasserstein diffusion in the sense described below). This terminology echoes the notion of Wasserstein space, defined as the space $\mathcal{P}_2(\mathbb{R}^d)$ of probability measures (on \mathbb{R}^d , for some $d \geq 1$) with finite second moment, equipped with the 2nd Wasserstein distance \mathcal{W}_2 . Many works from calculus of variations demonstrate the interest to endow the Wasserstein space with a kind of Riemannian structure, see [39, 62, 63] and the book [1]. In this approach, the tangent space at $\mu \in \mathcal{P}_2(\mathbb{R}^d)$ is the closure in $L^2(\mathbb{R}^d, \mathbb{R}^d; \mu)$ (μ -square integrable functions from \mathbb{R}^d into itself) of smooth compactly supported gradient vector fields on \mathbb{R}^d . Accordingly, the *Wasserstein derivative* or intrinsic gradient of a functional defined on $\mathcal{P}_2(\mathbb{R}^d)$ reads, at any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, as the gradient of a real-valued function (i.e., a potential) defined on \mathbb{R}^d . Roughly speaking, this potential corresponds to the so-called flat/functional derivative used to formulate the generator of the aforementioned Fleming-Viot process, see [26, 27, 70]. Wasserstein diffusions are usually expected to be valued in $\mathcal{P}_2(\mathbb{R})$ and consistent with \mathcal{W}_2 ; i.e. the small time large deviations having rate functional \mathcal{W}_2^2 and the local variance (or quadratic variation) in the corresponding chain rule (or Itô formula) is expected to derive from the Riemannian metric. Whilst the Fleming-Viot process is not a \mathcal{W}_2 -Wasserstein diffusion, examples are known. The most famous is the 1d Wasserstein diffusion constructed by von Renesse and Sturm, in [66], wherein they introduce a parametrised class of *entropy* probability measures on $\mathcal{P}([0, 1])$ - the space of probability measures on $[0, 1]$ - and then to consider, under each of these probability measures, the Markov process associated with the Dirichlet form generated by the Riemannian metric. The entropy probability measures are constructed by transferring Poisson-Dirichlet measures on the space of quantile functions on $[0, 1]$ onto $\mathcal{P}([0, 1])$, by means of the isometry that exists between the two spaces when the former is equipped with the L^2 -norm and the latter with \mathcal{W}_2 . The same isometry plays a key role in our work, however we use slightly different quantile functions.

Although the work [66] has had a great impact in the field, it is fair to say that this Wasserstein diffusion remains a difficult approach. In particular, definition *via* a Dirichlet form does not permit generic starting points and, to the best of our knowledge, there has not been any systematic analysis of the semigroup’s properties. We refer to [5, 71] for particle approximations of this Wasserstein diffusion and

[32] for a log-Sobolev inequality. Several works have been written in the wake of [66]. For example, in [42, 43], Konarovskyi proposed an alternative construction in one dimension, leading to another definition of the Wasserstein diffusion. From the particle system perspective, this approach aims at evolving a cloud of massive random particles, with the heavier particles having smaller fluctuations. The particles aggregate, becoming heavier as they collide. As opposed to the Dirichlet form construction, the model allows one to consider arbitrary initial conditions, but the collision rules force the dynamics to instantaneously take its values in the set of finitely supported probability measures. The analysis has been pushed further in [45], but many questions remain open, starting with uniqueness when the cloud of particles is initialised from a continuum. We refer to [44] and the references therein for an extension allowing for fragmentation and to [55] for a mollification of the coalescing dynamics, for which uniqueness holds true. Last but not least, the $1d$ dynamics constructed in [42, 43] are somehow extended to the higher dimensional setting in [28] but using the theory of Dirichlet forms in the spirit of [66].

Connection with the Dean-Kawasaki equation. The aforementioned works are connected with stochastic Fokker-Planck equations. In [42, 43, 66], these Wasserstein diffusions each induce generators (acting on functionals of probability measures) sharing similarities with the generator of the so-called Dean-Kawasaki equation. Formally, the latter is a stochastic version of a standard Fokker-Planck equation (of order 1 or 2 depending on the cases) including an additional noisy term whose local quadratic variation derives exactly from the Riemannian metric on $\mathcal{P}(\mathbb{R}^d)$ (with $d = 1$ in [42, 43, 66]). However, it has been proved in [46, 47] that the Dean-Kawasaki equation, in its strict version, cannot be solvable except in trivial cases where it reduces to a finite dimensional particle system (which requires the initial distribution to be finitely supported). This negative result has an interesting consequence: some extra correction is needed in the dynamics, which is exactly what is done in [42, 43, 66]. However, so far there has not been any canonical choice for such a correction.

The very spice of the Dean-Kawasaki equation may be explained as follows. When the solution is at some probability measure $\mu \in \mathcal{P}(\mathbb{R})$, a typical particle in the mean-field continuum, located at some point $x \in \mathbb{R}$, should be subjected to the value at this point x of a cylindrical Wiener noise on $L^2(\mathbb{R}, \mathbb{R}; \mu)$, which makes no sense in general. This suggests that Dean-Kawasaki dynamics can be approached by replacing the cylindrical noise by a coloured noise. To a certain extent, this idea is the basis of the two contributions [30] and [56].

In [56], the resulting semigroup is shown to have an (albeit weak) mollification effect on functions over $\mathcal{P}(\mathbb{R})$.

1.2 Our contribution

Smoothing properties of the Ornstein-Uhlenbeck process. Unlike many of the aforementioned works, our aim is not to provide another candidate Wasserstein diffusion. Our primary motivation in this contribution is to construct as explicitly as possible a probability-measure valued process having sufficiently strong smoothing properties. Although not discussed further within this text, our long-term goal is to

propose a corresponding theory of linear or nonlinear parabolic Partial Differential Equations (PDEs) on the space of probability measures and to exhibit, in this context, second order operators allowing to smooth singularities that may appear in the corresponding hyperbolic PDEs. Of course, such a process should share similarities with Wasserstein diffusions, but as we will see, the diffusion introduced in this paper does not satisfy the pre-requisites for being a Wasserstein diffusion.

Our approach is based on two observations. First, Lions [51] showed in his lectures on mean-field games at the Collège de France, that in the study of mean-field models, it can prove useful to *lift* probability measures into random variables, i.e., to invert the map sending a random variable to its statistical distribution. Although the inverse is multi-valued, it has been shown that Lions' lifting principle provides a clear picture of the Wasserstein derivative: in short, it can be represented as a Fréchet derivative on a Hilbert space of square-integrable random variables, see e.g. [36]. Our second remark is a well-known fact from stochastic analysis: we know how to construct a Hilbert-valued diffusion process with strong smoothing properties. A simple example is the Ornstein-Uhlenbeck process driven by an appropriate operator, see for instance [21, 22, 23]. This suggests the following procedure: we should project onto the space of probability measures, an Ornstein-Uhlenbeck process taking values in a space of square-integrable random variables. Whilst this looks very appealing, this idea has an obvious drawback. In general, the projection should destroy the Markov nature of the dynamics; transition probabilities started from two different random variables representing the same probability measure may not be the same.

Our construction is thus inspired from the Lie-Trotter-Kato formula and related splitting methods. We alternate between, one step in the space of random variables following some prescribed Ornstein-Uhlenbeck dynamics, and a projection operation to return back from the space of random variables to the space of probability measures. Choosing the probability space carrying the random variables is simple: we work on the circle, $\mathbb{S} \cong \mathbb{R}/\mathbb{Z}$, equipped with Lebesgue measure. The choice of projection is much more difficult. It is an essential aspect in implementing the splitting scheme and, as in the contributions [30, 42, 43, 55, 56, 66], it leads us to limit our study to the one-dimensional case, with the following two advantages. First, probability measures can be easily identified with quantile functions on the circle (or 'symmetric non-increasing functions', see Proposition 2.1), which makes the choice of projection easier as it suffices to send a function on the circle to an appropriate rearrangement. Second, the rearrangement operation is an easy way to transform a random variable on the circle into a quantile function whilst preserving its statistical law (under the Lebesgue measure).

The resulting scheme in which we combine 'flat' dynamics and rearrangement is very much inspired by earlier works of Brenier on discretisation schemes for conservation laws, see for instance [15, 16], with the main difference being that the works of Brenier are mostly for deterministic dynamics. Since we choose the Laplacian to be the driving operator in the Ornstein-Uhlenbeck dynamics, we call the resulting equation the 'rearranged stochastic heat equation'.

Rearranged and reflected equations. The presence of the noise raises many subtleties in our construction. One particular issue is that the rearrangement operation and the

Laplacian driving the Stochastic Heat Equation (SHE) do not marry well. Obviously, they do not commute. As a result, the smoothing effect of the Laplacian (acting on functions on \mathbb{S}) is weaker when the rearrangement is present. At least, this is what we observe in our computations. This has a rather dramatic consequence on the choice of the noise. One key feature of the SHE is that after convolution with the heat kernel, the cylindrical white noise driving the SHE gives a true random function. When the SHE is rearranged (as we do here), this no longer seems to be the case. In order to remedy this problem, we need to colour the noise driving the SHE. As expected, this impacts the generated semigroup's smoothing properties. Nevertheless we succeed to show that the rate at which the derivative of the semigroup blows up in small time is integrable, as we initially intended. It remains an open question whether the same construction can be achieved for the SHE driven by a cylindrical white noise.

Another difficulty is to obtain a suitable formulation of the rearranged SHE. Although Brenier's works [15, 16] quite clearly suggest to see the rearrangement as a reflection and indeed to write the rearranged SHE as a reflected equation, again, the presence of the noise requires additional precautions. The study of reflected differential equations is in general more complicated in the stochastic case than in the deterministic case because the solutions are no longer of bounded variation. We refer to the seminal article [52] in the case of finite dimensional equations. To the best of our knowledge, there is no general theory covering our infinite dimensional formulation of the rearranged SHE. We therefore propose a tailor-made interpretation in which the reflection term is constructed by hand. Schematically, the rearranged SHE is written as a stochastic partial differential equation (SPDE) on the space $L^2(\mathbb{S}) := L^2(\mathbb{S}, \text{Leb}_{\mathbb{S}})$ (of functions on the circle that are square-integrable with respect to the Lebesgue measure) subject to a reflection term forcing the solution to remain in the cone of our chosen quantile functions (symmetric non-increasing). This representation is reminiscent of the $1d$ reflected stochastic differential equation studied by Nualart and Pardoux [61] (and extended in [31]), in which the SHE is constrained to be positive. Although the latter positivity constraint may be interpreted as a constraint on the monotonicity of the primitive, the rearranged SHE that we study here is not the primitive of the Nualart-Pardoux reflected equation.

The form of the reflection in [61] was further specified in the later contributions [75, 76] due to Zambotti. These results provide a more refined description of the solution's behaviour at the domain's boundary. In our approach we are not able at this stage, to give a similar picture. Our construction of the reflection process and its associated integral is too elementary. In particular, we consider only the action of the reflection process on functions that are far more regular than the solution of the equation itself. Fortunately, this does not prevent us from obtaining a characterisation of the solutions, sufficient to carry out our program to the end. In fact, Zambotti's results are based on a formula of integration by parts that allows one to reinterpret the solutions of the Nualart-Pardoux equation by means of the theory of Dirichlet forms. The adaptation to our case remains completely open. We refer however to the papers [9, 10, 11, 68] for more general works that have been published subsequently on reflected stochastic differential equations in infinite dimension.

Description of the results. The rearranged SHE is proven well-posed in the strong sense. The main solvability result is Theorem 4.15 and the reader may find the notion of solution in Definition 4.13. The proof holds in two main steps. The first is to show existence of weak solutions and the second one is to prove that uniqueness holds in the strong sense. Strong existence then follows from a standard adaptation of Yamada-Watanabe's theorem. As is often the case, the first step is more challenging. Weak solutions are obtained as weak limits of linear interpolations of an Euler scheme: each iteration is a small time step of Ornstein-Uhlenbeck dynamics in $L^2(\mathbb{S})$ followed by rearrangement of the terminal random variable. Part of the challenge is to show that the scheme is tight (in the space of continuous functions). This is done in Section 3 by using several key properties of the rearrangement operation, as presented in Section 2. To complete the proof of the existence of a weak solution, we need to give an appropriate sense to the reflection process, which is one of the goals of Section 4. The main point in proving strong uniqueness is to impose, in the definition of a solution, a weak form of orthogonality between the solution and the reflection. The second main statement of the article is Theorem 5.9, which says that the semi-group generated by our rearranged SHE is strongly Feller, i.e., maps bounded measurable functions into continuous functions. Moreover, the semi-group returns Lipschitz continuous functions, with Lipschitz constant diverging integrably in small time. The proof of Theorem 5.9 draws heavily on previous works on the so-called Bismut-Elworthy-Li formula, an integration by parts formula for the transition probabilities of a diffusion process, see for instance [34, 35, 72] in the finite-dimensional framework and [24] and [21, Chapter 7] in infinite dimension. Such an integration by parts is strongly related to Malliavin calculus, see for instance Exercise 2.3.5 in the book [60], together with the papers [12] and [58]. Transposition of the Bismut-Elworthy-Li formula to the reflected setting is known however to raise some technical difficulties. A major obstacle, is to prove differentiability of the flow with respect to the initial condition. We refer to [29] for the first result in this direction (drifted Brownian motion with reflection in the orthant) and to [3, 4, 53, 54] for further results. None of these results (which are all in finite dimension) apply to our case. At this stage, we do not know if similar results hold for the rearranged SHE. Instead, in our analysis, we use the sole property that the flow (generated by the rearranged SHE) is Lipschitz continuous with respect to the initial condition and thus almost everywhere differentiable when the initial condition is restricted to a finite-dimensional space.

Comparison with recent literature and further prospects. A few weeks before we put this work on arXiv, another arXiv pre-publication was published ([65]) in which the authors introduce, on the space of probability measures, a Dirichlet form whose construction has some similarities with the construction of the rearranged SHE that we introduce here. Note that the results of the two papers do not overlap, but an in-depth study would be necessary to link the two constructions more properly. In short, the work [65] aims at projecting on the space of probability measures a Gaussian measure constructed on an L^2 space of random variables and then at considering, under this measure, the Dirichlet form generated by the Riemannian metric on $\mathcal{P}_2(\mathbb{R}^d)$ (with $d \geq 1$). For example, in $1d$, this Gaussian measure can be

the invariant measure of the SHE driven by a cylindrical white noise. Although this example (in $1d$) does not fit our assumptions (since we need the noise to be coloured), it is worth noting that, if we had to write formally the generator of the rearranged SHE in this case, it would be different from the one computed in [65, Theorem 4.1].

We also highlight that our construction has a simple particle interpretation. At each time step of the Euler scheme, we can indeed consider a particle approximation of the SHE, as given for example by a finite volume discretisation. Then, at the end of each time step, the rearrangement operation, when implemented on the particles, simply consists in ordering them. We do not discuss this further in the rest of the article (for obvious reasons of length).

The reader may wonder about higher dimensional extensions. Although this is indeed a natural equation, we think it is useful to recall that many of the aforementioned works (notably those concerning the construction of a Wasserstein diffusion) are also in one dimension. From this point of view, this limitation in our model should not come as a surprise. As for the possible ways to extend the construction to the case $d \geq 2$, one possibility is to use the tools of optimal transport ([14]), but this perspective is open at this stage. The reader may also worry about the fact that, in dimension $d \geq 2$, the stochastic heat equation (when driven by the Laplace operator) requires a coloured noise, of a higher regularity than what we use here. In fact, this would be only the case if we considered the stochastic heat equation on a space of dimension d (typically the d -dimensional torus). Actually, our belief is that we could define the stochastic heat equation on the $1d$ torus, but regard it as a system of d equations. That said, another possibility could be to replace the Laplacian by another operator.

Organisation of the paper. We introduce some preliminary material in Section 2, including some (known) results on the symmetric rearrangement on \mathbb{S} . Section 3 is dedicated to the analysis of the approximating scheme. In particular, the reader will find all the required assumptions on the noise in the introduction of Section 3. Tightness is established in Proposition 3.4. The definition of a solution to the rearranged SHE is clarified in Section 4, see Definition 4.13. Existence and uniqueness are guaranteed by Theorem 4.15. The smoothing properties of the semigroup is studied in Section 5, the main Lipschitz estimate being stated in Theorem 5.9.

2 Preliminary Material

2.1 The symmetric non-increasing rearrangement

Throughout, the circle \mathbb{S} is chosen to be parametrised by the interval $(-1/2, 1/2]$ and 0 is regarded as a privileged fixed point on the circle, i.e., $\mathbb{S} := (\mathbb{R} + 1/2)/\mathbb{Z}$.

Proposition 2.1 *Given a measurable function $f : \mathbb{S} \rightarrow \mathbb{R}$, there exists a unique function, called symmetric non-increasing rearrangement of f and denoted $f^* : \mathbb{S} \rightarrow [-\infty, +\infty]$, that satisfies the following two properties:*

1. f^* is symmetric (with respect to 0), is non-increasing and right-continuous on the interval $[0, 1/2)$, and is left-continuous at $1/2$ (left- and right-continuity being here understood for the topology on $[-\infty, +\infty]$),

2. **Cavalieri's principle:** the image of the Lebesgue measure $\text{Leb}_{\mathbb{S}}$ by f^* is the same as the image of the Lebesgue measure by f , namely, for all $a \in \mathbb{R}$, $\text{Leb}_{\mathbb{S}}(\{x \in \mathbb{S} : f^*(x) \leq a\}) = \text{Leb}_{\mathbb{S}}(\{x \in \mathbb{S} : f(x) \leq a\})$.

Intuitively, f^* should be regarded as a quantile function, the symmetrisation procedure here forcing an obvious form of ‘continuous periodicity’ (whose interpretation requires some care as f^* may have jumps). Indeed, it must be noted that the collection of functions f^* satisfying item 1 in the definition above are one-to-one with the set $\mathcal{P}_2(\mathbb{R})$ of probability measures on \mathbb{R} that have a finite-second moment. In fact, for f^* as in item 1 and for a probability measure $\mu \in \mathcal{P}_2(\mathbb{R})$, the measure $\text{Leb}_{\mathbb{S}} \circ (f^*)^{-1}$ is equal to μ if and only if $x \in [0, 1] \mapsto f^*((1-x)/2)$ coincides with the usual quantile function, i.e. the usual generalised inverse of the (right-continuous) cumulative distribution function. The reader is referred to Baernstein [8] for further details, see in particular Definition 1.29 therein for the general definition of symmetric rearrangements in the Euclidean setting and Chapter 7 in the same book for a specific treatment of spherical symmetric rearrangements. We use the following quite often:

Definition 2.2 A function $f : \mathbb{S} \rightarrow \mathbb{R}$ is said to be symmetric non-increasing if $f = f^*$. The collection of equivalence classes in $L^2(\mathbb{S})$ containing a symmetric non-increasing function is denoted by $U^2(\mathbb{S})$.

It is a cone.

Below, we often consider elements of $L^2_{\text{sym}}(\mathbb{S})$. They are defined as functions in $L^2(\mathbb{S})$ that are Lebesgue almost everywhere symmetric. One of these elements is said to be *non-increasing* (we refrain from tautological use of the word symmetric given the context of the circle) if it coincides almost everywhere with an element of $U^2(\mathbb{S})$. Notice that we may choose the latter representative to be uniquely defined as a symmetric non-increasing function. Indeed, two elements of $U^2(\mathbb{S})$ that coincide in $L^2(\mathbb{S})$ coincide in fact everywhere on \mathbb{S} (courtesy of the left- and right-continuity properties). Also, the following proposition is of clear importance.

Proposition 2.3 $L^2_{\text{sym}}(\mathbb{S})$ and $U^2(\mathbb{S})$ are closed subsets of $L^2(\mathbb{S})$ equipped with $\|\cdot\|_2$.

Proof Closedness of $L^2_{\text{sym}}(\mathbb{S})$ is obvious. Closedness of $U^2(\mathbb{S})$ follows from Lemma 2.6 below: if $(f_n)_{n \geq 1}$ in $U^2(\mathbb{S})$ converges to some $f \in L^2_{\text{sym}}(\mathbb{S})$, then $f = f^*$. \square

2.2 Reformulating the main results

Our diffusion process with suitable smoothing properties on $\mathcal{P}_2(\mathbb{R})$ arrives via the construction of a diffusion process with values in $U^2(\mathbb{S})$. The equivalence relies on the fact that the mapping $f^* \in U^2(\mathbb{S}) \mapsto \text{Leb}_{\mathbb{S}} \circ (f^*)^{-1} \in \mathcal{P}_2(\mathbb{R})$ is an isometry when $\mathcal{P}_2(\mathbb{R})$ is equipped with the \mathcal{W}_2 -Wasserstein distance, i.e., for any f^*, g^* in $U^2(\mathbb{S})$,

$$\|f^* - g^*\|_2 = \mathcal{W}_2(\text{Leb}_{\mathbb{S}} \circ (f^*)^{-1}, \text{Leb}_{\mathbb{S}} \circ (g^*)^{-1}),$$

$$\text{where } \mathcal{W}_2(\mu, \nu)^2 := \inf_{\pi \in \mathcal{P}(\mathbb{R}^2) : \pi \circ e_x^{-1} = \mu, \pi \circ e_y^{-1} = \nu} \int_{\mathbb{R}^2} |x - y|^2 \pi(dx, dy),$$

with $e_x : (x, y) \in \mathbb{R}^2 \mapsto x$ and $e_y : (x, y) \in \mathbb{R}^2 \mapsto y$ being the two evaluation mappings on \mathbb{R}^2 . This identity is a consequence of Lemma 2.6, since for any π as above, there exist two (measurable) functions f and g from \mathbb{S} to \mathbb{R} such that $\pi = \text{Leb}_{\mathbb{S}} \circ (f, g)^{-1}$.

In this framework, our main results can be (re)formulated as follows:

1. We introduce a stochastic differential equation on $U^2(\mathbb{S})$ in the form of a reflected (or rearranged) stochastic equation on $L^2(\mathbb{S})$ whose reflection term forces solutions to stay within the cone $U^2(\mathbb{S})$, whenever they are initialised from $U^2(\mathbb{S})$, see Theorem 4.15. Solutions induce a Lipschitz continuous flow with values in $U^2(\mathbb{S})$. The construction of the rearranged equation relies on an Euler scheme, in which we alternate some flat dynamics in the space $L^2_{\text{sym}}(\mathbb{S})$ with the rearrangement operation that projects back the solution onto $U^2(\mathbb{S})$.
2. The second main statement is Theorem 5.9, which says that the semigroup generated by our rearranged stochastic equation maps bounded measurable functions on $U^2(\mathbb{S})$ into Lipschitz continuous functions on $U^2(\mathbb{S})$. Recast on $\mathcal{P}_2(\mathbb{R})$ (through the isometry between $U^2(\mathbb{S})$ and $\mathcal{P}_2(\mathbb{R})$), we get in this way a semigroup that maps bounded measurable functions on $\mathcal{P}_2(\mathbb{R})$ into Lipschitz continuous functions (with respect to the 2-Wasserstein distance \mathcal{W}_2).

2.3 Key properties of the symmetric non-increasing rearrangement

In the subsection, we expand a list of useful properties that are satisfied by f^* . The first one just follows from item 2 in the statement of Proposition 2.1.

Lemma 2.4 (Preservation of L^p norms) *With the same notations as in Proposition 2.1, we have, for any $p \in [1, \infty]$, $\|f^*\|_p = \|f\|_p$.*

The next result, called the Hardy-Littlewood inequality, is fundamental.

Lemma 2.5 (Hardy-Littlewood inequality) *Let f and g be two measurable real-valued functions defined on \mathbb{S} such that $\|f\|_p < \infty$ and $\|g\|_q < \infty$, for $p, q \in [1, \infty]$, with $1/p + 1/q = 1$. Then,*

$$\int_{\mathbb{S}} f(x)g(x)dx \leq \int_{\mathbb{S}} f^*(x)g^*(x)dx.$$

We refer to [8, Corollary 2.16] for a general statement in the Euclidean setting, but stated under the conditions that f and g are non-negative, and to [8, Section 7.3] or [6, 7] for a version without non-negativity constraints that is specifically stated on the circle. We now turn to the well-known property of non-expansion:

Lemma 2.6 (Non-expansion property) *Let f and g be two measurable real-valued functions with $\|f\|_p < \infty$ and $\|g\|_p < \infty$, for $p \in [1, \infty]$. Then, $\|f^* - g^*\|_p \leq \|f - g\|_p$.*

We refer to [8, Corollary 2.23] for the Euclidean setting (which requires f and g to be positive valued) and to [8, Section 7.3] for the extension to the spherical setting (which no longer requires f and g to be positive valued).

The following statement is taken from [6, 7], see also [8, Theorem 8.1].

Lemma 2.7 (Riesz rearrangement inequality) *Let f , g and h be three measurable real-valued functions on \mathbb{S} , such that $\|f\|_p < \infty$, $\|g\|_q < \infty$ and $\|h\|_r < \infty$ for $p, q, r \in [1, \infty]$ with $1/p + 1/q + 1/r = 1$. Then,*

$$\int_{\mathbb{S}} \int_{\mathbb{S}} f(x)g(x-y)h(y)dx dy \leq \int_{\mathbb{S}} \int_{\mathbb{S}} f^*(x)g^*(x-y)h^*(y)dx dy.$$

2.4 The heat kernel and the rearrangement operator

We now address several basic properties of the composition of the rearrangement operator and the heat kernel. First, we recall that the periodic heat semigroup (with specific diffusivity parameter 1) on the circle \mathbb{S} , which we denote $(e^{t\Delta})_{t \geq 0}$, has the following kernel (see Dym and McKean p.63 [33]):

$$\Gamma_t(x) := \frac{1}{\sqrt{4\pi t}} \sum_{n \in \mathbb{Z}} \exp \left\{ -\frac{(x-n)^2}{4t} \right\}, \quad t > 0, x \in \mathbb{S}. \quad (2.1)$$

Lemma 2.8 *For any $t > 0$, the function $x \mapsto \Gamma_t(x)$ is non-increasing on $(0, 1/2)$ and non-decreasing on $(-1/2, 0)$. That is, $\Gamma_t(\cdot) = \Gamma_t(\cdot)^*$, the rearrangement acting on x .*

The proof of Lemma 2.8 is not trivial, due to the series underpinning the expression of Γ . The reader will find a general discussion on spherical heat kernels in the recent paper [59], but specific (and much easier) computations that suffice for the proof of the above statement can be found in [2]. We conclude this subsection with:

Lemma 2.9 *For f in $U^2(\mathbb{S})$ and $t > 0$, the convolution $f * \Gamma_t(\cdot)$ is also in $U^2(\mathbb{S})$.*

Proof Let $h = f * \Gamma_t(\cdot)$, for a given $t > 0$. Lemmas 2.7 and 2.8 yield

$$\|h\|_2^2 = \int_{\mathbb{S}} \int_{\mathbb{S}} f(x)\Gamma_t(x-y)h(y)dx dy \leq \int_{\mathbb{S}} \int_{\mathbb{S}} f(x)\Gamma_t(x-y)h^*(y)dx dy = \langle h, h^* \rangle_2.$$

By the preservation of L^p norms, $\|h - h^*\|_2^2 = \|h\|_2^2 + \|h^*\|_2^2 - 2\langle h, h^* \rangle_2 = 2\|h\|_2^2 - 2\langle h, h^* \rangle_2 \leq 0$, which yields $h = h^*$ almost everywhere. Since h is continuous (by convolution), so is h^* (see [8, Subsection 2.4]). Therefore, h and h^* coincide. \square

By combining Proposition 2.3 with Lemma 2.9, we obtain the following stronger closedness property:

Proposition 2.10 *For any $a > 0$, $\{f \in L_{\text{sym}}^2(\mathbb{S}) : \|f\|_2 \leq a\}$ and $\{f \in U^2(\mathbb{S}) : \|f\|_2 \leq a\}$ are closed subsets of $H_{\text{sym}}^{-1}(\mathbb{S})$ equipped with $\|\cdot\|_{2,-1}$.*

Proof Take a bounded sequence $(f^n)_{n \geq 1}$ in $L_{\text{sym}}^2(\mathbb{S})$ that converges (for $\|\cdot\|_{2,-1}$) to some $f \in H_{\text{sym}}^{-1}(\mathbb{S})$. By lower semi-continuity of the L^2 -norm with respect to the H^{-1} -norm, we deduce that f belongs to $L^2(\mathbb{S})$ with $\|f\|_2 \leq \liminf_{n \rightarrow \infty} \|f^n\|_2$.

Assume now that the sequence $(f^n)_{n \geq 1}$ takes values in $U^2(\mathbb{S})$. Then, for each $\varepsilon > 0$, $f * \Gamma_\varepsilon = e^{\varepsilon\Delta} f$ is in $U^2(\mathbb{S})$. This follows from the following two points. Firstly, for each $n \geq 1$, $e^{\varepsilon\Delta} f_n$ is in $U^2(\mathbb{S})$ (as a consequence of Lemma 2.9). Secondly, $\|e^{\varepsilon\Delta} f_n - e^{\varepsilon\Delta} f\|_2$ tends to 0 as n tends to ∞ . By closedness of $U^2(\mathbb{S})$ with respect to the L^2 norm, we get that $e^{\varepsilon\Delta} f \in U^2(\mathbb{S})$.

Finally, since f is in $L^2(\mathbb{S})$, $\|e^{\varepsilon\Delta} f - f\|_2$ tends to 0 with ε , and we can invoke again the fact $U^2(\mathbb{S})$ is closed with respect to the L^2 norm.

2.5 Some notation

We introduce a few notations related with functional and Fourier analysis. The space of continuous functions from one metric space \mathcal{X} to another, \mathcal{Y} , is denoted $\mathcal{C}(\mathcal{X}, \mathcal{Y})$. For $k \geq 1$, we denote by $\mathcal{C}_0^\infty(\mathbb{R}^k)$ the space of infinitely differentiable real-valued functions on \mathbb{R}^k with compact support.

We recall that \mathbb{S} is the circle parametrised by the interval of length 1. Also, we let

$$e_m^{\mathfrak{R}} : x \in \mathbb{S} \mapsto \sqrt{2} \cos(2m\pi x), \quad e_m^{\mathfrak{S}} : x \in \mathbb{S} \mapsto \sqrt{2} \sin(2m\pi x),$$

for any natural number m , together with $e_0^{\mathfrak{R}} := 1$ and $e_0^{\mathfrak{S}} := 0$, form the complete Fourier basis on $L^2(\mathbb{S})$, where $L^2(\mathbb{S})$ is the space of square integrable functions on \mathbb{S} . Usually, we just use the even (cosine) Fourier functions, which prompts us to use the shorter notation e_m for $e_m^{\mathfrak{R}}$.

The Lebesgue measure on \mathbb{S} is denoted $\text{Leb}_{\mathbb{S}}$ with $d\text{Leb}_{\mathbb{S}}(x)$ written as dx . For any $p \geq 1$, we call $\|\cdot\|_p$ the L^p norm on the space of measurable functions f on $(\mathbb{S}, \text{Leb}_{\mathbb{S}})$ with $\int_{\mathbb{S}} |f(x)|^p dx < \infty$. Similarly, when $p = \infty$, the notation $\|\cdot\|_\infty$ is used for the L^∞ (supremum) norm, i.e. $\|f\|_\infty := \text{esssup}\{f(x) : x \in \mathbb{S}\}$. The inner product between two elements f and g in $L^2(\mathbb{S})$ is denoted $\langle f, g \rangle_2$, or $\langle f, g \rangle$, or also $f \cdot g$ depending on the context. For an element $f \in L^1(\mathbb{S})$ and a non-negative integer m , we call $\hat{f}_m^{\mathfrak{R}} := \int_{\mathbb{S}} f(x) e_m^{\mathfrak{R}}(x) dx$ the cosine Fourier mode of f of index m and $\hat{f}_m^{\mathfrak{S}} := \int_{\mathbb{S}} f(x) e_m^{\mathfrak{S}}(x) dx$ the sine Fourier mode of f of index m . When f is Lebesgue almost everywhere (written a.e. hereafter) symmetric, i.e. $f(-x) = f(x)$ a.e., all the sine Fourier modes are 0 and we write $\hat{f}_m = \langle f, e_m \rangle$ instead of $\hat{f}_m^{\mathfrak{R}}$. In that case, \hat{f}_m is a real number. We denote by $L_{\text{sym}}^2(\mathbb{S})$ the set of functions f in $L^2(\mathbb{S})$ that are a.e. symmetric. More generally, for a parameter $\mu \in \mathbb{R}$, we denote by $H_{\text{sym}}^\mu(\mathbb{S})$ the Sobolev space of symmetric functions/distributions f such that $\|f\|_{2,\mu}^2 := \sum_{m \in \mathbb{N}_0} (m \vee 1)^{2\mu} \hat{f}_m^2 < \infty$, with the notation \hat{f}_m extending in an obvious manner to the distributional case (\mathbb{N} is the collection of natural numbers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$). Of course, $H_{\text{sym}}^0(\mathbb{S})$ is just $L_{\text{sym}}^2(\mathbb{S})$. The inner product on $H_{\text{sym}}^\mu(\mathbb{S})$ is denoted $\langle f, g \rangle_{2,\mu} := \sum_{m \in \mathbb{N}_0} (m \vee 1)^{2\mu} \hat{f}_m \hat{g}_m$.

For any integer $k \geq 1$, we denote by $\mathcal{C}^k(\mathbb{S})$ the space of k -times continuously differentiable functions on \mathbb{S} . For a real number x , we write $\lfloor x \rfloor$ for the floor of x , $\lceil x \rceil$ for the ceiling of x and $x_+ := \max(x, 0)$ (resp. $x_- = \min(-x, 0)$) for the positive (resp. negative) part of x . For two reals x and y , we let $x \vee y := \max(x, y)$ and $x \wedge y := \min(x, y)$. Moreover, for a differentiable real-valued function on \mathbb{S} , we write Df for the derivative of f . And, we let $\Delta := D^2$.

As for constants that are used in the various inequalities, they are usually written in the form $c_{a,b}$ or $C_{a,b}$, where the subscripts are quantities on which the current constant depends, and are implicitly allowed to vary from line to line.

3 Approximation Scheme and its Estimates

Our construction relies on a discretisation scheme in which we alternate one random move in the Hilbert space $L_{\text{sym}}^2(\mathbb{S})$ and rearrangement, forcing the output of the scheme to stay within the subset of symmetric non-increasing functions $U^2(\mathbb{S})$.

Definition of the noise. The randomisation in $L^2_{\text{sym}}(\mathbb{S})$ obeys an Euler scheme with Gaussian increments. We introduce the following Wiener process, $(W_t)_{t \geq 0}$:

$$W_t := B_t^0 e_0 + \sum_{m \in \mathbb{N}} m^{-\lambda} B_t^m e_m \equiv \sum_{m \in \mathbb{N}_0} \lambda_m B_t^m e_m, \quad t \geq 0, \quad (3.1)$$

where $\lambda > 1/2$ and the sequence $(\lambda_m)_{m \in \mathbb{N}_0}$ is given by $\lambda_0 := 1$ and $\lambda_m := m^{-\lambda}$ for $m \in \mathbb{N}$. Here, $\{(B_t^m)_{t \geq 0}\}_{m \in \mathbb{N}_0}$ are independent standard Brownian motions constructed on a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ (satisfying the usual conditions).

Our choice $\lambda > 1/2$ precludes the white noise and forces the sequence $(\lambda_m)_{m \in \mathbb{N}_0}$ to be square summable. In particular, the process $(W_t)_{t \geq 0}$ can be equivalently defined as an $L^2_{\text{sym}}(\mathbb{S})$ -valued Brownian motion with covariance function

$$Q : (f, g) \in (L^2_{\text{sym}}(\mathbb{S}))^2 \mapsto s \wedge t \sum_{m \in \mathbb{N}_0} \lambda_m^2 \hat{f}^m \hat{g}^m = s \wedge t \langle f, g \rangle_{2, -\lambda}. \quad (3.2)$$

Definition of the scheme. The approximation scheme is constructed via composition of the stochastic convolution associated with W and the rearrangement operator $*$ defined in Proposition 2.1. Given a stepsize $h \in (0, 1)$, we define $(X_n^h)_{n \in \mathbb{N}_0}$ by

$$X_{n+1}^h = \left(e^{h\Delta} X_n^h + \int_0^h e^{(h-s)\Delta} dW_s^{n+1} \right)^*, \quad X_0^h := X_0, \quad W_s^{n+1} := W_{s+nh} - W_{nh}, \quad (3.3)$$

where X_0 is a $U^2(\mathbb{S})$ -valued random variable assumed to be independent of $(W_t)_{t \geq 0}$ (see **Assumption on X_0** for a clear formulation). Note that measurability of X_{n+1}^h , seen as a random variable with values in $L^2_{\text{sym}}(\mathbb{S})$ (equipped with its Borel σ -field) is guaranteed by the continuity of the rearrangement operation (see Lemma 2.6).

In Subsection 3.1, the dependence of W^{n+1} on h is suppressed in the notation, since h is kept fixed. It is only in the forthcoming Subsection 3.2 that h becomes variable as we let the latter tend to 0.

Reminders about the stochastic convolution. For an \mathcal{F}_0 -measurable initial condition X_0 with values in $L^2_{\text{sym}}(\mathbb{S})$, the stochastic convolution provides a weak solution to the SHE (see Da Prato and Zabczyk [23, Ch.5] for a comprehensive introduction)

$$dX_t = \Delta X_t dt + dW_t, \quad t \geq 0,$$

written on $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$. That is to say, that for all $t \geq 0$ and $\varphi \in \mathcal{C}^2(\mathbb{S})$, the process $\hat{X} := (\hat{X}_t)_{t \geq 0}$ defined by

$$\hat{X}_t := e^{t\Delta} X_0 + \int_0^t e^{(t-s)\Delta} dW_s, \quad t \geq 0,$$

$$\text{satisfies } \mathbb{P}\text{-a.s., } \langle \hat{X}_t, \varphi \rangle = \langle X_0, \varphi \rangle + \int_0^t \langle \hat{X}_s, \Delta \varphi \rangle ds + \langle W_t, \varphi \rangle, \quad t \geq 0.$$

By [48, Theorem 2, p.146], the process \hat{X} has a version with continuous sample paths. From [77, Corollary 1, p.345], this version is adapted. Additionally, from [69,

Theorem 6, p.4], the following pathwise estimate holds for $p \geq 2$,

$$\|\hat{X}_t\|_2^p \leq \|\hat{X}_0\|_2^p + p \int_0^t \|\hat{X}_s\|_2^{p-2} \langle \hat{X}_s, dW_s \rangle + \frac{p(p-1)}{2} \int_0^t \|\hat{X}_s\|_2^{p-2} d[W]_s, \quad (3.4)$$

$$\text{with } [W]_t = \sum_{m \in \mathbb{N}_0} \lambda_m^2 t, \quad t \geq 0. \quad (3.5)$$

Due to [78, Theorem 2, p.147], see also [69, Theorem 5, p.4], for $p \geq 2$, $T > 0$,

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)\Delta} dW_s \right\|_2^p \right] \leq c_p \mathbb{E} [[W]_T^{p/2}]. \quad (3.6)$$

Subsection 3.1 is dedicated to proving estimates on the scheme that are uniform in the stepsize h . Tightness is addressed in Subsection 3.2.

Distributional derivative of the noise. For any $t \geq 0$, we let

$$w_t := DW_t = -2\pi \sum_{m \in \mathbb{N}} m^{1-\lambda} B_t^m e_m^{\mathcal{S}}, \quad (3.7)$$

which is a Brownian motion with values in $H_{\text{anti-sym}}^{-1}(\mathbb{S})$, the latter being defined as the dual of the space $H_{\text{anti-sym}}^1(\mathbb{S})$ of anti-symmetric periodic functions with a square-integrable generalised gradient.

Assumption on X_0 . Throughout the rest of the paper, we assume that X_0 is an \mathcal{F}_0 -measurable random variable with values in $U^2(\mathbb{S})$, satisfying

$$\forall p \geq 1, \quad \mathbb{E} \left[\|X_0\|_2^{2p} \right] < \infty. \quad (3.8)$$

3.1 L^p estimates of the solution

We start with some preliminary estimates for the L^p norm of the process $(X_n^h)_{n \in \mathbb{N}_0}$.

Lemma 3.1 For $p > 0$ (and for h being the stepsize of the scheme and λ the exponent colouring the noise),

$$\mathbb{E} \left[\left\| \int_0^h e^{(h-s)\Delta} dW_s \right\|_2^{2p} \right] \leq c_{p,\lambda} h^p; \text{ when } p = 1, \quad c_{1,\lambda} = \sum_{m \in \mathbb{N}_0} \lambda_m^2 = \frac{d}{dt} [W]_t. \quad (3.9)$$

Proof The proof is standard and follows from combining Theorem 4.36 in [23], p114 (refer to p.96 therein for related notation), with Fourier analysis and (3.1). \square

As a consequence, we have:

Lemma 3.2 For $T > 0$ and $p \geq 2$ (and for h being the stepsize of the scheme and λ the exponent colouring the noise),

$$\sup_{n \in \mathbb{N}_0: nh \leq T} \mathbb{E} \left[\|X_n^h\|_2^p \right] \leq c_{p,\lambda,T} (1 + \mathbb{E} [\|X_0\|_2^p]). \quad (3.10)$$

Proof The first step follows from the fact that the rearrangement preserves L^p norms.

$$\mathbb{E} \left[\|X_n^h\|_2^p \right] = \mathbb{E} \left[\left\| e^{h\Delta} X_{n-1}^h + \int_0^h e^{(h-s)\Delta} dW_s^n \right\|_2^p \right]. \quad (3.11)$$

The mild solution to the stochastic heat equation started from X_{n-1}^h and driven by $(W_r^n)_{0 \leq r \leq h}$ (see (3.3)) is denoted here by

$$\hat{X}_s^{h,n-1} := e^{s\Delta} X_{n-1}^h + \int_0^s e^{(s-r)\Delta} dW_r^n, \quad s \in [0, h]. \quad (3.12)$$

Then, by estimate (3.4),

$$\begin{aligned} & \mathbb{E} \left[\|X_n^h\|_2^p \right] \\ & \leq \mathbb{E} \left[\|X_{n-1}^h\|_2^p + p \int_0^h \|\hat{X}_s^{h,n}\|_2^{p-2} \langle \hat{X}_s^{h,n}, dW_s^n \rangle + \frac{p(p-1)}{2} \int_0^h \|\hat{X}_s^{h,n}\|_2^{p-2} d[W^n]_s \right]. \end{aligned} \quad (3.13)$$

One may remove the martingale terms (by induction over the index n in (3.11), the left-hand side therein is obviously finite, and then the left-hand side in (3.12) has a finite p -moment for any $p \geq 1$). It remains to control $\mathbb{E}[\|\hat{X}_s^{h,n}\|_2^q]$ for $q \geq 0$.

$$\|\hat{X}_s^{h,n}\|_2^q = \left\| e^{s\Delta} X_{n-1}^h + \int_0^s e^{(s-r)\Delta} dW_r^n \right\|_2^q \leq c_q \left(\|X_{n-1}^h\|_2^q + \left\| \int_0^s e^{(s-r)\Delta} dW_r^n \right\|_2^q \right), \quad (3.14)$$

using the contraction property of the heat semigroup. In light of Lemma 3.1,

$$\mathbb{E} \left[\|\hat{X}_s^{h,n}\|_2^q \right] \leq c_q \left(\mathbb{E} \left[\|X_{n-1}^h\|_2^q \right] + c_{q,\lambda} h^{q/2} \right). \quad (3.15)$$

Choosing $q = p - 2$ and injecting the above bound in (3.13), we obtain

$$\mathbb{E} \left[\|X_n^h\|_2^p \right] \leq \mathbb{E} \left[\|X_{n-1}^h\|_2^p \right] + c_{p,\lambda} h \mathbb{E} \left[\|X_{n-1}^h\|_2^{p-2} + h^{(p-2)/2} \right]. \quad (3.16)$$

The assumption $h < 1$ together with the bound $a^{p-2} \leq 1 + a^p$, for $a \geq 0$, gives

$$\mathbb{E} \left[\|X_n^h\|_2^p \right] \leq \left(1 + c_{p,\lambda} h \right) \mathbb{E} \left[\|X_{n-1}^h\|_2^p \right] + c_{p,\lambda} h.$$

The conclusion follows from the discrete version of Gronwall's lemma. \square

3.2 Tightness

For X_0 taking values in $U^2(\mathbb{S})$ and satisfying $\mathbb{E}[\|X_0\|_2^{2p}] < \infty$ for any $p \geq 1$, we address the tightness properties of the scheme, see Proposition 3.4 for the main statement. Whilst it would be possible to study tightness in $\mathcal{C}([0, \infty), L_{\text{sym}}^2(\mathbb{S}))$, it is in fact much simpler to work in $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}))$ (see for instance [38] for another use of H^{-1} in the analysis of McKean-Vlasov equation). To proceed, we define the following linear interpolation $(\tilde{X}_t^h)_{t \geq 0}$ of the scheme:

$$\tilde{X}_t^h := (\lceil t/h \rceil - t/h) X_{\lceil t/h \rceil}^h + (t/h - \lfloor t/h \rfloor) X_{\lfloor t/h \rfloor}^h. \quad (3.17)$$

Lemma 3.2 (applied with $T + 1$ instead of T) gives us the following bound:

Corollary 3.3 *For an initial condition $X_0 \in U^2(\mathbb{S})$ with finite moments of any order, for a real $T > 0$ and for any real $p \geq 1$, we have*

$$\sup_{t \leq T} \mathbb{E} \left[\|\tilde{X}_t^h\|_2^{2p} \right] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]} \quad (3.18)$$

Here is now the main result of this subsection:

Proposition 3.4 *For any finite time horizon $T > 1$, the linear interpolation schemes $\{\tilde{X}^h\}_{h \in (0,1)} := \{(\tilde{X}_t^h)_{t \geq 0}\}_{h \in (0,1)}$ have tight laws on $\mathcal{C}([0,T], H_{\text{sym}}^{-1}(\mathbb{S}))$. Moreover, for any $p \geq 1$, there exists a constant $C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}$, independent of h , such that*

$$\mathbb{E} \left[\sup_{n:nh \leq T+h} \|X_n^h\|_2^{2p} \right] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|_2^{2p}]}$$

Proof The proof is to verify Kolmogorov-Chentsov's criterion. Throughout the proof, we let $N_0 := \lceil T/h \rceil$.

First step. Consider the quantity $\sup_{n \in \{0, \dots, N_0\}} \|X_n^h - X_0\|_2$. By the triangle inequality,

$$\sup_{n \in \{0, \dots, N_0\}} \|X_n^h - X_0\|_2 \leq \sup_{n \in \{0, \dots, N_0\}} \left\{ \|X_n^h - e^{nh\Delta} X_0\|_2 + \|e^{nh\Delta} X_0 - X_0\|_2 \right\}.$$

To handle the first summand, one begins by use of Lemmas 2.6 and 2.9 and the contractive property of the heat semigroup:

$$\begin{aligned} \|X_n^h - e^{nh\Delta} X_0\|_2^2 &\leq \left\| e^{h\Delta} X_{n-1}^h - e^{nh\Delta} X_0 + \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\|_2^2 \\ &\leq \left\| X_{n-1}^h - e^{(n-1)h\Delta} X_0 \right\|_2^2 + \left\| \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\|_2^2 \\ &\quad + 2 \left\langle e^{h\Delta} X_{n-1}^h - e^{nh\Delta} X_0, \int_{(n-1)h}^{nh} e^{(nh-s)\Delta} dW_s \right\rangle. \end{aligned}$$

By iteration,

$$\begin{aligned} \|X_n^h - e^{nh\Delta} X_0\|_2^2 &\leq \sum_{k=1}^n \left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^2 \\ &\quad + 2 \sum_{k=1}^n \left\langle e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle \quad (3.19) \\ &=: T_n^1 + 2T_n^2, \end{aligned}$$

with the convention $T_0^1 = T_0^2 = 0$.

One now studies the regularity of the two discrete processes $(T_n^1)_{n \geq 0}$ and $(T_n^2)_{n \geq 0}$ (indexing by h is omitted).

For $(T_n^1)_{n \geq 0}$, observe that, for $p \geq 1$ and $0 \leq m < n$, by Lemma 3.1 and the generalised means inequality,

$$\mathbb{E} \left[|T_n^1 - T_m^1|^p \right] = \mathbb{E} \left[\left(\sum_{k=m+1}^n \left\| \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\|_2^2 \right)^p \right] \leq c_{p,\lambda} (h(n-m))^p.$$

We now turn to the process $(T_n^2)_{n \geq 0}$. It is a martingale. By the Burkholder-Davis-Gundy inequality,

$$\begin{aligned} \mathbb{E} \left[|T_n^2 - T_m^2|^p \right] &\leq \mathbb{E} \left[\left| \sum_{k=m+1}^n \left\langle e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle \right|^p \right] \\ &\leq \mathbb{E} \left[\left(\left[\sum_{k=m+1}^n \left\langle e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle \right]_n \right)^{p/2} \right], \end{aligned}$$

where the notation $[\cdot]_n$ denotes the quadratic variation up the n^{th} instant (note that here, this is from the $(m+1)^{\text{st}}$ instant). This may be estimated by writing

$$\begin{aligned} &\left[\sum_{k=m+1}^n \left\langle e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0, \int_{(k-1)h}^{kh} e^{(kh-s)\Delta} dW_s \right\rangle \right]_n \\ &= \sum_{k=m+1}^n \left(\left[e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0 \right]_0^2 h + \sum_{\ell \in \mathbb{N}} \left[e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0 \right]_{\ell}^2 \int_{(k-1)h}^{kh} \frac{e^{-8\pi^2(kh-s)\ell^2}}{\ell^{2\lambda}} ds \right) \\ &\leq h \sum_{k=m+1}^n \sum_{\ell \in \mathbb{N}_0} \left[e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0 \right]_{\ell}^2 = h \sum_{k=m+1}^n \left\| e^{h\Delta} X_{k-1}^h - e^{kh\Delta} X_0 \right\|_2^2. \end{aligned} \quad (3.20)$$

Applying the generalised means inequality and using, from Corollary 3.3, that $\sup_{k=0, \dots, N_0} \mathbb{E}[\|X_{kh}^h\|_2^p] \leq C_{p, \lambda, T, \mathbb{E}[\|X_0\|_2^2]}$, one obtains

$$\mathbb{E} \left[|T_n^2 - T_m^2|^p \right] \leq c_{p, \lambda, T, \mathbb{E}[\|X_0\|_2^2]} (h(n-m))^{\frac{p}{2}}.$$

Returning to (3.19), via application of the Kolmogorov-Chentsov continuity theorem [see Theorem 1.2.1 in [67]] (to the linear interpolation of the two processes, T^1 and T^2 in (3.19)), we deduce that, for $\alpha \in (0, (\frac{p}{2} - 1)/2p)$,

$$\|X_n^h - e^{nh\Delta} X_0\|_2 \leq \mathfrak{E}^h(nh)^\alpha, \quad n \in \{0, \dots, N_0\},$$

almost surely for a (non-negative) random variable \mathfrak{E}^h with a finite $L^{2p}(\mathbb{P})$ -moment that satisfies $\mathbb{E}[(\mathfrak{E}^h)^{2p}] \leq c_{p, \lambda, T, \mathbb{E}[\|X_0\|_2^2]}$. Consequently,

$$\|X_n^h - X_0\|_2 \leq \mathfrak{E}^h(nh)^\alpha + w(nh), \quad n \in \{0, \dots, N_0\},$$

where $w(x)$ is a random variable that depends on X_0 , that tends almost surely to 0 with x and that is dominated by $2\|X_0\|_2$. Notice that this implies in particular that

$$\mathbb{E} \left[\sup_{n: nh \leq T+h} \|X_n^h\|_2^{2p} \right] \leq c_{p, \lambda, T, \mathbb{E}[\|X_0\|_2^2]}^{2p}. \quad (3.21)$$

Second step. From here, we take $p \geq 2$. Differently from the first step, we now estimate the increments of the scheme in the space $H_{\text{sym}}^{-1}(\mathbb{S})$. To begin, apply the triangle and generalised means inequalities:

$$\begin{aligned} &\mathbb{E} \left[\left\| X_n^h - X_m^h \right\|_{2, -1}^{2p} \right] \\ &\leq 2^{2p-1} \mathbb{E} \left[\left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_{2, -1}^{2p} + \left\| X_m^h - e^{(n-m)h\Delta} X_m^h \right\|_{2, -1}^{2p} \right]. \end{aligned} \quad (3.22)$$

The second summand in the above right hand side is simpler to handle. By expanding in Fourier modes the left hand side below, one gets, for any $u \in L^2(\mathbb{S})$,

$$\|e^{(t-s)\Delta}u - u\|_{2,-1}^2 \leq c(t-s)\|u\|_2^2, \quad (3.23)$$

for a universal constant $c > 0$. This is enough for the desired estimation. For the first summand in (3.22), one proceeds via the following sequence of inequalities. Starting with Lemmas 2.6 and 2.9,

$$\begin{aligned} \left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_{2,-1}^{2p} &\leq \left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_2^{2p} \\ &\leq \left\| e^{h\Delta} X_{n-1}^h + \int_0^h e^{(h-s)\Delta} dW_s^n - e^{(n-m)h\Delta} X_m^h \right\|_2^{2p}, \end{aligned} \quad (3.24)$$

which may be estimated by means of (3.4), by considering the process

$$\hat{X}_s^{h,n-1} := e^{s\Delta} \left[X_{n-1}^h - e^{(n-1-m)h\Delta} X_m^h \right] + \int_0^s e^{(s-r)\Delta} dW_r^n, \quad s \in [0, h].$$

Following the same sequence of inequalities as in (3.13), (3.14), (3.15) and (3.16), we obtain¹

$$\begin{aligned} \mathbb{E} \left[\left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_2^{2p} \right] &\leq \mathbb{E} \left[\left\| X_{n-1}^h - e^{(n-1-m)h\Delta} X_m^h \right\|_2^{2p} \right] \\ &\quad + c_{p,\lambda} h \left(\mathbb{E} \left[\left\| X_{n-1}^h - e^{(n-1-m)h\Delta} X_m^h \right\|_2^{2p-2} \right] + h^{p-1} \right), \end{aligned}$$

which gives, by iteration,

$$\mathbb{E} \left[\left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_2^{2p} \right] \leq c_{p,\lambda} h \sum_{k=m+1}^{n-1} \left(\mathbb{E} \left[\left\| X_k^h - e^{(k-m)h\Delta} X_m^h \right\|_2^{2p-2} \right] + h^{p-1} \right).$$

We proceed by induction on p , assuming for a while that p is an integer (greater than or equal to 1). When $p = 1$, the above inequality yields $\mathbb{E}[\|X_n^h - e^{(n-m)h\Delta} X_m^h\|_2^2] \leq c_{1,\lambda} h(n-m)$. By induction, we get, for any $p \in \mathbb{N}$,

$$\mathbb{E} \left[\left\| X_n^h - e^{(n-m)h\Delta} X_m^h \right\|_2^{2p} \right] \leq c_{p,\lambda} (h(n-m))^p. \quad (3.25)$$

When $p \geq 1$, we apply (3.25) to $\lceil p \rceil$ and then get the conclusion for p by Hölder's inequality applied with exponent $\lceil p \rceil / p$. By (3.22), (3.23), (3.24) and (3.25), we obtain

$$\mathbb{E} \left[\left\| X_n^h - X_m^h \right\|_{2,-1}^{2p} \right] \leq c_{p,\lambda} (h(n-m))^p \left(1 + \mathbb{E} \left[\sup_{0 \leq k \leq n} \|X_k^h\|_2^{2p} \right] \right).$$

¹ Although the reader may find the computations reminiscent of (3.19), the objective is in fact different. In (3.19), the goal is to apply Kolmogorov-Chentsov's theorem to the process $(\|X_n^h - e^{nh\Delta} X_0\|_2^2)_{0 \leq \lfloor n/h \rfloor \leq T}$. The purpose here is obviously not the same.

Inserting the conclusion of the first step (see (3.21)), we have established that

$$\mathbb{E} \left[\|X_n^h - X_m^h\|_{2,-1}^{2p} \right] \leq c_{p,\lambda} (nh - mh)^p. \quad (3.26)$$

Conclusion. The conclusion of the first step (see (3.21) again) says that, for all $t \in [0, T]$, the family $\{\tilde{X}_t^h\}_{0 < h < 1}$ is tight on $H_{\text{sym}}^{-1}(\mathbb{S})$ (as bounded subsets of $L_{\text{sym}}^2(\mathbb{S})$ are relatively compact in $H_{\text{sym}}^{-1}(\mathbb{S})$). And the conclusion of the second step (see (3.26)) says that, for any $T > 0$ and any $\varepsilon \in (0, 1)$, the trajectories $((\tilde{X}_t^h)_{0 \leq t \leq T})_{0 < h < 1}$ live, with probability greater than $1 - \varepsilon$, in a set of equicontinuous trajectories from $[0, T]$ to $H_{\text{sym}}^{-1}(\mathbb{S})$. \square

4 Limiting Dynamics: Characterisation and Well-posedness

This section addresses the weak limits of the schemes. As discussed in the introduction, it is expected that those weak limits, say denoted by X , should satisfy a reflected stochastic differential equation in infinite dimension, understood in the sense,

$$dX_t = \Delta X_t dt + dW_t + d\eta_t, \quad t \geq 0, \quad (4.1)$$

for X_0 satisfying the standing assumption (3.8). Here, $(\eta_t)_{t \geq 0}$ should be understood as a forcing term that reflects X into $U^2(\mathbb{S})$. Equations of this type have been treated in [68], but in the absence of an integration by parts formula (analogous to [75, 76] for the models in [31, 61]) these results do not apply in our setting. Consequently, our approach follows the application of limit theorems to the tested/weak behaviour of the schemes. Below, we often refer to (4.1) - with accompanying conditions on the process η - as the **rearranged stochastic heat equation (or rearranged SHE)**. The reader impatient for the exact solution concept should skip momentarily to Definition 4.13, with caution that the fourth condition contains an integral that is defined en route.

The purpose of this section is thus to identify conditions satisfied by any weak limit that are, ultimately, sufficient to prove that the weak limit (of the schemes) is unique. This goal is reached in a series of five subsections. In Subsection 4.1, we prove that weak limits satisfy an equation of the form (4.1). Subsection 4.2 concerns the construction of an integral with respect to the reflection process $(\eta_t)_{t \geq 0}$. Notably this integral is non-decreasing with respect to $U^2(\mathbb{S})$ -valued integrand processes. Moreover, in Subsection 4.3 we establish an orthogonality property between X and η , key to proving uniqueness of the weak limit. The rigorous definition of a solution to (4.1) together with the main statement of its existence and uniqueness are given in Subsection 4.4, see Definition 4.13 and Theorem 4.15. We end the section with a proof of the Lipschitz regularity of the flow induced by the solution in Subsection 4.5.

4.1 Testing of the weak limits

Our analysis of the weak limits relies on the statement below, in which we use the notion of ‘time-locally bounded trajectories with respect to $\|\cdot\|_2$ (resp. $\|\cdot\|_{2,-2}$)’. For a normed vector space $(E, \|\cdot\|)$, a function $t \in [0, \infty) \mapsto f_t \in E$ is said to be ‘time-locally bounded with respect to $\|\cdot\|$ ’ if, for any $T > 0$, $\sup_{0 \leq t \leq T} \|f_t\| < \infty$.

Proposition 4.1 *Let $(X_t, W_t)_{t \geq 0}$ be a weak limit (over $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}) \times L_{\text{sym}}^2(\mathbb{S}))$ equipped with the topology of uniform convergence on compact subsets) of the processes $\{(\tilde{X}_t^h, W_t)_{t \geq 0}\}_{h>0}$ as h tends to 0, this weak limit being constructed on the same filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ as the scheme itself and the second component $(W_t)_{t \geq 0}$ of the weak limit abusively denoted the same as the noise in the scheme.*

Then, $(X_t, W_t)_{t \geq 0}$ is \mathbb{F} -adapted, $(X_t)_{t \geq 0}$ is $U^2(\mathbb{S})$ -valued (i.e., each X_t has symmetric and non-increasing values, see (2.2)) and has time-continuous trajectories with respect to $\|\cdot\|_{2,-1}$ and time-locally bounded trajectories with respect to $\|\cdot\|_2$, and $(W_t)_{t \geq 0}$ is an $L_{\text{sym}}^2(\mathbb{S})$ -valued Q -Brownian motion with respect to \mathbb{F} . Moreover, there exists an \mathbb{F} -adapted process $(\eta_t)_{t \geq 0}$ with values in $H_{\text{sym}}^{-2}(\mathbb{S})$, with time-continuous trajectories with respect to $\|\cdot\|_{2,-3}$ and time-locally bounded trajectories with respect to $\|\cdot\|_{2,-2}$, such that, with probability 1, for any $u \in H_{\text{sym}}^2(\mathbb{S})$:

1. $\forall t > s \geq 0, \quad \langle X_t - X_s, u \rangle = \int_s^t \langle X_r, \Delta u \rangle dr + \langle W_t - W_s, u \rangle + \langle \eta_t - \eta_s, u \rangle, \quad (4.2)$
2. *if u is non-increasing in the sense of Definition 2.2, then the path $(\langle \eta_t, u \rangle)_{t \geq 0}$ is non-decreasing (with t) and starts from 0 at time 0.*

The hypothesis that the weak limit can be constructed on the same space $(\Omega, \mathcal{A}, \mathbb{P})$ as in Section 3 can be made without loss of generality. In short, this just requires the probability space to be ‘rich enough’ (e.g., it is an atomless Polish probability space, on which we can construct arbitrarily distributed random variables with values in any other Polish space), which as additional assumption, is not a hindrance for us. The claim that $(\Omega, \mathcal{A}, \mathbb{P})$ can be equipped with the filtration \mathbb{F} requires a little more care: \mathbb{F} cannot be any given filtration, which is a common feature with weak limits of processes. We clarify the choice of \mathbb{F} in the proof below. We do this only for the convenience of using the same notation \mathbb{F} for this specific choice, as we are convinced that there is no risk of confusion for the reader. Similarly, denoting the second component of the weak limit by $(W_t)_{t \geq 0}$ is also rather abusive, but is justified by the fact that the second component’s law in any weak limit remains that of a Q -Brownian motion with values in $L_{\text{sym}}^2(\mathbb{S})$, see (3.1) and (3.2).

Remark 4.2 The proof of Proposition 4.1 shows that the shape of the process $(\eta_t)_{t \geq 0}$ in (4.2) can be further clarified. Indeed, denote by $(V_t)_{t \geq 0}$ the solution to the (usual) SHE with X_0 as initial condition and with $(W_t)_{t \geq 0}$ as driving noise (recalling that we keep this notation in the limit setting), i.e., the $L_{\text{sym}}^2(\mathbb{S})$ -valued process,

$$V_t := e^{t\Delta} X_0 + \int_0^t e^{(t-s)\Delta} dW_s, \quad t \geq 0, \quad (4.3)$$

and let $Y_t := X_t - V_t$. Then, for $t \geq 0$, one has, with probability 1, for any $v \in H_{\text{sym}}^2(\mathbb{S})$,

$$\forall t \geq 0, \quad \langle \eta_t, v \rangle = \langle Y_t, v \rangle - \int_0^t \langle Y_r, \Delta v \rangle dr. \quad (4.4)$$

The following formal argument (made rigorous below) gives intuition for item 2 of Proposition 4.1. For δ small, and v non-increasing.

$$\begin{aligned} \langle \eta_{t+\delta} - \eta_t, v \rangle &= \langle Y_{t+\delta} - Y_t, v \rangle - \int_t^{t+\delta} \langle Y_r, \Delta v \rangle dr \\ &\approx \langle Y_{t+\delta} - e^{\delta\Delta} Y_t, v \rangle = \left\langle X_{t+\delta} - e^{\delta\Delta} X_t - \int_t^{t+\delta} e^{(t+\delta-s)\Delta} dW_s, v \right\rangle \\ &= \left\langle \left(e^{\delta\Delta} X_t + \int_t^{t+\delta} e^{(t+\delta-s)\Delta} dW_s \right)^* - \left(e^{\delta\Delta} X_t + \int_t^{t+\delta} e^{(t+\delta-s)\Delta} dW_s \right), v \right\rangle \geq 0, \end{aligned}$$

the last line following from Lemma 2.5. To implement this argument onto the scheme, we let (with the same notation as in (3.3) for W^{n+1} and with V as in (4.3)):

$$V_n^h := V_{nh}, \quad \text{i.e.} \quad V_{n+1}^h = e^{h\Delta} V_n^h + \int_0^h e^{(h-s)\Delta} dW_s^{n+1}, \quad V_0^h = X_0. \quad (4.5)$$

The so-called shifted scheme, $X^h - V^h = (X_n^h - V_n^h)_{n \geq 0}$ is denoted $Y^h = (Y_n^h)_{n \geq 0}$, so that (3.3) may be rewritten as

$$\begin{aligned} X_{n+1}^h &= \left(V_{n+1}^h + e^{h\Delta} (X_n^h - V_n^h) \right)^*, \quad n \geq 0. \\ Y_{n+1}^h &= \left(V_{n+1}^h + e^{h\Delta} Y_n^h \right)^* - V_{n+1}^h, \quad n \geq 0. \end{aligned} \quad (4.6)$$

Following (3.17), we introduce the interpolations \tilde{Y}_t^h and \tilde{V}_t^h of Y^h and V^h . Then,

$$\tilde{Y}_t^h = \tilde{X}_t^h - \tilde{V}_t^h, \quad t \geq 0. \quad (4.7)$$

Proof (of Proposition 4.1.) Throughout the proof, we fix $T > 0$. It suffices to study the weak limits, as h tends to 0, of $\{\tilde{X}^h, W\}$ on $[0, T]$. For a given $h > 0$, with probability 1 and for any $u \in H_{\text{sym}}^3(\mathbb{S})$,

$$\begin{aligned} \langle Y_{n+1}^h - Y_n^h, u \rangle &= \langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u \rangle + \langle (e^{h\Delta} - I) Y_n^h, u \rangle \\ &= \langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u \rangle + \int_0^h \langle e^{s\Delta} Y_n^h, \Delta u \rangle ds, \end{aligned}$$

where we used the identity $\partial_s e^{s\Delta} = \Delta e^{s\Delta}$.

Rearranging, and working under the additional assumption that u is non-increasing, we use the rewritten shifted scheme (4.6) and Lemma 2.5 to show:

$$\begin{aligned} \langle Y_{n+1}^h - Y_n^h, u \rangle &- \int_0^h \langle e^{s\Delta} Y_n^h, \Delta u \rangle ds \\ &= \langle Y_{n+1}^h - e^{h\Delta} Y_n^h, u \rangle = \left\langle \left(V_{n+1}^h + e^{h\Delta} Y_n^h \right)^* - \left(V_{n+1}^h + e^{h\Delta} Y_n^h \right), u \right\rangle \geq 0, \end{aligned} \quad (4.8)$$

We rewrite the second term on the first line:

$$\int_0^h \langle e^{s\Delta} Y_n^h, \Delta u \rangle ds = \int_0^h \langle Y_n^h, (e^{s\Delta} - I) \Delta u \rangle ds + h \langle Y_n^h, \Delta u \rangle. \quad (4.9)$$

Summing over n and letting $N_r = \lfloor r/h \rfloor$, for $r > 0$, we get for any $(s, t) \in [0, T]^2$,

$$\begin{aligned} & \left| \sum_{n=N_s}^{N_t} \int_0^h \langle e^{r\Delta} Y_n^h, \Delta u \rangle dr - \int_s^t \langle \tilde{Y}_r^h, \Delta u \rangle dr \right| \\ & \leq \left| \sum_{n=N_s}^{N_t} \int_0^h \langle Y_n^h, (e^{r\Delta} - I) \Delta u \rangle dr \right| + \left| \sum_{n=N_s}^{N_t} h \langle Y_n^h, \Delta u \rangle - \int_s^t \langle \tilde{Y}_r^h, \Delta u \rangle dr \right| \\ & \leq c_T \sup_{0 \leq r \leq h} \| (e^{r\Delta} - I) \Delta u \|_2 \sup_{n \in \{0, \dots, N_r\}} \| Y_n^h \|_2 + \left| \sum_{n=N_s}^{N_t} h \langle Y_n^h, \Delta u \rangle - \int_s^t \langle \tilde{Y}_r^h, \Delta u \rangle dr \right| \\ & =: T_1^h(t) + T_2^h(s, t). \end{aligned} \quad (4.10)$$

Since $\Delta u \in L^2(\mathbb{S})$, we know that $\lim_{h \searrow 0} \sup_{0 \leq r \leq h} \| (e^{r\Delta} - I) \Delta u \|_2 = 0$. Together with Proposition 3.4 (recalling that $Y_n^h = (X_n^h - V_n^h)_{n \geq 0}$), we deduce that

$$\forall \varepsilon > 0, \quad \lim_{h \searrow 0} \mathbb{P} \left(\left\{ \sup_{0 \leq t \leq T} T_1^h(t) \geq \varepsilon \right\} \right) = 0. \quad (4.11)$$

Similarly, by tightness of $\{\tilde{X}^h\}_{h \in (0,1)}$ on $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}))$, we deduce that $\{\tilde{Y}^h\}_{h \in (0,1)}$ is also tight on $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}))$ (by (4.7)), from which we easily get that (since \tilde{Y}^h is the linear interpolation of Y^h and because $\Delta u \in H_{\text{sym}}^1(\mathbb{S})$)

$$\forall \varepsilon > 0, \quad \lim_{h \searrow 0} \mathbb{P} \left(\left\{ \sup_{0 \leq s < t \leq T} T_2^h(s, t) \geq \varepsilon \right\} \right) = 0. \quad (4.12)$$

Returning to (4.10), the last two displays (4.11) and (4.12) yield, for all $\varepsilon > 0$,

$$\lim_{h \searrow 0} \mathbb{P} \left(\sup_{0 \leq s < t \leq T} \left| \sum_{n=N_s}^{N_t} \int_0^h \langle e^{r\Delta} Y_n^h, \Delta u \rangle dr - \int_s^t \langle \tilde{Y}_r^h, \Delta u \rangle dr \right| \geq \varepsilon \right) = 0. \quad (4.13)$$

It remains to insert (4.13) into (4.8), by summing the (non-negative) left-hand side of (4.8) from N_s to N_t and subtracting this from the main term in (4.13). This supplies us with a term $T_3^h(s, t)$ such that

$$\forall (s, t) \in [0, T]^2 : s < t, \quad \langle \tilde{Y}_t^h - \tilde{Y}_s^h, u \rangle - \int_s^t \langle \tilde{Y}_r^h, \Delta u \rangle dr \geq T_3^h(s, t), \quad (4.14)$$

$$\text{and } \forall \varepsilon > 0, \quad \lim_{h \searrow 0} \mathbb{P} \left(\sup_{0 \leq s < t \leq T} |T_3^h(s, t)| \geq \varepsilon \right) = 0. \quad (4.15)$$

Now we let h tend to 0. Following the statement, we slightly abuse notation and write $(X_t, W_t)_{0 \leq t \leq T}$ a weak limit of $\{(\tilde{X}_t^h, W_t)_{0 \leq t \leq T}\}_{h \in (0,1)}$. We denote by \mathbb{F} , the usual augmentation of the filtration generated by $(X_t, W_t)_{0 \leq t \leq T}$. Clearly, $(W_t)_{0 \leq t \leq T}$ is an $L_{\text{sym}}^2(\mathbb{S})$ -valued Q -Brownian motion with respect to \mathbb{F} . Along the same subsequence,

$\{(\tilde{X}_t^h, W_t, \tilde{V}_t^h, \tilde{Y}_t^h)_{0 \leq t \leq T}\}_{h \in (0,1]}$ converges to $(X_t, W_t, V_t, Y_t)_{0 \leq t \leq T}$, where $(V_t)_{0 \leq t \leq T}$ solves the SHE (4.3) (driven by the limit process W) and $Y_t = X_t - V_t$, for $t \in [0, T]$ (in particular, $Y_0 = 0$). Obviously, $(X_t, W_t, V_t, Y_t)_{0 \leq t \leq T}$ is \mathbb{F} -adapted. By Proposition 2.10, $(X_t)_{0 \leq t \leq T}$ takes values in $U^2(\mathbb{S})$ (since $(\tilde{X}_t^h)_{0 \leq t \leq T}$ does, for any $h > 0$) in the sense that, with probability 1, for all $t \geq 0$, X_t belongs to $U^2(\mathbb{S})$. By construction, it is continuous with respect to $\|\cdot\|_{2,-1}$. Using Proposition 3.4 together with the fact that the mapping $(x_t)_{0 \leq t \leq T} \mapsto \sup_{0 \leq t \leq T} \|x_t\|_2$ is lower semi-continuous with respect to the uniform convergence topology on $\mathcal{C}([0, T], H_{\text{sym}}^{-1}(\mathbb{S}))$, we deduce that, with probability 1, $\sup_{0 \leq t \leq T} \|X_t\|_2 < \infty$. In turn, $(Y_t)_{0 \leq t \leq T}$ is valued in $L_{\text{sym}}^2(\mathbb{S})$, with bounded trajectories (the bound being random), because $(V_t)_{0 \leq t \leq T}$ has continuous trajectories with values in $L_{\text{sym}}^2(\mathbb{S})$. Moreover, using (4.14) and (4.15), we obtain, when $u \in H_{\text{sym}}^3(\mathbb{S})$ is non-increasing,

$$\forall (s, t) \in [0, T]^2 : s < t, \quad \langle Y_t - Y_s, u \rangle - \int_s^t \langle Y_r, \Delta u \rangle dr \geq 0. \quad (4.16)$$

The above is true, for any $u \in H_{\text{sym}}^3(\mathbb{S})$, with probability 1, for all $t > s \geq 0$. By a separability argument, it is true with probability 1, for all $u \in H_{\text{sym}}^3(\mathbb{S})$ and for all $t > s \geq 0$. And, then by a new density argument (using the fact that $(Y_t)_{t \geq 0}$ has time-locally bounded trajectories with respect to $\|\cdot\|_2$), it is true with probability 1, for all $u \in H_{\text{sym}}^2(\mathbb{S})$ and for all $t > s \geq 0$. This prompts us to let, for any $v \in H_{\text{sym}}^2(\mathbb{S})$,

$$\forall t \in [0, T], \quad \langle \eta_t, v \rangle := \langle Y_t, v \rangle - \int_0^t \langle Y_r, \Delta v \rangle dr.$$

Obviously, the processes $(\eta_t)_{0 \leq t \leq T}$ has bounded trajectories with respect to $\|\cdot\|_{2,-2}$ and continuous trajectories with respect to $\|\cdot\|_{2,-3}$. By (4.16), we deduce that, a.s., for any $u \in H_{\text{sym}}^2(\mathbb{S}) \cap U^2(\mathbb{S})$, $(\langle \eta_t, u \rangle)_{0 \leq t \leq T}$ is non-decreasing. This proves item 2 in the statement. Moreover, by replacing $(Y_t)_{0 \leq t \leq T}$ by $(X_t - V_t)_{0 \leq t \leq T}$ in the definition of $(\eta_t)_{0 \leq t \leq T}$ and by recalling from (4.3) that, for any $v \in H^2(\mathbb{S})$,

$$\langle V_t, v \rangle - \int_0^t \langle V_r, \Delta v \rangle dr = \langle e^{t\Delta} X_0, v \rangle + \int_0^t e^{(t-r)\Delta} dW_r,$$

we easily verify item 1 in the statement, completing the proof. \square

In fact, the local in time boundedness property of the trajectories of $(X_t)_{t \geq 0}$ (in $\|\cdot\|_2$) and $(\eta_t)_{t \geq 0}$ (in $\|\cdot\|_{2,-2}$) can be improved. Using again Proposition 3.4 and the notation defined in Remark 4.2 together with the fact that the mapping $(x_t)_{0 \leq t \leq T} \mapsto \sup_{0 \leq t \leq T} \|x_t\|_2$ is lower semi-continuous with respect to the uniform convergence topology on $\mathcal{C}([0, T], H_{\text{sym}}^{-1}(\mathbb{S}))$, we have

Proposition 4.3 *For any $p \geq 1$ and any $T > 0$, there exists constant $C_{p,\lambda,T,\mathbb{E}[\|X_0\|^{2p}]}$, such that for any weak limit X as in Proposition 4.1 and Y as in Remark 4.2,*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|X_t\|_2^{2p} \right] + \mathbb{E} \left[\sup_{t \in [0, T]} \|Y_t\|_2^{2p} \right] \leq C_{p,\lambda,T,\mathbb{E}[\|X_0\|^{2p}]}.$$

4.2 Integral with respect to the reflection process

Our next objective is to construct an integral with respect to the reflection process $(\eta_t)_{t \geq 0}$ identified in the statement of Proposition 4.1. We make use of the resulting integral in order to establish uniqueness of the weak limits obtained in Proposition 4.1.

In the construction, we use the fact that, with probability 1, the path $(\eta_t)_{t \geq 0}$ satisfies the forthcoming two assumptions **(E1)** and **(E2)**, which are spelled out as follows for a **deterministic** trajectory $(n_t)_{t \geq 0}$:

- (E1)** $t \mapsto n_t$ is a function from $[0, \infty)$ to $H_{\text{sym}}^{-2}(\mathbb{S})$, locally bounded with respect to $\|\cdot\|_{2,-2}$ and continuous with respect to $\|\cdot\|_{2,-3}$;
- (E2)** For any $u \in H_{\text{sym}}^2(\mathbb{S}) \cap U^2(\mathbb{S})$, the function $t \in [0, \infty) \mapsto \langle \eta_t, u \rangle$ is non-decreasing.

The integral we construct below holds for a path $(n_t)_{t \geq 0}$ satisfying only the two assumptions **(E1)** and **(E2)**. In particular, the integral with respect to $(\eta_t)_{t \geq 0}$ is then obtained pathwise, by choosing $(n_t)_{t \geq 0}$ as the current realisation of $(\eta_t)_{t \geq 0}$. Now, let $u \in H_{\text{sym}}^2(\mathbb{S}) \cap U^2(\mathbb{S})$. Even though $(n_t)_{t \geq 0}$ is continuous with respect to the weaker norm $\|\cdot\|_{2,-3}$, the fact that it also takes values in $H_{\text{sym}}^{-2}(\mathbb{S})$ implies that $t \in [0, \infty) \mapsto \langle \eta_t, u \rangle$ is continuous. Then, if we consider in addition another continuous **deterministic** trajectory $(z_t)_{t \geq 0}$ valued in $L_{\text{sym}}^2(\mathbb{S})$, **(E2)** allows us to define

$$\left(\int_0^t \langle z_r, u \rangle d\langle n_r, u \rangle \right)_{t \geq 0} \quad (4.17)$$

as a time-continuous Riemann-Stieltjes integral. From this, we want to give a meaning to the as yet informally written integrals $(\int_0^t z_r \cdot dn_r)_{t \geq 0}$, where the dot \cdot in the notation is intended to denote a form of duality presence between the integrand and the integrator. Our definition of the integral is done by analogy with Parseval's identity, setting u in (4.17) to be (cosine) elements in the Fourier basis. The next step is to expand $\int_0^t z_r \cdot dn_r$ along the (cosine) Fourier basis $(e_m)_{m \in \mathbb{N}_0}$, noticing that one may decompose each e_m as the difference of two elements of $U^2(\mathbb{S})$, e_m^+ and e_m^- , with

$$e_m^\pm(x) := e_m(0)t + \int_0^x \left[-\mathbb{1}_{(-1/2, 0]}(y) (De_m(y))_\mp + \mathbb{1}_{[0, 1/2)}(y) (De_m(y))_\pm \right] (y) dy,$$

with $\iota = 1$ if $\pm = +$ and 0 if $\pm = -$. The functions e_m^+ and e_m^- are in $U^2(\mathbb{S})$ (courtesy of the symmetry properties of e_m) and $e_m = e_m^+ - e_m^-$. Therefore, one may set:

$$\int_s^t \langle z_r, e_m \rangle d\langle n_r, e_m \rangle := \int_s^t \langle z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_s^t \langle z_r, e_m \rangle d\langle n_r, e_m^- \rangle. \quad (4.18)$$

For any $\varepsilon > 0$, $(z_r)_{r \geq 0}$ can be replaced by $(e^{\varepsilon \Delta} z_r)_{r \geq 0}$ in (4.18), which is a consequence of Lemma 2.9. The following statement is key in the construction of our integral (4.2).

Lemma 4.4 *For any $k \in \mathbb{N}$ and any $\varepsilon > 0$, there exists a constant $c_{k, \varepsilon}$ such that, for any two (deterministic) curves $(n_t)_{t \geq 0}$ and $(z_t)_{t \geq 0}$, with $(n_t)_{t \geq 0}$ satisfying **(E1)** and **(E2)** and with $(z_t)_{t \geq 0}$ a continuous path in $L_{\text{sym}}^2(\mathbb{S})$, and any $T \geq 0$ and $m \in \mathbb{N}_0$,*

$$\sup_{t \in [0, T]} \left| \int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^\pm \rangle \right| \leq \frac{c_{k, \varepsilon}}{m^k \vee 1} \|n_T\|_{2,-2} \sup_{t \in [0, T]} \|z_t\|_2. \quad (4.19)$$

When $(n_t)_{t \geq 0}$ is understood as a realisation of $(\eta_t)_{t \geq 0}$, the term $\|n_T\|_{2,-2}$ becomes $\|\eta_T\|_{2,-2}$ and, with $(Y_t)_{t \geq 0}$ as in (4.4), it can be upper bounded by

$$\|\eta_T\|_{2,-2} \leq c_T \sup_{t \in [0, T]} \|Y_t\|_2. \quad (4.20)$$

In (4.19), we use the notation e_m^\pm to indicate that the result holds true with both e_m^+ and e_m^- . Also, note that the L^2 contributions of e_m^+ and e_m^- diverge with m . This is precisely the reason why we consider integrands of the form $(e^{\varepsilon \Delta} z_t)_{t \geq 0}$, since the heat kernel forces the higher modes of the convolution to decay exponentially fast. In brief, for any k and ε as in the statement, we can find two constants c_k and $c_{k,\varepsilon}$ such that,

$$\forall m \in \mathbb{N}, \forall r \geq 0, \quad |\langle e^{\varepsilon \Delta} z_r, e_m \rangle| \leq c_k |\langle D^{2k} e^{\varepsilon \Delta} z_r, \frac{1}{m^{2k}} e_m \rangle| \leq c_{k,\varepsilon} \frac{1}{m^{2k}} \|z_r\|_2. \quad (4.21)$$

Obviously, the proof of Lemma 4.4 relies on the bound (4.21), whence appears the constant $c_{k,\varepsilon}$ in the statement.

Proof (Proof of Lemma 4.4.) We begin with the following simple observation. It is easy to verify that each e_m^\pm belongs to $H_{\text{sym}}^2(\mathbb{S})$ with $\|e_m^\pm\|_{2,2} \leq c(m^2 \vee 1)$.

By **(E2)**, one can use standard properties of the Riemann-Stieltjes integral:

$$\begin{aligned} \sup_{t \in [0, T]} \left| \int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^\pm \rangle \right| &\leq \sup_{t \in [0, T]} |\langle e^{\varepsilon \Delta} z_t, e_m \rangle| \times \langle n_T, e_m^\pm \rangle \\ &\leq c_{k,\varepsilon} \frac{m^2 \vee 1}{m^{2k} \vee 1} \|n_T\|_{2,-2} \sup_{t \in [0, T]} \|z_t\|_2, \end{aligned} \quad (4.22)$$

with the last line following from (4.21) together with the bound $\|e_m^\pm\|_{2,2} \leq c(m^2 \vee 1)$. This shows (4.19). As for the proof of (4.20), we just make use of (4.4). \square

Lemma 4.4 allows us to make the following definition:

Definition 4.5 For two (deterministic) curves $(n_t)_{t \geq 0}$ and $(z_t)_{t \geq 0}$, with $(n_t)_{t \geq 0}$ satisfying **(E1)** and **(E2)** and with $(z_t)_{t \geq 0}$ being a continuous function from $[0, \infty)$ to $L_{\text{sym}}^2(\mathbb{S})$, we can define, almost surely, for any $\varepsilon > 0$ the integral process $(\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r)_{t \geq 0}$ as the limit, for the uniform topology on compact subsets:

$$\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r := \lim_{M \rightarrow \infty} \sum_{m=0}^M \left(\int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^- \rangle \right).$$

It satisfies

$$\forall T \geq 0, \quad \sup_{t \in [0, T]} \left| \int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r \right| \leq c_\varepsilon \|n_T\|_{2,-2} \times \sup_{t \in [0, T]} \|z_t\|_2. \quad (4.23)$$

When $(n_t)_{t \geq 0}$ is understood as a realisation of $(\eta_t)_{t \geq 0}$, the term $\|n_T\|_{2,-2}$ becomes $\|\eta_T\|_{2,-2}$ and can be upper bounded as in (4.20).

Remark 4.6 The following two remarks are in order:

1. In Definition 4.5, not only is the convergence uniform in time in a fixed segment $[0, T]$, for some $T > 0$, but it is also uniform with respect to $(z_t)_{0 \leq t \leq T}$ when the latter is required to satisfy $\sup_{t \in [0, T]} \|z_t\|_2 \leq A$ for some given $A > 0$. This is a direct consequence of the form of the rate of convergence given by (4.19).
2. Lemma 4.4 and Definition 4.5 extend to the case when $(z_t)_{t \geq 0}$ is piecewise constant (i.e., there exists an increasing locally-finite sequence of time indices $(t_k)_{k \geq 0}$, with $t_0 = 0$, such that $t \in [t_k, t_{k+1}) \mapsto z_t \in L^2_{\text{sym}}(\mathbb{S})$ is constant for each $k \geq 0$).

The following lemma explains the interest of the second remark right above.

Lemma 4.7 *Within the same framework as in Definition 4.5 but with $(z_t)_{t \geq 0}$ therein being piecewise constant (with the same jumping times $(t_k)_{k \geq 0}$ as in Remark 4.6), $(\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r)_{t \geq 0}$ coincides with the process defined by standard Riemann sums, i.e.,*

$$\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r = \sum_{k \geq 0: t_k \leq t} \langle e^{\varepsilon \Delta} z_{t_k}, \eta_{t_{k+1} \wedge t} - \eta_{t_k} \rangle, \quad t \geq 0.$$

In particular, if z_{t_k} , for each $k \geq 0$, is symmetric non-increasing, then

$$\forall t \geq 0, \quad \int_0^t e^{\varepsilon \Delta} z_r \cdot d\eta_r \geq 0.$$

Before we prove Lemma 4.7, we state the following important corollary.

Corollary 4.8 *Within the same framework as in Definition 4.5, with $(z_t)_{t \geq 0}$ therein being continuous, we let, for any $k \in \mathbb{N}$, $z^k = (z_t^k)_{t \geq 0}$ be the piecewise constant approximation of $z = (z_t)_{t \geq 0}$ of stepsize $1/k$, namely $z_t^k := z_{\lfloor kt \rfloor / k}$, for $t \geq 0$. Then, the following convergence holds true, uniformly on compact subsets,*

$$\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r = \lim_{k \rightarrow \infty} \int_0^t e^{\varepsilon \Delta} z_r^k \cdot dn_r, \quad t \geq 0.$$

In particular, if z_t is in $U^2(\mathbb{S})$ for each $t \geq 0$, then, for all $t \geq 0$, $\int_0^t e^{\varepsilon \Delta} z_r \cdot dn_r \geq 0$.

Proof (of Lemma 4.7) Back to Definition 4.5 - but for a path of the type discussed in the second item of Remark 4.6 - we then observe that, for any integer $M \geq 1$,

$$\begin{aligned} & \sum_{m=0}^M \left(\int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^+ \rangle - \int_0^t \langle e^{\varepsilon \Delta} z_r, e_m \rangle d\langle n_r, e_m^- \rangle \right) \\ &= \sum_{m=0}^M \sum_{k \geq 0: t_k \leq t} \langle e^{\varepsilon \Delta} z_{t_k}, e_m \rangle \left[\left(\langle n_{t_{k+1} \wedge t}, e_m^+ \rangle - \langle n_{t_k}, e_m^+ \rangle \right) - \left(\langle n_{t_{k+1} \wedge t}, e_m^- \rangle - \langle n_{t_k}, e_m^- \rangle \right) \right] \\ &= \sum_{m=0}^M \sum_{k \geq 0: t_k \leq t} \langle e^{\varepsilon \Delta} z_{t_k}, e_m \rangle \left(\langle n_{t_{k+1} \wedge t}, e_m \rangle - \langle n_{t_k}, e_m \rangle \right). \end{aligned}$$

Calling $I_M(t)$ the sum in the first line and exchanging the sums in the last line, we get

$$I_M(t) = \sum_{k \geq 0: t_k \leq t} \sum_{m=0}^M \left(\langle e^{\varepsilon \Delta} z_{t_k}, e_m \rangle \left(\langle n_{t_{k+1} \wedge t}, e_m \rangle - \langle n_{t_k}, e_m \rangle \right) \right).$$

Since, for each $k \in \mathbb{N}$, n_{t_k} belongs to $H_{\text{sym}}^{-2}(\mathbb{S})$ and $e^{\varepsilon\Delta} z_{t_k}$ to $H_{\text{sym}}^2(\mathbb{S})$, we have

$$\forall k \in \mathbb{N}, \quad \lim_{M \rightarrow \infty} \sum_{m=0}^M \left(\langle e^{\varepsilon\Delta} z_{t_k}, e_m \rangle \left(\langle n_{t_{k+1} \wedge t}, e_m \rangle - \langle \eta_{t_k}, e_m \rangle \right) \right) = \langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1} \wedge t} - n_{t_k} \rangle,$$

from which we easily deduce that

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r := \lim_{M \rightarrow \infty} I_M(t) = \sum_{k:t_k \leq t} \langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1} \wedge t} - n_{t_k} \rangle,$$

the convergence being uniform with respect to t in compact subsets.

Whenever z_{t_k} is in $U^2(\mathbb{S})$, so is $e^{\varepsilon\Delta} z_{t_k}$, see Lemma 2.9. By the second item in Proposition 4.1, we then obtain that $\langle e^{\varepsilon\Delta} z_{t_k}, n_{t_{k+1} \wedge t} - n_{t_k} \rangle \geq 0$. \square

It remains to check Corollary 4.8.

Proof (Proof of Corollary 4.8) The first claim in the statement of Corollary 4.8 is a consequence of (4.23), using the linearity of the integral, which says that

$$\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r - \int_0^t e^{\varepsilon\Delta} z_r^k \cdot dn_r = \int_0^t e^{\varepsilon\Delta} (z_r - z_r^k) \cdot dn_r, \quad t \in [0, T],$$

together with the fact that $\lim_{k \rightarrow \infty} \sup_{0 \leq t \leq T} \|z_t - z_t^k\|_2 = 0$. As for the second claim, it follows from Lemma 4.7. \square

Remark 4.9 Notice that (with the presence of the factor 2 in the exponential below)

$$\begin{aligned} \int_0^t e^{2\varepsilon\Delta} z_r \cdot dn_r &= \int_0^t e^{\varepsilon\Delta} z_r \cdot d(e^{\varepsilon\Delta} n_r) = \lim_{M \rightarrow \infty} \sum_{m=0}^M \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle e^{\varepsilon\Delta} \eta_r, e_m \rangle \\ &= \lim_{M \rightarrow \infty} \sum_{m=0}^M \int_0^t \langle e^{\varepsilon\Delta} z_r, e_m \rangle d\langle \eta_r, e^{\varepsilon\Delta} e_m \rangle, \end{aligned}$$

where it must be stressed that $(e^{\varepsilon\Delta} n_t)_{t \geq 0}$ on the first line satisfies **(E1)** and **(E2)**. While **(E1)** follows from the contractive properties of the heat semigroup, **(E2)** follows from Lemma 2.9 (in words, $e^{\varepsilon\Delta} u \in U^2(\mathbb{S})$ if $u \in U^2(\mathbb{S})$).

A proof of the above identity is as follows. By Corollary 4.8 (and with the same notation), we can write $\int_0^t e^{2\varepsilon\Delta} z_r \cdot dn_r = \lim_{k \rightarrow \infty} \int_0^t e^{2\varepsilon\Delta} z_r^k \cdot dn_r$. Then, Lemma 4.7 allows one to write the right-hand side as a Riemann sum. The proof is completed by expanding the terms in the Riemann sum in Fourier coefficients, exactly as in the proof of Lemma 4.7.

Remark 4.10 Definition 4.5 supplies us with the integral $(\int_0^t e^{\varepsilon\Delta} z_r \cdot dn_r)_{t \geq 0}$, when $(z_t)_{t \geq 0}$ is a deterministic continuous path with values in $L_{\text{sym}}^2(\mathbb{S})$ and $(n_t)_{t \geq 0}$ satisfies **(E1)** and **(E2)**. Importantly, one can replace $(z_t)_{t \geq 0}$ by the realisation of a (stochastic) continuous process $(Z_t)_{t \geq 0}$ with values in $L_{\text{sym}}^2(\mathbb{S})$ and $(n_t)_{t \geq 0}$ by the same process $(\eta_t)_{t \geq 0}$ as in Proposition 4.1. The integral is denoted $(\int_0^t e^{\varepsilon\Delta} Z_r \cdot d\eta_r)_{t \geq 0}$. It is continuous in time. When $(Z_t)_{t \geq 0}$ is adapted to the filtration \mathbb{F} used in the statement of Proposition 4.1, the integral is also adapted to \mathbb{F} , due to Lemma 4.7 and Corollary 4.8.

4.3 Orthogonality of the reflection

We now come to the last property in the description of the weak limits:

Proposition 4.11 *Let $(X_t, W_t)_{t \geq 0}$ be a weak limit of the processes $\{(\tilde{X}_t^h, W_t)_{t \geq 0}\}_{h > 0}$ as h tends to 0, as given by Proposition 4.1. Then, for any $t \geq s \geq 0$,*

$$\lim_{\varepsilon \searrow 0} \mathbb{E} \left[\int_s^t e^{\varepsilon \Delta} X_r \cdot d\eta_r \right] = 0. \quad (4.24)$$

To appreciate the scope of the above statement, the reader should recall that $(X_t)_{t \geq 0}$ takes values in $U^2(\mathbb{S})$. Therefore, Corollary 4.8 yields, almost surely, $\int_s^t e^{\varepsilon \Delta} X_r \cdot d\eta_r \geq 0$, for any $t \geq s \geq 0$. In particular, Fatou's lemma (together with the time continuity of the integral) implies that, with probability 1, for any $t \geq s \geq 0$,

$$\liminf_{\varepsilon \searrow 0} \int_s^t e^{\varepsilon \Delta} X_r \cdot d\eta_r = 0.$$

We regard this property as a (weak) form of orthogonality between X_r and $d\eta_r$, recalling that the orthogonality property is standard in reflected equations (see for instance the seminal work [52]). Moreover, we obtain the following corollary as an important by-product of the proof of Proposition 4.11.

Corollary 4.12 *Let $(X_t, W_t)_{t \geq 0}$ be a weak limit of the processes $\{(\tilde{X}_t^h, W_t)_{t \geq 0}\}_{h > 0}$ as h tends to 0, as given by Proposition 4.1. Then, with probability 1, the process $(X_t)_{0 \leq t \leq T}$ has time-continuous trajectories with respect to $\|\cdot\|_2$ and, for any $T > 0$, it takes values in $L^2([0, T], H_{\text{sym}}^1(\mathbb{S}))$ with $\mathbb{E} \int_0^T \|DX_t\|_2^2 dt < \infty$. With probability 1, the process $(\eta_t)_{0 \leq t \leq T}$ has time-continuous trajectories with respect to $\|\cdot\|_{2,-2}$.*

Proof (of Proposition 4.11 and Corollary 4.12)

First step. We first prove that $(X_t)_{0 \leq t \leq T}$ takes values in $L^2([0, T], H_{\text{sym}}^1(\mathbb{S}))$ (which is not a direct corollary of Proposition 4.11, but which comes as a consequence of the global architecture of the proof). In order to do so, we return to the scheme (3.3). For a given $h \in (0, 1]$ and for any integer $n \geq 0$,

$$\mathbb{E} [\|X_{n+1}^h\|_2^2] \leq \mathbb{E} [\|e^{h\Delta} X_n^h\|_2^2] + \mathbb{E} \left[\left\| \int_0^h e^{(h-s)\Delta} dW_s^{n+1} \right\|_2^2 \right]. \quad (4.25)$$

Then, using the fact that $(e^{t\Delta} X_n^h)_{0 \leq t \leq h}$ solves the heat equation, one has the equality

$$\mathbb{E} \left[\|e^{h\Delta} X_n^h\|_2^2 + 2 \int_0^h \|De^{s\Delta} X_n^h\|_2^2 ds \right] = \mathbb{E} [\|X_n^h\|_2^2]. \quad (4.26)$$

Considering $h \leq \varepsilon$ for some $\varepsilon \in (0, 1)$, we have $\mathbb{E} [\|e^{\varepsilon \Delta} DX_n^h\|_2^2] \leq \mathbb{E} [\|De^{s\Delta} X_n^h\|_2^2]$ for $s \in (0, h]$. By (4.25), (4.26), and Lemma 3.1 (for the definition of $c_{1,\lambda}$), we get

$$\mathbb{E} [\|X_{n+1}^h\|_2^2] - \mathbb{E} [\|X_n^h\|_2^2] + 2h\mathbb{E} [\|De^{\varepsilon \Delta} X_n^h\|_2^2] \leq c_{1,\lambda} h.$$

And then, for $t \geq s \geq 0$ and $N_t := \lfloor t/h \rfloor$ and $N_s := \lfloor s/h \rfloor$, we have

$$\mathbb{E}[\|X_{N_t}^h\|_2^2] - \mathbb{E}[\|X_{N_s}^h\|_2^2] + 2h \sum_{n=N_s}^{N_t-1} \mathbb{E}[\|De^{\varepsilon\Delta} X_n^h\|_2^2] \leq c_{1,\lambda} h(N_t - N_s).$$

Choosing $s = 0$, lower bounding $\|X_{N_t}^h\|_2^2$ by $\|e^{\varepsilon\Delta} X_{N_t}^h\|_2^2$, recalling the notation (3.17) and combining tightness of the family $\{(\tilde{X}_t^h)_{t \geq 0}\}_{h \in (0,1]}$ in $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}))$ with Corollary 3.3 (which supplies us with uniform integrability properties), we obtain

$$\mathbb{E}[\|e^{\varepsilon\Delta} \tilde{X}_t^h\|_2^2] - \mathbb{E}[\|X_0\|_2^2] + 2 \int_0^t \mathbb{E}[\|De^{\varepsilon\Delta} \tilde{X}_r^h\|_2^2] dr \leq c_{1,\lambda} t + o_h(1), \quad (4.27)$$

with $\lim_{h \searrow 0} o_h(1) = 0$ (the rate possibly depending on ε). Noticing that the function $z \mapsto D[e^{\varepsilon\Delta} z]$ is continuous from $H_{\text{sym}}^{-1}(\mathbb{S})$ into $L_{\text{sym}}^2(\mathbb{S})$, we can easily take some weak limit as in the statement (as h tends to 0). We get

$$\mathbb{E}[\|e^{\varepsilon\Delta} X_t\|_2^2] - \mathbb{E}[\|X_0\|_2^2] + 2 \int_0^t \mathbb{E}[\|De^{\varepsilon\Delta} X_r\|_2^2] dr \leq c_{1,\lambda} t. \quad (4.28)$$

Since $c_{1,\lambda}$ is independent of ε , this establishes $\mathbb{E} \int_0^T \|DX_t\|_2^2 dt < \infty$.

Second step. Next, return to equation (4.2), with $u \in H_{\text{sym}}^2(\mathbb{S})$ replaced by $e^{\varepsilon\Delta} u$,

$$\langle X_t - X_s, e^{\varepsilon\Delta} u \rangle - \int_s^t \langle X_s, \Delta e^{\varepsilon\Delta} u \rangle ds = \langle W_t - W_s, e^{\varepsilon\Delta} u \rangle + \langle \eta_t - \eta_s, e^{\varepsilon\Delta} u \rangle. \quad (4.29)$$

The next step is to choose $u = e_m$ and then to apply Itô's formula in order to expand $(\langle X_t, e^{\varepsilon\Delta} e_m \rangle)_{t \geq 0}$. To do so, it is worth recalling from (4.18) that since $e_m = e_m^+ - e_m^-$ with $e_m^\pm \in U^2(\mathbb{S})$, the process $(\langle \eta_t, e^{\varepsilon\Delta} e_m \rangle)_{t \geq 0}$ may be written as the difference of two non-decreasing processes and consequently has finite variation. Therefore, due to Itô's formula,

$$\begin{aligned} d\langle X_t, e^{\varepsilon\Delta} e_m \rangle &= 2\langle e^{\varepsilon\Delta} X_t, e_m \rangle \langle \Delta e^{\varepsilon\Delta} X_t, e_m \rangle dt + 2\langle X_t, e^{\varepsilon\Delta} e_m \rangle d\langle \eta_t, e^{\varepsilon\Delta} e_m \rangle \\ &\quad + 2\langle X_t, e^{\varepsilon\Delta} e_m \rangle d\langle W_t, e^{\varepsilon\Delta} e_m \rangle + d\left[\langle W_\cdot, e^{\varepsilon\Delta} e_m \rangle\right]_t, \end{aligned} \quad (4.30)$$

where, as before, the symbol $[\cdot]_t$ is used to denote the bracket. Integrating over $[0, t]$, applying expectation, summing over $m \in \mathbb{N}_0$ and then integrating by parts,

$$\begin{aligned} &\mathbb{E}[\|e^{\varepsilon\Delta} X_t\|_2^2] + 2 \int_0^t \mathbb{E}[\|De^{\varepsilon\Delta} X_r\|_2^2] dr \\ &= \mathbb{E}[\|e^{\varepsilon\Delta} X_0\|_2^2] + 2\mathbb{E}\left[\sum_{m \in \mathbb{N}_0} \int_0^t \langle e^{\varepsilon\Delta} X_r, e_m \rangle d\langle \eta_r, e^{\varepsilon\Delta} e_m \rangle\right] + \mathbb{E}[\|e^{\varepsilon\Delta} W_t\|_2^2] \\ &= \mathbb{E}[\|e^{\varepsilon\Delta} X_0\|_2^2] + \mathbb{E}[\|e^{\varepsilon\Delta} W_t\|_2^2] + 2\mathbb{E}\left[\int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r\right], \end{aligned} \quad (4.31)$$

where we used Definition 4.5 and Remark 4.9 to get the last line. Combining with the inequality (4.28), recalling (3.9) and passing to the limit as $\varepsilon \rightarrow 0$, this implies that

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \mathbb{E} \left[\int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r \right] &= \mathbb{E}[\|X_t\|_2^2] + 2 \int_0^t \mathbb{E}[\|DX_r\|_2^2] dr - \mathbb{E}[\|X_0\|_2^2] - \mathbb{E}[\|W_t\|_2^2] \\ &\leq c_{1,\lambda} t - \mathbb{E}[\|W_t\|_2^2] = 0, \end{aligned}$$

which completes the proof of Proposition 4.11 (recall that $\int_s^t e^{\varepsilon\Delta} X_r \cdot d\eta_r \leq \int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r$ for $s \in [0, t]$).

Third step. We now prove the first part of Corollary 4.12 (time-continuity of the trajectories of $(X_t)_{t \geq 0}$ with respect to $\|\cdot\|_2$). To do so, we come back to (4.31), but without expectation. We have

$$\begin{aligned} & \|e^{\varepsilon\Delta} X_t\|_2^2 + 2 \int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr \\ &= \|e^{\varepsilon\Delta} X_0\|_2^2 + \sum_{m \in \mathbb{N}_0} \lambda_m^2 e^{-8\pi^2 m^2 \varepsilon t} + 2 \int_0^t e^{2\varepsilon\Delta} X_r \cdot dW_r + 2 \int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r. \end{aligned} \quad (4.32)$$

Fix $T > 0$ as in the first step and assume that $t \in [0, T]$. Writing the same identity as above but at another time $s \in [0, T]$, for $s \leq t$, we obtain

$$\begin{aligned} \left| \|e^{\varepsilon\Delta} X_t\|_2^2 - \|e^{\varepsilon\Delta} X_s\|_2^2 \right| &\leq 2 \int_s^t \|DX_r\|_2^2 dr + \sum_{m \in \mathbb{N}_0} \lambda_m^2 e^{-8\pi^2 m^2 \varepsilon} (t-s) \\ &\quad + 2 \left| \int_s^t e^{2\varepsilon\Delta} X_r \cdot dW_r \right| + 2 \int_s^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r. \end{aligned}$$

Upper bounding $\int_s^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r$ by $\int_0^T e^{2\varepsilon\Delta} X_r \cdot d\eta_r$, we observe from the second step that $\sup_{0 \leq s \leq t \leq T} \left| \int_s^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r \right|$ tends to 0 in probability. Similarly, we deduce from Doob's inequality that

$$\sup_{0 \leq s \leq t \leq T} \left| \int_s^t e^{2\varepsilon\Delta} X_r \cdot dW_r - \int_s^t X_r \cdot dW_r \right|$$

tends to 0 in probability. Therefore, by extracting a subsequence $(\varepsilon_n)_{n \geq 0}$ that tends to 0, we can easily pass to the limit in the right-hand side of (4.3), with probability 1, for all $0 \leq s \leq t \leq T$. As for the left-hand side, we can consider an event of probability 1, on which $\sup_{0 \leq r \leq T} \|X_r\|_2 < \infty$ (courtesy of Proposition 4.3). On this event, we can pass to the limit for all $s, t \in [0, T]$ in the left-hand side of (4.3).

We deduce that, \mathbb{P} -almost surely, for all $s, t \in [0, T]$ with $s \leq t$,

$$\left| \|X_t\|_2^2 - \|X_s\|_2^2 \right| \leq 2 \int_s^t \|DX_r\|_2^2 dr + c_{1,\lambda}(t-s) + 2 \left| \int_s^t X_r \cdot dW_r \right|.$$

By the second step, with probability 1, the right-hand side tends to 0 as $t-s$ tends to 0. This proves that there exists an event of probability 1 on which the trajectory $t \in [0, T] \mapsto \|X_t\|_2^2$ is continuous. Recalling that the trajectory $t \in [0, T] \mapsto X_t$ is already known to be continuous with respect to $\|\cdot\|_{2,-1}$, we easily deduce that it is continuous with respect to $\|\cdot\|_2$. By (4.2), we deduce that the trajectory $t \in [0, T] \mapsto \eta_t$ is continuous with respect to $\|\cdot\|_{2,-2}$. \square

4.4 Definition and uniqueness of solutions to the rearranged SHE

We now define a solution to the rearranged SHE studied in this paper.

Definition 4.13 On a given (filtered) probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q -Brownian motion $(W_t)_{t \geq 0}$ with values in $L^2_{\text{sym}}(\mathbb{S})$ (with respect to the filtration \mathbb{F}) and with an \mathcal{F}_0 -measurable initial condition X_0 with values in $U^2(\mathbb{S})$ (see (2.2)) and with finite moments of any order (see (3.8)), we say that a pair of processes $(X_t, \eta_t)_{t \geq 0}$ solves the rearranged SHE (4.1) driven by $(W_t)_{t \geq 0}$ and X_0 if

1. $(X_t)_{t \geq 0}$ is a continuous \mathbb{F} -adapted process with values in $U^2(\mathbb{S})$;
2. $(\eta_t)_{t \geq 0}$ is a continuous \mathbb{F} -adapted process with values in $H_{\text{sym}}^{-2}(\mathbb{S})$, starting from 0 at 0, such that, with probability 1, for any $u \in H^2_{\text{sym}}(\mathbb{S})$ that is non-increasing, the path $(\langle \eta_t, u \rangle)_{t \geq 0}$ is non-decreasing;
3. with probability 1, for any $u \in H^2_{\text{sym}}(\mathbb{S})$,

$$\forall t \geq 0, \quad \langle X_t, u \rangle = \int_0^t \langle X_r, \Delta u \rangle dr + \langle W_t, u \rangle + \langle \eta_t, u \rangle. \quad (4.33)$$

4. for any $t \geq 0$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E} \left[\int_0^t e^{\varepsilon \Delta} X_r \cdot d\eta_r \right] = 0. \quad (4.34)$$

Of course, the definition of the integral in (4.34) is understood as in Definition 4.5.

We now address pathwise uniqueness to the rearranged SHE.

Proposition 4.14 Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q -Brownian motion $(W_t)_{t \geq 0}$ with values in $L^2_{\text{sym}}(\mathbb{S})$ (with respect to the filtration \mathbb{F}) and with an \mathcal{F}_0 -measurable initial condition X_0 with values in $U^2(\mathbb{S})$ and with finite moments of any order, there exists at most one solution $(X_t, \eta_t)_{t \geq 0}$ to (4.1) that satisfies Definition 4.13.

Proof Consider two candidate solutions $(X_t^1, \eta_t^1)_{t \geq 0}$ and $(X_t^2, \eta_t^2)_{t \geq 0}$. Then, by Itô's formula, for any $\varepsilon > 0$, \mathbb{P} -a.s., for any $m \in \mathbb{N}_0$,

$$\begin{aligned} & d \langle e^{\varepsilon \Delta} (X_t^1 - X_t^2), e_m \rangle^2 \\ &= 2 \langle e^{\varepsilon \Delta} (X_t^1 - X_t^2), e_m \rangle \left[\langle \Delta e^{\varepsilon \Delta} (X_t^1 - X_t^2), e_m \rangle dt + d \langle \eta_t^1 - \eta_t^2, e^{\varepsilon \Delta} e_m \rangle \right], \quad t \geq 0. \end{aligned} \quad (4.35)$$

Summing over $m \in \mathbb{N}_0$ and integrating by parts, one obtains (\mathbb{P} -a.s)

$$\begin{aligned} & \|e^{\varepsilon \Delta} (X_t^1 - X_t^2)\|_2^2 + 2 \int_0^t \|De^{\varepsilon \Delta} (X_r^1 - X_r^2)\|_2^2 dr = 2 \int_0^t e^{2\varepsilon \Delta} (X_r^1 - X_r^2) \cdot d(\eta_r^1 - \eta_r^2) \\ & \leq 2 \left(\int_0^t e^{2\varepsilon \Delta} X_r^1 \cdot d\eta_r^1 + \int_0^t e^{2\varepsilon \Delta} X_r^2 \cdot d\eta_r^2 \right), \end{aligned} \quad (4.36)$$

where we used Corollary 4.8 to establish the last inequality.

Applying expectation and setting $\varepsilon \rightarrow 0$, we obtain from (4.34) that $X_t^1 = X_t^2$, \mathbb{P} -a.s. for any $t \geq 0$, which shows that X^1 and X^2 are indistinguishable. Since η^1 and η^2 are defined via X^1 and X^2 , there exists a unique solution to (4.1). \square

By combining Propositions 4.1 and 4.14 with Corollary 4.12, we deduce from an obvious adaptation of Yamada-Watanabe argument, the first main result of the paper:

Theorem 4.15 *Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q -Brownian motion $(W_t)_{t \geq 0}$ with values in $L^2_{\text{sym}}(\mathbb{S})$ (with respect to the filtration \mathbb{F}) and with an \mathcal{F}_0 -measurable initial condition X_0 with values in $L^2_{\text{sym}}(\mathbb{S})$, there exists a unique solution $(X_t, \eta_t)_{t \geq 0}$ to the rearranged SHE (4.1) that satisfies Definition 4.13.*

Moreover, the processes $\{(\tilde{X}_t^h, W_t)_{t \geq 0}\}_{h > 0}$, as defined in (3.3), are convergent in law (over $\mathcal{C}([0, \infty), H_{\text{sym}}^{-1}(\mathbb{S}) \times L^2_{\text{sym}}(\mathbb{S}))$) equipped with the topology of uniform convergence on compact subsets) and the limit is the law of $(X_t, W_t)_{t \geq 0}$.

Notice that, as in Proposition 4.14, we use the same noise for the scheme and for the limiting equation. However, in contrast to Proposition 4.14, there is no abuse in doing so: the first part of the statement allows us to construct the solution to the rearranged equation on the same filtered probability space (equipped with the same noise) as the scheme. It shall prove useful to note that item 4 in Definition 4.13 may be strengthened:

Proposition 4.16 *Let $(X_t, \eta_t)_{t \geq 0}$ satisfy Definition 4.13 except item 4 therein. Then, item 4 holds true if and only if one of the following two properties below is satisfied:*

- 4' *For any $t > 0$, there exists a sequence of positive reals $(\varepsilon_q)_{q \geq 1}$, with 0 as limit, such that, in \mathbb{P} -probability, $\lim_{q \rightarrow \infty} \int_0^t e^{\varepsilon_q \Delta} X_r \cdot d\eta_r = 0$.*
- 4'' *It holds that, for any $p \geq 1$, for any $t > 0$, $\lim_{\varepsilon \searrow 0} \mathbb{E}[(\int_0^t e^{\varepsilon \Delta} X_r \cdot d\eta_r)^p] = 0$.*

Proof We proceed as follows. We consider a process $(X_t, \eta_t)_{t \geq 0}$ satisfying Definition 4.13 except item 4 therein. Obviously 4'' implies 4, which implies in turn 4'. The only difficulty is to prove that 4' implies 4''.

Assuming 4', we recall that, by construction, the argument inside the power function in 4'' is non-negative. We then show that, for any $t \geq 0$,

$$\forall p \geq 1, \quad \sup_{\varepsilon > 0} \mathbb{E} \left[\left(\int_0^t e^{\varepsilon \Delta} X_r \cdot d\eta_r \right)^p \right] < \infty. \quad (4.37)$$

We restart from (4.30) and we follow the derivation of (4.31), but without taking the expectation therein. For a given $\varepsilon > 0$, we restart from (4.32) (which holds true under items 1, 2 and 3 in Definition 4.13, even though item 4 is not known yet). Recall from the contractivity of the heat semigroup that the function

$$\varepsilon \in (0, \infty) \mapsto \int_0^t \|De^{\varepsilon \Delta} X_r\|_2^2 dr$$

is non-increasing (for any given realisation). Choosing $\varepsilon = \varepsilon_q/2$ for some $q \in \mathbb{N}_0$ in (4.32) and taking the limit (in probability) as q tends to ∞ , we deduce from 4' that

$$\|X_t\|_2^2 + 2 \sup_{\varepsilon > 0} \int_0^t \|e^{\varepsilon \Delta} DX_r\|_2^2 dr = \|X_0\|_2^2 + \|W_t\|_2^2 + 2 \int_0^t X_s \cdot dW_s.$$

By taking power p and recalling (3.9) and Proposition 4.3 together with the fact that $\|X_t\|_2$ has finite moments of any order for any t (see Proposition 3.4), we deduce that

$$\mathbb{E} \left[\left(\sup_{\varepsilon > 0} \int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 \right)^p \right] < \infty. \quad (4.38)$$

Back to (4.32), we can express $\int_0^t e^{2\varepsilon\Delta} X_r \cdot d\eta_r$ in terms of all the other terms. Using the Burkholder-Davis-Gundy inequality to handle the stochastic integral, (4.37) follows.

Now that we have (4.37), it suffices to prove item 4'' with $p = 1$. The result for $p > 1$ then follows from a standard uniform integrability argument. In fact, by combining (4.37) and item 4', we already know that, for any $t > 0$,

$$\lim_{q \rightarrow \infty} \mathbb{E} \left[\int_0^t e^{\varepsilon_q \Delta} X_r \cdot d\eta_r \right] = 0. \quad (4.39)$$

It remains to observe that the limit $\lim_{\varepsilon \searrow 0} \mathbb{E} \int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r$ exists necessarily. Indeed, taking expectation in (4.32), we observe that

$$\begin{aligned} \lim_{\varepsilon \searrow 0} \mathbb{E} \left[\int_0^t e^{\varepsilon\Delta} X_r \cdot d\eta_r \right] &= \mathbb{E} [\|X_t\|_2^2] + 2 \sup_{\varepsilon > 0} \mathbb{E} \left[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr \right] \\ &\quad - \mathbb{E} [\|X_0\|_2^2] - \mathbb{E} [\|W_t\|_2^2], \end{aligned}$$

where we used once again the contractivity of the heat semigroup to write $\lim_{\varepsilon \rightarrow 0} \mathbb{E} \left[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr \right] = \sup_{\varepsilon > 0} \mathbb{E} \left[\int_0^t \|De^{\varepsilon\Delta} X_r\|_2^2 dr \right]$. The proof is complete. \square

4.5 Lipschitz regularity of the flow

We conclude this section with the following result, crucial for the rest of the paper. The proof is the same as that of uniqueness; we just need to retain the difference of the initial conditions in the argumentation.

Proposition 4.17 *Given $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ equipped with a Q -Brownian motion $(W_t)_{t \geq 0}$ with values in $L_{\text{sym}}^2(\mathbb{S})$ (with respect to \mathbb{F}), consider (X^x, η^x) and (X^y, η^y) the solutions to the (4.1) with $X_0^x = x \in U^2(\mathbb{S})$ and $X_0^y = y \in U^2(\mathbb{S})$ as initial conditions. Then,*

$$\mathbb{P}\text{-a.s.}, \quad \|X_t^x - X_t^y\|_2^2 + 2 \int_0^t \|De^{\varepsilon\Delta} (X_r^x - X_r^y)\|_2^2 dr \leq \|x - y\|_2^2, \quad t \geq 0. \quad (4.40)$$

As an obvious (but very useful) consequence of Proposition 4.17, we have, for all $T > 0$,

$$\sup_{0 \leq t \leq T} \left\| \eta_t^x - \eta_t^y \right\|_{2,-2} \leq c_T \|x - y\|_2, \quad (4.41)$$

for a constant c_T only depending on T . The proof follows from the identity (4.33).

Remark 4.18 Inequalities (4.40) and (4.41) say that, for each $x \in L_{\text{sym}}^2(\mathbb{S})$, we can find versions of $(X_t^x)_{t \geq 0}$ and $(\eta_t^x)_{t \geq 0}$ such that, for any $T > 0$, the mappings $(\omega, t, x) \in \Omega \times [0, T] \times L_{\text{sym}}^2(\mathbb{S}) \mapsto X_t^x(\omega) \in L_{\text{sym}}^2(\mathbb{S})$ and $(\omega, t, x) \in \Omega \times [0, T] \times L_{\text{sym}}^2(\mathbb{S}) \mapsto \eta_t^x(\omega) \in H_{\text{sym}}^{-2}(\mathbb{S})$ are measurable, continuous in t and Lipschitz in x .

5 Smoothing Effect

It is demonstrated below that the semigroup -

$$\{P_t\}_{t \geq 0} \quad \text{with} \quad P_t f(x) := \mathbb{E}[f(X_t^x)], \quad t \geq 0, x \in U^2(\mathbb{S}),$$

for f within the set of bounded measurable functions on $U^2(\mathbb{S})$ - maps bounded measurable functions into Lipschitz continuous functions on $U^2(\mathbb{S})$, at least when the parameter λ in (3.1) belongs to $(1/2, 1)$. Importantly, we prove that the rate at which the Lipschitz constant of $P_t f$ blows up as t decreases to 0 is integrable, see Theorem 5.9 below together with Remark 5.10 for possible applications to infinite dimensional PDEs.

We first consider a finite dimensional reduction of the problem. For a given truncation level $M \in \mathbb{N}_0$ and for any v in $L_{\text{sym}}^2(\mathbb{S})$, we let

$$v^M := \sum_{m=0}^M \langle v, e_m \rangle e_m(\cdot), \quad v^{*,M} := \left(\sum_{m=0}^M \langle v, e_m \rangle e_m(\cdot) \right)^*.$$

Clearly, $v^{*,M}$ is an element of $U^2(\mathbb{S})$ parametrised by the (finite) vector of Fourier modes $(\langle v, e_m \rangle)_{m=0, \dots, N}$. We let $E^M := \{v^M, v \in L_{\text{sym}}^2(\mathbb{S})\}$ and $E^{*,M} := \{v^{*,M}, v \in L_{\text{sym}}^2(\mathbb{S})\}$. Obviously, $E^M \cong \mathbb{R}^{M+1}$. The point is to prove that, for any $t > 0$, the mapping $x \in E^M \mapsto P_t f(x^{*,M})$ is Lipschitz continuous, with a Lipschitz constant independent of M . Reducing the dimensionality allows us to use many tools from finite dimensional analysis, notably Rademacher's theorem. Together with Proposition 4.17, the latter says that, for a given $t > 0$, the flow $x \in E^M \mapsto X_t^{x^*}$ is almost everywhere (for the Lebesgue measure on E^M) differentiable. In this way, we avoid having to establish the everywhere differentiability of the flow of solutions to (4.1) with respect to the initial condition, a property that is not clear to us at this stage. (See however, the references [3, 4, 29, 53, 54] cited in Subsection 1.2 for positive results in this direction when the reflected dynamics take values in a finite dimensional space.)

The second step is to consider, for $x, v \in L_{\text{sym}}^2(\mathbb{S})$, $\delta \in \mathbb{R}$ and $T > 0$, the difference

$$P_T f((x + \delta v)^{*,M}) - P_T f(x^{*,M}) = \mathbb{E} \left[f(X_T^{(x + \delta v)^{*,M}}) \right] - \mathbb{E} \left[f(X_T^{x^{*,M}}) \right], \quad (5.1)$$

and to represent it *via* use of a Girsanov transformation. This adapts earlier arguments from Malliavin calculus, see [12, 58], and from the proof of the so-called Bismut-Elworthy-Li formula, see [35, 72]. The key idea is to consider the *shifted process*

$$\left(X_t^{(x + \delta \frac{T-t}{T} v)^{*,M}} \right)_{0 \leq t \leq T},$$

which satisfies $X_0^{(x + \delta \frac{T-t}{T} v)^{*,M}} = (x + \delta v)^{*,M}$ and $X_T^{(x + \delta \frac{T-t}{T} v)^{*,M}} = X_T^{x^{*,M}}$. It is shown that, under a particular change of measure, the shifted process is the unique solution to (4.1) started from $(x + \delta v)^{*,M}$. To guarantee the conditions of Girsanov transformation, we need to localise the dynamics and to enact the *shifting* up to a stopping time, the form of which is clarified in (5.17). For $t \in [0, T]$, $y, v \in E^M$, $\delta \in \mathbb{R}$, we let

$$y_t(v, \delta) := y + \delta \frac{T-t}{T} v, \quad \text{and} \quad y_t^*(v, \delta) := \left(y + \delta \frac{T-t}{T} v \right)^*. \quad (5.2)$$

The time horizon T is implicitly understood (and omitted) in the two left-hand sides and the function y is manifested by the notations y_t and y_t^* . For say, $x \in E^M$, we write $x_t(v, \delta)$ and $x_t^*(v, \delta)$. Moreover, frequently we will take derivatives with respect to the finite-dimensional variable $y \in E^M$. The gradient is denoted ∂_y . Notice that for $y \in E^M$, both y and y^* can be regarded as elements of $L_{\text{sym}}^2(\mathbb{S})$. By Lemma 2.4 and Parseval identity, $\|y^*\|_2 = \|y\|_2 = |y|$ where $|\cdot|$ stands for the Euclidean norm on E^M .

5.1 Shifted state process and tilted reflection process

In this subsection, we address the dynamics of the shifted pair $(X_t^{y_t^*(v, \delta)}, \eta_t^{y_t^*(v, \delta)})_{0 \leq t \leq T}$, for y and v in E^M and $\delta \in \mathbb{R}$. Until further notice, the number M of low frequency modes to which the initial condition is truncated is fixed and the indexing with respect to M is omitted. To identify the dynamics, we expand the time evolution of the Fourier modes of the shifted process integrated against a test function $\varphi \in \mathcal{C}_0^\infty(E^M)$:

$$\left(\int_{E^M} \langle X_t^{y_t^*(v, \delta)}, e_m \rangle \varphi(y) dy \right)_{0 \leq t \leq T},$$

with $\langle X_t^{y_t^*(v, \delta)}, e_m \rangle$ standing for the m^{th} Fourier mode of $X_t^{y_t^*(v, \delta)}$.

Changing variables and recalling the dynamics (4.2), we have

$$\begin{aligned} \int_{E^M} \langle X_t^{y_t^*(v, \delta)}, e_m \rangle \varphi(y) dy &= \int_{E^M} \langle X_t^{y^*}, e_m \rangle \varphi(y_t(v, -\delta)) dy \\ &= \int_{E^M} \langle X_0^{y^*}, e_m \rangle \varphi(y - \delta v) dy + \int_{E^M} \int_0^t \langle X_s^{y^*}, \Delta e_m \rangle \varphi(y_s(v, -\delta)) ds dy \\ &\quad + \int_{E^M} \left(\int_0^t \varphi(y_s(v, -\delta)) d \langle \eta_s^{y^*}, e_m \rangle \right) dy + \int_{E^M} \left(\int_0^t \varphi(y_s(v, -\delta)) d \langle W_s, e_m \rangle \right) dy \\ &\quad + \frac{\delta}{T} \int_{E^M} \int_0^t \langle X_s^{y^*}, e_m \rangle \partial_y \varphi(y_s(v, -\delta)) \cdot v ds dy, \end{aligned} \quad (5.3)$$

where $\partial_y \varphi(y_s(v, -\delta)) \cdot v$ represents the gradient of φ in the direction of v at point $y_s(v, -\delta)$. Notice that the well-posedness of the second integral in the penultimate line is guaranteed by the stochastic version of Fubini's theorem, [23, Theorem 4.33].

Ideally, we would like to revert back the variables in the various integrals appearing in the expansion (5.3) and hence to compute the test function φ at the generic point y instead of $y_s(v, -\delta)$. The main difficulty is to handle the integral

$$\int_{E^M} \left(\int_0^t \varphi(y_s(v, -\delta)) d \langle \eta_s^{y^*}, e_m \rangle \right) dy. \quad (5.4)$$

The existence of the time integral in the right-hand side follows from (4.23) and (4.41), the latter ensuring in particular the measurability of the mapping $(\omega, t, y) \mapsto \langle \eta_t^{y^*}(\omega), e_m \rangle$. The combination of both guarantees that, almost surely,

$$\forall R > 0, \quad \sup_{0 \leq t \leq T} \sup_{|y| \leq R} \|\eta_t^{y^*}\|_{2, -2} \leq \sup_{0 \leq t \leq T} \|\eta_t^0\|_{2, -2} + cTR < \infty, \quad (5.5)$$

for a constant c_T only depending on T .

Before formulating a convenient change of variables for (5.4) in the forthcoming Proposition 5.4, we introduce some useful ingredients. This includes defining a so-called *tilted* version $(\tilde{\eta}_t^{y^*})_{0 \leq t \leq T}$ of the reflection process $(\eta_t^{y^*})_{0 \leq t \leq T}$. We proceed below as in Subsection 4.2 and formally replace (at least for the first result) the realisation of the field $((\eta_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ (here, y is directly assumed to be in $E^{*,M}$) by a **deterministic** flow $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ satisfying the following two properties:

- (F1) For any $y \in E^{*,M}$, the trajectory $t \in [0, T] \mapsto n_t^y$ satisfies (E1) and (E2) in Subsection 4.2, restricted in an obvious manner to the interval $[0, T]$;
- (F2) The flow satisfies the Lipschitz condition $\sup_{0 \leq t \leq T} \|n_t^x - n_t^y\|_{2,-2} \leq c_T |x - y|$, for any $x, y \in E^{*,M}$. Equivalently, $\sup_{0 \leq t \leq T} \|n_t^{x^*} - n_t^{y^*}\|_{2,-2} \leq c_T |x - y|$, for any $x, y \in E^M$.

The reader may reformulate (5.5) accordingly. The following definition clarifies the form of the corrected (or *tilted*) reflection term:

Definition 5.1 Let $v \in E^M$ and $\delta \in \mathbb{R}$ be given and M be as in (5.1). Moreover, let $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ satisfy (F1)–(F2), and $((z_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ be a deterministic jointly continuous $L_{\text{sym}}^2(\mathbb{S})$ -valued flow, i.e., the map $(t, y) \in [0, T] \times E^{*,M} \mapsto z_t^y \in L_{\text{sym}}^2(\mathbb{S})$ is continuous. Then, for any $m \in \mathbb{N}_0$, $y \in E^M$ and $t \in [0, T]$, define

$$\partial_y^* n_t^{m,y} := \begin{cases} \partial_y \langle n_t^{y^*}, e_m \rangle & \text{whenever the } \mathbb{R}^M\text{-valued gradient at} \\ & \text{point } y \in E^M \text{ in the right-hand side exists,} \\ 0 & \text{otherwise,} \end{cases} \quad (5.6)$$

$$\text{and } \langle \tilde{n}_t^{y,(v,\delta)}, e_m \rangle := \langle n_t^{y^*(v,\delta)}, e_m \rangle + \frac{\delta}{T} \int_0^t \left[\partial_w^* n_s^{m,w} \cdot v \right]_{|w=y_s(v,\delta)} ds. \quad (5.7)$$

Measurability of $(t, y) \mapsto \partial_y^* n_t^{m,y}$ is obvious and, in fact, (F2) says that the derivative in Definition 5.1 exists for any $t \in [0, T]$, for almost every $y \in E^M$, and thus for almost every $(t, y) \in [0, T] \times E^M$. When $((n_t^y)_{0 \leq t \leq T})_{y \in E^M}$ is replaced by $((\eta_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$, the mapping $(t, \omega, y) \in [0, T] \times E^M \times \Omega \mapsto \partial_y^* \eta_t^{m,y}$ is measurable with respect to $\mathcal{P} \times \mathcal{B}(E^M)$, where \mathcal{P} is the progressive σ -algebra on $[0, T] \times \Omega$ (Ω equipped with the same \mathbb{F} as before) and $\mathcal{B}(E^M)$ is the Borel σ -algebra on E^M . In the random setting, we define $\langle \tilde{\eta}_t^{y,(v,\delta)}, e_m \rangle$ by replacing the letter n with η throughout (5.7).

It is easy to check that $\partial_y^* n_t^y := \sum_{m \in \mathbb{N}_0} \partial_y^* n_t^{m,y} e_m$ as defined through the Fourier modes (5.6) is an element of $H_{\text{sym}}^{-2}(\mathbb{S})$, since the series $\sum_{m \in \mathbb{N}_0} m^{-4} |\partial_y^* n_t^{m,y}|^2$ is bounded. In particular, we have, for a constant C_c depending on c in (F2)

$$\|\partial_y^* n_t^y\|_{2,-2} \leq C_c. \quad (5.8)$$

As a corollary of (5.8), we get the following statement, which allows us to regard $(\tilde{n}_t^y)_{0 \leq t \leq T}$ as a continuous path with values in $H_{\text{sym}}^{-2}(\mathbb{S})$:

Corollary 5.2 *Within the framework of Definition 5.1, for any $R > 0$,*

$$\forall R > 0, \quad \sup_{0 \leq t \leq T} \sup_{|y| \leq R} \left\| \tilde{n}_t^{y,(v,\delta)} \right\|_{2,-2} \leq \sup_{0 \leq t \leq T} \|n_t^0\|_{2,-2} + c(R + \delta|v|) < \infty, \quad (5.9)$$

for the same constant c as in (F2).

Based on the process $(\tilde{\eta}_t^{y,(v,\delta)})_{0 \leq t \leq T}$ introduced prior, we provide here a useful change of variables for (5.4). It goes through the following notation. For $\varphi \in \mathcal{C}_0^\infty(E^M)$, $\delta \in \mathbb{R}$ and $y, v \in E^M$, we let

$$\tilde{n}_t^{\varphi,(v,\delta)} := \int_{E^M} \varphi(y) \tilde{\eta}_t^{y,(v,\delta)} dy, \quad t \in [0, T], \quad (5.10)$$

which is regarded as a path with values in $H_{\text{sym}}^{-2}(\mathbb{S})$. The following statement will be proven in Subsection 5.5:

Lemma 5.3 *Assume that the function φ is positive-valued. Then, for any $z \in U^2(\mathbb{S}) \cap H_{\text{sym}}^2(\mathbb{S})$, the process $(\langle \tilde{n}_t^{\varphi,(v,\delta)}, z \rangle)_{0 \leq t \leq T}$ is non-decreasing (in time).*

The above corollary shows that $(\tilde{n}_t^{\varphi,(v,\delta)})_{0 \leq t \leq T}$ satisfies Assumption **(E2)** in Subsection 4.2. Since **(E1)** (time continuity with respect to $\|\cdot\|_{2,-3}$) follows quite obviously from the joint continuity of the flow $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$, we can invoke Definition 4.5 to give meaning to the integral

$$\left(\int_0^t \langle e^{\varepsilon \Delta} z_s, d\tilde{n}_s^{\varphi,(v,\delta)} \rangle \right)_{0 \leq t \leq T},$$

for $\varepsilon > 0$ and for a continuous path $(z_t)_{0 \leq t \leq T}$ with values in $L_{\text{sym}}^2(\mathbb{S})$, at least when φ takes non-negative values, which will suffice for our purpose. In order to state our change of variable in a convenient way, we also let

$$\tilde{z}_t^{y,\varphi,(v,\delta)} := \varphi(y_t(v, -\delta)) z_t, \quad t \in [0, T]. \quad (5.11)$$

We claim (the proof, which is technical, is also deferred to Subsection 5.5)

Proposition 5.4 *Let $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ satisfy **(F1)**–**(F2)**, φ be a non-negative valued test function in $\mathcal{C}_0^\infty(E^M)$ and $(z_t)_{0 \leq t \leq T}$ is a continuous path with values in $L_{\text{sym}}^2(\mathbb{S})$. Then, one has*

$$\int_E \left(\int_0^T \langle e^{\varepsilon \Delta} \tilde{z}_t^{y,\varphi,(v,\delta)}, dn_t^{y,*} \rangle \right) dy = \int_0^T \langle e^{\varepsilon \Delta} z_t, d\tilde{n}_t^{\varphi,(v,\delta)} \rangle, \quad t \in [0, T]. \quad (5.12)$$

Remark 5.5 Following Remark 4.10, all results given in this subsection can be applied when $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ is the realisation of $((\eta_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ (which satisfies **(F1)** and **(F2)** thanks to Remark 4.18).

5.2 Integrating in time the shifted tilted reflection process

We here achieve two objectives; not only do we explicit the dynamics of the process $(X_t^{y,*(v,\delta)})_{0 \leq t \leq T}$ (for a given $y \in E^M$), but we also clarify the notion of a pathwise integral with respect to the *tilted* reflection process $(\tilde{\eta}_t^{y,(v,\delta)})_{0 \leq t \leq T}$ defined in (5.7). In comparison, the integral with respect to $(\tilde{\eta}_t^{\varphi,(v,\delta)})_{0 \leq t \leq T}$ (see (5.10), replacing $(\tilde{\eta}_t^{y,(v,\delta)})_{0 \leq t \leq T}$ by $(\tilde{\eta}_t^{y,(v,\delta)})_{0 \leq t \leq T}$) is just constructed for φ smooth. Somehow, we want to choose φ as a Dirac mass.

Proposition 5.6 Fix $\delta \in \mathbb{R}$ and $v \in E^M$. Then, with probability 1, for almost every $y \in E^M$, the path $(\tilde{\eta}_t^{y, (v, \delta)})_{0 \leq t \leq T}$ satisfies **(E1)** and **(E2)** in Subsection 4.2, so that Definition 4.5 can be invoked in order to define an integral with respect to $(\tilde{\eta}_t^{y, (v, \delta)})_{0 \leq t \leq T}$.

Moreover, with probability 1, for almost every $y \in E^M$, for any $u \in H_{\text{sym}}^2(\mathbb{S})$,

$$\begin{aligned} \langle X_t^{y, (v, \delta)}, u \rangle &= \langle X_0^{(y+\delta v)^*}, u \rangle + \int_0^t \langle X_s^{y, (v, \delta)}, \Delta u \rangle ds + \int_0^t d \langle \tilde{\eta}_s^{y, (v, \delta)}, u \rangle \\ &\quad + \int_0^t d \langle W_s, u \rangle - \frac{\delta}{T} \int_0^t \partial_y \langle X_s^{y, (v, \delta)}, u \rangle \cdot v ds. \end{aligned} \quad (5.13)$$

With probability 1, for almost every $y \in E^M$, almost every $t \in [0, T]$ and any $u \in H_{\text{sym}}^2(\mathbb{S})$, the derivative $\partial_y \langle X_s^{y, (v, \delta)}, u \rangle$ appearing in (5.13) exists and is jointly measurable on $\Omega \times [0, T] \times E^M$.

Proof The existence of the derivatives, as stated in the last sentence, follows from Rademacher's theorem. Indeed, by [13, Theorem 4], with probability 1, for any $t \in [0, T]$, the map $y \in E^M \mapsto X_t^{y, (v, \delta)} \in L_{\text{sym}}^2(\mathbb{S})$ is almost everywhere differentiable. By Fubini's theorem, the map $(\omega, t, y) \mapsto \partial_y X_t^{y, (v, \delta)}(\omega)$ is hence defined up to a negligible subset of $\Omega \times [0, T] \times E^M$ and induces a jointly measurable mapping on $\Omega \times [0, T] \times E^M$. Since, for any $t \in [0, T]$, the map $y \mapsto y_t(v, \delta) = y + \delta(T-t)/Tv$ preserves the Lebesgue measure, we deduce that $(\omega, t, y) \mapsto \partial_y X_t^{y, (v, \delta)}$ is also defined up to a negligible subset of $\Omega \times [0, T] \times E^M$ and also induces a jointly measurable mapping on $\Omega \times [0, T] \times E^M$.

First step. Notice that, once (5.13) has been proven to hold true, for a given u , with probability 1 and for almost every y , it is easy to get the result with probability 1, for almost every y and any $u \in H_{\text{sym}}^2(\mathbb{S})$. It suffices to use the separability of $H_{\text{sym}}^2(\mathbb{S})$.

In order to prove (5.13), we consider a function $\varphi \in \mathcal{C}_0^\infty(E^M)$ with values in $[0, +\infty)$. Repeating (5.3), we get, for any $u \in H_{\text{sym}}^2(\mathbb{S})$, with probability 1,

$$\begin{aligned} &\int_{E^M} \langle X_t^{y, (v, \delta)}, u \rangle \varphi(y) dy \\ &= \int_{E^M} \langle X_0^{y, (v, \delta)}, u \rangle \varphi(y - \delta v) dy + \int_{E^M} \int_0^t \langle X_s^{y, (v, \delta)}, \Delta u \rangle \varphi(y_s(v, -\delta)) ds dy \\ &\quad + \int_{E^M} \left(\int_0^t \varphi(y_s(v, -\delta)) d \langle \tilde{\eta}_s^{y, (v, \delta)}, u \rangle \right) dy + \int_{E^M} \left(\int_0^t \varphi(y_s(v, -\delta)) d \langle W_s, u \rangle \right) dy \\ &\quad + \frac{\delta}{T} \int_{E^M} \int_0^t \langle X_s^{y, (v, \delta)}, u \rangle \partial_y \varphi(y_s(v, -\delta)) \cdot v ds dy. \end{aligned}$$

Here is the key point. By (5.11) and (5.12), we can perform a change of variable in the penultimate line (with $z_t = u$). As for the last line, we can make an integration by parts, recalling the Lipschitz property of the flow, see Proposition 4.17. We get

$$\begin{aligned} &\int_{E^M} \langle X_t^{y, (v, \delta)}, u \rangle \varphi(y) dy \\ &= \int_{E^M} \langle X_0^{(y+\delta v)^*}, u \rangle \varphi(y) dy + \int_{E^M} \int_0^t \langle X_s^{y, (v, \delta)}, \Delta u \rangle \varphi(y) ds dy + \langle \tilde{\eta}_t^{\varphi, (v, \delta)}, u \rangle \\ &\quad + \langle W_t, u \rangle \int_{E^M} \varphi(y) dy - \frac{\delta}{T} \int_{E^M} \int_0^t \varphi(y) \partial_y \langle X_s^{y, (v, \delta)}, u \rangle \cdot v ds dy. \end{aligned} \quad (5.14)$$

Second step. Assume $\varphi \geq 0$. By invoking Lemma 5.7 below and then by taking the supremum over $u \in H_{\text{sym}}^2(\mathbb{S})$ in (5.14) right above, we claim

$$\|\tilde{\eta}_t^{\varphi, (v, \delta)} - \tilde{\eta}_s^{\varphi, (v, \delta)}\|_{2, -2} \leq \zeta_R(|t - s|) \int_{E^M} \varphi(y) dy, \quad (s, t) \in [0, T],$$

for a random field ζ_R , with values in $(0, \infty)$ and with $\lim_{\rho \rightarrow 0} \zeta_R(\rho) = 0$ almost surely. By the definition (5.10) of $\tilde{\eta}^{\varphi, (v, \delta)}$, we get that, for any $u \in H_{\text{sym}}^2(\mathbb{S})$, with $\|u\|_{2,2} \leq 1$,

$$\left| \int_{E^M} \varphi(y) \langle \tilde{\eta}_t^{y, (v, \delta)} - \tilde{\eta}_s^{y, (v, \delta)}, u \rangle dy \right| \leq \zeta_R(|t - s|) \int_{E^M} \varphi(y) dy, \quad (s, t) \in [0, T]^2.$$

We deduce that, with probability 1, for almost every $y \in E^M$, for s, t in a dense countable subset of $[0, T]$, for u in a dense countable subset of the unit ball of $H_{\text{sym}}^2(\mathbb{S})$,

$$\left| \langle \tilde{\eta}_t^{y, (v, \delta)}, u \rangle - \langle \tilde{\eta}_s^{y, (v, \delta)}, u \rangle \right| \leq \zeta_R(|t - s|).$$

Therefore, with probability 1, for almost every $y \in E^M$, for s, t in a dense countable subset of $[0, T]$,

$$\left\| \tilde{\eta}_t^{y, (v, \delta)} - \tilde{\eta}_s^{y, (v, \delta)} \right\|_{2, -2} \leq \zeta_R(|t - s|),$$

and then we have a continuous extension to the whole $[0, T]$.

Third step. We observe from Lemma 5.3 that, with probability 1, for almost every y , for s, t in a dense countable subset of $[0, T]$, for u in a countable subset of $U^2(\mathbb{S})$, $\langle \tilde{\eta}_t^{y, (v, \delta)} - \tilde{\eta}_s^{y, (v, \delta)}, u \rangle \geq 0$. By density, the continuous extension satisfies the same inequality for all $s, t \in [0, T]$ and $u \in H_{\text{sym}}^2(\mathbb{S})$. This completes the proof of the first part of the statement in Proposition 5.6. In particular, we can construct, almost surely, for almost every $y \in E^M$, an integral with respect to the process $(\tilde{\eta}_t^{y, (v, \delta)})_{0 \leq t \leq T}$ as the latter satisfies **(E1)** and **(E2)** in Subsection 4.2. Combining Lemma 4.7 and Corollary 5.2, we deduce that the identity (5.10) is preserved, despite the additional extension by continuity. Proposition 5.6 then follows by inserting (5.10) into (5.14). \square

In the proof of Proposition 5.6, we made use of the following statement:

Lemma 5.7 *Let $\delta \in \mathbb{R}$ and $v \in E^M$. Then, for any $R > 0$, there exists a random field ζ_R , with values in $(0, \infty)$ and with $\lim_{\rho \rightarrow 0} \zeta_R(\rho) = 0$, such that, for $|y| \leq R$,*

$$\left\| X_t^{y^* (v, \delta)} - X_s^{y^* (v, \delta)} \right\|_2 \leq \zeta_R(|t - s|), \quad (s, t) \in [0, T].$$

Proof By Remark 4.18, the flow $((X_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ is jointly continuous in (t, y) . \square

5.3 Dynamics under new probability measure

For $\delta \in \mathbb{R}$ and $v \in E^M$ as before (with $M \in \mathbb{N}$ fixed), Proposition 5.6 prompts us to implement a Girsanov transformation in such a way that, for almost every $y \in E^M$, under a new probability measure $\mathbb{Q}^{y, (v, \delta)}$ depending on y , the process $(\tilde{W}_t^{y, (v, \delta)})_{t \geq 0}$ defined via Fourier modes by (recalling the definition (3.1) of the weights $(\lambda_m)_{m \in \mathbb{N}_0}$):

$$\langle \tilde{W}_t^{y, (v, \delta)}, e_m \rangle := \langle W_t, e_m \rangle - \frac{\delta}{T} \int_0^t \lambda_m^{-1} \partial_y \langle X_s^{y, (v, \delta)}, e_m \rangle \cdot v ds, \quad t \geq 0, \quad m \in \mathbb{N}_0,$$

becomes a Q -Wiener process. Of course, it must be stressed that, in the definition of $((\tilde{W}_t^{y, (v, \delta)}, e_m))_{0 \leq t \leq T}$, the integral process only exists for almost every $\omega \in \Omega$ and for almost every $y \in E^M$, see the last line in Proposition 5.6. In order to remedy this issue, denote

$$\chi_t^{m, y, (v, \delta)} := \begin{cases} \partial_y \langle e_m, X_t^{y, (v, \delta)} \rangle \cdot v & \text{if the derivative exists,} \\ 0 & \text{otherwise,} \end{cases} \quad m \in \mathbb{N}_0, \quad y, v \in E^M,$$

which allows one to extend the derivative when it does not exist.

Intuitively, one expects that, for almost every $y \in E^M$, the process $(X_t^{y, (v, \delta)})_{0 \leq t \leq T}$ solves under the new probability measure, the conditions of Definition 4.13. As we will see next, the main challenge is in fact to verify the orthogonality (4.34), but a first difficulty is to define the new probability measure. Whilst we wish to let

$$\frac{d\mathbb{Q}^{y, (v, \delta)}}{d\mathbb{P}} := \mathcal{E}_T \left\{ \frac{\delta}{T} \sum_{m \in \mathbb{N}_0} \int_0^{\cdot} \lambda_m^{-1} \chi_s^{m, y, (v, \delta)} dB_s^m \right\}, \quad (5.15)$$

where \mathcal{E}_T is a shorter notation for the Doléans-Dade exponential at time T , it is however not immediate that this defines indeed a new probability measure. For this reason, we employ a localisation argument by introducing the stopping time

$$\tau_{y, (v, \delta)} := \inf \left\{ t \geq 0 : \left| \int_0^t \sum_{m \in \mathbb{N}_0} \lambda_m^{-1} \chi_s^{m, y, (v, \delta)} dB_s^m \right| \geq \frac{T}{\delta} \right\} \wedge T. \quad (5.16)$$

Therefore,

$$\frac{d\mathbb{Q}^{y, (v, \delta), \tau}}{d\mathbb{P}} := \mathcal{E}_T \left\{ \frac{\delta}{T} \int_0^{\cdot \wedge \tau_{y, (v, \delta)}} \sum_{m \in \mathbb{N}_0} \lambda_m^{-1} \chi_s^{m, y, (v, \delta)} dB_s^m \right\}. \quad (5.17)$$

is a probability density (where we have omitted the subscript $(y, (v, \delta))$ in the index τ in the left-hand side) and the process $(\tilde{W}_t^{y, (v, \delta), \tau})_{t \geq 0}$ defined in Fourier modes by

$$\langle \tilde{W}_t^{y, (v, \delta), \tau}, e_m \rangle := \langle W_t, e_m \rangle - \frac{\delta}{T} \int_0^{t \wedge \tau_{y, (v, \delta)}} \lambda_m^{-1} \chi_s^{m, y, (v, \delta)} ds, \quad t \geq 0 \quad m \in \mathbb{N}_0, \quad (5.18)$$

is a Q -Wiener process under $\mathbb{Q}^{y, (v, \delta), \tau}$. Here is the main statement of this subsection:

Proposition 5.8 *Let $y, v \in E^M$ and $\delta \in \mathbb{R}$. For $(\tilde{\eta}_t^{y, (v, \delta)})_{0 \leq t \leq T}$ as in (5.7) and $\tau_{y, (v, \delta)}$ as in (5.16), let (omitting the parameters (v, δ) in the notation $y^*(v, \delta)$)*

$$\begin{aligned} \tilde{X}_t^{y, (v, \delta), \tau} &:= \begin{cases} X_t^{y^*}, & t \in [0, \tau_{y, (v, \delta)}] \\ X_t^{y_{\tau_{y, (v, \delta)}}^*}, & t \in [\tau_{y, (v, \delta)}, T] \end{cases}, \\ \tilde{\eta}_t^{y, (v, \delta), \tau} &:= \begin{cases} \tilde{\eta}_t^{y, (v, \delta)}, & t \in [0, \tau_{y, (v, \delta)}] \\ \eta_t^{y_{\tau_{y, (v, \delta)}}^*} - \eta_{\tau_{y, (v, \delta)}}^{y_{\tau_{y, (v, \delta)}}^*} + \tilde{\eta}_{\tau_{y, (v, \delta)}}^{y, (v, \delta), \tau}, & t \in [\tau_{y, (v, \delta)}, T] \end{cases}. \end{aligned}$$

Then, for almost every $y \in E^M$, the process $(\tilde{X}_t^{y, (v, \delta), \tau}, \tilde{\eta}_t^{y, (v, \delta), \tau})_{0 \leq t \leq T}$ satisfies Definition 4.13 under the probability measure $\mathbb{Q}^{y, (v, \delta), \tau}$.

While the notation looks complicated, $(\tilde{X}_t^{y, (v, \delta), \tau}, \tilde{\eta}_t^{y, (v, \delta), \tau})_{0 \leq t \leq T}$ has in fact a quite simple interpretation: the shifting $y_t(v, \delta)$ is enacted up until time $t = \tau_{y, (v, \delta)}$.

Proof Item 1 in Definition 4.13 is easily checked by means of Proposition 4.17 and Remark 4.18. Items 2 and 3 follow from Proposition 5.6 and Remark 4.18. In both cases, the properties are proved on $[\tau_{y, (v, \delta)}, T]$ by applying Definition 4.13 itself for the solution restarted from the random initial condition $X_{\tau_{y, (v, \delta)}}^{y_{\tau_{y, (v, \delta)}}^*}$ (omitting the notation (v, δ) in $y(v, \delta)$) and driven by the shifted version $(W_{t+\tau_{y, (v, \delta)}} - W_{\tau_{y, (v, \delta)}})_{t \geq 0}$ of the noise.

The main difficulty is to check item 4 in Definition 4.13. As above, it is easily verified on $[\tau_{y, (v, \delta)}, T]$ by applying Definition 4.13 for the solution restarted from the random initial condition $X_{\tau_{y, (v, \delta)}}^{y_{\tau_{y, (v, \delta)}}^*}$ (for instance, we may invoke item 4 for the restarted solution on $[\tau_{y, (v, \delta)}, T + \tau_{y, (v, \delta)}]$ and then use the non-decreasing property of the integral to get the result on $[\tau_{y, (v, \delta)}, T]$). The key point is thus to prove that, for almost every $y \in E^M$,

$$\lim_{\varepsilon \searrow 0} \mathbb{E}_{\mathbb{Q}^{y, (v, \delta), \tau}} \left[\int_0^{\tau_{y, (v, \delta)}} e^{\varepsilon \Delta} X_s^{y_s^*} \cdot d\tilde{\eta}_s^{y, (v, \delta)} \right] = 0.$$

By Proposition 4.16, it suffices to prove that the above convergence holds, for almost every $y \in E^M$, in $\mathbb{Q}^{y, (v, \delta), \tau}$ probability, along a subsequence. Moreover, by the localisation procedure (5.16), we have a bound on the moments of the density $d\mathbb{Q}^{y, (v, \delta), \tau} / d\mathbb{P}$ and it suffices to establish the convergence in \mathbb{P} probability only. Actually, since the integral is non-decreasing in time, it is sufficient to address the convergence for the integral on the entire $[0, T]$. The argument is as follows. We claim that, for $\varphi \in \mathcal{C}_0^\infty(E^M)$ with non-negative values, with probability 1,

$$\begin{aligned} & \sum_{m \in \mathbb{N}_0} \int_{E^M} \int_0^T \left\langle e^{\varepsilon \Delta} X_t^{y_t^*}, e_m \right\rangle \varphi(y_t(v, -\delta)) d\langle \eta_t^{y_t^*}, e_m \rangle dy \\ &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{\eta}_{r_{i+1}}^{y, (v, \delta)}, e^{\varepsilon \Delta} X_{r_i}^{y_{r_i}^*} \right\rangle - \left\langle \tilde{\eta}_{r_i}^{y, (v, \delta)}, e^{\varepsilon \Delta} X_{r_i}^{y_{r_i}^*} \right\rangle \right] dy, \end{aligned} \quad (5.19)$$

which is a straightforward consequence of the forthcoming Proposition 5.12 (with $z_t^y = X_t^y$ therein). (Proposition 5.12 is a technical result, established in the final

Subsection 5.5 dedicated to the proofs of Lemma 5.3 and Proposition 5.4.) Now, one can use (5.5) to exchange the order of summation and integration in the left-hand side of (5.19). Thus, the latter is equal to $\int_{E^M} (\int_0^T [\varphi(y_t^*(v, -\delta)) e^{\varepsilon \Delta} X_t^{y^*}] \cdot d\eta_t^{y^*}) dy$. As for the right-hand side of (5.19), we can invoke Proposition 5.6 and Corollary 4.8, and regard the sum therein as a Riemann sum associated with $\int_0^T (e^{\varepsilon \Delta} X_t^{y_t^*(v, \delta)}) \cdot d\tilde{\eta}_t^{y, (v, \delta)}$. Recalling the inequality (4.23) together with Corollary 5.2 and the fact that, with probability 1, the flow $((X_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ is jointly continuous (with values in $L_{\text{sym}}^2(\mathbb{S})$), we can exchange the limit (over N) and the integral (in y). Therefore, with probability 1,

$$\begin{aligned} & \int_{E^M} \left(\int_0^T [\varphi(y_t^*(v, -\delta)) e^{\varepsilon \Delta} X_t^{y^*}] \cdot d\eta_t^{y^*} \right) dy \\ &= \int_{E^M} \varphi(y) \left(\int_0^T (e^{\varepsilon \Delta} X_t^{y_t^*(v, \delta)}) \cdot d\tilde{\eta}_t^{y, (v, \delta)} \right) dy. \end{aligned} \quad (5.20)$$

By following the proof of Proposition 4.16 (and in particular the proof of (4.37)), we can easily have a bound for $\mathbb{E}[\int_0^T [\varphi(y_t(v, -\delta)) e^{\varepsilon \Delta} X_t^{y^*}] \cdot d\eta_t^{y^*}]$ that is uniform with respect to $\varepsilon \in (0, 1)$ and to y in compact subsets of E^M . Therefore, by item 4 in Definition 4.13, we deduce that the expectation of the left-hand side in (5.20) tends to 0 (with ε). Then, the expectation of the right-hand side in (5.20) also tends to 0 (with ε). Recalling that φ is non-negative valued and assuming that φ matches 1 on a given compact subset of E^M , we deduce from Fatou's lemma that, for any $R > 0$,

$$\int_{E^M} \mathbf{1}_{\{|y| \leq R\}} \liminf_{\varepsilon \searrow 0} \mathbb{E} \left[\int_0^T (e^{\varepsilon \Delta} X_t^{y_t^*(v, \delta)}) \cdot d\tilde{\eta}_t^{y, (v, \delta)} \right] dy = 0.$$

Therefore, for almost every $y \in E^M$, there exists a subsequence $(\varepsilon_q)_{q \geq 1}$ (possibly depending on y), with 0 as limit, such that

$$\lim_{q \rightarrow \infty} \mathbb{E} \left[\int_0^T (e^{\varepsilon_q \Delta} X_t^{y_t^*(v, \delta)}) \cdot d\tilde{\eta}_t^{y, (v, \delta)} \right] = 0,$$

which implies convergence in probability, as we claimed. \square

5.4 Regularity

We arrive at the main statement of this section, which asserts that the semigroup generated by the solution to (4.1) maps bounded functions into Lipschitz functions:

Theorem 5.9 *Assume that λ in (3.1) is in $(1/2, 1)$. Let $((X_t^x)_{t \geq 0})_{x \in L_{\text{sym}}^2(\mathbb{S})}$ be the flow generated by (4.1), as defined in Remark 4.18. Then, there exists a constant c_λ , only depending on λ such that, for any $t > 0$ and any bounded (measurable) function $f : L_{\text{sym}}^2(\mathbb{S}) \rightarrow \mathbb{R}$, the function $x \in L_{\text{sym}}^2(\mathbb{S}) \mapsto \mathbb{E}[f(X_t^x)]$ is Lipschitz continuous with $c_\lambda t^{-(1+\lambda)/2}$ as Lipschitz constant.*

Remark 5.10 Notice that, for $\lambda \in (1/2, 1)$, the exponent $(1 + \lambda)/2$ is strictly less than 1. This guarantees that the rate at which the Lipschitz constant blows up when time becomes small is integrable. This is expected to have important applications for the analysis of PDEs on $\mathcal{P}(\mathbb{R})$ driven by the generator of the process $((X_t^x)_{t \geq 0})_{x \in L_{\text{sym}}^2(\mathbb{S})}$.

Proof First step. We start with a bounded measurable function $f : L_{\text{sym}}^2(\mathbb{S}) \rightarrow \mathbb{R}$. We are also given a threshold M as in (5.1), a time horizon $T > 0$ and a non-zero element $v \in E^M$. By Proposition 5.8, we know that for almost every $y \in E^M$, under $\mathbb{Q}^{y, (v, \delta), \tau}$, the process $(\tilde{X}_t^{y, (v, \delta), \tau}, \tilde{\eta}_t^{y, (v, \delta), \tau})_{0 \leq t \leq T}$ is the unique solution to the rearranged SHE started from $y + \delta v$ and driven by the tilted noise (5.18). By an obvious adaptation of the Yamada-Watanabe theorem (to which we already alluded before the proof of Theorem 4.15), we deduce that not only uniqueness holds in the strong sense (as guaranteed by Theorem 4.15) but it also holds in the weak sense. Therefore,

$$\mathbb{E} \left[f(X_T^{(y + \delta v)^*}) \right] = \mathbb{E}_{\mathbb{Q}^{y, (v, \delta), \tau}} \left[f(\tilde{X}_T^{y, (v, \delta), \tau}) \right], \quad (5.21)$$

where we recall the notations (5.16) and (5.17).

Second step.

$$\begin{aligned} \mathbb{Q}^{y, (v, \delta), \tau} \left(\{ \tau_{y, (v, \delta)} < T \} \right) &= \mathbb{Q}^{y, (v, \delta), \tau} \left(\left\{ \sup_{0 \leq t \leq T} \left| \int_0^t \sum_{m \in \mathbb{N}_0} \lambda_m^{-1} \chi_s^{m, y, (v, \delta)} dB_s^m \right| \geq \frac{T}{\delta} \right\} \right) \\ &\leq \frac{\delta^2}{T^2} \mathbb{E}^{y, (v, \delta), \tau} \int_0^T \sum_{m \in \mathbb{N}_0} |\lambda_m^{-1} \chi_s^{m, y, (v, \delta)}|^2 ds. \end{aligned}$$

Using the fact that $d\mathbb{Q}^{y, (v, \delta), \tau} / d\mathbb{P} \leq e$, we can rewrite this as

$$\mathbb{Q}^{y, (v, \delta), \tau} \left(\{ \tau_{y, (v, \delta)} < T \} \right) \leq \frac{\delta^2 e}{T^2} \mathbb{E} \int_0^T \sum_{m \in \mathbb{N}_0} |\lambda_m^{-1} \chi_s^{m, y, (v, \delta)}|^2 ds.$$

Third step. Returning to equation (5.21), one has that, for almost every $y \in E^M$,

$$\begin{aligned} &\mathbb{E} \left[f(X_T^{(y + \delta v)^*}) \right] \\ &= \mathbb{E}_{\mathbb{Q}^{y, (v, \delta), \tau}} \left[f(\tilde{X}_T^{y, (v, \delta), \tau}) \mathbf{1}_{\{T = \tau_{y, (v, \delta)}\}} \right] + \mathbb{E}_{\mathbb{Q}^{y, (v, \delta), \tau}} \left[f(\tilde{X}_T^{y, (v, \delta), \tau}) \mathbf{1}_{\{\tau_{y, (v, \delta)} < T\}} \right] \\ &= \mathbb{E}_{\mathbb{Q}^{y, (v, \delta), \tau}} \left[f(X_T^{y*}) \right] + \mathcal{O} \left(\mathbb{Q}^{y, (v, \delta), \tau} \left(\{ \tau_{y, (v, \delta)} < T \} \right) \right), \end{aligned} \quad (5.22)$$

where the (big) Landau symbol $\mathcal{O}(\cdot)$ is uniform with respect to y , M , δ and v . And then, for $v \in E^M$, and for almost every $y \in E^M$,

$$\begin{aligned} &\left| \mathbb{E} \left[f(X_T^{(y + \delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y*}) \right] \right| \\ &\leq \|f\|_{\infty} d_{TV}(\mathbb{Q}^{y, (v, \delta), \tau}, \mathbb{P}) + \mathcal{O} \left(\mathbb{Q}^{y, (v, \delta), \tau} \left(\{ \tau_{y, (v, \delta)} < T \} \right) \right). \end{aligned}$$

where d_{TV} is the distance in total variation (see [74, p. 22]) By Pinsker's inequality (see [74, Eq. (22.25)]), we get

$$\begin{aligned} & \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| \\ & \leq \|f\|_\infty \sqrt{2 \mathbb{E}_{\mathbb{Q}^{y,(\nu,\delta),\tau}} \left[\ln \left(\frac{d\mathbb{Q}^{y,(\nu,\delta),\tau}}{d\mathbb{P}} \right) \right]} + \mathcal{O} \left(\mathbb{Q}^{y,(\nu,\delta),\tau} (\{\tau_{y,(\nu,\delta)} < T\}) \right). \end{aligned}$$

By (5.16) and (5.17),

$$\begin{aligned} \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| & \leq \frac{\delta \sqrt{e} \|f\|_\infty}{T} \mathbb{E} \left[\int_0^{\tau_{y,(\nu,\delta)}} \sum_{m \in \mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right]^{1/2} \\ & \quad + \mathcal{O} \left(\mathbb{Q}^{y,(\nu,\delta),\tau} (\{\tau_{y,(\nu,\delta)} < T\}) \right). \end{aligned}$$

By the second step, we end up with

$$\begin{aligned} \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| & \leq \frac{\delta \sqrt{e} \|f\|_\infty}{T} \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right]^{1/2} \\ & \quad + \mathcal{O} \left(\frac{\delta^2 e}{T^2} \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right] \right). \quad (5.23) \end{aligned}$$

Fourth Step. By Hölder's inequality,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right] & = \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| (1 \vee m)^\lambda \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right] \\ & \leq \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right]^\lambda \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right]^{1-\lambda}. \end{aligned}$$

To estimate the above, return to Proposition 4.17. Changing therein (x, y) for $((y_t(\delta, \nu) + \mu \nu)^*, y_t^*(\delta, \nu))$ for $\mu > 0$, dividing by μ and letting μ tend to 0, we observe that $\sum_{m \in \mathbb{N}_0} \left| \chi_s^{m,y,(\nu,\delta)} \right|^2 \leq \|v\|_2^2$. Therefore,

$$\begin{aligned} \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} \left| \lambda_m^{-1} \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right] & \leq (T \|v\|_2^2)^{(1-\lambda)} [\mathfrak{E}_T^{y,(\nu,\delta)}]^\lambda \\ & \quad \text{with } \mathfrak{E}_T^{y,(\nu,\delta)} := \mathbb{E} \left[\int_0^T \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \chi_s^{m,y,(\nu,\delta)} \right|^2 ds \right]. \quad (5.24) \end{aligned}$$

Fifth step. By (5.23), and (5.24), there exists a constant C , independent of M , δ , ν and y , such that

$$\begin{aligned} & \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| \\ & \leq \frac{\delta \sqrt{e} \|v\|_2^{1-\lambda} \|f\|_\infty}{T^{(1+\lambda)/2}} [\mathfrak{E}_T^{y,(\nu,\delta)}]^{\lambda/2} + \frac{C \delta^2 \|v\|_2^{2(1-\lambda)}}{T^{1+\lambda}} [\mathfrak{E}_T^{y,(\nu,\delta)}]^\lambda. \end{aligned}$$

Integrating with respect to $y \in E^M$ with respect to a smooth compactly supported density ψ on E^M , using Jensen's inequality and dividing through by δ yields

$$\begin{aligned} & \int_{E^M} \frac{\Psi(y)}{\delta} \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| dy \\ & \leq \frac{\sqrt{e} \|v\|_2^{1-\lambda} \|f\|_\infty}{T^{(1+\lambda)/2}} [\mathfrak{E}_T^{\Psi, (v, \delta)}]^{\lambda/2} + \frac{C\delta \|v\|_2^{2(1-\lambda)}}{T^{1+\lambda}} [\mathfrak{E}_T^{\Psi, (v, \delta)}]^\lambda, \end{aligned} \quad (5.25)$$

where $\mathfrak{E}_T^{\Psi, (v, \delta)} := \int_{E^M} \mathfrak{E}_T^{y, (v, \delta)} \Psi(y) dy$, that is

$$\begin{aligned} \mathfrak{E}_T^{y, (v, \delta)} &= \int_0^T \int_{E^M} \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \partial_y \langle e_m, X_s^{y^*} \rangle \cdot v \right|^2 \Psi(y) dy ds \\ &= \int_0^T \int_{E^M} \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \partial_y \langle e_m, X_s^{y^*} \rangle \cdot v \right|^2 \Psi(y_s(v, -\delta)) dy ds, \end{aligned}$$

with the last line following from a change of variable (as done in (5.14)).

Assume for a while (the proof is given right below) that we have a deterministic bound for $\int_0^T \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \partial_y \langle e_m, X_t^{y^*} \rangle \cdot v \right|^2 dt$, independently of y (in a full subset of E^M). Then, by Lebesgue's dominated convergence theorem, we can let δ to 0 in (5.25):

$$\liminf_{\delta \rightarrow 0} \int_{E^M} \frac{\Psi(y)}{\delta} \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| dy \leq \frac{\sqrt{e} \|v\|_2^{1-\lambda} \|f\|_\infty}{T^{(1+\lambda)/2}} [\mathfrak{E}_T^{\Psi, (v, 0)}]^{\lambda/2}.$$

To estimate the right-hand side, we proceed as in the derivation of (5.24). We return back to (4.40) in Proposition 4.17, this time swapping (x, y) therein for $((y + \mu v)^*, y^*)$ for $\mu > 0$, divide by μ and let μ tend to 0. Subsequently, we let ε in (4.40) tend to 0 and deduce by Fatou's lemma that

$$\mathfrak{E}_T^{y, (v, 0)} = \int_0^T \sum_{m \in \mathbb{N}_0} (1 \vee m)^2 \left| \partial_y \langle e_m, X_s^{y^*} \rangle \cdot v \right|^2 ds \leq \|v\|_2^2,$$

for almost every $y \in E^M$, which gives

$$\liminf_{\delta \rightarrow 0} \int_{E^M} \frac{\Psi(y)}{\delta} \left| \mathbb{E} \left[f(X_T^{(y+\delta v)^*}) \right] - \mathbb{E} \left[f(X_T^{y^*}) \right] \right| dy \leq \frac{\sqrt{e} \|v\|_2 \|f\|_\infty}{T^{(1+\lambda)/2}}. \quad (5.26)$$

Last Step. We now assume that f itself is Lipschitz continuous. By Lipschitz continuity of the flow $(X_T^{y^*})_{y \in E^M}$, the mapping $P_T^M f : y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is Lipschitz continuous (with respect to the Euclidean norm on E^M , which coincides with the $L^2(\mathbb{S})$ -norm) and thus almost everywhere differentiable. Then, the left-hand side in (5.26) is equal to $\int_{E^M} \Psi(y) |\nabla_y P_T^M f(y) \cdot v| dy$. Since the bound is true for any density Ψ , we get that the almost everywhere gradient of $y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is less than $\sqrt{e} \|f\|_\infty T^{-(1+\lambda)/2}$. Therefore, the Lipschitz constant of $y \in E^M \mapsto \mathbb{E}[f(X_T^{y^*})]$ is less

than $\sqrt{\varepsilon}\|f\|_\infty T^{-(1+\lambda)/2}$, with E^M being equipped with the $L^2_{\text{sym}}(\mathbb{S})$ -norm. Approximating any $y \in L^2_{\text{sym}}(\mathbb{S})$ by a sequence $(y^M \in E^M)_{M \geq 1}$, we see that the result remains true when the function $y \mapsto \mathbb{E}[f(X_T^{y^*})]$ is considered on the entire $L^2_{\text{sym}}(\mathbb{S})$.

It remains to pass from a Lipschitz function f to a merely bounded (measurable) function, but this may be regarded as a consequence of standard results in measure theory, see for instance [64, Lemma 2.2 p.160]. \square

5.5 Proofs of Lemma 5.3 and Proposition 5.4

Throughout, we use the notations introduced in Subsection 5.1. We then start with the following lemma that exposes a subtlety when changing variables in the integral (5.4).

Lemma 5.11 *Let $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ satisfy **(F1)**–**(F2)** and $((z_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ be a deterministic jointly continuous $L^2_{\text{sym}}(\mathbb{S})$ -valued flow, i.e., the map $(t, y) \in [0, T] \times E^{*,M} \mapsto z_t^y \in L^2_{\text{sym}}(\mathbb{S})$ is continuous. Then, for any $\varepsilon > 0$, for any family $\{r_i^N\}_{i=0, \dots, N}$ of subdivision points of $[0, T]$ with $\lim_{N \rightarrow \infty} \sup_{i=1, \dots, N} |r_i^N - r_{i-1}^N| = 0$, the following identity holds true (for convenience we merely write r_i for r_i^N):*

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varphi} &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta_{z_{r_i}^*(v, \delta)}} \right\rangle - \left\langle n_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta_{z_{r_i}^*(v, \delta)}} \right\rangle \right] dy, \\ \text{with } \mathcal{I}_{\varepsilon, \varphi} &:= \sum_{m \in \mathbb{N}_0} \int_{E^M} \left\{ \int_0^T \langle e^{\varepsilon \Delta_{z_t^*}}, e_m \rangle \varphi(y_t(v, -\delta)) d \langle n_t^{y^*}, e_m \rangle \right\} dy. \end{aligned} \quad (5.27)$$

We remark that the first superscript in the first line above is not $y_{r_{i+1}}^*(v, \delta)$ as one might expect.

Proof We let

$$\mathcal{I}_{\varepsilon, \varphi}^m := \int_{E^M} \left\{ \int_0^T \langle e^{\varepsilon \Delta_{z_t^*}}, e_m \rangle \varphi(y_t(v, -\delta)) d \langle n_t^{y^*}, e_m \rangle \right\} dy.$$

By Corollary 4.8, one has for a fixed value of $m \in \mathbb{N}_0$:

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varphi}^m &= \int_{E^M} \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \langle e^{\varepsilon \Delta_{z_{r_i}^*}}, e_m \rangle \varphi(y_{r_i}(v, -\delta)) \left[\langle n_{r_{i+1}}^{y^*}, e_m \rangle - \langle n_{r_i}^{y^*}, e_m \rangle \right] dy \\ &= \lim_{N \rightarrow \infty} \int_{E^M} \sum_{i=0}^{N-1} \langle e^{\varepsilon \Delta_{z_{r_i}^*}}, e_m \rangle \varphi(y_{r_i}(v, -\delta)) \left[\langle n_{r_{i+1}}^{y^*}, e_m \rangle - \langle n_{r_i}^{y^*}, e_m \rangle \right] dy, \end{aligned}$$

the argument for exchanging the limit and the sum following from Lebesgue's dominated convergence theorem. From Lemma 4.4, it is indeed clear that the sum

over i on the second line is uniformly bounded in N . Therefore, performing for each $i \in \{0, \dots, N-1\}$ an obvious change of variable for the integral in y , we get

$$\mathcal{J}_{\varepsilon, \varphi}^m = \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left\langle e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle \left[\left\langle \eta_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle - \left\langle \eta_{r_i}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle \right] dy.$$

In fact, Lemma 4.4 says more: the argument inside the limit decays polynomially fast with m , uniformly in N . In particular, summing over $m \in \mathbb{N}_0$, one can exchange the sum over m and the limit over N . Since $\sum_{m \in \mathbb{N}_0} \mathcal{J}_{\varepsilon, \varphi}^m = \mathcal{J}_{\varepsilon, \varphi}$, we obtain

$$\begin{aligned} \mathcal{J}_{\varepsilon, \varphi} &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \sum_{m \in \mathbb{N}_0} \left\langle e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle \left[\left\langle \eta_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle - \left\langle \eta_{r_i}^{y_{r_i}^*(v, \delta)}, e_m \right\rangle \right] dy \\ &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)}, \eta_{r_{i+1}}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)}, \eta_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy, \end{aligned}$$

which is the desired result. \square

The bulk of the analysis carried out in this subsection is the following statement:

Proposition 5.12 *Let $((n_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ satisfy **(F1)**–**(F2)**, and $((z_t^y)_{0 \leq t \leq T})_{y \in E^{*,M}}$ be a deterministic $L_{\text{sym}}^2(\mathbb{S})$ -valued flow such that $(t, y) \in [0, T] \times E^{*,M} \mapsto z_t^y \in L_{\text{sym}}^2(\mathbb{S})$ is continuous. Then, with the same notation as in (5.27), for $\varepsilon > 0$ and for $\varphi \in \mathcal{C}_0^\infty(E^M)$,*

$$\begin{aligned} &\mathcal{J}_{\varepsilon, \varphi} \\ &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{n}_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle \tilde{n}_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy. \end{aligned} \quad (5.28)$$

Proof (of Proposition 5.12) It suffices to prove (5.28) for a flow $((z_t^y)_{0 \leq t \leq T})_{y \in E^M}$, differentiable in y , with derivative jointly continuous in (t, y) . Indeed, by a mollification argument in the variable y , we can approximate any $((z_t^y)_{0 \leq t \leq T})_{y \in E^M}$ that is only continuous in (t, y) by a flow that is regular in y (with jointly continuous derivatives) and use (4.23) in order to pass to the limit in (5.28). We thus assume below that $((z_t^y)_{0 \leq t \leq T})_{y \in E^M}$ is differentiable in y , with derivative jointly continuous in (t, y) .

Another key observation is that $(t, y) \in [0, T] \times E^M \mapsto n_t^{y*} \in H_{\text{sym}}^{-2}(\mathbb{S})$ is jointly continuous in (t, y) and thus uniformly continuous on $[0, T] \times \text{Supp}(\varphi)$, with $\text{Supp}(\varphi)$ denoting the support of φ . This follows from **(F2)** and the fact that the map $t \in [0, T] \mapsto n_t^{y*} \in \mathbb{H}_{\text{sym}}^{-2}(\mathbb{S})$ is continuous for each $y \in E^M$. By Lemma 5.11, one has

$$\begin{aligned} \mathcal{J}_{\varepsilon, \varphi} &= \lim_{N \rightarrow \infty} \int_{E^M} \varphi(y) \left\{ \sum_{i=0}^{N-1} \left[\left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)} \right\rangle \right. \right. \\ &\quad \left. \left. + \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_{i+1}}^*(v, \delta)} \right\rangle - \left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] \right. \\ &\quad \left. + \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle n_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right\} dy, \end{aligned}$$

which, by exchanging the first and third lines in the summand, can be rewritten

$$\begin{aligned} \mathcal{I}_{\varepsilon, \varphi} &= \lim_{N \rightarrow \infty} \left\{ \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle n_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy \right. \\ &\quad + \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} \left(z_{r_i}^{y_{r_{i+1}}^*(v, \delta)} - z_{r_i}^{y_{r_i}^*(v, \delta)} \right) \right\rangle dy \\ &\quad \left. + \sum_{i=0}^{N-1} \int_{EM} \left[\varphi(y) - \varphi\left(y + \delta \frac{r_{i+1} - r_i}{T} v\right) \right] \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle dy \right\} \\ &=: \lim_{N \rightarrow \infty} \left\{ T_1^N + T_2^N + T_3^N \right\}. \end{aligned}$$

Analysis of T_1^N . By applying Definition 5.1 (at point $y_t(v, \delta)$ instead of y) and by using the fact that $(\tilde{n}_t^y)_{0 \leq t \leq T}$ takes values in $H_{\text{sym}}^{-2}(\mathbb{S})$, we get

$$\begin{aligned} T_1^N &= \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{n}_{r_{i+1}}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle \tilde{n}_{r_i}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy \\ &\quad - \frac{\delta}{T} \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \sum_{m \in \mathbb{N}_0} \left[\int_{r_i}^{r_{i+1}} \left(\partial_w \left[\left\langle n_s^{w^*}, e_m \right\rangle \right]_{|w=y_s(v, \delta)} \cdot v \right) \left\langle z_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} e_m \right\rangle ds \right] dy \\ &= \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{n}_{r_{i+1}}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle \tilde{n}_{r_i}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy \\ &\quad - \frac{\delta}{T} \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \sum_{m \in \mathbb{N}_0} \left[\int_{r_i}^{r_{i+1}} \left(\partial_y \left[\left\langle n_s^{y_s^*(v, \delta)}, e_m \right\rangle \right] \cdot v \right) \left\langle z_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} e_m \right\rangle ds \right] dy. \end{aligned}$$

Exchanging the integral in y and the sum over m (which is possible thanks to Lemma 4.4) and then performing an integration by parts in the last line, we obtain

$$\begin{aligned} T_1^N &= \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left[\left\langle \tilde{n}_{r_{i+1}}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle \tilde{n}_{r_i}^{y, (v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy \\ &\quad + \frac{\delta}{T} \sum_{i=0}^{N-1} \int_{r_i}^{r_{i+1}} \int_{EM} \left(\partial_y \varphi(y) \cdot v \right) \left\langle n_s^{y_s^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle dy ds \\ &\quad + \frac{\delta}{T} \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \sum_{m \in \mathbb{N}_0} \left[\int_{r_i}^{r_{i+1}} \left\langle n_s^{y_s^*(v, \delta)}, e_m \right\rangle \left(\partial_y \left[\left\langle z_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} e_m \right\rangle \right] \cdot v \right) ds \right] dy \\ &=: T_{1,1}^N + T_{1,2}^N + T_{1,3}^N. \end{aligned} \tag{5.29}$$

Analysis of $T_{1,3}^N + T_2^N$. Using the regularity of the flow $((z_t^y)_{0 \leq t \leq T})_{y \in E^*, M}$, we write

$$\begin{aligned} T_2^N &= \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e^{\varepsilon \Delta} \left(z_{r_i}^{y_{r_{i+1}}^*(v, \delta)} - z_{r_i}^{y_{r_i}^*(v, \delta)} \right) \right\rangle dy \\ &= -\frac{\delta}{T} \int_{EM} \varphi(y) \sum_{i=0}^{N-1} (r_{i+1} - r_i) \sum_{m \in \mathbb{N}_0} \left\langle n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e_m \right\rangle \left(\partial_y \left[\left\langle z_{r_i}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} e_m \right\rangle \right] \cdot v \right) dy \\ &\quad + \sum_{i=0}^{N-1} \varphi(r_{i+1} - r_i), \end{aligned}$$

where \mathcal{O} is the little Landau symbol (and is here implicitly understood to be uniform in N and i). And, then using the joint regularity of $((n_t^y)_{0 \leq t \leq T})_{y \in E^*, M}$, we get

$$\begin{aligned} \lim_{N \rightarrow \infty} (T_2^N + T_{1,3}^N) &= \lim_{N \rightarrow \infty} \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \sum_{m \in \mathbb{N}_0} \left[\int_{r_i}^{r_{i+1}} \left\langle n_s^{y_{r_i}^*(v, \delta)} - n_{r_{i+1}}^{y_{r_{i+1}}^*(v, \delta)}, e_m \right\rangle \right. \\ &\quad \left. \times \left(\partial_y \left[\left\langle z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle, e^{\varepsilon \Delta} e_m \right] \cdot v \right) ds \right] dy = 0. \end{aligned}$$

Analysis of $T_{1,2}^N + T_3^N$. Adding and subtracting the quantity

$$\frac{\delta}{T} \sum_{i=0}^{N-1} \int_{EM} (\partial_y \varphi(y) \cdot v) (r_{i+1} - r_i) \left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle dy,$$

we have

$$\begin{aligned} &T_{1,2}^N + T_3^N \\ &= \sum_{i=0}^{N-1} \frac{\delta}{T} \int_{r_i}^{r_{i+1}} \int_{EM} (\partial_y \varphi(y) \cdot v) \left[\left\langle n_s^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle - \left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right] dy ds \\ &\quad + \sum_{i=0}^{N-1} \int_{EM} \left\{ \left[\varphi(y) + \left(\frac{\delta}{T} (r_{i+1} - r_i) \right) \partial_y \varphi(y) \cdot v - \varphi \left(y + \frac{\delta}{T} (r_{i+1} - r_i) v \right) \right] \right. \\ &\quad \left. \times \left\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \right\rangle \right\} dy \\ &=: T_{(1,2,3),1}^N + T_{(1,2,3),2}^N. \end{aligned} \tag{5.30}$$

Analysis of $T_{(1,2,3),1}^N + T_{(1,2,3),2}^N$. We claim that the limits of the two terms in the above argument are 0 as we can write both of them in the form $\sum_{i=0}^{N-1} \mathcal{O}(r_{i+1} - r_i)$.

The limit of $(T_{(1,2,3),2}^N)_{N \geq 1}$ is easily handled by using the fact that φ is smooth and by invoking the duality between $H_{\text{sym}}^2(\mathbb{S})$ and $H_{\text{sym}}^{-2}(\mathbb{S})$ (to handle terms of the form $\langle n_{r_{i+1}}^{y_{r_i}^*(v, \delta)}, e^{\varepsilon \Delta} z_{r_i}^{y_{r_i}^*(v, \delta)} \rangle$).

The limit of $(T_{(1,2,3),1}^N)_{N \geq 1}$ is shown to be 0 by invoking the fact that the mapping $(t, y) \in [0, T] \times E^M \mapsto \eta_t^y \in \mathbb{H}_{\text{sym}}^{-2}(\mathbb{S})$ is jointly continuous in (t, y) and thus uniformly continuous on $[0, T] \times \text{Supp}(\varphi)$.

Conclusion. Back to (5.30), we deduce from the above analysis that

$$\lim_{N \rightarrow \infty} \left\{ T_1^N + T_2^N + T_3^N \right\} = \lim_{N \rightarrow \infty} \left\{ T_{1,1}^N + T_{1,2}^N + T_{1,3}^N + T_2^N + T_3^N \right\} = \lim_{N \rightarrow \infty} T_{1,1}^N.$$

This completes the proof. \square

We now apply Proposition 5.12 to the proofs of Lemma 5.3 and Proposition 5.4. In order to do so, we assume that the flow $((z_t^y)_{0 \leq t \leq T})_{y \in E^*, M}$ in Proposition 5.12 reduces to one single trajectory $(z_t)_{0 \leq t \leq T}$. Recalling the notation (5.11) and following

the derivation of (5.20), we observe that the left-hand side in (5.28) (whose explicit form is given in (5.27)) can be rewritten as

$$\mathcal{I}_{\varepsilon, \varphi} = \int_{EM} \left(\int_0^T \langle e^{\varepsilon \Delta \tilde{z}_t^{y, \varphi, (v, \delta)}}, dn_t^{y*} \rangle \right) dy. \quad (5.31)$$

In order to handle the right-hand side of (5.28), we recall (5.10). We observe that the argument in the limit appearing in the right-hand side of (5.28) can be rewritten

$$\begin{aligned} & \int_{EM} \varphi(y) \sum_{i=0}^{N-1} \left[\langle \tilde{n}_{r_{i+1}}^{y, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle - \langle \tilde{n}_{r_i}^{y, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle \right] dy \\ &= \sum_{i=0}^{N-1} \left[\langle \tilde{n}_{r_{i+1}}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle - \langle \tilde{n}_{r_i}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle \right]. \end{aligned} \quad (5.32)$$

Proof (of Lemma 5.3 and Proposition 5.4) With $z \in U^2(\mathbb{S})$, we apply Proposition 5.12 with $(z_t := \mathbf{1}_{[r, s]}(t)z)_{0 \leq t \leq T}$ for a given pair $(r, s) \in [0, T]^2$ satisfying $r < s$. By (5.11), (5.31) and (5.32), we obtain

$$\int_{EM} \left(\int_0^T \langle e^{\varepsilon \Delta \tilde{z}_t^{y, \varphi, (v, \delta)}}, dn_t^{y*} \rangle \right) dy = \lim_{N \rightarrow \infty} \sum_{i=0}^{N-1} \left[\langle \tilde{n}_{r_{i+1}}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle - \langle \tilde{n}_{r_i}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle \right]. \quad (5.33)$$

Regardless the choice of the subdivision $\{r_i^N\}_{i=0, \dots, N}$ (recall that we omit the superscript N in the various equations), we have

$$\begin{aligned} & \sum_{i=0}^{N-1} \left[\langle \tilde{n}_{r_{i+1}}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle - \langle \tilde{n}_{r_i}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle \right] \\ &= \sum_{i=0}^{N-1} \left\{ \left[\langle \tilde{n}_{r_{i+1}}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle - \langle \tilde{n}_{r_i}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle \right] \mathbf{1}_{[r, s]}(r_i) \right\}. \end{aligned}$$

If we assume that r and s belong to the collection $\{r_i^N\}_{i=0, \dots, N}$, which can be done without any loss of generality, we get

$$\sum_{i=0}^{N-1} \left[\langle \tilde{n}_{r_{i+1}}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle - \langle \tilde{n}_{r_i}^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z_{r_i}} \rangle \right] = \langle \tilde{n}_s^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle - \langle \tilde{n}_r^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle.$$

Then, (5.33) yields

$$\langle \tilde{n}_s^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle - \langle \tilde{n}_r^{\varphi, (v, \delta)}, e^{\varepsilon \Delta z} \rangle = \int_{EM} \left(\int_0^T \langle e^{\varepsilon \Delta \tilde{z}_t^{y, \varphi, (v, \delta)}}, dn_t^{y*} \rangle \right) dy.$$

Recalling the definition (5.11) and using the fact that $\varphi \geq 0$, we observe that $(\tilde{z}_t^{y, \varphi, (v, \delta)})_{0 \leq t \leq T}$ takes values in $U^2(\mathbb{S})$. By Corollary 4.8, the right-hand side is non-negative. Assuming that $z \in H_{\text{sym}}^2(\mathbb{S})$ and letting ε tend to 0, we complete the proof of Lemma 5.3. In turn, this permits us to invoke Lemma 4.7 to let N tend to ∞ in the right-hand side of (5.32). This gives the right-hand side in (5.12). As for the left-hand side in (5.32), the limit is given by Proposition 5.12 and identifies with the right-hand side of (5.31). This gives the left-hand side in (5.12) and proves Proposition 5.4. \square

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