Limit trees for free group automorphisms: universality

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Abstract

To any free group automorphism, we associate a universal (cone of) limit tree(s) with three defining properties: first, the tree has a minimal isometric action of the free group with trivial arc stabilizers; second, there is a unique expanding dilation of the tree that represents the free group automorphism; and finally, the loxodromic elements are exactly the elements that weakly limit to dominating attracting laminations under forward iteration by the automorphism. So the action on the tree detects the automorphism's dominating exponential dynamics.

As a corollary, our previously constructed limit pretree that detects the exponential dynamics is canonical. We also characterize all very small trees that admit an expanding homothety representing a given automorphism. In the appendix, we prove a variation of Feighn–Handel's recognition theorem for atoroidal outer automorphisms.

Introduction

We previously constructed a limit pretree that detects the exponential dynamics for an arbitrary free group automorphism [22]. In this sequel, we prove the construction is canonical. This completes the existence and uniqueness theorem for a free group automorphism's limit pretree. Recall that if we record all the compact geodesics in an $\mathbb{R}$-tree but forget their lengths, then the resulting structure is a pretree; briefly, a pretree is a set with a structure that encodes the notion of closed intervals satisfying certain axioms. Pretrees are the baseline of our constructions; for instance, ($\mathbb{R}$-)trees will be defined as pretrees with convex metrics, and pseudotrees as pretrees with a certain hierarchy of convex pseudometrics.

In [22], we motivated the existence and uniqueness theorem of a limit pretree by describing it as a free group analogue to the Nielsen–Thurston theory for surface homeomorphisms, which in turn can be seen as the surface analogue to the Jordan canonical form for linear maps. We now give our own motivation for this result.

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Universal representation of an endomorphism

It feels rather odd to discuss my personal motivation while using the communal "we"; excuse me as I break this convention a bit for this section. In my doctoral thesis, I extended Brinkmann's hyperbolization theorem to mapping tori of free group endomorphisms. This required studying the dynamics of endomorphisms. Along the way, I proved that injective endomorphisms have canonical representatives. More precisely, suppose $\phi\colon F \to F$ is an injective endomorphism of a finitely generated free group; then there is:

1. a minimal simplicial F-action on a simplicial tree T with trivial edge stabilizers;
2. a ϕ-equivariant expanding embedding $f\colon T \to T$ (unique up to isotopy); and
3. an element in F is T-elliptic if and only if one of its forward ϕ-iterates is conjugate to an element in a $[\phi]$-periodic free factor of F.

Existence of the limit free splitting (i.e. T with its F-action) for the outer class $[\phi]$ was the core of my thesis (see also [21, Theorem 3.4.5]). Universality follows from bounded cancellation: any other simplicial tree T' satisfying these three condition will be uniquely equivariantly isomorphic to T [21, Proposition 3.4.6].

In a way, the limit free splitting detects and filters the "nonsurjective dynamics" of the (outer) endomorphism. When $\phi\colon F \to F$ is an automorphism, then T is a singleton and the free splitting provides no new information. On the other extreme, the F-action on T can be free; in this case, let $\Gamma := F\backslash T$ be the quotient graph. Then the outer endomorphism $[\phi]$ is represented by a unique expanding immersion $[f]\colon \Gamma \to \Gamma$ and $[\phi]$ is *expansive* — such outer endomorphisms are characterized by the absence of $[\phi]$-periodic (conjugacy classes of) nontrivial free factors [21, Corollary 3.4.8]. The most important thing is that the expanding immersion $[f]$ has nice dynamics and greatly simplifies the study of expansive outer endomorphisms.

After completing my thesis, I found myself in a paradoxical situation: I had a better "understanding" of nonsurjective endomorphisms than automorphisms — the main obstacle to studying the dynamics of nonsurjective endomorphisms was understanding the dynamics of automorphisms. The naïve expectation (when I started my thesis) had been that nonsurjective endomorphisms have more complicated dynamics as they are not invertible. The current project was born out of an obligation to correct this imbalance.

Universal representation of an automorphism

What follows is a direct analogue of the above discussion in the setting of automorphisms. The main theorem of [22] produces an action that detects and filters the "exponential" dynamics of an automorphism. Specifically, suppose $\phi\colon F \to F$ is an automorphism of a finitely generated free group. Then there is:

1. a minimal rigid F-action on a real pretree T with trivial arc stabilizers;

2. a ϕ-equivariant "expanding" pretree-automorphism $f\colon T \to T$; and

3. an element in F is T-elliptic if and only if it grows polynomially with respect to $[\phi]$.

The pair of the pretree T and its rigid F-action is called a *(forward) limit pretree* for the outer automorphism $[\phi]$. The theorem is stated properly in Chapter III as Theorem III.1. When $[\phi]$ is polynomially growing, then *the* limit pretree is a singleton (and hence unique) but provides no new information. We are mainly interested in exponentially growing $[\phi]$ as their limit pretrees are not singletons. On the other hand, the F-action on a limit pretree is free if and only if $[\phi]$ is *atoroidal*, i.e. there are no $[\phi]$-periodic (conjugacy classes of) nontrivial elements [22, Corollary III.5]. As with expanding immersions and expansive outer endomorphisms, the expanding "homeomorphism" $[f]$ (on the quotient space $F \backslash T$) has dynamics that could facilitate the study of atoroidal outer automorphisms.

Unlike the endomorphism case, uniqueness of limit pretrees requires a more involved argument. It was remarked in the epilogue of [22] that the only source of indeterminacy in the existence proof was [22, Proposition III.2]; this proposition is restated in Section I.4 as Proposition I.2 and a proof is sketched in Sections II.1 and II.4. The main result of this paper is a universal version of the proposition. It can also be thought of as an existence and uniqueness theorem for an action that detects and filters the "dominating" exponential dynamics of an outer automorphism:

Main Theorem (Theorems III.10–III.11)**.**

Let $\phi\colon F \to F$ be an automorphism of a finitely generated free group and $\{\mathcal{A}_j^{dom}[\phi]\}_{j=1}^k$ a (possibly empty) subset of $[\phi]$-orbits of dominating attracting laminations for $[\phi]$.

Then there is:

1. *a minimal factored F-tree $(Y, \Sigma_{j=1}^k \delta_j)$ with trivial arc stabilizers;*

2. *a unique ϕ-equivariant expanding dilation $f\colon (Y, \Sigma_{j=1}^k \delta_j) \to (Y, \Sigma_{j=1}^k \delta_j)$; and*

3. *for $1 \leq j \leq k$, a nontrivial element in F is δ_j-*loxodromic *if and only if its forward ϕ-iterates have axes that weakly limit to $\mathcal{A}_j^{dom}[\phi]$;*

moreover, the factored F-tree $(Y, \Sigma_{j=1}^k \delta_j)$ is unique up to a unique equivariant dilation.

Thus the factored tree (up to rescaling of its factors δ_j) is a universal construction for outer automorphisms of free groups, and we call it *the complete dominating (resp. topmost) tree* if we consider the whole set of orbits of dominating (resp. topmost) attracting laminations. As a corollary, the previously constructed limit pretrees are independent of the choices made in the proof of Theorem III.1, i.e. *the* limit pretree is canonical (Corollary III.9). Let us now briefly define the emphasized terms in the theorem's statement.

An *F-tree* is an ($\mathbb{R}$-)tree with an isometric F-action. Informally, an F-tree is *factored* if its metric has been equivariantly decomposed as a sum $\sum_{j=1}^k \delta_j$ of pseudometrics. For a factored F-tree $(Y, \Sigma_{j=1}^k \delta_j)$, an element in F is *δ_i-loxodromic* if it is it is Y-loxodromic and its axis has positive δ_i-diameter. An equivariant homeomorphism $(T, \Sigma_{j=1}^k d_j) \to (Y, \Sigma_{j=1}^k \delta_j)$ of factored F-trees is a *dilation* if it is a homothety of each pair of *factors* d_j and δ_j; a dilation is *expanding* if each factor-homothety is expanding.

A *lamination* in F is a nonempty closed subset in the space of lines in F. A sequence of lines (e.g. axes) *weakly limits* to a lamination if some subsequence converges to the lamination. Any $[\phi]$ has a finite set of *attracting laminations* which is empty if and only if $[\phi]$ is polynomially growing; this set is partially ordered by inclusion and has an order-preserving $[\phi]$-action. The maximal elements of the partial order are called *topmost*. An attracting lamination A for $[\phi]$ has an associated *stretch factor* $\lambda(A)$; it is *dominating* if any distinct attracting lamination A' for $[\phi]$ containing A has a strictly smaller stretch factor $\lambda(A') < \lambda(A)$. Topmost attracting laminations are vacuously dominating; moreover, the $[\phi]$-action permutes the dominating attracting laminations.

Remark. If one considers a subset $\{\mathcal{A}_j^{top}[\phi]\}_{j=1}^k$ of $[\phi]$-orbits of topmost attracting laminations, then we prove the topmost tree has the additional property that its factor-pseudometrics are pairwise *mutually singular*: for each i, there is an element that is δ_i-loxodromic but δ_j-elliptic for $j \neq i$ (see Section III.4). We highlight this feature by using the notation $(Y, \oplus_{j=1}^k \delta_j)$ for topmost trees.

Some applications of universal representations. Fix an automorphism $\phi\colon F \to F$; since $[\phi]$ has a unique equivariant dilation class $[Y, \Sigma_{j=1}^k \delta_j]$ of complete dominating limit trees, any invariant of the class is automatically an invariant of $[\phi]$. For instance, the Gaboriau–Levitt index $i(Y)$ (as defined in [11, Chapter III]) is the *dominating forward index for* $[\phi]$. In fact, since the limit pretree T for $[\phi]$ is canonical, its index $i(T)$ (defined in [22, Appendix A]) is the *exponential (forward) index for* $[\phi]$; when $[\phi]$ is atoroidal, the index $i(T)$ is closely related to the Gaboriau–Jaeger–Levitt–Lustig index for $[\phi]$ defined in [10, Section 6]. Each factor δ_j has an associated F-tree (Y_j^{dom}, δ_j); the pairing of δ_j with the orbit of dominating attracting lamination $\mathcal{A}_j^{dom}[\phi]$ means $i(Y_j^{dom})$ is an index for $\mathcal{A}_j^{dom}[\phi]$ respectively.

Our main application is a characterization of minimal F-trees with ϕ-equivariant expanding homotheties:

Main Corollary (Theorem V.3)**.**
Let $\phi\colon F \to F$ be an automorphism and (Y, δ) a minimal very small *F-tree. The F-tree (Y, δ) admits a ϕ-equivariant expanding homothety if and only if it is equivariantly isometric to the dominating tree for $[\phi]$ with respect to a subset of $[\phi]$-orbits of dominating attracting laminations with the same stretch factor.*

In the appendix, we prove a variation of Feighn–Handel's recognition theorem for atoroidal outer automorphisms.

Some historical context

This paper continues Gaboriau–Levitt–Lustig's philosophy of prioritizing limit trees in their alternative proof of the Scott conjecture [12]. In particular, our paper relies only on the existence of irreducible train tracks [4, Section 1] but none of the typical splitting paths analysis of relative train tracks [3, 9]. Zlil Sela gave another dendrogical proof the conjecture (now Bestvina–Handel's theorem) that used Rips' theorem in place of train track technology [25]. Frédéric Paulin gave yet another dendrological proof that avoids both train tracks and Rips' theorem [23].

About the same time, Bestvina–Fieghn–Handel used train tracks and trees to prove fully irreducible (outer) automorphisms have universal limit trees [2]. They used this to give a short dendrological proof of a special case of the Tits Alternative for $\mathrm{Out}(F)$; their later proof of the general case was much more involved due to the lack of such a universal limit construction [3]. Universal limit trees have been indispensable for studying fully irreducible automorphisms. In principle, a universal construction of limit trees for all automorphisms would lead to a dendrological proof of the Tits alternative and extend much of the theory for fully irreducible automorphisms to arbitrary automorphisms. Speaking of dendrological proofs of the Tits alternative, we mention that Camille Horbez gave such a proof with a very different approach [15].

Continuing the work started in [3], Feighn–Handel defined and proved the existence of completely split relative train tracks (CTs) in [9, Section 4]; they use CTs to characterize abelian subgroup of $\mathrm{Out}(F)$ [8]. The main obstacle when working with topological representatives is that they are not canonical, which can make defining invariants of the outer automorphism quite technical. This is the difficulty that we had to deal with in this paper; however, now that it is done, we can use our new universal representatives to define other invariants rather easily. A minor inconvenience when working with CTs is that they are only proven to exist for some (uniform) iterate of the outer automorphism; we were very careful (perhaps to a fault) in this paper to ensure our universal representatives exist for all outer automorphisms. Finally, a subtle advantage to our approach is that we find universal representatives for automorphisms and not just outer automorphisms!

In a sequel to [25], Sela used limit trees and Rips' theorem to give a canonical hierarchical decomposition of the free group F that is invariant under a given atoroidal automorphism [24]. This second paper was never published and a third announced paper that extends the canonical decomposition to arbitrary automorphisms was never released even as a preprint (as far as we know). We remark that the limit trees used in that paper were not (or rather, were never proven to be) canonical/universal. Perhaps, one could combine Sela's canonical decomposition with Bestvina–Feighn–Handel's work to give a universal construction of limit trees for atoroidal automorphisms — our approach is independent of Sela's work and applies more generally to exponentially growing automorphisms. Conversely, we suspect that a careful study of the structure of our topmost trees might recover Sela's canonical hierarchical decomposition.

Morgan–Shalen introduced the term "$\mathbb{R}$-trees" in [20]. They also defined "Λ-trees" for an ordered abelian group Λ. At first glance, the hierarchy of pseudometrics on a real pretree (defined in Section I.2) looks like a Λ-tree. But paths in our constructed hierarchies "exit" infinitesimal trees through metric completion points; whereas paths in a Λ-tree exit at infinity. Hierarchies appear to be a new construction to the best of our knowledge.

Proof outline for existence of topmost tree (Theorem III.7)

One method for constructing limit trees is iterating *expanding irreducible train tracks*. This is carried out in Section II.1 but it has two drawbacks: exponentially growing automorphisms do not always have expanding irreducible train tracks; and even when they do, the point stabilizers of the corresponding limit tree are not canonical as they can change with the choice of train tracks. We handle the first obstacle in Section II.4 by constructing a limit tree (Y_1, δ_1) using a descending sequence of irreducible train tracks, where only the last train track is expanding. Such descending sequences always exist for exponentially growing automorphisms.

Next, we construct in Section III.1 a pretree with an F-action whose point stabilizers are canonical. Set $G_1 \coloneqq F$, and let $\mathcal{G}_2$ be the $[\phi]$-invariant subgroup system determined by the point stabilizers of G_1 acting on Y_1. By restricting $[\phi]$ to $\mathcal{G}_2$ and inductively repeating the construction, we get a descending sequence of limit forests $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has ($[\phi]$-orbits of) attracting laminations $\mathcal{A}_i[\phi]$ for $[\phi]$ that are forward limits of $\mathcal{Y}_i$-loxodromic elements in $\mathcal{G}_i$. Starting with $X^{(1)} = Y_1$, equivariantly replace the points in X^i fixed by $\mathcal{G}_{i+1}$ with the pretrees $\mathcal{Y}_{i+1}$ to produce $X^{(i+1)}$ for $i < n$. The limit pretree $T = X^{(n)}$ has canonical point stabilizers: the maximal polynomially growing subgroups.

Everything we have mentioned so far is a rehash of [22]. From the blow-up construction, the limit pretree T inherits an F-invariant hierarchy $(\delta_i)_{i=1}^n$ of convex pseudometrics — the pseudometric δ_i is defined on maximal $\mathcal{G}_i$-invariant convex subsets of T of δ_{i-1}-diameter 0. The theorem is finally proven in Section III.4. The new insight for this proof: if attracting laminations $\mathcal{A}_i[\phi]$ are topmost, then the $\mathcal{G}_i$-invariant pseudometric δ_i can be extended to an F-invariant convex pseudometric, still denoted δ_i, on T. Let $\{\mathcal{A}_{\iota(j)}[\phi]\}_{j=1}^k$ be a subset of topmost attracting laminations. The sum of the corresponding F-invariant pseudometrics on T, denoted $\oplus_{j=1}^k \delta_{\iota(j)}$, is an F-invariant convex pseudometric on T. Let Y be the partition of T into its maximal subsets of $\oplus_{j=1}^k \delta_{\iota(j)}$-diameter 0; as these subsets are convex, Y inherits a pretree structure from T. The pseudometric $\oplus_{j=1}^k \delta_{\iota(j)}$ on T induces a convex metric, also denoted $\oplus_{j=1}^k \delta_{\iota(j)}$, on Y. The metric space $(Y, \oplus_{j=1}^k \delta_{\iota(j)})$ is our topmost tree. This concludes the outline.

At the end of Section III.5, we prove universality. The proof relies on Chapter IV: variations of Bestvina–Feighn–Handel's convergence criterion [2]; it boils down to bounded cancellation and Perron–Frobenius theory.

We use the results of [22] as black boxes and the two papers can be read in any order.

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Contents

I Preliminaries

In this paper, F denotes a free group with $2 \leq \mathrm{rank}(F) < \infty$. Subscripts never indicate the rank but instead are used as indices. For inductive arguments, we also work with a free group system of finite type: disjoint union $\bigsqcup_{j \in J} F_j$ of nontrivial finitely generated free groups F_j indexed by a possibly empty finite set J. In this paper, $\mathcal{F}$ is always a free group system of finite type with some component F_j that is not cyclic.

I.1 Group systems and actions

Nearly all statements and results about groups and connected spaces that we are interested in still hold when "connectivity" is relaxed and we work with "systems" componentwise. In general (almost categorical) terms, a system of [?-objects] is a disjoint union $\mathcal{O} = \bigsqcup_{j \in J} O_j$ of [?-objects] O_j indexed by some set J. An [?-isomorphism] of systems $\psi \colon \mathcal{O} \to \mathcal{O}'$ is a bijection $\sigma \colon J \to J'$ of the corresponding indexing sets and a union of [?-isomorphisms] $\psi_j \colon O_j \to O'_{\sigma \cdot j}$. The calligraphic font is reserved for systems.

In more concrete terms, here are some basic concepts that will show up in the paper:

1. an isomorphism of group systems $\psi \colon \mathcal{G} \to \mathcal{G}'$ is a bijection whose restriction to any component $G_j \subset \mathcal{G}$ is a group isomorphism of components; for group systems, we always assume (for convenience) components are nontrivial if the system is nonempty.

2. two isomorphisms of group systems $\psi, \psi' \colon \mathcal{G} \to \mathcal{G}'$ are in the same outer class $[\psi]$ if the component isomorphisms $\psi_j, \psi'_j \colon G_j \to G'_{\sigma \cdot j}$ differ only by post-composition with an inner automorphism of $G'_{\sigma \cdot j}$ for all $j \in J$.

3. a metric on a set system $\mathcal{X}$ is a disjoint union of metrics $d_j \colon X_j \times X_j \to \mathbb{R}_{\geq 0}$ on the components $X_j \subset \mathcal{X}$.

4. for a group system $\mathcal{G}$ indexed by J and object system $\mathcal{O}$ indexed by J', a $\mathcal{G}$-action on $\mathcal{O}$ (or $\mathcal{G}$-object system $\mathcal{O}$) is a union of component G_j-actions on $O_{\beta \cdot j}$ for some bijection $\beta \colon J \to J'$.

5. for an automorphism of a group system $\psi \colon \mathcal{G} \to \mathcal{G}$ and a $\mathcal{G}$-object system $\mathcal{O}$, the ψ-twisted $\mathcal{G}$-object system $\mathcal{O}\psi$ is given by precomposing the component $G_{\sigma \cdot j}$-action on $O_{\beta\sigma \cdot j}$ with the component isomorphism $\psi_j \colon G_j \to G_{\sigma \cdot j}$ to get a G_j-object $O_{\beta\sigma \cdot j}$.

I.2 Pretrees, trees, and hierarchies

Pretrees are what arises when one wants to discuss "treelike" objects without reference to a metric or topology. In this paper, the pretrees are the "primitive" objects and metrics/topologies are additional structures on the pretree — think of it the same way a Riemannian metric is a compatible addition to a manifold's smooth structure.

Fix a set T; an interval function on T is a function $[\cdot,\cdot]\colon T\times T\to\mathcal{P}(T)$, where $\mathcal{P}(T)$ is the power set of T, that satisfies the following axioms: for all $p,q,r\in T$,

1. (symmetric) $[p,q]=[q,p]$ contains $\{p,q\}$;
2. (thin) $[p,r]\subset[p,q]\cup[q,r]$; and
3. (linear) if $r\in[p,q]$ and $q\in[p,r]$, then $q=r$.

A pretree is a pair $(T,[\cdot,\cdot])$ of a nonempty set T and an interval function $[\cdot,\cdot]$ on T.

The subsets $[p,q]\subset T$ are called *closed intervals* and they should be thought of as the points between p and q (inclusive). We can similarly define *open* (resp. *half-open*) intervals by excluding both (resp. exactly one) of $\{p,q\}$. Generally, "interval" (with no qualifier) refers to any of the three types of intervals we have defined. An interval is degenerate if it is empty or a singleton. We usually omit the interval function and denote a pretree by T. Note that the real line $\mathbb{R}$ is a pretree.

Any subset $S\subset T$ of a pretree inherits an interval function: $[u,v]_S:=[u,v]\cap S$ for all $u,v\in S$. A subset $C\subset T$ is convex if $[p,q]\subset C$ for all $p,q\in C$; or equivalently, $[\cdot,\cdot]_C$ is the restriction of $[\cdot,\cdot]$ to $C\times C\subset T\times T$. A system of pretrees is a set system $\mathcal{T}=\bigsqcup_{j\in J}T_j$ and a disjoint union of interval functions on T_j; we call these systems *pretrees* for short.

Let $(T,[\cdot,\cdot])$ and $(T',[\cdot,\cdot]')$ be pretrees. A pretree-isomorphism is a bijection $f\colon T\to T'$ satisfying $f([p,q])=[f(p),f(q)]'$ for all $p,q\in T$. Similarly, a pretree-automorphism of $(T,[\cdot,\cdot])$ is a pretree-isomorphism $g\colon(T,[\cdot,\cdot])\to(T,[\cdot,\cdot])$. A pretree is real if its closed intervals are pretree-isomorphic to closed intervals of $\mathbb{R}$. By definition, the real line $\mathbb{R}$ is a real pretree. Note that being real is a property of a pretree, not an added structure like a metric! An arc of a real pretree T is a nonempty union of an ascending chain of nondegenerate intervals. A real pretree is degenerate if it is a singleton; and a system of real pretrees is degenerate if all components are degenerate.

Fix a real pretree T; a convex pseudometric on T is a function $d\colon T\times T\to\mathbb{R}_{\geq 0}$ satisfying the following axioms: for all $p,q,r\in T$,

1. (symmetric) $d(p,q)=d(q,p)$;
2. (convex) $d(p,r)=d(p,q)+d(q,r)$ if $q\in[p,r]$; and
3. (continuous) $d(p,q)=2\,d(p,q')$ for some $q'\in[p,q]$.

For any given convex pseudometric d on T, the preimage $d^{-1}(0)\subset T\times T$ is an equivalence relation on the real pretree T such that each equivalence class is convex and the set T_d of equivalence classes inherits a real pretree structure. A convex metric on T is a convex pseudometric whose equivalence relation $d^{-1}(0)$ is the equality relation on T. A (metric) tree (or $\mathbb{R}$-tree) is a real pretree with a convex metric; a forest is a system of trees. For example, the real line $\mathbb{R}$ is a tree with the standard metric $d_{\mathrm{std}}(p,q):=|p-q|$. Note that a convex pseudometric d on a real pretree T induces a convex metric, still denoted d, on the real pretree T_d; we refer to the tree (T_d, d) as the associated tree.

A λ-homothety of trees $h\colon (T, d) \to (Y, \delta)$ is a pretree-isomorphism $h\colon T \to Y$ that uniformly scales the metric d by λ:

$$\delta(h(p), h(q)) = \lambda\, d(p, q) \text{ for all } (p, q) \in \mathrm{dom}(d) = T \times T;$$

equivalently, $h^*\delta = \lambda d$, where $h^*\delta$ is the *pullback of* δ *via* h. A homothety is a λ-homothety for some $\lambda > 0$; it is expanding (resp. an isometry) if $\lambda > 1$ (resp. $\lambda = 1$). An isometry $\iota\colon (T, d) \to (T, d)$ is elliptic if it fixes a point of T; otherwise, it is loxodromic and acts by a nontrivial translation on its *axis*, the unique ι-invariant arc of (T, d) isometric to $(\mathbb{R}, d_{\mathrm{std}})$; the translation distance $\|\iota\|_d \in \mathbb{R}_{\geq 0}$ is 0 if ι is elliptic and equal to the displacement of points in ι's axis if ι is loxodromic. These definitions extend componentwise to forests.

Let d_1 be a nonconstant convex pseudometric on T and $d_{i+1}\colon d_i^{-1}(0) \to \mathbb{R}_{\geq 0}$ a nonconstant disjoint union of convex pseudometrics for $1 \leq i < n$. The sequence $(d_i)_{i=1}^n$ will be known as an n-level hierarchy of convex pseudometrics on T; We will say just *hierarchies* for short. A hierarchy $(d_i)_{i=1}^n$ has full support if d_n is a disjoint union of convex metrics. A pseudotree is a pair $(T, (d_i)_{i=1}^n)$ of a real pretree and a hierarchy with full support; a pseudoforest is a system of pseudotrees. A $(\lambda_i)_{i=1}^n$-homothety of n-level pseudoforests $h\colon (\mathcal{T}, (d_i)_{i=1}^n) \to (\mathcal{Y}, (\delta_i)_{i=1}^n)$ is a system of pretree-isomorphisms $h\colon \mathcal{T} \to \mathcal{Y}$ that scales each pseudometric d_i by λ_i:

$$\delta_i(h(p), h(q)) = \lambda_i\, d_i(p, q) \text{ for all } i \geq 1 \text{ and } (p, q) \in \mathrm{dom}(d_i);$$

a homothety is a $(\lambda_i)_{i=1}^n$-homothety for some $\lambda_i > 0$; it is expanding (resp. isometry) if each $\lambda_i > 1$ (resp. each $\lambda_i = 1$). As with trees, an isometry of a pseudotree is either elliptic (fixes a point) or loxodromic (translates a "pseudoaxes"). Hierarchies and pseudoforests are the fundamental (perhaps novel) tool in this paper. They are first used in Chapter III.

I.3 Simplicial actions and train tracks

For a pretree T, a direction at $p \in T$ is a maximal subset $D_p \subset T \setminus \{p\}$ not separated by p, i.e. $p \notin [q, r]$ for all $q, r \in D_p$. A branch point is a point with at least three directions, and a branch is a direction at a branch point. An endpoint is a point with at most one direction. A *simple pretree* is a pretree whose closed intervals are finite subsets. A pretree T is simplicial if it is real, its subset V of branch points and endpoints is a simple pretree, and no convex proper subset contains V; a vertex is a point in V. An (open) edge in a simplicial pretree T is a maximal convex subset $e \subset T$ that contains no vertex. By construction, edges are open intervals; the corresponding closed intervals in T are called closed edges.

Remark. Being simplicial is a property of a pretree, not an added structure! Besides that, our definition of a simplicial pretree is more general (with one exception) than the standard definition of a *simplicial tree* and has the advantage that it is independent of any choice of metric/topology. See [22, Interlude] for a discussion on this distinction. The one exception: the real line $\mathbb{R}$ is not a simplicial pretree!

An F*-pretree* is a pretree with an F-action by pretree-automorphisms. An F*-pseudotree* is a pair of a real F-pretree and an F-invariant hierarchy with full support; equivalently, an F-pseudotree (resp. F-tree) is a pseudotree with an isometric F-action. An F-pseudotree or F-tree is minimal if the underlying F-pretree has no proper nonempty F-invariant convex subset; in this case, the underlying F-pretree has no endpoints. We mostly consider minimal F-pseudotrees with trivial arc (pointwise) stabilizers.

Suppose an F-pseudotree $(T, (d_i)_{i=1}^n)$ has trivial arc stabilizers. For any nontrivial subgroup $G \leq F$, the characteristic convex subset (of T) for G is the unique minimal nonempty G-invariant convex subset $T(G) \subset T$. In an F-tree (T, d) with trivial arc stabilizers, the restriction of d to $T(G)$ is a G-invariant convex metric, still denoted d; the minimal G-tree $(T(G), d)$ is the characteristic subtree (of (T, d)) for G.

Remark. We do not really need an isometric action to define characteristic convex subsets and minimality. All we need is the F-action on the real pretree T to be *rigid/non-nesting*: no closed interval is sent properly into itself by the F-action [22, Section II.2]. While rigid actions are central to [22], they are superseded by isometric actions in this paper.

An F-pretree T is simplicial if T is simplicial and admits an F-invariant convex metric d; equivalently, a simplicial F-pretree is a simplicial pretree with a rigid F-action. Any simplicial F-pretree has an open cone (over a finite dimensional open simplex) worth of F-invariant convex metrics (up to an equivariant isometry isotopic to the identity map). The definitions given so far extend componentwise to systems.

Let $\mathcal{T}$ and $\mathcal{T}'$ be simplicial pretrees and $f\colon \mathcal{T} \to \mathcal{T}'$ a *tight cellular map*, i.e. a function that maps vertices to vertices and the restriction to any closed edge is a *pretree-embedding*, i.e. a pretree-isomorphism onto its image. For any choice of convex metrics d, d' on $\mathcal{T}, \mathcal{T}'$ respectively, there is a unique map $(\mathcal{T}, d) \to (\mathcal{T}', d')$ that is linear on edges and isotopic to f; whenever a choice of convex metrics is made, we implicitly replace f with this map.

Let $\mathcal{T}$ be a free splitting of $\mathcal{F}$, i.e. minimal simplicial $\mathcal{F}$-pretrees with trivial edge stabilizers, and suppose $\psi\colon \mathcal{F} \to \mathcal{F}$ is an automorphism of a free group system. The ψ-*twisted* free splitting $\mathcal{T}\psi$ is the same real pretrees $\mathcal{T}$ but the original simplicial $\mathcal{F}$-action is precomposed with ψ. A (relative) topological representative for ψ is a ψ*-equivariant* tight cellular map $f\colon \mathcal{T} \to \mathcal{T}$ on a nondegenerate free splitting $\mathcal{T}$ of $\mathcal{F}$: ψ-equivariance means $f(x \cdot p) = \psi(x) \cdot f(p)$ for all $x \in \mathcal{F}$ and $p \in \mathcal{T}$, or equivalently, $f\colon \mathcal{T} \to \mathcal{T}\psi$ is equivariant. Given a topological representative $f\colon \mathcal{T} \to \mathcal{T}$ for ψ, we let $[f]$ denote the induced map on the quotient $\mathcal{F}\backslash\mathcal{T}$; we say $[f]$ is a *topological representative* for the outer class $[\psi]$. A (relative) train track for ψ is a topological representative $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ whose iterates τ^m $(m \geq 1)$ are topological representatives for ϕ^m; or equivalently, whose iterates τ^m restrict to pretree-embeddings on closed edges.

For any free splitting $\mathcal{T}$ of $\mathcal{F}$, Bass-Serre theory gives a uniform bound on the number of $\mathcal{F}$-orbits of edges (linear in rank($\mathcal{F}$)) and relates the vertices with nontrivial stabilizers in a (componentwise) connected fundamental domain to a (possibly empty) *free factor system* $\mathcal{F}[\mathcal{T}]$ of $\mathcal{F}$ — take this as the working definition of free factor systems. The theory also gives a uniform bound on the complexity (e.g. ranks) of free factor systems. A free factor system $\mathcal{F}[\mathcal{T}]$ *proper* if $\mathcal{F}[\mathcal{T}] \neq \mathcal{F}$; equivalently, $\mathcal{F}[\mathcal{T}]$ is proper if and only if $\mathcal{T}$ is not degenerate. Any proper free factor system of $\mathcal{F}$ has strictly lower complexity than $\mathcal{F}$. The trivial free factor system of $\mathcal{F}$ is the (possibly empty) free factor system consisting of the cyclic $\mathcal{F}$-components; it is always proper since we assume $\mathcal{F}$ has a noncyclic component.

Remark. We will abuse notation and write $\mathcal{F}[\mathcal{T}] = \mathcal{F}[\mathcal{T}']$ for two free splittings $\mathcal{T}, \mathcal{T}'$ of $\mathcal{F}$ when we mean: an element of $\mathcal{F}$ is $\mathcal{T}$-elliptic if and only if it is $\mathcal{T}'$-elliptic.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and a topological representative $f\colon \mathcal{T} \to \mathcal{T}$ for ψ. By ψ-equivariance of f, the proper free factor system $\mathcal{F}[\mathcal{T}]$ is $[\psi]$-*invariant* — again, we can take this as the definition of $[\psi]$-invariance for proper free factor systems. Form a nonnegative integer square matrix $A[f]$ whose rows and columns are indexed by the $\mathcal{F}$-orbits of edges in $\mathcal{T}$; and the entry at row-$[e]$ and column-$[e']$ is given by the number of e-translates in the interval $f(e')$, where e, e' are edges in T. The topological representative f is irreducible if the matrix $A[f]$ is irreducible; or equivalently, if, for any pair of edges e, e' in $\mathcal{T}$, a translate of e is contained $f^m(e')$ for some $m = m(e, e') \geq 1$. It is a foundational theorem of Bestvina–Handel that automorphisms have irreducible train tracks.

Theorem I.1 (cf. [4, Theorem 1.7])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism of a free group system and $\mathcal{Z}$ a $[\psi]$-invariant proper free factor system of $\mathcal{F}$. Then there is an irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ, where the components of $\mathcal{Z}$ are $\mathcal{T}$-elliptic.*

The proof outline of [22, Theorem I.1] explains how to deduce the theorem as currently stated from the cited theorem.

Suppose $\psi\colon \mathcal{F} \to \mathcal{F}$ is an automorphism with an irreducible topological representative $f\colon \mathcal{T} \to \mathcal{T}$. Perron–Frobenius theory implies the matrix $A[f]$ has a unique real eigenvalue $\lambda = \lambda[f] \geq 1$ with a unique positive left eigenvector $\nu[f]$ whose entries sum to 1. From the eigenvector $\nu[f]$, we get an $\mathcal{F}$-invariant convex metric d_f on $\mathcal{T}$ (well-defined up to an equivariant isometry isotopic to the identity map). The restriction of f to any edge is a λ-homothetic embedding with respect to d_f; the metric d_f is the eigenmetric (on $\mathcal{T}$) for $[f]$. If $\lambda = 1$, then f is a ψ-equivariant simplicial automorphism of $\mathcal{T}$.

I.4 Growth types and limit trees

Since the introduction of train tracks, it has been standard to construct limit forests by iterating an expanding irreducible train track (Section II.1). Unfortunately, such a construction is not canonical as it can depend on the initial train track. The main idea of the paper: patch together a "descending" sequence of limit trees to get a limit pseudoforest and inductively "normalize" its hierarchy into a canonical limit pseudoforest.

Fix a free group system $\mathcal{G}$ of finite type (unlike $\mathcal{F}$, all components of $\mathcal{G}$ can be cyclic), an automorphism $\psi \colon \mathcal{G} \to \mathcal{G}$, and a metric free splitting $(\mathcal{T}, d)$ of $\mathcal{G}$ whose free factor system $\mathcal{Z} := \mathcal{F}[\mathcal{T}]$ is $[\psi]$-invariant. An element $x \in \mathcal{G}$ $[\psi]$-grows exponentially rel. d with rate λ_x if it is $\mathcal{T}$-loxodromic and the limit inferior of the sequence $\left(m^{-1} \log \|\psi^m(x)\|_d\right)_{m \geq 0}$ is $\log \lambda_x > 0$. If an element $[\psi]$-grows exponentially rel. d, then it $[\psi]$-grows exponentially rel. d' with the same rate for any metric free splitting $(\mathcal{T}', d')$ of $\mathcal{G}$ with $\mathcal{F}[\mathcal{T}'] = \mathcal{Z}$; say the element $[\psi]$-grows exponentially rel. $\mathcal{Z}$. An element $x \in \mathcal{G}$ $[\psi]$-grows polynomially rel. $\mathcal{Z}$ with degree $< n$ if the sequence $(m^{-n} \|\psi^m(x)\|_d)_{n \geq 0}$ converges to 0. Any element of $\mathcal{G}$ $[\psi]$-grows either exponentially or polynomially rel. $\mathcal{Z}$ [22, Corollary III.4]. The growth type of an element is preserved when passing to invariant subgroup systems of finite type.

The automorphism ψ is exponentially growing rel. $\mathcal{Z}$ if some element $[\psi]$-grows exponentially rel. $\mathcal{Z}$; otherwise, ψ is polynomially growing rel. $\mathcal{Z}$. The growth type of an outer class $[\psi]$ is also well-defined. The "rel. $\mathcal{Z}$" in our terminology may be omitted when $\mathcal{Z}$ is trivial. The next proposition deals with the first obstacle:

Proposition I.2 (cf. [22, Proposition III.2])**.** *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an automorphism of a free group system and $\mathcal{Z}$ a $[\psi]$-invariant proper free factor system. Then there is a:*

1. *a minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ with trivial arc stabilizers for which $\mathcal{Z}$ is elliptic; and*
2. *a unique ψ-equivariant expanding homothety $h \colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$.*

The forest $(\mathcal{Y}, \delta)$ is degenerate if and only if $[\psi]$ is polynomially growing rel. $\mathcal{Z}$.

The constructed $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is the *limit forest for* $[\psi]$ *rel.* $\mathcal{Z}'$, for some $[\psi]$-invariant proper free factor system $\mathcal{Z}'$ that supports $\mathcal{Z}$ (see Sections II.1 and II.4). Unfortunately, these limit forests depend on the choice of $\mathcal{Z}'$; our goal is to give a canonical contruction.

Given the central tool (hierarchies) and objective (universal limit trees), we outline again how these two fit together. Gaboriau–Levitt's index theory [11] gives a uniform bound on the complexity of the point stabilizers system $\mathcal{G}[\mathcal{Y}]$ for a minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ with trivial arc stabilizers — this is a partial generalization of Bass–Serre theory. When $\mathcal{Y}$ is not degenerate, the subgroup system $\mathcal{G}[\mathcal{Y}]$ has strictly lower complexity than $\mathcal{F}$. This allows us to induct on complexity (see Chapter III).

Suppose the automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ has a nondegenerate limit forest $(\mathcal{Y}_1, \delta_1)$ with nontrivial point stabilizers; the system of stabilizers $\mathcal{G} := \mathcal{G}[\mathcal{Y}]$ has strictly smaller complexity than $\mathcal{F}$. By ψ-equivariance of λ_1-homothety $h_1 \colon (\mathcal{Y}_1, \delta_1) \to (\mathcal{Y}_1, \delta_1)$, the $\mathcal{F}$-orbits of points with nontrivial stabilizers are permuted by $[h_1]$, the subgroup system $\mathcal{G}$ is $[\psi]$-invariant, and the restriction of ψ to $\mathcal{G}$ determines a unique outer automorphism $[\varphi]$ of $\mathcal{G}$.

Suppose $\varphi \colon \mathcal{G} \to \mathcal{G}$ has a nondegenerate limit forest $(\mathcal{Y}_2, \delta_2)$ with stretch factor λ_2. Using the blow-up construction from [22], we equivariantly blow-up $\mathcal{Y}_1$ with respect to $h_i \colon \mathcal{Y}_i \to \mathcal{Y}_i$ $(i = 1, 2)$ to get real pretrees $\mathcal{T}$ with a minimal rigid $\mathcal{F}$-action and a ψ-equivariant "$\mathcal{F}$-expanding" pretree-isomorphism $f \colon \mathcal{T} \to \mathcal{T}$ induced by h_1 and h_2. In fact, the blow-up construction implies the $\mathcal{F}$-pretrees $\mathcal{T}$ inherit an $\mathcal{F}$-invariant 2-level hierarchy (δ_1, δ_2) with full support and f is an expanding homothety with respect to this hierarchy. So we have a *limit pseudoforest* $(\mathcal{T}, (\delta_1, \delta_2))$ for $[\psi]$ (see Section III.1). Under what conditions can we construct an $\mathcal{F}$-invariant convex metric on $\mathcal{T}$ from (δ_1, δ_2)? The heart of the paper is the following observation: the two limit forests $(\mathcal{Y}_i, \delta_i)$ are paired with attracting laminations $\mathcal{L}_i^+[\psi]$ partially ordered by inclusion; an $\mathcal{F}$-invariant convex metric on $\mathcal{T}$ can be naturally constructed from (δ_1, δ_2) if $\mathcal{L}_2^+[\psi]$ is not in $\mathcal{L}_1^+[\psi]$ (see Section III.4) or $\lambda_1 < \lambda_2$ (see Section III.5)!

I.5 Bounded cancellation and laminations

Suppose a minimal $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is very small, i.e. nontrivial arc stabilizers are maximal cyclic subgroups and the fixed point subset for a nontrivial elliptic element is an arc. Let $(\mathcal{T}, d)$ be a metric free splitting of $\mathcal{F}$ and $[\cdot,\cdot]_T$ (resp. $[\cdot,\cdot]_Y$) denote the interval function for $\mathcal{T}$ (resp. $\mathcal{Y}$). A map $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ is piecewise-linear (PL) if the restriction to any closed edge is a linear map; an equivariant PL-map exists if and only if $\mathcal{T}$-elliptic elements in $\mathcal{F}$ are $\mathcal{Y}$-elliptic. Equivariant PL-maps $(\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ are surjective and Lipschitz since the isometric $\mathcal{F}$-action on $(\mathcal{Y}, \delta)$ is minimal and there are only finitely many $\mathcal{F}$-orbits of edges in $\mathcal{T}$; 1-Lipschitz maps are also known as metric maps. Generally, if $\mathcal{T}, \mathcal{Y}$ are free splittings of $\mathcal{F}$, then an equivariant function $f\colon \mathcal{T} \to \mathcal{Y}$ is a (simplicial) PL-map if its restrictions to any closed edge is isotopic to a linear map with respect to some/any $\mathcal{F}$-invariant convex metrics d, δ on $\mathcal{T}, \mathcal{Y}$ respectively.

Lemma I.3 (bounded cancellation)**.** *Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map. For some constant $C[f] \geq 0$ and all points $p, q \in \mathcal{T}$, the image $f([p,q]_T)$ is in the $C[f]$-neighbourhood of the interval $[f(p), f(q)]_Y$.*

Such a $C[f]$ is a cancellation constant for f. This proof is due to Bestvina–Feighn–Handel.

Sketch of proof [2, Lemma 3.1]. Let $\mathrm{Lip}(f)$ be the Lipschitz constant and $\mathrm{vol}(\mathcal{T}, d)$ the volume (mod $\mathcal{F}$). Then $f = g \circ h$ for some equivariant $\mathrm{Lip}(f)$-homothety h and metric PL-map g. Suppose f is *simple*: its target is a metric free splitting with free factor system $\mathcal{F}[\mathcal{T}]$. Then g factors as finitely many equivariant edge collapses and Stallings folds followed by an equivariant metric homeomorphism. The homeomorphism and each edge collapse have cancellation constants 0. A fold has a cancellation constant given by the length of folded segment. Finally, the metric PL-map g has a cancellation constant since cancellation constants are (sub)additive over compositions of metric maps. As cancellation constants are preserved by precomposition with homeomorphisms, the PL-map $f = g \circ h$ has a cancellation constant $C[f] < \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d)$.

Otherwise, the PL-map f is not simple. For a contradiction, suppose the image $f([p,q]_T)$ is not in the $\mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d)$-neighbourhood of $[f(p), f(q)]_Y$ for some $p, q \in \mathcal{T}$. Let $\delta(f(r_0), [f(p), f(q)]_Y) > \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ for some $\epsilon_0 > 0$ and point $r_0 \in [p,q]_T$.

For any $\epsilon > 0$, the PL-map f is approximated by an equivariant simple PL-map f_ϵ with $\mathrm{Lip}(f_\epsilon) < \mathrm{Lip}(f) + \epsilon$ and $C[f_\epsilon] \geq \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ (see [16, Theorem 6.1]). By the previous paragraph, $C[f_\epsilon] < \mathrm{Lip}(f_\epsilon)\,\mathrm{vol}(\mathcal{T}, d)$ for $\epsilon > 0$. So $C[f_\epsilon] < \mathrm{Lip}(f)\,\mathrm{vol}(\mathcal{T}, d) + \epsilon_0$ for small enough $\epsilon > 0$ — a contradiction. □

Remark. The results in this section apply to ψ-equivariant PL-maps $g\colon (\mathcal{T}, d) \to (\mathcal{T}, d)$ for any automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$: view g as an equivariant PL-map $(\mathcal{T}, d) \to (\mathcal{T}\psi, d)$ instead.

A line in a forest is an arc that is isometric to $(\mathbb{R}, d_{\mathrm{std}})$; a ray in a forest is an arc that is isometric to $(\mathbb{R}_{\geq 0}, d_{\mathrm{std}})$ and its origin is the point corresponding to 0 under the isometry. Two rays are *end-equivalent* if their intersection is a ray; an end of a forest is an end-equivalence class of rays in the forest. Note that there is a natural bijection between the set of lines in a forest and set of unordered pairs of distinct ends in the same component of the forest. For simplicial $\mathcal{F}$-pretrees $\mathcal{T}$, the notions of line/ray/end are well-defined for the cone of $\mathcal{F}$-invariant convex metrics on $\mathcal{T}$.

Corollary I.4 (cf. [10, Lemma 3.4])**.**
Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map.

1. *For any ray ρ in $(\mathcal{T}, d)$ with origin p_0, the image $f(\rho)$ is either bounded or in the $C[f]$-neighbourhood of a unique ray $f_*(\rho) \subset f(\rho)$ with origin $f(p_0)$; moverover, if ρ, ρ' represent the same end e and $f(\rho)$ is unbounded, then so is $f(\rho')$ and $f_*(\rho), f_*(\rho')$ are end-equivalent — denote their end-equivalence class by $f_*(e)$.*
2. *For any line l in $(\mathcal{T}, d)$, $f(l)$ is in a $C[f]$-neighbourhood of a unique line $f_*(l) \subset f(l)$ if both ends of l have unbounded f-images.*
3. *For any end ϵ of $(\mathcal{Y}, \delta)$, there is a unique end $f^*(\epsilon)$ of $(\mathcal{T}, d)$ with $\epsilon = f_*(f^*(\epsilon))$.*

Sketch of proof.
(1): Let ρ be a ray in $(\mathcal{T}, d)$, $p_0 \in \rho$ its origin, $f(\rho)$ unbounded, and $s_0 = f(p_0)$. Use Figure 1 for reference. By bounded cancellation and the Lipschitz property, $f(\rho)$ has at most one end of $(\mathcal{Y}, \delta)$. For some $n \geq 0$, assume $s_n \in [s_0, f(p)]_Y$ for all $p \in \rho \setminus [p_0, p_n]_T$. Set $C \coloneqq C[f]$. Since $f(\rho)$ is unbounded,

$$\delta(s_0, f(p_{n+1})) > 2\,\delta(s_0, s_n) + C$$

for some $p_{n+1} \in \rho \setminus [p_0, p_n]_T$. Pick $s_{n+1} \in [s_0, f(p_{n+1})]_Y$ satisfying $\delta(s_0, s_{n+1}) > 2\,\delta(s_0, s_n)$ and $\delta(s_{n+1}, f(p_{n+1})) > C$; so $s_n \in [s_0, s_{n+1}]_Y$. By bounded cancellation, the interval $[s_0, s_{n+1}]_Y \subset f([p_0, p_{n+1}]_T)$ is disjoint from $f(\rho \setminus [p_0, p_{n+1}]_T)$. So the union $\bigcup_{n \geq 0}[s_0, s_n]_Y$ is a ray $f_*(\rho)$ in $f(\rho)$ with origin s_0. By construction, $f(\rho)$ is in the C-neighbourhood of $f_*(\rho)$. Any bounded neighbourhood of a ray contains a unique ray, up to end-equivalence.

(2): Represent both ends of l with rays $\rho^\pm \subset l$ sharing the same origin. By Part 1 and bounded cancellation, we have rays $f_*(\rho^\pm)$ representing unique distinct ends $\epsilon^\pm$ of $(\mathcal{Y}, \delta)$;

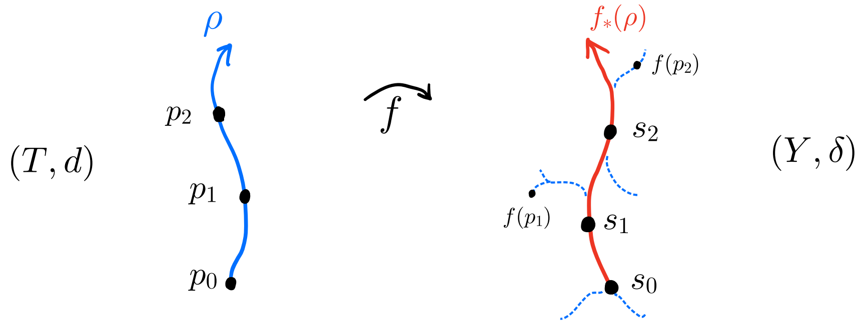

Figure 1: The ray $f_*(\rho)$ with origin $s_0 = f(p_0)$ is built inductively in the image $f(\rho)$.

moreover, $f(l) = f(\rho^-) \cup f(\rho^+)$ is in the C-neighbourhood of $f_*(\rho^-) \cup f_*(\rho^+) \subset f(l)$. Let $f_*(l) \subset f_*(\rho^-) \cup f_*(\rho^+)$ be the line determined by the ends $\epsilon^\pm$. Then $f(l)$ is in the C-neighbourhood of $f_*(l)$. Any bounded neighbourhood of a line contains a unique line.

(3): The argument is almost the same. Let ρ' be a ray representing ϵ and $s_0 = q_0$ its origin. Pick points $q_{n+1}, s_{n+1} \in \rho'$ with $\delta(s_0, s_{n+1}) > 2\,\delta(s_0, s_n)$, $\delta(s_0, q_{n+1}) > 2\,\delta(s_0, s_n) + C$, and $\delta(s_{n+1}, q_{n+1}) > C$. Since $f\colon \mathcal{T} \to \mathcal{Y}$ is surjective, we can lift q_n to $p_n \in T$. By bounded cancellation and K-Lipschitz property, the distance $d(p_0, [p_n, p_{n+m}]_T) > \frac{1}{K}\delta(s_0, s_n)$. Thus $(p_n)_{n\geq 0}$ determines an end e of $(\mathcal{T}, d)$ with unbounded f-image. Let ρ be a ray representing e with origin p_0. As $\rho' \subset f(\rho)$ by construction, we get $f_*(\rho) = \rho'$ by Part 1. By Part 2, the end e is the unique end with $f_*(e) = \epsilon$, and we denote it by $f^*(\epsilon)$. $\square$

The corollary defines the equivariant lifting (resp. projecting) function f^* (resp. f_*), where the domain $\mathrm{dom}(f^*)$ of f^* is the set of lines in $(\mathcal{Y}, \delta)$ and the domain $\mathrm{dom}(f_*)$ of f_* is the set of lines in $(\mathcal{T}, d)$ whose ends both have unbounded f-images. Note that the image $\mathrm{im}(f^*)$ is $\mathrm{dom}(f_*)$; moreover, f^* and f_* are inverses of each other. Both lifting and projecting functions are *canonical*: $f^* = g^*$ and $f_* = g_*$ for any equivariant maps $f, g\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ since f, g will be a bounded δ-distance from some equivariant PL-map; for lack of better notation, we still denote the functions by f^*, f_* despite this independence.

Alternatively, we view f^* and f_* as functions on the sets of $\mathcal{F}$-orbits of lines. We can equip these sets with a natural topology. The set $\mathbb{R}(\mathcal{Y}, \delta)$ of $\mathcal{F}$-orbits of lines in $(\mathcal{Y}, \delta)$ has the following topology: for any $p, q \in \mathcal{Y}$, let $U[p, q]$ be the $\mathcal{F}$-orbit of lines that contain a translate of $[p, q]$; the collection $\{U[p, q] : p, q \in \mathcal{Y}\}$ is a basis for the space of ($\mathcal{F}$-orbits of) lines. This space is well-defined for the equivariant homothetic class of $(\mathcal{Y}, \delta)$. The space of lines is also well-defined for the free splitting $\mathcal{T}$ and denoted $\mathbb{R}(\mathcal{T})$.

Claim I.5. *The canonical lifting function $f^*\colon \mathbb{R}(\mathcal{Y}, \delta) \to \mathbb{R}(\mathcal{T})$ is a topological embedding.*

Henceforth, we identify $\mathbb{R}(\mathcal{Y}, \delta)$ with a subspace of $\mathbb{R}(\mathcal{T})$ using the canonical embedding f^*.

Sketch of proof. We first prove the injection f^* is continuous. Let $\Lambda \subset \mathbb{R}(\mathcal{T})$ be a closed subset and $\Lambda_f := \Lambda \cap \mathrm{im}(f^*)$. Suppose $[\gamma]$ is in the closure of $f_*(\Lambda_f)$ in $\mathbb{R}(\mathcal{Y}, \delta)$. For continuity, it is enough to show $f^*[\gamma] \in \Lambda$. Fix a long interval $I_\gamma \subset \gamma$; then $I_\gamma \subset [f(p), f(q)]_Y$ for some $p, q \in f^*(\gamma)$. As $[\gamma]$ is in the closure of $f_*(\Lambda_f)$, the interval $I_\gamma \subset \gamma$ is in the line $f_*(l)$ for some $[l] \in \Lambda_f$. By bounded cancellation, the f-image of the intersection $I_l := f^*(\gamma) \cap l$ contains a long interval in I_γ. As the interval I_γ will exhaust γ, the interval I_l exhausts $f^*(\gamma)$; in particular, any interval in $f^*(\gamma)$ is contained in l for some $[l] \in \Lambda$. So $f^*[\gamma]$ is in the closed subset Λ.

We finally prove $f^* \colon \mathbb{R}(\mathcal{Y}, \delta) \to \mathrm{im}(f^*)$ is an open map, where the image $\mathrm{im}(f^*) \subset \mathbb{R}(\mathcal{T})$ has the subspace topology. Suppose $p, q \in \mathcal{Y}$ and $[\gamma] \in U[p, q]$, i.e. a line γ in $(\mathcal{Y}, \delta)$ contains an interval $[p, q]_Y$. There is an interval $[u, v]_T \subset f^*(\gamma)$ whose f-image covers a long neighbourhood of $[p, q]_Y$. By bounded cancellation, any line $f^*(\gamma')$ containing $[u, v]_T$ will have an f_*-image γ' containing $[p, q]_Y$. So $f^*[\gamma] \in U[u, v] \cap \mathrm{im}(f^*) \subset f^*(U[p, q])$. As $[\gamma] \in U[p, q]$ was arbitrary, the image $f^*(U[p, q])$ is open in $\mathrm{im}(f^*)$. $\square$

Now assume $\mathcal{T}'$ is a free splitting of $\mathcal{F}$ with $\mathcal{F}[\mathcal{T}] = \mathcal{F}[\mathcal{T}']$ and let $f \colon \mathcal{T} \to \mathcal{T}'$ be an equivariant PL-map. The folds in the factorization of f never identify points in the same $\mathcal{F}$-orbit. For $[l] \in \mathbb{R}(\mathcal{T})$, each end of l has unbounded f-image, i.e. $\mathrm{dom}(f_*) = \mathbb{R}(\mathcal{T})$; so $f_* \colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T}')$ is a canonical homeomorphism (with inverse f^*). Similarly, if $g \colon \mathcal{T} \to \mathcal{T}$ is a ψ-equivariant PL-map for some automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$, then $g_*, g^* \colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T})$ are canonical homeomorphisms for $[\psi]$.

Remark. We use ambiguous terminology and say "line" when we mean a line or an $\mathcal{F}$-orbit of a line; our notation remains distinct: "l" is always a line, while "$[l]$" is its $\mathcal{F}$-orbit.

A lamination in $(\mathcal{Y}, \delta)$ (resp. $\mathcal{T}$) is a nonempty closed subset of $\mathbb{R}(\mathcal{Y}, \delta)$ (resp. $\mathbb{R}(\mathcal{T})$); when the $\mathcal{F}$-forest in question is clear, we say *lamination* with no qualifier. An element of a lamination is called a leaf; a leaf segment of a lamination Λ is a nondegenerate closed interval in a line representing a leaf of Λ. A lamination is minimal if each leaf is dense in the lamination; a lamination is perfect if it has no isolated leaves.

Let $[l]$ be a line and Λ a lamination in $\mathbb{R}(\mathcal{Y}, \delta)$ (or $\mathbb{R}(\mathcal{T})$). A sequence $[l_m]_{m \geq 0}$ in the space of lines weakly limits to $[l]$ if some subsequence converges to $[l]$; we say $[l]$ is a *weak limit* of the sequence. The sequence $[l_m]_{m \geq 0}$ weakly limits to Λ if it weakly limits to every leaf of Λ. The "weak" terminology is used to highlight that the space of lines is not Hausdorff — a convergent sequence may have multiple limits!

More generally, a sequence $[p_m, q_m]_{m \geq 0}$ of intervals *converges* to $[l]$ if, for any closed interval $[a, b] \subset l$, $[p_m, q_m]$ contains a translate of $[a, b]$ for $m \gg 1$ (i.e. for large enough m) — precisely, there is an $M[a, b] \geq 1$ such that $U[a, b]$ contains $U[p_m, q_m]$ for $m \geq M[a, b]$. Again, a sequence of intervals weakly limits to $[l]$ if some subsequence converges to $[l]$ and it weakly limits to Λ if it weakly limits to every leaf of Λ.

II Limit forests

In this chapter, we sketch the proof of Proposition I.2 (existence of limit forests) and, in the process, introduce stable laminations. The first half deals with limit forests for expanding irreducible train tracks; then, in the second half, we extend the results to all limit forests.

II.1 Constructing limit forests (1)

This is a summary of [12, Appendix]; the reader is invited to read that appendix.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Set $\lambda := \lambda[\tau] > 1$ and let d_τ be the eigenmetric on $\mathcal{T}$ for $[\tau]$. For $m \geq 0$, let d_m be the pullback of $\lambda^{-m} d_\tau$ via τ^m:

$$d_m(p,q) := \lambda^{-m} d_\tau(\tau^m(p), \tau^m(q)) \leq d_\tau(p,q) \quad \text{for } p, q \text{ in a component of } T.$$

By definition, the pullback d_m is an $\mathcal{F}$-invariant (not necessarily convex) pseudometric on $\mathcal{T}$ whose quotient metric space is equivariantly isometric to $(\mathcal{T}\psi^m, \lambda^{-m} d_\tau)$. The λ-Lipschitz property for τ with respect to d_τ implies the sequence of pseudometrics d_m is (pointwise) monotone decreasing. The limit d_∞ is an $\mathcal{F}$-invariant pseudometric on $\mathcal{T}$, the quotient metric space $(\mathcal{T}_\infty, d_\infty)$ is an $\mathcal{F}$-forest, and the ψ-equivariant λ-Lipschitz train track τ induces a ψ-equivariant λ-homothety $h\colon (\mathcal{T}_\infty, d_\infty) \to (\mathcal{T}_\infty, d_\infty)$; in particular, the equivariant metric surjection $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}_\infty, d_\infty)$ semiconjugates τ to h: $\pi \circ \tau = h \circ \pi$.

As τ is a train track, the restriction of π to any edge of $\mathcal{T}$ is an isometric embedding. So $\mathcal{T}_\infty$ is not degenerate. In fact, the π-image of any edge of $\mathcal{T}$ can be extended to an axis for a $\mathcal{T}_\infty$-loxodromic element in $\mathcal{F}$. Thus the $\mathcal{F}$-forest $(\mathcal{T}_\infty, d_\infty)$ is minimal, and the uniqueness of h follows from [6, Theorem 3.7]. Finally, the minimal $\mathcal{F}$-forest $(\mathcal{T}_\infty, d_\infty)$ has trivial arc stabilizers. This sketches the first case of Proposition I.2. The $\mathcal{F}$-forest $(\mathcal{Y}_\tau, d_\infty) := (\mathcal{T}_\infty, d_\infty)$ is the (forward) limit forest for $[\tau]$.

II.2 Stable laminations (1)

The first part of this section is mostly adapted from Section 1 of [2]. The following definition of *stable* laminations is from [2, Definition 1.3].

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Set $\lambda := \lambda[\tau] > 1$, let d_τ be the eigenmetric on $\mathcal{T}$ for $[\tau]$, and pick an edge $e \subset \mathcal{T}$. Expanding irreducibility implies the interval $\tau^k(e)$ contains at least three translates of e for some $k \geq 1$. By the intermediate value theorem, $\tau^k(p) = x \cdot p$ for some $x \in \mathcal{F}$ and $p \in e$. Recall that edges are open intervals; since the restriction of $x^{-1} \cdot \tau^k$ to the edge e is an expanding λ^k-homothetic embedding $e \to \mathcal{T}$ (with respect to d_τ) that fixes p and has e in its image, we can extend e to a line $l_p \subset \mathcal{T}$ by iterating $x^{-1} \cdot \tau^k$. By construction, the restriction of $x^{-1} \cdot \tau^k$ to l_p is a λ^k-homothety $l_p \to l_p$ with respect to the eigenmetric d_τ for $[\tau]$; the $\mathcal{F}$-orbit $[l_p]$ is an eigenline of $[\tau^k]$ based at $[p]$ (in $\mathcal{F}\backslash\mathcal{T}$). A stable $\mathcal{T}$-lamination Λ^+ for $[\tau]$

is the closure of an eigenline of $[\tau^k]$ for some $k \geq 1$. By ϕ-equivariance of τ, the restriction of τ to l representing a leaf of a stable lamination Λ^+ is a λ-homothetic embedding. In fact, $[\tau]$ maps eigenlines to eigenlines, and the image $\tau_*(\Lambda^+) \coloneqq \{[\tau(l)] : [l] \in \Lambda^+\}$ is also a stable lamination for $[\tau]$.

As the transition matrix $A[\tau]$ is irreducible, it is a block transitive permutation matrix, and the "first return" matrix for each block is *primitive*, i.e. has a positive power. There is a bijective correspondence between the stable laminations for $[\tau]$ and the blocks of $A[\tau]$. In particular, there are finitely many stable laminations for $[\tau]$, these laminations are pairwise disjoint, and τ_* transitively permutes them [2, Lemma 1.2]. By finiteness, their union $\mathcal{L}^+[\tau]$ is a lamination and is called the *system of stable laminations* for $[\tau]$.

II.2.1 Quasiperiodic lines

A line $[l]$ in an $\mathcal{F}$-forest is *periodic* if it is the axis for the conjugacy class of some loxodromic element of $\mathcal{F}$. A line $[l]$ is quasiperiodic in an $\mathcal{F}$-forest if any closed interval I in l has an assigned number $L(I) \geq 0$ such that any interval in l of length $L(I)$ contains a translate of I; periodic lines are quasiperiodic. If $[l]$ is a quasiperiodic line, then any leaf of its closure Λ is quasiperiodic and hence dense in Λ (exercise), i.e. Λ is minimal. If $[l]$ is quasiperiodic but not periodic, then no leaf of its closure Λ is isolated (exercise), i.e. Λ is also perfect.

Remark. When the F-action on a free splitting T is free, then our definition of quasiperiodicity is equivalent to [2, Definition 1.7]; however, our definition is weaker when the action is not free.

Lemma II.1 (cf. [2, Proposition 1.8]). *The eigenlines of $[\tau^k]$ are quasiperiodic but not periodic for $k \geq 1$. Thus the stable laminations for $[\tau]$ are minimal and perfect.*

Proof. There is a length L_0 such that any interval in $\mathcal{T}$ of length L_0 contains an edge. Fix an $\mathcal{F}$-orbit $[I]$ of intervals in an eigenline $[l]$ of $[\tau^k]$. By construction, I is contained in $\tau^{km}(e)$ for some edge e in $\mathcal{T}$ and integer $m \geq 0$. As the blocks in $A[\tau^k]$ are primitive, there is an integer $m' \geq 1$ such that $\tau^{km'}(e')$ contains a translate of e for any edge e' in l. Altogether, any interval in l of length $\lambda[\tau]^{k(m+m')}L_0$ contains a translate of I. This proves quasiperiodicity.

Now assume, for a contradiction, that the eigenline $[l]$ were periodic, i.e. l is an axis for a $\mathcal{T}$-loxodromic element $x \in \mathcal{F}$. By construction, the $\mathcal{F}$-orbit $[l]$ is τ^k-invariant and hence the cyclic subgroup $\langle x \rangle$ is $[\psi^k]$-invariant. So x is $[\psi]$-periodic as ψ is an automorphism; yet x must $[\psi]$-grow exponentially since its axis is an eigenline and τ is expanding. $\square$

Fix an equivariant PL-map $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ and canonically embed $\mathbb{R}(\mathcal{Y}, \delta)$ into $\mathbb{R}(\mathcal{T})$ via f^* (Claim I.5). If a quasiperiodic line $[l] \in \mathbb{R}(\mathcal{T})$ is in the subspace $\mathbb{R}(\mathcal{Y}, \delta) = \mathrm{im}(f^*)$, then so its closure Λ in $\mathbb{R}(\mathcal{T})$ (exercise). Returning to limit forests, the equivariant metric PL-map $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ restricts to an isometric embedding on the leaves of $\mathcal{L}^+[\tau]$; therefore, the stable lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}_\tau, d_\infty) \subset \mathbb{R}(\mathcal{T})$.

II.2.2 Characterizing loxodromics

Let $(\mathcal{Y}_\tau, d_\infty)$ be the limit forest for $[\tau]$, $h : (\mathcal{Y}_\tau, d_\infty) \to (\mathcal{Y}_\tau, d_\infty)$ the unique ψ-equivariant λ-homothety, and $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ the constructed equivariant metric PL-map. By Lemma I.3, the map $\tau\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}, d_\tau)$ has a cancellation constant $C \coloneqq C[\tau]$. Set $C' \coloneqq \frac{C}{\lambda-1}$ and denote the interval functions for $\mathcal{T}$ by $[\cdot,\cdot]$. The sequence of equivariant metric maps $\tau^m\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}\psi^m, \lambda^{-m} d_\tau)$ have cancellation constants $\sum_{i=1}^m \lambda^{-i} C \le C'$; so their limit π has a cancellation constant $C[\pi] \coloneqq C'$.

Let $P \subset \mathcal{Y}_\tau$ be $\mathcal{F}$-orbit representatives of points with nontrivial stabilizers. Define the subgroup system $\mathcal{G}[\mathcal{Y}_\tau] \coloneqq \bigsqcup_{p \in P} G_p$, where $G_p \coloneqq \mathrm{Stab}_{\mathcal{F}}(p)$ is the stabilizer in $\mathcal{F}$ of $p \in P$. As the action on $\mathcal{Y}_\tau$ has trivial arc stabilizers, the system $\mathcal{G}[\mathcal{Y}_\tau]$ is malnormal: each component is malnormal (as a subgroup of the appropriate component of $\mathcal{F}$) and conjugates of distinct components (in the same component of $\mathcal{F}$) have trivial intersections. The ψ-equivariance of homothety h implies $\mathcal{G}[\mathcal{Y}_\tau]$ is $[\psi]$-invariant. By Gaboriau–Levitt's index theory, the complexity of $\mathcal{G}[\mathcal{Y}_\tau]$ is strictly less than that of $\mathcal{F}$ [11, Theorem III.2]. In particular, $\mathcal{G}[\mathcal{Y}_\tau]$ has finite type: P is finite, and each component G_p is finitely generated. The restriction of ψ to $\mathcal{G}[\mathcal{Y}_\tau]$ determines a unique outer automorphism of the system.

We now characterize the elliptic/loxodromic elements in $\mathcal{F}$ in the limit forest $(\mathcal{Y}_\tau, d_\infty)$:

Proposition II.2 (cf. [2, Proposition 1.6])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$.*

If $x \in \mathcal{F}$ is a $\mathcal{T}$-loxodromic element, then the following statements are equivalent:

1. *the element x is $\mathcal{Y}_\tau$-loxodromic;*
2. *the element x $[\psi]$-grows exponentially rel. $\mathcal{T}$: $\lim\limits_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{\mathcal{T}} = \log \lambda[\tau]$; and*
3. *the $\mathcal{T}$-axis for $\psi^m(x)$ has an arbitrarily long leaf segment of $\mathcal{L}^+[\tau]$ for $m \gg 1$.*

The restriction of ψ to the $[\psi]$-invariant subgroup system $\mathcal{G}[\mathcal{Y}_\tau]$ of $\mathcal{Y}_\tau$-point stabilizers has constant growth rel. $\mathcal{T}$: $\{\|\psi^m(x)\|_{\mathcal{T}} : m \ge 0\}$ is bounded for all $x \in \mathcal{G}[\mathcal{Y}_\tau]$.

Proof. Let $\lambda \coloneqq \lambda[\tau] > 1$, $C \coloneqq C[\tau]$ a cancellation constant for $\tau\colon (\mathcal{T}, d_\tau) \to (\mathcal{T}, d_\tau)$, and $C' \coloneqq \frac{C}{\lambda-1}$ a cancellation constant for $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$. Fix a line l in $\mathcal{T}$, and let $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ be the constructed equivariant metric PL-map.

Case 1: $d_\infty(\pi(p), \pi(q)) > 2C' + L$ for some $k \ge 0$, $p, q \in \tau_*^k(l)$, and $L > 0$. By definition of d_∞ (construction of π), $d_\tau(\tau^m(p), \tau^m(q)) > \lambda^m(2C' + L)$ for $m \gg 1$. Pick $m \gg 1$ and $r_m, s_m \in [\tau^m(p), \tau^m(q)]$ so that $d_\tau(\tau^m(p), r_m), d_\tau(s_m, \tau^m(q)) > \lambda^m C'$ and $d_\tau(r_m, s_m) > \lambda^m L$. By bounded cancellation (for τ^m), the interval $I_m \coloneqq [r_m, s_m]$ is disjoint from $\tau^m(\tau_*^k(l) \setminus [p,q])$ in (T, d_τ). So I_m is an interval in $\tau_*^{m+k}(l)$.

Let $N \coloneqq N(p,q)$ be the number of vertices in the interval (p,q). Then I_m is covered by $N+1$ leaf segments (of $\mathcal{L}^+[\tau]$) as τ is a train track. By the pigeonhole principle, I_m

(and hence $\tau_*^{m+k}(l)$) contains a leaf segment with d_τ-length $> \frac{\lambda^m L}{N+1}$; therefore, the line $\tau_*^n(l)$ in $\mathcal{T}$ contains arbitrarily long leaf segments for $m \gg 1$.

Case 2: $\pi(\tau_*^m(l))$ has diameter $\leq 2C'$ for all $m \geq 0$. We claim that any leaf segment in the line $\tau_*^m(l)$ $(m \geq 0)$ has d_τ-length $\leq 2C'$. For the contrapositive, suppose some $\tau_*^m(l)$ has a leaf segment with d_τ-length $L > 2C'$. By the train track property and bounded cancellation, $\tau_*^{m+1}(l)$ has a leaf segment with d_τ-length $\geq \lambda L - 2C > L$. By induction, $\pi(\tau_*^{m+m'}(l))$ has diameter $\geq \lambda^{m'}(L - 2C')$ for $m' \geq 0$ and $\lambda^{m'}(L-2C') > 2C'$ for $m' \gg 1$.

We finally return to the proof of the proposition. Fix a $\mathcal{T}$-loxodromic element $x \in \mathcal{F}$ and let $l \subset \mathcal{T}$ be its axis; in particular, $\pi(l)$ is x-invariant by equivariance of π. As τ is λ-Lipschitz with respect to d_τ, $\limsup\limits_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d_\tau} \leq \log \lambda$.

Case–i: $d_\infty(\pi(p), \pi(q)) > 2C'$ for some $k \geq 0$ and $p, q \in \tau_*^k(l)$. The line $\tau_*^m(l)$, the axis for $\phi^m(x)$ in $\mathcal{T}$, contains an arbitrarily long leaf segment for $m \gg 1$ by the Case 1 analysis. By bounded cancellation (for π), some nondegenerate interval I in $[\pi(p), \pi(q)]_\infty$ is disjoint from $\pi(\tau_*^k(l) \setminus [p,q])$. Since $\tau_*^k(l)$ is the axis for $\psi^k(x)$, it contains disjoint translates $[p,q]$, $\psi^k(x^{-n}) \cdot [p,q]$, $\psi^k(x^n) \cdot [p,q]$ for some $n \gg 1$. Then $\psi^k(x^{-n}) \cdot I$ and $\psi^k(x^n) \cdot I$ are in distinct components of $\mathcal{Y}_\tau \setminus I$ and $\psi^k(x)$ is $\mathcal{Y}_\tau$-loxodromic. Since $\|\cdot\|_{d_\infty} \leq \|\cdot\|_{d_\tau}$ and $\|\psi(\cdot)\|_{d_\infty} = \lambda \|\cdot\|_{d_\infty}$, we get $\log \lambda \leq \liminf\limits_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{d_\tau}$ and x is $\mathcal{Y}_\tau$-loxodromic. Finally, $\log \lambda = \lim\limits_{m\to\infty} \frac{1}{m} \log \|\psi^m(x)\|_{\mathcal{T}}$ since d_τ and the combinatorial metric on $\mathcal{T}$ are bilipschitz.

Case–ii: $\pi(\tau_*^m(l))$ has diameter $\leq 2C'$ for all $m \geq 0$. Any leaf segment in $\tau_*^m(l)$ $(m \geq 0)$ have d_τ-length $\leq 2C'$ by Case 2 analysis. Let N be the number of vertices in a fundamental domain of x acting on l. By the train track property, the fundamental domain of $\tau_*^m(l)$ is covered by $N+1$ leaf segments and $\|\psi^m(x)\|_{\mathcal{T}} \leq K \|\psi^m(x)\|_{d_\tau} \leq 2C'K(N+1)$ for some $K \geq 1$ and all $m \geq 0$. But x acts on $\mathcal{Y}_\tau$ by an isometry, and $\pi(l) \subset \mathcal{Y}_\tau$ is x-invariant; so x must be $\mathcal{Y}_\tau$-elliptic. $\square$

We now introduce the notion of *factored* forests. Suppose the stable laminations $\mathcal{L}^+[\tau]$ have components Λ_i^+ $(1 \leq i \leq k)$. The $\mathcal{F}$-orbits of edges in $\mathcal{T}$ can be partitioned into *blocks* $\mathcal{B}_i$ indexed by the components $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$. For $p, q \in \mathcal{T}$, let $d_\tau^{(i)}$ be the d_τ-length of the intersection of the interval $[p,q]$ and the subforest spanned by $\mathcal{B}_i$; this defines an $\mathcal{F}$-invariant convex pseudometric $d_\tau^{(i)}$ on $\mathcal{T}$. The metric d_τ is a sum of the pseudometrics $d_\tau^{(i)}$, denoted $\Sigma_{i=1}^k d_\tau^{(i)}$; we call $\Sigma_{i=1}^k d_\tau^{(i)}$ a factored $\mathcal{F}$-invariant convex metric and $(\mathcal{T}, \Sigma_{i=1}^k d_\tau^{(i)})$ a factored $\mathcal{F}$-forest. This factored metric is special: the *factors* $d_\tau^{(i)}$ $(1 \leq i \leq k)$ are pairwise mutually singular: for $i \neq j$, there are intervals (e.g. the leaf segments) with positive $d_\tau^{(i)}$-length and 0 $d_\tau^{(j)}$-length — such factored pseudometrics will be denoted by $\oplus_{i=1}^k d_\tau^{(i)}$ to invoke the idea of independence in direct sums. The limit pseudometrics $d_\infty^{(i)}$ are pairwise mutually singular since π is surjective and isometric on leaf segments; thus $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$. The next lemma is the cornerstone of our universality result:

Lemma IV.3 (cf. [2, Lemma 3.4])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ with eigenmetric d_τ, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$, and $\lambda \coloneqq \lambda[\tau]$.*

If $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and the k-component lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$, then the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m\geq 0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$, where $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ and $c_i > 0$.

Remark. Factored $\mathcal{F}$-forests are needed for this lemma when $k \geq 2$; moreover, the sequence $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)_{m\geq 0}$ will not converge in general (but is asymptotically periodic) when $k \geq 2$. Convergence is in the subspace of *translation distance functions* in $\mathbb{R}^{\mathcal{F}}_{\geq 0}$ with the product topology.

We give the proof in Section IV.1. In particular, if $\tau'\colon \mathcal{T}' \to \mathcal{T}'$ is another expanding irreducible train track for ψ and $\mathcal{F}[\mathcal{T}'] = \mathcal{F}[\mathcal{T}]$, then the limit forest for $[\tau']$ is equivariantly homothetic to $(\mathcal{Y}_\tau, d_\infty)$ — set $(\mathcal{Y}, \delta) \coloneqq (\mathcal{T}', d_{\tau'})$, apply the lemma, then observe that the sequence $(c_i)_{i=1}^k$ must be constant in this case. A minimal very small $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is an expanding forest for $[\psi]$ like $\mathcal{Y}_\tau$ if it is nondegenerate and there is:

1. a ψ-equivariant expanding homothety $(\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$; and
2. an equivariant PL-map $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$.

Corollary II.3. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$. Any expanding forests for $[\psi]$ like $\mathcal{Y}_\tau$ is uniquely equivariantly homothetic to $(\mathcal{Y}_\tau, d_\infty)$.*

Proof. Let $(\mathcal{Y}, \delta)$ be an expanding forest for $[\psi]$ like $\mathcal{Y}_\tau$, $f\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ an equivariant PL-map with cancellation constant $C \coloneqq C[f]$, $g\colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$ the ψ-equivariant expanding s-homothety, $x \in \mathcal{F}$ a $\mathcal{Y}$-loxodromic element. By equivariance of f, the element x is $\mathcal{T}$-loxodromic with axis $l_x \subset \mathcal{T}$. Let $[p_0, x \cdot p_0] \subset l_x$ be (the closure of) a fundamental domain of x acting on l_x. The interval $[p_0, x \cdot p_0]$ is a concatenation of $N \geq 1$ leaf segments (of $\mathcal{L}^+[\tau]$). Choose $m \gg 1$ so that $\|\psi^m(x)\|_\delta = s^m\|x\|_\delta > 2CN$. Note that the action of $\psi^m(x)$ on its axis has a fundamental domain $[p_m, \psi^m(x) \cdot p_m]$ covered by N leaf segments as τ is a train track. So $\delta(f(p_m), f(\psi^m(x) \cdot p_m)) > 2CN$ and, by the pigeonhole principle, $[p_m, \psi^m(x) \cdot p_m]$ contains a leaf segment $[q, r]$ with $\delta(f(q), f(r)) > 2C$.

Let $l \supset [q, r]$ represent some leaf $[l] \in \mathcal{L}^+[\tau]$. Bounded cancellation implies the components of $l \setminus [q, r]$ have f-images with disjoint closures. By quasiperiodicity of $[l]$ and equivariance of f, both ends of l have unbounded f-images, i.e. $[l] \in \mathrm{dom}(f_*) = \mathbb{R}(\mathcal{Y}, \delta)$ (Corollary I.4, Claim I.5). Finally, the closure of $[l]$ in $\mathbb{R}(\mathcal{T})$, a component $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$, is a subset of $\mathbb{R}(\mathcal{Y}, \delta)$ by quasiperiodicity of $[l]$. Note that the ψ-equivariant homothety g induces a homeomorphism $g_*\colon \mathbb{R}(\mathcal{Y}, \delta) \to \mathbb{R}(\mathcal{Y}, \delta)$ that is the restriction of the homeomorphism $\tau_*\colon \mathbb{R}(\mathcal{T}) \to \mathbb{R}(\mathcal{T})$. So $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{Y}, \delta)$ since τ_* acts transitively on the k components of $\mathcal{L}^+[\tau]$. Set $\lambda \coloneqq \lambda[\tau]$; by Lemma IV.3, the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m\geq 0}$ converges to the factored $\mathcal{F}$-forest $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$ for some $c_i > 0$. Yet $(\mathcal{Y}, \delta)$ is equivariantly isometric to $(\mathcal{Y}\psi, s^{-1}\delta)$; thus $s = \lambda$, $c_i = c_{i+1}$ $(i < k)$, $(\mathcal{Y}, \delta)$ is equivariantly isometric to $(\mathcal{Y}_\tau, c_1\, d_\infty)$, and the equivariant isometry is unique [6, Theorem 3.7]. $\square$

II.2.3 Iterated turns

We have already shown how iterating an edge in $\mathcal{T}$ by the train track τ produces the system of stable laminations $\mathcal{L}^+[\tau]$. Later, we will consider how $\mathcal{L}^+[\tau]$ determines laminations in (a free splitting of) the subgroup system $\mathcal{G}[\mathcal{Y}_\tau]$.

Let $\mathcal{T}'$ be a free splitting of $\mathcal{F}$ whose free factor system $\mathcal{F}[\mathcal{T}']$ is trivial. Then there is an equivariant PL-map $f\colon (\mathcal{T}', d') \to (\mathcal{T}, d_\tau)$. Let γ be a line in $(\mathcal{Y}_\tau, d_\infty)$, $\pi^*(\gamma)$ its lift to $(\mathcal{T}, d_\tau)$, and $f^*(\pi^*(\gamma))$ its lift to $(\mathcal{T}', d')$. Denote the ends of γ by ε_i $(i = 1, 2)$. Let $T \subset \mathcal{T}$ be the component containing $\pi^*(\gamma)$, and $T' \subset \mathcal{T}'$, $Y_\tau \subset \mathcal{Y}_\tau$, and $F \subset \mathcal{F}$ be the matching components. Denote the first return maps of τ, h, and ψ on T, Y_τ, and F by $\tilde{\tau}$, $\tilde{h}$, and φ respectively. For the rest of the section, redefine λ to be the stretch factor of the expanding homothety $\tilde{h}$.

Suppose $\circ$ is a point on the line γ with a nontrivial stabilizers $G_\circ \coloneqq \mathrm{Stab}_F(\circ)$. Let d_i $(i = 1, 2)$ be the direction at $\circ$ containing ε_i. By Gaboriau–Levitt index theory, $\tilde{h}^j(\circ) = y \cdot \circ$ and $\tilde{h}^j(d_i) = ys_i \cdot d_i$ for some $y \in F$, $s_i \in G_\circ$, and minimal $j \geq 1$. Since F acts on Y_τ with trivial arc stabilizers, the elements ys_1, ys_2 are unique and $s_1^{-1}s_2 \in G_\circ$ is independent of the chosen $y \in F$.

Set $y_0 \coloneqq \epsilon$ to be the trivial element and $y_{m+1} \coloneqq \varphi^{mj}(ys_1)y_m$ for $m \geq 0$. Let $T'(G)$ be the characteristic convex subset of T' for a nontrivial subgroup $G \leq F$. Since T' is simplicial, the characteristic convex subset $T'(G)$ is closed, and we have the closest point retraction $T' \to T'(G)$; it extends uniquely to the ends-completions. Let $q'_{i,m}$ be the closest point projection of $f^*(\pi^*(\tilde{h}_*^{mj}(\varepsilon_i)))$ to $T'(\varphi^{mj}(G_\circ))$. Set $\tau_\circ \coloneqq (ys_1)^{-1} \cdot \tilde{\tau}^j$ and $h_\circ \coloneqq (ys_1)^{-1} \cdot \tilde{h}^j$ to be $\psi_\circ$-equivariant maps for an automorphism $\psi_\circ\colon F \to F$ in the outer class $[\varphi^j]$. As $h_\circ$ fixes $\circ$, we get $\psi_\circ(G_\circ) = G_\circ$ and $y_m^{-1} \cdot T'(\varphi^{mj}(G_\circ))$ is the characteristic convex subset for $\psi_\circ^m(G_\circ) = G_\circ$. Thus $q_{i,m} \coloneqq y_m^{-1} \cdot q'_{i,m}$ is in $T'(G_\circ)$ for $i = 1, 2$ and $m \geq 0$. The interval $[q_{1,m}, q_{2,m}]$ in $T'(G_\circ)$, i.e. the closest point projection of $f^*(\pi^*(h_\circ^m(\gamma)))$, is the <u>turn</u> in $f^*(\pi^*(h_\circ^m(\gamma)))$ about $T'(G_\circ)$.

Since $h_\circ(d_1) = d_1$, the ends $h_{\circ *}^m(\varepsilon_1)$ $(m \geq 0)$ are in fact ends of d_1. If $h_{\circ *}(\varepsilon_1) = \varepsilon_1$, then the sequence $(q_{1,m})_{m \geq 0}$ is constant. Otherwise, the ends $h_{\circ *}^m(\varepsilon_1)$ are distinct for $m \geq 0$. Let $\gamma_{1,m}$ be the line in d_1 determined by $h_{\circ *}^{m+1}(\varepsilon_1)$ and $h_{\circ *}^m(\varepsilon_1)$. As $h_\circ$ is an expanding homothety, the distance $d_\infty(\circ, \gamma_{1,m}) > 0$ from $\circ$ to $\gamma_{1,m}$ grows exponentially in m. So $d_\infty(\circ, \gamma_{1,M_1}) > 2C[\pi \circ f]$ for some minimal $M_1 \geq 0$, and the line $f^*(\pi^*(\gamma_{1,m}))$ is disjoint from $T'(G_\circ)$ for $m \geq M_1$ by bounded cancellation (see Figure 2). In particular, the ends $f^*(\pi^*(h_{\circ *}^{m+1}(\varepsilon_1)))$ and $f^*(\pi^*(h_{\circ *}^m(\varepsilon_1)))$ have the same closest point projection to $T'(G_\circ)$ and the sequence $(q_{1,m})_{m \geq M_1}$ is constant.

Since $h_\circ(d_2) = s_1^{-1}s_2 \cdot d_2$, the ends $f^*(\pi^*(h_{\circ *}^{m+1}(\varepsilon_2)))$ and $\psi_\circ^m(s_1^{-1}s_2) \cdot f^*(\pi^*(h_{\circ *}^m(\varepsilon_2)))$ have the same closest point projection to $T'(G_\circ)$ for $m \gg 1$ by similar bounded cancellation

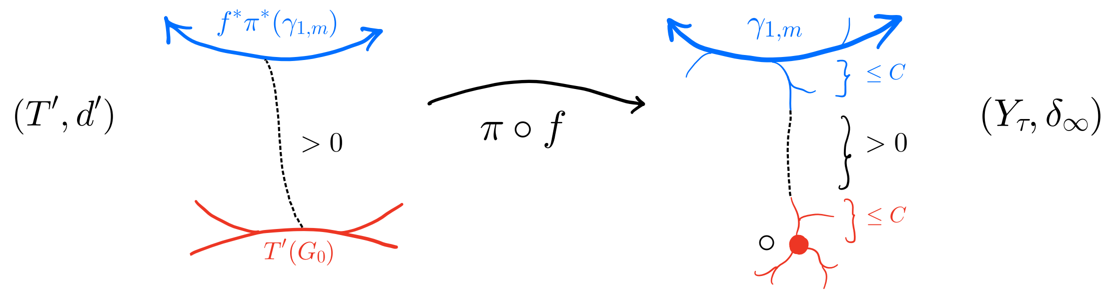

Figure 2: For $m \geq M_1$, the line $f^*(\pi^*(\gamma_{1,m}))$ cannot intersect $T'(G_\circ)$.

reasoning, i.e. $q_{2,m+1} = \psi_\circ^m(s_1^{-1}s_2) \cdot q_{2,m}$ for some minimal $M_2 \geq 0$ and all $m \geq M_2$.

Set $M = \max(M_1, M_2)$. The sequence $[q_{1,M+m}, q_{2,M+m}]_{m\geq 0}$ of intervals is well-defined for the line γ and point $\circ \in \gamma$ as M_1 and M_2 were chosen minimally. An *iterated turn over* $T'(G_\circ)$ *rel.* $\psi_\circ|_{G_\circ}$ is any such sequence of intervals. More generally, we define an iterated turn over T rel. φ: pick arbitrary points $p_i \in T$ $(i = 1, 2)$ and elements $x_i \in F$; set $p_{i,0} := p_i$ and $p_{i,m+1} := \varphi^m(x_i) \cdot p_{i,m}$ for $m \geq 0$; the sequence $[p_{1,m}, p_{2,m}]_{m\geq 0}$ is the iterated turn denoted by $(p_1, p_2 : x_1, x_2; \varphi)_T$. Any iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_T$ translates to a unique *normal* form $(p_1, p_2 : \epsilon, x_1^{-1}x_2; \tilde{\varphi})_T$ with $\tilde{\varphi}\colon y \mapsto x_1^{-1}\varphi(y)x_1$.

We now characterize the growth of an iterated turn over T rel. φ:

Proposition II.4. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ with eigenmetric d_τ, and $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$. Choose a nondegenerate component $T \subset \mathcal{T}$, corresponding components $F \subset \mathcal{F}$, $Y_\tau \subset \mathcal{Y}_\tau$, and a positive iterate ψ^k that preserves F. Let $\tilde{h}\colon (Y_\tau, d_\infty) \to (Y_\tau, d_\infty)$ be the φ-equivariant λ-homothety, where φ is in the outer automorphism $[\psi^k|_F]$ and $\lambda := (\lambda[\tau])^k$. Finally, for $i = 1, 2$, pick $p_i \in T$ and $x_i \in F$.*

The point $p_{i,m} := \varphi^{-1}(x_i) \cdots \varphi^{-m}(x_i) \cdot p_i$ in $(T\varphi^m, \lambda^{-m}d_\tau)$ converges to $\star_i$ in $(\overline{Y}_\tau, d_\infty)$ as $m \to \infty$, where $\star_i$ is the unique fixed point of $x_i^{-1} \cdot \tilde{h}$ in the metric completion $(\overline{Y}_\tau, d_\infty)$; concretely:

$$\lim_{m\to\infty} \lambda^{-m} d_\tau(p_{1,m}, p_{2,m}) = d_\infty(\star_1, \star_2).$$

If $x_1^{-1}x_2$ fixes $\star_1$, then $\star_1 = \star_2$ and the m^{th} term $[p_{1,m}, p_{2,m}]$ of the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_T$ has d_τ-length $\leq (m+1)A$ for some constant $A \geq 1$. Otherwise, $\star_1 \neq \star_2$ and the iterated turn has arbitrarily long leaf segments of $\mathcal{L}^+[\tau]$.

The limit $[\star_1, \star_2] \subset \overline{Y}_\tau$ of an iterated turn is independent of the points $p_1, p_2 \in T$. Thus we introduce the notion of an *algebraic* iterated turn over F rel. φ, denoted $(x_1, x_2; \varphi)_F$.

Proof. Let $p_1, p_2 \in T$, $x_1, x_2 \in F$, and $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ be the constructed equivariant metric PL-map. For $i = 1, 2$, set $p_{i,0} := p_i$ and $p_{i,m+1} := \varphi^m(x_i) \cdot p_{i,m}$ for $m \geq 0$. Recall that T and $T\varphi^m$ are the same pretrees but with different actions; thus, in $T\varphi^m$, we have $p_{i,m} = \varphi^{-1}(x_i)\cdots\varphi^{-m}(x_i)\cdot p_i$ for $m \geq 0$. As $\pi\colon (T, d_\tau) \to (Y_\tau, d_\infty)$ is an equivariant metric PL-map, so is the composition

$$\pi_m\colon (T\varphi^m, \lambda^{-m} d_\tau) \xrightarrow{\pi} (Y_\tau\varphi^m, \lambda^{-m} d_\infty) \xrightarrow{\tilde{h}^{-m}} (Y_\tau, d_\infty).$$

The point p_i in (T, d_τ) projects (via π) to $\pi(p_i)$ in (Y_τ, d_∞); the point $p_{i,m}$ in $(T\varphi^m, \lambda^{-m} d_\tau)$ projects (via π_m) to

$$\begin{aligned}
\pi_m(p_{i,m}) &\coloneqq \tilde{h}^{-m}(\pi(p_{i,m})) \\
&= \varphi^{-1}(x_i)\cdots\varphi^{-m}(x_i)\cdot \tilde{h}^{-m}(\pi(p_i)) \qquad (\, p_{i,m}, p_i \in T\varphi^m \,) \\
&= (x_i^{-1}\cdot \tilde{h})^{-m}(\pi(p_i)) \qquad\qquad\qquad\qquad (\, p_i \in T \,)
\end{aligned}$$

in (Y_τ, d_∞) for $m \geq 1$ — in the last line, $x_i^{-1}\cdot\tilde{h}$ is a λ-homothety $(Y_\tau, d_\infty) \to (Y_\tau, d_\infty)$. Since $(x_i^{-1}\cdot\tilde{h})^{-1}$ is contracting, the point $\pi_m(p_{i,m})$ converges (as $m \to \infty$) to the unique fixed point $\star_i$ of $(x_i^{-1}\cdot\tilde{h})^{-1}$ (and $x_i^{-1}\cdot\tilde{h}$) in the metric completion $(\overline{Y}_\tau, d_\infty)$ by the contraction mapping theorem; note that $x_1^{-1}x_2\cdot\star_1 = \star_1$ if and only if $\star_1 = \star_2$. Thus the π_m-projection of the point $p_{i,m}$ in $(T\varphi^m, \lambda^{-m} d_\tau)$ converges (as $m \to \infty$) to $\star_i$ in $(\overline{Y}_\tau, d_\infty)$; in particular,

$$\lim_{m\to\infty} \lambda^{-m} d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) = \lim_{m\to\infty} d_\infty(\pi_m(p_{1,m}), \pi_m(p_{2,m})) = d_\infty(\star_1, \star_2).$$

Let $\tilde{\tau}\colon T \to T$ be the φ-equivariant translate of a component of τ^k. The interval $[p_{1,m}, p_{2,m}] \subset T$, the m^{th} term in $(p_1, p_2 : x_1, x_2; \varphi)_T$, is covered by these $2m+1$ intervals:

$$\begin{aligned}
&\varphi^{m-1}(x_1)\cdots\varphi(x_1)\cdot[x_1\cdot p_1, \tilde{\tau}(p_1)], \ldots,\ \varphi^{m-1}(x_1)\cdot[\tilde{\tau}^{m-2}(x_1\cdot p_1), \tilde{\tau}^{m-1}(p_1)], \\
&[\tilde{\tau}^{m-1}(x_1\cdot p_1), \tilde{\tau}^m(p_1)],\ [\tilde{\tau}^m(p_1), \tilde{\tau}^m(p_2)],\ [\tilde{\tau}^m(p_2), \tilde{\tau}^{m-1}(x_2\cdot p_2)], \\
&\varphi^{m-1}(x_2)\cdot[\tilde{\tau}^{m-1}(p_2), \tilde{\tau}^{m-2}(x_2\cdot p_2)], \ldots,\ \varphi^{m-1}(x_2)\cdots\varphi(x_2)\cdot[\tilde{\tau}(p_2), x_2\cdot p_2].
\end{aligned}$$

Set $D \coloneqq \max\{d_\tau(x_i\cdot p_i, \tilde{\tau}(p_i) : i = 1, 2\}$ and $D' \coloneqq \frac{D}{\lambda-1}$.

Recall that $\lim\limits_{m'\to\infty} \lambda^{-m'} d_\tau(\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})) = d_\infty(\pi(p_{1,m}), \pi(p_{2,m}))$. For $m' \geq 0$, we get a similar covering of $[p_{1,m+m'}, p_{2,m+m'}]$ by $2m'+1$ intervals with the "middle" $[\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m})]$. Since $\tilde{\tau}$ is λ-Lipschitz with respect to d_τ, the sum of the d_τ-lengths of all intervals but the middle in this covering is $\leq \lambda^{m'} 2D'$. By the triangle inequality,

$$\lambda^{-(m+m')}\left| d_\tau(p_{1,m+m'}, p_{2,m+m'}) - d_\tau(\tilde{\tau}^{m'}(p_{1,m}), \tilde{\tau}^{m'}(p_{2,m}))\right| \leq \lambda^{-m} 2D'.$$

Fix $\epsilon > 0$; then $\lambda^{-m} 2D' < \epsilon$ and $|\lambda^{-m} d_\infty(\pi(p_{1,m}), \pi(p_{2,m})) - d_\infty(\star_1, \star_2)| < \epsilon$ for some $m \gg 1$.

Similarly,

$$\lambda^{-m}\left|\lambda^{-m'}d_\tau(\tilde{\tau}^{m'}(p_{1,m}),\tilde{\tau}^{m'}(p_{2,m}))-d_\infty(\pi(p_{1,m}),\pi(p_{2,m}))\right|<\epsilon$$

$$\text{and }\left|\lambda^{-(m+m')}d_\tau(p_{1,m+m'},p_{2,m+m'})-d_\infty(\star_1,\star_2)\right|<3\epsilon\text{ for }m'\gg 1,$$

$$\text{i.e. }\lim_{m\to\infty}\lambda^{-m}d_\tau(p_{1,m},p_{2,m})=d_\infty(\star_1,\star_2).$$

Let $N(u,v)$ be the number of vertices in an interval $(u,v)\subset T$; set N to be the maximum of $N(p_1,p_2)$, $N(x_1\cdot p_1,\tilde{\tau}(p_1))$, and $N(\tilde{\tau}(p_2),x_2\cdot p_2)$. As $\tilde{\tau}$ is a train track, the interval $[p_{1,m},p_{2,m}]$ is covered by $(2m+1)(N+1)$ leaf segments.

Suppose $\star_1=\star_2$. We claim that any leaf segment (of $\mathcal{L}^+[\tau]$) in $[p_{1,m},p_{2,m}]$ has uniformly (in $m\geq 0$) bounded d_τ-length — this implies $[p_{1,m},p_{2,m}]$ has d_τ-length $\leq(2m+1)(N+1)B$ for some bounding constant $B\geq 1$. We mimic Case 2 from the proof of Proposition II.2. For the contrapositive, suppose some term $[p_{1,m},p_{2,m}]$ has a leaf segment with d_τ-length $L>2(C[\pi]+D')$. By the train track property, bounded cancellation, and interval covering, $[p_{1,m+m'},p_{2,m+m'}]$ has a leaf segment with d_τ-length $\geq\lambda^{m'}(L-2C[\pi]-2D')$ for $m'\geq 0$; in $(T\varphi^{m+m'},\lambda^{-(m+m')}d_\tau)$, $[p_{1,m+m'},p_{2,m+m'}]$ has length $\geq\lambda^{-m}(L-2C[\pi]-2D')$. In the limit (as $m'\to\infty$), $d_\infty(\star_1,\star_2)\geq\lambda^{-m}(L-2C[\pi]-2D')>0$.

Suppose $\star_1\neq\star_2$. Set $L:=\frac{1}{2}d_\infty(\star_1,\star_2)>0$; then $\lambda^{-m}d_\tau(p_{1,m},p_{2,m})>L$ for some $m\gg 1$. By the pigeonhole principle, the interval $[p_{1,m},p_{2,m}]$ has a leaf segment with d_τ-length $\frac{\lambda^m L}{(2m+1)(N+1)}$, which can be arbitrarily large (in m). $\square$

II.2.4 Nested iterated turns

The first part of the previous subsection explains how a line in $(\mathcal{Y}_\tau,d_\infty)$ determines algebraic iterated turns over $\mathcal{G}[\mathcal{Y}_\tau]$. We now give a similar discussion for an iterated turn over $\mathcal{T}'$.

Recall how $f,T,T',Y_\tau,F,\tilde{\tau}$, $\tilde{h}$, and φ were chosen and λ was redefined in the previous subsection. Pick points $p_1',p_2'\in T'$ and elements $x_1,x_2\in F$. Set $T_m':=T'\varphi^m$, $T_m:=T\varphi^m$, $p_{i,0}':=p_i'$, $p_{i,m}':=\varphi^{-1}(x_i)\cdots\varphi^{-m}(x_i)\cdot p_i'$ in T_m', and $p_{i,m}=f(p_{i,m}')$ for $m\geq 1$ and $i=1,2$. By Proposition II.4, the point $p_{i,m}$ in $(T_m,\lambda^{-m}d_\tau)$ converges (as $m\to\infty$) to $\star_i$, the unique fixed point of $x_i^{-1}\cdot\tilde{h}$ in the metric completion $(\overline{Y}_\tau,d_\infty)$. The λ-homothety $h_i:=x_i^{-1}\cdot\tilde{h}$ is φ_i-equivariant for some automorphism $\varphi_i\colon F\to F$ in the outer class $[\varphi]$. Set $G_1:=\mathrm{Stab}_F(\star_1)$.

Case–a: $s:=x_1^{-1}x_2\in G_1$. Suppose G_1 is not trivial, and let $a_{i,m}$ be the closest point projection of $p_{i,m}'$ to $T'(\varphi^m(G_1))$ for $m\geq 0$. As $\tilde{h}(\star_1)=x_1\cdot\star_1$ and $\tilde{h}$ is φ-equivariant, we get $T'(\varphi^{m+1}(G_1))=\varphi^m(x_1)\cdot T'(\varphi^m(G_1))$, $a_{1,m+1}=\varphi^m(x_1)\cdot a_{1,m}$, and

$$\begin{aligned}a_{2,m+1}&=\varphi^m(x_1)\varphi^m(s)\cdot a_{2,m}\\&=\varphi^m(x_1)\cdots\varphi(x_1)x_1\varphi_1^m(s)\cdots\varphi_1^m(s)s\cdot a_{2,0}\quad\text{for }m\geq 0.\end{aligned}$$

Thus the closest point projection to $T'(\varphi^m(G_1))$ of the m^{th} term of the given iterated turn $(p_1',p_2':x_1,x_2;\varphi)_{T'}$ is a translate of the m^{th} term in $(a_{1,0},a_{2,0}:\epsilon,s;\varphi_1|_{G_1})_{T'(G_1)}$, where

$m \geq 0$ and ϵ is the trivial element. Hence we have an algebraic iterated turn $(\epsilon, s; \varphi_1|_{G_1})_{G_1}$ that is well-defined for the algebraic iterated turn $(x_1, x_2; \varphi)_F$.

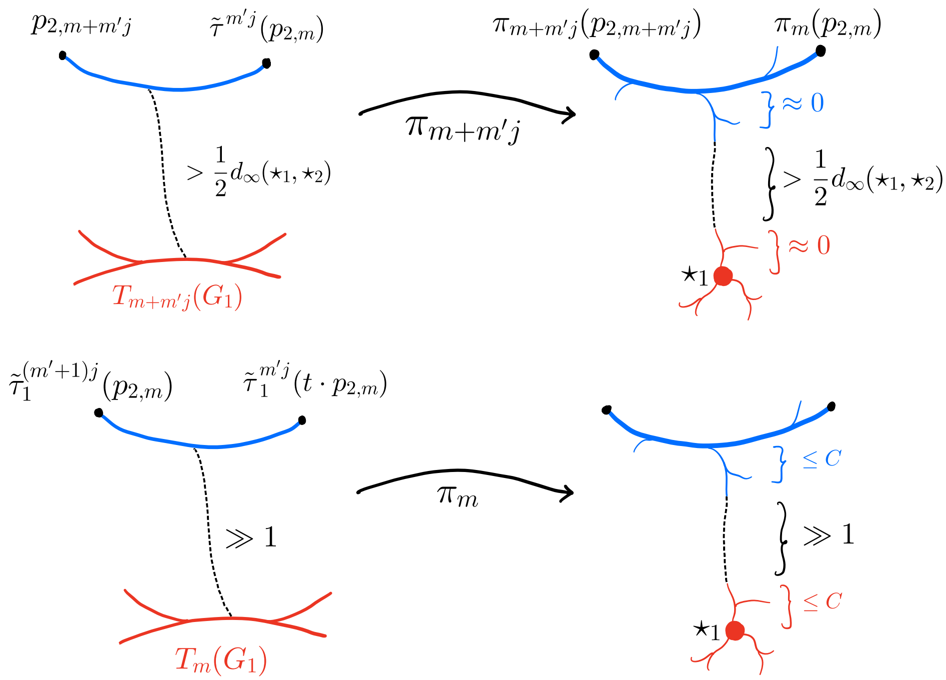

Figure 3: The two figures illustrating certain closest point projections are the same.

Case–b: $\star_1 \neq \star_2$. Suppose G_1 is not trivial — the argument is symmetric if $\mathrm{Stab}_F(\star_2)$ is not trivial — and let d be the direction at $\star_1$ containing $\star_2$. By Gaboriau–Levitt index theory, $h_1^j(d) = t \cdot d$ for some $t \in G_1$ and minimal $j \geq 1$. For $m \gg 1$, $\pi_m(p_{2,m}) = h_2^{-m}(\pi(p_2))$ is in the direction d since $h_2^{-m}(\pi(p_2)) \to \star_2$ in $(\overline{Y}_\tau, d_\infty)$. For $m \gg 1$ and $m' \geq 0$, the interval $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ in $(T_{m+m'j}, \lambda^{-m-m'j}d_\tau)$ is disjoint from $T_{m+m'j}(G_1)$ by bounded cancellation (see Figure 3, top); or equivalently, the interval $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ in T_m is disjoint from $T_m(\varphi^{m'j}(G_1))$. In fact, the $\lambda^{-m}d_\tau$-distance in T_m from $[p_{2,m+m'j}, \tilde{\tau}^{m'j}(p_{2,m})]$ to $T_m(\varphi^{m'j}(G_1))$ can be arbitrarily large for $m' \gg 1$.

Set $z_0 := \epsilon$ and $z_{m'+1} := \varphi^{m'}(x_1)z_{m'}$. Let $b'_{i,m'}$ $(i = 1, 2)$ be the closest point projection of $p'_{i,m+m'j}$ to $T'_m(\varphi^{m'j}(G_1)) = z_{m'j} \cdot T'_m(G_1)$ and set $b_{i,m'} := z_{m'j}^{-1} \cdot b'_{i,m'}$ in $T'_m(G_1)$. Following the definitions, $z_{m'j}^{-1} \cdot p'_{1,m+m'j} = p'_{1,m}$ in T'_m and $z_{m'j}^{-1} \cdot \tilde{\tau}^{m'j} = \tau_1^{m'j}$ in T_m, where $\tau_1 := x_1^{-1} \cdot \tilde{\tau}$; in particular, $b_{1,m'} = b_{1,0}$ for $m' \geq 0$. Since $h_1^j(d) = t \cdot d$, bounded cancellation implies the $\lambda^{-m}d_\tau$-distance in T_m from $[\tau_1^{(m'+1)j}(p_{2,m}), \varphi_1^{m'j}(t) \cdot \tau_1^{m'j}(p_{2,m})]$ to $T_m(G_1)$ is arbitrarily large for $m' \gg 1$ (see Figure 3, bottom).

So $[z_{(m'+1)j}^{-1} \cdot p_{2,m+(m'+1)j}, \varphi_1^{m'j}(t)z_{m'j}^{-1} \cdot p_{2,m+m'j}]$ is arbitrarily far from $T_m(G_1)$ by transitivity. By bounded cancellation, $[z_{(m'+1)j}^{-1} \cdot p'_{2,m+(m'+1)j}, \varphi_1^{m'j}(t) z_{m'j}^{-1} \cdot p'_{2,m+m'j}]$ is disjoint from $T'_m(G_1)$ for $m' \gg 1$, i.e. $b_{2,m'+1} = \varphi_1^{m'j}(t) \cdot b_{2,m'}$ for $m' \gg 1$. Thus, for some $M' \gg 1$, the sequence $[b_{1,M'+m'}, b_{2,M'+m'}]_{m' \geq 0}$ is an iterated turn over $T'_m(G_1)$ rel. $\varphi_1^j|_{G_1}$, denoted $(b_{1,M'}, b_{2,M'} : \epsilon, t; \varphi_1^j|_{G_1})_{T'_m(G_1)}$. The corresponding algebraic iterated turn $(\epsilon, t; \varphi_1^j|_{G_1})_{G_1}$ is well-defined for $(x_1, x_2; \varphi)_F$.

Now suppose $\circ \in (\star_1, \star_2)$ has a nontrivial stabilizer $G_\circ := \mathrm{Stab}_F(\circ)$. Let d_i $(i = 1, 2)$ be the direction at $\circ$ containing $\star_i$. By index theory again, $h_1^l(\circ) = x \cdot \circ$ and $h_1^l(d_i) = xs_i \cdot d_i$ for some $x \in F$, $s_i \in G_\circ$, and minimal $l \geq 1$. Since F acts on Y_τ with trivial arc stabilizers, the elements xs_1, xs_2 are unique and $s_1^{-1}s_2 \in G_\circ$ is independent of the chosen $x \in F$. For $m \gg 1$, $\pi_m(p_{i,m})$ is in the direction d_i since $\pi_m(p_{i,m}) \to \star_i$. A variation of the bounded cancellation argument used in the preceding paragraphs proves the following. For $m, m' \gg 1$, the interval $[p_{i,m+m'l}, \tilde{\tau}^{m'l}(p_{i,m})]$ in $(T_m, \lambda^{-m} d_\tau)$ is far from $T_m(\varphi^{m'l}(G_\circ))$.

Set $y_0 := \epsilon$, $y_{m'+1} := \varphi_\circ^{m'}(x) y_{m'}$, $\tau_\circ := x^{-1} \cdot \tau_1^l$, and $h_\circ := x^{-1} \cdot h_1^l$ to be $\varphi_\circ$-equivariant maps for an automorphism $\varphi_\circ \colon F \to F$ in the outer class $[\varphi_1^j]$. Let $c'_{i,m'} \in T'_m(\varphi^{m'l}(G_\circ))$ be the closest point projection of $p'_{i,m+m'l}$ and set $c''_{i,m'} := z_{m'l}^{-1} \cdot c'_{i,m'} \in T'_m(\varphi_1^{m'l}(G_\circ))$. Then $c_{i,m'} := y_{m'}^{-1} \cdot c''_{i,m'} \in T'_m(G_\circ)$ is the closest point projection of $y_{m'}^{-1} z_{m'l}^{-1} \cdot p'_{i,m+m'l}$. Since $h_\circ(d_i) = s_i \cdot d_i$, the interval $[\tau_\circ^{m'+1}(p_{i,m}), \varphi_\circ^{m'}(s_i) \cdot \tau_\circ^{m'}(p_{i,m})]$ is arbitrarily far from $T_m(G_\circ)$ for $m' \gg 1$. By transitivity, $[y_{m'+1}^{-1} z_{(m'+1)l}^{-1} \cdot p_{i,m+(m'+1)l}, \varphi_\circ^{m'}(s_i) y_{m'}^{-1} z_{m'}^{-1} \cdot p_{i,m+m'l}]$ is arbitrarily far from $T_m(G_\circ)$. As before, $[y_{m'+1}^{-1} z_{(m'+1)l}^{-1} \cdot p'_{i,m+(m'+1)l}, \varphi_\circ^{m'}(s_i) y_{m'}^{-1} z_{m'}^{-1} \cdot p'_{i,m+m'l}]$ is disjoint from $T'_m(G_\circ)$ for $m' \gg 1$, i.e. $c_{i,m'+1} = \varphi_\circ^{m'}(s_i) \cdot c_{i,m'}$ for $m' \gg 1$. Thus, for some $M'' \gg 1$, the sequence $[c_{1,M''+m'}, c_{2,M''+m'}]_{m' \geq 0}$ is an iterated turn over $T'_m(G_\circ)$ rel. $\varphi_\circ|_{G_\circ}$: $(c_{1,M''}, c_{2,M''} : s_1, s_2; \varphi_\circ|_{G_\circ})_{T'_m(G_\circ)}$. It is a "translate" of the normalized iterated turn: $(c_{1,M''}, c_{2,M''} : \epsilon, s_1^{-1}s_2; \varphi_\circ|_{G_\circ})_{T'_m(G_\circ)}$. The corresponding algebraic iterated turn $(\epsilon, s_1^{-1}s_2; \varphi_\circ|_{G_\circ})_{G_\circ}$ is well-defined for $(x_1, x_2; \varphi)_F$ and $\circ \in (\star_1, \star_2)$.

II.3 Coordinate-free laminations

We have only defined the stable laminations for an expanding irreducible train track $[\tau]$ representing $[\psi]$. The free splitting $\mathcal{T}$ of $\mathcal{F}$ can be seen as a coordinate system, and we need a coordinate-free definition of stable laminations that applies to all outer automorphisms.

Fix a proper free factor system $\mathcal{Z}$ of $\mathcal{F}$ and consider the set $scv(\mathcal{F}, \mathcal{Z})$ of all free splittings $\mathcal{T}'$ of $\mathcal{F}$ with $\mathcal{F}[\mathcal{T}'] = \mathcal{Z}$, i.e. an element of $\mathcal{F}$ is $\mathcal{T}'$-elliptic if and only if it is conjugate to an element of $\mathcal{Z}$; this set with some natural partial order is the *spine of relative outer space* [7]. For any pair of free splittings $\mathcal{T}_1, \mathcal{T}_2 \in scv(\mathcal{F}, \mathcal{Z})$, there are changes of coordinates, equivariant PL-maps $\mathcal{T}_1 \rightleftarrows \mathcal{T}_2$. We saw in the discussion following Claim I.5 that a change of coordinates $f \colon \mathcal{T}_1 \to \mathcal{T}_2$ induces a canonical homeomorphism $f_* \colon \mathbb{R}(\mathcal{T}_1) \to \mathbb{R}(\mathcal{T}_2)$ on the space of lines. Denote the canonical homeomorphism class of $\mathbb{R}(\mathcal{T}_1) \cong \mathbb{R}(\mathcal{T}_2)$ by $\mathbb{R}(\mathcal{F}, \mathcal{Z})$. If $\mathcal{Z}$ is the trivial free factor system of $\mathcal{F}$, then we denoted the canonical homeomorphism class by $\mathbb{R}(\mathcal{F})$ instead.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and a $[\psi]$-invariant proper free factor system $\mathcal{Z}$. Let $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ be the canonical induced homeomorphism on the space of lines: $f_* \circ g_{1*} = g_{2*} \circ f_*$ for any $\mathcal{T}_1, \mathcal{T}_2 \in scv(\mathcal{F}, \mathcal{Z})$, equivariant PL-map $f\colon \mathcal{T}_1 \to \mathcal{T}_2$, and ψ-equivariant PL-maps $g_i\colon \mathcal{T}_i \to \mathcal{T}_i$ $(i = 1, 2)$. A line $[l] \in \mathbb{R}(\mathcal{F}, \mathcal{Z})$ weakly ψ_*-limits to a lamination $\Lambda \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ if the sequence $(\psi_*^n[l])_{n\geq 0}$ weakly limits to Λ.

A coordinate-free definition of stable laminations boils down to characterizing the lines of a stable $\mathcal{T}$-lamination for $[\tau]$ in a way that is independent of coordinates. For the rest of the section, assume there is an equivariant PL-map $(\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ and consider the canonical embedding $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$. Note that a lamination $\Lambda \subset \mathbb{R}(\mathcal{Y}, \delta)$ is contained in a canonical lamination $\mathcal{L} \subset \mathbb{R}(\mathcal{T})$: set $\mathcal{L}$ to be the closure of Λ in $\mathbb{R}(\mathcal{T})$.

Claim (cf. [2, Lemma 1.9(2)]). *A line is quasiperiodic in $\mathbb{R}(\mathcal{Y}, \delta)$ if it is quasiperiodic in $\mathbb{R}(\mathcal{T})$.* (exercise) □

So quasiperiodicity is a well-defined property for a line in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$; moreover, the induced homeomorphism $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ preserves quasiperiodicity for any automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ that preserves $\mathcal{Z}$ (up to conjugacy).

Suppose there is an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$ for ψ with $\mathcal{F}[\mathcal{T}] = \mathcal{Z}$. Recall that the eigenlines of $[\tau^k]$ (for some $k \geq 1$) are constructed by iterating an expanding edge; more precisely, an eigenline $[l]$ of $[\tau^k]$ is the union $\bigcup_{n\geq 1} \tau^{kn}(\mathcal{F}\cdot e)$ for some edge $e \subset l$. The leaf segments $\tau^{kn}(e)$ determine a neighbourhood basis for $[l]$ in the space of lines.

For a line $[l] \in \mathbb{R}(\mathcal{F}, \mathcal{Z})$, a subset $U \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ is a *ψ_*^k-attracting neighbourhood* of $[l]$ if $\psi_*^k(U) \subset U$ and $\{\psi_*^{kn}(U) : n \geq 1\}$ is a neighbourhood basis for $[l]$ in the space of lines. A stable lamination for $[\psi]$ rel. $\mathcal{Z}$ is the closure of a quasiperiodic line in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. Note that the homeomorphism $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$ permutes the stable laminations for $[\psi]$ rel. $\mathcal{Z}$ and, by Lemma II.1, each stable $\mathcal{T}$-lamination for $[\tau]$ is identified with some stable lamination for $[\psi]$ rel. $\mathcal{Z}$. Let $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ be the union of stable laminations for $[\psi]$ rel. $\mathcal{Z}$.

Lemma II.5 (cf. [2, Lemma 1.12]). *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, and $\mathcal{Z} := \mathcal{F}[\mathcal{T}]$. The stable laminations $\mathcal{L}^+[\tau]$ for $[\tau]$ are identified with the stable laminations $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ for $[\psi]$ rel. $\mathcal{Z}$.*

So $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ is a lamination system whose finitely many components are the stable laminations for $[\psi]$ rel. $\mathcal{Z}$, and these are transitively permuted by $\psi_*\colon \mathbb{R}(\mathcal{F}, \mathcal{Z}) \to \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

Sketch of proof. Suppose a quasiperiodic line $[l]$ in $\mathcal{T}$ has a τ_*^k-attracting neighbourhood U for some $k \geq 1$. This forces any $\mathcal{T}$-loxodromic conjugacy class $[x]$ with axis in U to have a translation distance that (eventually) grows under forward $[\psi^k]$-iteration. In particular, the conjugacy class $[x]$ is $\mathcal{Y}_\tau$-loxodromic, and the line $[l]$, a weak ψ_*^k-limit of the $\mathcal{T}$-axis for $[x]$, is a leaf in $\mathcal{L}^+[\tau]$ by Proposition II.2. □

The stable laminations $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ are in the subspace $\mathbb{R}(\mathcal{Y}_\tau, d_\infty) \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

II.4 Constructing limit forests (2)

This chapter has thus far focused on automorphims with expanding irreducible train tracks. For the rest of the chapter, we extend our focus to all automorphisms.

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$, set $\mathcal{F}_1 \coloneqq \mathcal{F}$, $\psi_1 \coloneqq \psi$, and let $\mathcal{Z}$ be a $[\psi_1]$-invariant proper free factor system. By Theorem I.1, there is an irreducible train track $\tau_1\colon \mathcal{T}_1 \to \mathcal{T}_1$ for ψ_1. By ψ_1-equivariance of τ_1, the nontrivial vertex stabilizers of $\mathcal{T}_1$ determine a $[\psi_1]$-invariant proper free factor system $\mathcal{F}_2 \coloneqq \mathcal{F}[\mathcal{T}_1]$. The restriction of ψ_1 to $\mathcal{F}_2$ determines a unique outer class of automorphisms $\psi_2\colon \mathcal{F}_2 \to \mathcal{F}_2$. We can repeatedly apply Theorem I.1 to ψ_{i+1} $(i \geq 1)$ as long as $\lambda[\tau_i] = 1$ and $\mathcal{F}_{i+1} \coloneqq \mathcal{F}[\mathcal{T}_i]$ contains a noncyclic component. Bass-Serre theory implies this process must stop; we end up with a maximal sequence $(\tau_i)_{i=1}^n$ of irreducible train tracks with $\lambda[\tau_i] = 1$ for $1 \leq i < n$ — such a maximal sequence is called a descending sequence of irreducible train tracks for $[\psi]$ rel. $\mathcal{Z}$.

This leads to our working definition of growth type: $[\psi]$ is polynomially growing rel. $\mathcal{Z}$ if and only if $\lambda[\tau_n] = 1$ [22, Proposition III.1]. For automorphisms that are polynomially growing rel. $\mathcal{Z}$, define the limit forest to be degenerate.

Suppose $[\psi]$ is exponentially growing rel. $\mathcal{Z}$ and $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ is a descending sequence of irreducible train tracks for $[\psi]$ rel. $\mathcal{Z}$. Sections II.1–II.2 already cover the case $n = 1$, so we may assume $n > 1$ for the rest of the chapter. Set $\lambda \coloneqq \lambda[\tau_n] > 1$, $\mathcal{T}_n^\circ \coloneqq \mathcal{T}_n$, $\tau_n^\circ \coloneqq \tau_n$, and d_n° the eigenmetric on $\mathcal{T}_n^\circ$ for τ_n°. For $1 \leq i < n$, we inductively form an equivariant *simplicial blow-up* $\mathcal{T}_i^\circ$ of $\mathcal{T}_i$ rel. $\mathcal{T}_{i+1}^\circ$: the vertices with nontrivial stabilizers are equivariantly replaced by copies of corresponding components of $\mathcal{T}_{i+1}^\circ$ and arbitrary vertices in $\mathcal{T}_{i+1}^\circ$ are chosen as attaching points for the edges of $\mathcal{T}_i$. Let $\tau_i^\circ\colon \mathcal{T}_i^\circ \to \mathcal{T}_i^\circ$ be the topological representative for ψ_i induced by τ_i and τ_{i+1}°. As τ_i is a simplicial automorphism, we can make τ_i° a λ-Lipschitz map by assigning the same large enough length to the edges of $\mathcal{T}_i$ in the blow-up $\mathcal{T}_i^\circ$ when extending the metric d_{i+1}° on $\mathcal{T}_{i+1}^\circ$ to a metric d_i° on $\mathcal{T}_i^\circ$. The topological representative $\tau^\circ \coloneqq \tau_1^\circ$ on $\mathcal{T}^\circ \coloneqq \mathcal{T}_1^\circ$ is an equivariant blow-up of the descending sequence $(\tau_i)_{i=1}^n$. Set $d^\circ \coloneqq d_1^\circ$ and identify $(\mathcal{T}_i^\circ, d_i^\circ)$ with the characteristic subforest of $(\mathcal{T}^\circ, d^\circ)$ for $\mathcal{F}_i$. We will abuse terminology and refer to d° as the eigenmetric as well. Translates of edges in $\mathcal{T}^\circ$ coming from $\mathcal{T}_i$ form the i^{th} stratum of $\mathcal{T}^\circ$: the n^{th} stratum is exponential while the rest are (relatively) polynomial.

As in Section II.1, the maps $\tau^{\circ m}\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{T}^\circ\psi^m, \lambda^{-m}d^\circ)$ converge (as $m \to \infty$) to an equivariant metric surjection $\pi^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$. The map τ° induces a ψ-equivariant λ-homothety $h\colon (\mathcal{Y}, \delta) \to (\mathcal{Y}, \delta)$ and π° semiconjugates τ° to h. By restricting to $\mathcal{T}_i^\circ$, we have also constructed an equivariant metric surjection $\pi_i^\circ\colon (\mathcal{T}_i^\circ, d^\circ) \to (\mathcal{Y}_i, \delta)$ and ψ_i-equivariant λ-homothety h_i on $(\mathcal{Y}_i, \delta)$ for $2 \leq i \leq n$.

The $\mathcal{F}_n$-forest $(\mathcal{Y}_n, \delta)$ is the limit forest for $[\tau_n^\circ]$; so it is a nondegenerate minimal $\mathcal{F}_n$-forest with trivial arc stabilizers. For induction, assume $(\mathcal{Y}_i, \delta)$ is a nondegenerate minimal $\mathcal{F}_i$-forest with trivial arc stabilizers for $2 \leq i \leq n$. Equivariantly collapse $\mathcal{T}_2^\circ$ in $(\mathcal{T}^\circ, d^\circ)$ to get the $\mathcal{F}$-forest $(\mathcal{T}_1, d_1)$. For $m \geq 0$, the metric free splitting $(\mathcal{T}^\circ\psi^m, \lambda^{-m}d^\circ)$ is an equivariant metric blow-up of $(\mathcal{T}_1\psi^m, \lambda^{-m}d_1)$ rel. $(\mathcal{T}_2^\circ\psi_2^m, \lambda^{-m}d^\circ)$. Since $\tau_1\colon (\mathcal{T}_1, d_1) \to (\mathcal{T}_1\psi, d_1)$ is an equivariant isometry, the limit $(\mathcal{Y}, \delta)$ is equivariantly isometric to an equivariant metric blow-up of $(\mathcal{T}_1, d_1)$ rel. $(\mathcal{Y}_2, \delta)$ whose top stratum (edges coming from $\mathcal{T}_1$) have then been equivariantly collapsed, also known as a *graph of actions (with degenerate skeleton)* — more details are given in the next subsection. Thus $(\mathcal{Y}, \delta)$ is a nondegenerate minimal $\mathcal{F}$-forest with trivial arc stabilizers. See [22, Theorem IV.1] for a direct construction of $(\mathcal{Y}, \delta)$ as a graph of actions. This sketches the general case of Proposition I.2. The $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is the <u>limit forest</u> for $[\tau_i]_{i=1}^n$.

II.4.1 Decomposing limit forests

We now give a hierarchical decomposition of the limit forest $(\mathcal{Y}, \delta)$ and its space of lines.

Choose an iterate $[\tau_1^{k'}]$ that fixes all $\mathcal{F}$-orbits of branches in $\mathcal{T}_1$. Pick an edge e in $\mathcal{T}_1$ and one of its endpoints p. Replace $\psi^{k'}$ with an automorphism in its outer class $[\psi^{k'}]$ if necessary and assume $\tau_1^{k'}$ fixes p and e. Identify $(\mathcal{T}^\circ\psi^{mk'}, \lambda^{-mk'}d^\circ)$ with an equivariant metric blow-up of $(\mathcal{T}_1, \lambda^{-mk'}d_1)$ rel. $(\mathcal{T}_2^\circ\psi_2^{mk'}, \lambda^{-mk'}d^\circ)$ for $m \geq 0$, then let $p_m \in \mathcal{T}_2^\circ\psi_2^{mk'}$ be the attaching point of e to $\mathcal{T}_2^\circ\psi_2^{mk'}$ corresponding to the endpoint p. Since $\tau_1^{k'}$ fixes e and p, we get $p_m = p_0$ for $m \geq 1$. As in the first part of the proof for Proposition II.4, the sequence $(p_m)_{m\geq 0}$ converges to the unique fixed point $\star$ of $h_2^{k'}$ in the metric completion $(\overline{\mathcal{Y}}_2, \delta)$. So, in the description of $(\mathcal{Y}, \delta)$ as a graph of actions, the edge e is collapsed and identified with $\star$. Thus the closure $\widehat{\mathcal{Y}}_2$ of $\mathcal{Y}_2$ in $(\mathcal{Y}, \delta)$ is the union of $\mathcal{Y}_2$ with the $\mathcal{F}_2$-orbits of attaching points $\star$ as the pair (e, p) ranges over the $\mathcal{F}$-orbit representatives e of edges and their endpoints p. For the same reasons, we inductively get a similar description of the closure $\widehat{\mathcal{Y}}_{i+1}$ of $\mathcal{Y}_{i+1}$ in $(\mathcal{Y}_i, \delta)$ for $2 \leq i < n$.

Remark. Constructing $(\mathcal{Y}, \delta)$ directly by iterating τ° allows us to lift metric properties of $(\mathcal{Y}, \delta)$ to dynamical properties of τ° through the semiconjugacy $\pi^\circ \circ \tau^\circ = h \circ \pi^\circ$; this viewpoint is used in the Section II.5. On the other hand, constructing $(\mathcal{Y}, \delta)$ directly as we did in [22, Theorem IV.1] (and sketched in this subsection) gives us a nice structural description of intervals in the limit forest. This is explained in the next subsection and will be a key component of Chapter III!

For $1 < i \leq n$, any two translates of $\mathcal{T}_i^\circ \subset \mathcal{T}^\circ$ by elements of $\mathcal{F}$ either coincide or are disjoint by construction. This induces a canonical closed embedding of $\mathbb{R}(\mathcal{F}_i, \mathcal{Z})$ into $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ (exercise). Similarly, any two intersecting translates of $\mathcal{Y}_i \subset \mathcal{Y}$ by elements of $\mathcal{F}$ either coincide or have degenerate intersection. This also induces a canonical closed embedding $\mathbb{R}(\mathcal{Y}_i, \delta) \subset \mathbb{R}(\mathcal{Y}, \delta)$. Finally, the constructed equivariant metric map π° induces a canonical embedding of the topological pair $(\mathbb{R}(\mathcal{Y}, \delta), \mathbb{R}(\mathcal{Y}_i, \delta))$ into $(\mathbb{R}(\mathcal{F}, \mathcal{Z}), \mathbb{R}(\mathcal{F}_i, \mathcal{Z}))$.

II.4.2 Intervals in limit forests

Here is an inductive description of intervals in the limit forest $(\mathcal{Y}, \delta)$ in terms of the limit forest for $[\tau_n^\circ]$. For $1 \leq i \leq n$, the characteristic subforest $(\mathcal{Y}_i, \delta)$ of $(\mathcal{Y}, \delta)$ for $\mathcal{F}_i$ is the limit forest for $[\tau_j]_{j=i}^n$. For $1 < i \leq n$, let $\widehat{\mathcal{Y}}_i$ be the closure of $\mathcal{Y}_i$ in $(\mathcal{Y}_{i-1}, \delta)$.

It follows from the blow-up (and collapse) description of $\mathcal{Y}_{i-1}$ that its closed intervals are finite concatenations of closed intervals in translates of $\widehat{\mathcal{Y}}_i$. As shown in the previous subsection, the $\mathcal{F}_i$-orbits $[p]$ of points in $\widehat{\mathcal{Y}}_i \setminus \mathcal{Y}_i$ are fixed by the extension of $h_i^{k'}$ to $\widehat{\mathcal{Y}}_i$ for some $k' \geq 1$. As $p \notin \mathcal{Y}_i$, it has exactly one direction d_p in $\widehat{\mathcal{Y}}_i$. This direction's $\mathcal{F}_i$-orbit $[d_p]$ is also fixed (setwise) by the expanding homothety $h_i^{k'}$, and d_p determines a singular eigenray $\rho_p \subset \widehat{\mathcal{Y}}_i$ of $[h_i^{k'}]$ based at p. For any point $q \in \mathcal{Y}_i$, the closed interval $[p, q] \subset \widehat{\mathcal{Y}}_i$ is a concatenation of an initial segment of the singular eigenray ρ_p and a closed interval in $\mathcal{Y}_i$; therefore, closed intervals in $\mathcal{Y}_{i-1}$ are finite concatenations of translates of closed intervals in $\mathcal{Y}_i$ and initial segments of singular eigenrays of $[h_i^{k'}]$ for some $k' \geq 1$.

Let $\mathcal{L}^+_{\mathcal{Z}}[\psi_n] = \mathcal{L}^+[\tau_n]$ be the k-component stable laminations for $[\tau_n^\circ] = [\tau_n]$ and $\oplus_{j=1}^k \delta_j$ the factored $\mathcal{F}_n$-invariant convex metric on $\mathcal{Y}_n$ indexed by components $\Lambda_j^+ \subset \mathcal{L}^+_{\mathcal{Z}}[\psi_n]$. By the inductive description of intervals in $\mathcal{Y}$ and the fact h_n^k is a λ^k-homothety with respect to each factor δ_j, we get: δ_j equivariantly extends to $\mathcal{Y}$; $\delta = \oplus_{j=1}^k \delta_j$ is a factored $\mathcal{F}$-invariant convex metric on $\mathcal{Y}$; and h^k is a λ^k-homothety with respect to each factor δ_j.

The lamination $\mathcal{L}^+_{\mathcal{Z}}[\psi_n] \subset \mathbb{R}(\mathcal{Y}_n, \delta)$ can be seen as a $(\mathcal{Y}, \delta)$-lamination since $\mathbb{R}(\mathcal{Y}_n, \delta)$ is a closed subspace of $\mathbb{R}(\mathcal{Y}, \delta)$. Note that closed edges of $\mathcal{T}_n = \mathcal{T}_n^\circ$ are leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$); thus any closed interval in $\mathcal{T}_n^\circ$ is a finite concatenation of leaf segments. As the equivariant PL-map $\pi_n^\circ \colon (\mathcal{T}_n^\circ, d^\circ) \to (\mathcal{Y}_n, \delta)$ is surjective and isometric on leaf segments, we get:

Lemma II.6. *Let $\tau_n \colon \mathcal{T}_n \to \mathcal{T}_n$ be an expanding irreducible train track and $(\mathcal{Y}_n, \delta)$ its limit forest. Any closed interval in $\mathcal{Y}_n$ is a finite concatenation of leaf segments of $\mathcal{L}^+[\tau_n]$.* $\square$

This lemma no longer holds when $n \geq 2$ and we consider closed intervals in $\widehat{\mathcal{Y}}_n$. To account for this, let n^{th} level leaf blocks in $\mathcal{Y}$ be leaf segments. By the lemma, any interval of $\mathcal{Y}_n$ is a finite concatenation of n^{th} level leaf blocks.

Inductively define the $(i-1)^{st}$ level leaf blocks in $\mathcal{Y}$ $(1 < i \leq n)$ to be the i^{th} level leaf blocks or (translates of) closed intervals in singular eigenrays $\rho \subset \widehat{\mathcal{Y}}_i$ of $[h_i]$-iterates. By the earlier description of intervals and induction hypothesis, any interval of $\mathcal{Y}_{i-1}$ is a finite concatenation of $(i-1)^{st}$ level leaf blocks. The 1^{st} level leaf blocks are simply leaf blocks of $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$. Altogether, we have a generalization of Lemma II.6 in terms of leaf blocks:

Lemma II.7. *Let $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks for an automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ and $(\mathcal{Y}, \delta)$ the corresponding limit forest. Any closed interval in $\mathcal{Y}$ is a finite concatenation of leaf blocks of $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$, where $\mathcal{Z} := \mathcal{F}[\mathcal{T}_n]$.* $\square$

II.5 Stable laminations (2)

Fix an automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ with an invariant proper free factor system $\mathcal{Z}'$. Let $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks rel. $\mathcal{Z}'$ with $\lambda[\tau_n] > 1$, $(\mathcal{Y}, \delta)$ be the limit forest for $[\tau_i]_{i=1}^n$, $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$ with eigenmetric d°, and $\mathcal{Z} := \mathcal{F}[\mathcal{T}^\circ]$. The characteristic convex subsets of $\mathcal{T}^\circ$ for $\mathcal{F}_n := \mathcal{F}[\mathcal{T}_{n-1}]$ are identified with the free splitting $\mathcal{T}_n$.

Claim II.8. *The stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi_n]$ for $[\psi_n]$ in $\mathbb{R}(\mathcal{F}_n, \mathcal{Z})$ are identified with the stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$.*

Note that $\mathcal{L}^+_{\mathcal{Z}}[\psi] = \mathcal{L}^+_{\mathcal{Z}}[\psi_n]$ is in the subspace $\mathbb{R}(\mathcal{Y}_n, \delta) \subset \mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$.

Sketch of proof. Since $\lambda[\tau_i] = 1$ for $i < n$, no quasiperiodic line in $\mathbb{R}(\mathcal{F}, \mathcal{F}_n)$ has a ψ^k_*-attracting neighbourhood for any $k \geq 1$. Thus any stable lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ is contained in $\mathbb{R}(\mathcal{F}_n, \mathcal{Z})$ and corresponds to a stable lamination for $[\psi_n]$. □

We generalize Proposition II.4 by characterizing limits of iterated turns over $\mathcal{T}^\circ$:

Theorem II.9. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism with an invariant proper free factor system $\mathcal{Z}'$, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks rel. $\mathcal{Z}'$ with $\lambda[\tau_n] > 1$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, and $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$ with eigenmetric d°. Choose a nondegenerate component $T^\circ \subset \mathcal{T}^\circ$, corresponding components $F \subset \mathcal{F}$, $Y \subset \mathcal{Y}$, and a positive iterate ψ^k that preserves F. Let $\tilde{h}\colon (Y, \delta) \to (Y, \delta)$ be the φ-equivariant λ-homothety, where φ is in the outer class $[\psi^k|_F]$ and $\lambda := (\lambda[\tau_n])^k$. Finally, for $\iota = 1, 2$, pick $p_\iota \in T^\circ$ and $x_\iota \in F$.*

The point $p_{\iota,m} := \varphi^{-1}(x_\iota) \cdots \varphi^{-m}(x_\iota) \cdot p_\iota$ in $(T^\circ\varphi^m, \lambda^{-m} d^\circ)$ converges to $\star_\iota$ in $(\overline{Y}, \delta)$ as $m \to \infty$, where $\star_\iota$ is the unique fixed point of $x_\iota^{-1} \cdot \tilde{h}$ in the metric completion $(\overline{Y}, \delta)$.

If $x_1^{-1} x_2$ fixes $\star_1$, then $\star_1 = \star_2$ and the term $[p_{1,m}, p_{2,m}]$ $(m \geq 0)$ of the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$ has d°-length $\leq \alpha(m)$ for some (degree n) polynomial α. Otherwise, $\star_1 \neq \star_2$ and the iterated turn weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$, where $\mathcal{Z} := \mathcal{F}[\mathcal{T}^\circ]$.

An iterated turn $[p_{1,m}, p_{2,m}]_{m \geq 0}$ weakly limits to a component $\Lambda^+ \subset \mathcal{L}^+_{\mathcal{Z}}[\psi]$ if the term $[p_{1,m}, p_{2,m}]$ contains a leaf segment of Λ^+ with arbitrarily large d°-length as $m \to \infty$.

Sketch of proof. Let $\tilde{\tau}^\circ\colon (T^\circ, d^\circ) \to (T^\circ, d^\circ)$ be the φ-equivariant λ-Lipschitz topological representative induced by the irreducible train tracks $(\tau_i)_{i=1}^n$ and $\pi^\circ\colon (T^\circ, d^\circ) \to (Y, \delta)$ the equivariant metric map constructed using $\tilde{\tau}^\circ$-iteration. Even though π° may fail to be a PL-map, it still has a cancellation constant $C[\pi^\circ] \geq 0$ as a limit of equivariant metric maps with uniformly bounded cancellation constants. The proof of the first part is the same as in Proposition II.4 using π°, $\tilde{\tau}^\circ$, and the φ-equivariant λ-homothety $\tilde{h}$.

The interval $[p_{1,m}, p_{2,m}] \subset T^\circ$, a term in the sequence $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$, is covered by certain $2m+1$ intervals as in the proof of Proposition II.4. Since $\tilde{\tau}^\circ$ is induced by a descending sequence $(\tau_i)_{i=1}^n$ of irreducible train tracks, the intervals $[\tilde{\tau}^{\circ(l-1)}(x_1 \cdot p_1), \tilde{\tau}^{\circ l}(p_1)]$, $[\tau^{\circ l}(p_1), \tilde{\tau}^{\circ l}(p_2)]$, and $[\tilde{\tau}^{\circ l}(p_2), \tilde{\tau}^{\circ(l-1)}(x_2 \cdot p_2)]$ are covered by $\alpha(l)$ polynomial strata edges and leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$) for some degree $(n-1)$ polynomial α. So the interval $[p_{1,m}, p_{2,m}]$ is covered by $\alpha(m) + \sum_{l=1}^m 2\alpha(l)$ polynomial strata edges and leaf segments (of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$). Note that $\alpha(m) + \sum_{l=1}^m 2\alpha(l) \leq \beta(m)$ for some degree n polynomial β.

Assume $\star_1 = \star_2$, where $\star_\iota$ is the unique fixed point of $\tilde{h}_\iota \coloneqq x_\iota^{-1} \cdot \tilde{h}$ in metric completion $(\overline{Y}, \delta)$ for $\iota = 1, 2$. The proof given in Proposition II.4 implies there is a uniform bound on the d°-length of leaf segments in $[p_{1,m}, p_{2,m}]$. Consequently, the d°-length of $[p_{1,m}, p_{2,m}]$ is $\leq \beta(m)B$ for some constant $B \geq 1$.

Assume $\star_1 \neq \star_2$. Set $L \coloneqq \frac{1}{2}\delta(\star_1, \star_2) > 0$; then $\delta(\tilde{h}_1^{-m}(\pi^\circ(p_1)), \tilde{h}_2^{-m}(\pi^\circ(p_2))) > L$ and $d^\circ(p_{1,m}, p_{2,m}) > \lambda^m L$ for $m \gg 1$. The contribution of polynomial strata to the d°-length of $[p_{1,m}, p_{2,m}]$ is at most $\beta(m)B'$ for some constant $B' \geq 1$; the exponential stratum edges intersecting the interval are covered by $\beta(m)$ leaf segments. By the pigeonhole principle, the interval $[p_{1,m}, p_{2,m}]$, a term in the iterated turn $(p_1, p_2 : x_1, x_2; \varphi)_{T^\circ}$, has a leaf segment of d°-length $\geq \frac{\lambda^m L - \beta(m)B'}{\beta(m)} \gg 1$. Quasiperiodicity implies the iterated turn weakly limits to a component of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$. $\square$

Remark. The argument given in Subsection II.2.4 applies in this general context involving a descending sequence of irreducible train tracks; it describes how an iterated turn over $\mathcal{F}$ determines (nested) iterated turns over $\mathcal{G}[\mathcal{Y}]$.

As in Proposition II.2, we can characterize the elements in $\mathcal{F}$ that are $\mathcal{Y}$-loxodromic:

Theorem II.10. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism with an invariant proper free factor system $\mathcal{Z}'$, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks rel. $\mathcal{Z}'$ with $\lambda \coloneqq \lambda[\tau_n] > 1$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $\mathcal{T}^\circ$ an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$, and $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}^\circ]$.*

If $x \in \mathcal{F}$ is a $\mathcal{T}^\circ$-loxodromic element, then the following statements are equivalent:

1. *the element x is $\mathcal{Y}$-loxodromic;*
2. *the element x $[\psi]$-grows exponentially rel. $\mathcal{Z}$ with rate λ; and*
3. *the axis for the conjugacy class $[x]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$ weakly ψ_*-limits to $\mathcal{L}^+_{\mathcal{Z}}[\psi]$.*

The restriction of ψ to the $[\psi]$-invariant subgroup system $\mathcal{G}[\mathcal{Y}]$ of $\mathcal{Y}$-point stabilizers is polynomially growing rel. $\mathcal{Z}$ with degree $< n$.

Sketch of proof. Set $\lambda \coloneqq \lambda[\tau_n]$, $\mathcal{F}_1 \coloneqq \mathcal{F}$, and $\mathcal{F}_{i+1} \coloneqq \mathcal{F}[\mathcal{T}_i]$ for $1 \leq i < n$. Under the canonical embedding $\mathbb{R}(\mathcal{F}_i, \mathcal{Z}) \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$, we identify the stable laminations $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and $\mathcal{L}^+_{\mathcal{Z}}[\psi_i]$. Let $\mathcal{T}^\circ$ be an equivariant blow-up of free splittings $(\mathcal{T}_i)_{i=1}^n$ and $\mathcal{T}_i^\circ \subset \mathcal{T}^\circ$ the characteristic convex subsets for $\mathcal{F}_i$. Suppose $x \in \mathcal{F}_1$ is a $\mathcal{T}^\circ$-loxodromic element. The equivalence between Conditions 1–3 is given by Proposition II.2 if x is conjugate to an element of $\mathcal{F}_n$. Assume $n \geq 2$ and, up to conjugacy, $x \in \mathcal{F}_i$ is $\mathcal{T}_i$-loxodromic for some $i < n$.

Recall that $\tau^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{T}^\circ, d^\circ)$ is a ψ-equivariant λ-Lipschitz topological representative induced by the irreducible train tracks $(\tau_i)_{i=1}^n$ and $\pi^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$ is the constructed equivariant metric map. In particular, $\limsup_{m \to \infty} \frac{1}{m} \log \|\psi^m(x)\|_{d^\circ} \leq \log \lambda$.

Suppose $[\tau_i^{k'}]$ (for some $k' \geq 1$) fixes all $\mathcal{F}_i$-orbits of vertices and edges in $\mathcal{T}_i$. Let $l^\circ \subset \mathcal{T}_i^\circ$ be the axis for $x \in \mathcal{F}_i$. The axis l° projects to the axis l of x in $\mathcal{T}_i$; write l

as a biinfinite concatenation of edges $\cdots e_{-1} \cdot e_0 \cdot e_1 \cdots$ and identify $e_j \subset \mathcal{T}_i$ with its lift to $\mathcal{T}_i^\circ$. For $m \geq 0$ and any integer j, let $w_{j,m}$ be the closed interval in $\mathcal{T}_i^\circ$ between (lifts of) $\tau_i^m(e_j)$ and $\tau_i^m(e_{j+1})$; in fact, $w_{j,m}$ is in a component of $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ \subset \mathcal{T}_i^\circ$. Since $[\tau_i^{k'}]$ fixes the $\mathcal{F}_i$-orbits $[e], [e']$ and the vertex of $\mathcal{T}_i$ between them, the sequence $(w_{j,mk'+r})_{m\geq 0}$, up to translation, is an iterated turn over $\mathcal{T}_{i+1}^\circ$ rel. $\psi_{i+1}^{k'}$ for $0 \leq r < k'$; by Theorem II.9, the iterated turn limits to an interval $w_{j,r}^*$ in a translate of a component of $\widehat{\mathcal{Y}}_{i+1} \subset \mathcal{Y}_i$.

The intervals $w_{j,m}, w_{j+1,m}$ are always in distinct components of $\mathcal{F}_i \cdot \mathcal{T}_{i+1}^\circ$; therefore, the limit intervals $w_{j,r}^*, w_{j+1,r}^*$ have degenerate intersection. By the equivariance of the limits, the union $l_* \coloneqq \bigcup_j w_{j,0}^*$ is an x-invariant arc. If some limit interval $w_{j,0}^*$ is not degenerate, then x is $\mathcal{Y}_i$-loxodromic and l_* is its $\mathcal{Y}_i$-axis; otherwise, l_* is degenerate and x is $\mathcal{Y}_i$-elliptic.

Case 1: x is $\mathcal{Y}_i$-loxodromic, i.e. some limit interval $w_{j,0}^*$ is not degenerate. By Theorem II.9, the iterated turn $(w_{j,mk'})_{m\geq 0}$ over $\mathcal{T}_{i+1}^\circ$ rel. $\psi_{i+1}^{k'}$ weakly limits to a component of $\mathcal{L}_{\mathcal{Z}}^{+}[\psi_{i+1}]$. So $[l^\circ] \in \mathbb{R}(\mathcal{F}, \mathcal{Z})$ weakly $\psi_*^{k'}$-limits to a component of $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$. Finally, $[l^\circ]$ weakly ψ_*-limits to $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ since ψ_* acts transitively on the components of $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$. As π° is an equivariant metric map, $\|\cdot\|_\delta \leq \|\cdot\|_{d^\circ}$ and $\log\lambda \leq \liminf_{m\to\infty} \frac{1}{m}\log\|\psi^m(x)\|_{d^\circ}$.

Case 2: x is $\mathcal{Y}_i$-elliptic, i.e. each limit interval $w_{j,0}^*$ is degenerate. By Theorem II.9, the interval $w_{j,mk}$ has d°-length is bounded above by some degree $(n-i)$ polynomial (in m). Thus $\|\psi^{mk}(x)\|_{d^\circ}$ is bounded above by a degree $(n-i)$ polynomial. By ψ-equivariance of the homothety h_i, the elements $\psi(x), \ldots, \psi^{k-1}(x)$ are $\mathcal{Y}_i$-elliptic as well. The same argument implies $\|\psi^m(x)\|_{d^\circ}$ is bounded above by a degree $(n-i)$ polynomial. □

We conclude the chapter by stating the extension of Lemma IV.3 to all limit forests:

Lemma IV.5. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\mathcal{Z}'$ a $[\psi]$-invariant proper free factor system, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$ rel. $\mathcal{Z}'$ with $\lambda \coloneqq \lambda[\tau_n] > 1$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $(\mathcal{Y}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers, and $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$.*

If $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic and the k-component lamination $\mathcal{L}_{\mathcal{Z}}^{+}[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$, then the limit of $(\mathcal{Y}'\psi^{mk}, \lambda^{-mk}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \oplus_{j=1}^k c_j\,\delta_j)$, where $\delta = \oplus_{j=1}^k \delta_j$ and $c_j > 0$.

Again, we postpone the proof to Section IV.2. If $(\tau_i')_{i=1}^{n'}$ is another descending sequence for $[\psi]$ with $\mathcal{F}[\mathcal{T}_{n'}'] = \mathcal{Z}$, then its limit forest $(\mathcal{Y}', \delta')$ is equivariantly homothetic to $(\mathcal{Y}, \delta)$; therefore, $(\mathcal{Y}, \delta)$ is *the* limit forest for $[\psi]$ rel. $\mathcal{Z}$ (up to rescaling of δ). A nondegenerate minimal very small $\mathcal{F}$-forest $(\mathcal{Y}', \delta')$ is an expanding forest for $[\psi]$ rel. $\mathcal{Z}$ if:

1. the $\mathcal{F}$-forest $(\mathcal{Y}'\psi, \delta')$ is equivariantly isometric to $(\mathcal{Y}', s\delta')$ for some $s > 1$; and
2. the free factor system $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic.

Corollary II.11. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism and $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$ with $\lambda[\tau_n] > 1$. Any expanding forests for $[\psi]$ rel. $\mathcal{F}[\mathcal{T}_n]$ is the limit forest for $[\psi]$ rel. $\mathcal{F}[\mathcal{T}_n]$.*

We will end the paper with a complete generalization of this corollary (Theorem V.3).

Sketch of proof. Let $(\mathcal{Y}', \delta')$ be an expanding forest for $[\psi]$ rel. $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$ and $x \in \mathcal{F}$ a $\mathcal{Y}'$-loxodromic element. The proof is essentially the proof of Corollary II.3 with two main changes. First, choose $m \gg 1$ so that $\|\psi^m(x)\|_{\delta'} > \alpha(m)(2C[f]+B')$ for some polynomial α and constant $B' \geq 1$ determined by x; therefore, a fundamental domain of $\psi^m(x)$ acting on its axis has a leaf segment $[q, r]$ with $\delta'(f(q), f(r)) > 2C[f]$ by the pigeonhole principle. For the second change, we need $(\mathcal{Y}', \delta')$ to have trivial arc stabilizers in order to conclude the proof by invoking Lemma IV.5 instead of Lemma IV.3.

The minimal very small $\mathcal{F}$-forest $(\mathcal{Y}', \delta')$ has finitely many orbits of branch points [11]; it decomposes as some graph of actions whose skeleton is not degenerate in the forest if and only if the forest does not have dense orbits [18]. Any ψ-equivariant homothety must be an isometry if the skeleton were not degenerate. Since $(\mathcal{Y}', \delta')$ admits a ψ-equivariant expanding s-homothety, the skeleton must be degenerate and the forest has dense orbit. Very small $\mathcal{F}$-forests with dense orbits have trivial arc stabilizers [19, Lemma 4.2]. □

For a nondegenerate minimal $\mathcal{F}$-forest $(\mathcal{Y}', \delta')$, the projective stabilizer $\mathrm{Stab}[\mathcal{Y}', \delta']$ is the subgroup of automorphisms $\varphi \colon \mathcal{F} \to \mathcal{F}$ for which $\|\varphi(\cdot)\|_{\delta'} = s_\varphi \| \cdot \|_{\delta'}$ for some $s_\varphi > 0$. The function $\mathrm{SF} \colon \mathrm{Stab}[\mathcal{Y}', \delta'] \to \mathbb{R}_{>0}$ that maps $\varphi \mapsto s_\varphi$ is a homomorphism called the *stretch factor homomorphism* — $\mathbb{R}_{>0}$ is considered multiplicatively.

Corollary II.12. *Let* $\mathrm{SF} \colon \mathrm{Stab}[\mathcal{Y}', \delta'] \to \mathbb{R}_{>0}$ *be the stretch factor homomorphism for some nondegenerate minimal very small* $\mathcal{F}$*-forest* $(\mathcal{Y}', \delta')$*. The image of* SF *is cyclic.*

Proof. Suppose $\mathrm{SF}(\psi) > 1$ for some $\psi \in \mathrm{Stab}[\mathcal{Y}', \delta']$. Then ψ is exponentially growing since any $\mathcal{Y}'$-loxodromic element $[\psi]$-grows exponentialy with rate at least $\mathrm{SF}(\psi)$. Set $\mathcal{F}_1 \coloneqq \mathcal{F}$, $\psi_1 \coloneqq \psi$, and let $(\mathcal{Y}_1, \delta_1)$ be the limit forest for $[\psi_1]$ rel. some $[\psi_1]$-invariant proper free factor system $\mathcal{F}_2$. If $\mathcal{F}_2$ is not $\mathcal{Y}'$-elliptic, then the restrictions ψ_2 of ψ_1 to $\mathcal{F}_2$ are in the projective stabilizer of the nondegenerate characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{F}_2$ and have the same stretch factor $\mathrm{SF}(\psi)$.

By repeatedly considering limit forests and taking restrictions, we may assume some free factor system $\mathcal{F}_n$ is not $\mathcal{Y}'$-elliptic while a nested proper free factor system $\mathcal{F}_{n+1}$ is for some $n \geq 1$. Then the characteristic subforest of $(\mathcal{Y}', \delta')$ for the free factor system $\mathcal{F}_n$ is an expanding forest for $[\psi_n]$ rel. $\mathcal{F}_{n+1}$. By Corollary II.11, this subforest is equivariantly homothetic to the limit forest $(\mathcal{Y}_n, \delta_n)$ for $[\psi_n]$ rel. $\mathcal{F}_{n+1}$. In particular, $\mathrm{SF}(\psi)$ is the exponential growth rate for $[\psi_n]$ rel. $\mathcal{F}_{n+1}$ and is bounded away from 1 by a uniform constant that depends only on $\mathcal{F}$. Thus the image of SF is discrete, and discrete subgroups of $\mathbb{R}_{>0}$ are cyclic. □

III Main constructions

The limit forest produced by our proof of Proposition I.2 is universal for an outer automorphism and some choice of an invariant proper free factor system (Corollary II.11). Our goal is to remove the latter dependence on an invariant proper free factor system.

III.1 Assembling limit hierarchies

This section first summarizes the main result of the paper's prequel [22]. The general strategy follows closely the construction of limit forests sketched in Section II.4.

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and set $\mathcal{G}_1 \coloneqq \mathcal{F}$, $\psi_1 \coloneqq \psi$. By our proof of Proposition I.2, there is a nondegenerate limit forest $(\mathcal{Y}_1, \delta_1)$ for $[\psi_1]$ rel. $\mathcal{Z}_1$ (some proper free factor system of $\mathcal{G}_1$) and a unique ψ_1-equivariant expanding λ_1-homothety $h_1\colon (\mathcal{Y}_1, \delta_1) \to (\mathcal{Y}_1, \delta_1)$. Thus $\mathcal{Y}_1$-loxodromic elements in $\mathcal{F}$ $[\psi]$-grow exponentially rel. $\mathcal{Z}_1$ with rate λ_1. By Gaboriau–Levitt index theory and ψ_1-equivariance of τ_1, the nontrivial point stabilizers of $\mathcal{Y}_1$ determine a $[\psi_1]$-invariant malnormal subgroup system $\mathcal{G}_2 \coloneqq \mathcal{G}[\mathcal{Y}_1]$ with strictly lower complexity than $\mathcal{G}_1$. The restriction of ψ_1 to $\mathcal{G}_2$ determines a unique outer class of automorphisms $\psi_2\colon \mathcal{G}_2 \to \mathcal{G}_2$.

We can repeatedly apply Proposition I.2 to ψ_{i+1} ($i \geq 1$) as long as ψ_{i+1} is exponentially growing. This inductive invokation of Proposition I.2 eventually stops since the complexity of $\mathcal{G}_i$ is a strictly decreasing (in i) positive integer. In the end, we have a maximal sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of nondegenerate limit forests for $[\psi_i]$ rel. $\mathcal{Z}_i$ each with a unique ψ_i-equivariant expanding λ_i-homothety h_i on $(\mathcal{Y}_i, d_i)$ — such a maximal sequence of limit forests is a descending sequence of limit forests for $[\psi]$. By construction, an element $x \in \mathcal{F}$ has a conjugate in $\mathcal{G}_{n+1}$ if and only if x $[\psi]$-grows polynomially!

In Section II.4, the blow-ups of free splittings $(\mathcal{T}_i)_{i=1}^n$ were arbitary and done inductively upwards (i.e. started with $i = n$). We then used a limiting argument to produce the final limit forest $(\mathcal{Y}, \delta)$. For this section, the blow-ups of limit forests $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ will not be arbitrary but will make use of the expanding homotheties $(h_i)_{i=1}^n$; moreover, it will be done inductively downwards (i.e. starts with $i = 1$) to produce an $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$.

Set $(\mathcal{X}^{(1)}, \delta_1) \coloneqq (\mathcal{Y}_1, \delta_1)$ and $g^{(1)} \coloneqq h_1$. For $1 < i \leq n$, we inductively construct the *equivariant pseudoforest blow-up* $(\mathcal{X}^{(i)}, (\delta_j)_{j=1}^i)$ of the $\mathcal{F}$-pseudoforest $(\mathcal{X}^{(i-1)}, (\delta_j)_{j=1}^{i-1})$ rel. the $\mathcal{G}_i$-forest $(\mathcal{Y}_i, \delta_i)$ and expanding homotheties $g^{(i-1)}$ and h_i. Here is a sketch:

Let $(\overline{\mathcal{Y}}_i, \delta_i)$ be the metric completion and $\bar{h}_i$ the extension to the metric completion. For $1 \leq j < i$, assume that $(\mathcal{Y}_j, \delta_j)$ is equivariantly isometric to the associated $\mathcal{G}_j$-forest for the $\mathcal{G}_j$-invariant convex pseudometric δ_j restricted to $\mathcal{X}^{(i-1)}(\mathcal{G}_j)$, the characteristic convex subsets of $\mathcal{X}^{(i-1)}$ for $\mathcal{G}_j$. Since the hierarchy $(\delta_j)_{j=1}^{i-1}$ has full support, $(\mathcal{X}^{(i-1)}(\mathcal{G}_{i-1}), \delta_{i-1})$ is equivariantly isometric to $(\mathcal{Y}_{i-1}, \delta_{i-1})$ and the nontrivial point stabilizers of $\mathcal{X}^{(i-1)}$ are conjugates in $\mathcal{F}$ of $\mathcal{G}_i$-components. The points of $\mathcal{X}^{(i-1)}$ with nontrivial stabilizers are replaced by corresponding copies of $\overline{\mathcal{Y}}_i$-components; this produces a unique set system $\widehat{\mathcal{X}}^{(i)}$ with an $\mathcal{F}$-action that is the *equivariant set blow-up* of $\mathcal{X}^{(i-1)}$ rel. $\overline{\mathcal{Y}}_i$: it comes with an equivariant injection $\iota_i\colon \overline{\mathcal{Y}}_i \to \widehat{\mathcal{X}}^{(i)}$ and an equivariant surjection $\kappa_i\colon \widehat{\mathcal{X}}^{(i)} \to \mathcal{X}^{(i-1)}$ that is a bijection on the complement $\widehat{\mathcal{X}}^{(i)} \setminus \mathcal{F}\cdot\iota_i(\overline{\mathcal{Y}}_i)$. Consequently, there is a unique ψ-equivariant induced permutation $g^{(i)}\colon \widehat{\mathcal{X}}^{(i)} \to \widehat{\mathcal{X}}^{(i)}$ induced by $g^{(i-1)}$ and $\bar{h}_i$ — κ_i semiconjugates $\hat{g}^{(i)}$ to $g^{(i-1)}$ while ι_i conjugates $\bar{h}_i$ to the restriction $g^{(i)}\big|_{\iota_i(\overline{\mathcal{Y}}_i)}$.

There are plenty of equivariant interval functions $[\cdot,\cdot]^{(i)}$ on $\widehat{\mathcal{X}}^{(i)}$ compatible with $\mathcal{X}^{(i-1)}$ and $\mathcal{Y}_i$ — *compatibility* means the injection ι_i and surjection κ_i map intervals to intervals. Some compatible $\mathcal{F}$-pretrees $(\widehat{\mathcal{X}}^{(i)}, [\cdot,\cdot]^{(i)})$ are real [22, Proposition IV.3] and they naturally inherit an $\mathcal{F}$-invariant hierarchy $(\hat{\delta}_j)_{j=1}^{i}$ with full support: $(\hat{\delta}_j)_{j=1}^{i-1}$ is the pullback $\kappa_i^*(\delta_j)_{j=1}^{i-1}$ and $\hat{\delta}_i$ is the pushforward $\iota_{i*}\delta_i$ extended equivariantly to the orbit $\mathcal{F}\cdot\iota_i(\overline{\mathcal{Y}}_i)$; moreover, for $1 \le j \le i$, $(\mathcal{Y}_j, \delta_j)$ is equivariantly isometric to the associated $\mathcal{G}_j$-forest for the $\mathcal{G}_j$-invariant convex pseudometric $\hat{\delta}_j$ restricted to $\widehat{\mathcal{X}}^{(i)}(\mathcal{G}_j)$.

Claim ([22, Theorem IV.4])**.** *Since $\bar{h}_i$ is expanding, the permutation $g^{(i)}$ is a pretree-automorphism for a unique real compatible $\mathcal{F}$-pretree $(\widehat{\mathcal{X}}^{(i)}, [\cdot,\cdot]_g^{(i)})$.* □

Remark. This is the main technical result of [22]. Its proof uses Gaboriau–Levitt's index inequality and the contraction mapping theorem.

We now fix the interval function $[\cdot,\cdot]_g^{(i)}$ but omit it for brevity. By construction, the $\mathcal{F}$-pseudoforest $(\widehat{\mathcal{X}}^{(i)}, (\hat{\delta}_j)_{j=1}^{i})$ has trivial arc stabilizers and $g^{(i)}$ is an expanding homothety with respect to $(\hat{\delta}_j)_{j=1}^{i}$. Finally, let $\mathcal{X}^{(i)} \subset \widehat{\mathcal{X}}^{(i)}$ be the characteristic convex subsets for $\mathcal{F}$ and $(\delta_j)_{j=1}^{i}$ the restriction of the hierarchy $(\hat{\delta}_j)_{j=1}^{i}$ to $\mathcal{X}^{(i)}$, then replace the maps ι_i, κ_i, and $g^{(i)}$ with their restrictions to $\mathcal{X}^{(i)}$; so $(\mathcal{X}^{(i)}, (\delta_j)_{j=1}^{i})$ is a minimal $\mathcal{F}$-pseudoforest.

At the n^{th} iteration, we have a minimal $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n}) \coloneqq (\mathcal{X}^{(n)}, (\delta_i)_{i=1}^{n})$ with trivial arc stabilizers, unique for the descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$; the ψ-equivariant pretree-automorphism $h \coloneqq g^{(n)}$ on $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ is a $(\lambda_i)_{i=1}^{n}$-homothety, where $\lambda_i > 1$ is the scaling factor for the homothety h_i; lastly, an element $x \in \mathcal{F}$ is $\mathcal{T}$-elliptic if and only if x has a conjugate in $\mathcal{G}_{n+1}$. The real $\mathcal{F}$-pretrees $\mathcal{T}$ are the <u>limit pretrees</u> for $(\mathcal{Y}_i)_{i=1}^{n}$ and the $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ is the <u>limit pseudoforest</u> for $(\mathcal{Y}_i, \delta_i)_{i=1}^{n}$. To summarize,

Theorem III.1 (cf. [22, Theorem III.3])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism. Then there is:*

1. *a minimal $\mathcal{F}$-pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^{n})$ with trivial arc stabilizers;*
2. *a ψ-equivariant expanding homothety $h\colon (\mathcal{T}, (\delta_i)_{i=1}^{n}) \to (\mathcal{T}, (\delta_i)_{i=1}^{n})$; and*
3. *an element $x \in \mathcal{F}$ is $\mathcal{T}$-loxodromic if and only if x $[\psi]$-grows exponentially.*

The real pretrees $\mathcal{T}$ are degenerate if and only if $[\psi]$ is exponentially growing. □

Without metrics, there is not much one can do to compare limit pretrees. On the other hand, we do not expect limit pseudoforests to be well-defined (even up to homothety) for a given outer automorphism — this would be equivalent to the existence of a canonical descending sequence of limit forests. The new idea is to pick a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$ and normalize its hierarchy $(\delta_i)_{i=1}^n$ using the attracting laminations for $[\psi]$. For the normalized hierarchy, the associated top level forest will be universal; in particular, it is independent of any choices made in its construction.

III.2 Attracting laminations

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests. Let $\mathcal{G}_1 = \mathcal{F}$, $\mathcal{G}_{i+1} = \mathcal{G}[\mathcal{Y}_i]$, and $[\psi_i]$ be the restriction of $[\psi]$ to $\mathcal{G}_i$ for $i \geq 1$. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has matching stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ for $[\psi_i]$ rel. $\mathcal{Z}_i$, where $\mathcal{Z}_i$ is a $[\psi_i]$-invariant proper free factor system of $\mathcal{G}_i$. By Claim I.5, $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ is canonically identified with a subspace of $\mathbb{R}(\mathcal{G}_i)$ via a lifting map. As $\mathcal{G}_{i+1}$ is a malnormal subgroup system of $\mathcal{G}_i$, the space of lines $\mathbb{R}(\mathcal{G}_{i+1})$ is canonically identified with a closed subspace of $\mathbb{R}(\mathcal{G}_i)$ (exercise). By transitivity, $\mathbb{R}(\mathcal{G}_n) \subset \mathbb{R}(\mathcal{G}_{n-1}) \subset \cdots \subset \mathbb{R}(\mathcal{G}_0) = \mathbb{R}(\mathcal{F})$.

Consider this chain of canonical embeddings: $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i) \subset \mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$. Quasiperiodicity is not preserved by the first embedding but a weaker form of it is. A line $[l]$ is birecurrent in an $\mathcal{F}$-forest if any closed interval $I \subset l$ has infinitely many translates contained in both ends of l; quasiperiodic lines are birecurrent.

An attracting lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F})$ is the closure of a birecurrent line in $\mathbb{R}(\mathcal{F})$ with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. The set of all attracting laminations for $[\psi]$ is canonical as it is defined using canonical constructs: $\mathbb{R}(\mathcal{F})$ and the homeomorphism $\psi_*\colon \mathbb{R}(\mathcal{F}) \to \mathbb{R}(\mathcal{F})$. Note that ψ_* permutes the attracting laminations for $[\psi]$.

Remark. This definition is from [3, Definition 3.1.5]. Shortly, we will define topmost attracting laminations as done in [3, Section 6].

Lemma III.2 (cf. [3, Lemma 3.1.4])**.** *Let $f\colon (\mathcal{T}, d) \to (\mathcal{Y}, \delta)$ be an equivariant PL-map. A line is birecurrent in $\mathbb{R}(\mathcal{Y}, \delta)$ if and only if it is birecurrent in $\mathbb{R}(\mathcal{T})$.* (exercise) □

So leaves of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are birecurrent in $\mathbb{R}(\mathcal{G}_i)$ and hence $\mathbb{R}(\mathcal{F})$; moreover, a ψ_{i*}^k-attracting neighbourhood of a line in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ will lift to a ψ_*^k-attracting neighbourhood of the same line in $\mathbb{R}(\mathcal{F})$. (exercise) Thus the closure in $\mathbb{R}(\mathcal{F})$ of a stable lamination for $[\psi_i]$ rel. $\mathcal{Z}_i$, i.e. a component of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, is an attracting lamination for $[\psi]$.

Lemma III.3 (cf. [3, Lemma 3.1.10])**.** *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests. The components of stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ $(1 \leq i \leq n)$ determine all the attracting laminations for $[\psi]$.*

Sketch of proof. Suppose that $[l] \in \mathbb{R}(\mathcal{F})$ is a birecurrent line with a ψ_*^k-attracting neighbourhood for some $k \geq 1$. If $\mathcal{G}_{n+1} \neq \emptyset$, then either it consists of only cyclic components or the restriction of ψ_n to $\mathcal{G}_{n+1}$ is polynomially growing. Either way, $\mathcal{G}_{n+1}$ cannot support an attracting lamination of ψ_n. Let $i \leq n$ be the maximal index for which $\mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$ contains $[l]$. Birecurrence in $\mathbb{R}(\mathcal{F})$ and Lemma III.2 imply $[l]$ is birecurrent in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$

with a ψ_{i*}^k-attracting neighbourhood for some $k \geq 1$. Following the proof of Claim II.8, assume some descending chain $(\mathcal{F}_{i,j})_{j=2}^{n_i}$ of proper free factor systems of $\mathcal{F}_{i,1} \coloneqq \mathcal{G}_i$ was used to construct $(\mathcal{Y}_i, \delta_i)$; then any birecurrent line in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ with a ψ_{i*}^k-attracting neighbourhood is in $\mathbb{R}(\mathcal{F}_{i,n_i}, \mathcal{Z}_i)$. The proof of Lemma II.5 (with "birecurrence" in place of "quasiperiodicity") implies $[l] \in \mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$. $\square$

The finite set of all attracting laminations for $[\psi]$ is canonical (by definition) and partially ordered by inclusion; an attracting lamination for $[\psi]$ is topmost if it is maximal in this partial order. By Lemma II.5, ψ_{i*} transitively permutes the components of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$; so the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ is a ψ_*-orbit $\mathcal{L}_i^+[\psi]$ of attracting laminations for $[\psi]$. The goal is to normalize any limit pseudoforest $(\mathcal{T}, (d_i)_{i=1}^n)$ so that the levels are related to the partial order of the attracting laminations.

The next proposition is a repackaging of Theorem II.10 in the language of this chapter:

Proposition III.4. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$.*

For a nontrivial element $x \in \mathcal{F}$, the following statements are equivalent:

1. *the element x is $\mathcal{T}$-loxodromic;*
2. *the element x $[\psi]$-grows exponentially; and*
3. *the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to an attracting lamination.*

Proof. The equivalence between Conditions 1–2 is part of Theorem III.1. Suppose $x \in \mathcal{F}$ is $\mathcal{T}$-loxodromic and the limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$ is constructed from the descending sequence of limit forests $(\mathcal{Y}_i, \delta_i)$ for $1 \leq i \leq n$. By construction, the element x is conjugate to a $\mathcal{Y}_i$-loxodromic element $y \in \mathcal{G}_i$ for some $i \leq n$; in particular, x and y have the same axis in $\mathbb{R}(\mathcal{G}_i) \subset \mathbb{R}(\mathcal{F})$. The axis for y in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i) \subset \mathbb{R}(\mathcal{G}_i)$ weakly ψ_{i*}-limits to the stable laminations $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ by Theorem II.10; therefore, the shared axis for y and x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to the attracting laminations for $[\psi]$ determined by $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$, i.e. the closure of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$ in $\mathbb{R}(\mathcal{F})$.

Conversely, suppose $x \in \mathcal{F}$ is $\mathcal{T}$-elliptic. Then x is must be conjugate to a $\mathcal{Y}_n$-elliptic element $y \in \mathcal{G}_n$. If y is conjugate to an element of $\mathcal{Z}_i$, then the shared axis for y and x in the closed subspace $\mathbb{R}(\mathcal{Z}_i) \subset \mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to the attracting lamination for $[\psi]$ determined by a component of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$ — such an attracting lamination contains lines not in $\mathbb{R}(\mathcal{Z}_i)$. If y is not conjugate to an element of $\mathcal{Z}_i$, then the axis for y in $\mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ does not weakly ψ_{i*}-limit to $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$ by Theorem II.10; therefore, the shared axis for y and x in $\mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to the attracting lamination for $[\psi]$ determined by a component of $\mathcal{L}_{\mathcal{Z}_i}^+[\psi_i]$. By Lemma III.3, we have exhausted all possibilities when $1 \leq i \leq n$, and the axis for x in $\mathbb{R}(\mathcal{F})$ cannot weakly ψ_*-limit to an attracting lamination for $[\psi]$. $\square$

III.3 Pseudolaminations

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests, and let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ be the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Recall that $\mathcal{G}_1 = \mathcal{F}$, $\mathcal{G}_{i+1} = \mathcal{G}[\mathcal{Y}_i]$, and $[\psi_i]$ is the restriction of $[\psi]$ to $\mathcal{G}_i$ for $i \geq 1$. For $1 \leq i \leq n$, the stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are contained in $\mathbb{R}(\mathcal{Y}_i, \delta_i) \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$, where $\mathcal{Z}_i$ is some $[\psi_i]$-invariant proper free factor system of $\mathcal{G}_i$.

Let $\mathcal{T}_i \subset \mathcal{T}$ be the characteristic convex subsets for $\mathcal{G}_i$. By construction of $(\mathcal{T}, (\delta_i)_{i=1}^n)$, δ_i restricts to a $\mathcal{G}_i$-invariant convex pseudometric on $\mathcal{T}_i$ whose associated $\mathcal{G}_i$-forest can be equivariantly identified with $(\mathcal{Y}_i, \delta_i)$. Fix such an identification, and let $\kappa_i\colon \mathcal{T}_i \to \mathcal{Y}_i$ denote the natural equivariant collapse map. The stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ are in $\mathbb{R}(\mathcal{Y}_i, \delta_i)$; their leaves have unique lifts (via κ_i) to $\mathcal{T}_i \subset \mathcal{T}$; we call these <u>pseudoleaves</u> of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$. A <u>pseudoleaf segment</u> of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ is a closed interval in a (representative of a) pseudoleaf with nondegenerate κ_i-image in $\mathcal{Y}_i$.

Remarkably, the pseudoleaf segments detect weak ψ_*-limits of elements in attracting laminations. Let $\mathcal{L}^+_i[\psi]$ be the attracting laminations for $[\psi]$ determined by $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, i.e. the closure in $\mathbb{R}(\mathcal{F})$ of the stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$.

Proposition III.5. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an exponentially growing automorphism with a limit pseudoforest $(\mathcal{T}, (\delta_i)_{i=1}^n)$. For $1 \leq j \leq n$ and $\mathcal{T}$-loxodromic $x \in \mathcal{F}$, the axis for x in $\mathcal{T}$ contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ if and only if the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to the attracting laminations $\mathcal{L}^+_j[\psi]$.*

Proof. Let $(\mathcal{T}, (\delta_i)_{i=1}^n)$ be the limit pseudoforest for the descending sequence $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ of limit forests for $[\psi]$. For $i \leq n$, pick a descending sequence $(\tau_{i,j})_{j=1}^{n_i}$ of irreducible train tracks for $[\psi_i]$ rel. $\mathcal{Z}_i$; we can assume $\tau_{i+1,j}$ is defined on a free splitting of $\mathcal{Z}_i$ for some $j < n_{i+1}$ since $[\psi_{i+1}]$ is polynomially growing rel. $\mathcal{Z}_i$ (Theorem II.10). The train tracks $(\tau_{i,j})_{j=1}^{n_i}$ induce a ψ_i-equivariant λ_i-Lipschitz PL-map $\tau_i^\circ\colon (\mathcal{T}_i^\circ, d_i^\circ) \to (\mathcal{T}_i^\circ, d_i^\circ)$. Fix a metric free splitting $(\mathcal{T}^\star, d^\star)$ of $\mathcal{F}$ that is the metric blow-up of $(\mathcal{T}_1^\circ, d_1^\circ)$, $(\mathcal{T}_{i+1}^\circ(\mathcal{Z}_i), d_{i+1}^\circ)$ for $i < n$, and some metric free splitting $(\mathcal{T}_{n+1}^\circ, d_{n+1}^\circ)$ of $\mathcal{Z}_n$ whose free factor system $\mathcal{F}[\mathcal{T}_{n+1}^\circ]$ is trivial. As the $\mathcal{G}_i$-orbit of $\mathcal{T}_i^\circ(\mathcal{Z}_{i-1})$ is τ_i°-invariant, the maps $(\tau_i^\circ)_{i=1}^n$ induce a ψ-equivariant PL-map $\tau^\star$ on $(\mathcal{T}^\star, d^\star)$.

Let $x \in \mathcal{F}$ be a $\mathcal{T}$-loxodromic element. By construction, the element x is conjugate to a $\mathcal{Y}_i$-loxodromic $y_i \in \mathcal{G}_i$ for some $i \leq n$; let l_i° be the $\mathcal{T}_i^\circ$-axis for y_i. If $j = i$, then the equivalence in the proposition's statement follows from Theorem II.10. For the rest of the proof, we prove the equivalence when $j > i$. As we are going to invoke the same argument in the next proof, we mostly forget that l_i° is a $\mathcal{T}_i^\circ$-axis for a $\mathcal{Y}_i$-loxodromic element and only use the fact $[l_i^\circ] \in \mathbb{R}(\mathcal{Y}_i, \delta_i)$, i.e. l_i° projects to a line γ_i in $(\mathcal{Y}_i, \delta_i)$.

Suppose the $\mathcal{T}$-axis for x contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ for some $j > i$. Then the $\mathcal{T}$-axis for y_i contains a pseudoleaf segment σ_j of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ and $\kappa_i(\sigma_j)$ is a point $\circ_i \in \gamma_i$ with nontrivial point stabilizer $G_{\circ_i} := \mathrm{Stab}_{\mathcal{G}_i}(\circ_i)$. In Subsection II.2.3, we describe how the line γ_i in $(\mathcal{Y}_i, \delta_i)$ and point $\circ_i \in \gamma_i$ determine an algebraic iterated turn

$(\epsilon, s_{i+1,1}^{-1}s_{i+1,2}; \varphi_{i+1})_{\mathcal{G}_{i+1}}$. Any iterated turn $(\beta_{i+1,m})_{m\geq 0}$ over $\mathcal{T}_{i+1}^{\circ}$ realizing this algebraic iterated turn limits to an interval $[\star_{i+1,1}, \star_{i+1,2}]$ in the metric completion $(\overline{\mathcal{Y}}_{i+1}, \delta_{i+1})$ by Theorem II.9, and $[\star_{i+1,1}, \star_{i+1,2}]$ contains $\kappa_{i+1}(\sigma_j)$.

If $j = i+1$, then $[\star_{i+1,1}, \star_{i+1,2}] \supset \kappa_{i+1}(\sigma_{i+1})$ is not degenerate and $(\beta_{i+1,m})_{m\geq 0}$ weakly limits to a component of $\mathcal{L}^{+}_{\mathcal{Z}_{i+1}}[\psi_{i+1}]$ by Theorem II.9. Otherwise, for $k \geq i+1$, assume $\kappa_k(\sigma_j)$ is a point $\circ_k$ in the interval $[\star_{k,1}, \star_{k,2}] \subset \overline{\mathcal{Y}}_k$ corresponding to the algebraic iterated turn $(\epsilon, s_{k,1}^{-1}s_{k,2}; \varphi_k)_{\mathcal{G}_k}$, where $\circ_k$ has nontrivial stabilizer $G_{\circ_k}$. By the discussion in Subsection II.2.4 (and remark after Theorem II.9), the algebraic iterated turn over $\mathcal{G}_k$ and point $\circ_k$ in $[\star_{k,1}, \star_{k,2}]$ determine an algebraic iterated turn $(\epsilon, s_{k+1,1}^{-1}s_{k+1,2}; \varphi_{k+1})_{\mathcal{G}_{k+1}}$ that limits to $[\star_{k+1,1}, \star_{k+1,2}] \subset \overline{\mathcal{Y}}_j$; morevoer, $[\star_{k+1,1}, \star_{k+1,2}]$ contains $\kappa_{k+1}(\sigma_j)$. By induction, $[\star_{j,1}, \star_{j,2}]$ contains $\kappa_j(\sigma_j)$. Since $\kappa_j(\sigma_j)$ is not degenerate, any realization $(\beta_{j,m})_{m\geq 0}$ over $\mathcal{T}_j^{\circ}$ of the algebraic iterated turn $(\epsilon, s_{j,1}^{-1}s_{j,2}; \varphi_j)_{\mathcal{G}_j}$ weakly limits to a component of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$ by Theorem II.9.

In either case $(j \geq i+1)$, any realization over $\mathcal{T}^{\star}$ of $(\epsilon, s_{j,1}^{-1}s_{j,2}; \varphi_j)_{\mathcal{G}_j}$ weakly limits to (the closure in $\mathbb{R}(\mathcal{F})$ of) a component of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$ (bounded cancellation). If $j > i+1$, any realization over $\mathcal{T}^{\star}$ of $(\epsilon, s_{i+1,1}^{-1}s_{i+1,2}; \varphi_{i+1})_{\mathcal{G}_{i+1}}$ weakly limits to a component of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$ by transitivity. Hence the shared axis for y_i and x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to a component of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$. As $\psi_{j*}\colon \mathbb{R}(\mathcal{G}_j, \mathcal{Z}_j) \to \mathbb{R}(\mathcal{G}_j, \mathcal{Z}_j)$ acts transitively on the components of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$, the axis for x in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^{+}_{j}[\psi]$, the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$.

Conversely, suppose the axis $[l^{\star}]$ for y_i (and x) in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^{+}_{j}[\psi]$ for some $j > i$. Using $(\mathcal{T}^{\star}, d^{\star})$-coordinates, the axis $\tau_*^{\star m}(l^{\star})$ contains arbitrarily d_j°-long leaf segments of $\mathcal{L}^{+}_{j}[\psi]$ for $m \gg 1$. So $\tau_*^{\star M}(l^{\star})$ has a $\mathcal{L}^{+}_{j}[\psi]$-leaf segment $I^{\star} \subset \mathcal{T}^{\star}(\mathcal{Z}_{j-1})$ with d_j°-length $L > C' \coloneqq \frac{2C[\tau^{\star}]}{\lambda_j - 1}$ for $M \gg 1$. As τ_j° is a train track on leaves of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$, $\tau_*^{\star m}(l^{\star})$ has a $\mathcal{L}^{+}_{j}[\psi]$-leaf segment surviving from $I^{\star}$ with d_j°-length $> \lambda_j^{M-m}(L - C')$ for $m \geq M$.

Let $\rho_i\colon (\mathcal{T}^{\star}(\mathcal{G}_i), d^{\star}) \to (\mathcal{T}_i^{\circ}, d_i^{\circ})$ be an arbitrary equivariant PL-map. The ρ_i-image of $I^{\star} \subset \tau_*^{\star M}(l^{\star})$ is a vertex $v \in \tau_{i*}^{\circ M}(l_i^{\circ})$ with nontrivial stabilizer. Since a nondegenerate part of $I^{\star}$ survives in $\psi_*^m(l^{\star})$ for all $m \geq M$, we have $\tau_i^{\circ(m-M)}(v) \in \tau_{i*}^{\circ m}(l_i^{\circ})$ for all $m \geq M$ and $h_i^{-M}(\pi_i^{\circ}(v)) \in \gamma_i$ has a nontrivial stabilizer $G_v \coloneqq \mathrm{Stab}_{\mathcal{G}_i}(h_i^{-M}(\pi_i^{\circ}(v)))$, where h_i is the ψ_i-equivariant λ_i-homothety on $(\mathcal{Y}_i, \delta_i)$. As before, the line γ_i, point $h_i^{-M}(\pi_i^{\circ}(v)) \in \gamma_i$, and equivariant PL-maps $\rho_{i+1}, \ldots, \rho_j$ determine nested iterated turns over $\mathcal{T}_{i+1}^{\circ}, \ldots, \mathcal{T}_j^{\circ}$ limiting to intervals in $\overline{\mathcal{Y}}_{i+1}, \ldots, \overline{\mathcal{Y}}_j$. By the computation in the previous paragraph and quasiperiodicity of stable laminations, the last iterated turn over $\mathcal{T}_j^{\circ}$ weakly limits to a component of $\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j]$. So the corresponding interval $[\star_{j,1}, \star_{j,2}] \subset \overline{\mathcal{Y}}_j$ is not degenerate (Theorem II.9) and the $\mathcal{T}$-axis for y_i has an intersection with $\mathcal{T}_j$ whose κ_j-image is $[\star_{j,1}, \star_{j,2}]$. By the description of intervals in $\mathcal{Y}_j$, $[\star_{j,1}, \star_{j,2}]$ contains a leaf segment of $\pi_{j*}^{\circ}(\mathcal{L}^{+}_{\mathcal{Z}_j}[\psi_j])$; therefore, the $\mathcal{T}$-axes of y_i and x contain pseudoleaf segments of $\mathcal{L}^{+}_{\mathcal{T}}[\psi_j]$. $\square$

In fact, the containment relation on pseudoleaf segments detects the partial order on the set of attracting laminations:

Claim III.6. *For $1 \leq i, j \leq n$, a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ if and only if $\mathcal{L}^+_i[\psi]$ contains $\mathcal{L}^+_j[\psi]$.*

We only sketch the proof as it is almost identical to the proof of Proposition III.5.

Sketch of proof. There's nothing to show if $i = j$. Without loss of generality, assume $i < j$; certainly, $\mathcal{L}^+_j[\psi]$ does not contain $\mathcal{L}^+_i[\psi]$ and no pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$ can contain a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$. Let $[l^\circ_i]$ be an eigenline in $(\mathcal{T}^\circ_i, d^\circ_i)$ of $[\tau^{\circ k}_i]$ for some $k \geq 1$, and $l^\star$ be the lift of l°_i to $(\mathcal{T}^\star, d^\star)$. The projection γ_i (of l°_i) is a line in $(\mathcal{Y}_i, \delta_i)$, and we denote by $l_i \subset \mathcal{T}_i$ its lift via κ_i to a pseudoleaf of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$.

Suppose the pseudoleaf l_i of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ contains a pseudoleaf segment σ_j of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$. Then $\kappa_i(\sigma_j)$ is a point $\circ_i \in \gamma_i$ with nontrivial point stabilizer $G_{\circ_i}$. By the same argument as in the previous proof, the line $l^\star$ in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{L}^+_j[\psi]$. Note that $\psi^k_*[l^\star] = [l^\star]$ in $\mathbb{R}(\mathcal{F})$ as $[l^\circ_i]$ is an eigenline for $[\tau^{\circ k}_i]$; moreover, $\mathcal{L}^+_i[\psi]$ consists of the closures in $\mathbb{R}(\mathcal{F})$ of $[l^\star]$, …, $\psi^{k-1}_*[l^\star]$ since $\mathcal{L}^+_i[\psi]$ is a ψ_*-orbit of attracting laminations. So $\mathcal{L}^+_i[\psi] \supset \mathcal{L}^+_j[\psi]$.

Conversely, suppose $\mathcal{L}^+_i[\psi] \supset \mathcal{L}^+_j[\psi]$. As $\mathcal{L}^+_i[\psi]$ and $\mathcal{L}^+_j[\psi]$ are ψ_*-orbits of attracting laminations, the line $l^\star$ contains arbitrarily d°_j-long leaf segments of $\mathcal{L}^+_j[\psi]$. By the same argument as in the previous proof, the pseudoleaf l_i, and hence some pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$, contains a pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_j]$. □

III.4 Topmost forests

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with a descending sequence $(\mathcal{Y}_i, \delta_i)^n_{i=1}$ of limit forests, and let $(\mathcal{T}, (\delta_i)^n_{i=1})$ be the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)^n_{i=1}$. Each limit forest $(\mathcal{Y}_i, \delta_i)$ has stable laminations $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ for $[\psi_i]$ rel. $\mathcal{Z}_i$. Let $\mathcal{L}^+_{\mathcal{T}}[\psi_i]$ be the lifts to $\mathcal{T}$ of leaves in $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, $\mathcal{L}^+_i[\psi]$ the closure in $\mathbb{R}(\mathcal{F})$ of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$, and $\{\mathcal{A}^{top}_j[\psi_i]\}^{k_i}_{j=1}$ the subset of $\{\mathcal{L}^+_j[\psi]\}^n_{j=i}$ consisting of all topmost attracting laminations for $[\psi_i]$. So $\mathcal{A}^{top}_j[\psi_i] = \mathcal{L}^+_{\iota(i,j)}[\psi]$ for some subsequence $(\iota(i,j))^{k_i}_{j=1}$ of $(j)^n_{j=i}$ with $\iota(i,1) = i$, and $(\iota(i,j))^{k_i}_{j=2}$ is a subsequence of $(\iota(i+1,j))^{k_{i+1}}_{j=1}$ if $k_i \geq 2$.

For $i \geq 1$, we say the $\mathcal{G}_i$-invariant hierarchy $(\delta_j)^n_{j=i}$ on the characteristic convex subsets $\mathcal{T}_i \subset \mathcal{T}$ for $\mathcal{G}_i$ <u>normalizes</u> to a factored $\mathcal{G}_i$-invariant convex pseudometric $\Sigma^{k_i}_{j=1}\delta_{\iota(i,j)}$ if the $\mathcal{G}_{\iota(i,j)}$-invariant convex pseudometric $\delta_{\iota(i,j)}$ can be extended to a $\mathcal{G}_i$-invariant convex pseudometric, also denoted $\delta_{\iota(i,j)}$, on $\mathcal{T}_i$. The $\mathcal{F}$-invariant hierarchy $(\delta_i)^n_{i=1}$ normalizes to δ_1 if (and only if) $k_1 = 1$.

We may assume $k_1 \geq 2$ and the $\mathcal{G}_2$-invariant hierarchy $(\delta_i)^n_{i=2}$ normalizes to $\oplus^{k_2}_{j=1}\delta_{\iota(2,j)}$. Let $\widehat{\mathcal{T}}_2$ be the κ_1-preimage of the characteristic convex subsets $\mathcal{Y}_1(\mathcal{G}_2)$. Suppose $\mathcal{F}_{1,1} \coloneqq \mathcal{F}$, …, $\mathcal{F}_{1,m}$ are the proper free factor systems of $\mathcal{F}$ used to construct $(\mathcal{Y}_1, \delta_1)$ and let $\mathcal{T}_{1,1}$, …, $\mathcal{T}_{1,m}$ be their corresponding characteristic convex subsets in $\mathcal{T}$. By Lemma II.6, every closed interval in the characteristic convex subsets $\mathcal{Y}_1(\mathcal{F}_{1,m})$ is a finite concatenation of leaf segments of $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$. Thus every closed interval in $\mathcal{T}_{1,m}$ is a finite concatenation of pseudoleaf segments of $\mathcal{L}^+_{\mathcal{T}}[\psi_1]$ and closed intervals in $\mathcal{F}_{1,m} \cdot \widehat{\mathcal{T}}_2$.

Fix $j \in \{2, \ldots, k_1\}$. Since $\mathcal{L}^+_1[\psi]$ does not contain $\mathcal{L}^+_{\iota(1,j)}[\psi]$, Claim III.6 implies the intersection of any pseudoleaf segment of $\mathcal{L}^+_{\mathcal{T}}[\psi_1]$ with $\widehat{\mathcal{T}}_2$ has 0 diameter with respect to the convex pseudometric $\delta_{\iota(1,j)}$; we say that $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$ and $\delta_{\iota(1,j)}$ are independent. So the intersection of any closed interval in $\mathcal{T}_{1,m}$ with $\mathcal{F}_{1,m} \cdot \widehat{\mathcal{T}}_2$ has finitely many components that are translates of closed intervals in $\widehat{\mathcal{T}}_2$ with positive $\delta_{\iota(1,j)}$-diameter. Thus $\delta_{\iota(1,j)}$ can be extended to an $\mathcal{F}_{1,m}$-invariant convex pseudometric on $\mathcal{T}_{1,m}$ that is mutually singular with δ_1. By our inductive description of intervals in $\mathcal{Y}_1$ (Lemma II.7), the convex pseudometric $\delta_{\iota(1,j)}$ extends equivariantly to $\mathcal{T}$ as $\lambda_{\iota(1,j)} > 1$.

As j was arbitrary, the $\mathcal{F}$-invariant hierarchy $(\delta_i)_{i=1}^n$ normalizes to the factored convex pseudometric $\oplus_{j=1}^k \delta_{\iota(j)}$, where $k \coloneqq k_1$ and $\iota(j) \coloneqq \iota(1,j)$. Let $(\mathcal{Y}, \oplus_{j=1}^k \delta_{\iota(j)})$ be the associated factored $\mathcal{F}$-forest. The real $\mathcal{F}$-pretrees $\mathcal{Y}$ are minimal and have trivial arc stabilizers since the pseudometric $\oplus_{j=1}^k \delta_{\iota(j)}$ on $\mathcal{T}$ is convex. The ψ-equivariant $(\lambda_i)_{i=1}^n$-homothety h induces a ψ-equivariant $\oplus_{j=1}^k \lambda_{\iota(j)}$-dilation on $(\mathcal{Y}, \oplus_{j=1}^k \delta_{\iota(j)})$: a $\lambda_{\iota(j)}$-homothety with respect to each factor $\delta_{\iota(j)}$. By Proposition III.5, a nontrivial element of $\mathcal{F}$ is $\delta_{\iota(j)}$-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{A}^{top}_j[\psi_1]$ — here, $\delta_{\iota(j)}$-loxodromic means the element acts loxodromically on the associated $\mathcal{F}$-forest for $\delta_{\iota(j)}$. The factored $\mathcal{F}$-forest $(\mathcal{Y}, \oplus_{j=1}^k \delta_{\iota(j)})$ is the complete topmost limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. Given any subset of the ψ_*-orbits of topmost attracting laminations for $[\psi]$, then one may consider the associated factored $\mathcal{F}$-forest for the sum of corresponding pseudometrics:

Theorem III.7. *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an automorphism and $\{\mathcal{A}^{top}_j[\psi]\}_{j=1}^k$ a (possibly empty) subset of ψ_*-orbits of topmost attracting laminations for $[\psi]$.*

Then there is:

1. *a minimal factored $\mathcal{F}$-forest $(\mathcal{Y}, \oplus_{j=1}^k \delta_j)$ with trivial arc stabilizers;*
2. *a unique ψ-equivariant expanding dilation $f \colon (\mathcal{Y}, \oplus_{j=1}^k \delta_j) \to (\mathcal{Y}, \oplus_{j=1}^k \delta_j)$; and*
3. *for $1 \le j \le k$, a nontrivial element $x \in \mathcal{F}$ is δ_j-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{A}^{top}_j[\psi]$.* □

Fix some index $\iota(j) \neq 1$, and let $\mathcal{X}_{1,m}$ be the associated $\mathcal{F}_{1,m}$-forest for $\delta_1 \oplus \delta_{\iota(j)}$ on $\mathcal{T}_{1,m}$. Two lines in $(\mathcal{X}_{1,m}, \delta_1 \oplus \delta_{\iota(j)})$ representing leaves in $\mathcal{L}^+_{\mathcal{Z}_1}[\psi]$ overlap if they have a nondegenerate intersection; overlapping generates an equivalence relation and each overlapping class is identified with its union in $\mathcal{X}_{1,m}$. Let $\mathrm{supp}[\psi_1; \mathcal{Z}_1]$ denote the subgroup system corresponding to the (setwise) stabilizers of overlapping classes $L^+_{\mathcal{Z}_1}$ — this subgroup system, called the lower-support of $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$, is $[\psi]$-invariant. The system $\mathrm{supp}[\psi_1; \mathcal{Z}_1]$ is not empty as there are $\mathcal{Y}_1$-loxodromic elements whose axis in $\mathcal{Y}_1^*$ is contained in $L^+_{\mathcal{Z}_1}$. Note the number of components in $\underline{\mathrm{supp}}[\psi_1; \mathcal{Z}_1]$ is at most the number of components in $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$. Let $(\widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2), \delta_{\iota(j)})$ be the closure in $(\mathcal{X}_{1,m}, \delta_1 \oplus \delta_{\iota(j)})$ of the characteristic subforest for $\mathcal{G}_2$. By Lemma II.6 again, intervals in $\mathcal{X}_1(\mathcal{F}_{1,n_1})$ are finite concatenations of leaf segments of $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$ and closed intervals in $\widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$.

The overlapping classes $L^+_{\mathcal{Z}_1}$ and the $\mathcal{F}_{1,m}$-orbits of components of $\widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$ form an $\mathcal{F}_{1,m}$-invariant transverse covering of $\mathcal{X}_{1,m}$ (see [14, Definition 4.6]). Let $\mathcal{S}'$ be a simplicial $\mathcal{F}_{1,m}$-pretree: vertices ("component-vertices") in equivariant bijective correspondence with the components of the transverse covering (overlapping classes $L^+_{\mathcal{Z}_1}$ and translates of components of $\widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$); for each point in $\mathcal{X}_{1,m}$ contained in exactly two components of the transverse covering, there is an edge between the corresponding component-vertices; for each point contained in more than two components, there is a new vertex ("intersection-vertex") and an edge connecting it to each relevant component-vertex. By the blow-up construction, translates of components of $\widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$ either coincide or are disjoint. In particular, each intersection-vertex $v \in \mathcal{S}'$ with a nontrivial stabilizer is adjacent to a unique vertex $w \in \mathcal{S}'$ corresponding to a component of $\mathcal{F}_{1,m} \cdot \widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$ and the stabilizer of v fixes w; therefore, we can collapse all such edges $[v, w]$ to form a simplicial $\mathcal{F}_{1,m}$-pretree $\mathcal{S}$ whose intersection-vertices have trivial stabilizers.

The $\mathcal{F}_{1,m}$-forest $(\mathcal{X}_{1,m}, \delta_1 \oplus \delta_{\iota(j)})$ is a graph of actions with skeleton $\mathcal{S}$ and the nondegenerate "vertex trees" are the components of the transverse covering [14, Lemma 4.7]. As the $\psi_{1,m}$-equivariant expanding dilation on $(\mathcal{X}_{1,m}, \delta_1 \oplus \delta_{\iota(j)})$ permutes the overlapping classes (and components of $\mathcal{F}_{1,m} \cdot \widehat{\mathcal{X}}_{1,m}(\mathcal{G}_2)$), it induces a $\psi_{1,m}$-equivariant simplicial automorphism $\sigma \colon \mathcal{S} \to \mathcal{S}$ that preserves the "type" of a vertex.

Let $\mathcal{T}_1^\diamond$ be an equivariant blow-up of $(\mathcal{T}_{1,j})_{j=1}^{m-1}$, $\mathcal{S}$, and $\mathcal{X}_{1,m}(\mathcal{G}_2)$. When the metric $\delta_{\iota(j)}$ is extended appropriately to $\mathcal{T}_1^\diamond$, the simplicial automorphisms $(\tau_{1,j})_{j=1}^{m-1}$, σ, and the homothety f_2 on $\mathcal{X}_{1,m}(\mathcal{G}_2)$ induce a ψ-equivariant $\lambda_{\iota(j)}$-Lipschitz map $\tau^\diamond \colon (\mathcal{T}_1^\diamond, \delta^\diamond_{\iota(j)}) \to (\mathcal{T}_1^\diamond, \delta^\diamond_{\iota(j)})$ that linearly extends f_2. Using $\tau^\diamond$-iteration, we define the limit forest $(\mathcal{X}_1, \delta_{\iota(j)})$ for $[\tau_i]_{i=1}^{n-1}$, σ, and f_2. Like the previous convergence criteria, the proof of the following lemma is postponed to Section IV.3.

Lemma IV.7. *Let $\psi \colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$, $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$, $\mathcal{G}$ the nontrivial point stabilizer system for the limit forest for $[\psi]$ rel. $\mathcal{Z}$, $[\psi_\mathcal{G}]$ the $[\psi]$-restriction to $\mathcal{G}$, $(\mathcal{Y}_\mathcal{G}, \delta)$ a minimal $\mathcal{G}$-forest with trivial arc stabilizers such that $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ are independent, $h_\mathcal{G} \colon (\mathcal{Y}_\mathcal{G}, \delta) \to (\mathcal{Y}_\mathcal{G}, \delta)$ a $\psi_\mathcal{G}$-equivariant λ-homothety, $\mathcal{S}$ a minimal simplicial $\mathcal{F}[\mathcal{T}_{n-1}]$-forest that is the skeleton for the graph of actions for $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ, $\sigma \colon \mathcal{S} \to \mathcal{S}$ the corresponding simplicial automorphism, and $(\mathcal{X}, \delta)$ the limit forest for $[\tau_i]_{i=1}^{n-1}$, σ, and $h_\mathcal{G}$.*

If $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers, the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{G}$ is equivariantly isometric to $(\mathcal{Y}_\mathcal{G}, \delta)$, and the lower-support $\underline{\mathrm{supp}}[\psi; \mathcal{Z}]$ of $\mathcal{L}^+_\mathcal{Z}[\psi]$ is $\mathcal{Y}'$-elliptic, then the limit of $(\mathcal{Y}'\psi^m, \lambda^{-m}\delta')_{m \ge 0}$ is $(\mathcal{X}, \delta)$.

Fix a subset $\{\mathcal{A}_j^{top}[\psi]\}_{j=1}^k$ of ψ_*-orbits of topmost attracting laminations for $[\psi]$; a topmost forest for $[\psi]$ is a factored $\mathcal{F}$-forest satisfying the conclusion of Theorem III.7 with respect to this subset. Lemma IV.7 is enough to prove the universality of topmost forests:

Theorem III.8. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism and $\{\mathcal{A}_j^{top}[\psi]\}_{j=1}^k$ a (possibly empty) subset of ψ_*-orbits of topmost attracting laminations for $[\psi]$. Any topmost forest for $[\psi]$ with respect to the given subset has a unique equivariant dilation to any corresponding topmost limit forest for $[\psi]$.*

Thus the factored $\mathcal{F}$-forest $(\mathcal{Y}, \oplus_{j=1}^k \delta_{\iota(j)})$ is *the* complete topmost forest for $[\psi]$ (up to rescaling of the factors). We omit the proof since we are about to prove something stronger in the next section (see Theorem III.11).

Suppose $(\mathcal{T}, (\delta_i)_{i=1}^n)$ and $(\mathcal{T}', (\delta_i)_{i=1}^{n'})$ are two limit pseudoforests for $[\psi]$. Then $n = n'$ as they are exactly the number of ψ_*-orbits of attracting laminations for $[\psi]$. Using Theorem III.7, the hierarchies can be inductively normalized to $(\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^d$ and $(\oplus_{j=1}^{k_i} \delta'_{i,j})_{i=1}^d$ respectively, where d is the length of the longest chain in the partial order of attracting laminations for $[\psi]$ and $\delta_{i,j}, \delta'_{i,j}$ are indexed by the same ψ_*-orbit $\mathcal{A}_{i,j}[\psi]$ of attracting laminations. By inductively invoking Theorem III.8 and uniqueness of the blow-up construction, the normalized pseudoforests $(\mathcal{T}, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^d)$ and $(\mathcal{T}', (\oplus_{j=1}^{k_i} \delta'_{i,j})_{i=1}^d)$ are in the same equivariant dilation class and invariants of this class are invariants of $[\psi]$! In particular, $\mathcal{T}$ and $\mathcal{T}'$ are equivariantly pretree-isomorphic.

Corollary III.9. *Any two limit pretrees for an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ are equivariantly pretree-isomorphic.* $\square$

We can now define more invariants of an attracting lamination: let A be an attracting lamination for $[\psi]$, $\mathcal{A}[\psi]$ its ψ_*-orbit, and $(\delta_{i,j}, \lambda_{i,j})$ the corresponding pair of pseudometric and stretch factor in the normalized pseudoforest $(\mathcal{T}, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^d)$; then $\lambda(A) \coloneqq \lambda_{i,j}$ is a well-defined stretch factor for A. Now let A be topmost, $\{\mathcal{B}_{i',j'}\}$ be the whole subset of ψ_*-orbits of attracting laminations not contained in $\mathcal{A}[\psi]$, and $(\mathcal{T}_A, (\oplus_{j'=1}^{k'_i} \delta'_{i',j'})_{i'=1}^{d'})$ the associated normalized pseudoforest. Then the upper-support of $\mathcal{A}[\psi]$ is the subgroup system of point stabilizers $\overline{\operatorname{supp}}\, \mathcal{A}[\psi] \coloneqq \mathcal{G}[\mathcal{T}_A]$. Unlike the lower-support, the upper-support is always a malnormal subgroup system of finite type. Note that components of the lower-support are conjugate into components of the upper-support.

III.5 Dominating forests

Fix an exponentially growing automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$. Let $A \subset \mathbb{R}(\mathcal{F})$ be an attracting lamination for $[\psi]$ and $\lambda(A)$ its stretch factor. We say A is dominating if $\lambda(A) > \lambda(A')$ whenever A' is an attracting lamination for $[\psi]$ containing A and $A' \neq A$; topmost attracting laminations are vacuously dominating. We will extend Theorem III.7 to dominating attracting laminations by mimicking the reasoning in the previous section, focusing only on the changes needed for dominating attracting laminations.

Let $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ be a descending sequence of limit forests for $[\psi]$, $(\mathcal{L}_i^+[\psi])_{i=1}^n$ the corresponding sequence of ψ_*-orbits of attracting laminations for $[\psi]$, $(\mathcal{T}, (\delta_i)_{i=1}^n)$ the limit pseudoforest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$, and $\{\mathcal{A}_j^{dom}[\psi_i]\}_{j=1}^{k_i}$ the subset of $\{\mathcal{L}_j^+[\psi]\}_{j=i}^n$ consisting of all dominating attracting laminations for $[\psi_i]$ — recall that $\mathcal{Y}_i$ are $\mathcal{G}_i$-pretrees and $[\psi_i]$ is the restriction of $[\psi]$ to $\mathcal{G}_i$. As before, $\mathcal{A}_j^{dom}[\psi_i] = \mathcal{L}_{\iota(i,j)}^+[\psi]$ for some subsequence $(\iota(i,j))_{j=1}^{k_i}$ of $(j)_{j=i}^n$ with $\iota(i,1) = i$.

Suppose $k_1 \geq 2$ and the $\mathcal{G}_2$-invariant hierarchy $(\delta_i)_{i=2}^n$ normalizes to the factored $\mathcal{G}_2$-invariant convex pseudometric $\Sigma_{j=1}^{k_2}\delta_{\iota(2,j)}$ on the characteristic convex subsets $\mathcal{T}_2 \subset \mathcal{T}$ for $\mathcal{G}_2$. Fix some $j \in \{2, \ldots, k_1\}$. The previous section discusses how to equivariantly extend $\delta_{\iota(1,j)}$ to $\mathcal{T}$ when $\mathcal{L}_1^+[\psi]$ does not contain $\mathcal{L}_{\iota(1,j)}^+[\psi]$. Assume for the rest of this section that $\mathcal{L}_{\iota(1,j)}^+[\psi] \subset \mathcal{L}_1^+[\psi]$; thus $\lambda_1 < \lambda_{\iota(1,j)}$ as $\mathcal{L}_{\iota(1,j)}^+[\psi]$ is dominating. Let $(\mathcal{Y}^*, (\delta_1, \delta_{\iota(1,j)}))$ be the associated $\mathcal{F}$-pseudoforest for the $\mathcal{F}$-invariant 2-level hierarchy $(\delta_1, \delta_{\iota(1,j)})$ and h^* the ψ-equivariant pretree-automorphism on $\mathcal{Y}^*$ induced by $h \colon \mathcal{T} \to \mathcal{T}$.

Let $\tau_1 \colon (\Gamma_1, d_1) \to (\Gamma_1, d_1)$ be the λ_1-Lipschitz topological representative for ψ used to construct $(\mathcal{Y}_1, \delta_1)$ through iteration. Pick an equivariant blow-up Γ° of Γ_1 rel. $\mathcal{Y}^*(\mathcal{Z}) \subset \mathcal{Y}^*$, the characteristic convex subsets for the proper free factor system $\mathcal{Z} \coloneqq \mathcal{F}[\Gamma_1]$. Since $\mathcal{Z}$ is δ_1-elliptic, $\delta_{\iota(1,j)}$ is a metric on $\mathcal{Y}^*(\mathcal{Z})$. The blow-up inherits an $\mathcal{F}$-invariant 2-level hierarchy $(d_1, \delta_{\iota(1,j)})$ with full support. As Γ_1 is simplicial, this hierarchy extends to a factored $\mathcal{F}$-invariant convex metric $d_1 \oplus \delta_{\iota(1,j)}$ on Γ°.

Let $[\psi_{\mathcal{Z}}]$ be the restriction of $[\psi]$ to $\mathcal{Z}$ and $h_{\mathcal{Z}}^*$ the $\psi_{\mathcal{Z}}$-equivariant "restriction" of h^* to $(\mathcal{Y}^*(\mathcal{Z}), \delta_{\iota(1,j)})$. For a parameter $c > 0$, the topological representative τ_1 induces a ψ-equivariant map τ_c° on Γ° that extends $h_{\mathcal{Z}}^*$ and is linear with respect to $(c\, d_1) \oplus \delta_{\iota(1,j)}$ on edges from Γ_1. If $c \gg 1$, then τ_c° is $\lambda_{\iota(1,j)}$-Lipschitz with respect to $(c\, d_1) \oplus \delta_{\iota(1,j)}$ since $\lambda_1 < \lambda_{\iota(1,j)}$. Through τ_c°-iteration, we define a limit forest $(\mathcal{Y}, \delta_{\iota(1,j)})$ for τ_1 and $h_{\mathcal{Z}}^*$ whose characteristic subforest for $\mathcal{Z}$ is identified with $(\mathcal{Y}^*(\mathcal{Z}), \delta_{\iota(1,j)})$ — up to equivariant isometry, this limit forest is independent of the parameter c; moreover, there is an induced ψ-equivariant λ-homothety h on $(\mathcal{Y}, \delta_{\iota(1,j)})$ that restricts to $h_{\mathcal{Z}}^*$ on $\mathcal{Y}^*(\mathcal{Z})$.

We now refine this construction of a limit forest. For $n \geq 1$, set $d_n^\circ \coloneqq \lambda_1^{-n} d_1 \oplus \lambda_{\iota(1,j)}^{-n} \delta_{\iota(1,j)}$ and $\tau^\circ \coloneqq \tau_1^\circ$. The map $\tau^\circ \colon (\Gamma^\circ, d_0^\circ) \to (\Gamma^\circ\psi, d_1^\circ)$ is equivariant and $(1+D)$-Lipschitz for some $D \geq 0$. In fact, $\tau^\circ \colon (\Gamma^\circ, d_n^\circ) \to (\Gamma^\circ\psi, d_{n+1}^\circ)$ is $(1+Dr^n)$-Lipschitz, where $r \coloneqq \lambda_1 \lambda_{\iota(1,j)}^{-1}$. Set $p_n \coloneqq \prod_{i=0}^{n-1}(1+Dr^i)$; then $\tau^{\circ n} \colon (\Gamma^\circ, d_0^\circ) \to (\Gamma^\circ\psi^n, p_n^{-1} d_n^\circ)$ is equivariant and metric; moreover, the pullback of $p_n^{-1} d_n^\circ$ to Γ° along $\tau^{\circ n}$ converges to an $\mathcal{F}$-invariant pseudometric on Γ° as $n \to \infty$. Since $|r| < 1$, the sequence $(p_n)_{n=1}^\infty$ converges and the pullback of d_n° to Γ° along $\tau^{\circ n}$ converges to a factored $\mathcal{F}$-invariant pseudometric $\delta_1^\circ + \delta_{\iota(1,j)}^\circ$ on Γ°. Let $(\Gamma^*, \delta_1^\circ + \delta_{\iota(1,j)}^\circ)$ be the associated factored $\mathcal{F}$-forest for this factored pseudometric on Γ° — as $\mathcal{L}_{\iota(1,j)}^+[\psi] \subset \mathcal{L}_1^+[\psi]$, one can show that δ_1° and $\delta_{\iota(1,j)}^\circ$ are not mutually singular and $\delta_{\iota(1,j)}^\circ$ is actually a metric on Γ^*. By construction, the characteristic subforest for $\mathcal{Z}$ in $(\Gamma^*, \delta_1^\circ + \delta_{\iota(1,j)}^\circ)$ is equivariantly isometric to $(\mathcal{Y}^*(\mathcal{Z}), \delta_{\iota(1,j)})$. Similarly, the $\mathcal{F}$-forest $(\mathcal{Y}_1, \delta_1)$ is equivariantly isometric to the associated metric space for the pseudometric δ_1° on Γ^*

or Γ° (Corollary II.11).

Lemma IV.9. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\mathcal{Z}$ a $[\psi]$-invariant proper free factor system, $(\mathcal{Y}_\mathcal{Z}, \delta)$ a minimal $\mathcal{Z}$-forest with trivial arc stabilizers, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$ with $\mathcal{F}[\mathcal{T}_n] = \mathcal{Z}$, $h_\mathcal{Z}\colon (\mathcal{Y}_\mathcal{Z}, \delta) \to (\mathcal{Y}_\mathcal{Z}, \delta)$ a $\psi_\mathcal{Z}$-equivariant λ-homothety, and $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$ and $h_\mathcal{Z}$, where $\lambda > \lambda[\tau_n]$ and $[\psi_\mathcal{Z}]$ is the $[\psi]$-restriction to $\mathcal{Z}$.*

If $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers and the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{Z}$ is equivariantly isometric to $(\mathcal{Y}_\mathcal{Z}, \delta)$, then the limit of $(\mathcal{Y}'\psi^m, \lambda^{-m}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \delta)$.

Again, the proof is postponed to Section IV.4. Since the restriction of $[\psi]$ to $\mathcal{G}_1$ is polynomially growing rel. $\mathcal{Z}$, Lemma IV.9 implies the characteristic subforests $(\mathcal{Y}^*(\mathcal{G}_1), \delta_{\iota(1,j)})$ and $(\Gamma^*(\mathcal{G}_1), \delta^\circ_{\iota(1,j)})$ for $\mathcal{G}_1$ are equivariantly isometric. By uniqueness of the blow-up construction, Γ^* is equivariantly pretree-isomorphic to $\mathcal{Y}^*$; through this pretree-isomorphism, we can identify $\delta^\circ_{\iota(1,j)}$ with an extension of $\delta_{\iota(1,j)}$ to an $\mathcal{F}$-invariant convex pseudometric (in fact, metric) on $\mathcal{Y}^*$. Finally, we can lift $\delta_{\iota(1,j)}$ to an $\mathcal{F}$-invariant convex pseudometric on $\mathcal{T}$ since $\mathcal{T}$ is an equivariant blow-up of $\mathcal{Y}^*$.

As j was arbitrary, the $\mathcal{F}$-invariant hierarchy $(\delta_i)_{i=1}^n$ normalizes to the factored $\mathcal{F}$-invariant convex pseudometric $\Sigma_{j=1}^k \delta_{\iota(j)}$, where $k \coloneqq k_1$ and $\iota(j) \coloneqq \iota(1,j)$. We call the associated factored $\mathcal{F}$-forest $(\mathcal{Y}, \Sigma_{j=1}^k \delta_{\iota(j)})$ the complete dominating limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$. This proves the existence part of our main theorem:

Theorem III.10. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism and $\{\mathcal{A}_j^{dom}[\psi]\}_{j=1}^k$ a (possibly empty) subset of ψ_*-orbits of dominating attracting laminations for $[\psi]$.*

Then there is:

1. *a minimal factored $\mathcal{F}$-forest $(\mathcal{Y}, \Sigma_{j=1}^k \delta_j)$ with trivial arc stabilizers;*

2. *a unique ψ-equivariant expanding dilation $f\colon (\mathcal{Y}, \Sigma_{j=1}^k \delta_j) \to (\mathcal{Y}, \Sigma_{j=1}^k \delta_j)$; and*

3. *for $1 \leq j \leq k$, a nontrivial element $x \in \mathcal{F}$ is δ_j-loxodromic if and only if its axis in $\mathbb{R}(\mathcal{F})$ weakly ψ_*-limits to $\mathcal{A}_j^{dom}[\psi]$.* □

Fix a subset $\{\mathcal{A}_j^{dom}[\psi]\}_{j=1}^k$ of ψ_*-orbits of dominating attracting laminations for $[\psi]$; a dominating forest for $[\psi]$ is a factored $\mathcal{F}$-forest satisfying the conclusion of the previous theorem with respect to this subset. Finally, we prove universality:

Theorem III.11. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism and $\{\mathcal{A}_j^{dom}[\psi]\}_{j=1}^k$ a (possibly empty) subset of ψ_*-orbits of dominating attracting laminations for $[\psi]$. Any dominating forest for $[\psi]$ with respect to the given subset has a unique equivariant dilation to any corresponding dominating limit forest for $[\psi]$.*

Proof. Let $(\mathcal{Y}_i, \delta_i)_{i=1}^n$ be a descending sequence of limit forests for $[\psi]$, $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i] \subset \mathbb{R}(\mathcal{G}_i, \mathcal{Z}_i)$ the stable laminations for $(\mathcal{Y}_i, \delta_i)$, $\mathcal{L}^+_i[\psi]$ the closure of $\mathcal{L}^+_{\mathcal{Z}_i}[\psi_i]$ in $\mathbb{R}(\mathcal{F})$, $\{\mathcal{L}^+_{\iota(j)}[\psi]\}_{j=1}^k$ a subset of ψ_*-orbits of dominating attracting laminations, $(\mathcal{Y}^*, \Sigma_{j=1}^k \delta_{\iota(j)})$ the corresponding dominating limit forest for $(\mathcal{Y}_i, \delta_i)_{i=1}^n$, and $(\mathcal{Y}', \Sigma_{j=1}^k \delta'_j)$ a corresponding dominating forest for $[\psi]$. Turn the factored metrics into hierarchies, and consider the pseudoforests $(\mathcal{Y}^*, (\delta_{\iota(j)})_{j=1}^k)$ and $(\mathcal{Y}', (\delta'_j)_{j=1}^k)$. By Theorem III.10(3), $\delta_{\iota(1)}$ and δ'_1 have the same maximal elliptic subgroup system $\mathcal{G}$.

For induction, assume the $\mathcal{G}$-pseudoforests $(\mathcal{Y}^*(\mathcal{G}), (\delta_{\iota(j)})_{j=2}^k)$ and $(\mathcal{Y}'(\mathcal{G}), (\delta'_j)_{j=2}^k)$ are equivariantly homothetic. By uniqueness of the blow-up construction, it is enough to show that the associated $\mathcal{F}$-forests for $\delta_{\iota(1)}$ and δ'_1 (on $\mathcal{Y}^*$ and $\mathcal{Y}'$ respectively) are equivariantly homothetic. So we may assume $k = 1$ for the rest of the proof. If $\iota(1) = 1$, then $(\mathcal{Y}^*, \delta_{\iota(1)})$ and $(\mathcal{Y}', \delta'_1)$ are equivariantly homothetic by Lemma IV.5. Otherwise, $\iota(1) > 1$ and, for induction on complexity, we assume $(\mathcal{Y}^*(\mathcal{G}_2), \delta_{\iota(1)})$ and $(\mathcal{Y}'(\mathcal{G}_2), \delta'_1)$ are equivariantly homothetic. Either: 1) $\mathcal{L}^+_{\iota(1)}[\psi] \subset \mathcal{L}^+_1[\psi]$ and $\lambda_1 < \lambda_{\iota(1)}$ since $\mathcal{L}^+_{\iota(1)}[\psi]$ is dominating; or 2) the lower-support $\underline{\mathrm{supp}}[\psi_1; \mathcal{Z}_1]$ of $\mathcal{L}^+_{\mathcal{Z}_1}[\psi_1]$ is elliptic in $\mathcal{Y}^*$ and $\mathcal{Y}'$ by Theorem III.10(3). The $\mathcal{F}$-forests $(\mathcal{Y}^*, \delta_{\iota(1)})$ and $(\mathcal{Y}', \delta'_1)$ are equivariantly homothetic by Lemmas IV.9 and IV.7 respectively, and we are done. □

Thus the factored $\mathcal{F}$-forest $(\mathcal{Y}, \Sigma_{j=1}^k \delta_{\iota(j)})$ is *the* <u>complete dominating forest</u> for $[\psi]$.

IV Convergence criteria

This chapter adapts then extends Section 7 of Levitt–Lustig's paper [19]; they, in turn, gave complete details for the proof sketched by Bestvina–Feighn–Handel in [2, Lemma 3.4].

IV.1 Proof of Lemma IV.3

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an expanding irreducible train track $\tau\colon \mathcal{T} \to \mathcal{T}$. Let $\lambda := \lambda[\tau]$, $(\mathcal{Y}_\tau, d_\infty)$ be the limit forest for $[\tau]$, $\pi\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}_\tau, d_\infty)$ the constructed equivariant metric PL-map, $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{T})$ the stable lamination for $[\tau]$, and $k \geq 1$ the number of components of $\mathcal{L}^+[\tau]$. Suppose $f\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and $\mathcal{L}^+[\tau]$ is in the canonically embedded subspace $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$.

Claim IV.1 (cf. [19, Lemma 7.1])**.** *There is a sequence $c(f)$ of positive constants c_i indexed by the components $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$ such that*

$$\lim_{m\to\infty} \lambda^{-mk}\delta(f(\tau^{mk}(p)), f(\tau^{mk}(q))) = c_i\, d_\infty(\pi(p), \pi(q))$$

for any leaf segment $[p, q]$ of Λ_i^+.

Any two equivariant PL-maps $f, g\colon (\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ are a bounded δ-distance apart, and $c(f) = c(g)$. So we can define $c(\mathcal{Y}, \delta) := c(f)$; note that $c(\mathcal{Y}, s\,\delta) = s\,c(\mathcal{Y}, \delta)$ for $s > 0$. Without loss of generality, rescale the metric δ so that f is an equivariant metric PL-map.

Proof. Let $\nu^R := \nu^R[\tau]$ (resp. $\nu^L := \nu^L[\tau]$) be the unique positive right (resp. left) eigenvector for the irreducible transition matrix $A := A[\tau]$ whose sum of entries is 1 (resp. dot product $\langle \nu^L, \nu^R\rangle = k$). Suppose $[p, q]$ is a leaf segment (of a component $\Lambda_i^+ \subset \mathcal{L}^+[\tau]$) with endpoints at vertices of $\mathcal{T}$ and let $v := v[p, q]$ be the vector counting the occurrences of $[e]$ in $[p, q]$: $[e]$ is an $\mathcal{F}$-orbit of edges in $\mathcal{T}$; the entries of $v = (v_e)$ are indexed by the $\mathcal{F}$-orbits $[e]$; and v_e is the number of translates of e in $[p, q]$. The train track property gives us $v^{(m)} := v[\tau^m(p), \tau^m(q)] = A^m v$. Then, as $[p, q]$ is a leaf segment, the positive entries of $v^{(mk)}$ are indexed in the same block $\mathcal{B}_i = \mathcal{B}(\Lambda_i^+)$ for all $m \geq 0$. By Perron's theorem, if $[e]$ is in the block $\mathcal{B}_i$, then

$$\lim_{m\to\infty} \frac{v_e^{(mk)}}{\lambda^{mk}\langle \nu^L, v\rangle} = \nu_e^R.$$

For small $\epsilon > 0$, fix $m_\epsilon \gg 1$ such that $\delta_e(m_\epsilon) := \delta(f(\tau^{m_\epsilon k}(p_e)), f(\tau^{m_\epsilon k}(q_e))) > \epsilon^{-1}C[f]$ for every edge $e = [p_e, q_e]$ in $\mathcal{T}$ — we need the assumption $\mathcal{L}^+[\tau] \subset \mathbb{R}(\mathcal{Y}, \delta)$ for this. The interval $[\tau^{(m_\epsilon+m)k}(p), \tau^{(m_\epsilon+m)k}(q)]$ is a union of $v_e^{(mk)}$-many translates of $\tau^{m_\epsilon k}(e)$, as $[e]$ ranges over all the orbits of edges in $\mathcal{T}$. In $\mathcal{Y}$, we get

$$\sum_{[e]\subset\mathcal{T}} v_e^{(mk)}(\delta_e(m_\epsilon) - 2C[f]) \leq \delta(f(\tau^{(m_\epsilon+m)k}(p)), f(\tau^{(m_\epsilon+m)k}(q))) \leq \sum_{[e]\subset\mathcal{T}} v_e^{(mk)}\delta_e(m_\epsilon).$$

Divide by $\lambda^{(m_\epsilon+m)k}d_\infty(\pi(p),\pi(q)) = \lambda^{(m_\epsilon+m)k}\langle \nu^L, v\rangle$, and let $m\to\infty$:

$$(1-2\epsilon)\sum_{[e]\in\mathcal{B}_i}\nu_e^R\,\frac{\delta_e(m_\epsilon)}{\lambda^{m_\epsilon k}} \le \liminf_{m\to\infty}\frac{\delta(f(\tau^{mk}(p)),f(\tau^{mk}(q)))}{\lambda^{mk}d_\infty(\pi(p),\pi(q))}$$
$$\le \limsup_{m\to\infty}\frac{\delta(f(\tau^{mk}(p)),f(\tau^{mk}(q)))}{\lambda^{mk}d_\infty(\pi(p),\pi(q))} \le \sum_{[e]\in\mathcal{B}_i}\nu_e^R\,\frac{\delta_e(m_\epsilon)}{\lambda^{m_\epsilon k}}$$

Since f is a metric map, we have $\lambda^{-m_\epsilon k}\delta_e(m_\epsilon)\le \nu_e^L$. So the lim inf and lim sup above are real, equal, and depend only on the block $\mathcal{B}_i$ for Λ_i^+.

If ϵ is small, then $\epsilon^{-1}C[f] > 2C[f]+L$ for some $L>0$; by bounded cancellation,

$$c_i := \lim_{m\to\infty}\frac{\delta(f(\tau^{mk}(p)),f(\tau^{mk}(q)))}{\lambda^{mk}d_\infty(\pi(p),\pi(q))} \ge \lim_{m\to\infty}\frac{\|v^{(mk)}\|_1 L}{\lambda^{(m_\epsilon+m)k}\langle\nu^L,v\rangle} \ge \frac{\nu_e^R L}{\lambda^{m_\epsilon k}} > 0,$$

where $\|v^{(m)}\|_1$ is the sum of the entries in $v^{(m)}$ and $[e]$ is in the same block as $[p,q]$.

We now relax the restriction that $[p,q]$ is an edge-path, i.e. p,q need not be vertices. For $m\ge 0$, let $[\bar p_m,\bar q_m]$ be the shortest edge-path containing $[\tau^{mk}(p),\tau^{mk}(q)]$; for $m,m'\ge 0$,

$$\frac{\delta(f(\tau^{mk}(\bar p_{m'})),f(\tau^{mk}(\bar q_{m'}))) - \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))+2)} \le \frac{\delta(f(\tau^{(m+m')k}(p)),f(\tau^{(m+m')k}(q)))}{\lambda^{(m+m')k}d_\infty(\pi(p),\pi(q))}$$
$$\le \frac{\delta(f(\tau^{mk}(\bar p_{m'})),f(\tau^{mk}(\bar q_{m'}))) + \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))-2)}.$$

Both upper and lower bounds converge to c_i as $m',m\to\infty$: $[\bar p_{m'},\bar q_{m'}]$ is a leaf segment with endpoints at vertices of $\mathcal{T}$, so

$$\lim_{m'\to\infty}\lim_{m\to\infty}\frac{\delta(f(\tau^{mk}(\bar p_{m'})),f(\tau^{mk}(\bar q_{m'}))) \mp \lambda^{mk}2}{\lambda^{mk}(d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))\pm 2)}$$
$$= \lim_{m'\to\infty}\frac{c_i\, d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))\mp 2}{d_\infty(\pi(\bar p_{m'}),\pi(\bar q_{m'}))\pm 2} = c_i. \qquad \square$$

The next step is extending the claim to all intervals $[p,q]\subset\mathcal{T}$. Set $(c_i)_{i=1}^k := c(\mathcal{Y},\delta)$ and let $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ be the factorization indexed by the components $\Lambda_i^+\subset\mathcal{L}^+[\tau]$. For convenience, replace ψ with its iterate ψ^k, τ with τ^k, and λ with λ^k.

Claim IV.2 (cf. [19, Lemma 7.2])**.** *For any $p_1,p_2\in\mathcal{T}$,*

$$\lim_{m\to\infty}\lambda^{-m}\delta(f(\tau^m(p_1)),f(\tau^m(p_2))) = \sum_{i=1}^k c_i\, d_\infty^{(i)}(\pi(p_1),\pi(p_2)).$$

Proof. Let $[p_1, p_2]$ be an interval in $\mathcal{T}$ and $N(p_1, p_2)$ the number of vertices in (p_1, p_2). Suppose $\pi(p_1) = \pi(p_2)$, i.e. $d_\infty(\pi(p_1), \pi(p_2)) = 0$. Since f is a metric map, we get

$$0 \le \lambda^{-m}\delta(f(\tau^m(p_1)), f(\tau^m(p_2))) \le \lambda^{-m} d_\tau(\tau^m(p_1), \tau^m(p_2)),$$

and the limit of the middle term (as $m \to \infty$) is 0. So we may assume $d_\infty(\pi(p_1), \pi(p_2)) > 0$. For a given $m' \ge 0$, let $[\tau^{m'}(p_1), \tau^{m'}(p_2)]$ be a concatenation of $N' + 1$ leaf segments $[q_j, q_{j+1}]_{j=0}^{N'}$ (of $\Lambda^+_{i(j)} \subset \mathcal{L}^+[\tau]$) for some nonegative $N' \le N(p_1, p_2)$ and $i(j) \in \{1, \ldots, k\}$, where $q_0 = \tau^{m'}(p_1)$ and $q_{N'+1} = \tau^{m'}(p_2)$. Then, by Claim IV.1,

$$\limsup_{m\to\infty} \frac{\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2)))}{\lambda^m} \le \lim_{m\to\infty} \sum_{j=0}^{N'} \frac{\delta(f(\tau^m(q_j)), f(\tau^m(q_{j+1})))}{\lambda^m}$$

$$= \sum_{j=0}^{N'} c_{i(j)} d_\infty(\pi(q_j), \pi(q_{j+1})) = \sum_{i=1}^{k} c_i d_\tau^{(i)}(\tau^{m'}(p_1), \tau^{m'}(p_2)),$$

where the last equality comes from $d_\infty(\pi(q_j), \pi(q_{j+1})) = d_\tau^{(i(j))}(q_j, q_{j+1})$ since $[q_j, q_{j+1}]$ is a leaf segment. Divide by $\lambda^{m'}$, let $m' \to \infty$, and invoke the definition of $d_\infty^{(i)}$ to get

$$\limsup_{m+m'\to\infty} \frac{\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2)))}{\lambda^{m+m'}} \le \sum_{i=1}^{k} c_i d_\infty^{(i)}(\pi(p_1), \pi(p_2)).$$

Using bounded cancellation, we get a lower bound:

$$\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2))) \ge \sum_{j=0}^{N'} \delta(f(\tau^m(q_j)), f(\tau^m(q_{j+1}))) - 2N'C[f],$$

which, after dividing by $\lambda^{m+m'}$ and letting $m \to \infty$ then $m' \to \infty$, leads to

$$\liminf_{m+m'\to\infty} \frac{\delta(f(\tau^{m+m'}(p_1)), f(\tau^{m+m'}(p_2)))}{\lambda^{m+m'}} \ge \sum_{i=1}^{k} c_i d_\infty^{(i)}(\pi(p_1), \pi(p_2)). \qquad \square$$

Like in our construction of limit forests (Section II.1), let δ_m^* be the pullback of $\lambda^{-m}\delta$ via $f \circ \tau^m$ for $m \ge 0$. Then δ_m^* is an $\mathcal{F}$-invariant pseudometric on $\mathcal{T}$ whose associated metric space is equivariantly isometric to $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)$. By Claim IV.2, the (pointwise) limit $\lim_{m\to\infty} \delta_m^*$ is the pullback of $\oplus_{i=1}^k c_i\, d_\infty^{(i)}$ via π. In other words, the sequence $(\mathcal{Y}\psi^m, \lambda^{-m}\delta)_{m\ge0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$ and we are done:

Lemma IV.3 (cf. [2, Lemma 3.4]). *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for ψ, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\tau]$, and $\lambda := \lambda[\tau]$.*

If $(\mathcal{T}, d_\tau) \to (\mathcal{Y}, \delta)$ is an equivariant PL-map and the k-component lamination $\mathcal{L}^+[\tau]$ is in $\mathbb{R}(\mathcal{Y}, \delta) \subset \mathbb{R}(\mathcal{T})$, then the sequence $(\mathcal{Y}\psi^{mk}, \lambda^{-mk}\delta)_{m\ge0}$ converges to $(\mathcal{Y}_\tau, \oplus_{i=1}^k c_i\, d_\infty^{(i)})$, where $d_\infty = \oplus_{i=1}^k d_\infty^{(i)}$ and $c_i > 0$. $\square$

IV.2 Proof of Lemma IV.5

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an invariant proper free factor system $\mathcal{Z}'$ and a descending sequence of irreducible train tracks $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ rel. $\mathcal{Z}'$ with $\lambda \coloneqq \lambda[\tau_n] > 1$. Let $\mathcal{L}^+_{\mathcal{Z}}[\psi] \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$ be the k-component stable laminations for $[\psi]$ rel. $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}^\circ]$, $\mathcal{T}^\circ$ an equivariant blow-up of the free splittings $(\mathcal{T}_i)_{i=1}^n$, $\tau^\circ\colon \mathcal{T}^\circ \to \mathcal{T}^\circ$ a topological representative for $[\psi]$ induced by $[\tau_i]_{i=1}^n$, d° an $\mathcal{F}$-invariant convex metric on $\mathcal{T}^\circ$ that extends d_n on $\mathcal{T}_n$ such that τ° is λ-Lipschitz on $(\mathcal{T}^\circ, d^\circ)$, and $\pi^\circ\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$ the equivariant metric map to a limit forest constructed using τ°-iteration. We denote by d_n again the $\mathcal{F}$-invariant convex pseudometric on $\mathcal{T}^\circ$ that extends d_n on $\mathcal{T}_n$. Recall that the components $\Lambda^+_j \subset \mathcal{L}^+_{\mathcal{Z}}[\psi]$ index the factorizations $d_n = \oplus_{j=1}^k d_n^{(j)}$ and $\delta = \oplus_{j=1}^k \delta_j$. For convenience, set $\mathcal{F}_1 \coloneqq \mathcal{F}$ and $\mathcal{F}_{i+1} \coloneqq \mathcal{F}[\mathcal{T}_i]$, then replace ψ with ψ^k, τ° with $\tau^{\circ k}$, and λ with λ^k.

Suppose $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers, $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic, and $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$. Let $(\mathcal{Y}'_n, \delta')$ be the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{F}_n$ and $f_n\colon (\mathcal{T}_n, d_n) \to (\mathcal{Y}'_n, \delta')$ an equivariant PL-map. Extend f_n to an equivariant PL-map $f\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}', \delta')$. By Claim IV.1, we can set $(c_j)_{j=1}^k \coloneqq c(\mathcal{Y}'_n, \delta') > 0$.

Claim IV.4. *For any $p_1, p_2 \in \mathcal{T}^\circ$,*

$$\lim_{m\to\infty} \lambda^{-m} \delta'(f(\tau^{\circ m}(p_1)), f(\tau^{\circ m}(p_2))) = \sum_{j=1}^k c_j\, \delta_j(\pi^\circ(p_1), \pi^\circ(p_2)).$$

Proof. Let $[p_1, p_2]$ be an interval in $\mathcal{T}^\circ$ and assume $\delta(\pi^\circ(p_1), \pi^\circ(p_2)) > 0$ without loss of generality. Given Claim IV.2, we may assume $n \geq 2$. For $m' \geq 0$, the interval $[\tau^{\circ m'}(p_1), \tau^{\circ m'}(p_2)]$ is a concatenation of $\alpha(m')$ segments that are in $\mathcal{F} \cdot \mathcal{T}_n$ or edges from $\mathcal{T}_i$ $(i > 1)$, where $\alpha(m')$ is bounded by a polynomial in m' of degree $\leq n-2$. Set M to be the length of the longest edge from $\mathcal{T}_i$ $(i > 1)$ in $(\mathcal{T}^\circ, d^\circ)$. For $m' \gg 0$, let $[q_{m',l}, q_{m',l+1}]_{l=0}^{N(m')}$ be the nondegenerate $(\mathcal{F} \cdot \mathcal{T}_n)$-segments. As τ° and f are λ- and L-Lipschitz respectively,

$$\begin{aligned} &\delta'(f(\tau^{\circ(m+m')}(p_1)), f(\tau^{\circ(m+m')}(p_2)) \\ &\quad \leq \sum_{l=0}^{N(m')} \delta'(f_n(\tau_n^m(q_{m',l})), f_n(\tau_n^m(q_{m',l+1}))) + \alpha(m')\lambda^m LM. \end{aligned}$$

Divide by $\lambda^{m+m'}$, let $m \to \infty$ then $m' \to \infty$, and invoke Claim IV.2 and definition of δ_j:

$$\begin{aligned} &\limsup_{m+m'\to\infty} \frac{\delta'(f(\tau^{\circ(m+m')}(p_1)), f(\tau^{\circ m+m'}(p_2)))}{\lambda^{m+m'}} \\ &\leq \lim_{m'\to\infty} \sum_{l=0}^{N(m')} \sum_{j=1}^k \frac{c_j \delta_j(\pi^\circ(q_{m',l}), \pi^\circ(q_{m',l+1}))}{\lambda^{m'}} \\ &\leq \lim_{m'\to\infty} \sum_{j=1}^k \frac{c_j d_n^{(j)}(\tau^{\circ m'}(p_1), \tau^{\circ m'}(p_2))}{\lambda^{m'}} = \sum_{j=1}^k c_j \delta_j(\pi^\circ(p_1), \pi^\circ(p_2)), \end{aligned}$$

using the fact π° is a metric map. The intervals $[\pi^\circ(q_{m',l}), \pi^\circ(q_{m',l+1})]$ contribute at least

$$\lambda^{m'}\delta_j(\pi^\circ(p_1), \pi^\circ(p_2)) - \alpha(m')\,(M + 2C[\pi^\circ])$$

to the δ_j-length of $[\pi^\circ(\tau^{\circ m'}(p_1)), \pi^\circ(\tau^{\circ m'}(p_2))]$. As before, bounded cancellation gives us:

$$\begin{aligned}&\delta'(f(\tau^{\circ(m+m')}(p_1)), f(\tau^{\circ(m+m')}(p_2)))\\ &\quad\geq \sum_{l=0}^{N(m')} \delta'(f_n(\tau_n^m(q_{m',l})), f_n(\tau_n^m(q_{m',l+1}))) - 2\,\alpha(m')C[f].\end{aligned}$$

Divide by $\lambda^{m+m'}$ and let $m \to \infty$ then $m' \to \infty$ yields:

$$\begin{aligned}&\liminf_{m+m'\to\infty} \frac{\delta'(f(\tau^{\circ(m+m')}(p_1)), f(\tau^{\circ m+m'}(p_2)))}{\lambda^{m+m'}}\\ &\geq \lim_{m'\to\infty} \sum_{j=1}^{k} c_j \sum_{l=0}^{N(m')} \frac{\delta_j(\pi^\circ(q_{m',l}), \pi^\circ(q_{m',l+1}))}{\lambda^{m'}} \geq \sum_{j=1}^{k} c_j\delta_j(\pi^\circ(p_1), \pi^\circ(p_2)),\end{aligned}$$

where the last inequality comes from the contribution inequality above. □

The rest of the argument is the same as in the previous section. Let δ_m^* be pullback of $\lambda^{-m}\delta'$ via $f \circ \tau^{\circ m}$ for $m \geq 0$. By Claim IV.4, the limit $\lim_{m\to\infty} \delta_m^*$ is the pullback of $\oplus_{j=1}^k c_j\,\delta_j$ via π° and we are done:

Lemma IV.5. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\mathcal{Z}'$ a $[\psi]$-invariant proper free factor system, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$ rel. $\mathcal{Z}'$ with $\lambda \coloneqq \lambda[\tau_n] > 1$, $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$, $(\mathcal{Y}', \delta')$ a minimal $\mathcal{F}$-forest with trivial arc stabilizers, and $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$.*

If $\mathcal{Z}$ is $\mathcal{Y}'$-elliptic and the k-component lamination $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ is in $\mathbb{R}(\mathcal{Y}', \delta') \subset \mathbb{R}(\mathcal{F}, \mathcal{Z})$, then the limit of $(\mathcal{Y}'\psi^{mk}, \lambda^{-mk}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \oplus_{j=1}^k c_j\,\delta_j)$, where $\delta = \oplus_{j=1}^k \delta_j$ and $c_j > 0$. □

IV.3 Sketch of Lemma IV.7

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$. Let $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks for $[\psi]$, $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$, $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ the stable lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$, $(\mathcal{Y}_1, \delta_1)$ the limit forest for $[\psi]$ rel. $\mathcal{Z}$, $\mathcal{G} \coloneqq \mathcal{G}[\mathcal{Y}_1]$, $[\psi_{\mathcal{G}}]$ the restriction of $[\psi]$ to $\mathcal{G}$, $(\mathcal{Y}_{\mathcal{G}}, \delta)$ a minimal $\mathcal{G}$-forest with trivial arc stabilizers, and $h_{\mathcal{G}}\colon (\mathcal{Y}_{\mathcal{G}}, \delta) \to (\mathcal{Y}_{\mathcal{G}}, \delta)$ a $\psi_{\mathcal{G}}$-equivariant λ-homothety. Construct the equivariant psuedoforest blow-up $(\mathcal{Y}_1^*, (\delta_1, \delta))$ of $(\mathcal{Y}_1, \delta_1)$ rel. $(\mathcal{Y}_{\mathcal{G}}, \delta)$ and expanding homotheties representing $[\psi]$ and $[\psi_{\mathcal{G}}]$. For this section, we will assume assume $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ and δ are independent: the pseudoleaf segments for $\mathcal{L}_{\mathcal{Z}}^+[\psi]$ in $\mathcal{Y}_1^*$ have 0 δ-diameter intersections with $\mathcal{Y}_{\mathcal{G}}$. Set $\mathcal{F}_n \coloneqq \mathcal{F}[\mathcal{T}_{n-1}]$ and $[\psi_n]$ to be the restriction of $[\psi]$

to $\mathcal{F}_n$; the characteristic convex subset $\mathcal{Y}_1^*(\mathcal{F}_n) \subset \mathcal{Y}_1^*$ has a graph of actions decomposition with vertex forests $\widehat{\mathcal{Y}}_{\mathcal{G}}$ and the overlapping classes for $\mathcal{L}^+_{\mathcal{Z}}[\psi]$.

Let the minimal simplicial $\mathcal{F}_n$-forest $\mathcal{S}$ be the skeleton for the graph of actions for $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and δ. By construction, there is a ψ_n-equivariant simplicial automorphism $\sigma\colon \mathcal{S} \to \mathcal{S}$. The lower-support $\underline{\mathrm{supp}}[\psi;\mathcal{Z}]$ of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ is given by stabilizers of vertices in $\mathcal{S}$ corresponding to overlapping classes. Construct the equivariant blow-up $\mathcal{T}^\diamond$ of $(\mathcal{T}_i)$, $\mathcal{S}$, and $\mathcal{Y}_{\mathcal{G}}$; then extend the metric δ to an $\mathcal{F}$-invariant convex metric $d \oplus \delta$ on $\mathcal{T}^\diamond$ so that the ψ-equivariant map $\tau_c^\diamond\colon (\mathcal{T}^\diamond, (c\,d) \oplus \delta) \to (\mathcal{T}^\diamond, (c\,d) \oplus \delta)$ induced by $[\tau_i]_{i=1}^{n-1}$, σ and linearly extending $h_{\mathcal{G}}$ is λ-Lipschitz for any parameter $c \gg 1$. Let $d_c^\diamond \coloneqq (c\,d) \oplus \delta$; for $c \gg 1$, construct using $\tau_c^\diamond$-iteration an equivariant metric surjection $\pi_c^\diamond\colon (\mathcal{T}^\diamond, d_c^\diamond) \to (\mathcal{X}, \delta)$ that extends the identification of $(\mathcal{Y}_{\mathcal{G}}, \delta)$ and semiconjugates $\tau_c^\diamond$ to a ψ-equivariant λ-homothety on $(\mathcal{X}, \delta)$.

Suppose $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers and whose characteristic subforest for $\mathcal{G}$ is equivariantly isometric to $(\mathcal{Y}_{\mathcal{G}}, \delta)$. So if we also assume $\underline{\mathrm{supp}}[\psi;\mathcal{Z}]$ is $\mathcal{Y}'$-elliptic, then there is an equivariant map $f_c\colon (\mathcal{T}^\diamond, d_c^\diamond) \to (\mathcal{Y}', \delta')$ that linearly extends the identification of $(\mathcal{Y}_{\mathcal{G}}, \delta)$; this is necessarily a Lipschitz map. Pick any free splitting $\mathcal{T}$ of $\mathcal{F}$ with trivial $\mathcal{F}[\mathcal{T}]$. Then any equivariant PL-map $\mathcal{T} \to \mathcal{T}^\diamond$ is surjective (by minimality) and composes with f_c to give (up to an equivariant homotopy rel. the vertices) an equivariant PL-map with a cancellation constant. So f_c must have a cancellation constant. The proof of the next claim is a variation of Claim IV.4's proof:

Claim IV.6. *For any $p_1, p_2 \in \mathcal{T}^\diamond$,*

$$\lim_{m\to\infty} \lambda^{-m} \delta'(f_c(\tau_c^{\diamond m}(p_1)), f_c(\tau_c^{\diamond m}(p_2))) = \delta(\pi_c^\diamond(p_1), \pi_c^\diamond(p_2)).$$

Sketch of proof. For $m' \geq 0$, the interval $[\tau_c^{\circ m'}(p_1), \tau_c^{\circ m'}(p_2)]$ is a concatenation of $\alpha(m')$ segments that are in the orbit of $\mathcal{Y}_{\mathcal{G}}$ or edges from $\mathcal{T}_i$ ($i \geq 1$), where $\alpha(m')$ is bounded by a polynomial in m' of degree $\leq n-1$. With an almost identical argument, invoke the definition of π_c° to conclude

$$\lim_{m+m'\to\infty} \frac{\delta'(f_c(\tau_c^{\circ(m+m')}(p_1)), f_c(\tau_c^{\circ m+m'}(p_2)))}{\lambda^{m+m'}} = \delta(\pi_c^\circ(p_1), \pi_c^\circ(p_2)).$$

The set-up is simpler as τ_c° (resp. f_c) is a λ-homothety (resp. isometry) on $(\mathcal{Y}_{\mathcal{Z}}, \delta)$. □

As in the previous section, we have proven the following:

Lemma IV.7. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$, $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}_n]$, $\mathcal{G}$ the nontrivial point stabilizer system for the limit forest for $[\psi]$ rel. $\mathcal{Z}$, $[\psi_{\mathcal{G}}]$ the $[\psi]$-restriction to $\mathcal{G}$, $(\mathcal{Y}_{\mathcal{G}}, \delta)$ a minimal $\mathcal{G}$-forest with trivial arc stabilizers such that $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and δ are independent, $h_{\mathcal{G}}\colon (\mathcal{Y}_{\mathcal{G}}, \delta) \to (\mathcal{Y}_{\mathcal{G}}, \delta)$ a $\psi_{\mathcal{G}}$-equivariant λ-homothety, $\mathcal{S}$ a minimal simplicial $\mathcal{F}[\mathcal{T}_{n-1}]$-forest that is the skeleton for the graph of actions for $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and δ, $\sigma\colon \mathcal{S} \to \mathcal{S}$ the corresponding simplicial automorphism, and $(\mathcal{X}, \delta)$ the limit forest for $[\tau_i]_{i=1}^{n-1}$, σ, and $h_{\mathcal{G}}$.*

If $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers, the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{G}$ is equivariantly isometric to $(\mathcal{Y}_{\mathcal{G}}, \delta)$, and the lower-support $\underline{\mathrm{supp}}[\psi; \mathcal{Z}]$ of $\mathcal{L}^{+}_{\mathcal{Z}}[\psi]$ is $\mathcal{Y}'$-elliptic, then the limit of $(\mathcal{Y}'\psi^m, \lambda^{-m}\delta')_{m\geq 0}$ is $(\mathcal{X}, \delta)$. $\square$

IV.4 Sketch of Lemma IV.9

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an invariant proper free factor system $\mathcal{Z}$ and a minimal $\mathcal{Z}$-forest $(\mathcal{Y}_{\mathcal{Z}}, \delta)$ with trivial arc stabilizers. Let $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ be a descending sequence of irreducible train tracks for $[\psi]$ with $\mathcal{F}[\mathcal{T}_n] = \mathcal{Z}$, d_n the eigenmetric on $\mathcal{T}_n$ for $[\tau_n]$, and $h_{\mathcal{Z}}\colon (\mathcal{Y}_{\mathcal{Z}}, \delta) \to (\mathcal{Y}_{\mathcal{Z}}, \delta)$ a $\psi_{\mathcal{Z}}$-equivariant λ-homothety, where $\lambda > \lambda[\tau_n]$ and $[\psi_{\mathcal{Z}}]$ is the $[\psi]$-restriction to $\mathcal{Z}$. Set $\mathcal{F}_1 \coloneqq \mathcal{F}$ and $\mathcal{F}_{i+1} \coloneqq \mathcal{F}[\mathcal{T}_i]$.

Choose an arbitrary equivariant iterated blow-up $\mathcal{T}^*$ of $(\mathcal{T}_i)_{i=1}^n$ and let $\tau^*\colon \mathcal{T}^* \to \mathcal{T}^*$ be the ψ-equivariant topological representative induced by $(\tau_i)_{i=1}^n$. Extend the metric d_n on $\mathcal{T}_n$ to an $\mathcal{F}$-invariant convex metric d^* on $\mathcal{T}^*$ so that $\tau^*\colon (\mathcal{T}^*, d^*) \to (\mathcal{T}^*, d^*)$ is $\lambda[\tau_n]$-Lipschitz. Finally, choose an arbitrary equivariant metric blow-up $(\mathcal{T}^\circ, d^* \oplus \delta)$ of $(\mathcal{T}^*, d^*)$ rel. $(\mathcal{Y}_{\mathcal{Z}}, \delta)$. For a parameter $c > 0$, the topological representative τ^* induces a ψ-equivariant map τ_c° on $\mathcal{T}^\circ$ that linearly extends the λ-homothety $h_{\mathcal{Z}}$ with respect to the metric $d_c^\circ \coloneqq (c\, d^*) \oplus \delta$. As $\lambda > \lambda[\tau_n]$, the map τ_c° is λ-Lipschitz with respect to d_c° for $c \gg 1$. Let $(\mathcal{Y}, \delta)$ be the limit forest for $[\tau_c^\circ]$ and $\pi_c^\circ\colon (\mathcal{T}^\circ, d_c^\circ) \to (\mathcal{Y}, \delta)$ the equivariant metric surjection constructed through τ°-iteration.

Suppose $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers and whose characteristic subforest for $\mathcal{Z}$ is equivariantly isometric to $(\mathcal{Y}_{\mathcal{Z}}, \delta)$. Let $f_c\colon (\mathcal{T}^\circ, d_c^\circ) \to (\mathcal{Y}', \delta')$ be an equivariant map that linearly extends the identification of $(\mathcal{Y}_{\mathcal{Z}}, \delta)$.

Claim IV.8. *For any $p_1, p_2 \in \mathcal{T}^\circ$,*

$$\lim_{m\to\infty} \lambda^{-m}\delta'(f_c(\tau_c^{\circ m}(p_1)), f_c(\tau_c^{\circ m}(p_2))) = \delta(\pi_c^\circ(p_1), \pi_c^\circ(p_2)).$$

Sketch of proof. For $m' \geq 0$, the interval $[\tau_c^{\circ m'}(p_1), \tau_c^{\circ m'}(p_2)]$ is a concatenation of $\beta(m')$ segments that are in the orbit of $\mathcal{Y}_{\mathcal{Z}}$ or edges from $\mathcal{T}_i$ $(i \geq 1)$, where $\beta(m')$ has exponential growth rate $\lambda[\tau_n] < \lambda$. Proceed just as in the proof of Claim IV.6. $\square$

Altogether, we have proven the following:

Lemma IV.9. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\mathcal{Z}$ a $[\psi]$-invariant proper free factor system, $(\mathcal{Y}_{\mathcal{Z}}, \delta)$ a minimal $\mathcal{Z}$-forest with trivial arc stabilizers, $(\tau_i\colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$ a descending sequence of irreducible train tracks for $[\psi]$ with $\mathcal{F}[\mathcal{T}_n] = \mathcal{Z}$, $h_{\mathcal{Z}}\colon (\mathcal{Y}_{\mathcal{Z}}, \delta) \to (\mathcal{Y}_{\mathcal{Z}}, \delta)$ a $\psi_{\mathcal{Z}}$-equivariant λ-homothety, and $(\mathcal{Y}, \delta)$ the limit forest for $[\tau_i]_{i=1}^n$ and $h_{\mathcal{Z}}$, where $\lambda > \lambda[\tau_n]$ and $[\psi_{\mathcal{Z}}]$ is the $[\psi]$-restriction to $\mathcal{Z}$.*

If $(\mathcal{Y}', \delta')$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers and the characteristic subforest of $(\mathcal{Y}', \delta')$ for $\mathcal{Z}$ is equivariantly isometric to $(\mathcal{Y}_{\mathcal{Z}}, \delta)$, then the limit of $(\mathcal{Y}'\psi^m, \lambda^{-m}\delta')_{m\geq 0}$ is $(\mathcal{Y}, \delta)$. $\square$

V Expanding forests

We finally characterize the <u>expanding forests</u> for an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$, i.e. minimal very small $\mathcal{F}$-forests that admit ψ-equivariant expanding homotheties. By the last paragraph in the proof of Corollary II.11, expanding forests have trivial arc stabilizers. We start with a criterion of nonconvergence that complements Lemmas IV.7 and IV.9.

V.1 Nonconvergence criterion

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ with an invariant proper free factor system $\mathcal{Z}$ and a minimal $\mathcal{Z}$-forest $(\mathcal{Y}_{\mathcal{Z}}, \delta)$ with trivial arc stabilizers. Let $\tau\colon \mathcal{T} \to \mathcal{T}$ be an expanding irreducible train track for $[\psi]$ with $\mathcal{F}[\mathcal{T}] = \mathcal{Z}$, d_τ the eigenmetric on $\mathcal{T}$ for $[\tau]$, $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ the stable lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\psi]$ rel. $\mathcal{Z}$, and $h_{\mathcal{Z}}\colon (\mathcal{Y}_{\mathcal{Z}}, \delta) \to (\mathcal{Y}_{\mathcal{Z}}, \delta)$ a $\psi_{\mathcal{Z}}$-equivariant λ-homothety, where $1 < \lambda \le \lambda[\tau]$ and $[\psi_{\mathcal{Z}}]$ is the restriction of $[\psi]$ to $\mathcal{Z}$.

Set $\mathcal{G} \coloneqq \mathcal{G}[\mathcal{Y}_\tau]$, and denote the restriction of $[\psi]$ to $\mathcal{G}$ by $[\psi_{\mathcal{G}}]$. Since $[\psi_{\mathcal{G}}]$ is polynomially growing rel. $\mathcal{Z}$, we can equivariantly include $(\mathcal{Y}_{\mathcal{Z}}, \delta)$ in a minimal $\mathcal{G}$-forest $(\mathcal{Y}_{\mathcal{G}}, \delta)$ with trivial arc stabilizers and extend $h_{\mathcal{Z}}$ to a $\psi_{\mathcal{G}}$-equivariant λ-homothety $h_{\mathcal{G}}\colon (\mathcal{Y}_{\mathcal{G}}, \delta) \to (\mathcal{Y}_{\mathcal{G}}, \delta)$. Construct the equivariant psuedoforest blow-up $(\mathcal{Y}^*, (d_\infty, \delta))$ of $(\mathcal{Y}_\tau, d_\infty)$ rel. $(\mathcal{Y}_{\mathcal{G}}, \delta)$ and the expanding homotheties representing $[\psi]$ and $[\psi_{\mathcal{G}}]$. Finally, suppose $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and δ are *dependent*, i.e. the pseudoleaf segments for $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ in $\mathcal{Y}^*$ have some positive δ-diameter intersections with $\mathcal{Y}_{\mathcal{G}}$. We are essentially in the case not covered by Lemmas IV.7 and IV.9.

Choose an iterate $[\tau^k]$ that maps all $\mathcal{F}$-orbits of branches in $\mathcal{T}$ to $[\tau^k]$-fixed orbits. Pick a branch e^+ in $\mathcal{T}$; suppose its basepoint $p \in \mathcal{T}$ is a vertex with a nontrivial stabilizer. Without loss of generality, assume $\tau^k(e^+) = e^+$. Use the contraction mapping theorem to decide how to equivariantly attach $\tau^k(e^+)$ to $\mathcal{F} \cdot \mathcal{Y}_{\mathcal{Z}}$; then equivariantly attach e^+ to the same point. Now suppose the basepoint p has a trivial stabilizer but $\tau^k(p)$ has a nontrivial one. Then there are finitely many directions $e_1^+, \ldots, e_l^+$ at p. We have described how to attach their images $\tau^k(e_1), \ldots, \tau^k(e_l)$ to the $\mathcal{F}$-orbit of $\mathcal{Y}_{\mathcal{Z}}$; let $C_p \subset \mathcal{F} \cdot \mathcal{Y}_{\mathcal{Z}}$ be the *convex hull* of these attaching points. Equivariantly replace $p \in \mathcal{T}$ with a copy of $(C_p, \lambda^{-k}\delta)$ and attach e_j^+ to the copy of the attaching point for its τ^k-image. Finally, if $\tau^k(p)$ has a trivial stabilizer, then there is nothing to do. As $[e^+]$ ranges over all $\mathcal{F}$-orbits of branches in $\mathcal{T}$, this defines a preferred equivariant metric blow-up $(\mathcal{T}^\circ, d_\tau \oplus \delta)$ of $(\mathcal{T}, d_\tau)$ rel. $(\mathcal{Y}_{\mathcal{Z}}, \delta)$. The train track τ induces a ψ-equivariant map $\tau^\circ\colon (\mathcal{T}^\circ, d_\tau \oplus \delta) \to (\mathcal{T}^\circ, d_\tau \oplus \delta)$ that linearly extends the homothety $h_{\mathcal{Z}}$. The preferred construction guarantees τ° is a train track in a sense: $\tau^{\circ m}$ is injective on the edges from $\mathcal{T}$ for all $m \ge 1$.

Suppose $(\mathcal{Y}, \delta)$ is a minimal $\mathcal{F}$-forest with trivial arc stabilizers and whose characteristic subforest for $\mathcal{Z}$ is equivariantly isometric to $(\mathcal{Y}_{\mathcal{Z}}, \delta)$.

Claim V.1. *For some element x in $\mathcal{F}$, $\lambda^{-m}\|\psi^m(x)\|_\delta \to \infty$ as $m \to \infty$.*

Sketch of proof. A long leaf segment in $\mathcal{T}$ contains at least three (unoriented) translates

$x_i \cdot e$ $(1 \leq i \leq 3)$ of an edge e. So $x \coloneqq x_i^{-1} x_{i+1}$ is $\mathcal{T}$-loxodromic for some $i \pmod 3$. Choose a fundamental domain $[p,q]$ of x acting on its axis that is a leaf segment with endpoints at vertices. Set $d^\circ \coloneqq d_\tau \oplus \delta$, and let $f\colon (\mathcal{T}^\circ, d^\circ) \to (\mathcal{Y}, \delta)$ an equivariant map that linearly extends the identification of $(\mathcal{Y}_\mathcal{Z}, \delta)$.

The assumption that $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ were dependent implies the τ°-image of some edge e from $\mathcal{T}$ has a nondegenerate intersection with $\mathcal{F} \cdot \mathcal{Y}_\mathcal{Z}$. Fix $m' \gg 1$ so that, for some $L > 0$, $\tau^{\circ m'}(e) \cap \mathcal{F} \cdot \mathcal{Y}_\mathcal{Z}$ has a component with δ-length $\geq 2C[f] + L$ for all edges from $\mathcal{T}$. For $m \geq 0$, let $\beta(m)$ be the number of edges from $\mathcal{T}$ in $[\tau^{\circ m}(p), \tau^{\circ m}(q)]$; note that $\beta(m)$ grows exponentially in m with rate $\lambda[\tau]$ — the growth of $[\psi]$ rel. $\mathcal{Z}$. By the train track property of τ° and bounded cancellation for f, $\lambda^{-m}\|\psi^{m+m'}(x)\|_\delta \geq \sum_{i=0}^m \beta(i)\lambda^{-i}L$ tends to infinity as $m \to \infty$ since $\lambda \leq \lambda[\tau]$. $\square$

Thus there is no ψ-equivariant homothety of $(\mathcal{Y}, \delta)$:

Lemma V.2. *Let $\psi\colon \mathcal{F} \to \mathcal{F}$ be an automorphism, $\tau\colon \mathcal{T} \to \mathcal{T}$ an expanding irreducible train track for $[\psi]$, $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}]$, and $(\mathcal{Y}, \delta)$ an expanding forest for $[\psi]$ with stretch factor λ. If $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ are dependent, then $\lambda > \lambda[\tau]$.* $\square$

V.2 Expanding is dominating

Fix an automorphism $\psi\colon \mathcal{F} \to \mathcal{F}$ and an expanding forest $(\mathcal{Y}, \delta)$ for $[\psi]$. Our remaining goal is to generalize Corollary II.11: $(\mathcal{Y}, \delta)$ must be some dominating forest for $[\psi]$.

Let $\tau\colon \mathcal{T} \to \mathcal{T}$ be an expanding irreducible train track for $[\psi]$, $\mathcal{Z} \coloneqq \mathcal{F}[\mathcal{T}]$, $\mathcal{L}^+_\mathcal{Z}[\psi]$ the stable lamination for $[\psi]$ in $\mathbb{R}(\mathcal{F}, \mathcal{Z})$, $(\mathcal{Y}_\tau, d_\infty)$ the limit forest for $[\psi]$ rel. $\mathcal{Z}$, and $\mathcal{G} \coloneqq \mathcal{G}[\mathcal{Y}_\tau]$.

For induction, assume the characteristic subforest of $(\mathcal{Y}, \delta)$ for $\mathcal{G}$ is equivariantly isometric to the dominating forest for the restriction $[\psi_\mathcal{G}]$ (of $[\psi]$ to $\mathcal{G}$) with respect to some orbits $\{\mathcal{A}_i^{dom}[\psi_\mathcal{G}]\}_{i=1}^k$ with the same stretch factor $\lambda > 1$; denote the subforest by $(\mathcal{Y}_\mathcal{G}, \delta)$. Suppose $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ are dependent. By Lemma V.2, $\lambda > \lambda[\tau]$ and each $\mathcal{A}_i^{dom}[\psi_\mathcal{G}]$ is actually a ψ_*-orbit $\mathcal{A}_i^{dom}[\psi]$ of dominating attracting laminations for $[\psi]$. By Lemma IV.9, $(\mathcal{Y}, \delta)$ is equivariantly isometric to the dominating forest for $[\psi]$ with respect to $\{\mathcal{A}_i^{dom}[\psi]\}_{i=1}^k$.

We may now assume $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ are independent. So $\mathcal{A}_i^{dom}[\psi_\mathcal{G}]$ is a ψ_*-orbit $\mathcal{A}_i^{dom}[\psi]$ of dominating attracting laminations for $[\psi]$. Let $\mathcal{A}_0^{dom}[\psi] \subset \mathbb{R}(\mathcal{F})$ be the closure of $\mathcal{L}^+_\mathcal{Z}[\psi]$ and $(\mathcal{Y}^*, d_\infty \oplus \delta)$ the unique equivariant metric blow-up of $(\mathcal{Y}_\tau, d_\infty)$ rel. $(\mathcal{Y}_\mathcal{G}, \delta)$ that admits a ψ-equivariant expanding dilation. By construction, the blow-up is equivariantly isometric to the dominating forest for $[\psi]$ with respect to $\{\mathcal{A}_i^{dom}[\psi]\}_{i=0}^k$. Recall that independence of $\mathcal{L}^+_\mathcal{Z}[\psi]$ and δ implies $\mathcal{Y}^*$ is a graph of actions with vertex forests coming from $\widehat{\mathcal{Y}}_\mathcal{G}$ and overlapping classes for $\mathcal{L}^+_\mathcal{Z}[\psi]$ — these are $\mathcal{G}$- and $\underline{\mathrm{supp}}[\psi;\mathcal{Z}]$-forests respectively; let $\mathcal{S}$ be the skeleton for this graph of actions.

If the lower-support $\underline{\mathrm{supp}}[\psi;\mathcal{Z}]$ is $\mathcal{Y}$-elliptic, then $(\mathcal{Y}, \delta)$ is equivariantly isometric to the associated $\mathcal{F}$-forest for δ on $\mathcal{Y}$ by Lemma IV.7; in particular, $(\mathcal{Y}, \delta)$ is equivariantly isometric to the dominating forest for $[\psi]$ with respect to $\{\mathcal{A}_i^{dom}[\psi]\}_{i=1}^k$. Otherwise, $\underline{\mathrm{supp}}[\psi;\mathcal{Z}]$ is not $\mathcal{Y}$-elliptic. Let $\mathcal{T}' \subset \mathcal{T}$ be the characteristic convex subset for the upper-support $\overline{\mathrm{supp}}\,\mathcal{L}^+[\psi]$

of $\mathcal{L}^+[\psi]$ (defined at the end of Section III.4) and $[\psi']$ the restriction of $[\psi]$ to the upper-support. Independence of $\mathcal{L}^+_{\mathcal{Z}}[\psi]$ and δ implies $\mathcal{Z}' := \mathcal{F}[\mathcal{T}']$ is $\mathcal{Y}$-elliptic. So the characteristic subforests of $(\mathcal{Y}, \delta)$ and $(\mathcal{Y}_\tau, d_\infty)$ for the upper-support $\overline{\operatorname{supp}}\, \mathcal{L}^+[\psi]$ are expanding forests for $[\psi']$ rel. $\mathcal{Z}'$; by Corollary II.11, they are equivariantly homothetic and $\lambda[\tau] = \lambda$. Thus the characteristic subforests of $(\mathcal{Y}, \delta)$ and $(\mathcal{Y}_\tau, c\, d_\infty)$ for $\underline{\operatorname{supp}}[\psi; \mathcal{Z}]$ are equivariantly isometric for some $c > 0$. A minor modification of Lemma IV.7 implies $(\mathcal{Y}, \delta)$ is equivariantly isometric to $(\mathcal{Y}^*, c\, d_\infty \oplus \delta)$ — details are left to the reader; therefore, $(\mathcal{Y}, \delta)$ is equivariantly isometric to the dominating forest for $[\psi]$ with respect to $\{\mathcal{A}_i^{dom}[\psi]\}_{i=0}^k$.

Generally, $[\psi]$ has a descending sequence of irreducible train tracks $(\tau_i \colon \mathcal{T}_i \to \mathcal{T}_i)_{i=1}^n$. If $(\mathcal{Y}, \delta)$ is degenerate, then there is nothing to show. Otherwise, the ψ-expanding homothety on $(\mathcal{Y}, \delta)$ implies $\lambda[\tau_n] > 1$. Set $\mathcal{F}_1 := \mathcal{F}$ and $\mathcal{F}_{i+1} := \mathcal{F}[\mathcal{T}_i]$. The preceding discussion proves that the characteristic subforest of $(\mathcal{Y}, \delta)$ for $\mathcal{F}_n$ is equivariantly isometric to some dominating forest for the restriction $[\psi_n]$. Lemma IV.9 implies $(\mathcal{Y}, \delta)$ is equivariantly isometric to some dominating forest for $[\psi]$. Conversely, it follows from Theorem III.10(2) that the dominating forest for $[\psi]$ with respect to a subset of ψ_*-orbits of dominating attracting laminations with the same stretch factor is an expanding forest for $[\psi]$:

Theorem V.3. *An $\mathcal{F}$-forest $(\mathcal{Y}, \delta)$ is an expanding forest for an automorphism $\psi \colon \mathcal{F} \to \mathcal{F}$ if and only if it is equivariantly isometric to the dominating forest for $[\psi]$ with respect to a subset of ψ_*-orbits of dominating attracting laminations with the same stretch factor.* □

A Recognizing and centralizing atoroidal automorphisms

For a given outer automorphism, restrict it to point stabilizers of a complete topmost tree and inductively construct the descending sequence of complete topmost forests. The blow-up construction applied to this descending sequence produces the universal topmost pseudotree (whose underlying pretree is the limit pretree). For an application of this universal construction, we prove a recognition theorem for atoroidal outer automorphisms.

Corollary A.1. *If $[\phi]$ and $[\psi]$ are atoroidal outer automorphisms of F with the same universal topmost pseudotree, and the pseudotree admits a $\phi\psi^{-1}$-equivariant isometry fixing each orbit of branches, then $[\phi] = [\psi]$.*

The hypothesis is akin to assuming two pseudo-Anosov mapping classes have the same stable measured foliation, stretch factor, and action on singular leaves.

Proof. Let $(T, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^n)$ be the universal topmost pseudotree for $[\phi], [\psi]$ and denoted by ι the $\phi\psi^{-1}$-equivariant isometry on $(T, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^n)$ that fixes each orbit of branches. Choose $\phi' \in [\phi]$ such that the $\phi'\psi^{-1}$-equivariant isometry ι' on $(T, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^n)$ fixes a branch point. The F-action on the limit pretree T is free since $[\phi]$ is atoroidal. Adapting Kapovich–Lustig's Proposition 4.1 in [17] to pseudotrees, we conclude ι' fixes all points of T, i.e. ι' is the identity map on T and $\phi' = \psi$; therefore, $[\phi] = [\psi]$. □

We call this a recognition theorem because it lists a set of dynamical invariants (universal topmost pseudotree, stretch factors of the factored pseudometrics, and action on orbits of branches) that determine an atoroidal outer automorphism. Feighn–Handel's recognition theorem [9, Theorem 5.3] gives related dynamical invariants (attracting laminations, their stretch factors, non-repelling fixed points at infinity, and twist coordinates) that determine a forward rotationless outer automorphisms; their theorem can also be extended to all atoroidal outer automorphisms as in our corollary.

A minor change introducing twist coordinates extends our corollary (or Feighn–Handel's recognition theorem) to outer automorphisms whose limit pretrees have cyclic point stabilizers. With more care, the corollary should generalize to outer automorphisms whose restrictions to the point stabilizers of limit pretrees is linearly growing — linearly growing automorphisms have canonical representatives [5]. Generalizing to all outer automorphisms would require having canonical nondegenerate representatives for all polynomially growing automorphisms.

Corollary A.2. *If $\phi\colon F \to F$ is an atoroidal automorphism, then the centralizer of $[\phi]$ in the outer automorphism group $\mathrm{Out}(F)$ is virtually a free abelian group with rank at most the number of $[\phi]$-orbits of attracting laminations for $[\phi]$.*

Feighn–Handel do not explicitly state this corollary, but it follows from [8, Lemma 5.5]. Bestvina–Feighn–Handel previously proved that centralizers of fully irreducible outer automorphisms are virtually cyclic [2, Theorem 2.14]. In the first version of this paper, we claimed Corollary A.2 as a new result, and a referee told us the corollary follows from Feighn–Handel's work on CT maps. Our new proof uses the universal topmost pseudotree.

Proof. Let $(T, (\oplus_{j=1}^{k_i} \delta_{i,j})_{i=1}^{n})$ be the universal topmost pseudotree for $[\phi]$, $C[\phi]$ the centralizer for $[\phi]$ in $\mathrm{Out}(F)$, and $k \coloneqq \sum_{i=1}^{n} k_i$. Replace $C[\phi]$ with a finite index subgroup and assume it acts trivially on the attracting laminations for $[\phi]$. If $[\phi'] \in C[\phi]$, then the universal pseudotree supports a ϕ'-equivariant dilation by uniqueness of the pseudotree for $[\phi]$. Thus we can define a group homomorphism $\ell\colon C[\phi] \to \mathbb{R}^k_{>0}$ that maps $[\phi']$ to $(\lambda'_{i,j} : 1 \le i \le n, 1 \le j \le k_i)$. The image of $C[\phi]$ under each coordinate projection $\ell_{i,j}$ of ℓ is a cyclic subgroup of $\mathbb{R}_{>0}$ by Corollary II.12.

By index theory, we can replace $C[\phi]$ with a finite index subgroup again and assume it fixes the orbits of branches in T. As the F-action on T is free, the kernel $\ker(\ell)$ is trivial — see Proposition 4.2 in [17]. So $C[\phi]$ is free abelian with rank $\le k$. $\square$

Again, the corollary can be adapted to work for outer automorphisms whose limit pretrees have cyclic point stabilizers. Yassine Guerch recently gave another proof of this more general statement using different methods [13, Theorem 5.3]. With more care, our work or Feighn–Handel's can combine with Andrew–Martino's paper [1, Theorem 1.5] to characterize the centralizer of an outer automorphism whose restriction to point stabilizers of limit pretrees is linearly growing.

We think it is open whether arbitrary centralizers are finitely generated. For a complete description of arbitrary centralizers, one needs canonical nondegenerate representatives for polynomially growing automorphisms. Presumably, a polynomially growing automorphism of degree $d \geq 2$ has a canonical fixed free splitting whose loxodromics are exactly the elements that grow with degree d.

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