

LAGRANGIAN FIBRATION STRUCTURE ON THE COTANGENT BUNDLE OF A DEL PEZZO SURFACE OF DEGREE 4

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ABSTRACT. In this paper, we show that there is a natural Lagrangian fibration structure on the map Φ from the cotangent bundle of a del Pezzo surface X of degree 4 to \mathbb{C}^2 . Moreover, we describe explicitly all level surfaces of the above natural map Φ .

1. INTRODUCTION

Throughout this paper we will work over the field of complex numbers.

The cotangent bundle of a complex projective manifold carries a natural holomorphic symplectic 2-form. The existence of a natural Lagrangian fibration structure of these non-compact complex manifolds has not been studied very much.

There are two famous known examples in this direction, one is the Hitchin map [7] $h : T_X^* \rightarrow \mathbb{C}^{(r^2-1)(g-1)}$ where X is the moduli space $SU_C^s(r, d)$ of stable vector bundle of rank r with a fixed determinant of degree d coprime to r over a smooth projective curve C of genus g . This map has been used as a tool to derive results on the moduli spaces themselves in [1]. The other example is a rational homogeneous space G/P where G is a semisimple complex Lie group and P is a parabolic subgroup. The group G acts on the cotangent bundle $T_{G/P}^*$ as symplectic automorphisms. This induces the moment map $T_{G/P}^* \rightarrow \mathcal{G}^*$ to the dual of the Lie algebra of G (cf. Section 1.4 of [CG]). Both examples $SU_C^s(r, d)$ and G/P are Fano manifolds. This suggests that there may exist some interesting Lagrangian fibration structure in the cotangent bundles of Fano manifolds. J-M. Hwang [10] shows that the varieties of minimal rational tangents play an important role in the symplectic geometry of the cotangent bundles of uniruled projective manifolds.

The current paper is motivated by the fundamental work of the moduli spaces of vector bundles from the viewpoint of symplectic geometry of its cotangent bundle by Hitchin [7], and by the result of J-M. Hwang and Ramanan in [12], where they studied the Hitchin system and the Hitchin discriminant associated to the Hitchin map on the cotangent bundle of $SU_C^s(r, d)$.

Our computational result directly shows that the cotangent bundle of a del Pezzo surface X of degree 4 has also the Lagrangian fibration structure $\Phi : T_X^* \rightarrow \mathbb{C}^2$, and its level surfaces have some similar properties of the Hitchin discriminant in [12]. Similarly as Corollary 4.6 in [12], $\Phi^{-1}(\Delta)$ is the closure of the union of rational curves in T_X^* where $\Delta = \{b \in \mathbb{C}^2 \mid \Phi^{-1}(b) \text{ is singular}\}$. In the current paper, $\Delta = \{\text{five lines through the origin in } \mathbb{C}^2\}$.

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On the other hand, the positivity problem of the tangent bundle of a del Pezzo surface S of degree d is completely answered recently in [15] and in [9]. If S is a del Pezzo surface of degree d then

- T_S is big if and only if $d \geq 5$.
- $H^0(S, \text{Sym}^m T_S) = 0$ for all $m \geq 1$ if and only if $d \leq 3$.

So the case of $d = 4$ arouses special interest to us.

Let X be a del Pezzo surface of degree 4. Then X is a complete intersection of two hypersurfaces in $\mathbb{P}^4 = \mathbb{P}_{y_1, \dots, y_5}^4$ defined by homogeneous polynomials Q_1 and Q_2 of degree 2 in variables y_1, \dots, y_5 respectively. By a linear change of variables and multiplication by \mathbb{C}^* , we can assume that

$$(1.1) \quad Q_1 = \sum_{i=1}^5 y_i^2 \text{ and } Q_2 = \sum_{i=1}^5 a_i y_i^2$$

for some distinct $a_i \in \mathbb{C}$. (cf. Theorem 8.6.2 in [5]).

From Theorem 5.1 in [6] and the proof of Theorem 6.1 in [15], there is an isomorphism of graded rings:

$$(1.2) \quad \bigoplus_{m=0}^{\infty} H^0(X, \text{Sym}^m T_X) \simeq \mathbb{C}[Q_1, Q_2].$$

In particular Q_1 and Q_2 form a basis of $H^0(X, \text{Sym}^2 T_X)$.

Let

$$\Phi : T_X^* \rightarrow \mathbb{C}^2$$

be the natural morphism defined by the pair (Q_1, Q_2) . For each $e \in \mathbb{C}^2$, the fiber $\Phi^{-1}(e) \subset T_X^*$ will be called a level surface, and we will denote it by S_e . From the isomorphism in (1.2), we can see that $S_e \cong S_{\lambda e}$ for all $\lambda \in \mathbb{C}^*$.

First of all, in this paper, we show the following theorem. The proof will be given in Section 2.

Theorem 1.1. *The morphism $\Phi : T_X^* \rightarrow \mathbb{C}^2$ is a Lagrangian fibration.*

It means that the restriction $\omega|_{S_e}$ of the natural symplectic two form ω on T_X^* is zero. Remark 2.8 also explains the relation between the Lagrangian fibration structure of the Hitchin map for the case of $g = 2$ and the Lagrangian fibration structure of the cotangent bundle of a del Pezzo surface X of degree 4.

Meanwhile, we let $\zeta := \mathcal{O}(1)$ be the tautological line bundle on $\mathbb{P}(T_X)$ so that $\pi_* \zeta = T_X$ where $\pi : \mathbb{P}(T_X) \rightarrow X$ be the projection. By the isomorphism $H^0(\mathbb{P}(T_X), 2\zeta) \simeq H^0(X, \text{Sym}^2 T_X)$, the pencil $\{Q_e\}_{e \in \mathbb{P}^1}$ of quadric hypersurfaces in \mathbb{P}^4 induced by Q_1 and Q_2 gives the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$ defining a rational map $\tilde{\phi} : \mathbb{P}(T_X) \dashrightarrow \mathbb{P}^1$.

It is well known that there are exactly 16 lines ℓ_1, \dots, ℓ_{16} in \mathbb{P}^4 contained in X . The base locus B of the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$ consists of the disjoint union of 16 sections ℓ'_i of $\mathbb{P}(T_X|_{\ell_i}) \rightarrow \ell_i$ which are associated to quotients $T_X|_{\ell_i} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-1)$ (p.12 in [9]); Since $\ell'_i \cdot \zeta = -1$, we have $\ell'_i \subset B$.

After the blow-up $\mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$ along the base locus B , we have a morphism

$$\phi : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}^1,$$

and the following commutative diagram of morphisms and rational maps:

$$\begin{array}{ccccc}
 \mathrm{Bl}_B\mathbb{P}(T_X) & \xrightarrow{\mu_B} & \mathbb{P}(T_X) & \xleftarrow{\dots\dots\dots} & T_X^* \\
 & \searrow \phi & \downarrow \tilde{\phi} & \swarrow \pi & \swarrow \Pi \\
 & & \mathbb{P}^1 & & X \\
 & & & & \downarrow \Phi \\
 & & & & \mathbb{C}^2
 \end{array}$$

For each $\mathbf{e} \in \mathbb{P}^1$, we let $K_{\mathbf{e}}$ be the fiber $\phi^{-1}(\mathbf{e})$. In Lemma 3.5 we show that $K_{\mathbf{e}}$ is a double cover of X if $Q_{\mathbf{e}}$ is smooth. Due to the description of Section 2 in [6], for each $x \in X$, the points in $K_{\mathbf{e}}$ over x correspond to lines in $Q_{\mathbf{e}} \cap \mathbf{T}_x X$ through x where $\mathbf{T}_x X \subset \mathbb{P}^4$ denotes the embedded projective tangent plane to X at x . So if $Q_{\mathbf{e}}$ is smooth, the branch locus of the double cover from $K_{\mathbf{e}}$ to X is the locus of points x such that $Q_{\mathbf{e}} \cap \mathbf{T}_x X$ is a double line. The total dual VMRT theory (cf. [9]) also helps to give an explicit description of $K_{\mathbf{e}}$, especially when $K_{\mathbf{e}}$ is not irreducible. We will explain this description in Section 3.

Through an explicit description of $K_{\mathbf{e}}$, we get the following theorem (a) for a general $e \in \mathbb{C}^2$ and (b). Then by using the idea of the characteristic vector fields in [11], the proof of Theorem 1.2(=Theorem 3.13) can be completed. The proof will be given in Section 3.

Theorem 1.2. *Let $\mathbf{b}_1, \dots, \mathbf{b}_5$ be the points in \mathbb{P}^1 such that $Q_{\mathbf{b}_i}$ are singular. For each $i = 1, \dots, 5$, take one point b_i in the fiber at \mathbf{b}_i of the quotient map $\mathbb{C}^2 \setminus \{0\} \rightarrow \mathbb{P}^1$ and let $\mathbb{C} \cdot b_i$ be the line in \mathbb{C}^2 through b_i and the origin. We have the following description of level surfaces of $\Phi : T_X^* \rightarrow \mathbb{C}^2$.*

- (a) *For every $e \in \mathbb{C}^2 \setminus \cup_{i=1}^5 \mathbb{C} \cdot b_i$, S_e is $\bar{S}_e \setminus \{16 \text{ points}\}$ where \bar{S}_e is isomorphic to the Jacobian variety of a curve C_e of genus two.*
- (b) *For each $i = 1, \dots, 5$, we have the following description of S_{b_i} .*
 - (i) *S_{b_i} consists of two irreducible components $A_{i,1}$ and $A_{i,2}$.*
 - (ii) *Each $A_{i,j}$ for $j = 1, 2$ is a ruled surface $\setminus \{8 \text{ points}\}$ over an elliptic curve $\bar{E}_{i,j}$.*
 - (iii) *$A_{i,1} \cap A_{i,2}$ is an elliptic curve E'_{b_i} .*
 - (iv) *In the fibration $A_{i,j} \rightarrow \bar{E}_{i,j}$, E'_{b_i} intersects two distinct points at each fiber.*

We remark that X is isomorphic to the blow up of \mathbb{P}^2 at the five points which are the images of $\mathbf{b}_1, \dots, \mathbf{b}_5$ under the Veronese embedding (cf. [19]).

As a corollary, the map Φ is flat, and all elements of the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$ can be also fully described.

Theorem 1.3. *We have the following description of $K_{\mathbf{e}}$ for all $\mathbf{e} \in \mathbb{P}^1$.*

- (a) *For every $\mathbf{e} \in \mathbb{P}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is a K3 surface of degree 8 of Kummer type. It has 16 (-2)-curves $\ell_{\mathbf{e},i}$ which are intersection of $K_{\mathbf{e}}$ with the exceptional divisor D of the blow-up $\mu_B : \mathrm{Bl}_B\mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$.*
- (b) *For each $i = 1, \dots, 5$, we have the following description of $K_{\mathbf{b}_i}$.*
 - (i) *$K_{\mathbf{b}_i}$ consists of two irreducible components $\check{C}_{i,1}$ and $\check{C}_{i,2}$.*
 - (ii) *For each $j = 1, 2$, we have a conic fibration $\pi_{i,j} : X \rightarrow \mathbb{P}^1$ with four singular fibers such that $\check{C}_{i,j}$ is isomorphic to the blow-up of X at four distinct points which are the singular points of the four singular fibers of $\pi_{i,j}$.*
 - (iii) *$\check{C}_{i,1} \cap \check{C}_{i,2}$ is a smooth elliptic curve $E_{\mathbf{b}_i}$.*

- (iv) In the fibration $\check{\pi}_{i,j} : \check{\mathcal{C}}_{i,j} \rightarrow \mathbb{P}^1$ given by the composition of the blow-up $\check{\mathcal{C}}_{i,j} \rightarrow X$ in (ii) and the conic fibration $\pi_{i,j}$, $E_{\mathbf{b}_i}$ intersects two distinct points at each smooth fiber, and one point at the exceptional curve of each singular fiber.

This theorem is proved by Lemma 3.10 and Corollary 3.15.

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2. LAGRANGIAN FIBRATION STRUCTURE ON THE CONTANGENT BUNDLE

In order to show Theorem 1.1, we first describe the members of $H^0(X, \text{Sym}^2 T_X)$ in terms of local parameters of T_X^* .

Notation 2.1. Consider $\mathbb{P}^2 = \mathbb{P}_{x_0, x_1, x_2}^2$ which means that $[x_0, x_1, x_2]$ is a homogeneous coordinate system of \mathbb{P}^2 . Set $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$. Let $U_0 = \mathbb{A}_{x,y}^2 \subset \mathbb{P}^2$ be the affine open subset defined by $x_0 \neq 0$. Take one $H \in H^0(U_0, \text{Sym}^2 T_{\mathbb{P}^2})$. Then we can write H uniquely as

$$(2.1) \quad H = f(x, y) \left(\frac{\partial}{\partial x} \right)^2 + g(x, y) \left(\frac{\partial}{\partial y} \right)^2 + h(x, y) \left(\frac{\partial}{\partial x} \right) \left(\frac{\partial}{\partial y} \right)$$

where

$$f(x, y) = \sum_{i,j} f_{i,j} x^i y^j, \quad g(x, y) = \sum_{i,j} g_{i,j} x^i y^j, \quad h(x, y) = \sum_{i,j} h_{i,j} x^i y^j \in \mathbb{C}[x, y].$$

2.1. Description of $H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$.

Lemma 2.2. In the situation of Notation 2.1, H is in $H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$ if and only if $\deg f, \deg g, \deg h \leq 4$, and the following 18 linear forms of the coefficients of H vanish:

$$\begin{aligned} & h_{0,4}, \quad h_{1,3} - 2g_{0,4}, \quad h_{2,2} - 2g_{1,3}, \quad h_{3,1} - 2g_{2,2}, \quad h_{4,0} - 2g_{3,1}, \quad g_{4,0}, \\ & f_{0,3}, \quad f_{1,2} - h_{0,3}, \quad f_{2,1} + g_{0,3} - h_{1,2}, \quad f_{3,0} + g_{1,2} - h_{2,1}, \quad g_{2,1} - h_{3,0}, \quad g_{3,0}, \\ & f_{0,4}, \quad f_{1,3}, \quad f_{2,2} - g_{0,4}, \quad f_{3,1} - g_{1,3}, \quad f_{4,0} - g_{2,2}, \quad g_{3,1}. \end{aligned}$$

Proof. Assume that $H \in H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$.

Let $u = \frac{x_0}{x_2}$ and $v = \frac{x_1}{x_2}$. Then u and v form an affine coordinate system on the affine open subset $U_2 = \mathbb{A}_{u,v}^2 \subset \mathbb{P}^2$ given by $x_2 \neq 0$. Since $x = \frac{v}{u}$, $y = \frac{1}{u}$, $u = \frac{1}{y}$, $v = \frac{x}{y}$, we have

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x} \frac{\partial}{\partial u} + \frac{\partial v}{\partial x} \frac{\partial}{\partial v} = \frac{1}{y} \frac{\partial}{\partial v} = u \frac{\partial}{\partial v}$$

and

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y} \frac{\partial}{\partial u} + \frac{\partial v}{\partial y} \frac{\partial}{\partial v} = -\frac{1}{y^2} \frac{\partial}{\partial u} - \frac{x}{y^2} \frac{\partial}{\partial v} = -u^2 \frac{\partial}{\partial u} - uv \frac{\partial}{\partial v}.$$

Therefore

$$\begin{aligned}
H|_{U_2} &= f\left(\frac{v}{u}, \frac{1}{u}\right)u^2 \left(\frac{\partial}{\partial v}\right)^2 + g\left(\frac{v}{u}, \frac{1}{u}\right) \left(u^2 \frac{\partial}{\partial u} + uv \frac{\partial}{\partial v}\right) \\
&\quad - h\left(\frac{v}{u}, \frac{1}{u}\right) \left(u^3 \left(\frac{\partial}{\partial u}\right) \left(\frac{\partial}{\partial v}\right) + u^2 v \left(\frac{\partial}{\partial v}\right)^2\right) \\
&= g\left(\frac{v}{u}, \frac{1}{u}\right)u^4 \left(\frac{\partial}{\partial u}\right)^2 + \left\{f\left(\frac{v}{u}, \frac{1}{u}\right)u^2 + g\left(\frac{v}{u}, \frac{1}{u}\right)u^2 v^2 - h\left(\frac{v}{u}, \frac{1}{u}\right)u^2 v\right\} \left(\frac{\partial}{\partial v}\right)^2 \\
&\quad + \left\{2g\left(\frac{v}{u}, \frac{1}{u}\right)u^3 v - h\left(\frac{v}{u}, \frac{1}{u}\right)u^3\right\} \left(\frac{\partial}{\partial u}\right) \left(\frac{\partial}{\partial v}\right)
\end{aligned}$$

Since $H|_{U_2} \in H^0(U_2, \text{Sym}^2 T_{\mathbb{P}^2})$, the coefficients of

$$\left(\frac{\partial}{\partial u}\right)^2, \left(\frac{\partial}{\partial v}\right)^2, \text{ and } \left(\frac{\partial}{\partial u}\right) \left(\frac{\partial}{\partial v}\right)$$

appearing in $H|_{U_2}$ above are holomorphic functions in u, v . Therefore

$$g\left(\frac{v}{u}, \frac{1}{u}\right)u^4$$

which is the coefficient of $\left(\frac{\partial}{\partial u}\right)^2$ in $H|_{U_2}$ is holomorphic, and hence

$$g_{i,j} = 0 \text{ for all } i, j \text{ with } i + j \geq 5$$

which means that $\deg g \leq 4$.

We have the following equalities:

$$\begin{aligned}
&2g\left(\frac{v}{u}, \frac{1}{u}\right)u^3 v - h\left(\frac{v}{u}, \frac{1}{u}\right)u^3 \\
&= 2 \sum g_{i,j} \left(\frac{v}{u}\right)^i \left(\frac{1}{u}\right)^j u^3 v - \sum h_{i,j} \left(\frac{v}{u}\right)^i \left(\frac{1}{u}\right)^j u^3 \\
&= 2 \sum g_{i,j} u^{3-i-j} v^{i+1} - \sum h_{i,j} u^{3-i-j} v^i \tag{*}
\end{aligned}$$

From the same arguments as above, we can see that the coefficients of $u^{-l}v^k$ for $l > 0$ and $k \geq 0$ in (*) vanish which implies

$$h_{i,j} = 0 \text{ for all } i, j \text{ with } i + j \geq 5$$

and

$$h_{0,4} = 0, h_{1,3} = 2g_{0,4}, h_{2,2} = 2g_{1,3}, h_{3,1} = 2g_{2,2}, h_{4,0} = 2g_{3,1}, g_{4,0} = 0.$$

We have the following equalities:

$$\begin{aligned}
&f\left(\frac{v}{u}, \frac{1}{u}\right)u^2 + g\left(\frac{v}{u}, \frac{1}{u}\right)u^2 v^2 - h\left(\frac{v}{u}, \frac{1}{u}\right)u^2 v \\
&= \sum f_{i,j} \left(\frac{v}{u}\right)^i \left(\frac{1}{u}\right)^j u^2 + \sum g_{i,j} \left(\frac{v}{u}\right)^i \left(\frac{1}{u}\right)^j u^2 v^2 - \sum h_{i,j} \left(\frac{v}{u}\right)^i \left(\frac{1}{u}\right)^j u^2 v \\
&= \sum f_{i,j} u^{2-i-j} v^i + \sum g_{i,j} u^{2-i-j} v^{i+2} - \sum h_{i,j} u^{2-i-j} v^{i+1} \tag{**}
\end{aligned}$$

By the same reason as before, it follows that the coefficients of $u^{-l}v^k$ with $l > 0$ and $k \geq 0$ in (**) vanish. This shows that

$$f_{i,j} = 0 \text{ for all } i + j \geq 5$$

and

$$\begin{aligned} f_{0,3} = f_{1,2} - h_{0,3} = f_{2,1} + g_{0,3} - h_{1,2} = f_{3,0} + g_{1,2} - h_{2,1} = g_{2,1} - h_{3,0} = g_{3,0} = 0, \\ f_{0,4} = f_{1,3} - h_{0,4} = f_{2,2} + g_{0,4} - h_{1,3} = f_{3,1} + g_{1,3} - h_{2,2} = f_{4,0} + g_{2,2} - h_{3,1} = g_{3,1} - h_{4,0} = g_{4,0} = 0. \end{aligned}$$

So we obtained all the 18 linear relations in our lemma.

Coversely if H satisfies the conditions in this lemma, then H is holomorphic on $\mathbb{P}^2 \setminus \{(0 : 0 : 1)\}$ and hence it can be holomorphically extended to all \mathbb{P}^2 . \square

Remark 2.3. The above 18 linear relations in Lemma 2.2 are independent and thus $h^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2}) = 27$. The dimension can be also computed from the Euler sequence of $T_{\mathbb{P}^2}$. From the exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2} \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \rightarrow T_{\mathbb{P}^2} \rightarrow 0,$$

we have

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \rightarrow \text{Sym}^2(\mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}) \rightarrow \text{Sym}^2 T_{\mathbb{P}^2} \rightarrow 0.$$

We also remark that Lemma 2.2 is equivalent to the following statement: $H \in H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$ if and only if $\deg f, \deg g, \deg h \leq 4$, and

$$\begin{aligned} \frac{1}{x^2} f_4(x, y) &= \frac{1}{y^2} g_4(x, y) \\ f_4(x, y) &= x^2(f_{4,0}x^2 + f_{3,1}xy + f_{2,2}y^2) = \frac{x^2}{y^2} g_4(x, y) \\ g_4(x, y) &= y^2(g_{0,4}y^2 + g_{1,3}xy + g_{2,2}x^2) = \frac{y^2}{x^2} f_4(x, y) \\ h_4(x, y) &= xy(h_{1,3}y^2 + h_{2,2}xy + h_{3,1}x^2y) = \frac{2x}{y} g_4(x, y) = \frac{2y}{x} f_4(x, y) = \frac{y}{x} f_4(x, y) + \frac{x}{y} g_4(x, y). \end{aligned}$$

and

$$\begin{aligned} f_3(x, y) &= x(f_{3,0}x^2 + f_{2,1}xy + f_{1,2}y^2) \\ g_3(x, y) &= y(g_{2,1}x^2 + g_{1,2}xy + g_{0,3}y^2) \\ h_3(x, y) &= h_{3,0}x^3 + h_{2,1}x^2y + h_{1,2}xy^2 + h_{0,3}y^3 \\ &= g_{2,1}x^3 + (f_{3,0} + g_{1,2})x^2y + (f_{1,2} + g_{0,3})xy^2 + f_{1,2}y^3 \\ &= \frac{y}{x} f_3(x, y) + \frac{x}{y} g_3(x, y). \end{aligned}$$

2.2. Description of $H^0(X, \text{Sym}^2 T_X)$. Let $\mu_p : Y = \text{Bl}_p \mathbb{P}^2 \rightarrow \mathbb{P}^2 = \mathbb{P}_{x_0, x_1, x_2}^2$ be the blow-up at $p = (1 : a : b)$. Then

$$H^0(Y, \text{Sym}^2 T_Y) \subset H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$$

Take $H \in H^0(\mathbb{P}^2, \text{Sym}^2 T_{\mathbb{P}^2})$. For the affine open subset $U_0 \subset \mathbb{P}^2$ defined by $x_0 \neq 0$, write $H|_{U_0}$ as in Notation 2.1 so that it satisfies the properties in Lemma 2.2.

Lemma 2.4. *The above H is in $H^0(Y, \text{Sym}^2 T_Y)$ if and only if*

$$f(a, b) = g(a, b) = h(a, b) = g_x(a, b) = f_y(a, b) = 0$$

and

$$g_y(a, b) - h_x(a, b) = f_x(a, b) - h_y(a, b) = 0.$$

Proof. For simplicity we only prove that if $p = (1 : 0 : 0)$, then $H \in H^0(Y, \text{Sym}^2 T_Y)$ if and only if

$$f_{0,0} = g_{0,0} = h_{0,0} = g_{1,0} = f_{0,1} = g_{0,1} - h_{1,0} = f_{1,0} - h_{0,1} = 0$$

The proof for the general case can be done by the same argument.

Let E be the exceptional divisor of μ_p . Let $U_0 \subset \mathbb{P}^2 = \mathbb{P}_{x_0, x_1, x_2}^2$ be the affine open neighborhood of p defined by $x_0 \neq 0$. Consider $x = \frac{x_1}{x_0}$ and $y = \frac{x_2}{x_0}$ as an affine coordinate system on $U_0 = \mathbb{A}_{x,y}^2 \subset \mathbb{P}^2$ so that $p = (0, 0)$. Then $\mu_p^{-1}(U_0) \subset \mathbb{A}_{x,y}^2 \times \mathbb{P}_{z_0, z_1}^1$ is defined by $xz_1 = yz_0$. Let $W \subset \mu_p^{-1}(U_0)$ be the open subset given by $z_0 \neq 0$. Set $w = \frac{z_1}{z_0}$ and $r = x$. Then $y = xw$, $W = \mathbb{A}_{r,w}^2$, and $E \cap W$ is defined by $r = 0$ in W .

From the relations

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x} \frac{\partial}{\partial r} + \frac{\partial w}{\partial x} \frac{\partial}{\partial w} = \frac{\partial}{\partial r} - \frac{y}{x^2} \frac{\partial}{\partial w} = \frac{\partial}{\partial r} - \frac{w}{r} \frac{\partial}{\partial w},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y} \frac{\partial}{\partial r} + \frac{\partial w}{\partial y} \frac{\partial}{\partial w} = \frac{1}{r} \frac{\partial}{\partial w}$$

it follows that

$$\begin{aligned} H|_{W \setminus E} &= f(r, rw) \left(\frac{\partial}{\partial r} - \frac{w}{r} \frac{\partial}{\partial w} \right)^2 + g(r, rw) \left(\frac{1}{r} \frac{\partial}{\partial w} \right)^2 + h(r, rw) \left(\frac{\partial}{\partial x} - \frac{w}{r} \frac{\partial}{\partial w} \right) \left(\frac{1}{r} \frac{\partial}{\partial w} \right) \\ &= f(r, rw) \left(\frac{\partial}{\partial r} \right)^2 + \left\{ f(r, rw) \frac{w^2}{r^2} + g(r, rw) \frac{1}{r^2} - h(r, rw) \frac{w}{r^2} \right\} \left(\frac{\partial}{\partial w} \right)^2 \\ &\quad + \left\{ -2f(r, rw) \frac{w}{r} + h(r, rw) \frac{1}{r} \right\} \left(\frac{\partial}{\partial r} \right) \left(\frac{\partial}{\partial w} \right) \end{aligned}$$

The coefficient of $\left(\frac{\partial}{\partial w} \right)^2$ in $H|_{W \setminus E}$ satisfies the following equalities:

$$\begin{aligned} &f(r, rw) \frac{w^2}{r^2} + g(r, rw) \frac{1}{r^2} - h(r, rw) \frac{w}{r^2} \\ &= \sum f_{i,j} r^i (rw)^j \frac{w^2}{r^2} + \sum g_{i,j} r^i (rw)^j \frac{1}{r^2} - \sum h_{i,j} r^i (rw)^j \frac{w}{r^2} \\ &= \sum f_{i,j} r^{i+j-2} w^{j+2} + \sum g_{i,j} r^{i+j-2} w^j - \sum h_{i,j} r^{i+j-2} w^{j+1} \end{aligned}$$

Therefore if $H|_{W \setminus E}$ holomorphically extends to W ,

$$f_{0,0} = g_{0,0} = h_{0,0} = 0 \text{ and } g_{1,0} = g_{0,1} - h_{1,0} = f_{1,0} - h_{0,1} = f_{0,1} = 0$$

because $E \cap W$ is defined by $r = 0$ in W .

Similarly, from the following equalities

$$\begin{aligned} & -2f(r, rw)\frac{w}{r} + h(r, rw)\frac{1}{r} \\ &= -2\sum f_{i,j}r^i(rw)^j\frac{w}{r} + \sum h_{i,j}r^i(rw)^j\frac{1}{r} \\ &= -2\sum f_{i,j}r^{i+j-1}w^{j+1} + \sum h_{i,j}r^{i+j-1}w^j \end{aligned}$$

it follows that if $H|_{W \setminus E}$ holomorphically extends to W then $f_{0,0} = h_{0,0} = 0$.

Let $W' \subset \mu_p^{-1}(U)$ be the open subset defined by $z_1 \neq 0$. Let $w = \frac{z_0}{z_1}$. Then $x = yw$ and $W' = \mathbb{A}_{y,w}^2$. Using the following relations

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial}{\partial w} + \frac{\partial y}{\partial x} \frac{\partial}{\partial y} = \frac{1}{y} \frac{\partial}{\partial w},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial w}{\partial y} \frac{\partial}{\partial w} + \frac{\partial y}{\partial y} \frac{\partial}{\partial y} = -\frac{x}{y^2} \frac{\partial}{\partial w} + \frac{\partial}{\partial y} = -\frac{w}{y} \frac{\partial}{\partial w} + \frac{\partial}{\partial y},$$

we have

$$\begin{aligned} H|_{W' \setminus E} &= f(yw, y) \left(\frac{1}{y} \frac{\partial}{\partial w} \right)^2 + g(yw, y) \left(-\frac{w}{y} \frac{\partial}{\partial w} + \frac{\partial}{\partial y} \right)^2 \\ &\quad + h(yw, y) \left(\frac{1}{y} \frac{\partial}{\partial w} \right) \left(-\frac{w}{y} \frac{\partial}{\partial w} + \frac{\partial}{\partial y} \right) \\ &= \left\{ f(yw, y) \frac{1}{y^2} + g(yw, y) \frac{w^2}{y^2} - h(yw, y) \frac{w}{y^2} \right\} \left(\frac{\partial}{\partial w} \right)^2 + g(yw, y) \left(\frac{\partial}{\partial y} \right)^2 \\ &\quad + \left\{ -2g(yw, y) \frac{w}{y} + h(yw, y) \frac{1}{y} \right\} \left(\frac{\partial}{\partial y} \right) \left(\frac{\partial}{\partial w} \right). \end{aligned}$$

The coefficient of $\left(\frac{\partial}{\partial w} \right)^2$ in $H|_{W' \setminus E}$ satisfies the following equalities:

$$\begin{aligned} & f(yw, y) \frac{1}{y^2} + g(yw, y) \frac{w^2}{y^2} - h(yw, y) \frac{w}{y^2} \\ &= \sum f_{i,j}(yw)^i y^j \frac{1}{y^2} \sum g_{i,j}(yw)^i y^j \frac{w^2}{y^2} - \sum h_{i,j}(yw)^i y^j \frac{w}{y^2} \\ &= \sum f_{i,j} y^{i+j-2} w^i + \sum_{i,j} g_{i,j} y^{i+j-2} w^{i+2} - \sum h_{i,j} y^{i+j-2} w^{i+1} \end{aligned}$$

and thus if $H|_{W' \setminus E}$ holomorphically extends to W' then

$$f_{0,0} = g_{0,0} = h_{0,0} = 0 \text{ and } f_{0,1} = f_{1,0} - h_{0,1} = g_{0,1} - h_{1,0} = g_{1,0} = 0.$$

□

Lemma 2.5. *Let p_1, \dots, p_5 be five distinct points in \mathbb{P}^2 in general position, i.e., no three of them lie in a line. Then we can choose a homogeneous coordinate system on \mathbb{P}^2 so that $p_1 = (1 : 0 : 0)$, $p_2 = (1 : 1 : 0)$, $p_3 = (1 : 0 : 1)$, $p_4 = (1 : 1 : -1)$ or $(1 : 1 : -1/2)$, and $p_5 = (1 : a : b)$ for some $a, b \in \mathbb{C}$.*

Proof. Clearly we can choose a homogeneous coordinate system x_0, x_1, x_2 on \mathbb{P}^2 so that

$$p_1 = (1 : 0 : 0), p_2 = (1 : 1 : 0), p_3 = (1 : 0 : 1) \text{ and } p_4 = (1 : 1 : -1).$$

Assume that $p_5 = (0 : 1 : b)$. Set

$$M = \begin{bmatrix} x & y & z \\ 0 & x+y & 0 \\ 0 & 0 & x+z \end{bmatrix}.$$

Let us change the homogeneous coordinate system x_0, x_1, x_2 by the linear transform on \mathbb{P}^2 given by a matrix of the form M above such that

$$y + bz = x + y = -b(x + z) \neq 0.$$

In this new coordinates, we have

$$p_1 = (1 : 0 : 0), p_2 = (1 : 1 : 0), p_3 = (1 : 0 : 1), p_5 = (1 : 1 : -1)$$

and

$$p_4 = (x + y - z : x + y : -(x + z)).$$

Assume that $x + y - z = 0$. Then

$$x + y - z = bz - (b^2 + b)z - z = -(b^2 + b + 1)z = 0.$$

Since $z \neq 0$ we have

$$b^2 + b + 1 = 0.$$

Let us change the initial homogeneous coordinate system x_0, x_1, x_2 on \mathbb{P}^2 by a matrix of the form M above such that

$$y + bz = x + y = -2b(x + z) \neq 0.$$

In this new coordinates, we have

$$p_1 = (1 : 0 : 0), p_2 = (1 : 1 : 0), p_3 = (1 : 0 : 1), p_5 = (1 : 1 : -1/2)$$

and

$$p_4 = (x + y - z : x + y : -(x + z)).$$

Then

$$x + y - z = bz - (2b^2 + 3b)z - z = -(2b^2 + 2b + 1)z \neq 0$$

because $b^2 + b + 1 = 0$ and $z \neq 0$. We get our lemma. \square

2.3. Tangents of Lagrangian fibration. Let $\mu : X = \text{Bl}_{p_1, \dots, p_5} \mathbb{P}^2 \rightarrow \mathbb{P}^2$ be the blow up at five points $p_1, \dots, p_5 \in \mathbb{P}^2$ in general position. Let $E_i \subset X$ be the exceptional curve over p_i .

Take any two independent sections $H, G \in H^0(X, \text{Sym}^2 T_X)$. Then we can consider H and G as regular functions on T_X^* so that they define a morphism

$$\Phi : T_X^* \rightarrow \mathbb{C}^2, q \mapsto (H(q), G(q)).$$

We recall that for each $e \in \mathbb{C}^2$, the fiber $S_e = \Phi^{-1}(e)$ is called a level surface.

By Lemma 2.5, we can choose a homogeneous coordinate system x_0, x_1, x_2 on \mathbb{P}^2 so that

$$p_1 = (1 : 0 : 0), p_2 = (1 : 1 : 0), p_3 = (1 : 0 : 1), p_4 = (1 : \alpha : \beta), \text{ and } p_5 = (1 : a : b)$$

for some $a, b \in \mathbb{C}$, and $(\alpha, \beta) = (1, -1)$ or $(1, -1/2)$.

Let $U_0 \subset \mathbb{P}^2 = \mathbb{P}_{x_0, x_1, x_2}^2$ be the affine open subset defined by $x_0 \neq 0$. Set $x = \frac{x_1}{x_0}$, $y = \frac{x_2}{x_0}$, $u = \frac{\partial}{\partial x}$ and $v = \frac{\partial}{\partial y}$. Let $U := \Pi^{-1}(\mu^{-1}(U_0) \setminus \cup_{i=1}^5 E_i)$. Here $\Pi : T_X^* \rightarrow X$ is the projection morphism. We can consider the restrictions $H|_U$ and $G|_U$ as members in $\mathbb{C}[x, y, u, v]$.

Let us consider the canonical symplectic two form ω on T_X^* : $\omega|_U$ can be expressed as

$$\omega|_U = dx \wedge du + dy \wedge dv.$$

Lemma 2.6. *Take $e \in \mathbb{C}^2$ and $q \in S_e \cap U$. If $\dim T_q S_e = 2$ and*

$$H_y(q)G_v(q) - H_v(q)G_y(q) + H_x(q)G_u(q) - H_u(q)G_x(q) = 0$$

then $\omega|_{T_q S_e} = 0$. (Here $H_x(q) = \frac{\partial H}{\partial x}(q)$, $H_y(q) = \frac{\partial H}{\partial y}(q)$, and so on.)

Proof. We remark that the tangent space $T_q(S_e) \subset T_q(T_X^*)$ is defined by

$$(2.2) \quad d_q H = H_x(q)d_q x + H_y(q)d_q y + H_u(q)d_q u + H_v(q)d_q v = 0 \text{ and}$$

$$(2.3) \quad d_q G = G_x(q)d_q x + G_y(q)d_q y + G_u(q)d_q u + G_v(q)d_q v = 0.$$

Assume that $\dim T_q S_e = 2$ and

$$H_y(q)G_v(q) - H_v(q)G_y(q) + H_x(q)G_u(q) - H_u(q)G_x(q) = 0.$$

Then

$$A := -H_u(q) \frac{\partial}{\partial x} \Big|_q - H_v(q) \frac{\partial}{\partial y} \Big|_q + H_x(q) \frac{\partial}{\partial u} \Big|_q + H_y(q) \frac{\partial}{\partial v} \Big|_q$$

and

$$B := G_u(q) \frac{\partial}{\partial x} \Big|_q + G_v(q) \frac{\partial}{\partial y} \Big|_q - G_x(q) \frac{\partial}{\partial u} \Big|_q - G_y(q) \frac{\partial}{\partial v} \Big|_q$$

are tangent vectors in $T_q(S_e)$ because they satisfy the two equations (2.2) and (2.3). Since $\dim T_q(S_e) = 2$, they also form a basis of $T_q(S_e)$. By our assumption we have

$$\omega(A, B) = H_y(q)G_v(q) - H_v(q)G_y(q) + H_x(q)G_u(q) - H_u(q)G_x(q) = 0$$

which implies that $\omega|_{T_q(S_e)} = 0$. □

Proposition 2.7. *For all $q \in S_e^{sm} \cap U$ we have $\omega|_{T_q(S_e)} = 0$. Here S_e^{sm} denotes the smooth locus of S_e .*

Proof. By Lemma 2.6 it is enough to show that for all $q \in S_e^{sm} \cap U$,

$$H_y(q)G_v(q) - H_v(q)G_y(q) + H_x(q)G_u(q) - H_u(q)G_x(q) = 0.$$

Let us write $H|_U$ and $G|_U$ as in Notation 2.1 so that

$$H|_U = H(x, y, u, v) = f(x, y)u^2 + g(x, y)v^2 + h(x, y)uv \in \mathbb{C}[x, y, u, v]$$

and

$$G|_U = G(x, y, u, v) = c(x, y)u^2 + d(x, y)v^2 + e(x, y)uv \in \mathbb{C}[x, y, u, v]$$

where

$$f(x, y) = \sum_{i+j \leq 4} f_{i,j}x^i y^j, \quad g = \sum_{i+j \leq 4} g_{i,j}x^i y^j, \quad h = \sum_{i+j \leq 4} h_{i,j}x^i y^j$$

and

$$c(x, y) = \sum_{i+j \leq 4} c_{i,j}x^i y^j, \quad d = \sum_{i+j \leq 4} d_{i,j}x^i y^j, \quad e = \sum_{i+j \leq 4} e_{i,j}x^i y^j$$

are polynomials in variables x and y satisfying the conditions in Lemmas 2.2 and 2.4.

Set

$$\begin{aligned} R := & G_v(x, y, u, v)H_y(x, y, u, v) - H_v(x, y, u, v)G_y(x, y, u, v) \\ & + H_x(x, y, u, v)G_u(x, y, u, v) - H_u(x, y, u, v)G_x(x, y, u, v). \end{aligned}$$

Then R is a member of the polynomial ring

$$P := \mathbb{C}[f_{i,j}, g_{i,j}, h_{i,j}, c_{i,j}, d_{i,j}, e_{i,j}, x, y, u, v, a, b \mid i + j \leq 4].$$

Let I be the ideal of P generated by following polynomials given by the conditions in Lemmas 2.2 and 2.4.

$$\begin{aligned} & h_{0,4}, h_{1,3} - 2g_{0,4}, h_{2,2} - 2g_{1,3}, h_{3,1} - 2g_{2,2}, h_{4,0} - 2g_{3,1}, g_{4,0}, \\ & f_{0,3}, f_{1,2} - h_{0,3}, f_{2,1} + g_{0,3} - h_{1,2}, f_{3,0} + g_{1,2} - h_{2,1}, g_{2,1} - h_{3,0}, g_{3,0}, \\ & f_{0,4}, f_{1,3}, f_{2,2} - g_{0,4}, f_{3,1} - g_{1,3}, f_{4,0} - g_{2,2}, g_{3,1}, \end{aligned}$$

$$\begin{aligned} & e_{0,4}, e_{1,3} - 2d_{0,4}, e_{2,2} - 2d_{1,3}, e_{3,1} - 2d_{2,2}, e_{4,0} - 2d_{3,1}, d_{4,0}, \\ & c_{0,3}, c_{1,2} - e_{0,3}, c_{2,1} + d_{0,3} - e_{1,2}, c_{3,0} + d_{1,2} - e_{2,1}, d_{2,1} - e_{3,0}, d_{3,0}, \\ & c_{0,4}, c_{1,3}, c_{2,2} - d_{0,4}, c_{3,1} - d_{1,3}, c_{4,0} - d_{2,2}, d_{3,1}, \end{aligned}$$

$$\begin{aligned} & f(0,0), g(0,0), h(0,0), g_x(0,0), f_y(0,0), g_y(0,0) - h_x(0,0), f_x(0,0) - h_y(0,0), \\ & f(1,0), g(1,0), h(1,0), g_x(1,0), f_y(1,0), g_y(1,0) - h_x(1,0), f_x(1,0) - h_y(1,0), \\ & f(0,1), g(0,1), h(0,1), g_x(0,1), f_y(0,1), g_y(0,1) - h_x(0,1), f_x(0,1) - h_y(0,1), \\ & f(\alpha, \beta), g(\alpha, \beta), h(\alpha, \beta), g_x(\alpha, \beta), f_y(\alpha, \beta), g_y(\alpha, \beta) - h_x(\alpha, \beta), f_x(\alpha, \beta) - h_y(\alpha, \beta), \\ & f(a, b), g(a, b), h(a, b), g_x(a, b), f_y(a, b), g_y(a, b) - h_x(a, b), f_x(a, b) - h_y(a, b), \text{ and} \end{aligned}$$

$$\begin{aligned} & c(0,0), d(0,0), e(0,0), d_x(0,0), c_y(0,0), d_y(0,0) - e_x(0,0), c_x(0,0) - e_y(0,0), \\ & c(1,0), d(1,0), e(1,0), d_x(1,0), c_y(1,0), d_y(1,0) - e_x(1,0), c_x(1,0) - e_y(1,0), \\ & c(0,1), d(0,1), e(0,1), d_x(0,1), c_y(0,1), d_y(0,1) - e_x(0,1), c_x(0,1) - e_y(0,1), \\ & c(\alpha, \beta), d(\alpha, \beta), e(\alpha, \beta), d_x(\alpha, \beta), c_y(\alpha, \beta), d_y(\alpha, \beta) - e_x(\alpha, \beta), c_x(\alpha, \beta) - e_y(\alpha, \beta), \\ & c(a, b), d(a, b), e(a, b), d_x(a, b), c_y(a, b), d_y(a, b) - e_x(a, b), c_x(a, b) - e_y(a, b). \end{aligned}$$

For the proof it is enough to show the following claim.

Claim: If we fix a and b so that $p_5 = (1 : a : b) \neq p_i$ for all $i = 1, \dots, 4$, then R vanishes if H and G satisfy the relations in Lemmas 2.2 and 2.4.

By using Magma calculator, we can show that

$$(a-1)abR, (a-1)(b+1)aR, (a-1)abR, (b-\beta)abR, \text{ and } (\frac{1-\beta}{\alpha}a+b-1)bR$$

are members in I .

Assume that $a \neq 0$ and $b \neq 0$. Since $p_5 \neq p_4$, we have $a \neq 1$ or $b \neq \beta$ which implies that the claim because $(a-1)abR, (b-\beta)abR \in I$.

Assume that $a \neq 0$ and $b = 0$. Then $a \neq 1$ since $p_5 \neq p_2$. This implies the claim because $(a-1)(b+1)aR \in I$.

Assume that $a = 0$. Since $p_5 \neq p_1, p_3$, we can see that $b \neq 0, 1$. So we get the claim because $(\frac{1-\beta}{\alpha}a+b-1)bR \in I$. \square

Proof of Theorem 1.1. Let X be a del Pezzo surface of degree 4. By Lemma 2.5, we may assume that X is the blow-up of $\mathbb{P}_{x_0, x_1, x_2}^2$ at five distinct points $p_1 = (1 : 0 : 0)$, $p_2 = (1 : 1 : 0)$, $p_3 = (1 : 0 : 1)$, $p_4 = (1 : \alpha : \beta)$, and $p_5 = (1 : a : b)$ for some $a, b \in \mathbb{C}$, and $(\alpha, \beta) = (1, -1)$ or $(1, -1/2)$. Since the restriction $\Pi|_{S_e} : S_e \rightarrow X$ is surjective for all $e \in \mathbb{C}^2$, $S_e^{sm} \cap U$ forms a dense open subset of S_e^{sm} . From this and Proposition 2.7 we get the theorem. \square

Remark 2.8. The question on the relation between the Lagrangian fibration structure of the Hitchin map for the case of $g = 2$ and the Lagrangian fibration structure of the cotangent bundle of a del Pezzo surface X of degree 4 was raised by Beaville and Brambila-Paz when the second named author gave a talk at the the conference for Fabrizio Catanese's 70th birthday. Let $Z = SU_C^s(2, 1)$ where C is a smooth projective curve of genus 2. By the Hitchin map $h_Z : T_Z^* \rightarrow \mathbb{C}^3 = H^0(C, 2K_C)$,

$$\bigoplus_{m=0}^{\infty} H^0(Z, \text{Sym}^m T_Z) \simeq \mathbb{C}[F_1, F_2, F_3]$$

where $F_i \in H^0(Z, \text{Sym}^2 T_Z)$. This is an isomorphism of graded rings. It is well known that Z is a complete intersection of two smooth quadrics \bar{Q}_1 and \bar{Q}_2 in \mathbb{P}^5 . More precisely, if a genus two curve C is defined by six Weierstrass points λ_i for $i = 1, \dots, 6$ then Z is isomorphic to the complete intersection of two quadrics (cf. [16], [17], [3])

$$\bar{Q}_1 = \sum_{i=1}^6 X_i^2 = 0, \quad \bar{Q}_2 = \sum_{i=1}^6 \lambda_i X_i^2 = 0.$$

The above question is whether we can find a Z such that each fiber $\Phi^{-1}(e)$ of the Lagrangian fibration of a del Pezzo surface X can be embedded naturally into each fiber of the Hitchin map $h_Z : T_Z^* \rightarrow \mathbb{C}^3 = H^0(C, 2K_C)$ with $X = Z \cap H$ where H is a hyperplane in \mathbb{P}^5 .

There is a natural identification of the pencil \mathbb{P}_Q^1 of quadrics induced by \bar{Q}_1 and \bar{Q}_2 with $\mathbb{P}(H^0(C, K_C))$. Also from Theorem 5.1 in [6], there is an also isomorphism of graded rings:

$$\bigoplus_{m=0}^{\infty} H^0(Z, \text{Sym}^m[\Omega_Z^1(1)]) \simeq \mathbb{C}[\bar{Q}_1, \bar{Q}_2].$$

Then the preimage $h_Z^{-1}(W)$ of the Hitchin map of the image of a natural embedding $W := \text{Sym}^2 H^0(C, K_C)$ in $\mathbb{C}^3 = H^0(C, 2K_C)$ is the locus of singular spectral curves. This identification is explained in detail in the thesis of Sarbeswar Pal [18]. Also recently, Hitchin [8] studies explicitly the Hitchin map $h_Z : T_Z^* \rightarrow \mathbb{C}^3 = H^0(C, 2K_C)$.

Then the question is whether the restriction of this $h_Z^{-1}(W)$ over X , which is the intersection of Z with some hyperplane section H , is the cotangent bundle of X . Since we have a natural identification between

$$\bigoplus_{m=0}^{\infty} H^0(Z \cap H, \text{Sym}^m[\Omega_{Z \cap H}^1(1)]) \simeq \mathbb{C}[\bar{Q}_1 \cap H, \bar{Q}_2 \cap H] \quad \text{and}$$

$$\bigoplus_{m=0}^{\infty} H^0(Z, \text{Sym}^m[\Omega_Z^1(1)]) \simeq \mathbb{C}[\bar{Q}_1, \bar{Q}_2],$$

and due to the description of an irreducible component of a general fiber of the locus of singular spectral curves (Theorem 1.3 in [11]), if the question is true then a general fiber of the locus of singular spectral curves seems to be isomorphic either a \mathbb{P}^1 bundle or an elliptic fiber bundle over S_e (up to étale cover) in Theorem 1.2. We cannot answer on this question now because there is no enough study on the $h_Z : T_Z^* \rightarrow \mathbb{C}^3 = H^0(C, 2K_C)$. We leave it for the future study.

3. LEVEL SURFACES IN THE LAGRANGIAN FIBRATION

We use the same notations as in the introduction. Let X be a del Pezzo surface of degree 4. Let Q_1 and Q_2 be two quadratic forms in variables y_1, \dots, y_5 defining $X \subset \mathbb{P}^4 = \mathbb{P}_{y_1, \dots, y_5}^4$ such that $\det Q_1 = 1$. We define the characteristic polynomial $P(t) := \det(tQ_1 - Q_2)$, then it satisfies

$$P(t) = \prod_{i=1}^5 (t - \theta_i)$$

where all $\theta_i \in \mathbb{C}$ are distinct.

We have a pencil of quadric hypersurfaces in \mathbb{P}^4 :

$$\psi : \mathcal{Q} = \{Q_{\mathbf{e}}\}_{\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1} \rightarrow \mathbb{P}_{e_1, e_2}^1 = \mathbb{P}^1$$

such that its fiber at $\mathbf{e} = (e_1 : e_2) \in \mathbb{P}_{e_1, e_2}^1$ corresponds to the quadric hypersurface $Q_{\mathbf{e}}$ in \mathbb{P}^4 defined by $e_2 Q_1 - e_1 Q_2 = 0$.

For each $i = 1, \dots, 5$, set $\mathbf{a}_i = (1 : \theta_i) \in \mathbb{P}_{e_1, e_2}^1$. Then $Q_{\mathbf{e}}$ is singular exactly only when $\mathbf{e} = \mathbf{a}_i$ for some i .

Lemma 3.1 ([19]). *X is isomorphic to the blow-up of \mathbb{P}^2 at the images $p_i \in \mathbb{P}^2$ of $\mathbf{a}_i \in \mathbb{P}_{e_1, e_2}^1$ under the Veronese embedding $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$, and is isomorphic to the subscheme of $\mathbb{P}_{y_1, \dots, y_5}^4$ defined by*

$$(3.1) \quad \sum_{i=1}^5 P'(\theta_i)^{-1} y_i^2 = \sum_{i=1}^5 P'(\theta_i)^{-1} \theta_i y_i^2 = 0.$$

3.1. Surfaces in the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$. Let us consider X as the blow-up of \mathbb{P}^2 at $p_1, \dots, p_5 \in \mathbb{P}^2$ in Lemma 3.1 and denote by $\mu : X \rightarrow \mathbb{P}^2$ the blow-up morphism.

3.1.1. Description of lines in X . Let E_i be the exceptional curve on X over p_i and C the proper transform of the unique conic in \mathbb{P}^2 through all p_1, \dots, p_5 . For each $1 \leq i \neq j \leq 5$, let $\ell_{i,j} \subset X$ be the proper transform of the line in \mathbb{P}^2 connecting p_i and p_j . Then $C, \{E_i\}_i$ and $\{\ell_{i,j}\}_{i,j}$ are exactly the 16 lines ℓ_1, \dots, ℓ_{16} in X in the introduction. We denote by C', E'_i and $\ell'_{i,j} \subset \mathbb{P}(T_X)$ the sections of the respective lines associated quotients of the form: $T_X|_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1)$.

3.1.2. Linear system $|2\zeta|$. As seen in the introduction, the pencil $\{Q_{\mathbf{e}}\}_{\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1}$ of quadric hypersurfaces induced by Q_1 and Q_2 gives the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$. Let ℓ'_i be 16 sections of $\mathbb{P}(T_X)|_{\ell_i} \rightarrow \ell_i$ which are associated to quotients $T_X|_{\ell_i} = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-1)$. Let B be the base locus of the linear system $|2\zeta|$ in $\mathbb{P}(T_X)$. Using $\zeta \cdot \ell'_i = -1$ and the description of Section 2 in [6] we can show that B is supported on the disjoint union of 16 sections ℓ'_i so that $B = \sum_{i=1}^{16} a_i \ell'_i$ for some integers $a_i \geq 1$. Using the Grothendieck relation

$$\zeta^2 + \pi^* K_X \cdot \zeta + \pi^* c_2(T_X) = 0$$

we can calculate $\zeta^3 = -4$. Therefore

$$(2\zeta)^2 \cdot \zeta = \sum_{i=1}^{16} a_i \ell'_i \cdot \zeta = - \sum_{i=1}^{16} a_i = -16$$

which implies that $a_i = 1$ for all i .

Let

$$\mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$$

be the blow-up along B . We have a smooth member K_g of $|2\zeta|$ (see Corollary 2.4 in [4]). The exact sequence on normal bundles

$$0 \rightarrow N_{\ell'_i/K_g} = \mathcal{O}_{\ell'_i}(-2) \rightarrow N_{\ell'_i/\mathbb{P}(T_X)} \rightarrow N_{K_g/\mathbb{P}(T_X)}|_{\ell'_i} = \mathcal{O}_{\ell'_i}(-2) \rightarrow 0$$

shows that $N_{\ell'_i/\mathbb{P}(T_X)} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ because $\text{Ext}^1(\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$. Thus the exceptional divisor over ℓ'_i of the blow-up μ_B is isomorphic to $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)) \cong \mathbb{P}^1 \times \mathbb{P}^1$. This implies that, after blow-up, the rational map $\tilde{\phi} : \mathbb{P}(T_X) \dashrightarrow \mathbb{P}_{e_1, e_2}^1$ induced by the morphism $\Phi : T_X^* \rightarrow \mathbb{C}_{e_1, e_2}^2$ defined by the pair (Q_1, Q_2) can be extended to a morphism

$$\phi : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}_{e_1, e_2}^1 = \mathbb{P}^1$$

which is a family of members of $|2\zeta|$. We often consider each fiber $K_{\mathbf{e}} = \phi^{-1}(\mathbf{e})$ as a subscheme of $\mathbb{P}(T_X)$.

For each $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1$, we let

$$\pi_{\mathbf{e}} : K_{\mathbf{e}} \rightarrow X$$

be the restriction of the composition $\pi \circ \mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X) \rightarrow X$.

Lemma 3.2. *For each point x in X , $\pi_{\mathbf{e}}^{-1}(x)$ consists of two points with multiplicity except only when $Q_{\mathbf{e}}$ is singular and x is one of the intersection points of some two lines in X . In this exceptional case, $\pi_{\mathbf{e}}^{-1}(x)$ is isomorphic to \mathbb{P}^1 .*

Proof. Let Q be a smooth quadric hypersurface in \mathbb{P}^4 such that $X = Q_{\mathbf{e}} \cap Q$. The description of Section 2 in [6] says that each fiber $\pi_{\mathbf{e}}^{-1}(x)$ parametrizes lines in $Q_{\mathbf{e}} \cap \mathbf{T}_x X$ through x , where $\mathbf{T}_x X \subset \mathbb{P}^4$ denotes the embedded projective tangent plane to X at x . So we only need to show the next claim.

Claim: For a point x in X , $\mathbf{T}_x X \cap Q_{\mathbf{e}} = \mathbf{T}_x X$ if and only if $Q_{\mathbf{e}}$ is singular and x is the intersection point of some two lines in X .

Assume that $\mathbf{T}_x X \cap Q_{\mathbf{e}} = \mathbf{T}_x X$. Then $\mathbf{T}_x X \subset Q_{\mathbf{e}}$. Since any smooth quadric hypersurface in \mathbb{P}^4 contains no plane in \mathbb{P}^4 , $Q_{\mathbf{e}}$ is singular so that it is a cone over a quadric surface in \mathbb{P}^3 . We also have equalities $\mathbf{T}_x X \cap X = \mathbf{T}_x X \cap Q_{\mathbf{e}} \cap Q = \mathbf{T}_x X \cap Q$ as a set, which implies that $\mathbf{T}_x X \cap X$ is a union of some two lines in X and x is the intersection point of them.

Conversely, assume that $Q_{\mathbf{e}}$ is singular and $x \in X$ is the intersection point of some two lines ℓ_{l_1} and ℓ_{l_2} in X . Then the intersection $Q_{\mathbf{e}} \cap \mathbf{T}_x X$ contains ℓ_{l_1} , ℓ_{l_2} and some other line in the ruling of the cone structure on $Q_{\mathbf{e}}$ which implies that $Q_{\mathbf{e}} \cap \mathbf{T}_x X = \mathbf{T}_x X$. \square

3.1.3. Conic fibration on X . Let $\text{RatCurves}^n(X)$ be the normalized space of rational curves on X (see [14]). For each $i = 1, \dots, 5$, let $\mathcal{K}_{i,1}$ be the irreducible component of $\text{RatCurves}^n(X)$ containing the proper transform of a general line in \mathbb{P}^2 through p_i , and let $\mathcal{K}_{i,2}$ be that containing the proper transform of a general conic in \mathbb{P}^2 through $\{p_1, \dots, p_5\} \setminus \{p_i\}$. There is a conic fibration

$$\pi_{i,j} : X \rightarrow \mathbb{P}^1$$

whose general fiber is a member of $\mathcal{K}_{i,j}$. The conic fibration $\pi_{i,j} : X \rightarrow \mathbb{P}^1$ has four singular fibers. For each $k \in \{1, \dots, 5\} \setminus \{i\}$, there is a singular fiber of $\pi_{i,1}$ which is the union of $\ell_{k,i}$ and E_k . Three of the four singular fibers of $\pi_{i,2}$ are the unions of two lines of the forms ℓ_{ι_1, ι_2} and ℓ_{ι_3, ι_4} with $\{\iota_1, \iota_2, \iota_3, \iota_4\} = \{1, \dots, 5\} \setminus \{i\}$, and the last one is the union of C and E_i . We note that the union of singular fibers of $\pi_{i,1}$ and $\pi_{i,2}$ is exactly the union of 16 lines ℓ_1, \dots, ℓ_{16} in X .

3.1.4. Fibration on Total dual VMRT. Let $\check{\mathcal{C}}_{i,j}$ be the total dual VMRT associated to $\mathcal{K}_{i,j}$. We refer to the paper [9] for the total dual VMRT. Let L be the class of $\mu^* \mathcal{O}_{\mathbb{P}^2}(1)$. By Corollary 2.13 in [9], we have

$$\begin{aligned} [\check{\mathcal{C}}_{i,1}] &= \zeta - \pi^* L + \pi^*[E_1] + \dots + \pi^*[E_5] - 2\pi^*[E_i], \text{ and} \\ [\check{\mathcal{C}}_{i,2}] &= \zeta + \pi^* L - \pi^*[E_1] - \dots - \pi^*[E_5] + 2\pi^*[E_i] \end{aligned}$$

and thus

$$(3.2) \quad [\check{\mathcal{C}}_{i,1}] + [\check{\mathcal{C}}_{i,2}] = 2\zeta.$$

This shows that there are 5 points $\mathbf{b}_1, \dots, \mathbf{b}_5$ in \mathbb{P}_{e_1, e_2}^1 such that $K_{\mathbf{b}_i} = \check{\mathcal{C}}_{i,1} \cup \check{\mathcal{C}}_{i,2}$.

Lemma 3.3. *The 5 points $\mathbf{b}_i \in \mathbb{P}_{e_1, e_2}^1$ are the same as the 5 points $\mathbf{a}_i \in \mathbb{P}_{e_1, e_2}^1$ after reordering.*

Proof. We only need to show that each $Q_{\mathbf{b}_i}$ is singular. Suppose not. Then by Lemma 3.2, $\check{C}_{i,1}$ and $\check{C}_{i,2}$ are isomorphic to X . Clearly at least one of $\check{C}_{i,1}$ and $\check{C}_{i,2}$ contains some ℓ'_k .

Assume that $\check{C}_{i,1}$ contains some ℓ'_k . Let F be a general fiber of $\pi_{i,1} : X \rightarrow \mathbb{P}^1$. Then $F \cdot \zeta|_{\check{C}_{i,1}} = 0$ and $\ell'_k \cdot \zeta|_{\check{C}_{i,1}} = -1$. Since F is a conic and ℓ_k is a line in X , we have $[F] = 2[\ell'_k]$ in $\check{C}_{i,1}$, a contradiction. \square

We have a fibration $\check{\pi}_{i,j} : \check{C}_{i,j} \rightarrow \mathbb{P}^1$ of curves on $\check{C}_{i,j}$ given by the composition

$$\check{\pi}_{i,j} := \pi_{i,j} \circ \pi|_{\check{C}_{i,j}} : \check{C}_{i,j} \rightarrow X \rightarrow \mathbb{P}^1.$$

Lemma 3.4. *The restriction $\pi|_{\check{C}_{i,j}} : \check{C}_{i,j} \rightarrow X$ of $\pi : \mathbb{P}(T_X) \rightarrow X$ is the blow-up of four points of X which are the singular points of the singular fibers of the conic fibration $\pi_{i,j} : X \rightarrow \mathbb{P}^1$. Moreover the proper transform in $\check{C}_{i,j}$ of a line ℓ_k in a singular fiber of $\pi_{i,j}$ is equal to ℓ'_k .*

Proof. Let us assume that a singular fiber of $\pi_{i,1} : X \rightarrow \mathbb{P}^1$ consists of two lines ℓ_{ν_1} and ℓ_{ν_2} meeting at x . Then by Lemmas 3.2 and 3.3 the preimage $\pi_{\mathbf{b}_i}^{-1}(x)$ of $\pi_{\mathbf{b}_i} : K_{\mathbf{b}_i} \rightarrow X$ is isomorphic to \mathbb{P}^1 . Since $\pi_{\mathbf{b}_i}|_{\check{C}_{i,1}} = \pi|_{\check{C}_{i,1}}$ and $\pi_{\mathbf{b}_i}|_{\check{C}_{i,2}} = \pi|_{\check{C}_{i,2}}$ give isomorphisms in the outside of singular fibers of $\pi_{i,1}$ and $\pi_{i,2}$ respectively, the fiber $\pi_{\mathbf{b}_i}^{-1}(x)$ is contained in $\check{C}_{i,1}$. This shows the first statement in our lemma.

The fiber of $\check{\pi}_{i,1} : \check{C}_{i,1} \rightarrow X \rightarrow \mathbb{P}^1$ over the singular fiber $\ell_{\nu_1} \cup \ell_{\nu_2}$ of $\pi_{i,1}$ consists of the proper transforms $\hat{\ell}_{\nu_1}$ and $\hat{\ell}_{\nu_2}$ of ℓ_{ν_1} and ℓ_{ν_2} respectively, and 2ℓ where ℓ is the exceptional curve over x . Clearly ℓ is a fiber of $\pi : \mathbb{P}(X) \rightarrow X$ and hence $\ell \cdot \zeta|_{\check{C}_{i,1}} = 1$. From this and (fiber of $\check{\pi}_{i,1}) \cdot \zeta|_{\check{C}_{i,1}} = 0$, it follows that $\hat{\ell}_{\nu_1} \cdot \zeta|_{\check{C}_{i,1}} = -1$ and $\hat{\ell}_{\nu_2} \cdot \zeta|_{\check{C}_{i,1}} = -1$. This implies that $\hat{\ell}_{\nu_1} = \ell'_{\nu_1}$ and $\hat{\ell}_{\nu_2} = \ell'_{\nu_2}$. We are done. \square

We know that the five $K_{\mathbf{b}_i}$ are reducible. Next lemma shows that there is no other reducible $K_{\mathbf{e}}$.

Lemma 3.5. *For any $\mathbf{e} \in \mathbb{P}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is irreducible and the morphism $\pi_{\mathbf{e}} : K_{\mathbf{e}} \rightarrow X$ is a double cover, i.e., a finite morphism of degree 2.*

Proof. Since $Q_{\mathbf{e}}$ is smooth, Lemma 3.2 says that the morphism $\pi_{\mathbf{e}}$ is a finite morphism of degree 2. If $K_{\mathbf{e}}$ is reducible, then it is a union of two irreducible components which are isomorphic to X . We have a contradiction by the same reason as in the proof of Lemma 3.3. \square

For each $\mathbf{e} \in \mathbb{P}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, let $D_{\mathbf{e}} \subset X$ be the branch curve of the double covering $\pi_{\mathbf{e}} : K_{\mathbf{e}} \rightarrow X$ (see Lemma 3.5); the branch curve $D_{\mathbf{e}}$ is the locus of points x such that $Q_{\mathbf{e}} \cap \mathbf{T}_x X$ is a double line.

3.1.5. General $K_{\mathbf{e}}$. For a general $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is a K3 surface of degree 8 of Kummer type; Since $K_{\mathbb{P}(T_X)} = -2\zeta$, $K_{K_{\mathbf{e}}} = \mathcal{O}_{K_{\mathbf{e}}}$. So the branch curve $D_{\mathbf{e}} \subset X$ of the double covering $\pi_{\mathbf{e}} = \pi|_{K_{\mathbf{e}}} : K_{\mathbf{e}} \rightarrow X$ is in $|\mathcal{O}_X(2)|$ because $-K_X = \mathcal{O}_X(1)$ where $\mathcal{O}_X(1)$ is a hyperplane section of X in \mathbb{P}^4 . Therefore $D_{\mathbf{e}}$ is a nonsingular curve of genus 5 with degree

8 in \mathbb{P}^4 , and tangent to all 16 lines ℓ_i in X . And the lifts $\ell'_i \subset \mathbb{P}(T_X)$ of the 16 lines ℓ_i in X as in the introduction are (-2) -curves $\ell_{\mathbf{e},i}$ in $K_{\mathbf{e}} \subset \text{Bl}_B \mathbb{P}(T_X)$. These 16 (-2) -curves $\ell_{\mathbf{e},i}$ are the intersection of $K_{\mathbf{e}}$ with the exceptional divisor D of $\mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$. By the blow-down $\mu : X \rightarrow \mathbb{P}^2$, $D_{\mathbf{e}}$ goes to a plane sextic curve with five cusps at $\{p_1, \dots, p_5\}$.

By the above explanation, we obtain the following lemma.

Lemma 3.6. *For a general $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is a K3 surface of degree 8 of Kummer type. It has 16 (-2) -curves $\ell_{\mathbf{e},i}$ which are intersection of $K_{\mathbf{e}}$ with the exceptional divisor D of the blow-up $\mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$.*

Remark 3.7. We know

$$\chi_{\text{top}}(\text{Bl}_B \mathbb{P}(T_X)) = \chi_{\text{top}}(\mathbb{P}(T_X)) + 32 = \chi_{\text{top}}(X) \cdot 2 + 32 = 48.$$

Since $\chi_{\text{top}}(\text{K3 surface}) = 24$ and $\chi_{\text{top}}(K_{\mathbf{b}_i}) = 24$ for all $i = 1, \dots, 5$, $\chi_{\text{top}}(K_{\mathbf{e}}) = 24$ for a general $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1$ and $\{\mathbf{b}_1, \dots, \mathbf{b}_5\} \subset \mathbb{P}_{e_1, e_2}^1$. This seems to imply that for every $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is a K3 surface of degree 8 of Kummer type. In Corollary 3.15, we prove that this is true by considering on the Lagrangian fibration of the map $T_X^* \rightarrow \mathbb{C}_{e_1, e_2}^2$.

Remark 3.8. By the result by Skorobogatov (Theorem 3.1 in [19]), we have more explicit description of $K_{\mathbf{e}}$ for a general $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$. There exists an embedding $K_{\mathbf{e}} \subset \mathbb{P}_{y_1, \dots, y_6}^5$ so that it is defined by

$$\sum_{i=1}^6 Q'(\theta_i)^{-1} y_i^2 = \sum_{i=1}^6 Q'(\theta_i)^{-1} \theta_i y_i^2 = \sum_{i=1}^6 Q'(\theta_i)^{-1} \theta_i^2 y_i^2 = 0$$

where $\theta_1, \dots, \theta_5$ are the same θ_i s in Lemma 3.1, θ_6 is determined by $K_{\mathbf{e}}$, and

$$Q(t) := \prod_{i=1}^6 (t - \theta_i).$$

Furthermore the restriction of the projection map

$$\mathbb{P}_{y_1, \dots, y_6}^5 \dashrightarrow \mathbb{P}_{y_1, \dots, y_5}^4, (y_1 : \dots : y_6) \mapsto (y_1 : \dots : y_5)$$

gives a double cover $\pi_{\mathbf{e}} : K_{\mathbf{e}} \rightarrow X$ branched on a degree 8 curve $D_{\mathbf{e}}$ in X defined by

$$\sum_{i=1}^5 Q'(\theta_i)^{-1} y_i^2 = \sum_{i=1}^5 Q'(\theta_i)^{-1} \theta_i y_i^2 = \sum_{i=1}^5 Q'(\theta_i)^{-1} \theta_i^2 y_i^2 = 0.$$

3.1.6. Reducible $K_{\mathbf{e}}$. When $K_{\mathbf{e}}$ goes to $K_{\mathbf{b}_i} = \check{\mathcal{C}}_{i,1} \cup \check{\mathcal{C}}_{i,2}$, the branch curve $D_{\mathbf{e}}$ of $\pi_{\mathbf{e}} : K_{\mathbf{e}} \rightarrow X$ goes to $2E_{\mathbf{b}_i}$ where $E_{\mathbf{b}_i}$ is an elliptic curve which is a hyperplane section of X in \mathbb{P}^4 . The image of $E_{\mathbf{b}_i}$ of the blow-up $\mu : X \rightarrow \mathbb{P}^2$ is a cubic curve in \mathbb{P}^2 tangent to the line $\ell_{i,k}$ at p_k for each $k \in \{1, \dots, 5\} \setminus \{i\}$. For each i , this cubic plane curve is uniquely determined by this property.

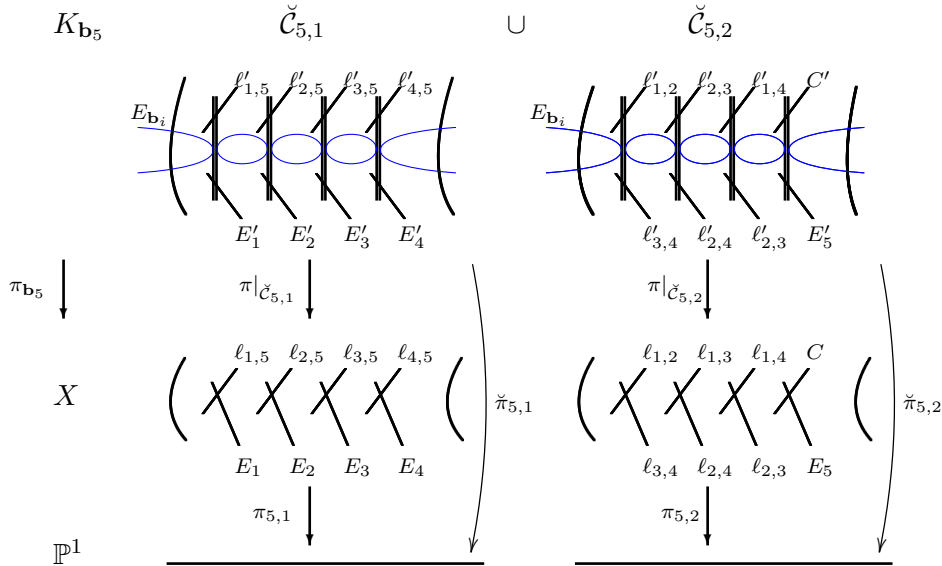
We can observe that $E_{\mathbf{b}_i}$ is the closure of the locus of x in X such that some two conics in X which are members of $\mathcal{K}_{i,1}$ and $\mathcal{K}_{i,2}$ respectively tangentially intersect at x .

Lemma 3.9. *$E_{\mathbf{b}_i}$ meet smooth fibers of $\pi_{i,j} : X \rightarrow \mathbb{P}^1$ at two distinct points, and the singular fibers of it at the singular points.*

Proof. Given a smooth conic curve in \mathbb{P}^2 through four points $\{p_1, \dots, p_5\} \setminus \{p_i\}$, there are two distinct lines in \mathbb{P}^2 through the point p_i which are tangent lines of the given conic curve. When this smooth conic specializes to a singular conic in X , the above two distinct lines in \mathbb{P}^2 goes to the unique double line in \mathbb{P}^2 through p_i and the singular point of that singular conic curve. This shows our lemma for $\pi_{i,2}$. The proof for the fibers of $\pi_{i,1}$ can be done in a similar method. \square

From Lemma 3.9, it follows that the restriction $\pi_{i,j}|_{E_{\mathbf{b}_i}} : E_{\mathbf{b}_i} \rightarrow \mathbb{P}^1$ is a double cover branched at the four singular values of $\pi_{i,j} : X \rightarrow \mathbb{P}^1$.

The intersection curve between two components $\check{C}_{i,1}$ and $\check{C}_{i,2}$ is the proper transform of $E_{\mathbf{b}_i}$ of the blow-up $\pi|_{\check{C}_{i,j}} : \check{C}_{i,j} \rightarrow X$. We will denote it by the same notation $E_{\mathbf{b}_i}$. By Lemma 3.9, $E_{\mathbf{b}_i}$ intersects two distinct points at each smooth fiber of $\check{\pi}_{i,j} : \check{C}_{i,j} \rightarrow \mathbb{P}^1$, and one point with multiplicity two at the exceptional curve of singular fibers of it; We note that the multiplicity of this exceptional curve is two in a singular fiber.



So we obtain the following lemma.

Lemma 3.10. *For each $i = 1, \dots, 5$, we have the following description of $K_{\mathbf{b}_i}$.*

- (i) $K_{\mathbf{b}_i}$ consists two irreducible components $\check{C}_{i,1}$ and $\check{C}_{i,2}$.
- (ii) Each $\check{C}_{i,j}$ for $j = 1, 2$ is isomorphic to the blow-up of four distinct points of X . These four points are singular points of the four singular fibers of the conic fibration $\pi_{i,j} : X \rightarrow \mathbb{P}^1$.
- (iii) $\check{C}_{i,1} \cap \check{C}_{i,2}$ is a smooth elliptic curve $E_{\mathbf{b}_i}$.
- (iv) In the fibration $\check{\pi}_{i,j} : \check{C}_{i,j} \rightarrow \mathbb{P}^1$, $E_{\mathbf{b}_i}$ intersects two distinct points at each smooth fiber, and one point at the exceptional curve of each singular fiber.

3.2. Description of level surfaces S_e . From now on, we want to describe level surfaces S_e . As seen in the introduction, S_e is defined by $\Phi^{-1}(e)$ where

$$\Phi : T_X^* \rightarrow \mathbb{C}_{e_1, e_2}^2 = \mathbb{C}^2$$

is the morphism defined by (Q_1, Q_2) . Here we consider Q_i as sections in $H^0(X, \text{Sym}^2 T_X)$.

Let

$$\Pi_e : S_e \rightarrow X$$

be the restriction of $\Pi : T_X^* \rightarrow X$. Take $e \neq 0 \in \mathbb{C}_{e_1, e_2}^2$ and denote by $\mathbf{e} \in \mathbb{P}^1$ the image point of e under the quotient map $\mathbb{C}_{e_1, e_2}^2 \setminus \{0\} \rightarrow \mathbb{P}_{e_1, e_2}^1$. There is a morphism

$$\tau_e : S_e \rightarrow K_{\mathbf{e}}$$

induced by the quotient map $T_X^* \dashrightarrow \mathbb{P}(T_X)$. So we have the following commutative diagram:

$$\begin{array}{ccc} S_e & \xrightarrow{\tau_e} & K_{\mathbf{e}} \\ & \searrow \Pi_e & \swarrow \pi_{\mathbf{e}} \\ & & X \end{array}$$

Since there is a graded ring isomorphism:

$$\bigoplus_{m=0}^{\infty} H^0(X, \text{Sym}^m T_X) \simeq \mathbb{C}[Q_1, Q_2],$$

$S_{\lambda e} \cong S_e$ for all $\lambda \in \mathbb{C}^*$. We have the following diagram of maps.

$$\begin{array}{ccccc} \ell_{\mathbf{e}, i} & \simeq & \ell'_i & & \\ \cap & & \cap & & \\ K_{\mathbf{e}} & \simeq & K_{\mathbf{e}} & \xleftarrow{\tau_e} & S_e \\ \cap & & \cap & & \cap \\ \text{Bl}_B \mathbb{P}(T_X) & \rightarrow & \mathbb{P}(T_X) & \xleftarrow{\dots} & T_X^* \\ & & \searrow \pi & & \searrow \Pi \\ & & X & & \\ & & \cup & & \downarrow \Phi \\ & & \ell_i & & \mathbb{C}_{e_1, e_2}^2 \\ & & \downarrow \tilde{\phi} & & \downarrow \\ \phi & \searrow & \mathbb{P}_{\mathbf{e}}^1 & \xleftarrow{\dots} & \mathbb{C}_e^2 \\ & & \downarrow & & \downarrow \\ & & \mathbf{e} & \xleftarrow{\dots} & e \end{array}$$

Lemma 3.11. *For every $e \in \mathbb{C}_{e_1, e_2}^2 \setminus \{(0)\}$, there is an involution ι on S_e acting freely and the morphism $\tau_e : S_e \rightarrow K_{\mathbf{e}}$ factors through the quotient map $S \rightarrow S/\iota$, i.e.,*

$$\tau_e : S_e \rightarrow S_e/\iota \hookrightarrow K_{\mathbf{e}}$$

so that $S_e/\iota = K_{\mathbf{e}} \setminus \cup_{i=1}^{16} \ell_{\mathbf{e}, i}$

Proof. For each $e = (e_1, e_2)$ and each point in X , there is an open neighborhood $U \cong \mathbb{C}_{x, y}^2$ of that point such that $S_e|_{\Pi^{-1}(U)} \subset \Pi^{-1}(U) \cong \mathbb{C}_{x, y, u, v}^4$ is locally defined by equations

$$Q_1 = f(x, y)u^2 + g(x, y)v^2 + h(x, y)uv = e_1 \text{ and}$$

$$Q_2 = c(x, y)u^2 + d(x, y)v^2 + e(x, y)uv = e_2.$$

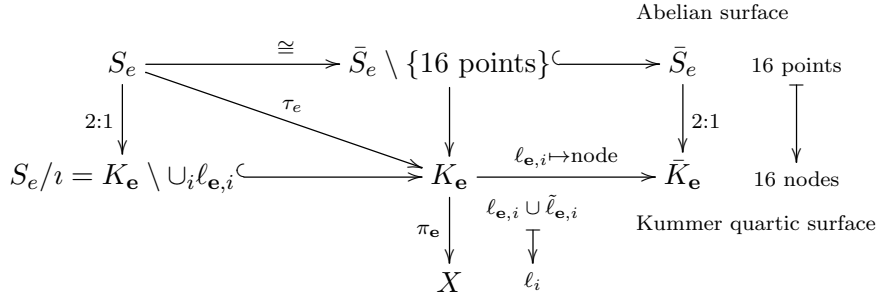
Here $Q_1 = H$ and $Q_2 = G$ in the notations in Section 2.3. So for a general point in X , there are four points in the preimage of the map $\Pi_e : S_e \rightarrow X$. And there is a natural involution $\iota : (x, y, u, v) \mapsto (x, y, -u, -v)$ acting freely on S_e .

Let $t = \frac{u}{v}$. Then given (x, y) , the solution of the equation

$$e_2(ft^2 + g + ht) = e_1(ct^2 + d + et)$$

gives a fiber of the map $\pi_e : K_e \rightarrow X$ and a fiber of the map $S_e/\iota \rightarrow X$. Therefore $S_e/\iota \hookrightarrow K_e$. Since the base locus of the linear system $|2\zeta|$ consists exactly of 16 sections ℓ'_i which are $\ell_{e,i}$ in K_e , we have $S_e/\iota = K_e \setminus \cup_{i=1}^{16} \ell_{e,i}$. \square

3.2.1. *General S_e .* Take general $e \in \mathbb{C}_{e_1, e_2}^2$ so that S_e is smooth. The preimage $\pi_e^{-1}(\ell_i)$ of $\ell_i \subset X$ splits into two curves in K_e , one is (-2) -curves $\ell_{e,i}$ and the other is a conic, denoted by $\tilde{\ell}_{e,i}$, cut by a trope (Remark 8.6.9 in [5]). Let \bar{K}_e be a Kummer quartic surface with 16 nodes obtained by contracting 16 (-2) -curves $\ell_{e,i}$ in K_e . It is well known that \bar{K}_e has a double cover \bar{S}_e which is an abelian surface. We note that \bar{S}_e does not contain any rational curve because it is an abelian surface. The level surface S_e for a general e is $\bar{S}_e \setminus \{16 \text{ points}\}$ where these 16 points are the preimage of 16 nodes of the double cover $\bar{S}_e \rightarrow \bar{K}_e$. Next figure shows these situations.



Remark 3.12. Take $\bar{\mathbf{e}} \neq \mathbf{b}_i \in \mathbb{P}_{e_1, e_2}^1$. Let $C_{\bar{\mathbf{e}}}$ be the smooth curve of genus 2 with 6 Weierstrass points over $\mathbf{b}_1, \dots, \mathbf{b}_5, \bar{\mathbf{e}} \in \mathbb{P}_{e_1, e_2}^1$ under the hyperelliptic involution $C_{\bar{\mathbf{e}}} \rightarrow \mathbb{P}_{e_1, e_2}^1$. If K_e is smooth then S_e can be embedded into the Jacobian variety $J_{\bar{\mathbf{e}}}$ of $C_{\bar{\mathbf{e}}}$ for some $\bar{\mathbf{e}}$ so that $J_{\bar{\mathbf{e}}} \setminus S_e$ consists of 16 disjoint points. It is not clear to us that $\bar{\mathbf{e}} = \mathbf{e}$.

3.2.2. *Reducible S_e .* For each i , we take one point $b_i \in \mathbb{C}_{e_1, e_2}^2$ over \mathbf{b}_i under the quotient map $\mathbb{C}_{e_1, e_2}^2 \setminus \{0\} \rightarrow \mathbb{P}_{e_1, e_2}^1$. We recall that $S_{b_i} \cong S_{\lambda b_i}$ for all $\lambda \in \mathbb{C}^*$. Now let us describe S_{b_i} by using the explicit description of $K_{\mathbf{b}_i}$ in Lemma 3.10. Let $A_{i,j}$ be the preimage $\tau_{b_i}^{-1}(\check{C}_{i,j})$ so that $S_{b_i} = A_{i,1} \cup A_{i,2}$. The restriction $\tau_{i,j} = \tau_{b_i}|_{A_{i,j}} : A_{i,j} \rightarrow \check{C}_{i,j}$ is a finite morphism of degree 2 whose image is equal to $\check{C}_{i,j} \setminus \cup_{k=1}^{16} \ell_{e,k}$. We remark that each $\check{C}_{i,j}$ contains only 8 (-2) curves and the multiplicity of the exceptional curves of $\pi|_{\check{C}_{i,j}} : \check{C}_{i,j} \rightarrow X$ is two in the singular fiber of $\check{\pi}_{i,j} : \check{C}_{i,j} \rightarrow \mathbb{P}^1$.

We have a fibration

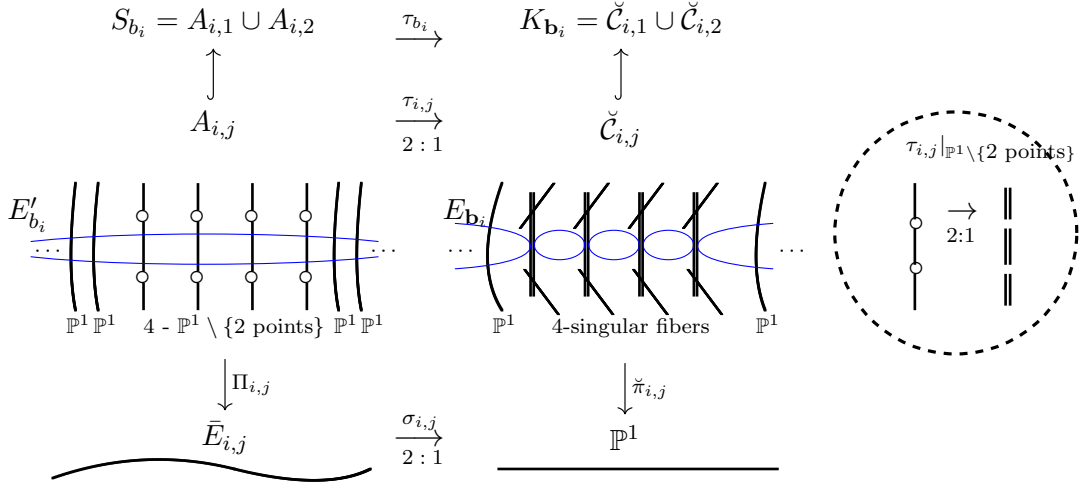
$$\Pi_{i,j} : A_{i,j} \rightarrow \bar{E}_{i,j}$$

over an elliptic curve $\bar{E}_{i,j}$, and a double cover $\sigma_{i,j} : \bar{E}_{i,j} \rightarrow \mathbb{P}^1$ branched on four singular values of $\check{\pi}_{i,j}$ and making the following commutative diagram:

$$\begin{array}{ccc} A_{i,j} & \xrightarrow{\tau_{i,j}} & \check{C}_{i,j} \\ \Pi_{i,j} \downarrow & & \downarrow \check{\pi}_{i,j} \\ \bar{E}_{i,j} & \xrightarrow[\text{2:1}]{\sigma_{i,j}} & \mathbb{P}^1 \end{array}$$

The preimage $E'_{b_i} = \tau_{i,j}^{-1}(E_{b_i}) \subset A_{i,j}$ intersect two distinct points on each fiber of $\Pi_{i,j}$. Every fiber of $\Pi_{i,j}$ is either \mathbb{P}^1 or $\mathbb{P}^1 \setminus \{\text{two points}\}$, and there are four fibers which are $\mathbb{P}^1 \setminus \{\text{two points}\}$. The restriction of $\tau_{i,j}$ to a fiber of $\Pi_{i,j}$ of the form $\mathbb{P}^1 \setminus \{\text{two points}\}$ gives a degree 2 morphism to the exceptional curve in a singular fiber of $\pi_{i,j}$.

Therefore S_{b_i} has two components $A_{i,1}$ and $A_{i,2}$, both are ruled surface $\setminus \{8 \text{ points}\}$ over an elliptic curve. The intersecting curve between $A_{i,1}$ and $A_{i,2}$ is the elliptic curve E'_{b_i} .



Now we are ready to prove Theorem 1.2.

Theorem 3.13. *We have the following description of level surfaces of the map $\Phi : T_X^* \rightarrow \mathbb{C}_{e_1, e_2}^2$.*

- For every $e \in \mathbb{C}_{e_1, e_2}^2 \setminus \cup_{i=1}^5 \mathbb{C} \cdot b_i$, S_e is $\bar{S}_e \setminus \{16 \text{ points}\}$ where \bar{S}_e is isomorphic to the Jacobian variety of a curve of genus two. Here, $\mathbb{C} \cdot b_i$ is the line in \mathbb{C}_{e_1, e_2}^2 through b_i and the origin 0 .
- For each $i = 1, \dots, 5$, we have the following description of S_{b_i} .
 - S_{b_i} consists two irreducible components $A_{i,1}$ and $A_{i,2}$.
 - Each $A_{i,j}$ is a ruled surface $\setminus \{8 \text{ points}\}$ over an elliptic curve $\bar{E}_{i,j}$.
 - $A_{i,1} \cap A_{i,2}$ is an elliptic curve E'_{b_i} .

(iv) *In the fibration $A_{i,j} \rightarrow \bar{E}_{i,j}$, E'_{b_i} intersects two distinct points at each fiber.*

Proof. By the above argument, we prove (a) for a general S_e and (b). So it is enough to prove that every S_e satisfies (a). If S_e has non-isolated singularities then K_e has also non-isolated singularities. But we know that K_e has at most isolated singularities if \mathbf{e} does not belong to $\{\mathbf{b}_1, \dots, \mathbf{b}_5\}$; Since the corresponding quadric Q_e is smooth, we have K_e is irreducible and $\pi_e : K_e \rightarrow X$ is a double cover (see Lemma 3.5) which implies that K_e has at worst isolated singularities.

Therefore S_e has at worst isolated singularities. We also note that S_e is isomorphic to $S_{\lambda e}$ for all $\lambda \in \mathbb{C}^*$, so S_e is a general singular fiber. Then by using the idea of the characteristic vector fields in [11], S_e should be smooth by the following reason.

Suppose q is an isolated singularity of S_e . Let $\mathbf{S} = \cup_{\lambda \in \mathbb{C}^*} S_{\lambda e}$, which is called a vertical surface in [11]. Let z be a local coordinate in a neighborhood of $\Phi(S_e)$ in $\Phi(\mathbf{S}) = \mathbb{C}^*$ and consider the Hamiltonian vector fields

$$\nu_i := \iota_\omega(\Phi^* dz)$$

by identification $\iota_\omega : T_M^* \rightarrow T_M$ where $M = T_X^*$ via using the natural symplectic 2-form ω . Since Φ is a Lagrangian fibration, these vector fields are tangent to S_e . So we have a flow of singularities in S_e coming from the singularity q . Therefore S_e cannot have an isolated singularity.

It also implies that K_e in $\mathbb{P}(T_X)$ corresponding to S_e is also smooth on $K_e \setminus \cup_{i=1}^{16} \ell_{e,i} = S_e/\iota$ because ι acts freely on S_e . Furthermore we can check that K_e is smooth along each $\ell_{e,i}$ which implies that K_e is smooth. \square

By the above theorem, we get the following corollaries.

Corollary 3.14. *The map $\Phi : T_X^* \rightarrow \mathbb{C}_{e_1, e_2}^2$ is flat.*

Proof. Clearly, T_X^* is irreducible and \mathbb{C}_{e_1, e_2}^2 is reduced. Then by using our description of level surfaces and Lemma 10.48 in [13], the map $\Phi : T_X^* \setminus \Phi^{-1}(0) \rightarrow \mathbb{C}_{e_1, e_2}^2 \setminus \{0\}$ is flat because Φ is essentially of finite type, pure dimensional, and its fibers are geometrically reduced. So it is enough to check Φ is flat over $\{0\}$.

We recall the following diagram of maps.

$$\begin{array}{ccccc}
 & & \ell'_i & & S_0 \\
 & & \cap & & \cap \\
 \text{Bl}_B \mathbb{P}(T_X) & \rightarrow & \mathbb{P}(T_X) & \xleftarrow{\dots\dots\dots} & T_X^* \\
 & \searrow \phi & \downarrow \tilde{\phi} & \swarrow \pi & \downarrow \Phi \\
 & & \mathbb{P}^1_{e_1, e_2} & \xleftarrow{\dots\dots\dots} & \mathbb{C}_{e_1, e_2}^2 \\
 & & & & \uparrow \Pi \\
 & & & & X \cup \ell_i
 \end{array}$$

Since $\tilde{\phi}$ is defined outside $\cup_{i=1}^{16} \ell'_i$, $S_0 = \Phi^{-1}(0)$ is contained in the union of the zero section of the map Φ and the preimage of $\cup_{i=1}^{16} \ell'_i$ of the quotient map $T_X^* \dashrightarrow \mathbb{P}(T_X)$. This implies that S_0 has dimension two.

For any smooth affine curve $R \subset \mathbb{C}_{e_1, e_2}^2$ through 0, S_0 is not an associated point of $\Phi^{-1}(R)$. This implies that the flatness of the map Φ . \square

In the proof of Theorem 3.13 it is proved that $K_{\mathbf{e}}$ is smooth for all $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$. From this and Lemma 3.6 we get the following corollary.

Corollary 3.15. *For every $\mathbf{e} \in \mathbb{P}_{e_1, e_2}^1 \setminus \{\mathbf{b}_1, \dots, \mathbf{b}_5\}$, $K_{\mathbf{e}}$ is a K3 surface of degree 8 of Kummer type. It has 16 (-2)-curves $\ell_{\mathbf{e}, i}$ which are intersection of $K_{\mathbf{e}}$ with the exceptional divisor D of $\mu_B : \text{Bl}_B \mathbb{P}(T_X) \rightarrow \mathbb{P}(T_X)$.*

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