

Double series for π and their q -analogues

Chuanan Wei¹, Guozhu Ruan²

¹School of Biomedical Information and Engineering
Hainan Medical University, Haikou 571199, China

²Medical Simulation Education Center
Hainan Medical University, Haikou 571199, China

Abstract. With the help of the partial derivative operator and several summation formulas for hypergeometric series, we find three double series for π . In terms of the operator just stated and several summation formulas for basic hypergeometric series, we also establish q -analogues of these double series.

Keywords: double series for π ; hypergeometric series; partial derivative operator; basic hypergeometric series; q -analogue

AMS Subject Classifications: 33D15; 05A15

1 Introduction

For a complex variable x , define the well-known Gamma function to be

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \quad \text{with } \operatorname{Re}(x) > 0.$$

Three important properties of this function can be expressed as

$$\Gamma(x+1) = x\Gamma(x), \quad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}, \quad \lim_{n \rightarrow \infty} \frac{\Gamma(x+n)}{\Gamma(y+n)} n^{y-x} = 1,$$

which will often be used without explanation in this paper. Subsequently, we may give the definition of the shifted-factorial:

$$(x)_n = \Gamma(x+n)/\Gamma(x),$$

where n is an integer and x is a complex number. Then the hypergeometric series can be defined by

$${}_rF_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix} ; z \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_r)_k}{(b_1)_k (b_2)_k \cdots (b_s)_k} \frac{z^k}{k!}.$$

Email addresses: weichuanan78@163.com (C. Wei), ythainmc@163.com (G. Ruan).

The research of π -formulas has a long history. In 1859, Bauer [2] discovered a simple result:

$$\sum_{k=0}^{\infty} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} = \frac{2}{\pi}. \quad (1.1)$$

In 1914, Ramanujan [24] displayed 17 series for $1/\pi$ without proof. Decades later, Borweins [4] proved all of them firstly. Two of Ramanujan's formulas are stated as

$$\sum_{k=0}^{\infty} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 4^k} = \frac{4}{\pi}, \quad (1.2)$$

$$\sum_{k=0}^{\infty} (8k+1) \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{k!^3 9^k} = \frac{2\sqrt{3}}{\pi}. \quad (1.3)$$

In 2011, Long [23] proposed the following conjecture : for any odd prime p ,

$$\sum_{k=0}^{(p-1)/2} (-1)^k (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 8^k} \sum_{j=1}^k \left(\frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right) \equiv 0 \pmod{p}, \quad (1.4)$$

which was certified by Swisher [27] after several years. Recently, Guo and Lian [15] conjectured two interesting double series for π related to (1.2) and (1.4):

$$\sum_{k=1}^{\infty} (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 4^k} \sum_{j=1}^k \left\{ \frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right\} = \frac{\pi}{12}, \quad (1.5)$$

$$\sum_{k=1}^{\infty} (-1)^k (6k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3 8^k} \sum_{j=1}^k \left\{ \frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right\} = -\frac{\sqrt{2}}{48} \pi. \quad (1.6)$$

which have been proved by Wei [29]. For more known series on π , we refer the reader to the papers [3, 5, 12, 22, 25, 28, 30].

Inspired by the work just mentioned, we shall established the following two theorems associated with (1.1) and (1.3).

Theorem 1.1.

$$\sum_{k=1}^{\infty} (-1)^k (4k+1) \frac{\left(\frac{1}{2}\right)_k^3}{k!^3} \sum_{i=1}^{2k} \frac{(-1)^i}{i^2} = \frac{\pi}{12}. \quad (1.7)$$

Theorem 1.2.

$$\sum_{k=1}^{\infty} (8k+1) \frac{\left(\frac{1}{2}\right)_k \left(\frac{1}{4}\right)_k \left(\frac{3}{4}\right)_k}{k!^3 9^k} \sum_{i=1}^k \left\{ \frac{1}{(2i-1)^2} - \frac{1}{36i^2} \right\} = \frac{\sqrt{3}}{54} \pi. \quad (1.8)$$

Furthermore, we shall provide the following double series for π^3 .

Theorem 1.3.

$$\sum_{k=1}^{\infty} (8k-1) \frac{(1)_k (-\frac{1}{4})_k (-\frac{3}{4})_k}{(\frac{1}{2})_k^2 (\frac{3}{2})_k} 9^k \sum_{i=1}^k \left\{ \frac{1}{(2i-1)^2} - \frac{9}{4i^2} \right\} = -\frac{\sqrt{3}\pi^3}{108}. \quad (1.9)$$

For an integer n and two complex numbers x, q with $|q| < 1$, define the q -shifted factorial to be

$$(x; q)_{\infty} = \prod_{i=0}^{\infty} (1 - xq^i), \quad (x; q)_n = \frac{(x; q)_{\infty}}{(xq^n; q)_{\infty}}.$$

For convenience, we shall also adopt the following notation:

$$(x_1, x_2, \dots, x_r; q)_m = (x_1; q)_m (x_2; q)_m \cdots (x_r; q)_m,$$

where $r \in \mathbb{Z}^+$ and $m \in \mathbb{Z}^+ \cup \{0, \infty\}$. Then following Gasper and Rahman [9], the basic hypergeometric series can be defined as

$${}_r\phi_s \left[\begin{matrix} a_1, a_2, \dots, a_r \\ b_1, b_2, \dots, b_s \end{matrix}; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_r; q)_k}{(q, b_1, b_2, \dots, b_s; q)_k} \left\{ (-1)^k q^{\binom{k}{2}} \right\}^{1+s-r} z^k.$$

Let $[n] = 1 + q + \cdots + q^{n-1}$ be the q -integer. Recently, Guo and Liu [16] and Guo and Zudilin [17] obtained the q -analogues of (1.1)-(1.3):

$$\begin{aligned} \sum_{k=0}^{\infty} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} &= \frac{(q, q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2}, \\ \sum_{k=0}^{\infty} q^{k^2} [6k+1] \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^4; q^4)_k^3} &= \frac{(1+q)(q^2, q^6; q^4)_{\infty}}{(q^4; q^4)_{\infty}^2}, \\ \sum_{k=0}^{\infty} q^{2k^2} [8k+1] \frac{(q; q^2)_k^2 (q; q^2)_{2k}}{(q^2; q^2)_{2k} (q^6; q^6)_k^2} &= \frac{(q^3; q^2)_{\infty} (q^3; q^6)_{\infty}}{(q^2; q^2)_{\infty} (q^6; q^6)_{\infty}}. \end{aligned}$$

Wei [29] got the q -analogues of (1.5) and (1.6):

$$\begin{aligned} \sum_{k=1}^{\infty} q^{k^2} [6k+1] \frac{(q; q^2)_k^2 (q^2; q^4)_k}{(q^4; q^4)_k^3} \sum_{j=1}^k \left\{ \frac{q^{2j-1}}{[2j-1]^2} - \frac{q^{4j}}{[4j]^2} \right\} \\ = \frac{(q^2; q^4)_{\infty}^2 (q^5; q^4)_{\infty}}{(q; q^4)_{\infty} (q^4; q^4)_{\infty}^2} \sum_{i=1}^{\infty} (-1)^{i-1} \frac{q^{2i}}{[2i]^2}, \\ \sum_{k=1}^{\infty} (-1)^k q^{3k^2} [6k+1] \frac{(q; q^2)_k^3}{(q^4; q^4)_k^3} \sum_{j=1}^k \left\{ \frac{q^{2j-1}}{[2j-1]^2} - \frac{q^{4j}}{[4j]^2} \right\} \\ = -\frac{(q^3, q^5; q^4)_{\infty}}{(q^4; q^4)_{\infty}^2} \sum_{i=1}^{\infty} \frac{q^{4i}}{[4i]^2}. \end{aligned}$$

More q -analogues of π -formulas can be seen in the papers [14, 18–21, 26].

Inspired by the work just mentioned, we shall derive the following q -analogues of Theorems 1.1-1.3.

Theorem 1.4.

$$\sum_{k=1}^{\infty} (-1)^k q^{k^2} [4k+1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \sum_{i=1}^{2k} (-1)^i \frac{q^i}{[i]^2} = \frac{(q, q^3; q^2)_{\infty}}{(q^2; q^2)_{\infty}^2} \sum_{j=1}^{\infty} \frac{q^{2j}}{[2j]^2}. \quad (1.10)$$

Theorem 1.5.

$$\begin{aligned} & \sum_{k=1}^{\infty} q^{2k^2} [8k+1] \frac{(q; q^2)_k^2 (q; q^2)_{2k}}{(q^2; q^2)_{2k} (q^6; q^6)_k^2} \sum_{i=1}^k \left\{ \frac{q^{2i-1}}{[2i-1]^2} - \frac{q^{6i}}{[6i]^2} \right\} \\ &= \frac{(q^3; q^2)_{\infty} (q^3; q^6)_{\infty}}{(q^2; q^2)_{\infty} (q^6; q^6)_{\infty}} \sum_{j=1}^{\infty} (-1)^{j-1} \frac{q^{3j}}{[3j]^2}. \end{aligned} \quad (1.11)$$

Theorem 1.6.

$$\begin{aligned} & \sum_{k=1}^{\infty} q^{2k^2+2k} [8k-1] \frac{(q^2; q^2)_k^2 (q^{-3}; q^2)_{2k}}{(q^4; q^2)_{2k} (q^3; q^6)_k^2} \sum_{i=1}^k \left\{ \frac{q^{6i-3}}{[6i-3]^2} - \frac{q^{2i}}{[2i]^2} \right\} \\ &= \frac{(q; q^2)_{\infty} (q^6; q^6)_{\infty}^3}{(q^4; q^2)_{\infty} (q^3; q^6)_{\infty}^3} \sum_{j=1}^{\infty} (-1)^j \frac{q^{3j-1}}{[3j]^2}. \end{aligned} \quad (1.12)$$

For a multivariable function $f(x_1, x_2, \dots, x_m)$, define the partial derivative operator \mathcal{D}_{x_i} by

$$\mathcal{D}_{x_i} f(x_1, x_2, \dots, x_m) = \frac{d}{dx_i} f(x_1, x_2, \dots, x_m) \quad \text{with } 1 \leq i \leq m.$$

Then we have the following four relations: for $n > 0$,

$$\begin{aligned} \mathcal{D}_x (x+y)_n &= (x+y)_n \sum_{i=1}^n \frac{1}{x+y-1+i}, \\ \mathcal{D}_x \sum_{i=1}^n \frac{1}{x+y+i} &= - \sum_{i=1}^n \frac{1}{(x+y+i)^2}, \\ \mathcal{D}_x (xy; q)_n &= -(xy; q)_n \sum_{i=1}^n \frac{yq^{i-1}}{1-xyq^{i-1}}, \\ \mathcal{D}_x \sum_{i=1}^n \frac{yq^i}{1-xyq^i} &= \sum_{i=1}^n \frac{y^2 q^{2i}}{(1-xyq^i)^2}, \end{aligned}$$

which will frequently be utilized without indication in this paper.

The structure of the paper is arranged as follows. We shall verify Theorems 1.1-1.3 via the partial derivative operator and some summation formulas for hypergeometric series in Section 2. Similarly, we shall prove Theorems 1.4-1.6 through the partial derivative operator and some summation formulas for basic hypergeometric series in Section 3.

2 Proof of Theorems 1.1-1.3

Firstly, we shall prove Theorem 1.1.

Proof of Theorem 1.1. Recall Dougall's ${}_5F_4$ summation formula (cf. [1, P. 71]):

$${}_5F_4 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, c, -n \\ \frac{a}{2}, 1 + a - b, 1 + a - c, 1 + a + n \end{matrix} ; 1 \right] = \frac{(1+a)_n(1+a-b-c)_n}{(1+a-b)_n(1+a-c)_n}.$$

The $c = 1 - b$ case of it reads

$${}_5F_4 \left[\begin{matrix} a, 1 + \frac{a}{2}, b, 1 - b, -n \\ \frac{a}{2}, 1 + a - b, a + b, 1 + a + n \end{matrix} ; 1 \right] = \frac{(a)_n(1+a)_n}{(a+b)_n(1+a-b)_n}. \quad (2.1)$$

Apply the partial derivative operator \mathcal{D}_b to both sides of (2.1) to obtain

$$\begin{aligned} & \sum_{k=1}^n \frac{(a)_k(1+\frac{a}{2})_k(b)_k(1-b)_k(-n)_k}{(1)_k(\frac{a}{2})_k(1+a-b)_k(a+b)_k(1+a+n)_k} \\ & \times \left\{ \sum_{i=1}^k \frac{1}{b-1+i} - \sum_{i=1}^k \frac{1}{-b+i} + \sum_{i=1}^k \frac{1}{a-b+i} - \sum_{i=1}^k \frac{1}{a+b-1+i} \right\} \\ & = \frac{(a)_n(1+a)_n}{(a+b)_n(1+a-b)_n} \left\{ \sum_{j=1}^n \frac{1}{a-b+j} - \sum_{j=1}^n \frac{1}{a+b-1+j} \right\}. \end{aligned}$$

Employing the operator \mathcal{D}_b to both sides of the last equation, there holds

$$\begin{aligned} & \sum_{k=1}^n \frac{(a)_k(1+\frac{a}{2})_k(b)_k(1-b)_k(-n)_k}{(1)_k(\frac{a}{2})_k(1+a-b)_k(a+b)_k(1+a+n)_k} \\ & \times \left\{ \left[\sum_{i=1}^k \frac{1}{b-1+i} - \sum_{i=1}^k \frac{1}{-b+i} + \sum_{i=1}^k \frac{1}{a-b+i} - \sum_{i=1}^k \frac{1}{a+b-1+i} \right]^2 \right. \\ & \left. - \left[\sum_{i=1}^k \frac{1}{(b-1+i)^2} + \sum_{i=1}^k \frac{1}{(-b+i)^2} - \sum_{i=1}^k \frac{1}{(a-b+i)^2} - \sum_{i=1}^k \frac{1}{(a+b-1+i)^2} \right] \right\} \\ & = \frac{(a)_n(1+a)_n}{(a+b)_n(1+a-b)_n} \left\{ \left[\sum_{j=1}^n \frac{1}{a-b+j} - \sum_{j=1}^n \frac{1}{a+b-1+j} \right]^2 \right. \\ & \left. + \left[\sum_{j=1}^n \frac{1}{(a-b+j)^2} + \sum_{j=1}^n \frac{1}{(a+b-1+j)^2} \right] \right\}. \quad (2.2) \end{aligned}$$

The $a = b = \frac{1}{2}$ case of (2.2) can be manipulated as

$$\sum_{k=1}^n (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3} \frac{(-n)_k}{(\frac{3}{2}+n)_k} \sum_{i=1}^{2k} \frac{(-1)^i}{i^2} = \frac{\Gamma(\frac{1}{2}+n)\Gamma(\frac{3}{2}+n)}{\Gamma(1+n)\Gamma(1+n)} \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \sum_{j=1}^n \frac{1}{4j^2}.$$

Letting $n \rightarrow \infty$ in the above identity, we arrive at

$$\sum_{k=1}^{\infty} (-1)^k (4k+1) \frac{(\frac{1}{2})_k^3}{k!^3} \sum_{i=1}^{2k} \frac{(-1)^i}{i^2} = \frac{1}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} \sum_{j=1}^{\infty} \frac{1}{4j^2}.$$

Calculating the series, which is on the right-hand side, by Euler's formula:

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}, \quad (2.3)$$

we are led to (1.7). □

Secondly, we shall give the proof of Theorem 1.2.

Proof of Theorem 1.2. An original Gosper Conjecture (cf. [10, p. 307]) is

$${}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, a + \frac{1}{2}, b, 1 - b, \frac{2a+1}{3} + n, -n \\ \frac{a}{2}, \frac{1}{2}, \frac{2a-b+3}{3}, \frac{2a+b+2}{3}, -3n, 1 + 2a + 3n \end{matrix} ; 1 \right] = \frac{(\frac{1+b}{3})_n (\frac{2-b}{3})_n (\frac{2a+2}{3})_n (\frac{2a+3}{3})_n}{(\frac{1}{3})_n (\frac{2}{3})_n (\frac{2a-b+3}{3})_n (\frac{2a+b+2}{3})_n}, \quad (2.4)$$

the nonterminating form of which can be seen in Gasper and Rahman [8, Equation (1.6)]. By means of the partial derivative operator \mathcal{D}_b and the $a \rightarrow a/2$ case of (2.4), we have

$$\begin{aligned} & \sum_{k=1}^n \frac{(\frac{a}{2})_k (1 + \frac{a}{4})_k (\frac{a+1}{2})_k (b)_k (1-b)_k (\frac{a+1}{3} + n)_k (-n)_k}{(1)_k (\frac{a}{4})_k (\frac{1}{2})_k (\frac{a-b+3}{3})_k (\frac{a+b+2}{3})_k (-3n)_k (1+a+3n)_k} \\ & \times \left\{ \sum_{i=1}^k \frac{3}{b-1+i} - \sum_{i=1}^k \frac{3}{-b+i} + \sum_{i=1}^k \frac{1}{\frac{a-b}{3}+i} - \sum_{i=1}^k \frac{1}{\frac{a+b-1}{3}+i} \right\} \\ & = \frac{(\frac{1+b}{3})_n (\frac{2-b}{3})_n (\frac{a+2}{3})_n (\frac{a+3}{3})_n}{(\frac{1}{3})_n (\frac{2}{3})_n (\frac{a-b+3}{3})_n (\frac{a+b+2}{3})_n} \\ & \times \left\{ \sum_{j=1}^n \frac{1}{\frac{b-2}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{-b-1}{3}+j} + \sum_{j=1}^n \frac{1}{\frac{a-b}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{a+b-1}{3}+j} \right\}. \end{aligned}$$

According to the operator \mathcal{D}_b and the last equation, it is routine to understand that

$$\begin{aligned} & \sum_{k=1}^n \frac{(\frac{a}{2})_k (1 + \frac{a}{4})_k (\frac{a+1}{2})_k (b)_k (1-b)_k (\frac{a+1}{3} + n)_k (-n)_k}{(1)_k (\frac{a}{4})_k (\frac{1}{2})_k (\frac{a-b+3}{3})_k (\frac{a+b+2}{3})_k (-3n)_k (1+a+3n)_k} \\ & \times \left\{ \left[\sum_{i=1}^k \frac{3}{b-1+i} - \sum_{i=1}^k \frac{3}{-b+i} + \sum_{i=1}^k \frac{1}{\frac{a-b}{3}+i} - \sum_{i=1}^k \frac{1}{\frac{a+b-1}{3}+i} \right]^2 \right. \\ & \left. - \left[\sum_{i=1}^k \frac{9}{(b-1+i)^2} + \sum_{i=1}^k \frac{9}{(-b+i)^2} - \sum_{i=1}^k \frac{1}{(\frac{a-b}{3}+i)^2} - \sum_{i=1}^k \frac{1}{(\frac{a+b-1}{3}+i)^2} \right] \right\} \end{aligned}$$

$$\begin{aligned}
&= \frac{(\frac{1+b}{3})_n (\frac{2-b}{3})_n (\frac{a+2}{3})_n (\frac{a+3}{3})_n}{(\frac{1}{3})_n (\frac{2}{3})_n (\frac{a-b+3}{3})_n (\frac{a+b+2}{3})_n} \\
&\times \left\{ \left[\sum_{j=1}^n \frac{1}{\frac{b-2}{3} + j} - \sum_{i=1}^n \frac{1}{\frac{-b-1}{3} + j} + \sum_{j=1}^n \frac{1}{\frac{a-b}{3} + j} - \sum_{j=1}^n \frac{1}{\frac{a+b-1}{3} + j} \right]^2 \right. \\
&\quad \left. - \left[\sum_{j=1}^n \frac{1}{(\frac{b-2}{3} + j)^2} + \sum_{j=1}^n \frac{1}{(\frac{-b-1}{3} + j)^2} - \sum_{j=1}^n \frac{1}{(\frac{a-b}{3} + j)^2} - \sum_{j=1}^n \frac{1}{(\frac{a+b-1}{3} + j)^2} \right] \right\}. \quad (2.5)
\end{aligned}$$

The $a = b = \frac{1}{2}$ case of (2.5) produces

$$\begin{aligned}
&\sum_{k=1}^n (8k+1) \frac{(\frac{1}{2})_k (\frac{1}{4})_k (\frac{3}{4})_k}{k!^3} \frac{(\frac{1}{2} + n)_k (-n)_k}{(\frac{3}{2} + 3n)_k (-3n)_k} \sum_{i=1}^k \left\{ \frac{1}{(2i-1)^2} - \frac{1}{36i^2} \right\} \\
&= \frac{1}{9} \frac{\Gamma(\frac{1}{2} + n)^2 \Gamma(\frac{5}{6} + n) \Gamma(\frac{7}{6} + n)}{\Gamma(1+n)^2 \Gamma(\frac{1}{3} + n) \Gamma(\frac{2}{3} + n)} \frac{\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{2})^2 \Gamma(\frac{5}{6}) \Gamma(\frac{7}{6})} \sum_{j=1}^{2n} \frac{(-1)^{j-1}}{j^2}.
\end{aligned}$$

Letting $n \rightarrow \infty$ in the upper identity, we get

$$\begin{aligned}
&\sum_{k=1}^{\infty} (8k+1) \frac{(\frac{1}{2})_k (\frac{1}{4})_k (\frac{3}{4})_k}{k!^3 9^k} \sum_{j=1}^k \left\{ \frac{1}{(2j-1)^2} - \frac{1}{16j^2} \right\} \\
&= \frac{1}{9} \frac{\Gamma(\frac{1}{3}) \Gamma(\frac{2}{3})}{\Gamma(\frac{1}{2})^2 \Gamma(\frac{5}{6}) \Gamma(\frac{7}{6})} \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^2}. \quad (2.6)
\end{aligned}$$

On the base of Euler's formula (2.3), we can find that

$$\sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} = \sum_{j=1}^{\infty} \frac{1}{j^2} - \sum_{j=1}^{\infty} \frac{1}{(2j)^2} = \frac{\pi^2}{8}.$$

So there is the formula

$$\sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j^2} = \sum_{j=1}^{\infty} \frac{1}{(2j-1)^2} - \sum_{i=1}^{\infty} \frac{1}{(2j)^2} = \frac{\pi^2}{12}. \quad (2.7)$$

Substituting (2.7) into (2.6), we catch hold of (1.8). \square

Thirdly, we shall display the proof of Theorem 1.3.

Proof of Theorem 1.3. A known ${}_7F_6$ summation formula ((cf. [6, p. 37]) can be written as

$$\begin{aligned}
&{}_7F_6 \left[\begin{matrix} a, 1 + \frac{a}{2}, a - \frac{1}{2}, b, 2 - b, \frac{2a+2}{3} + n, -n \\ \frac{a}{2}, \frac{3}{2}, \frac{2a-b+3}{3}, \frac{2a+b+1}{3}, -1 - 3n, 1 + 2a + 3n \end{matrix} ; 1 \right] \\
&= \frac{(\frac{2+b}{3})_n (\frac{4-b}{3})_n (\frac{2a+1}{3})_n (\frac{2a+3}{3})_n}{(\frac{2}{3})_n (\frac{4}{3})_n (\frac{2a-b+3}{3})_n (\frac{2a+b+1}{3})_n}, \quad (2.8)
\end{aligned}$$

the nonterminating form of which can be observed in Gasper and Rahman [8, Equation (4.7)]. Apply the partial derivative operator \mathcal{D}_b to the $a \rightarrow a/2$ case of (2.8) to deduce

$$\begin{aligned} & \sum_{k=1}^n \frac{(\frac{a}{2})_k (1 + \frac{a}{4})_k (\frac{a-1}{2})_k (b)_k (2-b)_k (\frac{a+2}{3} + n)_k (-n)_k}{(1)_k (\frac{a}{4})_k (\frac{3}{2})_k (\frac{a-b+3}{3})_k (\frac{a+b+1}{3})_k (-1-3n)_k (1+a+3n)_k} \\ & \times \left\{ \sum_{i=1}^k \frac{3}{b-1+i} - \sum_{i=1}^k \frac{3}{1-b+i} + \sum_{i=1}^k \frac{1}{\frac{a-b}{3}+i} - \sum_{i=1}^k \frac{1}{\frac{a+b-2}{3}+i} \right\} \\ & = \frac{(\frac{2+b}{3})_n (\frac{4-b}{3})_n (\frac{a+1}{3})_n (\frac{a+3}{3})_n}{(\frac{2}{3})_n (\frac{4}{3})_n (\frac{a-b+3}{3})_n (\frac{a+b+1}{3})_n} \\ & \times \left\{ \sum_{j=1}^n \frac{1}{\frac{b-1}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{1-b}{3}+j} + \sum_{j=1}^n \frac{1}{\frac{a-b}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{a+b-2}{3}+j} \right\}. \end{aligned}$$

Employing the operator \mathcal{D}_b to both sides of the last equation, it is easy to realize that

$$\begin{aligned} & \sum_{k=1}^n \frac{(\frac{a}{2})_k (1 + \frac{a}{4})_k (\frac{a-1}{2})_k (b)_k (2-b)_k (\frac{a+2}{3} + n)_k (-n)_k}{(1)_k (\frac{a}{4})_k (\frac{3}{2})_k (\frac{a-b+3}{3})_k (\frac{a+b+1}{3})_k (-1-3n)_k (1+a+3n)_k} \\ & \times \left\{ \left[\sum_{i=1}^k \frac{3}{b-1+i} - \sum_{i=1}^k \frac{3}{1-b+i} + \sum_{i=1}^k \frac{1}{\frac{a-b}{3}+i} - \sum_{i=1}^k \frac{1}{\frac{a+b-2}{3}+i} \right]^2 \right. \\ & \quad \left. - \left[\sum_{i=1}^k \frac{9}{(b-1+i)^2} + \sum_{i=1}^k \frac{9}{(1-b+i)^2} - \sum_{i=1}^k \frac{1}{(\frac{a-b}{3}+i)^2} - \sum_{i=1}^k \frac{1}{(\frac{a+b-2}{3}+i)^2} \right] \right\} \\ & = \frac{(\frac{2+b}{3})_n (\frac{4-b}{3})_n (\frac{a+1}{3})_n (\frac{a+3}{3})_n}{(\frac{2}{3})_n (\frac{4}{3})_n (\frac{a-b+3}{3})_n (\frac{a+b+1}{3})_n} \\ & \times \left\{ \left[\sum_{j=1}^n \frac{1}{\frac{b-1}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{1-b}{3}+j} + \sum_{j=1}^n \frac{1}{\frac{a-b}{3}+j} - \sum_{j=1}^n \frac{1}{\frac{a+b-2}{3}+j} \right]^2 \right. \\ & \quad \left. - \left[\sum_{j=1}^n \frac{1}{(\frac{b-1}{3}+j)^2} + \sum_{j=1}^n \frac{1}{(\frac{1-b}{3}+j)^2} - \sum_{j=1}^n \frac{1}{(\frac{a-b}{3}+j)^2} - \sum_{j=1}^n \frac{1}{(\frac{a+b-2}{3}+j)^2} \right] \right\}. \quad (2.9) \end{aligned}$$

The $a = -\frac{1}{2}, b = 1$ case of (2.9) engenders

$$\begin{aligned} & \sum_{k=1}^n (8k-1) \frac{(1)_k (-\frac{1}{4})_k (-\frac{3}{4})_k}{(\frac{1}{2})_k^2 (\frac{3}{2})_k} \frac{(\frac{1}{2}+n)_k (-n)_k}{(\frac{1}{2}+3n)_k (-1-3n)_k} \sum_{i=1}^k \left\{ \frac{1}{(2i-1)^2} - \frac{9}{4i^2} \right\} \\ & = \frac{\Gamma(1+n)^2 \Gamma(\frac{1}{6}+n) \Gamma(\frac{5}{6}+n)}{\Gamma(\frac{1}{2}+n)^2 \Gamma(\frac{2}{3}+n) \Gamma(\frac{4}{3}+n)} \frac{\Gamma(\frac{1}{2})^2 \Gamma(\frac{2}{3}) \Gamma(\frac{4}{3})}{\Gamma(\frac{1}{6}) \Gamma(\frac{5}{6})} \sum_{j=1}^{2n} \frac{(-1)^j}{j^2}. \end{aligned}$$

Letting $n \rightarrow \infty$ in this identity and using (2.7), we discover (1.9). \square

3 Proof of Theorems 1.4-1.6

For proving Theorem 1.4, we need the q -analogue of Dougall's ${}_5F_4$ summation formula (cf. [9, Equation (2.4.2)]):

$${}_6\phi_5 \left[\begin{matrix} a, qa^{\frac{1}{2}}, -qa^{\frac{1}{2}}, b, c, q^{-n} \\ a^{\frac{1}{2}}, -a^{\frac{1}{2}}, aq/b, aq/c, aq^{n+1} \end{matrix} ; q, \frac{aq^{n+1}}{bc} \right] = \frac{(aq, aq/bc; q)_n}{(aq/b, aq/c; q)_n}. \quad (3.1)$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. Apply the partial derivative operator \mathcal{D}_b to the $c = q/b$ case of (3.1) to obtain

$$\sum_{k=1}^n \frac{1 - aq^{2k}}{1 - a} \frac{(a, b, q/b, q^{-n}; q)_k}{(q, aq/b, ab, aq^{n+1}; q)_k} (aq^n)^k A_k(a, b) = \frac{(a, aq; q)_n}{(ab, aq/b; q)_n} B_n(a, b),$$

where

$$A_k(a, b) = \sum_{i=1}^k \frac{q^{i-1}}{1 - bq^{i-1}} - \sum_{i=1}^k \frac{q^i/b^2}{1 - q^i/b} + \sum_{i=1}^k \frac{aq^i/b^2}{1 - aq^i/b} - \sum_{i=1}^k \frac{aq^{i-1}}{1 - abq^{i-1}},$$

$$B_n(a, b) = \sum_{j=1}^n \frac{aq^j/b^2}{1 - aq^j/b} - \sum_{j=1}^n \frac{aq^{j-1}}{1 - abq^{j-1}}.$$

Employing the operator \mathcal{D}_b to both sides of the last equation, there holds

$$\sum_{k=1}^n \frac{1 - aq^{2k}}{1 - a} \frac{(a, b, q/b, q^{-n}; q)_k}{(q, aq/b, ab, aq^{n+1}; q)_k} (aq^n)^k \left\{ A_k(a, b)^2 - C_k(a, b) \right\}$$

$$= \frac{(a, aq; q)_n}{(ab, aq/b; q)_n} \left\{ B_n(a, b)^2 - D_n(a, b) \right\}, \quad (3.2)$$

where

$$C_k(a, b) = \sum_{i=1}^k \frac{q^{2i-2}}{(1 - bq^{i-1})^2} - \sum_{i=1}^k \frac{(q^i/b - 2)q^i/b^3}{(1 - q^i/b)^2}$$

$$+ \sum_{i=1}^k \frac{(aq^i/b - 2)aq^i/b^3}{(1 - aq^i/b)^2} - \sum_{i=1}^k \frac{a^2q^{2i-2}}{(1 - abq^{i-1})^2},$$

$$D_n(a, b) = \sum_{j=1}^n \frac{(aq^j/b - 2)aq^j/b^3}{(1 - aq^j/b)^2} - \sum_{j=1}^n \frac{a^2q^{2j-2}}{(1 - abq^{j-1})^2}.$$

The $a \rightarrow q, b \rightarrow q, q \rightarrow q^2$ case of (3.2) reads

$$\sum_{k=1}^n [4k + 1] \frac{(q; q^2)_k^3}{(q^2; q^2)_k^3} \frac{(q^{-2n}; q^2)_k}{(q^{3+2n}; q^2)_k} q^{(1+2n)k} \sum_{i=1}^{2k} (-1)^i \frac{q^i}{[i]^2} = \frac{(q, q^3; q^2)_n}{(q^2; q^2)_n^2} \sum_{j=1}^n \frac{q^{2j}}{[2j]^2}.$$

Letting $n \rightarrow \infty$ in the above identity, we arrive at (1.10). \square

In order to prove Theorem 1.5, we require the summation formula for basic hypergeometric series (cf. [7, p. 65]):

$$\begin{aligned} & \sum_{k=0}^n \frac{1 - a^{\frac{2}{3}} q^{\frac{4k}{3}} (a^{\frac{2}{3}}; q^{\frac{1}{3}})_{2k} (b^{\frac{1}{3}}, q^{\frac{1}{3}} b^{-\frac{1}{3}}; q^{\frac{1}{3}})_k (a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n}; q)_k}{1 - a^{\frac{2}{3}} (q^{\frac{1}{3}}; q^{\frac{1}{3}})_{2k} (a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n}; q^{\frac{1}{3}})_k (q a^{\frac{2}{3}} b^{-\frac{1}{3}}, q^{\frac{2}{3}} a^{\frac{2}{3}} b^{\frac{1}{3}}; q)_k} q^{\frac{k}{3}} \\ &= \frac{(q^{\frac{1}{3}} a^{\frac{2}{3}}; q^{\frac{1}{3}})_n (q, q^{\frac{1}{3}} b^{\frac{1}{3}}, q^{\frac{2}{3}} b^{-\frac{1}{3}}; q)_n}{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_n (q^{\frac{1}{3}} a^{\frac{2}{3}}, q a^{\frac{2}{3}} b^{-\frac{1}{3}}, q^{\frac{2}{3}} a^{\frac{2}{3}} b^{\frac{1}{3}}; q)_n}, \end{aligned} \quad (3.3)$$

where we have replaced

$$\frac{(b^{\frac{1}{3}}, q^{\frac{1}{3}} b^{-\frac{1}{3}}; q)_k}{(a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n}; q)_k} \quad \text{by} \quad \frac{(b^{\frac{1}{3}}, q^{\frac{1}{3}} b^{-\frac{1}{3}}; q^{\frac{1}{3}})_k}{(a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n}; q^{\frac{1}{3}})_k}$$

for correction. The nonterminating form of (3.3) can be seen in Gasper and Rahman [8, Equation (1.8)].

Subsequently, we start to prove Theorem 1.5.

Proof of Theorem 1.5. Via the partial derivative operator \mathcal{D}_b and the $a \rightarrow a^{3/2}, b \rightarrow b^3, q \rightarrow q^3$ of (3.3), we get

$$\begin{aligned} & \sum_{k=1}^n \frac{1 - a q^{4k} (a; q)_{2k} (b, q/b; q)_k (a q^{3n+1}, q^{-3n}; q^3)_k}{1 - a (q; q)_{2k} (a q^{3n+1}, q^{-3n}; q)_k (a q^3/b, a b q^2; q^3)_k} q^k E_k(a, b) \\ &= \frac{(a q; q)_{3n} (q^3, b q, q^2/b; q^3)_n}{(q; q)_{3n} (a q, a q^3/b, a b q^2; q^3)_n} F_n(a, b), \end{aligned}$$

where

$$\begin{aligned} E_k(a, b) &= \sum_{i=1}^k \frac{q^{i-1}}{1 - b q^{i-1}} - \sum_{i=1}^k \frac{q^i/b^2}{1 - q^i/b} + \sum_{i=1}^k \frac{a q^{3i}/b^2}{1 - a q^{3i}/b} - \sum_{i=1}^k \frac{a q^{3i-1}}{1 - a b q^{3i-1}}, \\ F_n(a, b) &= \sum_{j=1}^n \frac{q^{3j-2}}{1 - b q^{3j-2}} - \sum_{j=1}^n \frac{q^{3j-1}/b^2}{1 - q^{3j-1}/b} + \sum_{j=1}^n \frac{a q^{3j}/b^2}{1 - a q^{3j}/b} - \sum_{j=1}^n \frac{a q^{3j-1}}{1 - a b q^{3j-1}}. \end{aligned}$$

Through the operator \mathcal{D}_b and the last equation, it is clear that

$$\begin{aligned} & \sum_{k=1}^n \frac{1 - a q^{4k} (a; q)_{2k} (b, q/b; q)_k (a q^{3n+1}, q^{-3n}; q^3)_k}{1 - a (q; q)_{2k} (a q^{3n+1}, q^{-3n}; q)_k (a q^3/b, a b q^2; q^3)_k} q^k \left\{ E_k(a, b)^2 - G_k(a, b) \right\} \\ &= \frac{(a q; q)_{3n} (q^3, b q, q^2/b; q^3)_n}{(q; q)_{3n} (a q, a q^3/b, a b q^2; q^3)_n} \left\{ F_n(a, b)^2 - H_n(a, b) \right\}, \end{aligned} \quad (3.4)$$

where

$$G_k(a, b) = \sum_{i=1}^k \frac{q^{2i-2}}{(1 - b q^{i-1})^2} - \sum_{i=1}^k \frac{(q^i/b - 2) q^i/b^3}{(1 - q^i/b)^2}$$

$$\begin{aligned}
& + \sum_{i=1}^k \frac{(aq^{3i}/b - 2)aq^{3i}/b^3}{(1 - aq^{3i}/b)^2} - \sum_{i=1}^k \frac{a^2q^{6i-2}}{(1 - abq^{3i-1})^2}, \\
H_n(a, b) &= \sum_{j=1}^n \frac{q^{6j-4}}{(1 - bq^{3j-2})^2} - \sum_{j=1}^n \frac{(q^{3j-1}/b - 2)q^{3j-1}/b^3}{(1 - q^{3j-1}/b)^2} \\
& + \sum_{j=1}^n \frac{(aq^{3j}/b - 2)aq^{3j}/b^3}{(1 - aq^{3j}/b)^2} - \sum_{j=1}^n \frac{a^2q^{6j-2}}{(1 - abq^{3j-1})^2}.
\end{aligned}$$

The $a \rightarrow q, b \rightarrow q, q \rightarrow q^2$ case of (3.4) can be manipulated as

$$\begin{aligned}
& \sum_{k=1}^n q^{2k}[8k + 1] \frac{(q; q^2)_k^2 (q; q^2)_{2k}}{(q^2; q^2)_{2k} (q^6; q^6)_k^2} \frac{(q^{3+6n}, q^{-6n}; q^6)_k}{(q^{3+6n}, q^{-6n}; q^2)_k} \sum_{i=1}^k \left\{ \frac{q^{2i-1}}{[2i-1]^2} - \frac{q^{6i}}{[6i]^2} \right\} \\
&= \frac{(q^3; q^2)_{3n} (q^3; q^6)_n}{(q^2; q^2)_{3n} (q^6; q^6)_n} \sum_{j=1}^{2n} (-1)^{j-1} \frac{q^{3j}}{[3j]^2}.
\end{aligned}$$

Letting $n \rightarrow \infty$ in the upper identity, we are led to Theorem 1.5. \square

For the aim to prove Theorem 1.6, we shall draw support from the summation formula for basic hypergeometric series (cf. [7, p. 65]):

$$\begin{aligned}
& \sum_{k=0}^n \frac{1 - a^{\frac{2}{3}} q^{\frac{4k}{3}}}{1 - a^{\frac{2}{3}}} \frac{(q^{-n}, a^{\frac{2}{3}} q^{n+\frac{2}{3}}; q)_k}{(qa^{\frac{2}{3}} b^{-\frac{1}{3}}, q^{\frac{1}{3}} a^{\frac{2}{3}} b^{\frac{1}{3}}; q)_k} \frac{(b^{\frac{1}{3}}, q^{\frac{2}{3}} b^{-\frac{1}{3}}; q^{\frac{1}{3}})_k}{(a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n-\frac{1}{3}}; q^{\frac{1}{3}})_k} \frac{(a^{\frac{2}{3}}; q^{\frac{1}{3}})_{2k}}{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_{2k}} q^{\frac{k}{3}} \\
&= \frac{(q^{\frac{2}{3}} a^{\frac{2}{3}}; q^{\frac{1}{3}})_n}{(q^{\frac{1}{3}}; q^{\frac{1}{3}})_n} \frac{(q, q^{\frac{2}{3}} b^{\frac{1}{3}}, q^{\frac{4}{3}} b^{-\frac{1}{3}}; q)_n}{(q^{\frac{2}{3}} a^{\frac{2}{3}}, qa^{\frac{2}{3}} b^{-\frac{1}{3}}, q^{\frac{1}{3}} a^{\frac{2}{3}} b^{\frac{1}{3}}; q)_n}, \tag{3.5}
\end{aligned}$$

where we have replaced

$$\frac{(b^{\frac{1}{3}}, q^{\frac{2}{3}} b^{-\frac{1}{3}}; q)_k}{(a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n-\frac{1}{3}}; q)_k} \quad \text{by} \quad \frac{(b^{\frac{1}{3}}, q^{\frac{2}{3}} b^{-\frac{1}{3}}; q^{\frac{1}{3}})_k}{(a^{\frac{2}{3}} q^{n+\frac{1}{3}}, q^{-n-\frac{1}{3}}; q^{\frac{1}{3}})_k}$$

for correction. The nonterminating form of (3.5) can be observed in Gasper and Rahman [8, Equation (4.5)].

Finally, we begin to prove Theorem 1.6.

Proof of Theorem 1.6. Apply the partial derivative operator \mathcal{D}_b to the $a \rightarrow a^{3/2}, b \rightarrow b^3, q \rightarrow q^3$ case of (3.5) to deduce

$$\begin{aligned}
& \sum_{k=1}^n \frac{1 - aq^{4k}}{1 - a} \frac{(a/q; q)_{2k}}{(q^2; q)_{2k}} \frac{(b, q^2/b; q)_k}{(aq^{3n+1}, q^{-1-3n}; q)_k} \frac{(aq^{3n+2}, q^{-3n}; q^3)_k}{(aq^3/b, abq; q^3)_k} q^k R_k(a, b) \\
&= \frac{(aq; q)_{3n}}{(q^2; q)_{3n}} \frac{(q^3, bq^2, q^4/b; q^3)_n}{(aq^2, aq^3/b, abq; q^3)_n} S_n(a, b),
\end{aligned}$$

where

$$R_k(a, b) = \sum_{i=1}^k \frac{q^{i-1}}{1 - bq^{i-1}} - \sum_{i=1}^k \frac{q^{i+1}/b^2}{1 - q^{i+1}/b} + \sum_{i=1}^k \frac{aq^{3i}/b^2}{1 - aq^{3i}/b} - \sum_{i=1}^k \frac{aq^{3i-2}}{1 - abq^{3i-2}},$$

$$S_n(a, b) = \sum_{j=1}^n \frac{q^{3j-1}}{1 - bq^{3j-1}} - \sum_{j=1}^n \frac{q^{3j+1}/b^2}{1 - q^{3j+1}/b} + \sum_{j=1}^n \frac{aq^{3j}/b^2}{1 - aq^{3j}/b} - \sum_{j=1}^n \frac{aq^{3j-2}}{1 - abq^{3j-2}}.$$

Employing the operator \mathcal{D}_b to both sides of the last equation, it is obvious that

$$\begin{aligned} & \sum_{k=1}^n \frac{1 - aq^{4k}}{1 - a} \frac{(a/q; q)_{2k}}{(q^2; q)_{2k}} \frac{(b, q^2/b; q)_k}{(aq^{3n+1}, q^{-1-3n}; q)_k} \frac{(aq^{3n+2}, q^{-3n}; q^3)_k}{(aq^3/b, abq; q^3)_k} q^k \left\{ R_k(a, b)^2 - U_k(a, b) \right\} \\ &= \frac{(aq; q)_{3n}}{(q^2; q)_{3n}} \frac{(q^3, bq^2, q^4/b; q^3)_n}{(aq^2, aq^3/b, abq; q^3)_n} \left\{ S_n(a, b)^2 - V_n(a, b) \right\}, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} U_k(a, b) &= \sum_{i=1}^k \frac{q^{2i-2}}{(1 - bq^{i-1})^2} - \sum_{i=1}^k \frac{(q^{i+1}/b - 2)q^{i+1}/b^3}{(1 - q^{i+1}/b)^2} \\ &\quad + \sum_{i=1}^k \frac{(aq^{3i}/b - 2)aq^{3i}/b^3}{(1 - aq^{3i}/b)^2} - \sum_{i=1}^k \frac{a^2q^{6i-4}}{(1 - abq^{3i-2})^2}, \\ V_n(a, b) &= \sum_{j=1}^n \frac{q^{6j-2}}{(1 - bq^{3j-1})^2} - \sum_{j=1}^n \frac{(q^{3j+1}/b - 2)q^{3j+1}/b^3}{(1 - q^{3j+1}/b)^2} \\ &\quad + \sum_{j=1}^n \frac{(aq^{3j}/b - 2)aq^{3j}/b^3}{(1 - aq^{3j}/b)^2} - \sum_{j=1}^n \frac{a^2q^{6j-4}}{(1 - abq^{3j-2})^2}. \end{aligned}$$

The $a \rightarrow q^{-1}, b \rightarrow q^2, q \rightarrow q^2$ case of (3.6) engenders

$$\begin{aligned} & \sum_{k=1}^n q^{2k} [8k - 1] \frac{(q^2; q^2)_{2k}^2 (q^{-3}; q^2)_{2k}}{(q^4; q^2)_{2k} (q^3; q^6)_{2k}^2} \frac{(q^{3+6n}, q^{-6n}; q^6)_k}{(q^{1+6n}, q^{-2-6n}; q^2)_k} \sum_{i=1}^k \left\{ \frac{q^{6i-3}}{[6i-3]^2} - \frac{q^{2i}}{[2i]^2} \right\} \\ &= \frac{(q; q^2)_{3n} (q^6; q^6)_n^3}{(q^4; q^2)_{3n} (q^3; q^6)_n^3} \sum_{j=1}^{2n} (-1)^j \frac{q^{3j-1}}{[3j]^2}. \end{aligned}$$

Letting $n \rightarrow \infty$ in this identity, we catch hold of Theorem 1.6. \square

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 12071103).

References

- [1] G.E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge University Press, Cambridge, 2000.
- [2] G. Bauer, Von den Coefficienten der Reihen von Kugelfunctionen einer Variabeln, *J. Reine Angew. Math.* 56 (1859), 101–121.
- [3] A. Berkovich, H.H. Chan, M.J. Schlosser, Wronskians of theta functions and series for $1/\pi$, *Adv. Math.* 338 (2018), 266–304.
- [4] J.M. Borwein, P.B. Borwein, π and the AGM: A Study in Analytic Number Theory and Computational Complexity, Wiley, New York, 1987.
- [5] H.H. Chan, S.H. Chan, Z. Liu, Domb’s numbers and Ramanujan–Sato type series for $1/\pi$, *Adv. Math.* 186 (2004), 396–410.
- [6] W. Chu, Inversion Techniques and Combinatorial Identities: A unified treatment for the ${}_7F_6$ -series identities, *Collect. Math.* 45 (1994), 13–43.
- [7] W. Chu, Inversion Techniques and Combinatorial Identities: Jackson’s q -analogue of the Dougall–Dixon Theorem and the dual formulae, *Compos. math.* 95 (1995), 43–68.
- [8] G. Gasper, M. Rahman, An indefinite bibasic summation formula and some quadratic, cubic and quartic summation and transformation formulas, *Canad. J. Math.* 42 (1990), 1–27.
- [9] G. Gasper, M. Rahman, *Basic Hypergeometric Series* (2nd edition), Cambridge University Press, Cambridge, 2004.
- [10] I. Gessel, D. Stanton, Strange evaluations of hypergeometric series, *SIAM. J. Math. Anal.* 13 (1982), 295–308.
- [11] W. Gosper, Strip mining in the abandoned orefields of nineteenth century mathematics, in: *Computers in Mathematics*, D.V. Chudnovsky and R.D. Jenks, Eds., Dekker, New York, 1990, pp. 261–284.
- [12] J. Guillera, Generators of some Ramanujan formulas, *Ramanujan J.* 11 (2006), 41–48.
- [13] V.J.W. Guo, A q -analogue of the (I.2) supercongruence of Van Hamme, *Int. J. Number Theory* 15 (2019), 29–36.
- [14] V.J.W. Guo, q -Analogues of three Ramanujan-type formulas for $1/\pi$, *Ramanujan J.* 52 (2020), 123–132.
- [15] V.J.W. Guo, X. Lian, Some q -congruences on double basic hypergeometric sums, *J. Difference Equ. Appl.* 27 (2021), 453–461.
- [16] V.J.W. Guo, J.-C. Liu, q -Analogues of two Ramanujan-type formulas for $1/\pi$, *J. Difference Equ. Appl.* 24 (2018), 1368–1373.
- [17] V.J.W. Guo, W. Zudilin, Ramanujan-type formulae for $1/\pi$: q -analogues, *Integral Transforms Spec. Funct.* 29 (2018), 505–513.
- [18] V.J.W. Guo, W. Zudilin, A common q -analogue of two supercongruences, *Results Math.* 75 (2020), Art. 46
- [19] B. He, H. Zhai, Two q -summation formulas and q -analogues of series expansions for certain constants, Preprint (2018), arXiv: 1804.08210v4 [math. NT].
- [20] Q.-H. Hou, C. Krattenthaler, Z.-W. Sun, On q -analogues of some series for π and π^2 , *Prop. Amer. Math. Soc.* 147 (2019), 1953–1961.

- [21] Q.-H. Hou, Z.-W. Sun, q -Analogues of some series for powers of π , *Ann. Comb.* 25 (2021), 167–177.
- [22] Z.-G. Liu, Gauss summation and Ramanujan type series for $1/\pi$, *Int. J. Number Theory* 8 (2012), 289–297.
- [23] L. Long, Hypergeometric evaluation identities and supercongruences, *Pacific J. Math.* 249 (2011), 405–418.
- [24] S. Ramanujan, Modular equations and approximations to π , *Quart. J. Math. (Oxford)* 45 (1914), 350–372.
- [25] Z.-W. Sun, A new series for π^3 and related congruences, *Internat. J. Math.* 26 (2015), #1550055.
- [26] Z.-W. Sun, Two q -analogues of Euler’s formula $\zeta(2) = \pi^2/6$, *Colloq. Math.* 158 (2019), 313–320.
- [27] H. Swisher, On the supercongruence conjectures of van Hamme, *Res. Math. Sci.* (2015) 2:18.
- [28] L. Wang, Y. Yang, Ramanujan-type $1/\pi$ -series from bimodular forms, *Ramanujan J.* (2022). <https://doi.org/10.1007/s11139-021-00532-6>.
- [29] C. Wei, On two double series for π and their q -analogues, *Ramanujan J.* (2022). <https://doi.org/10.1007/s11139-022-00615-y>.
- [30] W. Zudilin, More Ramanujan-type formulae for $1/\pi^2$, *Russian Math. Surveys* 62 (2007), 634–636.