

# Averaging principle for slow-fast systems of rough differential equations via controlled paths <sup>\*†‡</sup>

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## Abstract

In this paper we prove the strong averaging principle for a slow-fast system of rough differential equations. The slow and the fast component of the system are driven by a rather general random rough path and Brownian rough path, respectively. These two driving noises are assumed to be independent. A prominent example of the driver of the slow component is fractional Brownian rough path with Hurst parameter between  $1/3$  and  $1/2$ . We work in the framework of controlled path theory, which is one of the most widely-used frameworks in rough path theory. To prove our main theorem, we carry out Khas'minskiĭ's time-discretizing method in this framework.

## 1 Introduction

We study the averaging principle for slow-fast systems of stochastic differential equations (SDEs). Although a few different limit theorems are called by the same name, the one we focus on in this paper is most typically formulated in the following way.

Let  $(w_t)$  and  $(b_t)$  be two independent standard (finite-dimensional) Brownian motions (BMs). A slow-fast system of (finite-dimensional) SDEs are given by

$$\begin{cases} X_t^\varepsilon &= x_0 + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t \sigma(X_s^\varepsilon, Y_s^\varepsilon) db_s, \\ Y_t^\varepsilon &= y_0 + \varepsilon^{-1} \int_0^t g(X_s^\varepsilon, Y_s^\varepsilon) ds + \varepsilon^{-1/2} \int_0^t h(X_s^\varepsilon, Y_s^\varepsilon) dw_s, \end{cases}$$

where  $0 < \varepsilon \ll 1$  is a small parameter. The processes  $X^\varepsilon$  and  $Y^\varepsilon$  are called the slow component and the fast component, respectively. Suitable conditions are imposed on  $g$  and  $h$  so that the following frozen SDE satisfies certain ergodicity for every  $x$ .

$$Y_t^{x,y} = y + \int_0^t g(x, Y_t^{x,y}) dt + \int_0^t h(x, Y_t^{x,y}) dw_t,$$

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An associated unique invariant probability measure is denoted by  $\mu^x$ . Set  $\bar{f}(x) = \int f(x, y)\mu^x(dy)$  and  $\bar{\sigma}(x) = \int \sigma(x, y)\mu^x(dy)$  and consider the following averaged SDE:

$$\bar{X}_t = x_0 + \int_0^t \bar{f}(\bar{X}_s)ds + \int_0^t \bar{\sigma}(\bar{X}_s)db_s.$$

The averaging principle of this type, which was initiated by Khas'minskiĭ [17], claims that  $X^\varepsilon$  converges to  $\bar{X}$  in an appropriate sense as  $\varepsilon \searrow 0$ . Even though the history is old and many papers have been written (see [7, 11, 12, 13, 21, 23, 26, 29, 33] for example), this research topic seems still quite active.

It should also be recalled that this averaging principle was generalized to various kind of stochastic equations. Examples include jump-type SDEs [10, 22, 28, 35, 36], distribution-dependent SDEs [27, 34], manifold-valued SDEs [19], functional-type SDEs such as SDEs with delay [1, 16, 31, 32] among others. (After a pioneering work [5], the case of stochastic partial differential equations has also been studied extensively. But, we do not discuss it in this paper.)

In all the preceding works mentioned above, the driving noises are (semi)martingales. One naturally wonders what happens to the averaging principle when the driving noise does not have (semi)martingale property. A prominent example of such noises is fractional Brownian motion (fBM). This research direction is fairly new and there are not many papers at the moment of writing.

When  $(b_t)$  is fBM with Hurst parameter  $H \in (1/2, 1)$  and  $(w_t)$  is the usual BM, the averaging principle for slow-fast systems like (2.1) was recently proved in [14, 24, 30, 15] under various settings. In these works, the integral  $db_s$  is understood as a Young (i.e. a generalized Riemann-Stieltjes) integral. Concerning this kind of slow-fast systems driven by fBM with  $H \in (1/2, 1)$  and BM, a few related problems have already been studied in [2, 3, 4]. Though it looks quite difficult to study the case where  $(w_t)$  is also fBM, a recent preprint [20] made a first attempt in that direction.

When  $(b_t)$  is fBM with Hurst parameter  $H < 1/2$ , the problem becomes quite difficult, mainly because neither Young integration nor Itô integration is available. In [25] the authors proved the averaging problem when  $(b_t)$  is fBM with  $1/3 < H \leq 1/2$  by using rough path (RP) theory. In the above mentioned paper,  $db_s$  is actually understood as a RP integral along a fractional Brownian RP.

The present paper is a continuation of [25] and generalizes its main theorem to a considerable extent (see Remark 2.2 below for details). The framework of RP theory adopted in this paper and that in [25] are different. In this paper the controlled path (CP) theory is exclusively used, while in [25] a fractional calculus approach to RPs is mainly used.

The organization of this paper is as follows: In Section 2, we introduce assumptions on the coefficients and on the driving RP and then state our main theorem. A comparison with preceding works and examples are also given. In Section 3, following the textbook [8], we review CP theory. We also slightly generalize well-known results on the well-posedness of a rough differential equation (RDE) so that we can deal with the slow

component of the slow-fast system (2.1) of RDEs and the averaged RDE (2.2). All arguments in this section are deterministic. The first half of Section 4 is devoted to a rigorous introduction of the slow-fast system of RDEs in a deterministic way. An Itô-Stratonovich correction formula for the fast component is also given. The second half is devoted to showing non-explosion of the slow-fast system of RDEs when the driving RP is random. Some moment estimates of the solution, which will be used in the proof of our main theorem, are also obtained. In Section 5 we prove our main theorem by carrying out Khas'minskii's time-discretizing method in the framework of CP theory. In appendices some important known results are summarized. The most important ones among them are basic facts on the frozen SDE associated with (the fast component of) the slow-fast system.

Before closing Introduction, we introduce the notation which will be used throughout the paper. We denote the set of integers by  $\mathbb{N} = \{0, 1, 2, \dots\}$ . Let  $T > 0$  be arbitrary and we work on the time interval  $[0, T]$  unless otherwise specified. and  $[a, b] \subset [0, T]$  is a subinterval. We write  $\Delta_{[a,b]} = \{(s, t) \in \mathbb{R}^2 \mid a \leq s \leq t \leq b\}$ . When  $[a, b] = [0, T]$ , we simply write  $\Delta_T$  for this set. (The time horizon  $T$  and the starting point  $(x_0, y_0)$  are arbitrary but fixed. We will not keep track of the dependence on  $T, x_0, y_0$ .)

Below,  $\mathcal{V}$  and  $\mathcal{W}$  are (real) Banach spaces. The set of bounded linear maps from  $\mathcal{V}$  to  $\mathcal{W}$  is denoted by  $L(\mathcal{V}, \mathcal{W})$ . When  $\mathcal{V} = \mathbb{R}^n$  and  $\mathcal{W} = \mathbb{R}^m$ ,  $L(\mathcal{V}, \mathcal{W})$  coincides with the set of all real  $m \times n$  matrices and is equipped with Hilbert-Schmidt norm instead of the operator norm.

- The set of all continuous paths  $\varphi: [a, b] \rightarrow \mathcal{V}$  is denoted by  $\mathcal{C}([a, b], \mathcal{V})$ . With the usual sup-norm  $\|\varphi\|_{\infty, [a,b]}$  on the  $[a, b]$ -interval,  $\mathcal{C}([a, b], \mathcal{V})$  is a Banach space. The difference of  $\varphi$  is denoted by  $\varphi^1$ , that is,  $\varphi_{s,t}^1 := \varphi_t - \varphi_s$  for  $(s, t) \in \Delta_{[a,b]}$ .
- Let  $0 < \gamma \leq 1$ . For a path  $\varphi: [a, b] \rightarrow \mathcal{V}$ , the  $\gamma$ -Hölder seminorm is defined by

$$\|\varphi\|_{\gamma, [a,b]} := \sup_{a \leq s < t \leq b} \frac{|\varphi_t - \varphi_s|_{\mathcal{V}}}{(t - s)^\gamma}.$$

If the right hand side is finite, we say  $\varphi$  is  $\gamma$ -Hölder continuous on  $[a, b]$ . The space of all  $\gamma$ -Hölder continuous paths on  $[a, b]$  is denoted by  $\mathcal{C}^\gamma([a, b], \mathcal{V})$ . The Banach norm on this space is  $|\varphi_a|_{\mathcal{V}} + \|\varphi\|_{\gamma, [a,b]}$ .

- Let  $0 < \gamma \leq 1$ . For a continuous map  $\eta: \Delta_{[a,b]} \rightarrow \mathcal{V}$ , we set

$$\|\eta\|_{\gamma, [a,b]} := \sup_{a \leq s < t \leq b} \frac{|\eta_{s,t}|_{\mathcal{V}}}{(t - s)^\gamma}.$$

If this is finite, then  $\eta$  vanishes on the diagonal. The set of all such  $\eta$  with  $\|\eta\|_{\gamma, [a,b]} < \infty$  is denoted by  $\mathcal{C}_2^\gamma([a, b], \mathcal{V})$ , which is a Banach space with  $\|\eta\|_{\gamma, [a,b]}$ .

- When  $[a, b] = [0, T]$ , we write  $\mathcal{C}(\mathcal{V})$ ,  $\mathcal{C}^\gamma(\mathcal{V})$ ,  $\mathcal{C}_2^\gamma(\mathcal{V})$  for these spaces and  $\|\cdot\|_\infty$ ,  $\|\cdot\|_\gamma$ ,  $\|\cdot\|_\gamma$  for the corresponding (semi)norms for simplicity of notation.

- Let  $U$  be an open set of  $\mathbb{R}^m$ . For  $k \in \mathbb{N}$ ,  $C^k(U, \mathbb{R}^n)$  stands for the set of  $C^k$ -functions from  $U$  to  $\mathbb{R}^n$ . (When  $k = 0$ , we simply write  $C(U, \mathbb{R}^n)$  instead of  $C^0(U, \mathbb{R}^n)$ .) The set of bounded  $C^k$ -functions  $f: U \rightarrow \mathbb{R}^n$  whose derivatives up to order  $k$  are all bounded is denoted by  $C_b^k(U, \mathbb{R}^n)$ , which is a Banach space with the norm  $\|f\|_{C_b^k} := \sum_{i=0}^k \|\nabla^i f\|_\infty$ . (Here,  $\|\cdot\|_\infty$  stands for the usual sup-norm on  $U$ .)

## 2 Assumptions and main result

In this section we first introduce natural assumptions on the coefficients and driving random RP of the following slow-fast system. Then, we state our main theorem.

Our slow-fast system of RDEs, defined on the time interval  $[0, T]$ , is given by

$$(2.1) \quad \begin{cases} X_t^\varepsilon &= x_0 + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t \sigma(X_s^\varepsilon) dB_s, \\ Y_t^\varepsilon &= y_0 + \varepsilon^{-1} \int_0^t g(X_s^\varepsilon, Y_s^\varepsilon) ds + \varepsilon^{-1/2} \int_0^t h(X_s^\varepsilon, Y_s^\varepsilon) dW_s. \end{cases}$$

Here,  $0 < \varepsilon \leq 1$  is a small constant and (the first level path of)  $(X^\varepsilon, Y^\varepsilon)$  takes values in  $\mathbb{R}^m \times \mathbb{R}^n$ . The starting point  $(x_0, y_0)$  is always deterministic and arbitrary. At the first stage, (2.1) is a deterministic system of RDEs driven by an  $(d+e)$ -dimensional RP which is denoted by  $(B, W)$ . A precise definition of the system (2.1) will be given in Section 4.

When we consider this slow-fast system of RDEs, we always impose the following conditions:

- $\sigma \in C^3(\mathbb{R}^m, L(\mathbb{R}^d, \mathbb{R}^m))$  and  $h \in C^3(\mathbb{R}^m \times \mathbb{R}^n, L(\mathbb{R}^e, \mathbb{R}^n))$ ,
- $f \in C(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m)$  is locally Lipschitz continuous and so is  $g \in C(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$ .

These guarantee that the slow-fast system of RDEs always has a unique local solution. (This fact is well-known. See also Remark 3.4 below.) Since we show the strong version of the averaging principle in this work, we assume that  $\sigma$  depends only on the slow variable.

To formulate our main result, we introduce several more assumptions on these coefficients.

**(H1)**  $\sigma$  is of  $C_b^3$ .

**(H2)**  $f$  is bounded and globally Lipschitz continuous.

**(H3)**  $h$  is globally Lipschitz continuous.

**(H4)** There exist constants  $\eta_1 \geq 0$  and  $C > 0$  such that, for all  $x \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$ ,

$$|g(x, y)| \leq C(|x|^{\eta_1} + |y|^{\eta_1} + 1).$$

We set the following condition for  $r \geq 0$ :

**(H5)<sub>r</sub>** There exist constants  $\eta_2 \geq 0$  and  $C > 0$  such that, for all  $x_1, x_2 \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$ ,

$$|g(x_1, y) - g(x_2, y)| \leq C|x_1 - x_2|(1 + |x_1|^{\eta_2} + |x_2|^{\eta_2} + |y|^r).$$

Next, we set the following condition for  $q \geq 2$ . If  $q' > q$ , then **(H6)<sub>q'</sub>** obviously implies **(H6)<sub>q</sub>** without changing the constants  $\gamma_1, \eta_3, C$ .

**(H6)<sub>q</sub>** There exist constants  $\gamma_1 > 0$ ,  $C > 0$  and  $\eta_3 \geq 0$  such that, for all  $x \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$ ,

$$2\langle y, g(x, y) \rangle + (q - 1)|h(x, y)|^2 \leq -\gamma_1|y|^2 + C(|x|^{\eta_3} + 1).$$

**(H7)** There exists a constant  $\gamma_2 > 0$  such that, for all  $x \in \mathbb{R}^m$  and  $y_1, y_2 \in \mathbb{R}^n$ ,

$$2\langle y_1 - y_2, g(x, y_1) - g(x, y_2) \rangle + |h(x, y_1) - h(x, y_2)|^2 \leq -\gamma_2|y_1 - y_2|^2.$$

Let  $T \in (0, \infty)$  and  $\frac{1}{3} < \alpha_0 \leq \frac{1}{2}$ . Let  $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{0 \leq t \leq T})$  be a filtered probability space satisfying the usual condition. On this probability space, the following two independent random variables  $w$  and  $B = (B^1, B^2)$  are defined. The former,  $w = (w_t)_{0 \leq t \leq T}$ , is a standard  $e$ -dimensional  $\{\mathcal{F}_t\}$ -BM. The Itô RP lift of  $w$  is denoted by  $W = (W^1, W^2)$ . The latter,  $B = \{(B_{s,t}^1, B_{s,t}^2)\}_{0 \leq s \leq t \leq T}$ , is an  $\Omega_\alpha(\mathbb{R}^d)$ -valued random variable (i.e., random RP) for every  $\alpha \in (1/3, \alpha_0)$ . Here,  $\Omega_\alpha(\mathbb{R}^d)$  is the space of  $\alpha$ -Hölder RPs over  $\mathbb{R}^d$ . We assume that  $(B_{s,t}^1, B_{s,t}^2)$  is  $\mathcal{F}_t$ -measurable for every  $(s, t)$  with  $0 \leq s \leq t \leq T$ . Note that  $B$  need not be (weakly) geometric.

We assume the following condition on the integrability of  $B$ . Below,  $\|B\|_\alpha := \|B^1\|_\alpha + \|B^2\|_{2\alpha}^{1/2}$  denotes the  $\alpha$ -Hölder homogeneous RP norm over the time interval  $[0, T]$ .

**(A)** For every  $\alpha \in (1/3, \alpha_0)$  and  $p \in [1, \infty)$ , we have  $\mathbb{E}[\|B\|_\alpha^p] < \infty$ .

Under this assumption, the mixed random RP  $(B, W)$  and the slow-fast system (2.1) of RDEs driven by it can be defined in a natural way (see Section 4 for precise definitions). We will show the averaging principle for (2.1) when it is driven by this random RP.

Before we provide our main theorem, we introduce the frozen SDE and the averaged RDE associated with the slow-fast system (2.1) in the usual way. The frozen SDE is given as follows:

$$Y_t^{x,y} = y + \int_0^t g(x, Y_t^{x,y})dt + \int_0^t h(x, Y_t^{x,y})d^l w_t,$$

Here,  $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$  are deterministic and arbitrary and  $d^l w_t$  stands for the standard Itô integral with respect to a standard  $e$ -dimensional BM  $(w_t)$ . (A more precise definition will be given in Eq. (C.1).) Under suitable conditions, the Markov semigroup  $(P_t^x)_{t \geq 0}$  defined by  $P_t^x \varphi(y) = \mathbb{E}[\varphi(Y_t^{x,y})]$  has a unique invariant probability measure, which is denoted by  $\mu^x$  (see Lemma C.3 below for details). It should be noted that we are only interested in the law of  $Y^{x,y}$  and hence any realization of BM will do.

Define the averaged drift by  $\bar{f}(x) = \int_{\mathbb{R}^n} f(x, y) \mu^x(dy)$  for  $x \in \mathbb{R}^m$ . The averaged RDE is given as follows:

$$(2.2) \quad \bar{X}_t = x_0 + \int_0^t \bar{f}(\bar{X}_s) ds + \int_0^t \sigma(\bar{X}_s) dB_s$$

Here,  $x_0 \in \mathbb{R}^m$  is the same as in (2.1). It will be shown that under suitable conditions, this RDE has a unique global solution (see Propositions 3.6 and C.5 below for details).

Now we are in a position to state our main result. It claims that (the first level path of) the slow component of the slow-fast system (2.1) of RDEs converges as  $\varepsilon \searrow 0$  to (the first level path of) the averaged RDE (2.2) in  $L^p$ -sense. This generalizes the main result of [25]. Here,  $\|\cdot\|_\beta$  stands for the  $\beta$ -Hölder (semi)norm of a usual path over the time interval  $[0, T]$ .

**Theorem 2.1.** *Assume (A), (H1)–(H4), (H5)<sub>r</sub>, (H6)<sub>q</sub> and (H7) for some  $q \geq 2$  and  $r \geq 0$  such that  $q > 2r$ . Then, for every  $p \in [1, \infty)$  and  $\beta \in (\frac{1}{3}, \alpha_0)$ , we have*

$$\lim_{\varepsilon \searrow 0} \mathbb{E}[\|X^\varepsilon - \bar{X}\|_\beta^p] = 0.$$

**Remark 2.2.** (1) In this paper Brownian RP which drives the fast component of the slow-fast RDEs is of Itô type, while in [25] it is of Stratonovich type. However, as will be explained in Lemma 4.3 and Remark 4.4, one can easily switch between the two formulations by adding or subtracting a standard correction term (even at the deterministic level). So, this is just a superficial difference.

(2) Our main theorem above is stronger than [25, Theorem 1.2]. The main differences are as follows:

- In [25],  $B$  is fractional Brownian RP with  $(\frac{1}{3}, \frac{1}{2}]$ . In this paper,  $B$  is a much more general random RP. (See also Example 2.4.)
- The conditions on the coefficients  $\sigma, h, f, g$  are relaxed in this paper. (See also Example 2.5.)
- In [25, Theorem 1.2],  $L^1$ -convergence of  $\|X^\varepsilon - \bar{X}\|_\infty$  was proved. In this paper,  $L^p$ -convergence of  $\|X^\varepsilon - \bar{X}\|_\beta$  is proved for every  $1 \leq p < \infty$ .

Also, as was mentioned earlier, the framework of RDE theory used in this paper is different from that in [25].

**Remark 2.3.** The law of  $X^\varepsilon - \bar{X}$  is uniquely determined by the law of  $B = (B^1, B^2)$  and the  $e$ -dimensional Wiener measure. So, the choice of a filtered probability space that carries  $B$  and  $w$  does not matter.

Verifying the existence of such a filtered probability space is not difficult. First, a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  that supports independent  $B$  and  $w$  clearly exists. Then, we set  $\mathcal{F}_t = \sigma\{w_u \mid 0 \leq u \leq t\} \vee \sigma\{B\} \vee \mathcal{N}$ ,  $0 \leq t \leq T$ , for example. Here,  $\mathcal{N}$  is the collection of  $\mathbb{P}$ -zero sets. Then,  $w$  is an  $\{\mathcal{F}_t\}$ -BM. The standard proof of the right continuity of Brownian filtration still works for  $\{\mathcal{F}_t\}$  after trivial modifications. Thus,  $\{\mathcal{F}_t\}$  satisfies the usual condition and  $(\Omega, \mathcal{F}, \mathbb{P}; \{\mathcal{F}_t\}_{0 \leq t \leq T})$  is a desired example.

**Example 2.4.** We provide some examples of  $B = (B^1, B^2)$  satisfying Assumption (A) in Theorem 2.1.

- (1) A deterministic RP  $B \in \Omega_{\alpha_0}(\mathbb{R}^d)$  with  $\frac{1}{3} < \alpha_0 \leq \frac{1}{2}$ . (In this case, we can actually take  $\beta = \alpha_0$  in Theorem 2.1.)
- (2) Fractional Brownian RP  $B^H$  with Hurst parameter  $\frac{1}{3} < H \leq \frac{1}{2}$ . In this case  $\alpha_0 = H$ . Note that when  $H = 1/2$ ,  $B^H$  is Brownian RP of Stratonovich type.
- (3) Brownian RP  $\tilde{B}$  of Itô type. In this case  $\alpha_0 = 1/2$ .
- (4) Mixture of fractional Brownian RP and Brownian RP of Itô type. Let  $B^H$  and  $\tilde{B}$  be as in (2) and (3) above, respectively and assume they are independent. Then, the mixed RP  $(B^H, \tilde{B})$  becomes an example with  $\alpha_0 = H$ . (The precise definition of mixture will be given in Definition 4.5.) It should also be noted that mixture of  $(B^H, \tilde{B})$  and  $W$  equals mixture of  $B^H$  and  $(\tilde{B}, W)$ . Here,  $W$  is another Brownian RP of Itô type and  $\{B^H, \tilde{B}, W\}$  are assumed to be independent.
- (5) In [9, Chapter 15] and [8, Chapters 10–11], there are examples of Gaussian RP whose Hölder regularity is between  $1/3$  and  $1/2$ . Since they satisfy an integrability theorem of Fernique type, their RP norms have moments of all orders. So, these Gaussian RP are nice examples.

A slow-fast system of usual SDEs basically corresponds to (3). The driving process of a slow-fast system in [25] is that of (2). In [30], the slow component is driven by BM and fBM with Hurst parameter  $H \in (\frac{1}{2}, 1)$ . So, the case (4) above can be viewed as a generalization of [30] to a RP setting.

The conditions on the coefficients  $\sigma, h, f, g$  in Theorem 2.1 are strictly weaker than those in the main theorem of the preceding work [25, Theorem 1.2]. For example, the following “superlinear” example satisfies the conditions of the former, but does not satisfy those of the latter.

**Example 2.5.** Let  $\kappa > 0$  and  $\lambda, \phi: \mathbb{R}^m \rightarrow [\kappa, \infty)$  be  $C^1$ -functions such that  $\nabla \lambda$  and  $\nabla \phi$  are of at most polynomial growth in  $x$ . Set

$$g(x, y) = -\lambda(x)y|y|^2 - \phi(x)y, \quad x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

Let  $\sigma, h, f$  satisfy (H1)–(H3). Then, (H4), (H5)<sub>r</sub> for some  $r \geq 0$ , and (H6)<sub>q</sub> for every  $q \geq 2$  are all satisfied. If we assume in addition that  $\|\nabla_y h\|_\infty < 2\kappa$ , then (H7) is also satisfied. Here,  $\nabla_y h$  is the partial gradient of  $h$  in the  $y$ -variable. Therefore, these  $\sigma, h, f, g$  satisfy the condition of Theorem 2.1.

### 3 Review of controlled path theory

This section is devoted to recalling CP theory. We basically follow the exposition in [8]. However, we slightly generalize the setting and improve some of the results for our

purpose. Note that everything in this section is deterministic. Throughout this section,  $\mathcal{V}$  and  $\mathcal{W}$  are Euclidean spaces and we assume  $\frac{1}{3} < \alpha \leq \frac{1}{2}$ .

### 3.1 Rough paths and controlled paths

First, we recall the definition of  $\alpha$ -Hölder rough path ( $\alpha$ -RP or RP). A continuous map  $X = (X^1, X^2): \Delta_T \rightarrow \mathcal{V} \oplus (\mathcal{V} \otimes \mathcal{V})$  is called  $\mathcal{V}$ -valued  $\alpha$ -RP if  $\|X^i\|_{i\alpha} < \infty$  for  $i = 1, 2$  and

$$(3.1) \quad X_{s,t}^1 = X_{s,u}^1 + X_{u,t}^1, \quad X_{s,t}^2 = X_{s,u}^2 + X_{u,t}^2 + X_{s,u}^1 \otimes X_{u,t}^1, \quad s \leq u \leq t.$$

The set of all  $\mathcal{V}$ -valued  $\alpha$ -RPs is denoted by  $\Omega_\alpha(\mathcal{V})$ , which is a complete metric space with the natural distance  $d_\alpha(X, \hat{X}) := \sum_{i=1}^2 \|X^i - \hat{X}^i\|_{i\alpha}$ . (Note that  $X^1$  and  $X^2$  vanish on the diagonal.) The homogeneous norm of  $X$  is denoted by  $\|X\|_\alpha := \|X^1\|_\alpha + \|X^2\|_{2\alpha}^{1/2}$ . Obviously,  $\Omega_\alpha(\mathcal{V}) \subset \Omega_\beta(\mathcal{V})$  if  $\frac{1}{3} < \beta \leq \alpha \leq \frac{1}{2}$ . The dilation by  $\delta \in \mathbb{R}$  is defined by  $\delta X = (\delta X^1, \delta^2 X^2)$ . Clearly,  $\|\delta X\|_\alpha = |\delta| \cdot \|X\|_\alpha$ .

Now we recall the definition of a controlled path (CP) with respect to a reference RP  $X = (X^1, X^2) \in \Omega_\alpha(\mathcal{V})$ . Let  $[a, b] \subset [0, T]$  be a subinterval. We say that  $(Y, Y^\dagger, Y^\sharp)$  is a  $\mathcal{W}$ -valued CP with respect to  $X$  on  $[a, b]$  if

$$(Y, Y^\dagger, Y^\sharp) \in \mathcal{C}^\alpha([a, b], \mathcal{W}) \times \mathcal{C}^\alpha([a, b], L(\mathcal{V}, \mathcal{W})) \times \mathcal{C}_2^{2\alpha}([a, b], \mathcal{W})$$

and

$$(3.2) \quad Y_t - Y_s = Y_s^\dagger X_{s,t}^1 + Y_{s,t}^\sharp, \quad (s, t) \in \Delta_{[a,b]}.$$

The set of all such CPs with respect to  $X$  is denoted by  $\mathcal{Q}_X^\alpha([a, b], \mathcal{W})$ . For simplicity,  $(Y, Y^\dagger, Y^\sharp)$  will often be written as  $(Y, Y^\dagger)$ . A natural seminorm of a CP is defined by

$$\|(Y, Y^\dagger, Y^\sharp)\|_{\mathcal{Q}_X^\alpha, [a,b]} = \|Y^\dagger\|_{\alpha, [a,b]} + \|Y^\sharp\|_{2\alpha, [a,b]}$$

$\mathcal{Q}_X^\alpha([a, b], \mathcal{W})$  is a Banach space with the norm  $|Y_a|_{\mathcal{W}} + |Y_a^\dagger|_{L(\mathcal{V}, \mathcal{W})} + \|(Y, Y^\dagger, Y^\sharp)\|_{\mathcal{Q}_X^\alpha, [a,b]}$ . When  $[a, b] = [0, T]$ , we simply write  $\mathcal{Q}_X^\alpha(\mathcal{W})$  and  $\|\cdot\|_{\mathcal{Q}_X^\alpha}$  instead.

**Example 3.1.** Here are a few typical examples of CPs for a given RP  $X \in \Omega_\alpha(\mathcal{V})$ . (In the first three examples the time interval is  $[0, T]$  just for simplicity. It can be replaced by any subinterval  $[a, b]$ .)

1. For  $\xi \in \mathcal{W}$  and  $\sigma \in L(\mathcal{V}, \mathcal{W})$ ,  $t \mapsto (\xi + \sigma X_{0,t}^1, \sigma)$  belongs to  $\mathcal{Q}_X^\alpha(\mathcal{W})$ . Note that  $\mathcal{Q}_X^\alpha$ -seminorm of this element is zero.
2. If  $\varphi \in \mathcal{C}^{2\alpha}(\mathcal{W})$ , then obviously  $(\varphi, 0) \in \mathcal{Q}_X^\alpha(\mathcal{W})$  with  $\|\varphi\|_{2\alpha} = \|(\varphi, 0)\|_{\mathcal{Q}_X^\alpha}$ . In this way, we have a natural continuous embedding  $\mathcal{C}^{2\alpha}(\mathcal{W}) \hookrightarrow \mathcal{Q}_X^\alpha(\mathcal{W})$ .



3. Suppose that  $(Y, Y^\dagger) \in \mathcal{Q}_X^\alpha(\mathcal{W})$  and  $g: \mathcal{W} \rightarrow \mathcal{W}'$  is a  $C^2$ -function to another Euclidean space  $\mathcal{W}'$ . Then  $(g(Y), g(Y)^\dagger) \in \mathcal{Q}_X^\alpha(\mathcal{W}')$  if we set  $g(Y)_t := g(Y_t)$  and  $g(Y)_t^\dagger := \nabla g(Y_t)Y_t^\dagger$ , where the right hand side is the composition of the two linear maps  $\nabla g(Y_t) \in L(\mathcal{W}, \mathcal{W}')$  and  $Y_t^\dagger \in L(\mathcal{V}, \mathcal{W})$ . Using Taylor's expansion, we can verify this fact as follows:

$$(3.3) \quad \begin{aligned} g(Y_t) - g(Y_s) &= \nabla g(Y_s)\langle Y_{s,t}^1 \rangle + \int_0^1 d\theta(1-\theta)\nabla^2 g(Y_s + \theta Y_{s,t}^1)\langle Y_{s,t}^1, Y_{s,t}^1 \rangle \\ &= \nabla g(Y_s)Y_s^\dagger X_{s,t}^1 + g(Y)_{s,t}^\sharp \end{aligned}$$

with

$$g(Y)_{s,t}^\sharp := \nabla g(Y_s)Y_{s,t}^\sharp + \int_0^1 d\theta(1-\theta)\nabla^2 g(Y_s + \theta Y_{s,t}^1)\langle Y_{s,t}^1, Y_{s,t}^1 \rangle.$$

From the assumptions that  $Y \in \mathcal{C}^\alpha(\mathcal{W})$  and  $Y^\sharp \in \mathcal{C}^{2\alpha}(\mathcal{W})$ , one can easily see that  $g(Y)^\sharp \in \mathcal{C}^{2\alpha}(\mathcal{W}')$ .

4. Concatenation of two CPs is also a CP. Let  $0 \leq a < b < c \leq T$ . For  $(Y, Y^\dagger) \in \mathcal{Q}_X^\alpha([a, b], \mathcal{W})$  and  $(\hat{Y}, \hat{Y}^\dagger) \in \mathcal{Q}_X^\alpha([b, c], \mathcal{W})$  with  $(Y_b, Y_b^\dagger) = (\hat{Y}_b, \hat{Y}_b^\dagger)$ , their concatenation  $(Z, Z^\dagger) := (Y * \hat{Y}, Y^\dagger * \hat{Y}^\dagger)$  can naturally be defined and belongs to  $\mathcal{Q}_X^\alpha([a, c], \mathcal{W})$ . (Here,  $*$  stands for the usual concatenation operation for two continuous paths.) It is clear that  $Z^\dagger \in \mathcal{C}^\alpha([a, c], \mathcal{W})$ . To prove that  $Z^\sharp \in \mathcal{C}^{2\alpha}(\mathcal{W})$ , it is sufficient to observe the following: For  $a \leq s \leq b \leq t \leq c$ ,

$$(3.4) \quad \begin{aligned} Z_{s,t}^\sharp &= Z_t - Z_s - Z_s^\dagger X_{s,t}^1 \\ &= (Y_b - Y_s - Y_s^\dagger X_{s,b}^1) + (\hat{Y}_t - \hat{Y}_b - \hat{Y}_b^\dagger X_{b,t}^1) + (Y_b^\dagger - Y_s^\dagger)X_{b,t}^1 \\ &= Y_{s,b}^\sharp + \hat{Y}_{b,t}^\sharp + (Y_b^\dagger - Y_s^\dagger)X_{b,t}^1. \end{aligned}$$

The right hand side is clearly dominated by a constant multiple of  $(t-s)^{2\alpha}$ .

## 3.2 Rough path integration of controlled paths

Now we discuss integration of a CP  $(Y, Y^\dagger) \in \mathcal{Q}_X^\alpha([a, b], L(\mathcal{V}, \mathcal{W}))$  against a RP  $X \in \Omega_\alpha(\mathcal{V})$ . It should be noted that  $Y^\dagger$  takes values in

$$L(\mathcal{V}, L(\mathcal{V}, \mathcal{W})) \cong L^{(2)}(\mathcal{V} \times \mathcal{V}, \mathcal{W}) \cong L(\mathcal{V} \otimes \mathcal{V}, \mathcal{W}),$$

where  $L^{(2)}(\mathcal{V} \times \mathcal{V}, \mathcal{W})$  stands for the vector space of bounded bilinear maps from  $\mathcal{V} \times \mathcal{V}$  to  $\mathcal{W}$ .

First, we define

$$J_{s,t} = Y_s X_{s,t}^1 + Y_s^\dagger X_{s,t}^2, \quad (s, t) \in \Delta_{[a,b]}.$$

From (3.1) and (3.2) we can easily see that

$$(3.5) \quad J_{s,u} + J_{u,t} - J_{s,t} = Y_{s,u}^\sharp X_{u,t}^1 + Y_{s,u}^{\dagger,1} X_{u,t}^2, \quad a \leq s \leq u \leq t \leq b,$$

where we set  $Y_{s,u}^{\dagger,1} = Y_u^\dagger - Y_s^\dagger$ . Let  $\mathcal{P} = \{s = t_0 < t_1 < \dots < t_N = t\}$  be a partition of  $[s, t] \subset [a, b]$ . Its mesh size is denoted by  $|\mathcal{P}|$ . We define  $J_{s,t}(\mathcal{P}) = \sum_{i=1}^N J_{t_{i-1}, t_i}$  and

$$(3.6) \quad \int_s^t Y_u dX_u = \lim_{|\mathcal{P}| \searrow 0} J_{s,t}(\mathcal{P}), \quad (s, t) \in \Delta_{[a,b]}.$$

The limit above is known to exist. This is called the RP integration. It will turn out in the next proposition that the RP integral against  $X$  is again a CP with respect to  $X$ . By the way it is defined, this RP integral clearly has additivity with respect to the interval  $[s, t]$ .

**Proposition 3.2.** *Let  $\frac{1}{3} < \alpha \leq \frac{1}{2}$  and  $[a, b] \subset [0, T]$ . Suppose that  $X \in \Omega_\alpha(\mathcal{V})$  and  $(Y, Y^\dagger) \in \mathcal{Q}_X^\alpha([a, b], L(\mathcal{V}, \mathcal{W}))$ . Then, the limit in (3.6) converges for all  $(s, t)$ . Moreover, we have*

$$(3.7) \quad \left( \int_a^\cdot Y_u dX_u, Y \right) \in \mathcal{Q}_X^\alpha([a, b], \mathcal{W})$$

with the following estimate:

$$(3.8) \quad \left| \int_s^t Y_u dX_u - (Y_s X_{s,t}^1 + Y_s^\dagger X_{s,t}^2) \right|_{\mathcal{W}} \\ \leq \kappa_\alpha (t-s)^{3\alpha} (\|Y^\sharp\|_{2\alpha, [a,b]} \|X^1\|_{\alpha, [a,b]} + \|Y^\dagger\|_{\alpha, [a,b]} \|X^2\|_{2\alpha, [a,b]}), \quad (s, t) \in \Delta_{[a,b]}.$$

Here, we set  $\kappa_\alpha = 2^{3\alpha} \zeta(3\alpha)$  with  $\zeta$  being the usual Riemann zeta function.

*Proof.* In this proof the norm of  $\mathcal{W}$  is denoted by  $|\cdot|$  for brevity. First we prove the convergence. For  $\mathcal{P}$  given as above, we can find  $i$  ( $1 \leq i \leq N-1$ ) such that  $t_{i+1} - t_{i-1} \leq 2(t-s)/(N-1)$  (see [8, p. 52] for instance). Then, we see from (3.5) that

$$\begin{aligned} |J_{s,t}(\mathcal{P}) - J_{s,t}(\mathcal{P} \setminus \{t_i\})| &= |J_{t_{i-1}, t_i} + J_{t_i, t_{i+1}} - J_{t_{i-1}, t_{i+1}}| \\ &= |Y_{t_{i-1}, t_i}^\sharp X_{t_i, t_{i+1}}^1 + Y_{t_{i-1}, t_i}^{\dagger,1} X_{t_i, t_{i+1}}^2| \\ &\leq (\|Y^\sharp\|_{2\alpha, [a,b]} \|X^1\|_{\alpha, [a,b]} + \|Y^\dagger\|_{\alpha, [a,b]} \|X^2\|_{2\alpha, [a,b]}) \left( \frac{2(t-s)}{N-1} \right)^{3\alpha}. \end{aligned}$$

Extracting points one by one from  $\mathcal{P}$  in this way until it becomes the trivial partition  $\{s, t\}$ , we have

$$(3.9) \quad |J_{s,t}(\mathcal{P}) - J_{s,t}| \leq \kappa_\alpha (\|Y^\sharp\|_{2\alpha, [a,b]} \|X^1\|_{\alpha, [a,b]} + \|Y^\dagger\|_{\alpha, [a,b]} \|X^2\|_{2\alpha, [a,b]}) (t-s)^{3\alpha}.$$

Note that the condition  $3\alpha > 1$  is used here. By a standard argument, (3.9) implies that  $\{J_{s,t}(\mathcal{P})\}_{\mathcal{P}}$  is Cauchy as  $|\mathcal{P}| \searrow 0$ . Therefore, the limit in (3.6) exists and (3.8) holds. Since

$$|Y_s^\dagger X_{s,t}^2| \leq (|Y_a^\dagger| + \|Y^\dagger\|_{\alpha, [a,b]}(b-a)) \|X^2\|_{2\alpha, [a,b]} (t-s)^{2\alpha},$$

(3.8) implies (3.7). □

### 3.3 Rough differential equations: The case of bounded and globally Lipschitz drift vector field

Now we discuss RDEs in the framework of controlled path theory. In this subsection we assume that  $\frac{1}{3} < \beta < \alpha \leq \frac{1}{2}$  and  $X \in \Omega_\alpha(\mathcal{V}) \subset \Omega_\beta(\mathcal{V})$ .

We set conditions on the coefficients of our RDE. Let  $\sigma: \mathcal{W} \rightarrow L(\mathcal{V}, \mathcal{W})$  be of  $C_b^3$  and let  $f: \mathcal{W} \times \mathcal{S} \rightarrow \mathcal{W}$  be a continuous map, where  $\mathcal{S}$  is a metric space, satisfy the following condition:

$$(3.10) \quad \sup_{y \in \mathcal{W}, z \in \mathcal{S}} |f(y, z)|_{\mathcal{W}} + \sup_{y, y' \in \mathcal{W}, y \neq y', z \in \mathcal{S}} \frac{|f(y, z) - f(y', z)|_{\mathcal{W}}}{|y - y'|_{\mathcal{W}}} < \infty.$$

The first and the second terms will be denoted by  $\|f\|_\infty$  and  $L_f$ , respectively.

For an  $\mathcal{S}$ -valued continuous path  $\psi: [0, T] \rightarrow \mathcal{S}$ , we consider the following RDE driven by  $X$  with the initial point  $\xi \in \mathcal{W}$ :

$$(3.11) \quad Y_t = \xi + \int_0^t f(Y_s, \psi_s) ds + \int_0^t \sigma(Y_s) dX_s, \quad Y_t^\dagger = \sigma(Y_t), \quad t \in [0, T].$$

For every  $(Y, Y^\dagger) \in \mathcal{Q}_X^\beta(\mathcal{W})$ , the right hand side of this system of equations also belongs to  $\mathcal{Q}_X^\beta(\mathcal{W})$ , due to Example 3.1 and Proposition 3.2. Therefore, (3.11) should be understood as an equality in  $\mathcal{Q}_X^\beta(\mathcal{W})$ . (Following [8, Sections 8.5–8.6], we slightly relax the Hölder topology of the space of CPs for quick proofs. The estimate of  $\|Y\|_\beta$  in the next proposition will be improved in Proposition 3.6.)

**Proposition 3.3.** *Let the assumptions be as above. Then, for every  $X \in \Omega_\alpha(\mathcal{V})$ ,  $\xi \in \mathcal{W}$  and  $\psi$ , there exists a unique global solution  $(Y, Y^\dagger) \in \mathcal{Q}_X^\beta(\mathcal{W})$  of RDE (3.11). Moreover, it satisfies the following estimate: there exist positive constants  $c$  and  $\nu$  independent of  $X, \xi, \psi, \sigma, f$  such that*

$$\|Y\|_\beta \leq c\{(K+1)(\|X\|_\alpha + 1)\}^\nu, \quad X \in \Omega_\alpha(\mathcal{V}).$$

Here, we set  $K := \|\sigma\|_{C_b^3} \vee \|f\|_\infty \vee L_f$ .

*Proof.* In this proof the norm of  $\mathcal{W}$  is denoted by  $|\cdot|$  for brevity. Without loss of generality we may assume  $T = 1$ . Let  $\tau \in (0, 1]$  and  $\xi \in \mathcal{W}$ .

We define  $\mathcal{M}_{[0, \tau]}^\xi, \mathcal{M}_{[0, \tau]}^1, \mathcal{M}_{[0, \tau]}^2: \mathcal{Q}_X^\beta([0, \tau], \mathcal{W}) \rightarrow \mathcal{Q}_X^\beta([0, \tau], \mathcal{W})$  by

$$\mathcal{M}_{[0, \tau]}^1(Y, Y^\dagger) = \left( \int_0^\cdot \sigma(Y_s) dX_s, \sigma(Y) \right), \quad \mathcal{M}_{[0, \tau]}^2(Y, Y^\dagger) = \left( \int_0^\cdot f(Y_s, \psi_s) ds, 0 \right),$$

and  $\mathcal{M}_{[0, \tau]}^\xi = (\xi, 0) + \mathcal{M}_{[0, \tau]}^1 + \mathcal{M}_{[0, \tau]}^2$ . If  $(Y, Y^\dagger)$  starts at  $(\xi, \sigma(\xi))$ , so does  $\mathcal{M}_{[0, \tau]}^\xi(Y, Y^\dagger)$ . A fixed point of  $\mathcal{M}_{[0, \tau]}^\xi$  is a solution of RDE (3.11) on the interval  $[0, \tau]$ .

We also set

$$B_{[0, \tau]}^\xi = \{(Y, Y^\dagger) \in \mathcal{Q}_X^\beta([0, \tau], \mathcal{W}) \mid \|(Y, Y^\dagger)\|_{\mathcal{Q}_X^\beta, [0, \tau]} \leq 1, Y_0 = \xi, Y_0^\dagger = \sigma(\xi)\}.$$

This is something like a ball of radius 1 centered at  $t \mapsto (\xi + \sigma(\xi)X_{0,t}^1, \sigma(\xi))$ . Since the initial point  $(Y_0, Y_0^\dagger)$  is fixed,  $\|\cdot\|_{\mathcal{Q}_X^\beta, [0, \tau]}$  works as a distance on this set.

For a while from now, we will work only on  $[0, \tau]$  and therefore omit  $[0, \tau]$  from the subscript for notational simplicity. We will often write  $Y_{s,t}^1 := Y_t - Y_s$ .

For  $(Y, Y^\dagger) \in B^\xi$  and  $(\tilde{Y}, \tilde{Y}^\dagger) \in B^{\tilde{\xi}}$ , the following estimates hold:

$$(3.12) \quad \|Y^\dagger\|_\infty \leq |\sigma(\xi)| + \sup_{s \leq \tau} |Y_s^\dagger - Y_0^\dagger| \leq K + \|Y^\dagger\|_{\beta\tau^\beta} \leq K + 1,$$

$$(3.13) \quad \|Y^\dagger - \tilde{Y}^\dagger\|_\infty \leq |Y_0^\dagger - \tilde{Y}_0^\dagger| + \|Y^\dagger - \tilde{Y}^\dagger\|_\beta \leq K|\xi - \tilde{\xi}| + \|Y^\dagger - \tilde{Y}^\dagger\|_\beta,$$

$$(3.14) \quad \begin{aligned} |Y_{s,t}^1| &\leq |Y_s^\dagger X_{s,t}^1| + |Y_{s,t}^\#| \\ &\leq (K+1)\|X^1\|_\alpha (t-s)^\alpha + \|Y^\#\|_{2\beta} (t-s)^{2\beta} \\ &\leq (K+1)(\|X^1\|_\alpha + 1)(t-s)^\alpha, \quad (s, t) \in \Delta_\tau, \end{aligned}$$

$$(3.15) \quad \begin{aligned} |Y_{s,t}^1 - \tilde{Y}_{s,t}^1| &\leq |(Y_s^\dagger - \tilde{Y}_s^\dagger)X_{s,t}^1| + |Y_{s,t}^\# - \tilde{Y}_{s,t}^\#| \\ &\leq (K|\xi - \tilde{\xi}| + \|Y^\dagger - \tilde{Y}^\dagger\|_\beta)\|X^1\|_\alpha (t-s)^\alpha + \|Y^\# - \tilde{Y}^\#\|_{2\beta} (t-s)^{2\beta} \\ &\leq \{K\|X^1\|_\alpha |\xi - \tilde{\xi}| \\ &\quad + (1 + \|X^1\|_\alpha)\|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}\}(t-s)^\alpha, \quad (s, t) \in \Delta_\tau. \end{aligned}$$

Note that  $Y$  is in fact  $\alpha$ -Hölder continuous. Hence, if  $\tau$  is small,  $\beta$ -Hölder seminorms of  $Y$  and  $Y - \tilde{Y}$  can be made very small. From (3.15) we can easily see that

$$(3.16) \quad \begin{aligned} \|Y - \tilde{Y}\|_\infty &\leq |Y_0 - \tilde{Y}_0| + \|Y_0^1 - \tilde{Y}_0^1\|_\infty \\ &\leq (1 + K\|X^1\|_\alpha \tau^\alpha)|\xi - \tilde{\xi}| + (1 + \|X^1\|_\alpha)\|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta} \tau^\alpha. \end{aligned}$$

Let us consider  $(\sigma(Y), \sigma(Y)^\dagger, \sigma(Y)^\#)$ , whose precise definition is given in the third item of Example 3.1. First,  $\sigma(Y)_t^\dagger = \nabla\sigma(Y_t)Y_t^\dagger \in L(\mathcal{V} \otimes \mathcal{V}, \mathcal{W})$ . More precisely, it is defined by

$$(3.17) \quad \nabla\sigma(Y_t)Y_t^\dagger \langle v \otimes v' \rangle := \nabla\sigma(Y_t) \langle Y_t^\dagger v, v' \rangle \quad \text{for } v, v' \in \mathcal{V}.$$

Then, we can see that

$$(3.18) \quad \sigma(Y)_t^\dagger - \sigma(Y)_s^\dagger = \int_0^1 d\theta \nabla^2 \sigma(Y_s + \theta Y_{s,t}^1) \langle Y_{s,t}^1, Y_t^\dagger \rangle + \nabla\sigma(Y_s) \langle Y_t^\dagger - Y_s^\dagger \rangle.$$

The second term on the right hand side is defined as in (3.17). Similarly, the precise meaning of the first term is given by  $v \otimes v' \mapsto \int_0^1 d\theta \nabla^2 \sigma(Y_s + \theta Y_{s,t}^1) \langle Y_{s,t}^1, Y_t^\dagger v, v' \rangle$ .

The remainder part reads:

$$(3.19) \quad \begin{aligned} \sigma(Y)_{s,t}^\# &= \nabla\sigma(Y_s) \langle Y_{s,t}^\#, \cdot \rangle \\ &\quad + \int_0^1 d\theta (1 - \theta) \nabla^2 \sigma(Y_s + \theta Y_{s,t}^1) \langle Y_{s,t}^1, Y_{s,t}^1, \cdot \rangle \in L(\mathcal{V}, \mathcal{W}). \end{aligned}$$

From (3.12), (3.14), (3.18) and (3.19), we can easily see that

$$\begin{aligned}\|\sigma(Y)^\dagger\|_\beta &\leq (K+1)^3(\|X^1\|_\alpha + 1), \\ \|\sigma(Y)^\sharp\|_{2\beta} &\leq (K+1)^3(\|X^1\|_\alpha + 1)^2.\end{aligned}$$

Next, we calculate the RP integral  $(\int_0^\cdot \sigma(Y)dX, \sigma(Y))$  using Proposition 3.2. It is easy to see that

$$\|\sigma(Y)\|_\beta \leq (K+1)^2(\|X^1\|_\alpha + 1)\tau^{\alpha-\beta}.$$

We can see from (3.8) and the above two estimates that

$$\begin{aligned}\left\|\left(\int_0^\cdot \sigma(Y)dX\right)^\sharp\right\|_{2\beta} &\leq \|\sigma(Y)^\dagger\|_\infty \|X^2\|_{2\beta} \\ &\quad + \kappa_\beta(\|\sigma(Y)^\sharp\|_{2\beta}\|X^1\|_\beta + \|\sigma(Y)^\dagger\|_\beta \|X^2\|_{2\beta}) \\ &\leq (1+2\kappa_\beta)(K+1)^3(\|X\|_\alpha + 1)^3\tau^{\alpha-\beta}.\end{aligned}$$

Thus, we have an estimate of  $\|\mathcal{M}^1(Y, Y^\dagger)\|_{\mathcal{Q}_X^\beta}$ . It is almost obvious from Example 3.1 that

$$\|\mathcal{M}^2(Y, Y^\dagger)\|_{\mathcal{Q}_X^\beta} \leq \left\|\int_0^\cdot f(Y_s, \psi_s)ds\right\|_{2\beta} \leq K\tau^{1-2\beta} \leq K\tau^{\alpha-\beta}.$$

Combining these three estimates we obtain that

$$\|\mathcal{M}^\xi(Y, Y^\dagger)\|_{\mathcal{Q}_X^\beta} \leq (2+2\kappa_\beta)(K+1)^3(\|X\|_\alpha + 1)^3\tau^{\alpha-\beta}.$$

Hence, if

$$(3.20) \quad \tau \leq \lambda \quad \text{with} \quad \lambda := \{8\kappa_\beta(K+1)^3(\|X\|_\alpha + 1)^3\}^{-1/(\alpha-\beta)},$$

then  $\mathcal{M}^\xi$  leaves  $B^\xi$  invariant. Note also that  $\kappa_\beta \geq 2$ . (For the rest of the proof we will assume  $\tau$  satisfies this inequality. The constant ‘‘8’’ in (3.20) has no particular meaning.)

Now we will prove that  $\mathcal{M}^\xi$  is a contraction on  $B^\xi$  for  $\tau$  as in (3.20). We will calculate  $(\sigma(Y) - \sigma(\tilde{Y}), \sigma(Y)^\dagger - \sigma(\tilde{Y})^\dagger)$  for  $(Y, Y^\dagger) \in B^\xi$  and  $(\tilde{Y}, \tilde{Y}^\dagger) \in B^\xi$ .

By straightforward (and slightly cumbersome) computations we obtain from (3.12)–(3.16) and (3.18) that

$$\begin{aligned}&|\{\sigma(Y)_t^\dagger - \sigma(\tilde{Y})_t^\dagger\} - \{\sigma(Y)_s^\dagger - \sigma(\tilde{Y})_s^\dagger\}| \\ &\leq \int_0^1 d\theta |\nabla^2 \sigma(Y_s + \theta Y_{s,t}^1) \langle Y_{s,t}^1, Y_t^\dagger \rangle - \nabla^2 \sigma(\tilde{Y}_s + \theta \tilde{Y}_{s,t}^1) \langle \tilde{Y}_{s,t}^1, \tilde{Y}_t^\dagger \rangle| \\ &\quad + |\sigma(Y_s)(Y_t^\dagger - Y_s^\dagger) - \sigma(\tilde{Y}_s)(\tilde{Y}_t^\dagger - \tilde{Y}_s^\dagger)| \\ &\leq \{2(K+1)^3(\|X^1\|_\alpha + 1)^2\tau^{\alpha-\beta}(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}) \\ &\quad + K(|\xi - \tilde{\xi}| + \|Y^\dagger - \tilde{Y}^\dagger\|_\beta)\}(t-s)^\beta.\end{aligned}$$

This estimate and (3.20) imply that

$$(3.21) \quad \|\sigma(Y)^\dagger - \sigma(\tilde{Y})^\dagger\|_\beta \leq (K+1)(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}).$$

In a very similar way, using (3.12)–(3.16) and (3.19), we can estimate the difference of the remainder parts as follows:

$$\begin{aligned} |\sigma(Y)_{s,t}^\# - \sigma(\tilde{Y})_{s,t}^\#| &\leq |\nabla\sigma(Y_s)\langle Y_{s,t}^\#, \cdot \rangle - \nabla\sigma(\tilde{Y}_s)\langle \tilde{Y}_{s,t}^\#, \cdot \rangle| \\ &\quad + \int_0^1 d\theta |\nabla^2\sigma(Y_s + \theta Y_{s,t}^1)\langle Y_{s,t}^1, Y_{s,t}^1, \cdot \rangle - \nabla^2\sigma(\tilde{Y}_s + \theta \tilde{Y}_{s,t}^1)\langle \tilde{Y}_{s,t}^1, \tilde{Y}_{s,t}^1, \cdot \rangle| \\ &\leq (K+1)(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta})(t-s)^{2\beta}, \end{aligned}$$

which immediately implies

$$(3.22) \quad \|\sigma(Y)^\# - \sigma(\tilde{Y})^\#\|_{2\beta} \leq (K+1)(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}).$$

Using Proposition 3.2 again, we estimate  $(\int_0^\cdot \{\sigma(Y) - \sigma(\tilde{Y})\}dX, \sigma(Y) - \sigma(\tilde{Y}))$ . From (3.21), (3.22), (3.15) and the fact  $\sigma(Y_t) - \sigma(Y_s) = \int_0^1 d\theta \nabla\sigma(Y_s + \theta Y_{s,t}^1)\langle Y_{s,t}^1, \cdot \rangle$ , we have

$$\|\sigma(Y) - \sigma(\tilde{Y})\|_\beta \leq (K+1)^2(\|X\|_\alpha + 1)^2\tau^{\alpha-\beta}(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta})$$

and

$$\begin{aligned} &\left\| \left( \int_0^\cdot \{\sigma(Y) - \sigma(\tilde{Y})\}dX \right)^\# \right\|_{2\beta} \\ &\leq \|\sigma(Y)^\dagger - \sigma(\tilde{Y})^\dagger\|_\infty \|X^2\|_{2\beta} \\ &\quad + \kappa_\beta (\|\sigma(Y)^\# - \sigma(\tilde{Y})^\#\|_{2\beta} \|X^1\|_\beta + \|\sigma(Y)^\dagger - \sigma(\tilde{Y})^\dagger\|_\beta \|X^2\|_{2\beta}) \\ &\leq 2(1 + \kappa_\beta)(K+1)(\|X\|_\alpha + 1)^2\tau^{\alpha-\beta}(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}). \end{aligned}$$

Using (3.20) we obtain from the above estimates that

$$\|\mathcal{M}^1(Y, Y^\dagger) - \mathcal{M}^1(\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta} \leq \frac{1}{2}(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}).$$

Since  $f$  is Lipschitz in the first argument, we can easily see from (3.16) that

$$\begin{aligned} \|\mathcal{M}^2(Y, Y^\dagger) - \mathcal{M}^2(\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta} &\leq \left\| \int_0^\cdot \{f(Y_s, \psi_s) - f(\tilde{Y}_s, \psi_s)\}ds \right\|_{2\beta} \\ &\leq K\|Y - \tilde{Y}\|_\infty \tau^{1-2\beta} \\ &\leq K(2|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta})\tau^{\alpha-\beta} \\ &\leq \frac{1}{4}(|\xi - \tilde{\xi}| + \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}). \end{aligned}$$

Summing up, we have

$$(3.23) \quad \|\mathcal{M}^\xi(Y, Y^\dagger) - \mathcal{M}^{\tilde{\xi}}(\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta} \leq \frac{3}{4}|\xi - \tilde{\xi}| + \frac{3}{4}\|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta}.$$

If  $\xi = \tilde{\xi}$  in particular, this estimate implies that  $\mathcal{M}^\xi$  is a contraction on  $B^\xi = B_{[0,\tau]}^\xi$  and has a unique fixed point in this set, which is a local solution of the RDE. Thus, we have obtained a solution on  $[0, \lambda]$ . Note that  $\tau$  (and  $\lambda$ ) is determined by  $\|X\|_\alpha$  and  $K$ , but is chosen independent of  $\xi$  and  $\psi$ .

Next, we do the same thing on the second interval  $[\lambda, 2\lambda]$  with the initial condition  $\xi$  at time 0 being replaced by  $Y_\lambda$  at time  $\lambda$ . Since all the estimates above is independent of  $\xi$  and  $\psi$ ,  $(Y_s, Y_s^\dagger)_{s \in [\lambda, 2\lambda]}$  satisfies the same estimates as those for  $(Y_s, Y_s^\dagger)_{s \in [0, \lambda]}$ . Concatenating them as in Example 3.1, we obtain a solution on  $[0, 2\lambda]$

We can continue this procedure to obtain a global  $(Y_s, Y_s^\dagger)_{s \in [0, 1]}$ . There are  $\lfloor \lambda^{-1} \rfloor + 1$  subintervals, where  $\lfloor \cdot \rfloor$  stands for the integer part. Except the last one, the length of each interval equals  $\lambda$ . On each subinterval,  $(Y, Y^\dagger)$  satisfies the same estimates. In particular, Inequality (3.14) implies that  $\beta$ -Hölder norm of  $Y$  on each subinterval is dominated by  $\{8\kappa_\beta(K+1)^2(\|X\|_\alpha + 1)^2\}^{-1}$ . By Hölder's inequality for finite sums, we can easily see that

$$\begin{aligned} \|Y\|_{\beta, [0, 1]} &\leq \{8\kappa_\beta(K+1)^2(\|X\|_\alpha + 1)^2\}^{-1} (\lfloor \lambda^{-1} \rfloor + 1)^{1-\beta} \\ &\leq c_{\alpha, \beta} \{(K+1)(\|X\|_\alpha + 1)\}^\nu, \end{aligned}$$

where  $\nu := 3(1 - \beta)/(\alpha - \beta) - 2 > 0$  and  $c_{\alpha, \beta} > 0$  is a constant which depends only on  $\alpha, \beta$ .

The uniqueness for this type of RDEs is well-known. So we just give a quick explanation. The uniqueness is a time-local issue, so it suffices to prove that any two solutions,  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$ , of RDE (3.11) must coincide near  $t = 0$ . Take any  $\beta' \in (1/3, \beta)$  and we work in  $\beta'$ -Hölder topology instead of  $\beta$ -Hölder topology. If  $\tau$  is small enough, both  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$  restricted to  $[0, \tau]$  belong to  $B_{[0, \tau]}^\xi$ . But, we have already proved that there is only one fixed point of  $\mathcal{M}_{[0, \tau]}^\xi$  in this ball. Hence,  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$  must be identically equal on  $[0, \tau]$ .  $\square$

**Remark 3.4.** (i) By examining the proof of Proposition 3.3, one naturally realize the following: Just to prove the existence of a unique global solution RDE (3.11) for any given  $\psi, X$  and  $\xi$ , it suffices to assume that  $\sigma$  is of  $C_b^3$  and  $f$  satisfies that

$$\sup_{y \in \mathcal{W}, t \in [0, T]} |f(y, \psi_t)|_{\mathcal{W}} + \sup_{y, y' \in \mathcal{W}, y \neq y', t \in [0, T]} \frac{|f(y, \psi_t) - f(y', \psi_t)|_{\mathcal{W}}}{|y - y'|_{\mathcal{W}}} < \infty.$$

(ii) By a standard cut-off argument, it immediately follows from (i) above that if  $\sigma$  is of  $C^3$  and  $f$  is locally Lipschitz continuous in the following sense

$$\sup_{|y| \vee |y'| \leq N, y \neq y', t \in [0, T]} \frac{|f(y, \psi_t) - f(y', \psi_t)|_{\mathcal{W}}}{|y - y'|_{\mathcal{W}}} < \infty, \quad N \in \mathbb{N},$$

then RDE (3.11) has a unique local solution for any given  $\psi, X$  and  $\xi$ . Hence, a unique solution exists up to either the explosion time or the time horizon  $T$ .

Together with RDE (3.11), we also consider the following RDE with the same  $X$  and  $\xi$ :

$$(3.24) \quad \tilde{Y}_t = \xi + \int_0^t \tilde{f}(\tilde{Y}_s, \tilde{\psi}_s) ds + \int_0^t \sigma(\tilde{Y}_s) dX_s, \quad \tilde{Y}_t^\dagger = \sigma(\tilde{Y}_t), \quad t \in [0, T].$$

We assume that  $\tilde{f}: \mathcal{W} \times \mathcal{S} \rightarrow \mathcal{W}$  is also continuous and satisfies Condition (3.10). Let  $\tilde{\psi}: [0, T] \rightarrow \mathcal{S}$  be another continuous path in  $\mathcal{S}$ .

**Proposition 3.5.** *Let  $\sigma, f, \tilde{f}, \xi$  be as above. For  $X \in \Omega_\alpha(\mathcal{V})$ ,  $\xi \in \mathcal{W}$  and  $\psi, \tilde{\psi}$ , denote by  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$  the unique solutions of RDEs (3.11) and (3.24) on  $[0, T]$ , respectively. For a bounded, globally Lipschitz map  $g: \mathcal{W} \rightarrow \mathcal{W}$ , set*

$$(3.25) \quad M_t := (Y_t - \tilde{Y}_t) - \int_0^t \{g(Y_s) - g(\tilde{Y}_s)\} ds - \int_0^t \{\sigma(Y_s) - \sigma(\tilde{Y}_s)\} dX_s, \quad t \in [0, T].$$

Then,  $M \in \mathcal{C}^1(\mathcal{W})$  and we have the following estimate for every  $\beta \in (\frac{1}{3}, \alpha)$ : there exist positive constants  $c$  and  $\nu$  such that

$$(3.26) \quad \|Y - \tilde{Y}\|_\beta \leq c \exp[c(K' + 1)^\nu (\|X\|_\alpha + 1)^\nu] \|M\|_{2\beta}.$$

Here, we set  $K' = \max\{\|\sigma\|_{C_b^3}, \|f\|_\infty, L_f, \|\tilde{f}\|_\infty, L_{\tilde{f}}, \|g\|_\infty, L_g\}$  and the constants  $c$  and  $\nu$  are independent of  $X, \xi, \psi, \tilde{\psi}, \sigma, f, \tilde{f}, g, M$ .

*Proof.* Without loss of generality we assume  $T = 1$ . For simplicity we write  $(Y, Y^\dagger)$  and  $(\tilde{Y}, \tilde{Y}^\dagger)$  for  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$ , respectively. It is easy to see that  $M \in \mathcal{C}^1(\mathcal{W})$ . Hence, (3.25) is in fact an equality in  $\mathcal{Q}_X^\beta(\mathcal{W})$  with the  $\dagger$ -parts being clearly equal.

Mimicking (3.20), we set  $\lambda' := \{8\kappa_\beta(K' + 1)^3 (\|X\|_\alpha + 1)^3\}^{-1/(\alpha-\beta)}$ . Set  $s_j := j\lambda'$  for  $0 \leq j \leq \lfloor 1/\lambda' \rfloor$  and  $s_N := 1$  with  $N := \lfloor 1/\lambda' \rfloor + 1$ . Then, on each subinterval  $[s_{j-1}, s_j]$ ,  $(Y, Y^\dagger) \in B_{[s_{j-1}, s_j]}^{\xi_j}$ ,  $(\tilde{Y}, \tilde{Y}^\dagger) \in B_{[s_{j-1}, s_j]}^{\tilde{\xi}_j}$  and the estimates in the proof of Proposition 3.3 are available (with  $K$  being replaced by  $K'$ ). From (3.23) we have for all  $j$  that

$$\begin{aligned} & \left\| \int_{s_{j-1}}^{\cdot} \{g(Y_s) - g(\tilde{Y}_s)\} ds - \int_{s_{j-1}}^{\cdot} \{\sigma(Y_s) - \sigma(\tilde{Y}_s)\} dX_s \right\|_{\mathcal{Q}_X^\beta, [s_{j-1}, s_j]} \\ & \leq \frac{3}{4} |\xi_{j-1} - \tilde{\xi}_{j-1}| + \frac{3}{4} \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta, [s_{j-1}, s_j]}, \end{aligned}$$

where we write  $\xi_j = Y_{s_j}$  and  $\tilde{\xi}_j = \tilde{Y}_{s_j}$ . Taking the seminorms of both sides of (3.25) on each subinterval, we can easily see that

$$(3.27) \quad \|(Y, Y^\dagger) - (\tilde{Y}, \tilde{Y}^\dagger)\|_{\mathcal{Q}_X^\beta, [s_{j-1}, s_j]} \leq 4\|M\|_{2\beta} + 3|\xi_{j-1} - \tilde{\xi}_{j-1}|.$$

Plugging this into (3.15), we obtain for all  $j$  that

$$(3.28) \quad |Y_{s,t}^1 - \tilde{Y}_{s,t}^1| \leq \{K'\|X^1\|_\alpha |\xi_{j-1} - \tilde{\xi}_{j-1}|$$



$$\begin{aligned}
& + (1 + \|X^1\|_\alpha)(4\|M\|_{2\beta} + 3|\xi_{j-1} - \tilde{\xi}_{j-1}|)\}(t-s)^\alpha \\
& \leq |\xi_{j-1} - \tilde{\xi}_{j-1}| + \|M\|_{2\beta}, \quad (s, t) \in \Delta_{[s_{j-1}, s_j]}
\end{aligned}$$

and in particular

$$|\xi_j - \tilde{\xi}_j| \leq 2|\xi_{j-1} - \tilde{\xi}_{j-1}| + \|M\|_{2\beta}.$$

By mathematical induction we have

$$|\xi_j - \tilde{\xi}_j| \leq (1 + 2^1 + \dots + 2^{j-1})\|M\|_{2\beta} = (2^j - 1)\|M\|_{2\beta}, \quad 1 \leq j \leq N.$$

Then, we see from (3.28) that

$$\begin{aligned}
(3.29) \quad \|Y^1 - \tilde{Y}^1\|_{\beta, [s_{j-1}, s_j]} & \leq \{K'\|X^1\|_\alpha|\xi_{j-1} - \tilde{\xi}_{j-1}| \\
& + (1 + \|X^1\|_\alpha)(4\|M\|_{2\beta} + 3|\xi_{j-1} - \tilde{\xi}_{j-1}|)\}(\lambda')^{\alpha-\beta} \\
& \leq 4(1 + K')(1 + \|X^1\|_\alpha)2^N(\lambda')^{\alpha-\beta}\|M\|_{2\beta}, \quad 1 \leq j \leq N.
\end{aligned}$$

By Hölder's inequality for finite sums and the trivial inequality  $N^{1-\beta}2^N \leq 2^{2N}$ , we see that

$$\begin{aligned}
(3.30) \quad \|Y^1 - \tilde{Y}^1\|_{\beta, [0,1]} & \leq 4(1 + K')(1 + \|X^1\|_\alpha)2^N(\lambda')^{\alpha-\beta}\|M\|_{2\beta}N^{1-\beta} \\
& \leq c_{\alpha,\beta}(K' + 1)(\|X\|_\alpha + 1) \\
& \quad \times \exp[c_{\alpha,\beta}\{(K' + 1)(\|X\|_\alpha + 1)\}^{3/(\alpha-\beta)}]\|M\|_{2\beta},
\end{aligned}$$

where  $c_{\alpha,\beta} > 0$  is a constant which depends only on  $\alpha, \beta$ . By adjusting the constant  $c_{\alpha,\beta}$ , we can easily obtain (3.26) from (3.30).  $\square$

### 3.4 Rough differential equations: The case of bounded and locally Lipschitz drift vector field

In this subsection we continue to study RDE (3.11), where  $\psi: [0, T] \rightarrow \mathcal{S}$  is a continuous path in  $\mathcal{S}$ . We still assume that  $\frac{1}{3} < \beta < \alpha \leq \frac{1}{2}$  and  $\sigma$  is of  $C_b^3$ , but relax the global Lipschitz condition (3.10) on  $f$ .

**Proposition 3.6.** *Suppose that  $\sigma$  is of  $C_b^3$ ,  $\|f\|_\infty := \sup_{y \in \mathcal{W}, z \in \mathcal{S}} |f(y, z)|_{\mathcal{W}} < \infty$  and*

$$\sup_{|y|_{\mathcal{V}}|y'| \leq N, y \neq y', t \in [0, T]} \frac{|f(y, \psi_t) - f(y', \psi_t)|_{\mathcal{W}}}{|y - y'|_{\mathcal{W}}} < \infty, \quad N \in \mathbb{N},$$

*then RDE (3.11) has a unique global solution  $(Y, Y^\dagger) \in \mathcal{Q}_X^\beta(\mathcal{W})$  for every  $X \in \Omega_\alpha(\mathcal{V})$ ,  $\xi \in \mathcal{W}$  and  $\psi$ . Moreover, there exists a constant  $C > 0$  such that*

$$(3.31) \quad \|Y\|_\beta \leq C\{(\|\sigma\|_{C_b^2}\|X\|_\alpha)^{1/\beta} + \|\sigma\|_{C_b^2}\|X\|_\alpha + \|f\|_\infty\}, \quad X \in \Omega_\alpha(\mathcal{V}).$$

*Here,  $C$  is independent of  $X, \xi, \psi, f, \sigma$ .*

*Proof.* We mimic the proof of a priori estimates in [8, Section 8.4]. In this proof positive constants  $c_i$ 's are independent of  $X, \xi, \psi, f, \sigma$ . Without loss of generality we may assume  $T = 1$ . We will write  $L := \|f\|_\infty$  and  $L' := \|\sigma\|_{C_b^2}$  for simplicity. The norms of finite dimensional vector spaces are simply denoted by  $|\cdot|$ .

By Remark 3.4, a unique local solution exists, which is denoted by  $\{(Y_t, Y_t^\dagger)\}_{0 \leq t \leq S}$ ,  $S \in (0, 1]$ . If  $\|Y\|_{\beta, [0, S]}$  is dominated by the right hand side of (3.31), which is independent of  $S$ , then  $Y$  does not explode and therefore extends to a global solution.

First, we consider the case  $L' \leq 1$ . For  $(s, t) \in \Delta_S$ ,

$$\begin{aligned} |Y_{s,t}^\#| &= |Y_{s,t} - \sigma(Y_s)X_{s,t}^1| \\ &\leq \left| \int_s^t \sigma(Y_u) dX_u - \sigma(Y_s)X_{s,t}^1 - \nabla \sigma(Y_s) \sigma(Y_s) X_{s,t}^2 \right| \\ &\quad + |\nabla \sigma(Y_s) \sigma(Y_s) X_{s,t}^2| + \left| \int_s^t f(Y_u, \psi_u) du \right| \\ &\leq \kappa_\beta (\|\sigma(Y)^\# \|_{2\beta, [s,t]} \|X^1\|_{\beta, [s,t]} + \|\sigma(Y)\|_{\beta, [s,t]} \|X^2\|_{2\beta, [s,t]}) (t-s)^{3\beta} \\ &\quad + \|X^2\|_{2\beta, [s,t]} (t-s)^{2\beta} + L(t-s), \end{aligned}$$

where Proposition 3.2 is used for the last inequality. Note that  $\|\sigma(Y)\|_{\beta, [s,t]} \leq \|Y\|_{\beta, [s,t]}$ . For  $h \in (0, S]$ , we set  $\|\cdot\|_{\beta; h} := \sup_{0 < t-s \leq h} \|\cdot\|_{\beta, [s,t]}$ . Then, it immediately follows that

$$(3.32) \quad \begin{aligned} \|Y^\# \|_{2\beta; h} &\leq \kappa_\beta (\|\sigma(Y)^\# \|_{2\beta; h} \|X^1\|_{\beta; h} + \|Y\|_{\beta; h} \|X^2\|_{2\beta; h}) h^\beta \\ &\quad + \|X^2\|_{2\beta; h} + Lh^{1-2\beta}. \end{aligned}$$

Next we calculate  $\sigma(Y)^\#$ . By definition, we have

$$\begin{aligned} \sigma(Y)_{s,t}^\# &= \sigma(Y_t) - \sigma(Y_s) - \nabla \sigma(Y_s) Y_s^\dagger X_{s,t}^1 \\ &= \sigma(Y_t) - \sigma(Y_s) - \nabla \sigma(Y_s) \langle Y_{s,t}^1 \rangle + \nabla \sigma(Y_s) \langle Y_{s,t}^\# \rangle. \end{aligned}$$

From this and Taylor's formula,

$$\|\sigma(Y)^\# \|_{2\beta; h} \leq \frac{1}{2} \|Y\|_{\beta; h}^2 + \|Y^\# \|_{2\beta; h}.$$

We put the above inequality back into (3.32). Then we can easily see that

$$(3.33) \quad \begin{aligned} \|Y^\# \|_{2\beta; h} &\leq \kappa_\beta \left( \frac{1}{2} \|Y\|_{\beta; h}^2 + \|Y^\# \|_{2\beta; h} \right) \|X^1\|_{\beta; h} h^\beta + \kappa_\beta \|Y\|_{\beta; h} \|X^2\|_{2\beta; h} h^\beta \\ &\quad + \|X^2\|_{2\beta; h} + Lh^{1-2\beta}. \end{aligned}$$

Hence, if  $h$  is so small that  $\kappa_\beta \|X\|_\beta h^\beta = \kappa_\beta (\|X^1\|_\beta + \|X^2\|_{2\beta}^{1/2}) h^\beta \leq 1/2$  holds, we see from (3.33) that

$$(3.34) \quad \|Y^\# \|_{2\beta; h} \leq \kappa_\beta \|Y\|_{\beta; h}^2 \|X^1\|_{\beta; h} h^\beta + 2\kappa_\beta \|Y\|_{\beta; h} \|X^2\|_{2\beta; h} h^\beta + 2\|X^2\|_{2\beta; h} + 2Lh^{1-2\beta}$$

$$\begin{aligned}
&\leq \frac{1}{2} \|Y\|_{\beta;h}^2 + \|Y\|_{\beta;h} \|X^2\|_{2\beta;h}^{1/2} + 2 \|X^2\|_{2\beta;h} + 2Lh^{1-2\beta} \\
&\leq \|Y\|_{\beta;h}^2 + 3 \|X^2\|_{2\beta;h} + 2Lh^{1-2\beta}.
\end{aligned}$$

Since  $Y_{s,t}^1 = Y_s^\dagger X_{s,t}^1 + Y_{s,t}^\sharp = \sigma(Y_s) X_{s,t}^1 + Y_{s,t}^\sharp$  by definition, we see from (3.34) and  $\kappa_\beta \geq 2$  that

$$\|Y\|_{\beta;h} h^\beta \leq \|X^1\|_{\beta;h} h^\beta + \|Y^\sharp\|_{2\beta;h} h^{2\beta} \leq \|X\|_\beta h^\beta + 2Lh + (\|Y\|_{\beta;h} h^\beta)^2.$$

If we set  $\lambda_h = \|X\|_\beta h^\beta + 2Lh$  and  $\varphi_h = \|Y\|_{\beta;h} h^\beta$  for  $h \in (0, S]$ , then we have

$$(3.35) \quad 0 \leq \varphi_h \leq \lambda_h + \varphi_h^2 \quad \text{if } \|X\|_\beta h^\beta \leq 1/(2\kappa_\beta).$$

Note that  $\varphi_h$  is left-continuous and non-decreasing in  $h$ . Moreover,  $\lim_{h \searrow 0} \varphi_h = 0$ . So,  $\varphi_h \leq 1/4$  if  $h \in (0, \eta)$  for some very small  $\eta > 0$ . However, this  $\eta$  seems to depend on  $Y$ . We will show that there exists  $\eta > 0$  which depends only on  $\|X\|_\beta$ .

It is easy to see that  $(1/2) - \sqrt{(1/4) - u} \leq 2u$  for  $u \in [0, 1/8]$ . Hence, it immediately follows from (3.35) that one and only one of the following two conditions, (3.36) and (3.37), must hold if  $\lambda_h \leq 1/8$ :

$$(3.36) \quad \varphi_h \geq \frac{1}{2} + \sqrt{\frac{1}{4} - \lambda_h} \geq \frac{3}{4},$$

$$(3.37) \quad \varphi_h \leq \frac{1}{2} - \sqrt{\frac{1}{4} - \lambda_h} \leq 2\lambda_h \leq \frac{1}{4}.$$

If  $0 < \delta < h$ , then  $\|Y\|_{\beta;h+\delta} \leq 2^{1-\beta} \|Y\|_{\beta;h}$ . This implies that the relative jump size of  $\varphi$  is at most  $2^{1-\beta}$ , i.e.  $\lim_{\delta \searrow 0} \varphi_{h+\delta} \leq 2^{1-\beta} \varphi_h$ . Hence, as  $h$  increases from 0 under the conditions that  $\lambda_h \leq 1/8$  and  $\|X\|_\beta h^\beta \leq 1/(2\kappa_\beta)$ ,  $\varphi_h$  cannot jump from  $[0, 1/4]$  to  $[3/4, \infty)$ . So  $\varphi_h$  must always satisfy (3.37) for all such  $h$ . By the above argument, we see that if

$$h \leq \Lambda, \quad \text{where we set } \Lambda := \min\{[16 \vee (2\kappa_\beta)]^{-1/\beta} \|X\|_\beta^{-1/\beta}, (32L)^{-1}, S\}$$

(this includes the case  $\|X\|_\beta = 0$  or  $L = 0$ ), then (3.37) holds and hence

$$\begin{aligned}
(3.38) \quad \|Y\|_{\beta;h} &\leq \varphi_h h^{-\beta} \leq 2\lambda_h h^{-\beta} \leq 2\|X\|_\beta + 4Lh^{1-\beta} \\
&\leq \begin{cases} 2\|X\|_\beta + c_1 L \min\{\|X\|_\beta^{-(1-\beta)/\beta}, 1\} & (\text{if } L \leq 1), \\ 2\|X\|_\beta + c_1 L \min\{\|X\|_\beta^{-(1-\beta)/\beta}, 1/L^{1-\beta}\} & (\text{if } L \geq 1), \end{cases}
\end{aligned}$$

also holds.

Set  $s_j := j\Lambda$  for  $0 \leq j \leq \lfloor S/\Lambda \rfloor$  and  $s_N := S$  with  $N := \lfloor S/\Lambda \rfloor + 1$ . Then,

$$(3.39) \quad N \leq c_2 \max\{\|X\|_\beta^{1/\beta}, L, 1\} + 1 \leq \begin{cases} c_3(\|X\|_\beta^{1/\beta} + 1) & (\text{if } L \leq 1), \\ c_3(\|X\|_\beta^{1/\beta} + L) & (\text{if } L \geq 1). \end{cases}$$

On each subinterval  $[s_{j-1}, s_j]$ , Inequality (3.38) is available.

First we calculate the case  $L \leq 1$ . Using (3.38), (3.39) and Hölder's inequality for finite sums again, we have

$$\begin{aligned} \|Y\|_{\beta,[0,S]} &\leq \|Y\|_{\beta;\Lambda} N^{1-\beta} \\ &\leq c_4(\|X\|_{\beta} + L \min\{\|X\|_{\beta}^{-(1-\beta)/\beta}, 1\})(\|X\|_{\beta}^{1/\beta} + 1)^{1-\beta} \\ &\leq c_5(\|X\|_{\beta}^{1/\beta} + \|X\|_{\beta} + L). \end{aligned}$$

The case  $L \geq 1$  is quite similar;

$$\begin{aligned} \|Y\|_{\beta,[0,S]} &\leq \|Y\|_{\beta;\Lambda} N^{1-\beta} \\ &\leq c_6(\|X\|_{\beta} + L \min\{\|X\|_{\beta}^{-(1-\beta)/\beta}, 1/L^{1-\beta}\})(\|X\|_{\beta}^{1/\beta} + L)^{1-\beta} \\ &\leq c_7(\|X\|_{\beta}^{1/\beta} + \|X\|_{\beta} L^{1-\beta} + L) \\ &\leq c_8(\|X\|_{\beta}^{1/\beta} + L) \leq c_8(\|X\|_{\beta}^{1/\beta} + \|X\|_{\beta} + L). \end{aligned}$$

In the second to the last inequality, we used Young's inequality. Either way, we have proved that  $Y$  does not explode and

$$(3.40) \quad \|Y\|_{\beta,[0,1]} \leq c_9(\|X\|_{\beta}^{1/\beta} + \|X\|_{\beta} + L).$$

Thus, the case  $L' \leq 1$  is done.

Finally, we consider the general case. Since the  $L' = 0$  is trivial, we may assume  $L' > 0$ . Note that  $(Y, \sigma(Y))$  solves RDE (3.11) if and only if  $(Y, \sigma(Y)/L')$  solves RDE (3.11) with  $\sigma$  and  $X$  being replaced by  $\sigma/L'$  and  $L'X$ , respectively. Here,  $L'X$  is the dilation of  $X$  by  $L'$ . Therefore, we have

$$\|Y\|_{\beta,[0,1]} \leq c_9\{(L'\|X\|_{\beta})^{1/\beta} + L'\|X\|_{\beta} + L\}.$$

Noting that  $\|X\|_{\beta} \leq \|X\|_{\alpha}$  since  $\beta < \alpha$ , we complete the proof.  $\square$

Next we generalize Proposition 3.5 to the case of bounded and locally Lipschitz drift vector fields. We consider RDEs (3.11) and (3.24) again and then define  $M$  as in (3.25) for a continuous map  $g: \mathcal{W} \rightarrow \mathcal{W}$ . This time we do not impose the global Lipschitz property on  $f, \tilde{f}, g$ , however. For  $N \in \mathbb{N}$ , we set

$$L_f^N := \sup_{|y| \vee |y'| \leq N, y \neq y', z \in \mathcal{S}} \frac{|f(y, z) - f(y', z)|_{\mathcal{W}}}{|y - y'|_{\mathcal{W}}}.$$

In the essentially same way we set  $L_{\tilde{f}}^N$  and  $L_g^N$ , too.

**Proposition 3.7.** *For  $X \in \Omega_{\alpha}(\mathcal{V})$ ,  $\xi \in \mathcal{W}$  and continuous paths  $\psi$  and  $\tilde{\psi}$ , we denote by  $(Y, \sigma(Y))$  and  $(\tilde{Y}, \sigma(\tilde{Y}))$  the unique solutions of RDEs (3.11) and (3.24) on  $[0, T]$ , respectively. We assume that  $\sigma$  is of  $C_b^3$  and that  $f, \tilde{f}, g$  are all bounded and satisfy*

$$L_f^N + L_{\tilde{f}}^N + L_g^N = O(N^r) \quad \text{as } N \rightarrow \infty$$

for some  $r > 0$ . Here,  $O$  stands for Landau's large  $O$ .

Then,  $M \in C^1(\mathcal{W})$  and we have the following estimate for every  $\beta \in (\frac{1}{3}, \alpha)$ : there exist positive constants  $C$  and  $\nu$  such that

$$\|Y - \tilde{Y}\|_\beta \leq C \exp(C \|X\|_\alpha^\nu) \|M\|_{2\beta}.$$

Here,  $C$  and  $\nu$  are independent of  $X, \xi, \psi, \tilde{\psi}, M$ . Recall that  $M$  was defined in (3.25).

*Proof.* In this proof,  $c_i$ 's are positive constants independent of  $X, \xi, \psi, \tilde{\psi}, M$ .

We use Proposition 3.6. From (3.31) we see that

$$\|Y\|_\beta \vee \|\tilde{Y}\|_\beta \leq c_1 (\|X\|_\alpha^{1/\beta} + \|X\|_\alpha + 1)$$

for some constant  $c_1 \geq 1$ . We denote the right hand side by  $R^X (\geq 1)$ .

Take a Lipschitz continuous function  $\chi: \mathcal{W} \rightarrow [0, 1]$  such that (i)  $\chi(y) = 1$  if  $|y| \leq R^X$ , (ii)  $\chi(y) = 0$  if  $|y| \geq 2R^X$  and (iii) the Lipschitz constant of  $\chi$  is at most 1. Then,  $\chi f$  is globally Lipschitz (in  $y$ ) since

$$\begin{aligned} |\chi(y)f(y, z) - \chi(y')f(y', z)| &\leq |\chi(y)| |f(y, z) - f(y', z)| + |\chi(y) - \chi(y')| |f(y', z)| \\ &\leq \{c_2 (R^X)^r + \|f\|_\infty\} |y - y'|_{\mathcal{W}} \\ &\leq c_3 \{(R^X)^r + 1\} |y - y'|_{\mathcal{W}}. \end{aligned}$$

It is obvious that  $\chi \tilde{f}$  and  $\chi g$  also satisfy the same property.

Noting that  $Y$  (resp.  $\tilde{Y}$ ) solves RDEs (3.11) (resp. (3.24)) whose drift vector field is replaced by  $\chi f$  (resp.  $\chi \tilde{f}$ ). Hence, we can use Inequality (3.26) in Proposition 3.5 with

$$K' \leq c_3 \{(R^X)^r + 1\} \leq c_4 (\|X\|_\alpha^{1/\beta} + 1).$$

This proves our assertion.  $\square$

**Remark 3.8.** In a recent preprint [6], well-posedness of an RDE with globally Lipschitz drift is proved (in the framework of CP theory). If one uses this result, one may be able to drop the boundedness assumption on  $f$  in this section and in our main theorem (Theorem 2.1). Though this problem looks interesting, we do not pursue it in this paper.

## 4 Slow-fast system of rough differential equations

In this section we define the slow-fast system (2.1) of RDEs precisely and study it in details. In this and the next sections, we always assume that  $\sigma$  and  $h$  are of  $C^3$  and  $f$  and  $g$  are locally Lipschitz continuous. Under these assumptions, the slow-fast system always has a unique time-local solution.

## 4.1 Deterministic aspects

As before,  $1/3 < \beta < \alpha \leq 1/2$  is assumed. Respecting the direct sum decomposition  $\mathbb{R}^{d+e} = \mathbb{R}^d \oplus \mathbb{R}^e$ , a generic path taking values in this space is denoted by  $(b, w) = (b_t, w_t)_{0 \leq t \leq T}$ . Similarly, a generic element  $\Xi = (\Xi^1, \Xi^2)$  of  $\Omega_\alpha(\mathbb{R}^{d+e})$  is denoted by

$$(4.1) \quad \Xi^1 = (B^1, W^1), \quad \Xi^2 = \begin{pmatrix} B^2 & I[B, W] \\ I[W, B] & W^2 \end{pmatrix}.$$

It is clear that  $B = (B^1, B^2) \in \Omega_\alpha(\mathbb{R}^d)$  and  $W = (W^1, W^2) \in \Omega_\alpha(\mathbb{R}^e)$ .  $I[B, W]$  takes values in  $\mathbb{R}^d \otimes \mathbb{R}^e$  and, loosely speaking, plays the role of the iterated integral  $(s, t) \mapsto \int_s^t B_{s,u}^1 \otimes d_u W_{s,u}^1$ . Also,  $I[W, B]$  can be explained in an analogous way.

In the same manner, respecting the direct sum decomposition  $\mathbb{R}^{m+n} = \mathbb{R}^m \oplus \mathbb{R}^n$ , a generic element of  $\mathbb{R}^{m+n}$  is denoted by  $z = (x, y)$ . We set

$$F_\varepsilon(x, y) = \begin{pmatrix} f(x, y) \\ \varepsilon^{-1}g(x, y) \end{pmatrix}, \quad \Sigma_\varepsilon(x, y) = \begin{pmatrix} \sigma(x) & O \\ O & \varepsilon^{-1/2}h(x, y) \end{pmatrix}.$$

Then,  $F_\varepsilon: \mathbb{R}^{m+n} \rightarrow \mathbb{R}^{m+n}$  and  $\Sigma_\varepsilon: \mathbb{R}^{m+n} \rightarrow L(\mathbb{R}^{d+e}, \mathbb{R}^{m+n})$ .

Let  $(Z, Z^\dagger) \in \mathcal{Q}_\Xi^\alpha(\mathbb{R}^{m+n})$  be a CP with respect to  $\Xi \in \Omega_\alpha(\mathbb{R}^{d+e})$ . We write  $Z = (X, Y)$ . Since  $Z^\dagger$  takes values in  $L(\mathbb{R}^{d+e}, \mathbb{R}^{m+n})$ , it can be written as a block matrix;

$$Z^\dagger = \begin{pmatrix} Z^{\dagger,11} & Z^{\dagger,12} \\ Z^{\dagger,21} & Z^{\dagger,22} \end{pmatrix}.$$

Using this notation, we can write the remainder part  $Z^\sharp$  as follows;

$$Z_{s,t}^\sharp = \begin{pmatrix} X_{s,t}^1 \\ Y_{s,t}^1 \end{pmatrix} - \begin{pmatrix} Z_s^{\dagger,11} & Z_s^{\dagger,12} \\ Z_s^{\dagger,21} & Z_s^{\dagger,22} \end{pmatrix} \begin{pmatrix} B_{s,t}^1 \\ W_{s,t}^1 \end{pmatrix}.$$

The precise definition of the slow-fast system (2.1) of RDEs (in the deterministic sense) is given by

$$(4.2) \quad Z_t^\varepsilon = z_0 + \int_0^t F_\varepsilon(Z_s^\varepsilon) ds + \int_0^t \Sigma_\varepsilon(Z_s^\varepsilon) d\Xi_s, \quad (Z^\varepsilon)_t^\dagger = \Sigma(Z_t^\varepsilon), \quad t \in [0, T].$$

We consider this RDE in the  $\beta$ -Hölder topology.

**Remark 4.1.** An element of a direct sum space is denoted by both a ‘‘column vector’’ and a ‘‘row vector.’’ These are not precisely distinguished.

If  $Z^\varepsilon$  solves the above slow-fast system (4.2), then its slow component  $X^\varepsilon$  solves an RDE driven by  $B$  alone on  $\mathbb{R}^m$  with  $Y^\varepsilon$  viewed as a parameter. (See (4.3) below. This type was introduced in (3.11) and discussed in Subsection 3.3.) Since a solution of an RDE is not a usual path, this fact is not obvious.

**Lemma 4.2.** *Let  $(Z^\varepsilon, \Sigma_\varepsilon(Z^\varepsilon)) \in \mathcal{Q}_{\Xi}^\beta([0, \tau], \mathbb{R}^{m+n})$ ,  $0 < \tau \leq T$ , be a unique solution of RDE (4.2) on  $[0, \tau]$ . Then,  $(X^\varepsilon, \sigma(X^\varepsilon))$  belongs to  $\mathcal{Q}_B^\beta([0, \tau], \mathbb{R}^m)$  and is a unique local solution of the following RDE driven by  $B$  on  $[0, \tau]$ :*

$$(4.3) \quad X_t^\varepsilon = x_0 + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t \sigma(X_s^\varepsilon) dB_s, \quad (X_t^\varepsilon)^\dagger = \sigma(X_t^\varepsilon), \quad t \in [0, T].$$

*Proof.* Applying Proposition 3.2 to the right hand side of RDE (4.2), we see that

$$Z_{s,t}^{\varepsilon,1} - \Sigma(Z_s^\varepsilon) \Xi_{s,t}^1 = \begin{pmatrix} X_{s,t}^{\varepsilon,1} \\ Y_{s,t}^{\varepsilon,1} \end{pmatrix} - \begin{pmatrix} \sigma(X_s^\varepsilon) & O \\ O & \varepsilon^{-1/2} h(X_s^\varepsilon, Y_s^\varepsilon) \end{pmatrix} \begin{pmatrix} B_{s,t}^1 \\ W_{s,t}^1 \end{pmatrix}$$

is of  $2\beta$ -Hölder as a function of  $(s, t) \in \Delta_\tau$  and so is its first component  $X_{s,t}^{\varepsilon,1} - \sigma(X_s^\varepsilon) B_{s,t}^1$ . This means  $(X^\varepsilon, \sigma(X^\varepsilon)) \in \mathcal{Q}_B^\beta([0, \tau], \mathbb{R}^m)$ .

It suffices to show that the first component of  $\int_0^t \Sigma_\varepsilon(Z_s^\varepsilon) d\Xi_s$  equals  $\int_0^t \sigma(X_s^\varepsilon) dB_s$ . A straightforward but slightly cumbersome computation of block matrices yields that

$$(4.4) \quad \Sigma_\varepsilon(Z_s^\varepsilon) \Xi_{s,t}^1 + \{\Sigma_\varepsilon(Z_s^\varepsilon)\}_s^\dagger \Xi_{s,t}^2 = \begin{pmatrix} \sigma(X_s^\varepsilon) B_{s,t}^1 + \nabla_x \sigma(X_s^\varepsilon) \sigma(X_s^\varepsilon) \langle B_{s,t}^2 \rangle \\ \varepsilon^{-1/2} h(X_s^\varepsilon, Y_s^\varepsilon) W_{s,t}^1 + \star \end{pmatrix},$$

where we set

$$(4.5) \quad \star = \varepsilon^{-1/2} \nabla_x h(X_s^\varepsilon, Y_s^\varepsilon) \sigma(X_s^\varepsilon) \langle I[B, W]_{s,t} \rangle + \varepsilon^{-1} \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle W_{s,t}^2 \rangle.$$

In above,  $\nabla_x$  and  $\nabla_y$  stands for the partial gradient operator in  $x$  and  $y$ , respectively. The precise meaning of  $\nabla_x \sigma(x) \sigma(x)$  was already essentially explained in (3.17). Also,  $\nabla_x h(x, y) \sigma(x)$  and  $\nabla_y h(x, y) h(x, y)$  should be understood in a similar way.

The left hand side of (4.4) is a summand in the modified Riemann sum that approximates  $\int_a^b \Sigma_\varepsilon(Z_s^\varepsilon) d\Xi_s$ ,  $(a, b) \in \Delta_\tau$ . The first component of the right hand side of (4.4) equals a summand in the modified Riemann sum that approximates  $\int_a^b \sigma(X_s^\varepsilon) dB_s$ . This verifies our assertion.  $\square$

Now we calculate the Itô-Stratonovich correction at a deterministic level. Set  $\text{Id}_e := \sum_{i=1}^e \mathbf{a}_i \otimes \mathbf{a}_i$  for an orthonormal basis  $\{\mathbf{a}_i\}_{i=1}^e$  of  $\mathbb{R}^e$ . Note that this definition is independent of the choice of  $\{\mathbf{a}_i\}$ . For  $\lambda \in \mathbb{R}$ , we set

$$\begin{aligned} \tilde{g}(x, y) &= g(x, y) - \lambda \nabla_y h(x, y) h(x, y) \langle \text{Id}_e \rangle \\ &= g(x, y) - \lambda \text{Trace}[\nabla_y h(x, y) h(x, y) \langle \bullet, \star \rangle] \end{aligned}$$

and  $\tilde{F}_\varepsilon(x, y) = (f(x, y), \varepsilon^{-1} \tilde{g}(x, y))^T$ . We also define  $\tilde{\Xi} = (\tilde{\Xi}^1, \tilde{\Xi}^2)$  by  $\tilde{\Xi}^1 = \Xi^1$  and

$$\tilde{\Xi}_{s,t}^2 = \begin{pmatrix} B_{s,t}^2 & I[B, W]_{s,t} \\ I[W, B]_{s,t} & \tilde{W}_{s,t}^2 \end{pmatrix}, \quad \text{where} \quad \tilde{W}_{s,t}^2 := W_{s,t}^2 + \lambda \text{Id}_e(t - s).$$

Since the definition of CP depends only on the first level of the reference RP,  $(Y, Y^\dagger)$  belongs to  $\mathcal{Q}_{\Xi}^\beta([a, b], \mathcal{W})$  if and only if it belongs to  $\mathcal{Q}_{\tilde{\Xi}}^\beta([a, b], \mathcal{W})$ .

**Lemma 4.3.** *Let the notation be as above and let  $\tau \in (0, T]$ . Then, the following are equivalent:*

- (1)  $(Z^\varepsilon, \Sigma_\varepsilon(Z^\varepsilon)) \in \mathcal{Q}_{\Xi}^\beta([0, \tau], \mathbb{R}^{m+n})$  and it solves RDE (4.2) on  $[0, \tau]$ .
- (2)  $(Z^\varepsilon, \Sigma_\varepsilon(Z^\varepsilon)) \in \mathcal{Q}_{\tilde{\Xi}}^\beta([0, \tau], \mathbb{R}^{m+n})$  and it solves the following RDE on  $[0, \tau]$ .

$$Z_t^\varepsilon = z_0 + \int_0^t \tilde{F}_\varepsilon(Z_s^\varepsilon) ds + \int_0^t \Sigma_\varepsilon(Z_s^\varepsilon) d\tilde{\Xi}_s, \quad (Z^\varepsilon)_t^\dagger = \Sigma(Z_t^\varepsilon).$$

*Proof.* We only prove (1) implies (2). (The proof of the converse is essentially the same.) The second term on the right hand side of (4.5) equals

$$\varepsilon^{-1} \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle \tilde{W}_{s,t}^2 \rangle + \varepsilon^{-1} \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle \text{Id}_e \rangle (t-s).$$

The second term above is a summand in the Riemann sum which approximates

$$\varepsilon^{-1} \int_0^\cdot \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle \text{Id}_e \rangle ds.$$

Thus, we have proved that (1) implies (2).  $\square$

**Remark 4.4.** If  $\tilde{W}^2$  and  $W^2$  are the second level paths of Stratonovich and Itô Brownian RP, respectively, then it is well-known that  $\tilde{W}_{s,t}^2 = W_{s,t}^2 + (1/2)\text{Id}_e(t-s)$  and Lemma 4.3 can be used with  $\lambda = 1/2$ . Therefore, we can also use Stratonovich Brownian RP to formulate the slow-fast system (2.1) of random RDEs. (Indeed, in the previous paper [25], the Stratonovich-type formulation is used.)

## 4.2 Probabilistic aspects

Let  $\frac{1}{3} < \alpha_0 \leq \frac{1}{2}$  and  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $w = (w_t)_{0 \leq t \leq T}$  be a standard  $e$ -dimensional Brownian motion and let  $B = \{(B_{s,t}^1, B_{s,t}^2)\}_{0 \leq s \leq t \leq T}$  be an  $\Omega_\alpha(\mathbb{R}^d)$ -valued random variable (i.e., random RP) defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  for every  $\alpha \in (1/3, \alpha_0)$ . We assume that  $w$  and  $B$  are independent. As for the integrability of  $B$ , Condition **(A)** is assumed. Let  $\{\mathcal{F}_t\}_{0 \leq t \leq T}$  be a filtration satisfying the usual condition as well as the following two conditions: (i)  $w$  is an  $\{\mathcal{F}_t\}$ -BM and (ii)  $t \mapsto (B_{0,t}^1, B_{0,t}^2)$  is  $\{\mathcal{F}_t\}$ -adapted.

Our random RP  $\Xi$  is defined in the following way. The symbols for the components of  $\Xi$  were set in (4.1).

**Definition 4.5.**  $B = (B^1, B^2)$  is given a priori in our assumptions. First, define  $W = (W^1, W^2)$  as the Itô-type Brownian RP, that is,

$$W_{s,t}^1 = w_t - w_s, \quad W_{s,t}^2 = \int_s^t (w_u - w_s) \otimes d^l w_u,$$



where  $d^I w$  stands for the standard Itô integral. It is well-known that  $W \in \Omega_{(1/2)-\delta}(\mathbb{R}^e)$  for every (sufficiently small)  $\delta > 0$ . Next, we set

$$(4.6) \quad I[B, W]_{s,t} = \int_s^t B_{s,u}^1 \otimes d^I w_u, \quad I[W, B]_{s,t} = W_{s,u}^1 \otimes B_{s,u}^1 - \int_s^t (d^I w_u) \otimes B_{s,u}^1.$$

Thus, we have set all components of  $\Xi = (\Xi^1, \Xi^2)$ .

**Lemma 4.6.** *Suppose  $\alpha \in (1/3, \alpha_0)$  and let  $\Xi$  be as in Definition 4.5. Under **(A)** it holds that  $\Xi \in \Omega_\alpha(\mathbb{R}^{d+e})$ , a.s. Moreover,  $\mathbb{E}[\|\Xi\|_\alpha^q] < \infty$  for every  $q \in [1, \infty)$ .*

*Proof.* We use a RP-version of Kolmogorov-Čentsov's continuity criterion (see [8, Theorem 3.1]). Take any  $\alpha' \in (\alpha, \alpha_0)$ . By the criterion it suffices to show that for every sufficiently large  $q \geq 1/\alpha'$ , there exists a constant  $C = C_q > 0$  such that

$$(4.7) \quad \mathbb{E}[|\Xi_{s,t}^1|^q]^{1/q} \leq C(t-s)^{\alpha'}, \quad \mathbb{E}[|\Xi_{s,t}^2|^{q/2}]^{2/q} \leq C(t-s)^{2\alpha'}, \quad (s, t) \in \Delta_T$$

holds. We will verify (4.7) componentwise.

Due to **(A)**,  $B = (B^1, B^2)$  clearly satisfies (4.7). It is well-known that Brownian RP  $W = (W^1, W^2)$  satisfies (4.7), too. Now, we estimate the  $(i, j)$ -component of  $I[B, W]$  ( $1 \leq i \leq d, 1 \leq j \leq e$ ). By Burkholder's inequality, we see that

$$\begin{aligned} \mathbb{E}\left[\left|\int_s^t B_{s,u}^{1,i} d^I w_u^j\right|^q\right] &\leq C_1 \mathbb{E}\left[\left|\int_s^t (B_{s,u}^{1,i})^2 du\right|^{q/2}\right] \\ &\leq C_1 \mathbb{E}[\|B^1\|_{\alpha'}^q] \left|\int_s^t (u-s)^{2\alpha'} du\right|^{q/2} \leq C_2 (u-s)^{(1+2\alpha')q/2} \end{aligned}$$

for certain positive constants  $C_1, C_2$  independent of  $(s, t)$ . Hence,  $I[B, W]$  satisfies (4.7). Due to (4.6), the proof for  $I[W, B]$  is essentially the same.  $\square$

In the sequel we work under **(A)** and assume that  $1/3 < \beta < \alpha < \alpha_0 (\leq 1/2)$ . For the rest of this section,  $\Xi$  is as in Definition 4.5. The precise meaning of the random RDE in our main theorem is RDE (4.2) driven by this  $\Xi$ .

We extend the time interval of the filtration  $\{\mathcal{F}_t\}$  by setting  $\mathcal{F}_t = \mathcal{F}_{t \wedge T}$  for  $t \geq 0$ . Denote by  $\hat{\mathbb{R}}^{m+n} := \mathbb{R}^{m+n} \cup \{\infty\}$  the one-point compactification of  $\mathbb{R}^{m+n}$ . If a global solution  $(Z_t^\varepsilon)_{t \in [0, T]}$  exists, then we set  $Z_t^\varepsilon = Z_{t \wedge T}^\varepsilon$  for  $t \geq 0$ . Otherwise, denote by  $(Z_t^\varepsilon)_{t \in [0, u^\varepsilon]}$ ,  $0 < u^\varepsilon \leq T$ , be a maximal local solution and set  $Z_t^\varepsilon = \infty$  for  $t \in [u^\varepsilon, \infty)$ . Either way,  $(Z_t^\varepsilon)$  is constant in  $t$  on  $[T, \infty)$ , a.s.

Define  $\tau_N^\varepsilon = \inf\{t \geq 0 \mid |Z_t^\varepsilon| \geq N\}$  for each  $N \in \mathbb{N}$  and  $\tau_\infty^\varepsilon = \lim_{N \rightarrow \infty} \tau_N^\varepsilon$ . (As usual  $\inf \emptyset := \infty$ .) These are  $\{\mathcal{F}_t\}$ -stopping times. Then, the following are equivalent:

- A global solution  $(Z_t^\varepsilon)_{t \in [0, T]}$  of RDE (4.2) exists.
- $(Z_t^\varepsilon)_{t \in [0, T]}$  defined as above is bounded in  $\mathbb{R}^{m+n}$ .
- $\tau_N^\varepsilon = \infty$  for some  $N$ .

- $\tau_\infty^\varepsilon > T$ .

It should be noted that while a solution of RDE (4.2) moves in a bounded set, its trajectory is uniformly continuous in  $t$  (because its Hölder norm is bounded). Hence, if  $(Z_t^\varepsilon)_{t \in [0, s]}$ ,  $0 < s \leq T$ , is bounded, then  $(Z_t^\varepsilon)_{t \in [0, s]}$  solves RDE (4.2).

On the other hand, if no global solution exists, then we have  $u^\varepsilon = \tau_\infty^\varepsilon \in (0, T]$  and  $\limsup_{t \nearrow \tau_\infty^\varepsilon} |Z_t| = \infty$ . Moreover,  $\lim_{t \nearrow \tau_\infty^\varepsilon} |Z_t| = \infty$  because of the uniform continuity mentioned above. Therefore,  $(Z_t^\varepsilon)_{t \geq 0}$  is a continuous process that takes values in  $\hat{\mathbb{R}}^{m+n}$ .

**Proposition 4.7.** *Let the notation be as above and assume (A). Then, for every  $\varepsilon \in (0, 1]$ ,  $Y^\varepsilon$  satisfies the following Itô SDE up to the explosion time of  $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$ :*

$$Y_t^\varepsilon = y_0 + \frac{1}{\varepsilon} \int_0^{t \wedge T} g(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^{t \wedge T} h(X_s^\varepsilon, Y_s^\varepsilon) d^l w_s, \quad 0 \leq t < \tau_\infty^\varepsilon.$$

*Proof.* Let  $\mathcal{P} = \{0 = t_0 < t_1 < \dots < t_K = t\}$  be a partition of  $[0, t]$  for  $0 < t \leq T$ . The summand of the Riemann sum for the RP integral in RDE (4.2) was given in (4.4) and (4.5).

First, we prove the lemma when  $h, \sigma$  are of  $C_b^3$  and  $f, g$  are bounded and globally Lipschitz continuous. In this case the solution never explodes, i.e.  $\tau_\infty^\varepsilon = \infty$ , a.s. It is easy to see that

$$(4.8) \quad \lim_{|\mathcal{P}| \searrow 0} \sum_{i=1}^K h(X_{t_{i-1}}^\varepsilon, Y_{t_{i-1}}^\varepsilon) W_{t_{i-1}, t_i}^1 = \int_0^t h(X_s^\varepsilon, Y_s^\varepsilon) d^l w_s \quad \text{in } L^2(\mathbb{P}).$$

Denote by  $\{\mathbf{e}_k\}_{k=1}^e$  and  $\{\mathbf{f}_p\}_{p=1}^n$  the canonical orthonormal basis of  $\mathbb{R}^e$  and  $\mathbb{R}^n$ , respectively. We write  $A_s^{k,l} := \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle \mathbf{e}_k \otimes \mathbf{e}_l \rangle$ ,  $A_s^{p;k,l} := \langle A_s^{k,l}, \mathbf{f}_p \rangle_{\mathbb{R}^n}$  and  $W_{s,t}^2 = \sum_{k,l=1}^e W_{s,t}^{2;k,l} \mathbf{e}_k \otimes \mathbf{e}_l$ . Then, we obviously see that the  $p$ th component of  $\nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon) \langle W_{s,t}^2 \rangle$  equals  $\sum_{k,l=1}^e A_s^{p;k,l} W_{s,t}^{2;k,l}$ . Noting that  $\mathbb{E}[W_{s,t}^{2;k,l} | \mathcal{F}_s] = 0$  for all  $k, l$  and  $s \leq t$ , we have for all  $i < j$  and  $p$  that

$$(4.9) \quad \begin{aligned} & \mathbb{E} \left[ \sum_{k,l=1}^e A_{t_{i-1}}^{p;k,l} W_{t_{i-1}, t_i}^{2;k,l} \cdot \sum_{k',l'=1}^e A_{t_{j-1}}^{p;k',l'} W_{t_{j-1}, t_j}^{2;k',l'} \right] \\ &= \sum_{k,l} \sum_{k',l'} \mathbb{E} [A_{t_{i-1}}^{p;k,l} W_{t_{i-1}, t_i}^{2;k,l} A_{t_{j-1}}^{p;k',l'} \mathbb{E}[W_{t_{j-1}, t_j}^{2;k',l'} | \mathcal{F}_{t_{j-1}}]] = 0. \end{aligned}$$

Using  $\mathbb{E}[(W_{s,t}^{2;k,l})^2] = (t-s)^2/2$ , we can easily see from (4.9) that

$$(4.10) \quad \begin{aligned} & \mathbb{E} \left[ \left| \sum_{i=1}^K \nabla_y h(X_{t_{i-1}}^\varepsilon, Y_{t_{i-1}}^\varepsilon) h(X_{t_{i-1}}^\varepsilon, Y_{t_{i-1}}^\varepsilon) \langle W_{t_{i-1}, t_i}^2 \rangle \right|^2 \right] \\ &= \sum_{p,k,l} \sum_{i=1}^K \mathbb{E} [ |A_{t_{i-1}}^{p;k,l} W_{t_{i-1}, t_i}^{2;k,l}|^2 ] \end{aligned}$$

$$\leq \frac{1}{2} \sum_{p,k,l} \sum_{i=1}^K \|h\|_\infty \|\nabla h\|_\infty (t_i - t_{i-1})^2 \leq \frac{1}{2} n e^2 \|h\|_\infty \|\nabla h\|_\infty |\mathcal{P}| \rightarrow 0$$

as the mesh size  $|\mathcal{P}|$  tends to 0.

In a similar way, we write  $\tilde{A}_s^{k,l} := \nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) \sigma(X_s^\varepsilon, Y_s^\varepsilon) \langle \tilde{\mathbf{e}}_k \otimes \mathbf{e}_l \rangle$ ,  $\tilde{A}_s^{p;k,l} := \langle \tilde{A}_s^{k,l}, \mathbf{f}_p \rangle_{\mathbb{R}^n}$  and  $I[B, W]_{s,t} = \sum_{k=1}^d \sum_{l=1}^e I[B, W]_{s,t}^{k,l} \tilde{\mathbf{e}}_k \otimes \mathbf{e}_l$ , where  $\{\tilde{\mathbf{e}}_k\}_{k=1}^d$  is the canonical orthonormal basis of  $\mathbb{R}^d$ . Then, the  $p$ th component of  $\nabla_y h(X_s^\varepsilon, Y_s^\varepsilon) \sigma(X_s^\varepsilon, Y_s^\varepsilon) \langle I[B, W]_{s,t} \rangle$  equals  $\sum_{k=1}^d \sum_{l=1}^e \tilde{A}_s^{p;k,l} I[B, W]_{s,t}^{k,l}$ . Since  $I[B, W]_{s,t}$  is an Itô integral,  $\mathbb{E}[I[B, W]_{s,t}^{k,l} | \mathcal{F}_s] = 0$  for all  $k, l$  and  $s \leq t$ . Hence, in the same way as in (4.9) we have

$$\mathbb{E} \left[ \sum_{k,l} \tilde{A}_{t_{i-1}}^{p;k,l} I[B, W]_{t_{i-1}, t_i}^{k,l} \cdot \sum_{k',l'} \tilde{A}_{t_{j-1}}^{p;k',l'} I[B, W]_{t_{j-1}, t_j}^{k',l'} \right] = 0$$

for all  $i < j$  and  $p$ . Since  $\mathbb{E}[(I[B, W]_{s,t}^{k,l})^2] = (t - s)^{2\alpha'+1} / (2\alpha' + 1)$ , we can calculate in the same way as in (4.10) to see that

$$\begin{aligned} (4.11) \quad & \mathbb{E} \left[ \left| \sum_{i=1}^K \nabla_y h(X_{t_{i-1}}^\varepsilon, Y_{t_{i-1}}^\varepsilon) \sigma(X_{t_{i-1}}^\varepsilon, Y_{t_{i-1}}^\varepsilon) \langle I[B, W]_{t_{i-1}, t_i} \rangle \right|^2 \right] \\ &= \sum_{p,k,l} \sum_{i=1}^K \mathbb{E} [|\tilde{A}_{t_{i-1}}^{p;k,l} I[B, W]_{t_{i-1}, t_i}^{k,l}|^2] \\ &\leq \sum_{p,k,l} \sum_{i=1}^K \|\sigma\|_\infty \|\nabla h\|_\infty \frac{(t_i - t_{i-1})^{2\alpha'+1}}{2\alpha' + 1} \leq \frac{n e^2}{2\alpha' + 1} \|\sigma\|_\infty \|\nabla h\|_\infty |\mathcal{P}|^{2\alpha'} \rightarrow 0 \end{aligned}$$

as the mesh size  $|\mathcal{P}|$  tends to 0.

Combining (4.8), (4.10) and (4.11), we have shown that

$$(4.12) \quad Y_t^\varepsilon = y_0 + \frac{1}{\varepsilon} \int_0^{t \wedge T} g(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^{t \wedge T} h(X_s^\varepsilon, Y_s^\varepsilon) d^l w_s, \quad t \geq 0$$

holds almost surely in this case.

From here only the standing assumption is assumed on the coefficients  $h, \sigma, f, g$ . Take any sufficiently large  $N$ . Let  $\phi_N: \mathbb{R}^{m+n} \rightarrow [0, 1]$  be a smooth function with compact support such that  $\phi_N \equiv 1$  on the ball  $\{z \in \mathbb{R}^{m+n} \mid |z| \leq N\}$  and set  $\hat{h} := h\phi_N$ . Also,  $\hat{\sigma}, \hat{f}, \hat{g}$  are defined in the same way. We replace the coefficients of RDE (4.2) by these corresponding data with “hat” and denote a unique solution by  $\hat{Z}^\varepsilon = (\hat{X}^\varepsilon, \hat{Y}^\varepsilon)$ . Then, (4.12) holds with  $\hat{X}^\varepsilon, \hat{Y}^\varepsilon, \hat{h}, \hat{g}$  in place of  $X^\varepsilon, Y^\varepsilon, h, g$ . By the uniqueness of the RDE, it holds that  $\hat{Z}_{t \wedge \tau_N^\varepsilon}^\varepsilon = Z_{t \wedge \tau_N^\varepsilon}^\varepsilon$  for all  $0 \leq t \leq T$ . Therefore, we almost surely have

$$\begin{aligned} (4.13) \quad & Y_{t \wedge \tau_N^\varepsilon \wedge T}^\varepsilon = \hat{Y}_{t \wedge \tau_N^\varepsilon \wedge T}^\varepsilon \\ &= y_0 + \frac{1}{\varepsilon} \int_0^{t \wedge \tau_N^\varepsilon \wedge T} \hat{g}(\hat{X}_s^\varepsilon, \hat{Y}_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^{t \wedge \tau_N^\varepsilon \wedge T} \hat{h}(\hat{X}_s^\varepsilon, \hat{Y}_s^\varepsilon) d^l w_s, \end{aligned}$$

$$= y_0 + \frac{1}{\varepsilon} \int_0^{t \wedge \tau_N^\varepsilon \wedge T} g(X_s^\varepsilon, Y_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^{t \wedge \tau_N^\varepsilon \wedge T} h(X_s^\varepsilon, Y_s^\varepsilon) d^I w_s, \quad t \geq 0.$$

Since  $\tau_N^\varepsilon \nearrow \tau_\infty^\varepsilon$  as  $N \rightarrow \infty$  a.s. on the set  $\{\tau_\infty^\varepsilon \leq T\}$ , we finish the proof by letting  $N \rightarrow \infty$ .  $\square$

**Proposition 4.8.** *Assume **(A)**, **(H1)**–**(H4)** and **(H6)**<sub>q</sub> for some  $q \geq 2$ . Then, the probability that  $Z^\varepsilon = (X^\varepsilon, Y^\varepsilon)$  explodes on  $[0, T]$  is zero. Moreover, we have*

$$(4.14) \quad \sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|X^\varepsilon\|_{\beta, [0, T]}^p] < \infty, \quad 1 \leq p < \infty,$$

$$(4.15) \quad \sup_{0 < \varepsilon \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^\varepsilon|^q] < \infty.$$

*Proof.* In this proof,  $c_i$  ( $i \geq 1$ ) denotes a positive constant independent of  $\varepsilon, N, s, t$  and the sample  $\omega \in \Omega$ .

Set  $\lambda_N^\varepsilon = \inf\{t \geq 0 \mid |Y_t^\varepsilon| \geq N\}$  for each  $N \in \mathbb{N}$  and  $\lambda_\infty^\varepsilon = \lim_{N \rightarrow \infty} \lambda_N^\varepsilon$ . Clearly,  $\lambda_N^\varepsilon \geq \tau_N^\varepsilon$ , a.s. We first show  $\lambda_\infty^\varepsilon = \tau_\infty^\varepsilon$ , a.s. (In other words,  $X^\varepsilon$  does not explode before  $Y^\varepsilon$  does.) Suppose that  $\lambda_\infty^\varepsilon > \tau_\infty^\varepsilon$  holds for some RP  $\Xi$ , which automatically implies  $\tau_\infty^\varepsilon \leq T$  and  $Y^\varepsilon$  stay bounded on  $[0, \tau_\infty^\varepsilon)$ . Due to Lemma 4.2 and Proposition 3.6,  $X^\varepsilon$  also stay bounded on  $[0, \tau_\infty^\varepsilon)$  and hence so does  $Z^\varepsilon$ . However, this is a contradiction.

Lemma 4.2 and Proposition 3.6 also imply the following. If  $\tau_\infty^\varepsilon \leq T$  (i.e.  $Z^\varepsilon$  explodes), then  $X^\varepsilon$  stay bounded on  $[0, \tau_\infty^\varepsilon)$  and hence  $\lim_{t \nearrow \tau_\infty^\varepsilon} |Y_t^\varepsilon| = \infty$ . In particular, there exists  $c_1, c_2 > 0$  such that  $\|X^\varepsilon\|_{\beta, [0, \tau_\infty^\varepsilon)} \leq c_1(\|B\|_\alpha^{c_2} + 1)$ .

For a while we use **(H6)**<sub>2</sub> instead of **(H6)**<sub>q</sub>. First, we prove non-explosion for any fixed  $\varepsilon$  by using Proposition 4.7, in particular (4.13). Lemma B.2 (with  $q = 2$ ) and **(H6)**<sub>2</sub> yield that

$$(4.16) \quad \begin{aligned} \mathbb{E}[|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^2] &= |y_0|^2 + \varepsilon^{-1} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} \{2\langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle + |h(X_s^\varepsilon, Y_s^\varepsilon)|^2\} ds \right] \\ &\leq |y_0|^2 + \varepsilon^{-1} \mathbb{E} \left[ \int_0^t \{-\gamma_1 |Y_s^\varepsilon|^2 + C |X_s^\varepsilon|^{\eta_3} + C\} \mathbf{1}_{\{s \leq \tau_N^\varepsilon\}} ds \right] \\ &\leq |y_0|^2 + \varepsilon^{-1} CT \{c_1^{\eta_3} \mathbb{E}[(\|B\|_\alpha^{c_2} + 1)^{\eta_3}] + 1\} \\ &\leq |y_0|^2 + \varepsilon^{-1} c_3, \quad t \in [0, T]. \end{aligned}$$

Since we stopped the time by  $\tau_N^\varepsilon$ , the martingale part in the Itô formula is a true martingale and its expectation vanishes. Recall that, as  $N \rightarrow \infty$ ,  $|Y_{T \wedge \tau_N^\varepsilon}^\varepsilon| \rightarrow \infty$  if  $\tau_\infty^\varepsilon \leq T$  and  $|Y_{T \wedge \tau_N^\varepsilon}^\varepsilon| \rightarrow |Y_T^\varepsilon|$  if  $\tau_\infty^\varepsilon = \infty$ . Applying Fatou's lemma to (4.16) with  $t = T$ , we see that  $\{\tau_\infty^\varepsilon \leq T\}$  is a zero set, i.e. explosion never occurs. From this (4.14) immediately follows. Again by Fatou's lemma,  $\sup_{t \leq T} \mathbb{E}[|Y_t^\varepsilon|^2] \leq |y_0|^2 + \varepsilon^{-1} c_3$  holds.

In a similar way as above, we use Lemma B.2 again to obtain that, for every  $p \geq 2$ ,

$$(4.17) \quad \begin{aligned} \mathbb{E}[|Y_t^\varepsilon|^p \mathbf{1}_{\{t \leq \tau_N^\varepsilon\}}] &\leq \mathbb{E}[|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^p] \\ &= |y_0|^p + \varepsilon^{-1} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} p \{|Y_s^\varepsilon|^{p-2} \langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle \right. \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2}|Y_s^\varepsilon|^{p-2}|h(X_s^\varepsilon, Y_s^\varepsilon)|^2 \\
& + \frac{p-2}{2}|Y_s^\varepsilon|^{p-4} \cdot (Y_s^\varepsilon)^\top h(X_s^\varepsilon, Y_s^\varepsilon)h(X_s^\varepsilon, Y_s^\varepsilon)^\top Y_s^\varepsilon \} ds \Big] \\
\leq & |y_0|^p + \varepsilon^{-1} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} \frac{p}{2}|Y_s^\varepsilon|^{p-2} \right. \\
& \quad \left. \times \{2\langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle + (p-1)|h(X_s^\varepsilon, Y_s^\varepsilon)|^2\} ds \right] \\
\leq & |y_0|^p + \varepsilon^{-1} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} c_4 |Y_s^\varepsilon|^{p-2} \{|Y_s^\varepsilon|^2 + |X_s^\varepsilon|^{\eta_3 \vee 2} + 1\} ds \right] \\
\leq & |y_0|^p + \varepsilon^{-1} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} c_5 \{|Y_s^\varepsilon|^p + |X_s^\varepsilon|^{(\eta_3 \vee 2)p/2} + 1\} ds \right] \\
\leq & |y_0|^p + \varepsilon^{-1} c_5 \int_0^t \mathbb{E}[|Y_s^\varepsilon|^p \mathbf{1}_{\{s \leq \tau_N^\varepsilon\}}] ds + \varepsilon^{-1} c_6, \quad t \in [0, T].
\end{aligned}$$

Here, we have also used **(H3)**, **(H6)**<sub>2</sub>, (4.14) and Young's inequality.

Using Lemma A.1 (1) (Gronwall's inequality), we have

$$\mathbb{E}[|Y_t^\varepsilon|^p \mathbf{1}_{\{t \leq \tau_N^\varepsilon\}}] \leq (|y_0|^p + \varepsilon^{-1} c_6) e^{\varepsilon^{-1} c_5 T}, \quad t \in [0, T].$$

Putting this inequality on the right hand side of (4.17), we have

$$\mathbb{E}[|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^p] \leq C_\varepsilon, \quad t \in [0, T],$$

where  $C_\varepsilon > 0$  is a constant which depends on  $\varepsilon$ , but not on  $N, t$ . Since  $p$  can be arbitrarily large, this implies that  $\{|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^p\}_{N=1}^\infty$  is uniformly integrable. Since  $|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon| \rightarrow |Y_t^\varepsilon|$  as  $N \rightarrow \infty$ , a.s., it follows that  $\mathbb{E}[|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^p] \rightarrow \mathbb{E}[|Y_t^\varepsilon|^p]$  as  $N \rightarrow \infty$  and

$$(4.18) \quad \sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^\varepsilon|^p] < \infty \quad \text{for every } \varepsilon \in (0, 1] \text{ and } 2 \leq p < \infty.$$

On the other hand, it immediately follows from **(H3)**, **(H4)** and **(H6)**<sub>2</sub> that

$$\begin{aligned}
& \left| |Y_s^\varepsilon|^{p-2} \langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle + \frac{1}{2}|Y_s^\varepsilon|^{p-2}|h(X_s^\varepsilon, Y_s^\varepsilon)|^2 \right. \\
& \quad \left. + \frac{p-2}{2}|Y_s^\varepsilon|^{p-4} \cdot (Y_s^\varepsilon)^\top h(X_s^\varepsilon, Y_s^\varepsilon)h(X_s^\varepsilon, Y_s^\varepsilon)^\top Y_s^\varepsilon \right| \mathbf{1}_{\{s \leq \tau_N^\varepsilon\}} \\
& \leq |Y_s^\varepsilon|^{p-1} C (|X_s^\varepsilon|^{\eta_1} + |Y_s^\varepsilon|^{\eta_1} + 1) + \frac{p-1}{2}|Y_s^\varepsilon|^{p-2} (|X_s^\varepsilon| + |Y_s^\varepsilon| + |h(0, 0)|)^2 \\
& \leq c_7 (|Y_s^\varepsilon|^{c_7} + |X_s^\varepsilon|^{c_7} + 1).
\end{aligned}$$

The right hand side is independent of  $N$  and integrable on  $\Omega \times [0, t]$  with respect to the measure  $\mathbb{P} \otimes ds$ , thanks to (4.18). Hence, we can use the dominated convergence theorem to (4.17) to obtain that

$$\mathbb{E}[|Y_t^\varepsilon|^p] = \lim_{N \rightarrow \infty} \mathbb{E}[|Y_{t \wedge \tau_N^\varepsilon}^\varepsilon|^p]$$

$$\begin{aligned}
&= |y_0|^p + \frac{1}{\varepsilon} \lim_{N \rightarrow \infty} \mathbb{E} \left[ \int_0^{t \wedge \tau_N^\varepsilon} \{p|Y_s^\varepsilon|^{p-2} \langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle + \frac{1}{2}|Y_s^\varepsilon|^{p-2} |h(X_s^\varepsilon, Y_s^\varepsilon)|^2 \right. \\
&\quad \left. + \frac{p-2}{2} |Y_s^\varepsilon|^{p-4} \cdot (Y_s^\varepsilon)^\top h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon)^\top Y_s^\varepsilon \} ds \right] \\
&= |y_0|^p + \frac{1}{\varepsilon} \mathbb{E} \left[ \int_0^t \{p|Y_s^\varepsilon|^{p-2} \langle Y_s^\varepsilon, g(X_s^\varepsilon, Y_s^\varepsilon) \rangle + \frac{1}{2}|Y_s^\varepsilon|^{p-2} |h(X_s^\varepsilon, Y_s^\varepsilon)|^2 \right. \\
&\quad \left. + \frac{p-2}{2} |Y_s^\varepsilon|^{p-4} \cdot (Y_s^\varepsilon)^\top h(X_s^\varepsilon, Y_s^\varepsilon) h(X_s^\varepsilon, Y_s^\varepsilon)^\top Y_s^\varepsilon \} ds \right].
\end{aligned}$$

After differentiating both sides with respect to  $t$ , we set  $p = q$  and use **(H6)** <sub>$q$</sub>  as follows:

$$\begin{aligned}
\frac{d}{dt} \mathbb{E}[|Y_t^\varepsilon|^q] &= \frac{1}{\varepsilon} \mathbb{E} \left[ q|Y_t^\varepsilon|^{q-2} \langle Y_t^\varepsilon, g(X_t^\varepsilon, Y_t^\varepsilon) \rangle + \frac{1}{2}|Y_t^\varepsilon|^{q-2} |h(X_t^\varepsilon, Y_t^\varepsilon)|^2 \right. \\
&\quad \left. + \frac{q-2}{2} |Y_t^\varepsilon|^{q-4} \cdot (Y_t^\varepsilon)^\top h(X_t^\varepsilon, Y_t^\varepsilon) h(X_t^\varepsilon, Y_t^\varepsilon)^\top Y_t^\varepsilon \right] \\
&\leq \frac{q}{2\varepsilon} \mathbb{E} [ |Y_t^\varepsilon|^{q-2} \{ 2 \langle Y_t^\varepsilon, g(X_t^\varepsilon, Y_t^\varepsilon) \rangle + (q-1) |h(X_t^\varepsilon, Y_t^\varepsilon)|^2 \} ] \\
&\leq \frac{q}{2\varepsilon} \mathbb{E} [ |Y_t^\varepsilon|^{q-2} \{ -\gamma_1 |Y_t^\varepsilon|^2 + C |X_t^\varepsilon|^{\eta_3} + C \} ] \\
&\leq -\frac{q\gamma_1}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon|^q] + \frac{c_8}{\varepsilon} \mathbb{E}[|X_t^\varepsilon|^{\eta_3 q/2} + 1] \\
&\leq -\frac{q\gamma_1}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon|^q] + \frac{c_9}{\varepsilon}, \quad t \in [0, T].
\end{aligned}$$

Here, we have also used Young's inequality and (4.14). We can now use Lemma A.1 (2) (the differential version of Gronwall's inequality) with  $b = -q\gamma_1/(4\varepsilon)$  and  $c = c_9/\varepsilon$  to obtain that

$$\mathbb{E}[|Y_t^\varepsilon|^q] \leq |y_0|^q e^{-tq\gamma_1/(4\varepsilon)} + \frac{4c_9}{q\gamma_1} (1 - e^{-tq\gamma_1/(4\varepsilon)}) \leq |y_0|^q + \frac{4c_9}{q\gamma_1}, \quad t \in [0, T].$$

Note that the right hand side is independent of  $\varepsilon$  as desired. Thus, we have obtained (4.15). This completes the proof of Proposition 4.8.  $\square$

## 5 Proof of main result

This section is devoted to proving our main result (Theorem 2.1).

First, we introduce a new parameter  $\delta$  with  $0 < \varepsilon < \delta \leq 1$ . (In spirit,  $0 < \varepsilon \ll \delta \ll 1$ . Later, we will set  $\delta := \varepsilon^{1/(4\beta)} \log \varepsilon^{-1}$ .) We divide  $[0, T]$  into subintervals of equal length  $\delta$ . For  $s \geq 0$ , set  $s(\delta) := \lfloor s/\delta \rfloor \delta$ , which is the nearest breaking point preceding (or equal to)  $s$ .

We set

$$(5.1) \quad \hat{Y}_t^\varepsilon = y_0 + \frac{1}{\varepsilon} \int_0^t g(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t h(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) d^1 w_s, \quad t \in [0, T].$$

Note that  $\hat{Y}^\varepsilon$ 's dependence on  $\delta$  is suppressed in the notation. This approximation process satisfies the following two estimates.

**Lemma 5.1.** *Under the same assumptions as in Proposition 4.8, we have the following: For every  $\delta$  and  $\varepsilon$  with  $0 < \varepsilon < \delta \leq 1$ , the above process  $\hat{Y}^\varepsilon$  does not explode and satisfies*

$$(5.2) \quad \sup_{0 < \varepsilon < \delta \leq 1} \sup_{0 \leq t \leq T} \mathbb{E}[|\hat{Y}_t^\varepsilon|^q] < \infty.$$

*Proof.* The proof is essentially the same as that of Proposition 4.8. (In fact, this one is easier because we already know  $X_{s(\delta)}^\varepsilon$  exists and satisfies the estimate (4.14).)  $\square$

**Lemma 5.2.** *Assume (A), (H1)–(H4), (H5)<sub>r</sub>, (H6)<sub>q</sub> and (H7) for some  $q \geq 2$  and  $r \geq 0$  such that  $q > 2r$ . Then, there exists a positive constant  $C$  independent of  $\delta$  such that*

$$\sup_{\varepsilon \in (0, \delta)} \sup_{0 \leq t \leq T} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] \leq C\delta^{2\beta}.$$

*Proof.* By Itô's formula, we can easily see that

$$\begin{aligned} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] &= \frac{2}{\varepsilon} \mathbb{E} \left[ \int_0^t \langle g(X_s^\varepsilon, Y_s^\varepsilon) - g(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon), Y_s^\varepsilon - \hat{Y}_s^\varepsilon \rangle ds \right] \\ &\quad + \frac{1}{\varepsilon} \mathbb{E} \left[ \int_0^t |h(X_s^\varepsilon, Y_s^\varepsilon) - h(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)|^2 ds \right]. \end{aligned}$$

Note that, due to (4.14), (4.18), (5.2) and (H3) the martingale part is actually a true martingale. (Therefore, stopping times are not needed here.)

By differentiating with respect to  $t$ , we get

$$\begin{aligned} \frac{d}{dt} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] &= \frac{2}{\varepsilon} \mathbb{E} \left[ \langle g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon), Y_t^\varepsilon - \hat{Y}_t^\varepsilon \rangle \right] \\ &\quad + \frac{1}{\varepsilon} \mathbb{E} \left[ |h(X_t^\varepsilon, Y_t^\varepsilon) - h(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon)|^2 \right] \\ &= \frac{1}{\varepsilon} \mathbb{E} \left[ \{ 2 \langle g(X_t^\varepsilon, Y_t^\varepsilon) - g(X_t^\varepsilon, \hat{Y}_t^\varepsilon), Y_t^\varepsilon - \hat{Y}_t^\varepsilon \rangle \right. \\ &\quad \left. + |h(X_t^\varepsilon, Y_t^\varepsilon) - h(X_t^\varepsilon, \hat{Y}_t^\varepsilon)|^2 \right] \\ &\quad + \frac{2}{\varepsilon} \mathbb{E} \left[ \langle g(X_t^\varepsilon, \hat{Y}_t^\varepsilon) - g(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon), Y_t^\varepsilon - \hat{Y}_t^\varepsilon \rangle \right] \\ &\quad + \frac{2}{\varepsilon} \mathbb{E} \left[ \langle h(X_t^\varepsilon, Y_t^\varepsilon) - h(X_t^\varepsilon, \hat{Y}_t^\varepsilon), h(X_t^\varepsilon, \hat{Y}_t^\varepsilon) - h(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon) \rangle \right] \\ &\quad + \frac{1}{\varepsilon} \mathbb{E} \left[ |h(X_t^\varepsilon, \hat{Y}_t^\varepsilon) - h(X_{t(\delta)}^\varepsilon, \hat{Y}_t^\varepsilon)|^2 \right] \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

We will estimate  $I_i$  ( $1 \leq i \leq 4$ ). Below  $c_i$  ( $i \geq 1$ ) are positive constants independent of  $t, \varepsilon, \delta$ . We will often use Young's inequality. It is clear from (H7) that

$$(5.3) \quad I_1 \leq -\frac{\gamma_2}{\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2].$$

Using **(H3)** and (4.14), we have

$$\begin{aligned}
(5.4) \quad I_3 + I_4 &\leq \frac{C_1}{\varepsilon} \text{Lip}(h)^2 \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon| \cdot |X_t^\varepsilon - X_{t(\delta)}^\varepsilon| + |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2] \\
&\leq \frac{\gamma_2}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_2}{\varepsilon} \mathbb{E}[|X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2] \\
&\leq \frac{\gamma_2}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_3}{\varepsilon} \delta^{2\beta},
\end{aligned}$$

where  $\text{Lip}(h)$  stands for the Lipschitz constant of  $h$ .

Next, using **(H5)**<sub>r</sub> with  $2r < q$ , Proposition 4.8 and Lemma 5.1, we estimate  $I_2$  as follows:

$$\begin{aligned}
(5.5) \quad I_2 &\leq \frac{2C}{\varepsilon} \mathbb{E}[(1 + \|X^\varepsilon\|_\infty^{\eta_2} + |\hat{Y}_t^\varepsilon|^r) |X_t^\varepsilon - X_{t(\delta)}^\varepsilon| \cdot |Y_t^\varepsilon - \hat{Y}_t^\varepsilon|] \\
&\leq \frac{\gamma_2}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_4}{\varepsilon} \mathbb{E}[\{(1 + \|X^\varepsilon\|_\infty)^{2\eta_2} + |\hat{Y}_t^\varepsilon|^{2r}\} |X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^2] \\
&\leq \frac{\gamma_2}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_4}{\varepsilon} \mathbb{E}[(1 + \|X^\varepsilon\|_\infty)^{4\eta_2}]^{1/2} \mathbb{E}[|X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^4]^{1/2} \\
&\quad + \frac{C_4}{\varepsilon} \mathbb{E}[|\hat{Y}_t^\varepsilon|^q]^{2r/q} \mathbb{E}[|X_t^\varepsilon - X_{t(\delta)}^\varepsilon|^{2q/(q-2r)}]^{(q-2r)/q} \\
&\leq \frac{\gamma_2}{4\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_5}{\varepsilon} \delta^{2\beta}.
\end{aligned}$$

Note that we also used Hölder's inequality above.

Combining (5.3)–(5.5), we see that

$$\frac{d}{dt} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] \leq -\frac{\gamma_2}{2\varepsilon} \mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] + \frac{C_6}{\varepsilon} \delta^{2\beta}, \quad t \in [0, T].$$

Applying the differential version of Gronwall's inequality (Lemma A.1 (2)) with  $f_0 = 0$ ,  $b = -\gamma_2/(2\varepsilon)$  and  $c = c_6\delta^{2\beta}/\varepsilon$ , we have

$$\mathbb{E}[|Y_t^\varepsilon - \hat{Y}_t^\varepsilon|^2] \leq \frac{c_6\delta^{2\beta}}{\gamma_2} \{1 - e^{-t\gamma_2/(2\varepsilon)}\} \leq c_7\delta^{2\beta}, \quad t \in [0, T].$$

This completes the proof of the lemma.  $\square$

It is easy to see that, if we define

$$\begin{aligned}
(5.6) \quad M_t &= \int_0^t \{f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon)\} ds + \int_0^t \{f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)\} ds \\
&\quad + \int_0^t \{f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{s(\delta)}^\varepsilon)\} ds + \int_0^t \{\bar{f}(X_{s(\delta)}^\varepsilon) - \bar{f}(X_s^\varepsilon)\} ds,
\end{aligned}$$

then

$$(5.7) \quad X_t^\varepsilon - \bar{X}_t = \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t \sigma(X_s^\varepsilon) dB_s - \int_0^t \bar{f}(\bar{X}_s) ds - \int_0^t \sigma(\bar{X}_s) dB_s$$



$$= M_t + \left( \int_0^t \{\bar{f}(X_s^\varepsilon) - \bar{f}(\bar{X}_s)\} ds + \int_0^t \{\sigma(X_s^\varepsilon) - \sigma(\bar{X}_s)\} dB_s \right)$$

holds as an equality of CPs with respect to  $B$ . We will apply Proposition 3.7 to (5.7). The sum of the first four terms on the right hand side of (5.7) corresponds to  $M$  in the proposition.

In the next lemma we estimate their Hölder norms.

**Lemma 5.3.** *Assume the same condition as in Lemma 5.2. Then, there exists a positive constant  $C$  independent of  $\varepsilon, \delta$  such that*

$$\begin{aligned} \mathbb{E} \left[ \left\| \int_0^\cdot \{f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon)\} ds \right\|_1^2 + \left\| \int_0^\cdot \{\bar{f}(X_{s(\delta)}^\varepsilon) - \bar{f}(X_s^\varepsilon)\} ds \right\|_1^2 \right. \\ \left. + \left\| \int_0^\cdot \{f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)\} ds \right\|_1^2 \right] \leq C\delta^{2\beta}. \end{aligned}$$

Here,  $\|\cdot\|_1$  stands for the 1-Hölder (i.e. Lipschitz) norm.

*Proof.* We use the globally Lipschitz property of  $f$ . The first term on the left hand side is dominated by

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^T |f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon)| ds \right)^2 \right] &\leq T \int_0^T \mathbb{E}[|f(X_s^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon)|^2] ds \\ &\leq T^2 \text{Lip}(f)^2 \sup_{0 \leq s \leq T} \mathbb{E}[|X_s^\varepsilon - X_{s(\delta)}^\varepsilon|^2] \\ &\leq T^2 \text{Lip}(f)^2 \mathbb{E}[\|X^\varepsilon\|_\beta^2] \delta^{2\beta} \leq C\delta^{2\beta}. \end{aligned}$$

Here, we used (4.14). Similarly, the third term is dominated by

$$\mathbb{E} \left[ \left( \int_0^T |f(X_{s(\delta)}^\varepsilon, Y_s^\varepsilon) - f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon)| ds \right)^2 \right] \leq T^2 \text{Lip}(f)^2 \sup_{0 \leq s \leq T} \mathbb{E}[|Y_s^\varepsilon - \hat{Y}_s^\varepsilon|^2] \leq C\delta^{2\beta}.$$

Here, we used Lemma 5.2.

The estimate of the second term is more difficult since  $\bar{f}$  may not be globally Lipschitz continuous. Since  $q \geq 2\eta_2$ , we see from Proposition C.5 that

$$|\bar{f}(X_{s(\delta)}^\varepsilon) - \bar{f}(X_s^\varepsilon)| \leq C' |X_{s(\delta)}^\varepsilon - X_s^\varepsilon| (1 + |X_{s(\delta)}^\varepsilon|^\xi + |X_s^\varepsilon|^\xi)$$

for certain positive constants  $C'$  and  $\xi$  which do not depend on  $\varepsilon, \delta, s$ . (Below, the value of  $C'$  may change from line to line.) Then, the second term is dominated by

$$\begin{aligned} \mathbb{E} \left[ \left( \int_0^T |\bar{f}(X_{s(\delta)}^\varepsilon) - \bar{f}(X_s^\varepsilon)| ds \right)^2 \right] &\leq T^2 C' \sup_{0 \leq s \leq T} \mathbb{E}[|X_{s(\delta)}^\varepsilon - X_s^\varepsilon|^2 (1 + |X_{s(\delta)}^\varepsilon|^\xi + |X_s^\varepsilon|^\xi)^2] \\ &\leq T^2 C' \mathbb{E}[\|X^\varepsilon\|_\beta^4]^{1/2} \delta^{2\beta} \sup_{0 \leq s \leq T} \mathbb{E}[1 + |X_s^\varepsilon|^{4\xi}] \leq C'\delta^{2\beta}. \end{aligned}$$

Here, we used (4.14) again. This completes the proof.  $\square$

**Lemma 5.4.** *Assume the same condition as in Lemma 5.2 and let  $0 < \gamma < 1$ . Then, there exists a positive constant  $C$  independent of  $\varepsilon, \delta$  such that*

$$\mathbb{E} \left[ \left\| \int_0^{\cdot} \{f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{s(\delta)}^\varepsilon)\} ds \right\|_\gamma^2 \right] \leq C(\delta^{2(1-\gamma)} + \delta^{-2\gamma}\varepsilon).$$

*Proof.* We write  $G_t = \int_0^t \{f(X_{s(\delta)}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{s(\delta)}^\varepsilon)\} ds$  and  $G_{s,t}^1 = G_t - G_s$  for simplicity. It suffices to prove the case  $1/2 < \gamma < 1$ . Below,  $c_i$  ( $i \geq 1$ ) are positive constants independent of  $s, t, \delta, \varepsilon, k$ .

First, consider the case  $0 < t - s \leq 2\delta$ . Since both  $f$  and  $\bar{f}$  are bounded, we can easily see that

$$(5.8) \quad |G_{s,t}^1| \leq (\|f\|_\infty + \|\bar{f}\|_\infty)(2\delta)^{1-\gamma}(t-s)^\gamma.$$

Next, consider the case  $t - s > 2\delta$ . In this case, there exists  $k \in \mathbb{N}$  such that  $[k\delta, (k+1)\delta] \subset [s, t]$ . In the following estimates, we suppose that  $s\delta^{-1}, t\delta^{-1} \notin \mathbb{N}$  (because the other cases are actually easier to be dealt with). Using Schwarz's inequality and noting that the number of subintervals that are contained in  $[s, t]$  does not exceed  $(t-s)/\delta$ , we have

$$\begin{aligned} |G_{s,t}^1|^2 &\leq \left| G_{s, ([s/\delta]+1)\delta}^1 + \sum_{k=[s/\delta]+1}^{\lfloor t/\delta \rfloor - 1} G_{k\delta, (k+1)\delta}^1 + G_{\lfloor t/\delta \rfloor \delta, t}^1 \right|^2 \\ &\leq c_1 \delta^2 + 2 \sum_{k=[s/\delta]+1}^{\lfloor t/\delta \rfloor - 1} 1^2 \times \sum_{k=[s/\delta]+1}^{\lfloor t/\delta \rfloor - 1} |G_{k\delta, (k+1)\delta}^1|^2 \\ &\leq c_1 \delta^2 + \frac{2(t-s)}{\delta} \sum_{k=0}^{\lfloor T/\delta \rfloor - 1} |G_{k\delta, (k+1)\delta}^1|^2 \end{aligned}$$

and therefore

$$(5.9) \quad \frac{|G_{s,t}^1|^2}{(t-s)^{2\gamma}} \leq c_2 \delta^{2(1-\gamma)} + c_2 \delta^{-2\gamma} \sum_{k=0}^{\lfloor T/\delta \rfloor - 1} |G_{k\delta, (k+1)\delta}^1|^2.$$

Combining (5.8) and (5.9), we obtain that

$$(5.10) \quad \mathbb{E}[\|G\|_\gamma^2] \leq c_2 \delta^{2(1-\gamma)} + c_2 T \delta^{-(1+2\gamma)} \max_{0 \leq k \leq \lfloor T/\delta \rfloor - 1} \mathbb{E} \left[ \left\| \int_{k\delta}^{(k+1)\delta} \{f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)\} ds \right\|_\gamma^2 \right].$$

As will be seen in (5.13) below, the expectation above is dominated by  $c_3 \varepsilon \delta$ . Hence, we have

$$\mathbb{E}[\|G\|_\gamma^2] \leq c_2 \delta^{2(1-\gamma)} + c_4 \delta^{-2\gamma} \varepsilon,$$

which is the desired estimate.

Now, using a result of the frozen SDE in Appendix C, we estimate the expectation on the right hand side of (5.10). Let  $\bar{w} = (\bar{w}_t)_{t \geq 0}$  be another standard  $e$ -dimensional BM which is independent of  $(B, w)$  and let  $(Y_t^{x,y})_{t \geq 0}$  be a unique solution of the frozen SDE (C.1) driven by  $(\bar{w}_t)_{t \geq 0}$ . As will be shown in Proposition C.6, it holds that

$$(5.11) \quad \mathbb{E} \left[ \left| \int_0^t \{f(x, Y_s^{x,y}) - \bar{f}(x)\} ds \right|^2 \right] \leq C'(1 + |x|^{\eta_3/2} + |y|)t, \quad (x, y) \in \mathbb{R}^{m+n}, t \geq 0$$

for some constant  $C' > 0$  independent of  $x, y, t$ . Recall that  $\eta_3 \geq 0$  is the constant in **(H6)**<sub>q</sub>.

We define  $\bar{Y}_t = Y_t^{x,y}|_{(x,y)=(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)}$ . (This means that  $(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)$  is plugged into the superscript of  $Y_t^{x,y}$ .) The starting point  $(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)$  and  $\bar{w}$  are independent. Then,  $(\bar{Y}_{t/\varepsilon})$  satisfies the following SDE:

$$(5.12) \quad \begin{aligned} \bar{Y}_{t/\varepsilon} &= \hat{Y}_{k\delta}^\varepsilon + \int_0^{t/\varepsilon} g(X_{k\delta}^\varepsilon, \bar{Y}_s) ds + \int_0^{t/\varepsilon} h(X_{k\delta}^\varepsilon, \bar{Y}_s) d^l \bar{w}_s \\ &= \hat{Y}_{k\delta}^\varepsilon + \frac{1}{\varepsilon} \int_0^t g(X_{k\delta}^\varepsilon, \bar{Y}_{s/\varepsilon}) ds + \frac{1}{\sqrt{\varepsilon}} \int_0^t h(X_{k\delta}^\varepsilon, \bar{Y}_{s/\varepsilon}) d^l (\sqrt{\varepsilon} \bar{w}_{s/\varepsilon}). \end{aligned}$$

Note that  $(\sqrt{\varepsilon} \bar{w}_{s/\varepsilon})_{t \geq 0}$  is again a standard  $e$ -dimensional BM independent of  $(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)$ . Compare SDE (5.12) with SDE (5.1). Then, we see that, restricted to the interval  $[0, \delta]$ ,  $(\hat{Y}_{k\delta+t}^\varepsilon)$  and  $(\bar{Y}_{t/\varepsilon})$  satisfy the same SDE with the same starting point (and the same frozen variable). Hence, by the uniqueness of law of the frozen SDE, it holds that

$$(X_{k\delta}^\varepsilon, (\hat{Y}_{k\delta+t}^\varepsilon)_{t \in [0, \delta]}) = (X_{k\delta}^\varepsilon, (\bar{Y}_{t/\varepsilon})_{t \in [0, \delta]}) \quad \text{in law.}$$

Hence, we have

$$\begin{aligned} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} \{f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)\} ds \right|^2 \right] &= \mathbb{E} \left[ \left| \int_0^\delta \{f(X_{k\delta}^\varepsilon, \bar{Y}_{s/\varepsilon}) - \bar{f}(X_{k\delta}^\varepsilon)\} ds \right|^2 \right] \\ &= \varepsilon^2 \mathbb{E} \left[ \left| \int_0^{\delta/\varepsilon} \{f(X_{k\delta}^\varepsilon, \bar{Y}_s) - \bar{f}(X_{k\delta}^\varepsilon)\} ds \right|^2 \right]. \end{aligned}$$

Under the conditional expectation  $\mathbb{E}[\cdot | \sigma\{X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon\}]$ ,  $\bar{w}$  is still a standard BM and  $(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)$  can be viewed as constants, i.e. non-random. (Here,  $\sigma\{X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon\}$  stands for the sub- $\sigma$ -field generated by  $(X_{k\delta}^\varepsilon, \hat{Y}_{k\delta}^\varepsilon)$ .) Hence, we can use (5.11) to see that the right hand side above is dominated by

$$\varepsilon^2 \mathbb{E}[C'(1 + |X_{k\delta}^\varepsilon|^{\eta_3/2} + |\hat{Y}_{k\delta}^\varepsilon|)(\delta/\varepsilon)] \leq c_3 \varepsilon \delta,$$

where Proposition 4.8 and Lemma 5.1 were used. Thus, we have seen that

$$(5.13) \quad \max_{0 \leq k \leq \lfloor T/\delta \rfloor - 1} \mathbb{E} \left[ \left| \int_{k\delta}^{(k+1)\delta} \{f(X_{k\delta}^\varepsilon, \hat{Y}_s^\varepsilon) - \bar{f}(X_{k\delta}^\varepsilon)\} ds \right|^2 \right] \leq c_3 \varepsilon \delta,$$

which completes the proof of the lemma.  $\square$

Now we prove our main theorem.

*Proof of Theorem 2.1.* Applying Proposition 3.7 to (5.6) and (5.7), we obtain that

$$\|X^\varepsilon - \bar{X}\|_\beta \leq C \exp(C\|B\|_\alpha^\nu) \|M\|_{2\beta}$$

for certain positive constant  $C$  and  $\nu$  which are independent of  $\varepsilon, \delta$ . (Below,  $C$  and  $\nu$  may vary from line to line.) By Lemmas 5.3 and 5.4, we have

$$\mathbb{E}[\|M\|_{2\beta}^2] \leq C(\delta^{2\beta} + \delta^{2(1-2\beta)} + \delta^{-4\beta}\varepsilon).$$

Therefore, if we set  $\delta := \varepsilon^{1/(4\beta)} \log \varepsilon^{-1}$  for example, then  $\|M\|_{2\beta}$  converges to 0 in  $L^2$ -sense as  $\varepsilon \searrow 0$ . It immediately follows that  $\|X^\varepsilon - \bar{X}\|_\beta^p$  converges to 0 in probability as  $\varepsilon \searrow 0$  for every  $p \in [1, \infty)$ .

On the other hand, we see from Proposition 3.6 that

$$\sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|X^\varepsilon - \bar{X}\|_\beta^p] \leq 2^{p-1} \sup_{0 < \varepsilon \leq 1} \mathbb{E}[\|X^\varepsilon\|_\beta^p] + 2^{p-1} \mathbb{E}[\|\bar{X}\|_\beta^p] \leq C(\mathbb{E}[\|B\|_\alpha^{\nu p}] + 1) < \infty$$

for every  $p \in [1, \infty)$ . This implies that  $\{\|X^\varepsilon - \bar{X}\|_\beta^p\}_{0 < \varepsilon \leq 1}$  are uniformly integrable for each fixed  $p$ . Hence, we have  $\mathbb{E}[\|X^\varepsilon - \bar{X}\|_\beta^p] \rightarrow 0$  as  $\varepsilon \searrow 0$ . This completes the proof of the main theorem.  $\square$

## A Gronwall's inequality

In this appendix we recall two versions of Gronwall's inequality. Note that in the differential version,  $b$  can be negative.

**Lemma A.1.** *Let  $T > 0$ .*

(1) *If a Lebesgue-integrable Borel measurable function  $f: [0, T] \rightarrow [0, \infty)$  satisfies for  $a, b \in [0, \infty)$  that*

$$f_t \leq a + b \int_0^t f_s ds, \quad t \in [0, T],$$

*then we have*

$$f_t \leq ae^{bt}, \quad t \in [0, T].$$

(2) *Let  $f: [0, T] \rightarrow \mathbb{R}$  be an absolutely continuous function. Suppose that for  $b, c \in \mathbb{R}$  we have*

$$f'_t \leq bf_t + c, \quad \text{for almost all } t \in [0, T].$$

*Then, we have*

$$f_t \leq (f_0 + \frac{c}{b})e^{bt} - \frac{c}{b}, \quad t \in [0, T].$$

*Proof.* (1) is well-known. See Kusuoka [18, p. 214] for example. We now prove (2). First, consider the case  $c = 0$ . For a.a.  $t$ , we have  $(f_t e^{-bt})' = (f'_t - bf_t)e^{-bt} \leq 0$ . By integrating, we have  $f_t e^{-bt} \leq f_0$  for every  $t$ . If  $c \neq 0$ , just set  $g_t = f_t + (c/b)$ . Then,  $g'_t \leq bg_t$  and hence  $g_t e^{-bt} \leq g_0$  for every  $t$ . This proves (2).  $\square$

## B A simple application of Itô's formula

In this appendix we calculate  $|Y_t|^q$ ,  $q \in [2, \infty)$ , for a multi-dimensional Itô process  $(Y_t)$  by using Itô's formula.

Set  $F(y) = |y|^q = \{(y^1)^2 + \dots + (y^n)^2\}^{q/2}$  for  $y \in \mathbb{R}^n$  and  $q \in [2, \infty)$ . It is well-known  $F \in C^2(\mathbb{R}^n, \mathbb{R})$ . First, we calculate its first and second order partial derivatives. We write  $\partial_i = \partial/\partial y^i$  for simplicity.

**Lemma B.1.** *Let  $F$  be as above. Then, for all  $1 \leq i, j \leq n$  and  $y \in \mathbb{R}^n$ , we have the following:*

$$\begin{aligned}\partial_i F(y) &= q|y|^{q-2}y^i, \\ \partial_i^2 F(y) &= q|y|^{q-2} + q(q-2)|y|^{q-4}(y^i)^2, \\ \partial_i \partial_j F(y) &= q(q-2)|y|^{q-4}y^i y^j \quad \text{for } i \neq j.\end{aligned}$$

(Note that the right hand sides are well-defined for every  $q \geq 2$  and  $y \in \mathbb{R}^n$ . When  $q > 2$ ,  $\partial_i \partial_j F(0) = 0$  for all  $(i, j)$ . When  $q = 2$ ,  $(q-2)$  times “any quantity” is understood to be 0.)

*Proof.* The proof is quite straightforward and therefore is omitted.  $\square$

Let  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{0 \leq t \leq T})$  be a filtered probability space satisfying the usual condition and  $(w_t)_{0 \leq t \leq T}$  be an  $e$ -dimensional standard  $\{\mathcal{F}_t\}$ -BM defined on it. Let  $(G_t)_{0 \leq t \leq T}$  and  $(H_t)_{0 \leq t \leq T}$  be progressively measurable processes which take values in  $\mathbb{R}^n$  and  $L(\mathbb{R}^e, \mathbb{R}^n)$ , respectively. ( $L(\mathbb{R}^e, \mathbb{R}^n)$  is the set of  $n \times e$  real matrices.) We assume that

$$\mathbb{P}\left(\int_0^T |G_s| ds < \infty\right) = 1 = \mathbb{P}\left(\int_0^T |H_s|^2 ds < \infty\right)$$

and define

$$Y_t = Y_0 + \int_0^t G_s ds + \int_0^t H_s dw_s, \quad t \in [0, T],$$

where  $Y_0$  is an  $\mathcal{F}_0$ -measurable  $\mathbb{R}^n$ -valued random variable. Equivalently, in the component form, it reads

$$Y_t^i = Y_0^i + \int_0^t G_s^i ds + \sum_{k=1}^e \int_0^t H_s^{ik} dw_s^k, \quad t \in [0, T], \quad 1 \leq i \leq n.$$

One can easily see that  $d\langle Y^i, Y^j \rangle_t = (H_t H_t^\top)^{ij} dt$  for all  $i, j$ . Here,  $H^\top$  stands for the transpose of  $H$  and  $(HH^\top)^{ij}$  is the  $(i, j)$ -component of the matrix  $HH^\top$ .

**Lemma B.2.** *Let  $(Y_t)_{0 \leq t \leq T}$  be as above and  $q \geq 2$ . Then, we almost surely have*

$$|Y_t|^q = |Y_0|^q + (\text{a local martingale})$$

$$\begin{aligned}
& + q \int_0^t \{ |Y_s|^{q-2} \langle Y_s, G_s \rangle + \frac{1}{2} |Y_s|^{q-2} |H_s|^2 + \frac{q-2}{2} |Y_s|^{q-4} (Y_s^\top H_s H_s^\top Y_s) \} ds \\
& \leq |Y_0|^q + (\text{a local martingale}) \\
& + \frac{q}{2} \int_0^t |Y_s|^{q-2} \{ 2 \langle Y_s, G_s \rangle + (q-1) |H_s|^2 \} ds, \quad t \in [0, T].
\end{aligned}$$

Here,  $|H_s|$  stands for the Hilbert-Schmidt norm of the matrix  $H_s$ .

*Proof.* Since  $0 \leq Y_s^\top H_s H_s^\top Y_s \leq |H_s|^2 |Y_s|^2$ , the inequality is trivial. Now we prove the equality by Itô's formula.

$$\begin{aligned}
dF(Y_t) &= \sum_i \partial_i F(Y_t) dY_t^i + \frac{1}{2} \sum_{i,j} \partial_i \partial_j F(Y_t) d\langle Y^i, Y^j \rangle_t \\
&= \sum_i q |Y_t|^{q-2} Y_t^i \sum_k H_t^{ik} dw_t^k + \sum_i q |Y_t|^{q-2} Y_t^i G_t^i dt \\
&\quad + \frac{1}{2} \sum_i q |Y_t|^{q-2} \sum_k (H_t^{ik})^2 dt + \frac{1}{2} \sum_{i,j} q(q-2) |Y_t|^{q-4} Y_t^i Y_t^j \sum_k H_t^{ik} H_t^{jk} dt.
\end{aligned}$$

The first term on the right hand side is clearly a local martingale. In the second term,  $\sum_i Y_t^i G_t^i = \langle Y_s, G_s \rangle$ . In the third term,  $\sum_{i,k} (H_t^{ik})^2 = |H_t|^2$ . In the fourth term,  $\sum_{i,k,j} Y_t^i H_t^{ik} H_t^{jk} Y_t^j = Y_t^\top H_t H_t^\top Y_t$ . Noting these, we complete the proof.  $\square$

## C Frozen SDE

In this appendix, we recall basic facts on the frozen SDE associated with the slow-fast system (2.1). We basically follow [23, Subsection 3.3]. Let  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$  be a filtered probability space satisfying the usual condition and  $(w_t)_{t \geq 0}$  be an  $e$ -dimensional standard  $\{\mathcal{F}_t\}$ -BM defined on it. (The time interval is  $[0, \infty)$ . The setting in this appendix is not exactly the same as that in the main part of this paper.) Since we are interested only in the law of the frozen SDE, the choice of  $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$  that supports  $(w_t)$  does not matter. In this appendix, we simply write  $dw_t$  for the standard Itô integral instead of  $d^l w_t$ .

For any given  $x \in \mathbb{R}^m$ , we consider the following  $\mathbb{R}^n$ -valued SDE on  $[0, \infty)$ :

$$(C.1) \quad dY_t^{x,y} = g(x, Y_t^{x,y}) dt + h(x, Y_t^{x,y}) dw_t, \quad Y_0^{x,y} = y \in \mathbb{R}^n.$$

We assume throughout this appendix that  $h \in C(\mathbb{R}^m \times \mathbb{R}^n, L(\mathbb{R}^e, \mathbb{R}^n))$  is globally Lipschitz continuous and  $g \in C(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^n)$  is locally Lipschitz continuous.

Under **(H6)**<sub>2</sub>, SDE (C.1) has a unique strong (global) solution  $(Y_t^{x,y})_{t \geq 0}$  for every  $x$  and  $y$ . (The proof of this fact is essentially the same as the corresponding part of Proposition 4.8.) For every  $x \in \mathbb{R}^m$ , the family of processes  $\{Y_t^{x,y}\}_{y \in \mathbb{R}^n}$  indexed by  $y$

becomes a diffusion process on  $\mathbb{R}^n$ . The corresponding semigroup is denoted by  $(P_t^x)_{t \geq 0}$ , that is,

$$P_t^x \varphi(y) := \mathbb{E}[\varphi(Y_t^{x,y})], \quad t \geq 0, y \in \mathbb{R}^n$$

for every bounded Borel-measurable function  $\varphi$  on  $\mathbb{R}^n$ . The next lemma (with  $q = 2$ ) and the standard Krylov-Bogoliubov argument yield the existence of an invariant probability measure for  $(P_t^x)_{t \geq 0}$  for every  $x$ .

**Lemma C.1.** *Assume  $(\mathbf{H6})_q$  for some  $q \geq 2$ . Then, there exists a positive constants  $C_q$  (which is independent of  $t, x, y$ ) such that*

$$\mathbb{E}[|Y_t^{x,y}|^q] \leq e^{-tq\gamma_1/4}|y|^q + C_q(1 + |x|^{q\eta_3/2}), \quad t \geq 0, x \in \mathbb{R}^m, y \in \mathbb{R}^n.$$

Here,  $\gamma_1 > 0, \eta_3 \geq 0$  are the constant in  $(\mathbf{H6})_q$ .

*Proof.* This is a special case of [23, Lemma 3.6]. Alternatively, we can prove this lemma by slightly modifying the proof of Proposition 4.8.  $\square$

**Lemma C.2.** *Assume  $(\mathbf{H6})_2$  and  $(\mathbf{H7})$ . Then, it holds that*

$$\mathbb{E}[|Y_t^{x,y_1} - Y_t^{x,y_2}|^2] \leq e^{-\gamma_2 t}|y_1 - y_2|^2, \quad t \geq 0, x \in \mathbb{R}^m, y_1, y_2 \in \mathbb{R}^n.$$

Here,  $\gamma_2 > 0$  is the constant in  $(\mathbf{H7})$ .

*Proof.* This is a special case of [23, Lemma 3.7]. Alternatively, we can prove this lemma by slightly modifying the proof of Proposition 4.8.  $\square$

**Lemma C.3.** *Assume  $(\mathbf{H6})_q$  for some  $q \geq 2$  and  $(\mathbf{H7})$ . Then, for every  $x \in \mathbb{R}^m$ , the semigroup  $(P_t^x)_{t \geq 0}$  has a unique invariant probability measure  $\mu^x$ . Furthermore, the following two estimates hold:*

(i) *There exists a positive constant  $C_q$  (which is independent of  $x$ ) such that*

$$\int_{\mathbb{R}^n} |z|^q \mu^x(dz) \leq C_q(1 + |x|^{q\eta_3/2}), \quad x \in \mathbb{R}^m.$$

Here,  $\eta_3 \geq 0$  is the constant in  $(\mathbf{H6})_q$ .

(ii) *There exists a positive constant  $C'$  such that for every  $t \geq 0, x \in \mathbb{R}^m, y \in \mathbb{R}^n$  and Lipschitz function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ ,*

$$\left| P_t^x \varphi(y) - \int_{\mathbb{R}^n} \varphi(z) \mu^x(dz) \right| \leq C' \text{Lip}(\varphi) e^{-\gamma_2 t/2} (1 + |x|^{\eta_3/2} + |y|).$$

Here,  $\text{Lip}(\varphi)$  is the Lipschitz constant of  $\varphi$  and  $\gamma_2 > 0$  is the constant in  $(\mathbf{H7})$ . The constant  $C'$  is independent of  $t, x, y, \varphi$ .

*Proof.* This is a special case of [23, Proposition 3.8].  $\square$

**Lemma C.4.** *Assume  $(\mathbf{H5})_r$  and  $(\mathbf{H6})_{2(r\vee 1)}$  for some  $r \geq 0$ . Then, there exist positive constants  $C$  and  $\xi$  independent of  $t, x_1, x_2, y$  such that*

$$\mathbb{E}[|Y_t^{x_1, y} - Y_t^{x_2, y}|^2] \leq C|x_1 - x_2|^2(1 + |x_1|^{2\xi} + |x_2|^{2\xi} + |y|^{2(r\vee 1)})$$

for all  $t \geq 0$ ,  $x_1, x_2 \in \mathbb{R}^m$  and  $y \in \mathbb{R}^n$ .

*Proof.* This is a special case of [23, Lemma 3.10].  $\square$

Under the assumptions of Lemma C.3, we set

$$\bar{f}(x) = \int_{\mathbb{R}^n} f(x, y) \mu^x(dy), \quad x \in \mathbb{R}^m$$

for  $f: \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}^m$  whenever the integral on the right hand side is well-defined.

**Proposition C.5.** *We assume  $(\mathbf{H2})$ ,  $(\mathbf{H5})_r$  and  $(\mathbf{H6})_{2(r\vee 1)}$  for some  $r \geq 0$ . Then, the map  $\bar{f}: \mathbb{R}^m \rightarrow \mathbb{R}^m$  defined as above is bounded and locally Lipschitz continuous. More precisely, we have  $\|\bar{f}\|_\infty \leq \|f\|_\infty$  and*

$$|\bar{f}(x_1) - \bar{f}(x_2)| \leq C|x_1 - x_2|(1 + |x_1|^\xi + |x_2|^\xi), \quad x_1, x_2 \in \mathbb{R}^m$$

for some constant  $C > 0$  independent of  $x_1, x_2$ . Here,  $\xi > 0$  is the constant which appears in Lemma C.4.

*Proof.* Clearly,  $\|\bar{f}\|_\infty \leq \|f\|_\infty$  holds. We now show the local Lipschitz property. For every Lipschitz function  $\varphi: \mathbb{R}^n \rightarrow \mathbb{R}$ , we see from Lemmas C.3 and C.4 that

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \varphi(y) \mu^{x_1}(dy) - \int_{\mathbb{R}^n} \varphi(y) \mu^{x_2}(dy) \right| &\leq \lim_{t \rightarrow \infty} |P_t^{x_1} \varphi(0) - P_t^{x_2} \varphi(0)| \\ &\leq \lim_{t \rightarrow \infty} |\mathbb{E}[\varphi(Y_t^{x_1, 0})] - \mathbb{E}[\varphi(Y_t^{x_2, 0})]| \\ &\leq \text{Lip}(\varphi) \limsup_{t \rightarrow \infty} \mathbb{E}[|Y_t^{x_1, 0} - Y_t^{x_2, 0}|] \\ &\leq \text{Lip}(\varphi) C|x_1 - x_2|(1 + |x_1|^\xi + |x_2|^\xi). \end{aligned}$$

Then, we have

$$\begin{aligned} |\bar{f}(x_1) - \bar{f}(x_2)| &\leq \int_{\mathbb{R}^n} |f(x_1, y) - f(x_2, y)| \mu^{x_1}(dy) \\ &\quad + \left| \int_{\mathbb{R}^n} f(x_2, y) \mu^{x_1}(dy) - \int_{\mathbb{R}^n} f(x_2, y) \mu^{x_2}(dy) \right| \\ &\leq \text{Lip}(f)|x_1 - x_2| + \text{Lip}(f)C|x_1 - x_2|(1 + |x_1|^\xi + |x_2|^\xi). \end{aligned}$$

This completes the proof of the proposition.  $\square$



**Proposition C.6.** *Assume **(H2)**, **(H7)** and **(H6)**<sub>q</sub> for some  $q \geq 2$ . Then, we have*

$$\mathbb{E} \left[ \left| \int_0^t \{f(x, Y_s^{x,y}) - \bar{f}(x)\} ds \right|^2 \right] \leq C(1 + |x|^{\eta_3/2} + |y|)t, \quad (x, y) \in \mathbb{R}^{m+n}, t \geq 0$$

for some constant  $C > 0$  independent of  $x, y, t$ . Here,  $\eta_3 \geq 0$  is the constant in **(H6)**<sub>q</sub>.

*Proof.* The expectation with respect to the law of  $Y^{x,y}$  is denoted by  $\hat{\mathbb{E}}^{x,y}$ . For  $s \leq u$ , we see from the Markov property that

$$\begin{aligned} & |\mathbb{E}[\langle f(x, Y_u^{x,y}) - \bar{f}(x), f(x, Y_s^{x,y}) - \bar{f}(x) \rangle]| \\ &= |\hat{\mathbb{E}}^{x,y}[\langle f(x, Y_u) - \bar{f}(x), f(x, Y_s) - \bar{f}(x) \rangle]| \\ &= |\hat{\mathbb{E}}^{x,y}[\langle \hat{\mathbb{E}}^{x,Y_s}[f(x, Y_{u-s}) - \bar{f}(x)], f(x, Y_s) - \bar{f}(x) \rangle]| \\ &\leq 2\|f\|_\infty \hat{\mathbb{E}}^{x,y} [|\hat{\mathbb{E}}^{x,Y_s}[f(x, Y_{u-s}) - \bar{f}(x)]|] \\ &\leq 2\|f\|_\infty \hat{\mathbb{E}}^{x,y} [C' \text{Lip}(f) e^{-\gamma_2(u-s)/2} (1 + |x|^{\eta_3/2} + |Y_s|)] \\ &\leq 2C' \|f\|_\infty \text{Lip}(f) e^{-\gamma_2(u-s)/2} \mathbb{E}[1 + |x|^{\eta_3/2} + |Y_s^{x,y}|] \\ &\leq C_1 e^{-\gamma_2(u-s)/2} (1 + |x|^{\eta_3/2} + |y|) \end{aligned}$$

for some constant  $C_1 > 0$  which is independent of  $x, y, t$ . Here, we used Lemma C.3 (ii) to the third to the last inequality and Lemma C.1 to the last inequality.

A simple application of Fubini's theorem yields that

$$\begin{aligned} & \mathbb{E} \left[ \left| \int_0^t \{f(x, Y_s^{x,y}) - \bar{f}(x)\} ds \right|^2 \right] \\ &= 2 \int_0^t ds \int_s^t du \mathbb{E}[\langle f(x, Y_u^{x,y}) - \bar{f}(x), f(x, Y_s^{x,y}) - \bar{f}(x) \rangle] \\ &\leq C_2 (1 + |x|^{\eta_3/2} + |y|) \int_0^t ds \int_s^t du e^{-\gamma_2(u-s)/2} \\ &\leq C_3 (1 + |x|^{\eta_3/2} + |y|)t \end{aligned}$$

for some constant  $C_3 > 0$  which is independent of  $x, y, t$ . □

## References

- [1] J. Bao, Q. Song, G. Yin and C. Yuan, Ergodicity and strong limit results for two-time-scale functional stochastic differential equations, *Stoch. Anal. Appl.* 35 (2017), no. 6, 1030–1046.
- [2] S. Bourguin, S. Gailus and K. Spiliopoulos, Typical dynamics and fluctuation analysis of slow-fast systems driven by fractional Brownian motion, *Stoch. Dyn.* 21 (2021), no. 7, Paper No. 2150030, 30 pp.

- [3] S. Bourguin, S. Gailus and K. Spiliopoulos, Discrete-time inference for slow-fast systems driven by fractional Brownian motion, *Multiscale Model. Simul.* 19 (2021), no. 3, 1333–1366.
- [4] S. Bourguin, T. Dang and K. Spiliopoulos, Moderate deviation principle for multiscale systems driven by fractional Brownian motion, Preprint (2022). arXiv:2206.06794.
- [5] S. Cerrai and M. Freidlin, Averaging principle for a class of stochastic reaction-diffusion equations, *Probab. Theory Related Fields* 144 (2009), no. 1-2, 137–177.
- [6] L. H. Duc, Controlled differential equations as rough integrals, Preprint (2020). arXiv: 2007.06295.
- [7] M. Freidlin and A. D. Wentzell, Random perturbations of dynamical systems, Translated from the 1979 Russian original by Joseph Szücs, Third edition, *Grundlehren der mathematischen Wissenschaften* 260, Springer, Heidelberg, 2012.
- [8] P. Friz and M. Hairer, A course on rough paths, Springer, Cham, 2014.
- [9] P. Friz and N. Victoir, Multidimensional stochastic processes as rough paths, Cambridge University Press, Cambridge, 2010.
- [10] D. Givon, Strong convergence rate for two-time-scale jump-diffusion stochastic differential systems, *Multiscale Model. Simul.* 6 (2007), no. 2, 577–594.
- [11] D. Givon, I. G. Kevrekidis and R. Kupferman, Strong convergence of projective integration schemes for singularly perturbed stochastic differential systems, *Commun. Math. Sci.* 4 (2006), no. 4, 707–729.
- [12] J. Golec, Stochastic averaging principle for systems with pathwise uniqueness, *Stochastic Anal. Appl.* 13 (1995), no. 3, 307–322.
- [13] J. Golec and G. Ladde, Averaging principle and systems of singularly perturbed stochastic differential equations, *J. Math. Phys.* 31 (1990), no. 5, 1116–1123.
- [14] M. Hairer and X. Li, Averaging dynamics driven by fractional Brownian motion, *Ann. Probab.* 48 (2020), no. 4, 1826–1860.
- [15] M. Han, Y. Xu, B. Pei and J.-L. Wu, Two-time-scale stochastic differential delay equations driven by multiplicative fractional Brownian noise: averaging principle, *J. Math. Anal. Appl.* 510 (2022), no. 2, Paper No. 126004, 31 pp.
- [16] J. Hu and C. Yuan, Strong convergence of neutral stochastic functional differential equations with two time-scales, *Discrete Contin. Dyn. Syst. Ser. B* 24 (2019), no. 11, 5831–5848.

- [17] R. Z. Khas'minskiĭ, On the principle of averaging the Itô's stochastic differential equations, *Kybernetika* 4 (1968), 260–279.
- [18] S. Kusuoka, *Stochastic analysis*, Springer, Singapore, 2020.
- [19] X. Li, Perturbation of conservation laws and averaging on manifolds, *Computation and combinatorics in dynamics, stochastics and control*, 499–550, *Abel Symp.* 13, Springer, Cham, 2018.
- [20] X. Li and J. Sieber, Slow-fast systems with fractional environment and dynamics, Preprint (2022). arXiv:2012.01910.
- [21] D. Liu, Strong convergence of principle of averaging for multiscale stochastic dynamical systems, *Commun. Math. Sci.* 8 (2010), no. 4, 999–1020.
- [22] D. Liu, Strong convergence rate of principle of averaging for jump-diffusion processes, *Front. Math. China* 7 (2012), no. 2, 305–320.
- [23] W. Liu, M. Röckner, X. Sun and Y. Xie, Averaging principle for slow-fast stochastic differential equations with time dependent locally Lipschitz coefficients, *J. Differential Equations* 268 (2020), no. 6, 2910–2948.
- [24] B. Pei, Y. Inahama and Y. Xu, Averaging Principles for Mixed Fast-Slow Systems Driven by Fractional Brownian Motion, To appear in *Kyoto J. Math.* (2023). arXiv: 2001.06945.
- [25] B. Pei, Y. Inahama and Y. Xu, Averaging principle for fast-slow system driven by mixed fractional Brownian rough path, *J. Differential Equations* 301 (2021), 202–235.
- [26] M. Röckner, X. Sun and Y. Xie, Strong and weak convergence in the averaging principle for SDEs with Hölder coefficients, Preprint (2019). arXiv: 1907.09256.
- [27] M. Röckner, X. Sun and Y. Xie, Strong convergence order for slow-fast McKean-Vlasov stochastic differential equations, *Ann. Inst. Henri Poincaré Probab. Stat.* 57 (2021), no. 1, 547–576.
- [28] X. Sun, L. Xie and Y. Xie, Strong and weak convergence rates for slow-fast stochastic differential equations driven by  $\alpha$ -stable process, Preprint (2021). arXiv: 2004.02595.
- [29] A. Yu. Veretennikov, On an averaging principle for systems of stochastic differential equations, *Math. USSR-Sb.* 69 (1991), no. 1, 271–284.
- [30] R. Wang, Y. Xu and H. Yue, Stochastic averaging for the non-autonomous mixed stochastic differential equations with locally Lipschitz coefficients, *Statist. Probab. Lett.* 182 (2022), Paper No. 109294, 11 pp.

- [31] F. Wu and G. Yin, An averaging principle for two-time-scale stochastic functional differential equations, *J. Differential Equations* 269 (2020), no. 1, 1037–1077.
- [32] F. Wu and G. Yin, Fast-slow-coupled stochastic functional differential equations, *J. Differential Equations* 323 (2022), 1–37.
- [33] J. Xu, J. Liu and Y. Miao, Strong averaging principle for two-time-scale SDEs with non-Lipschitz coefficients, *J. Math. Anal. Appl.* 468 (2018), no. 1, 116–140.
- [34] J. Xu, J. Liu, J. Liu and Y. Miao, Strong averaging principle for two-time-scale stochastic McKean-Vlasov equations, *Appl. Math. Optim.* 84 (2021), suppl. 1, S837–S867.
- [35] J. Xu and Y. Miao,  $L^p$  ( $p > 2$ )-strong convergence of an averaging principle for two-time-scales jump-diffusion stochastic differential equations, *Nonlinear Anal. Hybrid Syst.* 18 (2015), 33–47.
- [36] B. Zhang, H. Fu, L. Wan and J. Liu, Weak order in averaging principle for stochastic differential equations with jumps, *Adv. Difference Equ.* 2018, Paper No. 197, 20 pp.

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